Falling Dominoes

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Introduction

The link to the problem statement from Zach Wissner-Gross can be found at
https://thefiddler.substack.com/p/can-you-tip-the-dominoes.

Original problem

This scenario can be modeled with a geometric distribution, where $p = .01$. Using the version of the distribution which includes the "success" trial, the median value of this distribution is

$$M = \left\lceil \frac{-1}{\log_2(1-p)} \right\rceil = \left\lceil \frac{-\ln 2}{\ln(1-p)} \right\rceil. \tag{1}$$

For $p = 0.01$, we have

$$\frac{-\ln 2}{\ln 0.99} \approx 68.97 \Rightarrow \left\lceil \frac{-\ln 2}{\ln 0.99} \right\rceil = 69. \tag{2}$$

Extra credit problem

Setting $p = 10^{-k}$, the expression $M/10^k$ equals

$$\frac{M}{10^k} = 10^{-k} \left\lceil \frac{-\ln 2}{\ln(1-10^{-k})} \right\rceil. \tag{3}$$

To simplify the computations, first notice that

$$\begin{aligned}
&0 < \left\lceil \frac{-\ln 2}{\ln 0.99} \right\rceil - \frac{-\ln 2}{\ln(1-10^{-k})} < 1 && (4)\\
&\Rightarrow 0 < 10^{-k} \left\lceil \frac{-\ln 2}{\ln(1-10^{-k})} \right\rceil - 10^{-k} \frac{-\ln 2}{\ln(1-10^{-k})} < 10^{-k} && (5)\\
&\Rightarrow 0 \le \lim_{k\to\infty} 10^{-k} \left\lceil \frac{-\ln 2}{\ln(1-10^{-k})} \right\rceil - \lim_{k\to\infty} 10^{-k} \frac{-\ln 2}{\ln(1-10^{-k})} \le \lim_{k\to\infty} 10^{-k} = 0 && (6)\\
&\Rightarrow \lim_{k\to\infty} 10^{-k} \left\lceil \frac{-\ln 2}{\ln(1-10^{-k})} \right\rceil = \lim_{k\to\infty} 10^{-k} \frac{-\ln 2}{\ln(1-10^{-k})}, && (7)
\end{aligned}$$

so we can ignore the ceiling function within $M/10^k$.

The Taylor series expansion for $\ln(1+x)$ centered at $x = 0$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \tag{8}$$

and it converges for $x \in (-1, 1]$. Substituting $x = -10^{-k}$, which lies in the interval of convergence for $k > 0$, into (8) we get,

$$\ln(1-10^{-k}) = -10^{-k} - \frac{1}{2}10^{-2k} - \frac{1}{3}10^{-3k} - \frac{1}{4}10^{-4k} - \frac{1}{5}10^{-5k} - \cdots. \tag{9}$$

By plugging this into the denominator of (7), we get

$$
\begin{aligned}
\lim_{k\to\infty} 10^{-k}\frac{-\ln 2}{\ln(1-10^{-k})} &= \lim_{k\to\infty} 10^{-k}\frac{-\ln 2}{-10^{-k}-\frac{1}{2}10^{-2k}-\frac{1}{3}10^{-3k}-\cdots} && (10)\\
&= \lim_{k\to\infty}\frac{\ln 2}{10^{k}\left(10^{-k}+\frac{1}{2}10^{-2k}+\frac{1}{3}10^{-3k}+\cdots\right)} && (11)\\
&= \lim_{k\to\infty}\frac{\ln 2}{1+\frac{1}{2}10^{-k}+\frac{1}{3}10^{-2k}+\cdots} && (12)\\
&= \frac{\ln 2}{1+0} && (13)\\
&= \ln 2. && (14)
\end{aligned}
$$

Therefore, as $k\to\infty$, the ratio $M/10^k$ approaches $\ln 2 \approx 0.6931$.