SPIRALS

HISTORY: The investigation of Spirals began at least with the ancient Greeks. The famous Equiangular Spiral was discovered by Descartes, its properties of self-reproduction by James (Jacob) Bernoulli (1654-1705) who requested that the curve be engraved upon his tomb with the phrase "Eadem mutata resurgo" ("I shall arise the same, though changed").*

1. EQUIANGULAR SPIRAL: $\boxed{r = a \cdot e^{\theta \cdot \cot \alpha}}$. (Also called <u>Logarithmic</u> from an equivalent form of its equation.) Discovered by Descartes in 1638 in a study of dynamics.

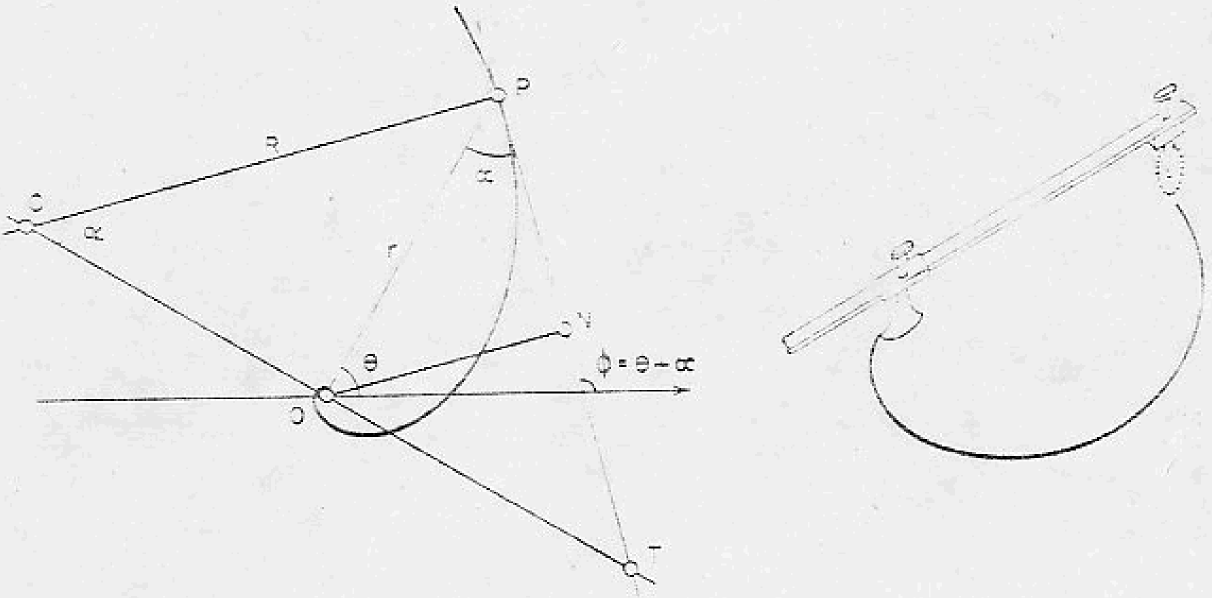

Fig. 183

(a) The curve cuts all radii vectores at a <u>constant angle</u> α. ($\frac{r}{r'} = \tan \alpha$).

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*Lietzman, W.: Lustiges und Merkwurdiges von Zahlen und Formen, p. 40, gives a picture of the tombstone.

(b) Curvature: Since $p = r \cdot \sin\alpha$, $R = r \cdot \frac{dr}{dp} = r \cdot \csc\alpha = CP$ (the polar normal). $R = s \cdot \cot\alpha$.

(c) Arc Length: $\frac{dr}{ds} = \left(\frac{dr}{d\theta}\right)\left(\frac{d\theta}{ds}\right) = (r \cdot \cot\alpha)\left(\frac{\sin\alpha}{r}\right) = \cos\alpha$, and thus $s = r \cdot \sec\alpha = PT$, where s is measured from the point where $r = 0$. Thus, the arc length is equal to the polar tangent (Descartes).

(d) Its pedal and thus all successive pedals with respect to the pole are equal Equiangular Spirals.

(e) Evolute: PC is tangent to the evolute at C and angle $PCO = \alpha$. OC is the radius vector of C. Thus the first and all successive evolutes are equal Equiangular Spirals.

