Kater's Pendulum:
An Easy Method to Accurately Estimate Acceleration due to Gravity

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We experimentally determine the local value of acceleration due to gravity, g, using Kater's pendulum. Kater's pendulum is a reversible free swinging pendulum that can be used to give an accurate estimate for g with relative ease. An equation relating the period of the pendulum to g is derived and an experimental value for the period of the pendulum is found. Our experiment found that $g = 9.801 \pm 0.015 \frac{m}{s^2}$. By taking averaging the period over 50 oscillations we were able to obtain a measure of g that deviates only $0.006 \frac{m}{s^2}$ from the conventional value of g.

I. INTRODUCTION

In this lab we used Kater's pendulum to experimentally find the value for acceleration due to gravity, g. Kater's pendulum is a reversible free swinging pendulum invented by British physicist Henry Kater in 1817 [1].

Kater's pendulum provided early physicists an instrument that would measure g with remarkable accuracy without requiring any modern advancements of science. Friedrich Bessel then laid the theoretical groundwork used in this experiment in 1826 which made the task of finding g even easier because it greatly reduced the number of measurements needed to estimate g [2]. It is a useful experiment for any physics student to perform because it tests their ability to gather accurate data and analyze the data properly. This is because in order to successfully obtain a good measure of g the student must take accurate measurements of distance and time and possess a solid understanding of the governing equations which should be consistent with the data.

II. THEORY

The period of a simple pendulum can be derived from its differential equation and by assuming that the amplitude of oscillation is small [3]. A simple pendulum is an ideal point mass, m, hanging a distance L from its axis of rotation.

The period of oscillation for a simple pendulum is:

$$T = 2\pi\sqrt{\frac{I}{mgL}} \tag{1}$$

Here, $I = mL^2$. Physical pendulums are not point masses however; their rotational inertia cannot be so simply described because their center of mass may not be easily located and the pendulum may not have a perfectly symmetric shape.

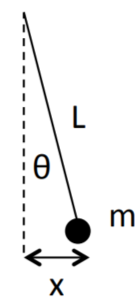

FIG. 1. A simple pendulum.

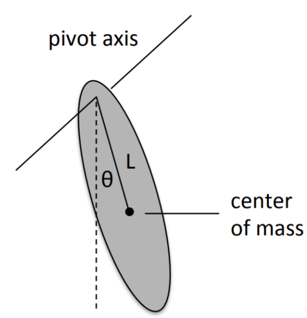

FIG. 2. A physical pendulum.

With good design, reasonable assumptions and clever mathematical thinking, the expression for the rotational inertia of a physical pendulum may be simplified and eventually ignored altogether. In this experiment for instance, Kater's pendulum allows us to neglect the burden of finding the pendulum's exact center of mass. To see why, we must examine the structure of Kater's pendulum in Fig. 3.

As can be seen from Fig. 3, natural symmetry is built into Kater's pendulum. The center of mass can be safely assumed to lie along a vertical axis regardless

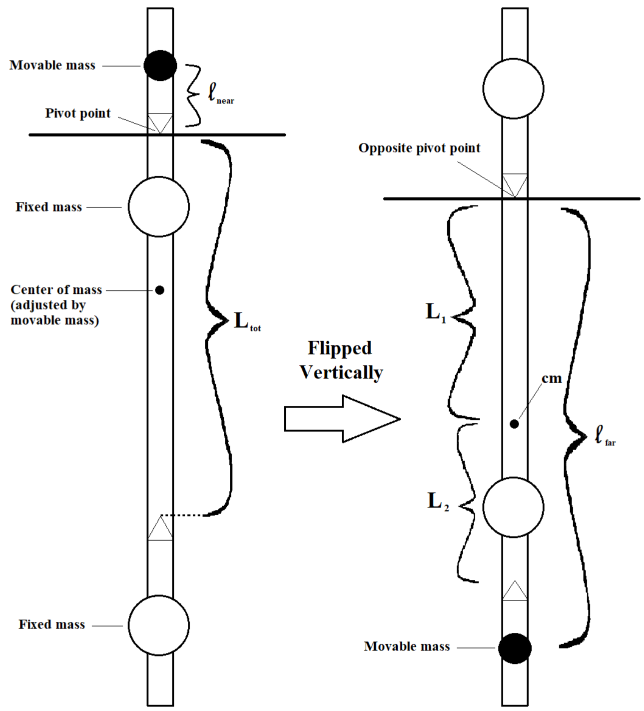

FIG. 3. Kater's Pendulum.

of the movable mass's position. The pivot points, rather than the center of mass, are where the axes of rotation lie. l_{near} and l_{far} are distances from the movable mass to whichever pivot is being used as the rotational axis. If we considered an axis directly though the center of mass then we can use the parallel axis theorem to express the rotational inertia when the axis of rotation is through either of the pivot points instead of the center of mass.

