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THE TOPOLOGY OF REGULAR CURVE FAMILIES WITH MULTIPLE SADDLE POINTS.* [1]

By William M. Boothby.

Introduction. It is known that the level curves of any function $f(x, y)$ which is harmonic in a simply connected domain form a curve family which is regular (locally homeomorphic to parallel lines) in the neighborhood of every point of the domain with the exception at most of a set of isolated points at each of which the curve family has a singularity of the multiple saddle point type (Figure 1). The central task of this and a later paper [2] is to

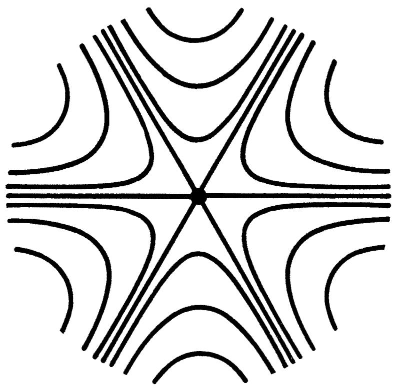

FIGURE 1
MULTIPLE SADDLE POINT

prove that these local properties are sufficient to characterize topologically the level curves of such harmonic functions, and to develop certain general topological properties of these families of curves which should prove useful in the study of harmonic functions and analytic functions. These investi-

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[1] The material in this paper and the one to follow was taken from the author's Ph. D. thesis at the University of Michigan. The author wishes to express his gratitude to Professor Wilfred Kaplan for his guidance in this research and his advice in the preparation of this paper.

[2] The Topology of the Level Curves of Harmonic Functions with Critical Points, to appear in this Journal.

gations generalize some of the results of several papers of W. Kaplan in which curve families which were regular (without singularities) throughout a simply connected domain were considered. In particular it was proved by Kaplan that (1) every such curve family is homeomorphic to the level curves of a function harmonic in a simply connected domain [VIII]; (2) every such family is homeomorphic to the solution family of a system of differential equations $dx/dt = f(x, y)$, $dy/dt = g(x, y)$ [VII]; and (3) that every such family can be decomposed into the sum of a denumerable collection of non-overlapping subfamilies each homeomorphic to the parallel lines of a half-plane. These results are all extended in this and the following paper to the more general type of curve family with isolated singularities of the multiple saddle point type. More precisely, the generalization of (1), (2), and (3) is achieved in the following paper; in this paper the necessary preliminary theorems are proved about the general topological structure of such families, the principal theorem being that the curve family can always be considered as the family of level curves of a continuous function without relative extrema.

Section **1** is devoted to definitions and to enumeration without proof of certain important properties of curve families in general. The most important of these deal with the sets of limit points $L(C+)$, $L(C-)$, $L(C)$, of the individual (directed) curves C of the family. These theorems stem essentially from the work of Poincaré and Bendixson. They depend, however, on the regularity only, as noted by Kaplan.

In Section **2** we restrict our curve families to have isolated singularities only and note that this implies that the set of singularities must then be closed and denumerable. The important notion of index (relative to the curve family) of a closed curve and of a singularity is defined and the theorems we will need stated, in particular the fact that the index of a closed curve is equal to the sum of the indices of the singularities in its interior. This notion of index was defined for curve families by Hamburger [see IX] and it coincides with the Poincaré index when the curve family is the family of integral curves of a vector field. It was studied in detail by Y. S. Chin [II] from which source the theorem mentioned was taken.

Finally in Section **3** we restrict our curve families to be regular in a simply connected domain except for multiple saddle points, which, for reasons explained below, will henceforth be called branch points. Such a family will be called a branched regular curve family and is the principal object of interest in what follows. The singularities will by their very nature be isolated, hence form a closed denumerable set. The index forms a useful tool for ruling out many undesirable configurations of curves, such as closed curves, " polygons " of curves (with singular points as vertices), and so on. Moreover it enables us to investigate some of the relations between the curve family and cross-sectional arcs (i. e. arcs which locally intersect each curve only once). Finally it enables us to show that each curve either extends to infinity in a given direction on it or ends at a branch point.

From this it follows easily that the curves are of two types, one the regular curves which extend to infinity in each direction, the other branched curves which end in a branch point at one or both ends. These latter then naturally form configurations which are trees (without ends) in the topological sense, i. e. they are locally finite, one dimensional simplicial complexes, with curves of the family as 1-cells, branch points as 0-cells. It is shown by use of a theorem of Adkisson and MacLane [I] that any such tree may be straightened out to a rectilinear one by a homeomorphism of the entire plane.