(f) Its inverse with respect to the pole is an Equiangular Spiral.

(g) It is, Fig. 184, the stereographic projection ($x = k \tan\frac{\varphi}{2}\cos\theta$, $y = k \tan\frac{\varphi}{2}\cdot\sin\theta$) of a Loxodrome (the curve cutting all meridians at a constant angle: the course of a ship holding a fixed direction on the compass), from one of its poles onto the equator (Halley 1696).

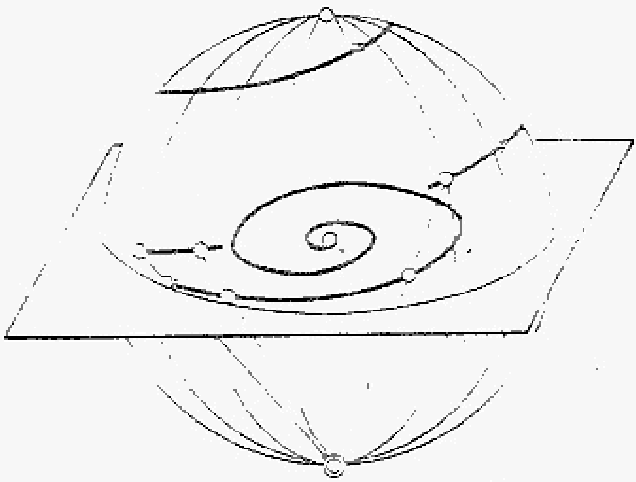


Fig. 184

(h) Its Catacaustic and Diacaustic with the light source at the pole are Equiangular Spirals.

(i) Lengths of radii drawn at equal angles to each other form a geometric progression.

(j) Roulette: If the spiral be rolled along a line, the path of the pole, or of the center of curvature of the point of contact, is a straight line.

Fig. 185

(k) <u>The septa of the Nautilus are Equiangular Spirals</u>. The curve seems also to appear in the arrangement of seeds in the sunflower, the formation of pine cones, and other growths.

(l) <u>The limit of a succession of Involutes of any given curve is an Equiangular Spiral</u>.

Let the given curve be $\sigma = f(\theta)$ and denote by s_n the arc length of an nth involute. Then all first involutes are given by

$$s_1 = \int_0^\theta (c + f)d\theta = c\theta + \int_0^\theta f(\theta)d\theta,$$

where <u>c</u> represents the distance measured along the tangent to the given curve. Selecting a particular value for <u>c</u> for all successive involutes:

$$s_2 = \int_0^\theta [c + c\theta + \int_0^\theta f(\theta)d\theta]\, d\theta$$

$$\vdots$$

$$s_n = c\theta + c\theta^2/2! + c\theta^3/3! + \ldots + [\int_0^\theta f(\theta)d\theta]^{nth},$$

where this nth iterated integral may be shown to approach zero. (See Byerly.) Accordingly,

$$s = \lim_{n \to \infty} s_n = c\left(\theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots + \frac{\theta^n}{n!} + \dots\right)$$

or $$\boxed{s = c(e^\theta - 1)} ,$$

an Equiangular Spiral.

(m) It is the development of a Conical Helix (See Spiral of Archimedes.)

2. THE SPIRALS: $\boxed{r = a\theta^n}$ include as special cases the following: $\boxed{n = 1}$: $\boxed{r = a\theta}$ Archimedean (due to Conan but studied particularly by Archimedes in a tract still extant. He probably used it to square the circle).

(a) Its polar subnormal is constant.

(b) Arc length from O to θ: $s = \frac{a}{2}[\theta\sqrt{1+\theta^2} + \ln(\theta+\sqrt{1+\theta^2})]$ (Archimedes).

(c) $A = \frac{r^3}{6a}$. (from $\theta = 0$ to $\theta = r/a$).

