1 Introduction

This week's problem statement, found in Zach Wissner-Gross's weekly Fiddler on the Proof column at https://thefiddler.substack.com/p/can-you-sweep-the-series, concerns a prediction made earlier this week by a TV sports personality that the Boston Celtics would defeat the New York Knicks in 5 games.

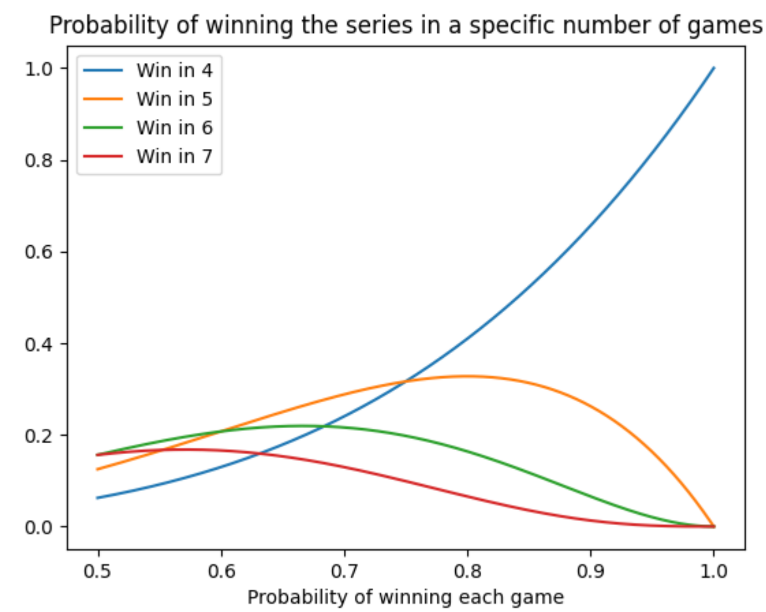
At the time of publication, this prediction had already aged poorly (which the author hints at in the problem write-up); the Knicks won the first two games of the series. For the rest of the problem, assume the math represents a hypothetical series and not the actual one.

2 When is winning in 5 games most likely?

Since the series must end with a win, the probability of winning a series in exactly n games is equivalent to the probability of winning 3 out of the first $n-1$ games multiplied by the probability p of winning the final game. This means that the probability of winning a series in n games, where p is the probability of winning a single game, is

n	$C(n-1,3)$	Probability of winning in n games
4	1	p^4
5	4	$4p^4(1-p)$
6	10	$10p^4(1-p)^2$
7	20	$20p^4(1-p)^3$

A plot of these probabilities for $p \geq 0.5$ is given here. For $p < 0.5$, it is more likely to lose the series in n games than to win in n games for any valid given value of n.



From this plot, we can see that a, the lower bound for the region where winning in 5 games is the most likely outcome, is the non-trivial value of p where it is equally likely for a team to win in 5 games and in 6 games. Likewise, b, the upper bound for this region, is the non-trivial value of p where it is equally likely for a team to win in 5 games and in 4 games.

Therefore, a (which does not equal 0 or 1, so it is safe to divide by a or $1-a$) is a solution to

$$\begin{aligned} 4a^4(1-a) &= 10a^4(1-a)^2 \\ \frac{2}{5} &= 1-a \\ a &= \frac{3}{5}. \end{aligned}$$

Similarly b (which is also not 0 or 1) satisfies

$$\begin{aligned} b^4 &= 4b^4(1-b) \\ \frac{1}{4} &= 1-b \\ b &= \frac{3}{4}. \end{aligned}$$

3 Is a sweep more likely than a 7 game series?

Now, we assume that p is randomly selected from the uniform distribution on the interval (a, b), for the a and b found in the previous section. We want to know whether the probability of the Celtics winning in a sweep is greater than the probability of the series going to seven games, with either team winning.

The probability of the series going to 7 games, regardless of who wins, is

$$p_7 = \binom{6}{3} p^3(1-p)^3;$$

in other words, it is the probability of each team winning exactly 3 times if they play six games. The break-even point, where $p_4 = p^4$ is equal to p_7, is a solution to the equation

$$\begin{aligned} p^4 &= \binom{6}{3} p^3(1-p)^3 \\ p &= 20(1-p)^3 \\ 0 &= 20p^3 - 60p^2 + 61p - 20. \end{aligned}$$

This function has one root in the interval $(a, b) = (0.6, 0.75)$. Since p_4 is increasing and p_7 is decreasing on this range, a sweep is more likely than a seven-game series when p is greater than this root. Through an implementation of Newton's method found in NewtonsMethod.py, it was found that the root is located at $p \approx 0.67658$. By plugging this value into the formula $(b-p)/(b-a)$, we find that $p_4 > p_7$ with a probability of approximately 0.48945.