Finally in Section **4** it is shown that the trees are distributed in such a fashion on the plane that, if it is done with sufficient care, we may remove certain curves of each tree with their endpoints, i. e. make cuts in the plane extending to infinity along the tree, in such a fashion that the plane with these curves removed is simply connected and the remaining curves of the family form a regular curve family filling this simply connected subset of the plane. It follows from Kaplan [IV] that these curves are the level curves of a continuous function without relative extrema, and by extending this function to the entire plane, we obtain our family too as the level curves of a continuous function without relative extrema.

1. General Properties of Regular Curve Families in the Plane.

1. 1. Curve families filling a region. An *open curve* will mean a homeomorphic image of an open interval, a *closed curve* a homeomorphic image of a circle, a *half-open curve* a homeomorphic image of a half-open interval, and an *arc* of a closed interval. A *curve* will mean any one of these four. A family F of curves will be said to *fill* a subset R of the Euclidean plane π if every curve of F is in R and every point of R lies on one and only one curve of F. If U is a subset of π such that each curve C of a family F filling R intersects U in a set $U \cap C$ each of whose components is a curve, then we denote by $F[U]$ the curve family filling $U \cap R$ whose curves are the components of $C \cap U$ for all C in F. If the curve family F fills R and the curve family G fills S, then F and G will be called *homeomorphic* if there is a homeomorphism of R onto S such that the image of each curve in F is a curve in G. If p is a point of R, R filled by a curve family F, then C_p will denote the curve of F through p.

1.2. Regularity. Henceforth, F will denote a curve family filling a subset R of π, the *oriented* Euclidean plane. We shall use R_0 to denote the rectangle $|x| \leqq 1$, $|y| \leqq 1$, of the xy-plane, and F_0 the family of lines $y = \text{constant}$ filling R_0. F will be said to be *regular* at a point p of R if there is a set $U(p)$ in R to which p is interior (relative to R) and such that $F[U(p)]$ is homeomorphic to F_0. F is then *regular in R* if F is regular at every point of R. A *cross-section* of F (*through* the point r) is an arc pq (to which r is interior), such that pq lies in a subset R' of R which is open relative to R, and such that each curve of $F[R']$ meets pq at most once. An *r-neighborhood* of a point p of R will mean a subset $U(p)$, $\overline{U(p)} \subset R$, which contains p and is open relative to R, and is moreover such that $F[\overline{U(p)}]$ is homeomorphic to the family F_0, filling the rectangle R_0 (above) of the xy-plane, in such a way that the inverse images of the lines $|x| = 1$ are cross-sections. The following properties of regular curve families will be useful later.

THEOREM 1.2-1. *If a family F fills an open region R and is regular in R, then each curve of F is either open or closed in π.* [IV, 1]

THEOREM 1.2-2. *If a family F fills any region R and is regular in R, then every point p of R has an arbitrarily small r-neighborhood $U(p)$, and there is a cross-section pq with p as endpoint. If p is in the interior of R, then there is a cross-section through p. Moreover if st is any arc lying on a curve C of F, then there is, within any ϵ-neighborhood $U_\epsilon(st)$, an r-neighborhood containing st.* [IV, 8 and VI, 1]

THEOREM 1.2-3. *Let F be a curve family filling R where $R = R_1 \cup R_2$ and $F[R_1]$ and $F[R_2]$ are both defined. If p is an interior point of R, and $F[R_1]$ and $F[R_2]$ are both regular at p, then F is regular at p.* [VI, 2]

1.3. The sets $L(C+)$ and $L(C-)$. From this point on we assume every curve of F directed, although we may later find it at times convenient to reverse directions on a given curve. If C is any open curve in F, then by a *positive* (*negative*) *limit point* of C will be meant any point q which is the limit of a sequence $p_n = f(t_n)$, where C is the image under f of $0 < t < 1$ with increasing t corresponding to the positive direction on C and $t_n \to 1$ ($t_n \to 0$). The set of all positive (negative) limit points of the directed curve C will be denoted by $L(C+)$ (by $L(C-)$). $L(C)$ is defined by $L(C) = L(C+) \cup L(C-)$ and $C \cup L(C)$ is the point set closure of C. Clearly, $L(C) \cap C$ is empty since C is homeomorphic to $0 < t < 1$. The theorems below involving $L(C+)$ hold equally well for $L(C-)$.