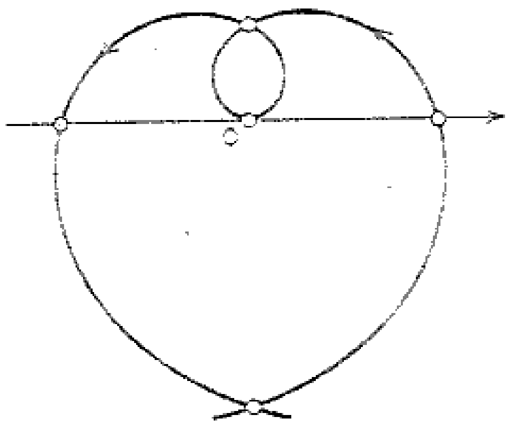
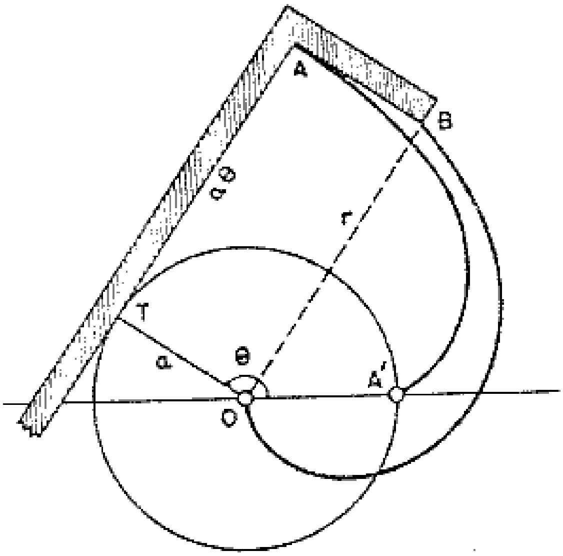
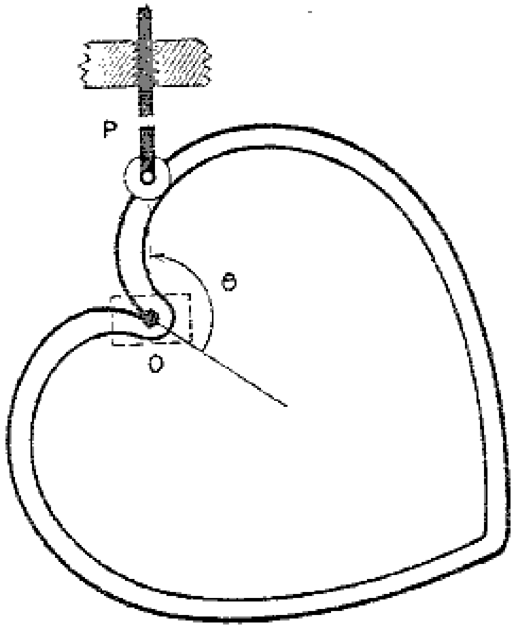


Fig. 186

(d) It is the Pedal of the Involute of a Circle with respect to its center. This suggests the description by a carpenter's square rolling without slipping upon a circle, Fig. 187(a). Here OT = AB = a. Let A start at A', B at O. Then AT = arc A'T = r = aθ. Thus B describes the Spiral of Archimedes while A traces an Involute of the Circle. Note that the center of rotation is T. Thus TA and TB, respectively, are normals to the paths of A and B.





(a) Fig. 187 (b)

(e) Since $r = a\theta$ and $\dot{r} = a\dot{\theta}$, this spiral has found wide use as a cam, Fig. 187(b) to produce uniform linear motion. The cam is pivoted at the pole and rotated with constant angular velocity. The piston, kept in contact with a spring device, has uniform reciprocating motion.

(f) It is the <u>Inverse</u> of a <u>Reciprocal</u> <u>Spiral</u> with respect to the Pole.

(g) "The <u>casings</u> <u>of</u> <u>centrifugal</u> <u>pumps</u>, such as the German supercharger, follow this spiral to allow air which increases uniformly in volume with each degree of rotation of the fan blades to be conducted to the outlet without creating back-pressure." - P. S. Jones, 18th Yearbook, N.C.T.M. (1945) 219.

(h) The orthographic projection of a Conical Helix on a plane perpendicular to its axis is a Spiral of Archimedes. The development of this Helix, however, is an Equiangular Spiral (Fig. 188).

