Bicycle Sprints

Andrew Ford

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1 Introduction

This week's problem statement can be found in Zach Wissner-Gross's weekly Fiddler on the Proof column at https://thefiddler.substack.com/p/can-you-sprint-to-the-finish

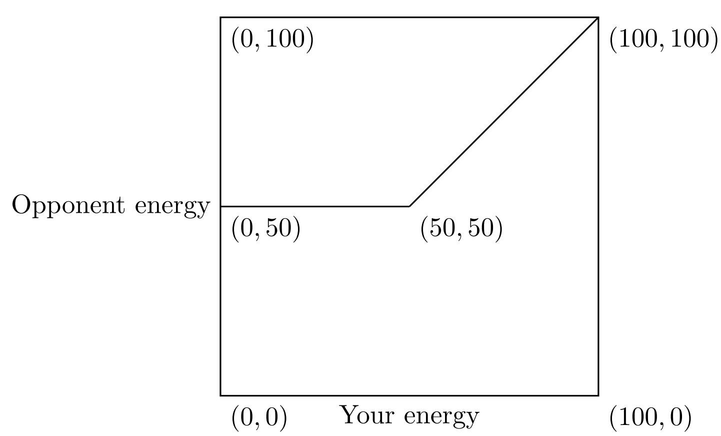
2 One other rider

You and a competitor are far ahead of the rest of the pack and will be the first two finishers of the current stage. Your opponent will sprint to the finish if and only if they are at 50 percent or better, and you need to both determine the best strategy to determine when to sprint and discover your probability of winning the stage under this strategy.

First, note the following three details:

1. This stage is assumed to exist in a vacuum, and there is no apparent long-term benefit to saving your energy for future stages.
2. If you decide not to sprint, it is only possible to win if your opponent also decides not to sprint. Likewise, if you do decide to sprint, you cannot lose unless your opponent also decides to sprint.
3. Your opponent does not take your strategy or your level of energy into account in their decision to sprint or not.

In this situation, it is actually always optimal to sprint to the finish- no matter what your opponent decides, you cannot get a worse result by sprinting to the finish line, even if your legs are feeling terrible.



The plot above represents the space of all possible energy levels you and your opponent can have; the area below the segmented line represents cases where you win, and the area above the segmented line represents cases where your opponent wins. Since both your energy level and your opponent's energy level are uniformly distributed between 0% and 100%, and the area where you win represents 5/8 of the total square, you have a 5/8 probability of winning.

3 Two other riders

In the second scenario, you have received directions from your team to sprint to the finish if your energy level is above some cutoff c. For the purposes of writing up later math in a readable way, c is written as a decimal instead of a percentage. Because the radio message from your team manager was a bit garbled, you don't actually know what c is, so you randomly pick a uniformly distributed percentage from 0% to 100% to use as a cutoff percentage instead.

Additionally, you have two opponents who will each sprint to the finish if their energy level is 50% or better. There are now three possible scenarios to consider based on the value of c and your own energy level.

If your energy level is lower than c, which occurs with a probability of c, then you will lose if either opponent decides to sprint and win with a probability of 1/3 if neither opponent sprints. Each opponent has a 50% chance of sprinting, so your probability of winning in this scenario is 1/12.

Next, consider the case where your energy level is higher than c but lower than 50%. If $c < 50\%$, this occurs with probability $0.5 - c$. Otherwise, it occurs with probability 0. In this case, you will lose if either opponent decides to sprint (because they would guaranteed to have a higher energy level than you) and win if neither opponent decides to sprint. Again, each opponent has a 50% chance of sprinting, so your probability of winning in this scenario is 1/4.

Finally, consider the case where your energy level is higher than c and at least 50%. If $c \leq 50\%$, this occurs with probability 50%. Otherwise, it occurs with probability $1 - c$. In this case, let $x \geq c$ be your energy level. If either of your opponents has an energy level higher than x, then it is also higher than 50% and they will sprint and finish before you. If not, then you will win. In this scenario, your probability of winning for a given value of x is x^2. For a given value of c, your probability of winning is

$$\frac{\int_c^1 x^2\,dx}{1-c} = \frac{1}{1-c}\left(\left.\frac{x^3}{3}\right|_c^1\right) = \frac{1}{1-c}\left(\frac{1-c^3}{3}\right) = \frac{c^2+c+1}{3}$$

if $c > 50\%$ and

$$\frac{\int_{0.5}^1 x^2\,dx}{1/2} = 2\left.\frac{x^3}{3}\right|_{0.5}^1 = 2\cdot\frac{1-0.5^3}{3} = \frac{7}{12}$$

if $c \leq 50\%$.

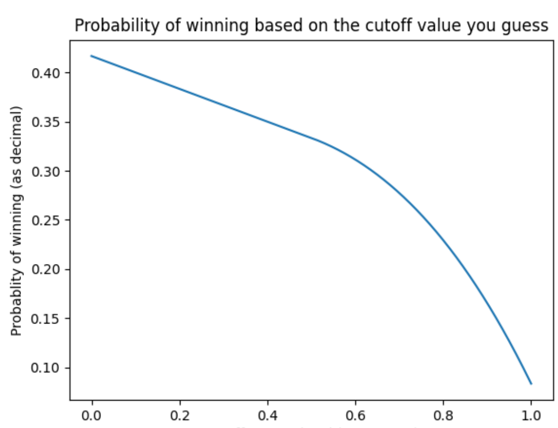
Combining these cases together, we find that, if $c \leq 50\%$, your probability of winning the stage is

$$P = c\cdot\frac{1}{12} + (0.5-c)\cdot\frac{1}{4} + 0.5\cdot\frac{7}{12} = \frac{c}{12} + \frac{1}{8} - \frac{c}{4} + \frac{7}{24} = \frac{5}{12} - \frac{c}{6}. \tag{1}$$

If $c > 50\%$, your probability of winning the stage is

$$P = c\cdot\frac{1}{12} + (1-c)\cdot\frac{c^2+c+1}{3} = \frac{c}{12} + \frac{1-c^3}{3}. \tag{2}$$

A plot of the resulting piecewise function is included here.



This means that the overall probability of winning the stage, by (1) and (2), is

$$
\begin{aligned}
&\int_0^{0.5} \frac{5}{12} - \frac{c}{6}\, dc + \int_{0.5}^{1} \frac{c}{12} + \frac{1-c^3}{3}\, dc && (3)\\
&= \left(\frac{5c}{12} - \frac{c^2}{12}\right)\bigg|_0^{0.5} + \left(\frac{c^2}{24} + \frac{c}{3} - \frac{c^4}{12}\right)\bigg|_{0.5}^{1} && (4)\\
&= \left(\frac{5}{24} - \frac{1}{48}\right) - 0 + \left(\frac{1}{24} + \frac{1}{3} - \frac{1}{12}\right) - \left(\frac{1}{96} + \frac{1}{6} - \frac{1}{192}\right) && (5)\\
&= \frac{3}{16} + \frac{7}{24} - \frac{11}{64} && (6)\\
&= \frac{59}{192}. && (7)
\end{aligned}
$$

Therefore, right before you select a cut-off energy percentage at random, you have a 15/64 probability of winning the stage.