THEOREM 1. 3-1. *If C is an element of a regular curve family F, and $L(C+)$ contains a closed curve D of F, then $L(C+) = D$.* [IV]

THEOREM 1. 3-2. *If F is a regular curve family filling R and p is in R and, moreover, in $L(C+)$ for some curve C of F, then every point of the curve D_p of F through p is in $L(C+)$.* [IV, 7]

2. Isolated Singularities and Index.

2. 1. Isolated singularities. Let F be a curve family regular in R and b be an isolated boundary point of R. Then, if for some neighborhood $U(b)$ in π, F is regular in $U(b) - b$ but not in $U(b)$, we shall call b an *isolated singularity* of F. In [VI] W. Kaplan has completely classified the structure of a regular curve family in the neighborhood of an isolated singularity. Below we shall actually be interested in a very restricted type of isolated singularity; however, in this section no restrictions will be made.

THEOREM 2. 1-1. *Let F be a regular curve family filling the region R, where R consists of the entire plane minus a set of points B, each of which is an isolated singularity of F. Then B is a closed and denumerable point set.*

Proof. From the fact that each point of B has a neighborhood containing no other point of B it follows at once that B is denumerable and that no point of B is a limit point of B. Now consider a point p at which F is regular. We must show that p is not in $\overline{B}$, whence $\overline{B} = B$. By the definition of regularity there is a set $U \subset R$ such that p is interior to U relative to R and there is a homeomorphism $f\colon U \to R_0 = \{(x,y) \mid |x| \leqq 1; |y| \leqq 1\}$. It is clear that U is a Jordan domain bounded by the inverse image of the closed curve which forms the boundary of R_0. Moreover we can, as is well-known, extend f to map all of π on the xy-plane. Hence, for simplicity, we shall deal with the homeomorphic image of π, R, F, etc. on the xy-plane. If $p' = f(p)$ is interior to $R_0 = f(U)$, then p' is not in $\overline{B'} = \overline{f(B)} = f(\overline{B})$ since the complete description of U, the r-neighborhood of p, makes it clear that the interior of U (in π) is in R. Now assume p' lies on the boundary of R_0. Since p' is by definition of U, an interior point of R_0 relative to R', there is a small circle which with its interior intersects R' in a set contained in R_0. Then clearly, if this circle is of radius less than one, at least two quadrants of it lie outside R_0 hence outside R', i. e. in B'. It follows that B' cannot be a set of isolated points. Therefore our assumption that there was such a point on the boundary of R_0 is false, and p' is interior to R_0 and not in $\bar{B}'$. Hence $\dot{B}' = B'$ and $B = B$, which completes the proof.

2. 2. Index. Following the definition given in Kerékjártó [IX, p. 251 ff.], we define the *index* of an isolated singularity b of a family F as follows: Let K be any simple closed curve containing the isolated singularity but no other singularities on it or in its interior and let $U_1, \cdots, U_n$ be a covering of K by r-neighborhoods. Then, if the neighborhoods have been chosen sufficiently small, it is clear that we may replace K by a simple closed curve K' in $\bigcup_{i=1}^{n} U_i$ and such that (i) K' is a "polygon" composed of sides which are alternately (1) arcs of curves of F and (2) cross-sections of F, and (ii) K, K' each contain in their interiors only the singularity b. Every vertex of the polygon K' is the intersection of a cross-section and a curve of F; we call it an *internal* vertex, if the curve of F which forms the side at that vertex enters the interior of K' at the vertex, and in the other case we call it an *external* vertex. If we denote the number of internal vertices by e and external vertices by a, then let $\rho(K') = 1 - (a - e)/4$, it is shown by Kerékjártó that $\rho(K')$ so defined is independent of the particular K' chosen, depending only on the singularity b. We may then denote this number by $\rho(b)$ and define it to be the index of b.

More generally, given any closed curve K in the region R filled by the regular curve family F, we may in the same fashion define a curve K' which is close to K, which contains in its interior exactly those singularities contained interior to K, and which consists itself of alternate arcs and cross-sections. Exactly as above we may define $\rho(K') = 1 - (a - e)/4$ and show that this quantity depends only on K, not on the choice of K'. Hence we denote it $\rho(K)$ and call it the index of the curve K. Since we have restricted ourselves to families with isolated singularities, there will of course be only a finite number of singularities interior to any closed curve. The index of a closed curve relative to a given curve family has been studied in detail by Y. S. Chin [II] and he proves the following:

THEOREM 2. 2-1. *Let K be a simple closed curve in R, then $\rho(K) = \sum_{i=1}^{n} \rho(b_i)$ where b_i, $i = 1, \cdots, n$ are the singularities of F interior to K.*

This theorem will not be proved since a detailed proof may be found in [II]. We also state the following useful theorem from Kerékjártó.

THEOREM 2. 2-2. *Let b be an isolated singularity of the regular curve family F and let b be the positive endpoint of exactly p curves of F and the negative endpoint of exactly q curves of F. Then* $\rho(b) = 1 - (p+q)/2$.