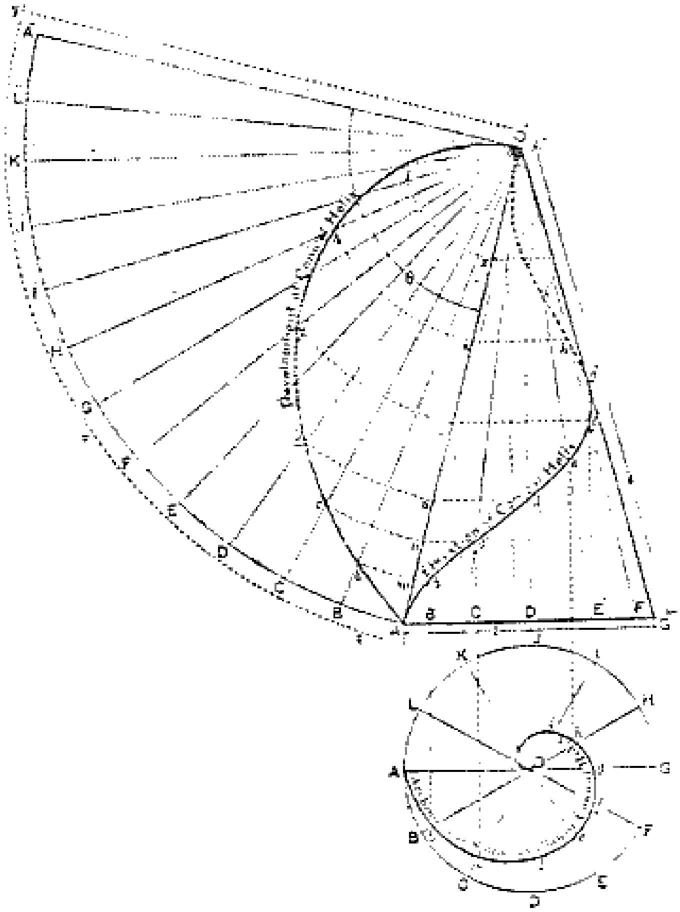


Fig. 188

$\boxed{n = -1}$: $\boxed{r\theta = a}$ Reciprocal (Varignon 1704). (Sometimes called Hyperbolic because of its analogy to the equation $xy = a$).

(a) Its polar subtangent is constant.

(b) Its asymptote is a units from the initial line.

$$\lim_{\theta \to 0} r \cdot \sin\theta =$$

$$\lim_{\theta \to 0} a \cdot \frac{\sin\theta}{\theta} = a.$$

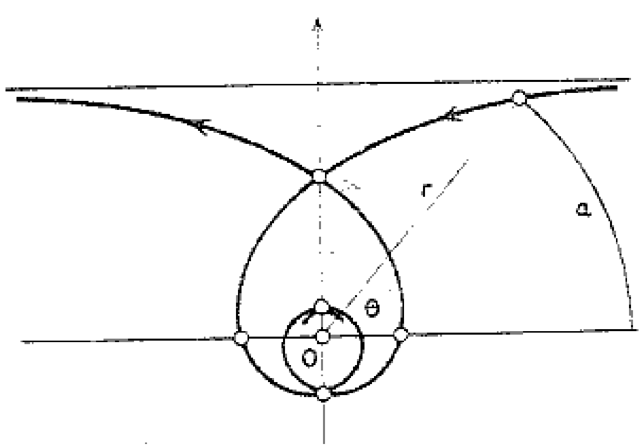


Fig. 189

(c) Arc Lengths of all circles (centers at the pole) measured from the curve to the axis are constant (= a).

(d) The area bounded by the curve and two radii is proportional to the difference of these radii.

(e) It is the inverse with respect to the pole of an Archimedean Spiral.

(f) Roulette: As the curve rolls upon a line, the pole describes a Tractrix.

(g) It is a path of a Particle under a central force which varies as the cube of the distance. (See Lemniscate 4h and Spirals 3f.)

$\boxed{n = 1/2}$: $\boxed{r^2 = a^2\theta}$ Parabolic (because of its analogy to $y^2 = a^2x$) (Fermat 1636).

(a) It is the inverse with respect to the pole of a Lituus.

