Hw6 pg1

Monday, May 16, 2022 11:34 AM

Exercise 1 (10 points) $|E| = m$ $|V| = n$

$$\underset{m\times n}{[A]}\ \underset{n\times 1}{[y]} = \underset{n\times 1}{\begin{bmatrix} y_{u_1} \\ \vdots \\ y_{u_n} \end{bmatrix}} \geq \begin{bmatrix} 1 \end{bmatrix}$$

Let $G = (V,E)$ be a bipartite graph with parts $V = V_1 \dot\cup V_2$. Consider the linear program:

A ≈ Matrix of edges x Nodes indicating/storing info for which edges incident to vertices

$$\min \sum_{u \in V} y_u$$
$$y_u + y_v \geq 1 \quad \forall \{u,v\} \in E$$
$$y_u \geq 0 \quad \forall u \in V$$

constraint that either u and/or v must be incident to $\{u,v\}$ in final calculation for all edges $\{u,v\} \in E$

a) If you write the problem in the matrix form $\min\{\underline{\mathbf{1}^T y} \mid Ay \geq \mathbf{1}; y \geq \mathbf{0}\}$, how is the matrix A defined?

$1^T y$ = sum[entries of y] i.e. → y is $n \times 1$ vector corresponding to a vertex $\in V$

$1^T y$ sums entries of y_u where sum(y) = # edges incident to vertex u

b) Prove that all extreme points of $P = \{y \in \mathbb{R}^V \mid Ay \geq \mathbf{1}; y \geq \mathbf{0}\}$ are integral (A defined as in a)).

$y \in \mathbb{R}$ $n \times 1$

c) Which problem that you know from the lecture, does the above LP solve?

a) Lets Matrix $A := \underbrace{\{0, 1\}}_{\text{possible values}}^{\underbrace{E \times V}_{}}$ dimensions of $A = |E| \times |V|$ where all possible pairs edge, vertex make up entries of A ... mb

where entry $A_{E,V}(e,v) := \begin{cases} 1 & e \text{ incident to } v \\ 0 & \text{otherwise} \end{cases}$

b) $P := \{y \in \mathbb{R}^V \mid Ay \geq 1 ; y \geq 0\}$

A ≈ (Node, edge) Incident indication Matrix of Bipartite graph G w/ dimensions $m = |E|$ $n = |V|$

Every entry of A is 0 or 1; A is totally unimodular because determinant of all submatrices either 0, 1, -1

constraints for each row

Then by Lemma 54, all extreme points of P are integral (A is totally unimodular & b = vector of $1 \Rightarrow b \in \mathbb{Z}^m$) ... corollary 56

c) Problem LP solves = ___

from observation, looks like we want to find min subset $S \subseteq V$ such that at least every edge has at least one endpoint to a vertex $\in S$. An optimum extreme point solution := $\underline{y} \in \{0, 1\}^V$;

y from Ay & $1^T y$ → indicates whether or not vertex will be used or not when minimizing $\sum y_u$ 0 · (column = u) ⇒ vertex not included. 1 · (column = u) ⇒ vertex included

$S := \{u \in V \mid y_u = 1\}$;

$y_u \in y; y_u = 1 \Rightarrow$ vertex u included in final selection; summation

From my interpretation, this looks like the $\boxed{\text{minimum vertex cover problem}}$

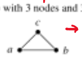
Hw6 pg2

Tuesday, May 17, 2022 3:09 PM

Exercise 2 (10 points)

a) Consider the triangle graph $G = (V, E)$ with 3 nodes and 3 edges

← Not Bipartite (i.e. Not 2 colorable)



→ each node incident to 2 edges

Section 6.1

and the matching polytope $P_M = \{x \in \mathbb{R}^E \mid \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V;\ x_e \geq 0 \ \forall e \in E\}$ associated with it. Write it in form $P_M = \{x \in \mathbb{R}^E \mid Ax \leq b\}$ and give the 6×3 constraint matrix A. What 3×3 submatrix A_I of A has $\det(A_I) \notin \{-1, 0, 1\}$? What is $\det(A_I)$? Compute A_I^{-1}. Which is the extreme point $x = A_I^{-1} b_I$ that belongs to this submatrix?

b) Which of the following matrices is TU? Argue why or why not!

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

hmm...

for $P_M = \{x \in \mathbb{R}^E \mid \underbrace{\sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V}_{\text{for every vertex, sum of edge costs in its cut is } \leq 1};\ \underbrace{x_e \geq 0 \ \forall e \in E}_{\text{all edge weights non-negative}}\}$

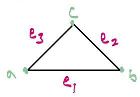
x 3×1

$$\begin{matrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{matrix}$$

a)

i) write in form $P_M = \{x \in \mathbb{R}^E \mid Ax \leq b\}$; $A \in 6 \times 3$ constraints × objective variables

Define x_1, x_2, x_3 as edge costs for edges $e \in \{e_1, e_2, e_3\}$



i.e. x_1 = cost of edge e_2 = edge$\{a, b\}$

$x = [x_1, x_2, x_3]^T$

$$A := \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{array} \begin{bmatrix} 1 & & 1 \\ 1 & 1 & \\ & 1 & 1 \\ -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} x \leq \begin{array}{c} b \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

constraints:

$c_1 : e \in \delta(a) := \{e_1, e_3\}$; $\sum_{e \in \delta(a)} x_e = x_1 + x_3$ is ≤ 1

c_2 " "

c_3 " "

c_4, c_5, c_6 := Non-negativity constraints for edge costs

ii) 3x3 Submatrix A_I of A with $\det(A_I) \notin \{-1, 0, 1\}$ → i.e. which $A_I^{3\times3}$ is not Totally Unimodular

consider submatrix A_I : $\boxed{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$

first 3 rows →

$A_I \cong$ Node-edge incident Matrix of G

$\det(A_I) = 1(1-0) - 0(1-0) + 1(1-0) = 1 + 0 + 1 = 2 \notin \{-1, 0, 1\}$

$$A_I^{-1} : \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

A^{-1}

$$A_I^{-1} = \frac{1}{2} \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}}_{= A_I^{-1'} \text{ prime}} ;\ b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x = A_I^{-1} b_I = \frac{1}{2} A_I^{-1'} b_I = \frac{1}{2}\left[1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right] = \boxed{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}} = x$$

a bit confused about this

b)

A_1 Not TU:

consider submatrix A_1' 3x3 with rows 1, 2, 4, cols 1, 2, 3

$$A_1' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\det(A_1') = 1(-1) - 0 + 1(0-1) = -2 \notin \{-1, 0, 1\}$

⇒ Not TU

or A_1'' = rows 1, 2, 3 cols 3, 4, 5

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix};\ \det(A_1'') = 1(1-0) - 0 + 1(1-0) = 2 \notin \{-1, 0, 1\}$$

⇒ Not TU

A_2 Not TU:

∄ operation from Lemma 55 such that $A_2 = \left(B \,\middle|\, \frac{C_1}{C_2}\right)$

There always exists column ∈ B with more than one nonzero entry