Hw5 pg1

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Exercise 1 (8 points)
Consider the linear program

$$\begin{array}{rcll} \max\ x_1 - 2x_2 & & & (P) \\ x_1 + x_2 & \le & 4 & x_2 \le -x_1 + 4 \\ x_1 - 3x_2 & \le & 6 & x_2 \ge \frac{1}{3}x_1 - 2 \\ -2x_1 - x_2 & \le & -3 & x_2 \ge -2x_1 + 3 \\ 4x_1 - 3x_2 & \le & 15 & x_2 \ge \frac{4}{3}x_1 - 5 \end{array}$$

$c = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $b^T = [4, 6, -3, 15]$

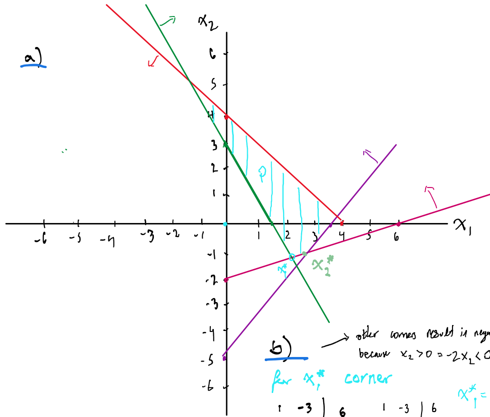
a) Draw the feasible region of LP (P).

b) Determine the optimum solution x^* by inspecting the picture.

c) State the dual program (to (P)).

d) Find the optimum solution to the dual. Check that it is feasible and the objective function value matches the objective function value for the dual.
Hint: Your knowledge about duality will tell you very quickly how the dual solution has to look like. You do not need to run the simplex algorithm to find the dual solution!

a)



b) → other corners result in negative / smaller obj values because $x_2 > 0 \Rightarrow -2x_2 < 0$

for x_1^* corner

$\left[\begin{array}{cc|c} 1 & -3 & 6 \\ -2 & -1 & -3 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & -3 & 6 \\ 0 & -7 & 9 \end{array}\right]$ $x_1^* = \left(\frac{15}{7}, -\frac{9}{7}\right)$

$obj_1 = \frac{15}{7} - 2\left(-\frac{9}{7}\right) = \frac{33}{7}$

$x_2 = -\frac{9}{7}$ $x_1 + \frac{27}{7} = 6$ $x_1 = \frac{15}{7}$

for x_2^* corner

$\left[\begin{array}{cc|c} 1 & -3 & 6 \\ 4 & -3 & 15 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & -3 & 6 \\ 0 & 9 & -9 \end{array}\right]$ $x_2^* = (3, -1)$

$obj_2 = 3 - 2(-1) = 5$

$x_2 = -1$ $x_1 = 3$ $\frac{35}{7} > \frac{33}{7}$

$\boxed{\text{so } x_2^* = (3, -1) \text{ is optimum w/ obj value} = 5}$

c) dual (D): $\min\{b^T y \mid y^T A = c^T, y \ge 0\}$

$b = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 15 \end{bmatrix}$ $y^T = [y_1\ y_2\ y_3\ y_4]$ $c^T = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 \\ 1 & -3 \\ -2 & -1 \\ 4 & -3 \end{bmatrix}$ $\underset{1\times 4\ \ 4\times 2}{y^T A} = [$

Then dual:

$$\boxed{\begin{array}{ll} \min & 4y_1 + 6y_2 - 3y_3 + 15y_4 \\ \text{st} & y_1 + y_2 - 2y_3 + 4y_4 = 1 \\ & y_1 - 3y_2 - y_3 - 3y_4 = -2 \\ & y_{[1,4]} \ge 0 \end{array}}$$

d) optimum solution for dual: ← and (P) feasible

tight inequalities: $a_1, \dots a_m$: rows of A $x^* = (3, -1)$ $I := \overbrace{\{i \mid a_i^T x^* = b_i\}}^{\text{index of row where opt soln} = b \text{ of that row}}$

Then $I = \{2, 4\}$ b/c intersection of constraints 2 & 4 gave us the opt soln $x^* = (3, -1)$

$\Rightarrow y_1 = y_3 = 0$ by complementary slackness theorem. "cannot have slack in both constraint row and it's associated dual variable
i.e. in 2 places @ same time (primal variable, dual constraint) or (primal constraint, dual variable)
i.e. dual var $> 0 \Rightarrow$ associated primal constraint no slack; constraint is tight/binding

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So we get:

$$\begin{array}{ll} \min & 4y_1 + 6y_2 - 3y_3 + 15y_4 \\ \text{st} & y_2 + 4y_4 = 1 \\ & -3y_2 - 3y_4 = -2 \\ & y_{[1,4]} \ge 0 \end{array}$$

$\Rightarrow \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 1 & 1 & 2/3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 3 & 1/3 \end{array}\right]$ $y_4 = \frac{1}{9}$; $y_2 + 4\left(\frac{1}{9}\right) = 1 \Rightarrow y_2 = \frac{5}{9}$

