Picnics on a semicircular π-land

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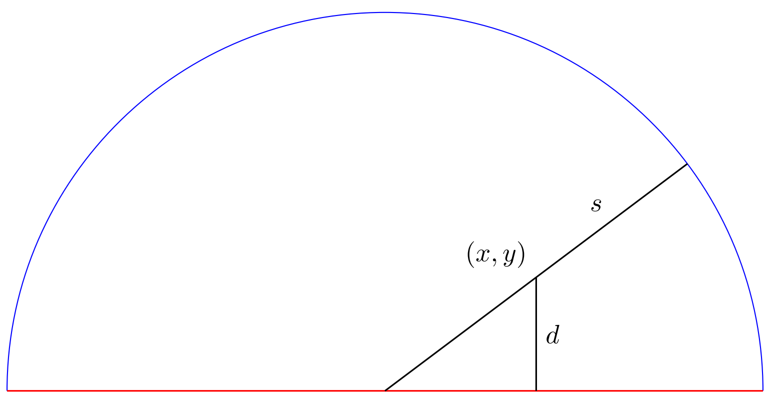
1 Introduction

This article contains a solution to the March 14, 2025 (Pi Day) puzzle from Zach Wissner-Gross's Fiddler on the Proof article series. The original problem can be found at https://thefiddler.substack.com/p/a-pi-day-puzzle.

2 Original Problem

Note that the radius of π-land is irrelevant for the original problem. To maintain consistency with the extra credit problem, assume that π-land is a semicircle of radius 1, centered at $(0, 0)$, containing the portion of the unit circle above the x-axis. A diagram depicting a sample point (x, y) within the semicircular island, as well as its distances d and s from Diametric Beach (red) and Semicircular Beach (blue) respectively, is shown below.



Since Diametric Beach lies on the x-axis of the coordinate system, $d = y$. Note that the radius of the semicircle which passes through (x, y) also passes through the point on Semicircular Beach which lies closest to (x, y). Since the distance from the origin to (x, y) is $\sqrt{x^2 + y^2}$, the distance s equals $1 - \sqrt{x^2 + y^2}$.

Therefore, a point (x, y) is closer to Diametric Beach than to Semicircular Beach when

$$y < 1 - \sqrt{x^2 + y^2} \tag{1}$$
$$\sqrt{x^2 + y^2} < 1 - y \tag{2}$$
$$x^2 + y^2 < 1 - 2y + y^2 \tag{3}$$
$$x^2 < 1 - 2y \tag{4}$$
$$2y < 1 - x^2 \tag{5}$$
$$y < \frac{1 - x^2}{2}. \tag{6}$$

Since any point (x, y) within the semicircle is equally likely to be chosen, the probability that the picnic will be set up closer to Diametric Beach than to Semicircular Beach is equivalent to

$$P = \frac{\int_{-1}^{1} \frac{1-x^2}{2}\, dx}{\pi/2}; \tag{7}$$

the denominator equals the area of the semicircular island. The integral in the numerator equals

$$\int_{-1}^{1} \frac{1-x^2}{2}\, dx = \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1} = \frac{1}{2}\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{2}{3}. \tag{8}$$

Plugging this result into (7), the probability that the picnic will be closer to Diametric Beach than to Semicircular Beach is

$$P = \frac{2/3}{\pi/2} = \frac{4}{3\pi} \approx 0.4244. \tag{9}$$

3 Extra Credit

Denote the region of π-land below the curve $y = (1-x^2)/2$ as D and the region above the semicircle above the curve $y = (1-x^2)/2$ as S. Since the area of π-land is still $\pi/2$, the distance from a point (x, y) in D to Diameter Beach is y, and the distance from a point (x, y) in S to Semicircular Beach is $1 - \sqrt{x^2+y^2}$, the average distance from a random point in π-land to the nearest beach is

$$\text{Average distance} = \frac{\iint_D y\, dA + \iint_S 1 - \sqrt{x^2+y^2}\, dA}{\pi/2}. \tag{10}$$

The first of these two double integrals is relatively straightforward.

$$\begin{aligned}
\iint_D y\, dA &= \int_{-1}^{1}\int_0^{\frac{1-x^2}{2}} y\, dy\, dx && (11)\\
&= \int_{-1}^{1} \frac{y^2}{2}\Bigg|_0^{\frac{1-x^2}{2}} dx && (12)\\
&= \int_{-1}^{1} \frac{1}{2}\left(\frac{\left(1-x^2\right)^2}{4}\right) dx && (13)\\
&= \frac{1}{8}\int_{-1}^{1} 1 - 2x^2 + x^4\, dx && (14)\\
&= \frac{1}{8}\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Bigg|_{-1}^{1} && (15)\\
&= \frac{1}{8}\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right] && (16)\\
&= \frac{2}{15}. && (17)
\end{aligned}$$

For the second, it is more convenient to switch to polar coordinates. For any given value of θ, the lower bound for r lies along the curve $y = (1-x^2)/2$. Note that this curve also satisfies the property that $d = s$, where d and s are defined as in the original problem's solution. This means

$$polarlowboundd = s \Rightarrow y = 1 - \sqrt{x^2+y^2} \Rightarrow r\sin\theta = 1 - r \Rightarrow r(1+\sin\theta) = 1 \Rightarrow r = \frac{1}{1+\sin\theta}. \tag{18}$$

Therefore, the second double integral in the numerator of (10) equals

$$\iint_S 1 - \sqrt{x^2+y^2}\,dA = \int_0^\pi \int_{\frac{1}{1+\sin\theta}}^1 (1-r)r\,dr\,d\theta \tag{19}$$

$$= \int_0^\pi \int_{\frac{1}{1+\sin\theta}}^1 r - r^2\,dr\,d\theta \tag{20}$$

$$= \int_0^\pi \left(\frac{r^2}{2} - \frac{r^3}{3}\right)\Bigg|_{\frac{1}{1+\sin\theta}}^1 d\theta \tag{21}$$

$$= \int_0^\pi \frac{1}{6} - \left(\frac{1}{2(1+\sin\theta)^2} - \frac{1}{3(1+\sin\theta)^3}\right) d\theta \tag{22}$$

$$= \int_0^\pi \frac{1}{6}\,d\theta - \int_0^\pi \left(\frac{1}{2(1+\sin\theta)^2} - \frac{1}{3(1+\sin\theta)^3}\right) d\theta \tag{23}$$

$$= \frac{\pi}{6} - \int_0^\pi \left(\frac{1}{2(1+\sin\theta)^2} - \frac{1}{3(1+\sin\theta)^3}\right) d\theta \tag{24}$$

At this point I decided to use a numerical approximation to the remaining integral, which is coded in PicnicIslandNumericalIntegration.py. I also double-checked my result with Wolfram Alpha, which found that the integral equals 16/45; this is consistent with the result from the numerical integration code.

Plugging these results into (10), we find that the average distance equals

$$\frac{\frac{2}{15} + \left(\frac{\pi}{6} - \frac{16}{45}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}\left(\frac{\pi}{6} - \frac{10}{45}\right) = \frac{1}{3} - \frac{4}{9\pi} \approx 0.19186. \tag{25}$$