Tenzi and Threezi

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1 Link to problem statement

The link to the problem statement from Zach Wissner-Gross can be found at https://thefiddler.substack.com/p/can-you-solve-the-tricky-mathematical.

2 Original problem

2.1 Code-based solution

Define the events A and B as follows:

- A: There are exactly x pieces of candy corn in the bag.
- B: The first three pieces of candy that you draw are peanut butter cups.

By Bayes' Theorem, the conditional probability of A given B for a specific value of n is equal to

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \tag{1}$$

Note that there is no maximum number of pieces of candy corn in the bag, but that the number of pieces of candy corn must be a nonnegative integer. To work around this problem, assume that there is some theoretical maximum number of pieces of candy corn in the bag, denoted n; after finding an expression for the expected value of the number of pieces of candy corn in the bag in terms of n, the problem can be solved by taking the limit of this expression as $n \to \infty$.

To compute $P(A|B)$, we must compute the following three probabilities. Note that $P(B)$ is simply the arithmetic mean of all of the possible cases for $P(B|A)$, since all values of x in $\{0, 1, \ldots, n\}$ are equally likely.

$$P(A) = \frac{1}{n+1}, \tag{2}$$

$$P(B|A) = \frac{3}{x+3} \cdot \frac{2}{x+2} \cdot \frac{1}{x+1} = \frac{6}{(x+1)(x+2)(x+3)}, \tag{3}$$

$$P(B) = \frac{1}{n+1} \sum_{i=0}^{n} \frac{6}{(i+1)(i+2)(i+3)}. \tag{4}$$

The expected value of the number of pieces of candy corn in the bag is given by

$$E(x) = \sum_{j=0}^{n} jP((x=j)|B). \tag{5}$$

By (1), (2), (3), and (4), the expected value of the number of pieces of candy corn in the bag is

$$E(x) = \sum_{j=0}^{n} j \frac{\frac{6}{(j+1)(j+2)(j+3)} \cdot \frac{1}{n+1}}{\frac{1}{n+1} \sum_{i=0}^{n} \frac{6}{(i+1)(i+2)(i+3)}} \tag{6}$$

$$= \sum_{j=0}^{n} j \frac{\frac{1}{(j+1)(j+2)(j+3)}}{\sum_{i=0}^{n} \frac{1}{(i+1)(i+2)(i+3)}} \tag{7}$$

The code in the file TrickOrTreatSimple.py contains a solution to this series through numerical methods, and the limit of this sum as $n \to \infty$ appears to be 1.

2.2 Analytical solution

To find an analytical solution, we use the partial fraction decomposition given here to find telescoping sums:

$$\frac{1}{(x+1)(x+2)(x+3)} = \frac{1/2}{x+1} + \frac{-1}{x+2} + \frac{1/2}{x+3}. \tag{8}$$

By this partial fraction decomposition, the sum (4) corresponding to $P(B)$ equals, after canceling terms with the same denominator which sum to 0,

$$\sum_{i=0}^{n} \frac{1}{(i+1)(i+2)(i+3)} = \frac{1/2}{1} + \frac{-1}{2} + \frac{1/2}{2} + \frac{1/2}{n+2} + \frac{-1}{n+2} + \frac{1/2}{n+3}. \tag{9}$$

The limit of this expression as $n \to 0$ is therefore

$$\lim_{n\to\infty} \frac{1/2}{1} + \frac{-1}{2} + \frac{1/2}{2} + \frac{1/2}{n+2} + \frac{-1}{n+2} + \frac{1/2}{n+3} = \frac{1}{4}. \tag{10}$$

Combining (7) with (10), we get

$$E(x) = \sum_{j=0}^{n} j \frac{\frac{1}{(j+1)(j+2)(j+3)}}{\frac{1}{4}} = 4 \sum_{j=0}^{n} \frac{j}{(j+1)(j+2)(j+3)}. \tag{11}$$

To find this sum, the partial fraction decomposition (8) can be modified to

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{x/2}{x+1} + \frac{-x}{x+2} + \frac{x/2}{x+3}. \tag{12}$$

After computing the associated telescoping sum, the non-canceling terms become

$$\sum_{j=0}^{n} \frac{j}{(j+1)(j+2)(j+3)} = \frac{1/2}{2} + \frac{-1}{3} + \frac{2/2}{3} + \ldots + \frac{-n}{n+2} + \frac{n/2}{n+3} + \frac{(n-1)/2}{n+2}. \tag{13}$$

Taking the limit of the terms containing n as $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{j=0}^{n} \frac{j}{(j+1)(j+2)(j+3)} = \frac{1}{4} - \frac{1}{3} + \frac{1}{3} - 1 + \frac{1}{2} + \frac{1}{2} = \frac{1}{4}. \tag{14}$$

Plugging this result into (11) confirms the result apparent from the numerical solution that

$$E(x) = 4 \cdot \frac{1}{4} = 1. \tag{15}$$