Making a fair die-rolling game

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1 Link to problem statement

The link to the problem statement from Zach Wissner-Gross can be found at
https://thefiddler.substack.com/p/can-you-even-the-odds.

2 Original problem

You (player A) and your friend (player B) are rolling a die, and the first player to roll a 5 wins the game. To mitigate the advantage that the first player has, you alternate who goes first in each round, rolling the die in the order AB|BA|AB|BA, and so on, until one of you rolls a 5. As in the original problem statement, the vertical bars mark gaps between rounds.

Let P be the probability that player A wins the game. If neither A nor B rolls a 5, then the game essentially resets, with B becoming the first player. In that situation, the conditional probability that B eventually wins the game given that A and B both fail to roll a 5 on their first roll must become P. Because of this, the probability that A wins given that both A and B fail to roll a 5 on their first roll must be $1 - P$.

The probability of A winning on the first roll is $1/6$, and the probability that both A and B fail to roll a 5 on their first rolls is $(5/6)^2$. In the only other case (A doesn't roll a 5 on their first roll, but B does), A immediately loses. Therefore, the probability P must satisfy the equation

$$P = \frac{1}{6} + \left(\frac{5}{6}\right)^2 (1 - P) \tag{1}$$

$$P = \frac{1}{6} + \frac{25}{36}(1 - P) \tag{2}$$

$$\frac{61}{36}P = \frac{31}{36} \tag{3}$$

$$P = \frac{31}{61} \approx 50.820\%. \tag{4}$$

3 Extra credit problem

In this case, round 1 consists of player A rolling the die, round 2 consists of player B rolling the die, and every round after that consists of a sequence of die rolls that is the same as all previous die rolls, but with the players swapped.

A|B|BA|BAAB|BAABABBA|BAABABBAABBABAAB|...

Define A_n and B_n as the probabilities that player A and player B win in round n, respectively. Then the probability P that A wins is

$$P = \sum_{n=1}^{\infty} A_n, \tag{5}$$

and we also should have

$$1 - P = \sum_{n=1}^{\infty} B_n. \tag{6}$$

Using the first two single-roll rounds as a base case for this recursive relation, we have

$$A_1 = 1/6, \tag{7}$$

$$B_1 = 0, \tag{8}$$

$$A_2 = 0, \tag{9}$$

$$B_2 = (5/6)(1/6) = 5/36. \tag{10}$$

We can develop a formula to determine A_n and B_n recursively for $n \geq 3$. Note that each round consists of the entirety of the game's previous rolls, with the players swapped. This also means that each round after the second is twice as long as the previous round. The probability of the game reaching a given round n without anyone rolling a 5 is

$$R_n = \left(\frac{5}{6}\right)^{\left(2^{n-1}\right)}. \tag{11}$$

Combining the probability of reaching round n with the structure of round n relative to previous rounds, we find that

$$A_n = \left(\sum_{i=1}^{\infty} B_i\right) R_n, \tag{12}$$

$$B_n = \left(\sum_{i=1}^{\infty} A_i\right) R_n. \tag{13}$$

The Python file ThueMorseRollFive.py contains code to compute P and $1 - P$. The algorithm stops once the computed values of P and $1 - P$ sum to a value greater than $1 - \varepsilon$, where ε is set to 10^{-16}. The algorithm therefore stops after the $n = 9$ term, and we find that the probability of player A winning is approximately

$$P \approx 50.159\%. \tag{14}$$

This sequence represents an improvement over the snake draft format from the original problem.