Collaborative Random Card Game

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Link to problem statement

The link to the problem statement from Zach Wissner-Gross can be found at https://thefiddler.substack.com/p/can-you-win-the-collaborative-card.

Mathematical answer - original problem

Assume there is some subset of n out of the 52 cards which are located in the same position in both players' decks. Then there are $\binom{52}{n}$ ways to choose such a subset of n cards, and the probability of that set of n cards being located in the same spot in both decks is $\frac{1}{52\cdot 51\cdot \ldots \cdot (52-n+1)}$. Note that this formula does not exclude the possibility that some card outside of the subset also is located in the same position in both players' decks. Therefore, the probability of at least n cards being located in the same position is

$$\begin{aligned}
P &= \binom{52}{n} \cdot \frac{1}{52 \cdot 51 \cdot \ldots \cdot (52-n+1)} \\
&= \frac{52 \cdot 51 \cdot \ldots \cdot (52-n+1)}{n!} \cdot \frac{1}{52 \cdot 51 \cdot \ldots \cdot (52-n+1)} \\
&= \frac{1}{n!}
\end{aligned}$$

The probability of a loss P_l is equivalent to the probability that at least one card is located in the same position in both players' decks. By the Inclusion-Exclusion principle, this probability is equal to

$$P_l = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots + \frac{1}{51!} - \frac{1}{52!} = \sum_{n=1}^{52} \frac{(-1)^{n+1}}{n!}.$$

This formula bears a resemblance to the power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Substituting $x = -1$ and performing some algebra with the terms of this series, we get

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$
$$e^{-1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$
$$e^{-1} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$$

This implies that $1 - e^{-1}$ is very close to P_l, and is the limit of P_l as the number of cards in the deck approaches infinity. The probability of winning P_w is therefore very close to e^{-1}; by the Alternating Series Estimation Theorem,

$$P_w = e^{-1} + \varepsilon, \text{ where } |\varepsilon| < \frac{1}{53!}.$$

Monte Carlo simulation - original and extra credit

To approximate the solution to the extra credit problem, I wrote a simulation in Python, extended the range of deck sizes from 1 to 100, and ran 10000 simulations for each value of n for both the original and extra credit problems. We would expect the probability of winning in the original problem to quickly converge to e^{-1}. The probability of winning in the extra credit problem should be higher, as both copies of a card could end up in the same player's deck- it would be impossible to lose the game when one of these cards comes up.

Using the Monte Carlo method, the probability of winning in the original scenario was found to be approximately 0.3647 and the probability of winning in the extra credit scenario was found to be approximately 0.6015.

Given that $e^{-1} \approx 0.3679$, and the variance of the binomial distribution is $\sigma^2 = n \cdot p \cdot (1-p)$, the standard deviation of the success rate is approximately 0.0048. This means that there is no evidence to suggest that the simulation deviated from the mathematical answer.

Conjecture on mathematical answer for extra credit

The simulated results for the extra credit problem also quickly converged to some value. After a bit of trial and error under the assumption that the extra credit solution was somehow related to the original solution, I discovered that squaring P_l resulted in the value $P_l^2 = (1 - e^{-1})^2 \approx 0.3996$. Under this conjecture, the probability of winning would approach 0.6004 as $n \to \infty$, which is consistent with the results in the Monte Carlo simulation. While I have not found a mathematical explanation yet, it is reasonable to conjecture that the actual answer to the extra credit version of the problem is

$$P = 1 - \left(1 - e^{-1}\right)^2 + \varepsilon = 1 - \left(1 - 2e^{-1} + e^{-2}\right) + \varepsilon = 2e^{-1} - e^{-2} + \varepsilon \approx 0.6004.$$