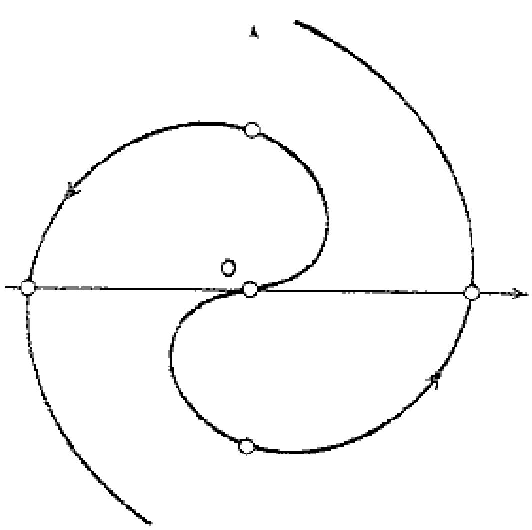


Fig. 190

$\boxed{n = -1/2}$: $\boxed{r^2\theta = a^2}$ Lituus (Cotes, 1722). (Similar in form to an ancient Roman trumpet.)

(a) The areas of all circular sectors OPA are constant $(\frac{r^2\theta}{2} = \frac{a^2}{2})$.

(b) It is the inverse with respect to the pole of a Parabolic Spiral.

(c) Its asymptote is the initial line.

$$\lim_{\theta \to 0} r \cdot \sin\theta = \lim_{\theta \to 0} a\sqrt{\theta}\,\frac{\sin\theta}{\theta} = 0.$$

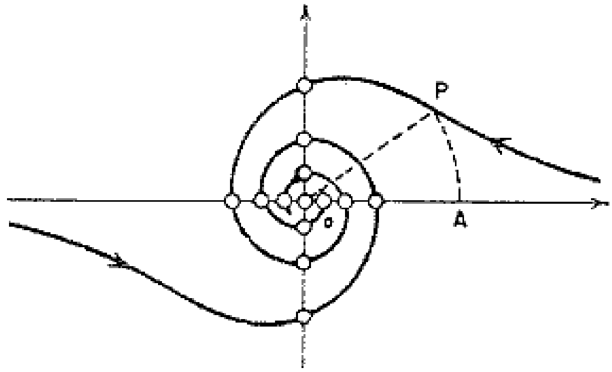


Fig. 191

(d) The Ionic Volute: Together with other spirals, the Lituus is used as a volute in architectural design. In practice, the Whorl is made with the curve emanating from a circle drawn about the pole.

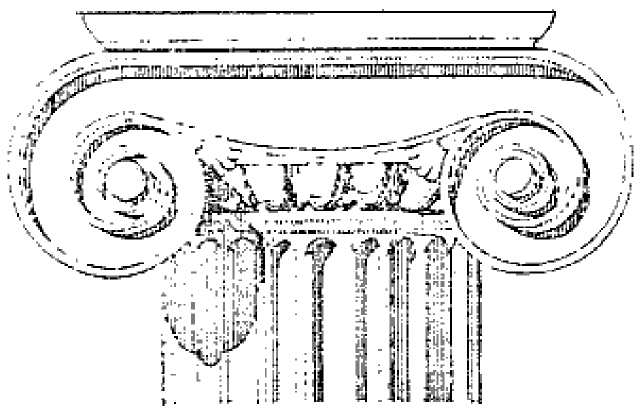


Fig. 192

3. THE SINUSOIDAL SPIRALS: $r^n = a^n \cos n\theta$ or $r^n = a^n \sin n\theta$. (n a rational number). Studied by Maclaurin in 1718.

(a) Pedal Equation: $r^{n+1} = a^n p$.

(b) Radius of Curvature: $R = \dfrac{a^n}{(n+1)r^{n-1}} = \dfrac{r^2}{(n+1)p}$ which affords a simple geometrical method of constructing the center of curvature.

(c) Its Isoptic is another Sinusoidal Spiral.

(d) It is rectifiable if $\frac{1}{n}$ is an integer.

(e) All positive and negative pedals are again Sinusoidal Spirals.

(f) A body acted upon by a central force inversely proportional to the (2n + 3) power of its distance moves upon a Sinusoidal Spiral.

(g) Special Cases:

n	Curve
-2	Rectangular Hyperbola
-1	Line
-1/2	Parabola
-1/3	Tschirnhausen Cubic
1/3	Cayley's Sextic
1/2	Cardioid
1	Circle
2	Lemniscate

(In connection with this family see also Pedal Equations 6 and Pedal Curves 3).

