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ECE C147 HW1

(1) Let A be a square matrix, and $AA^T = I$

i) Construct a 2×2 example of A.

Lets say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Then, $AA^T = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$

In order for this to be equal to $AA^T = I$, then:

$ac+bd = 0$

$a^2+b^2 = c^2+d^2 = 1$

Notice this looks a bit like trigonometry.

$a = \cos\theta \quad b = -\sin\theta$

$c = \sin\theta \quad d = \cos\theta$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Notice the reflection symmetry.

Eigenvalues:

$$A - \lambda I = \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$$

$$\det(|A - \lambda I|) = (\cos\theta - \lambda)^2 + \sin^2\theta$$
$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$
$$= \lambda^2 - 2\cos\theta\,\lambda + 1$$

$$|A - \lambda I| = 0 = \lambda^2 - 2\cos\theta\,\lambda + 1$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

Eigenvalues $\boxed{\lambda_0 = \cos\theta + j\sin\theta,\ \lambda_1 = \cos\theta - j\sin\theta}$

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Eigenvalues:

Notice the reflection symmetry.

$$A - \lambda I = \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$$

$$\det(|A-\lambda I|) = (\cos\theta - \lambda)^2 + \sin^2\theta$$
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Eigenvalues $\boxed{\lambda_0 = \cos\theta + j\sin\theta,\ \lambda_1 = \cos\theta - j\sin\theta}$

Eigenvectors. $A-\lambda_1 I = \begin{bmatrix} -j\sin\theta & -\sin\theta \\ \sin\theta & -j\sin\theta \end{bmatrix}$

$(A-\lambda I)v_1 = 0$

$\left[\begin{array}{cc|c} -j\sin\theta & -\sin\theta & 0 \\ \sin\theta & -j\sin\theta & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -j & 0 \\ 1 & -j & 0 \end{array}\right]$

Vector 1 $= v_1 \begin{bmatrix} 1 \\ -j \end{bmatrix}$

$A-\lambda_2 = \begin{bmatrix} +j\sin\theta & -\sin\theta \\ \sin\theta & +j\sin\theta \end{bmatrix}$

$(A-\lambda_2 I)v_2 = 0$

$\left[\begin{array}{cc|c} +j\sin\theta & \sin\theta & 0 \\ \sin\theta & +j\sin\theta & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} j & -1 & 0 \\ 1 & j & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & j & 0 \\ 1 & j & 0 \end{array}\right]$

vector 2 $= v_2 \begin{bmatrix} 1 \\ j \end{bmatrix}$

Eigenvectors Are: $\begin{bmatrix} 1 \\ -j \end{bmatrix}$ and $\begin{bmatrix} 1 \\ +j \end{bmatrix}$

$v_1 \cdot v_2^* = 1-1=0$, and so these eigenvectors are orthogonal, and seen to be reflections across the imaginary axis.

ii) $Av = \lambda v$

$||Av||^2 = ||\lambda v||$

$(Av)^T(Av) = |\lambda|^2 ||v||^2 = v^T A^T A v = v^T v = ||v||^2$

$\lambda^2 ||v||^2 = ||v||^2$

$\boxed{\text{so } |\lambda|^2 = 1}$

iii) Eigenvectors are orthogonal when $v_1^T v_2^* = 0$

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$

$Av_2^* = \lambda_2^* v_2^*$

$(Av_1)^T (Av_2^*) = (\lambda_1 v_1)^T (\lambda_2^* v_2^*)$

$v_1^T A^T A v_2^* = \lambda_1 \lambda_2^* v_1^T v_2^* = v_1^T v_2^*$; $v_1^T v_2^* I(1-\lambda_1\lambda_2^*) = 0$

λ_1 is not equal to λ_2, so $1-\lambda_1\lambda_2^* \neq 0$, $v_1^T v_2^* = 0$

Hence, v_1 and v_2 are orthogonal.

iv) The matrix A can be considered a rotation transformation matrix that will rotate x in the coordinate plane by θ.

b) let A be a matrix
$A = UDV^T$ (Decomposition)
$= VDU^T$ where V and D are orthogonal.

$AA^T = UDV^T VDU^T$
$= UD^2U^T$

$A^TA = VDU^T UDV$
$= VD^2V^T$

The form of AA^T and A^TA is the eigen decomposition where U has the eigenvectors of AA^T in its columns and v for A^TA.
Hence: the left singular vectors of A are the eigenvectors of AA^T
The right singular vectors of A are the eigenvectors of A^TA.

From $AA^T = UD^2U^T$ and $A^TA = VD^2V^T$, we notice that AA^T and AA^T have the same eigenvalues (the diagonal elements of D^2.)
So then the non-zero singular values of A are the square roots of the eigenvalues of AA^T and A^TA.

c) i) False, every linear operator has up to n distinct eigenvalues.

ii) False, wont be linearly independent.
iii) $x^TAx \geq 0 \rightarrow x_i \geq 0 \rightarrow$ True
iv) True, it's the number of linearly independent column vectors.
v) $A(\vec{v}_1 + \vec{v}_2) = \lambda \vec{v}_1 + \lambda \vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$
True