$$I = I_{cm} + mL^2 \tag{2}$$

I_{cm} is the amount of inertia about an axis running through the center of mass while mL^2 is a correction term for a parallel axis a distance L from the center of mass; this other axis is the actual axis about which the object is rotating. We can see these parallel axes are the two pivot points on Fig. 3. Now we replace I in (1) with (2) to get an expression for the period of Kater's pendulum.

$$T_{Kater} = 2\pi\sqrt{\frac{I_{cm} + mL^2}{mgL}}$$

Rearranging, we arrive at our starting equation to relate the period of Kater's pendulum to g. We get two equations, one for each pivot point which corresponds to a different L, named L_1 and L_2 in Fig. 3.

$$T_1 = 2\pi\sqrt{\frac{\frac{I_{cm}}{m} + L_1^2}{gL_1}} \tag{3}$$

and

$$T_2 = 2\pi\sqrt{\frac{\frac{I_{cm}}{m} + L_2^2}{gL_2}} \tag{4}$$

From here, we are ready to derive an expression for $g(T)$ which will be used in the experiment to find g from the collected data.

Derivation of $g(T)$

To start off, we use the fact that I_{cm} appears in both (3) and (4). First, we will isolate $I_{cm}(T, L)$ and then relate it to both T_1 and L_1 and T_2 and L_2. We get:

$$I_{cm}(T, L) = \frac{gmLT^2}{4\pi} - mL^2 \tag{5}$$

Now we set (5) expressed for each pivot point equal to one another and then solve for g – an algebraically tedious process.

$$\frac{gmL_1T_1^2}{4\pi} - mL_1^2 = \frac{gmL_2T_2^2}{4\pi} - mL_2^2$$

$$\frac{g}{4\pi^2}(L_1T_1^2 - L_2T_2^2) = L_1^2 - L_2^2$$

$$L_1^2 - L_2^2 = (L_1 + L_2)(L_1 - L_2)$$

$$\frac{4\pi^2}{g} = \frac{L_1T_1^2 - L_2T_2^2}{(L_1 + L_2)(L_1 - L_2)}$$

$$= \frac{(L_1 - L_2)(T_1^2 + T_2^2) + (L_1 + L_2)(T_1^2 - T_2^2)}{2(L_1 + L_2)(L_1 - L_2)}$$

$$= \frac{T_1^2 + T_2^2}{2(L_1 + L_2)} + \frac{T_1^2 - T_2^2}{2(L_1 - L_2)}$$

$$g = 8\pi^2 \left[\frac{T_1^2 + T_2^2}{L_1 + L_2} + \frac{T_1^2 - T_2^2}{L_1 - L_2}\right]^{-1} \tag{6}$$

From Fig. 3 we observe that,

$$L_{tot} \equiv L_1 + L_2 \tag{7}$$

Substituting (7) into (6) gives,

$$g = 8\pi^2 \left[\frac{T_1^2 + T_2^2}{L_{tot}} + \frac{T_1^2 - T_2^2}{L_1 - L_2}\right]^{-1} \tag{8}$$

Now comes the genius of Kater's pendulum: if we can find an L_1 and L_2 such that (3) is equal to (4) (or at least very near in value) then (8) may be further simplified because $T_1 = T_2$ causes the second term inside the bracket $(T_1^2 - T_2^2)$ to disappear (or become negligibly small). This would give,

$$g = 4\pi^2 \frac{L_{tot}}{T^2} \tag{9}$$

Here T is that special period such that (3) is equal to (4). This condition will only occur at specific L_1 and L_2 lengths but if we can find those points and the associated period there then we can find g. This final theoretical result guided our experimental approach.

III. METHODS

We measured L_{tot} and timed the period, T, of the pendulum at various lengths, twice at each length because the pendulum is flipped to use each pivot point as a point of rotation. Each time we moved the movable mass by a measured amount, we measured T for both orientations pictured in Fig. 3. We measured l_{near} instead of L_1 or L_2 because those lengths depend on the location of the center of mass which we did not know. We planned to then plot period measurements against their corresponding l_{near} measurements. This plot would depict the data for both pivot points and a spline fitted line would follow the data associated with each pivot point. We found where these fitted lines crossed by inspection.

Points of intersection would correspond to lengths of L_1 and L_2 where $T_1 = T_2$. This would allow us to legitimately use (9) instead of (8) to find g. We used a ruler to measure L_{tot} and a stop watch to time 50 periods to get an average of T_1 and T_2.

IV. RESULTS

As expected, our results gave an exceptionally accurate estimate of g with only a pendulum, stopwatch and a ruler. All plots were rendered in MATLAB. Fig. 4 is a plot of T_1 and T_2 against l_{near} (refer to Fig. 3 to recall how l_{near} is defined).

Strictly following the spline fit of each curve, we observed two points of intersection between T_1 and T_2. Because the spline fit does not indicate exactly how the curves behave (since it is based on a small sample of 10 data points), we averaged these two points of intersection to find an estimate of the T in (9). Thus, we treated the region between these two points of intersection of l_{near} and l_{far} as a granular image of a single point of intersection between the two lines. In theory, even if multiple points of intersection are possible they should each yield the same T value because g is a constant.