(h) Tangent Construction: Since $r^{n-1}\, r' = -\, a^n \sin n\theta$,

$$\frac{r}{r'} = -\cot n\theta = \cot(\pi - n\theta) = \tan\psi$$

and $$\psi = n\theta - \frac{\pi}{2}$$

which affords an immediate construction of an arbitrary tangent.

4. EULER'S SPIRAL: (Also called Clothoid or Cornu's Spiral). Studied by Euler in 1781 in connection with an investigation of an elastic spring.

Definition:

$$\begin{cases} \sqrt{2t}\cdot dx = a\cdot \sin t\cdot dt \\ \sqrt{2t}\cdot dy = a\cdot \cos t\cdot dt, \end{cases}$$

or $\boxed{R\cdot s = a^2}$,

$$s^2 = 2a^2 t$$

Asymptotic Points:

$$x_0, y_0 = \pm \frac{a\sqrt{\pi}}{2}.$$

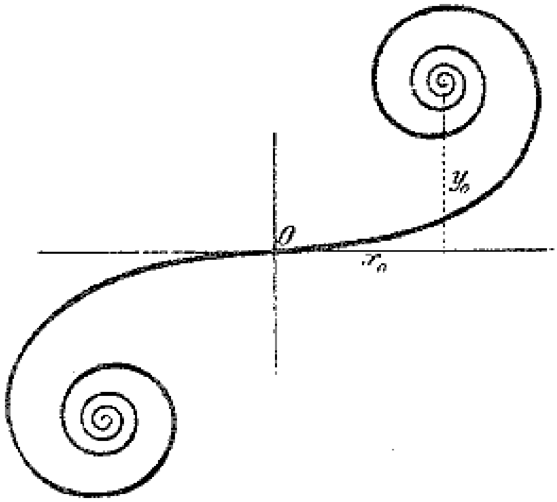


Fig. 193

(a) It is involved in certain problems in the diffraction of light.

(b) It has been advocated as a transition curve for railways. (Since arc length is proportional to curvature. See AMM.)

5. COTES' SPIRALS: These are the paths of a particle subject to a central force proportional to the cube of the distance. The five varieties are included in the equation:

$$\frac{1}{p^2} = \frac{A}{r^2} + B.$$

They are:

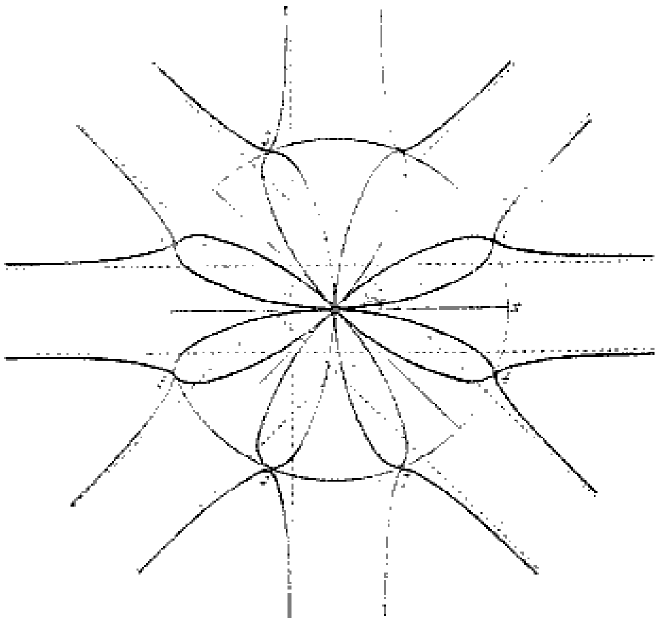


Fig. 194

1. $B = 0$: the Equiangular Spiral;
2. $A = 1$: the Reciprocal Spiral;
3. $\frac{1}{r} = a \cdot \sinh n\theta$;
4. $\frac{1}{r} = a \cdot \cosh n\theta$;
5. $\frac{1}{r} = a \cdot \sin n\theta$ (the inverse of the Roses).

The figure is that of the Spiral $r \cdot \sin 4\theta = a$ and its inverse Rose.

The Glissette traced out by the focus of a Parabola sliding between two perpendicular lines is the Cotes' Spiral: $r \cdot \sin 2\theta = a$.

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