Canonical integrals

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Andrew, I have a few challenge problems for your NSD code. They are to do with evaluating contour integral solutions $A(x, y)$ of the parabolic wave equation in "inner" coordinates (x, y), and the associated rightward-propagating solutions $A(x(x_0), y(y_0))e^{ikx_0}$ of the Helmholtz equation in "outer" coordinates (x_0, y_0). I can explain the application in due course!

Challenge 1

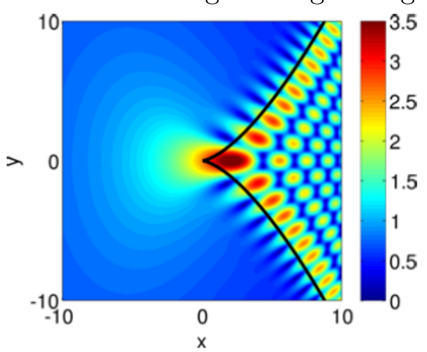
Pearcey-type integral with quartic phase:

$$A_{31} = \int_{\Gamma_{31}} e^{i(-yt - xt^2/2 + t^4/4)}\, dt, \tag{1}$$

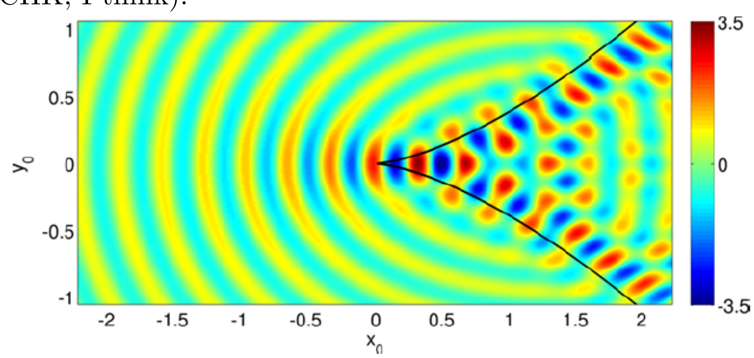
where Γ_{31} is any contour going from $e^{i9\pi/8}\infty$ to $e^{i\pi/8}\infty$. Plot

- $|A_{31}(x, y)|$ on $[-10, 10] \times [10, 10]$,
- $\mathrm{Re}[A_{31}(k^{1/2}x_0, k^{3/4}y_0)e^{ikx_0}]$ on $[-10/k^{1/2}, 10/k^{1/2}] \times [-10/k^{3/4}, 10/k^{3/4}]$,

for $k = 20$ and (if possible) $k = 40$. The $k = 20$ results should look something like this (which I obtained a long time ago using CHK, I think):



(a) $|A_{31}|$



(b) $\mathrm{Re}\left[A_{31}e^{ikx_0}\right]$

Challenge 2

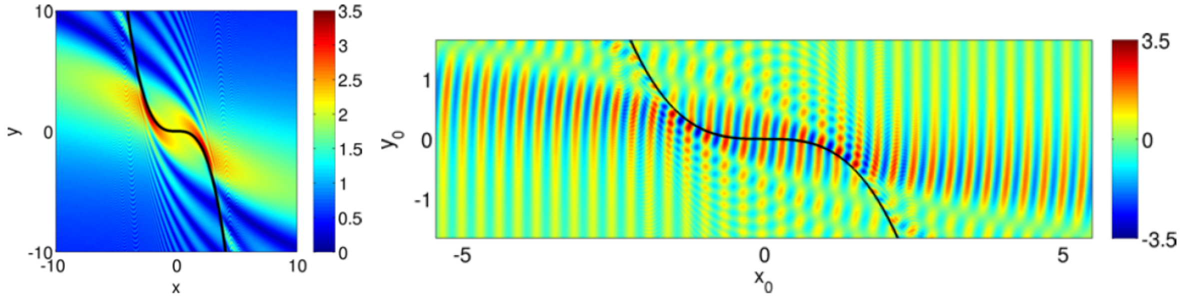
A related integral with quintic phase:

$$A_{31} = \int_{\Gamma_{31}} e^{i(-yt^2 - xt^4/2 + 2t^5/5)}\, dt, \tag{2}$$

where Γ_{31} is now any contour going from $e^{i9\pi/10}\infty$ to $e^{i\pi/10}\infty$. Plot

- $A_{31}(x,y)|$ on $[-10,10] \times [10,10]$,
- $\mathrm{Re}[A_{31}(k^{1/5}x_0, k^{3/5}y_0)\mathrm{e}^{\mathrm{i}kx_0}]$ on $[-10/k^{1/5}, 10/k^{1/5}] \times [-10/k^{3/5}, 10/k^{3/5}]$,

for $k = 20$ and (if possible) $k = 40$. The $k = 20$ results should look something like this (again, obtained using CHK, I think):



(a) $|A_{31}|$ (b) $\mathrm{Re}\left[A_{31}\mathrm{e}^{\mathrm{i}kx_0}\right]$

Challenge 3

The same integrand as in Challenge 2 but with a different contour:

$$A_{32} = \int_{\Gamma_{32}} \mathrm{e}^{\mathrm{i}(-yt^2 - xt^4/2 + 2t^5/5)}\,\mathrm{d}t, \tag{3}$$

where Γ_{32} is now any contour going from $\mathrm{e}^{\mathrm{i}9\pi/10}\infty$ to $\mathrm{i}\infty$. Plot

- $|A_{32}(x,y)|$ on $[-10,10] \times [10,10]$,
- $\mathrm{Re}[A_{32}(k^{1/5}x_0, k^{3/5}y_0)\mathrm{e}^{\mathrm{i}kx_0}]$ on $[-10/k^{1/5}, 10/k^{1/5}] \times [-10/k^{3/5}, 10/k^{3/5}]$,

for $k = 20$ and (if possible) $k = 40$. Now I have no results to compare to!

Challenge 4

The same integrand as in Challenges 2 and 3 but with a different contour:

$$A_{52} = \int_{\Gamma_{52}} \mathrm{e}^{\mathrm{i}(-yt^2 - xt^4/2 + 2t^5/5)}\,\mathrm{d}t, \tag{4}$$

where Γ_{52} is now any contour going from $\mathrm{e}^{\mathrm{i}17\pi/10}\infty$ to $\mathrm{i}\infty$. Plot

- $|A_{52}(x,y)|$ on $[-10,10] \times [10,10]$,
- $\mathrm{Re}[A_{52}(k^{1/5}x_0, k^{3/5}y_0)\mathrm{e}^{\mathrm{i}kx_0}]$ on $[-10/k^{1/5}, 10/k^{1/5}] \times [-10/k^{3/5}, 10/k^{3/5}]$,

for $k = 20$ and (if possible) $k = 40$. Again, no results to compare to I'm afraid!