2 a) $P(H \mid H50) = 0.5$ $P(T \mid H50) = 0.5$
$P(H \mid H60) = 0.6$ $P(T \mid H60) = 0.4$

$$P(H50 \mid T) = \frac{P(T \mid H50)P(H50)}{P(T \mid H50)P(H50) + P(T \mid H60)P(H60)}$$
$$= \frac{0.5}{0.5 + 0.4} = \boxed{0.56 = \frac{0.5}{0.9}}$$

b) T H H H

$P(A|H50) = 0.5^4$

$P(A|H60) = 0.6^3 \cdot 0.4$

$$P(H50|A) = \frac{P(A|H50)P(H50)}{P(A|H50)P(H50) + P(A|H60)P(H60)}$$

$\boxed{= 0.4197}$

iii) $P(H50) = P(H55) = P(H60) = \frac{1}{3}$

$P(H|H50) = 0.5$ $\quad$ $P(A|H50) = 0.5^{10}$

$P(H|H55) = 0.55$ $\quad$ $P(A|H55) = 0.55^9 \cdot 0.45$

$P(H|H60) = 0.6$ $\quad$ $P(A|H60) = 0.6^9 \cdot 0.4$

$$\sum P(B|A_j)P(A_j) = P(A|H50)P(H50) + P(A|H55)P(H55) + P(A|H60) \cdot P(H60)$$
$$= 0.0024$$

$$P(H50|A) = \frac{P(A|H50)P(H50)}{0.0024} = 0.1379$$

$$P(H55|A) = \frac{P(A|H55)P(H55)}{0.0024} = 0.2927$$

$$P(H60|A) = \frac{P(A|H60)P(H60)}{0.0024}$$

6) Pregnancy test with following stats:

P(women pregnant | Positive test)

$P(B|A) = 0.99$, $P(B|A^c) = 0.1$

$P(A^c) = 0.99 \rightarrow P(A) = 0.01$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$
$$= \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.1 \cdot 0.99} = \boxed{0.0909}$$

If a woman has a positive test, there is a 9% chance she's pregnant, which makes sence since there's a false positive chance of 9.9%, and a true positive chance of 1% for any woman since most women are not pregnant.

c)

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$E(Ax+b) = \sum_x (Ax+b)P(x)$$

$$= \sum_x Ax\,p(x) + \sum_x b\,p(x)$$

$$= A\sum_x x\,p(x) + b\sum_x p(x)$$

$$E(Ax+b) = A\,E(x) + b$$

d) $$\operatorname{cov}(x) = E\left((x-E(x))(x-E(x))^T\right)$$

$$\operatorname{cov}(Ax+b) = E\left((Ax+b-E(Ax+b))(Ax+b-E(Ax+b))^T\right)$$

$$= E\left((Ax+b-AE(x)-b)(Ax+b-AE(x)-b)^T\right)$$

$$= E\left((Ax-AE(x))(Ax-AE(x))^T\right)$$

$$= E\left(A(x-E(x))(x-E(x))^T A^T\right)$$

$$= A\,E\left((x-E(x))(x-E(x))^T\right)A^T$$

$$\operatorname{cov}(Ax+b) = A\operatorname{cov}(x)A^T$$

Multivariable calculus

a) $\nabla_x x^T A y = \boxed{Ay}$

b) $\nabla_y x^T A y = \boxed{A^T x}$ → Let $B = x^T A = [B_1, B_2, \dots, B_m]$, $By = B \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$

then $\nabla_y By = \begin{bmatrix} \frac{\partial By}{\partial y_1} \\ \vdots \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} = B^T = (x^T A)^T = A^T x$

c) $\nabla_A x^T A y = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$

$= \begin{bmatrix} x_1 \cdots x_n \end{bmatrix} \begin{bmatrix} a_{11}y_1 + \cdots + a_{1m}y_m \\ \vdots \\ a_{n1}y_1 + \cdots + a_{nm}y_m \end{bmatrix}$

$\nabla_A x^T A y = \begin{bmatrix} \frac{\partial x^T A y}{\partial A_{11}} & \frac{\partial x^T A y}{\partial a_{1n}} \\ \frac{\partial x^T A y}{\partial a_{n1}} & \frac{\partial x^T A y}{\partial a_{nm}} \end{bmatrix} = \boxed{x y^T}$

d) $f = x^T A x + b^T x$, what is $\nabla_x f$?

$\nabla_x f = \nabla_x x^T A x + \nabla_x b^T x$

$= (A + A^T) x + b$

e) $f = tr(AB) = \sum_i (AB)_{i,i} = \sum_i \sum_j A_{i,j} B_{j,i}$

$\frac{\partial f}{\partial A_{i,j}} = B_{j,i}$

$\frac{\partial f}{\partial A} = B^T$

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4) $\hat{y} = Wx$

$$\min_W \frac{1}{2} \sum_{i=1}^{n} \|y^i - Wx^i\|^2$$

$$L = \frac{1}{2} \sum_{i=1}^{n} \|y^i - Wx^i\|^2$$

we know $\|A\|^2 = tr(AA^T)$

$$L = \frac{1}{2} \sum_{i=1}^{n} tr[(y^i - Wx^i)(y^i - Wx^i)^T]$$

$$= \frac{1}{2} tr((Y - WX)(Y - WX)^T)$$

$$= \frac{1}{2} tr((Y - WX)(Y^T - X^T W^T))$$

$$= \frac{1}{2} tr(YY^T) - \frac{1}{2} tr(YX^T W^T) - \frac{1}{2} tr(WXY^T) + \frac{1}{2} tr(WXX^T W^T)$$

$$\frac{\partial L}{\partial W} = -(XY^T)^T + \frac{1}{2} WXX^T + \frac{1}{2} WXX^T$$

$YX^T = WXX^T$

$W = \boxed{YX^T(XX^T)^{-1}}$