$\boxed{y^* = \left(0, \frac{5}{9}, 0, \frac{1}{9}\right)}$

check obj (P) = obj (D)

$\boxed{obj(P) = 5;\ obj(D) = 4(0) + 6\left(\frac{5}{9}\right) - 3(0) + 15\left(\frac{1}{9}\right) = \frac{10}{3} + \frac{5}{3} = \frac{15}{3} = 5 \Rightarrow obj(P) = 5 = obj(D)}$:)

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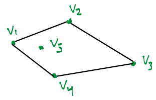
Exercise 2 (5 points)
Let $x_1, \dots, x_k \in \mathbb{R}^n$ be vectors and let $K := \text{conv}\{x_1, \dots, x_k\}$. Prove that if the origin $\mathbf{0} = (0, \dots, 0)$ is not in K, then there is a vector $c \in \mathbb{R}^n$ with $c^T x_i \geq 1$ for all $i = 1, \dots, k$.
Hint: Apply the hyperplane separation theorem.



- Let $P = \{\text{origin}\}$; P is convex
- P convex $\wedge$ K convex $\Rightarrow$ $\exists$ hyperplane $\underbrace{c^T x = \beta}_{\text{hyperplane that separates } P \text{ \& } K}$ w/ $c^T x > \beta > c^T y \ \forall x \in K \ \forall y \in P$ By Hyperplane Separation theorem
- $P = \{\text{origin}\} \Rightarrow y = \text{origin}$ is only element of $P \Rightarrow c^T x > \beta > c^T \cdot 0 \Rightarrow c^T x > \beta > 0$
- Now want to show $\exists$ vector $c \in \mathbb{R}^n$ $c^T x_i \geq 1$ for $i = [1, K]$... Thinking... use $c^T x > \beta$
- $\boxed{\text{Define a vector } d := \frac{1}{\beta} c.}$ then $d^T x_i > \beta \Rightarrow \underbrace{\left(\sum_{i=1}^{K} \frac{1}{\beta}\right)}_{\text{factor out } \frac{1}{n} \text{ from } d} c^T x_i > \beta \Rightarrow \beta c^T x_i > \beta \Rightarrow c^T x_i > \frac{\beta}{\beta} = 1 \Rightarrow c^T x_i > 1$ for $i = [1, n]$

Exercise 3 (7 points)
In this exercise, we want to study a system of a primal and dual LP that is in a different form than the one in the lecture. Consider

$$\begin{array}{rcll} \max c^T x & & & (P) \\ Ax & \leq & b & \\ x & \geq & \mathbf{0} & \end{array} \qquad \text{and} \qquad \begin{array}{rcll} \min b^T y & & & (D) \\ y^T A & \geq & c^T & \\ y & \geq & \mathbf{0} & \end{array}$$

a) Give a direct proof that $(P) \leq (D)$. More concrete, suppose that $Ax \leq b, x \geq \mathbf{0}, y^T A \geq c^T, y \geq \mathbf{0}$ and then directly prove that $c^T x \leq y^T b$ without relying on any theorem from the lecture.

Thinking about example from notes where (D) is an upperbound for (P)

$y^T A \geq c^T$ by assumption

$, \geq 0\}$

$y^T A x \geq c^T x$ Multiply by x on right; x non-negative for all its entries... $x \geq 0$ → doesn't flip inequality

$y^T A x - c^T x \geq 0$ Move $c^T x$ to other side

$y^T b - c^T x \geq y^T Ax - c^T x \geq 0$ By assumption $Ax \leq b$; $y \geq 0$

$\boxed{\text{Then } y^T b - c^T x \geq 0 \Rightarrow y^T b \geq c^T x \ \square \dots}$

b) Show that the optimum values for (P) and (D) are the same, given that both systems are feasible.
Hint: The easiest way to prove this is to rewrite (P) and (D) so that you arrive at systems $\max\{\tilde{c}^T \tilde{x} \mid \tilde{A}\tilde{x} \leq \tilde{b}\}$ and $\min\{\tilde{b}^T \tilde{y} \mid \tilde{y}^T \tilde{A} = \tilde{c}^T;\ \tilde{y} \geq \mathbf{0}\}$. Then you can use the result from the lecture that both systems have the same optimum value (again, given that both are feasible).

given LP Pair above

$\hat{D} = \min\{b^T y \mid y^T A + \underbrace{s}_{\text{slack for equality}} = c^T;\ s \geq 0\}$

Let $c^T - s := \hat{c}^T$; $\hat{b} := y$; $\hat{A}\hat{x} := Ax + s$ ... Maybe.. idk

Dual of $\hat{D} = \max\{\hat{c}^T \hat{x} \mid \hat{A}\hat{x} \leq \hat{b}\}$

$D(\hat{D}) = \max\{\hat{c}^T \hat{x} \mid \hat{A}\hat{x} \leq \hat{b}\} \approx$ results in optimal value for P, D when feasible...