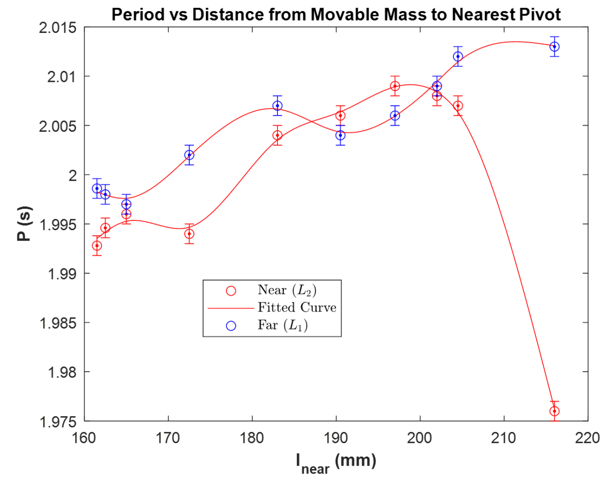

FIG. 4. Plot of two curves following T_1 and T_2 respectively.

From the left, we encounter the first and then the second point of intersection as shown in Figures 5 and 6.

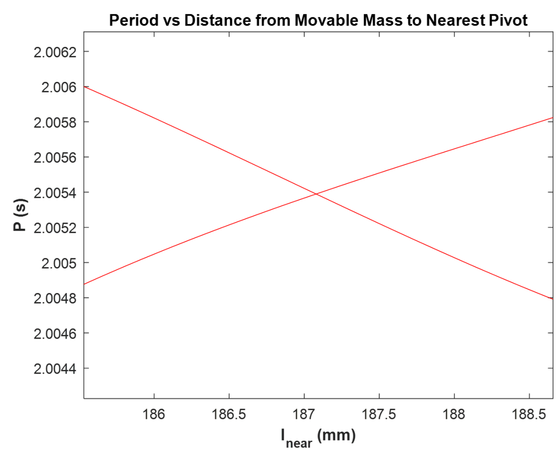

FIG. 5. Magnified image of first intersection point.

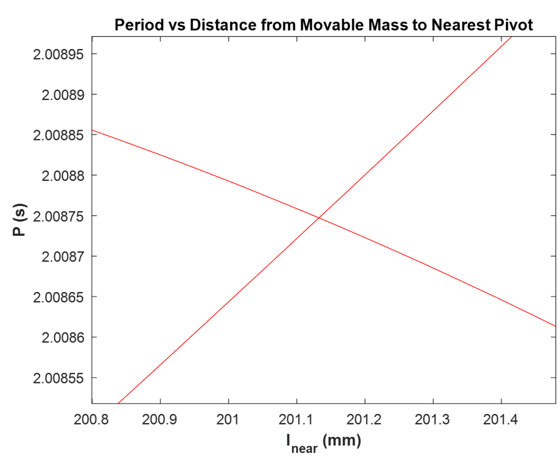

FIG. 6. Magnified image of second intersection point.

The average T between these two points of intersection was found to be $2.0071 \pm 0.001s$. Using (9) we calculated the associated g to be $9.801 \pm 0.015 \frac{m}{s^2}$. Uncertainty in g was calculated using (10) which is the error propagation formula.

$$\sigma_g = \pm \sqrt{\left(\frac{\partial g}{\partial T}\,\sigma_T\right)^2 + \left(\frac{\partial g}{\partial L_{tot}}\,\sigma_{L_{tot}}\right)^2} \qquad (10)$$

V. CONCLUSION

In this experiment we determined the local value of gravitational acceleration using Kater's pendulum and found an estimate very close to the conventional value of g. Our experimental value of g was $9.801 \frac{m}{s^2}$ while the conventional value is $9.807 \frac{m}{s^2}$ – a mere $0.006 \frac{m}{s^2}$ difference or 0.006% discrepancy which may simply be attributed to air resistance and local variance of g [4].

Initially, when the two points of intersection on Fig. 4 were observed, it was attempted to ascertain whether two pairs of L_1 and L_2 could give two or more points of equality between T_1 and T_2. However, this lead to a complicated analysis of either (8) which could be reduced to a function of three variables at minimum after substituting (7) into (8) or by comparing (3) to (4) and again substituting (7) into the either (3) or (4). The second approach seems to be a more efficient one but invariably requires knowledge of what I_{cm} is since I_{cm} varies with L_1 and L_2. While an interesting theoretical study, determining I_{cm} would be a laborious task because it requires knowledge of where the center of mass is and contradicts one of the intended purposes of Kater's pendulum, namely to find g without the need to find the center of mass of the pendulum.

Our findings reveal that the experiment was successfully performed and that for most applications, only a Kater's pendulum, stopwatch and measuring stick or tape is needed to determine a sufficiently accurate local value of g. It also reveals the impressive accuracy with which 19th century physicists were able to calculate g with ease and reproducibility.

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