3. Regular Curve Families Whose Singularities Are Branch Points.

3. 1. Branch points. Let b be a boundary point of $R \subset \pi$ and F a curve family regular in R. If there exists a neighborhood $U(b)$ such that $F[U(b) - b]$ is homeomorphic to the level curves of $\Re(z^n)$, the real part of $f(z) = z^n$, $n > 1$, under a homeomorphism g carrying $U(b)$ onto $|z| < 1$ with $g(b) = 0$, then we say that b is a *branch point* of F with multiplicity n. This configuration is probably better known as a multiple saddle point, but the term branch point will be adopted below for two reasons: first, these points appear as branch points on trees and, second, they will have branch points of the inverse function as images if we are concerned with the level curves of the real part of an analytic function. The term saddle point was used in the title because it seems to be more suggestive of the type of curve family considered. (See Figure 1.)

The neighborhood $U(b)$ above will be called an *admissible* neighborhood; more precisely $U(b)$ plus the homeomorphism g, which shall always be taken as an orientation preserving homeomorphism, will constitute an admissible neighborhood. In the case of a branch point b of multiplicity n, there are exactly $2n$ curves of $F[U(b)]$ which end at b, i.e. which may be directed so that $L(C+) = b$. It follows that the multiplicity is independent of the choice of $U(b)$. It is clear that b is an isolated singularity of F.

From now on we shall consider only curve families F which are regular in a region R which in general will contain all of π except for a set B of branch points of F. At times it may be expedient to let $R \cup B$ be not all of π but a simply connected domain in π. Of course, topologically the two are equivalent and for convenience we shall consider almost always the first case. Such families will be called *branched regular curve families* filling [3] π (or an open, simply-connected subset of π). The following theorem is well-known:

THEOREM 3. 1-1. *The family F of level curves of a function $f(x,y)$ harmonic in a simply connected* [4] *domain D is a branched regular curve family filling D. The critical points of $f(x,y)$ are the branch points of F.*

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[3] Strictly speaking, according to the definition in 1. 1, the curve family fills not π but π *minus the branch points.* For convenience the latter phrase will often be dropped below.

[4] Simple connectedness is not needed for this theorem, but is needed for the converse.

It is this theorem which motivates in large part this study of such curve families. A converse to this theorem will be proved in a subsequent paper.

3. 2. General properties of branched regular curve families. Restricting ourselves now to branched regular curve families F filling π, i. e. more precisely $R = \pi - B$, B the set of branch points, we make a number of preliminary definitions. The curves of F are assumed directed, so that there shall be no ambiguity in the use of the symbols $L(C+)$ and $L(C-)$. We note that since R is open by Theorem 2. 1-1, every curve of F is by Theorem 1. 2-1 either open or closed. If $L(C) = 0$ we call C a regular curve, and if $L(C+) = b \varepsilon B$, i. e., a branch point, we shall say that C is a *branched curve*, branched at the positive end at b; we also call b the *positive endpoint* of C in this case. Similarly if $L(C-) = b' \varepsilon B$. It will subsequently be shown that the only possibility is that $L(C+) = 0$ or $= b$, a single branch point, and similarly $L(C-)$, so we shall not give any name to the as yet possible type of curve which might have more than one positive (or negative) limit point).

If $b \varepsilon B$, the curves of F which have b as endpoint, together with their endpoints, form a set called the *star of* b, $St(b)$; and the same curves without their endpoints, except b, form the *open star of* b. If b is of multiplicity n, then there are at most $2n$ curves in $St(b)$; it will be shown later in this section that there are exactly $2n$, i. e., the two endpoints of a curve cannot coincide. It is useful to note that since $St(b)$ contains a finite number of curves, plus the fact that B is denumerable, there are at most a denumerable number of branched curves in F.

If $C_1, \cdots, C_n$ are $n \geqq 2$ distinct branched curves of F with their endpoints, and may be so directed that $L(C_i+) = b_i = L(C_{i+1}-)$, $b_i \varepsilon B$ and b_i distinct for $i = 1, \cdots, n-1$, and if in addition $L(C_1-) \neq b_i \neq L(C_n+)$, $i = 1, \cdots, n-1$, then we call $C_1, \cdots, C_n$ either a *simple polygon* of branched curves or a *chain* of branched curves according to whether or not there is a $b_0 \varepsilon B$ such that $L(C_1-) = b_0 = L(C_n+)$. In brief, the curves $C_1, \cdots, C_n$, together with their endpoints will form a chain if the collection is homeomorphic to an arc, and a simple polygon if homeomorphic to a circle. For convenience we shall call a single curve a polygon if it has coincident endpoints. The theorem which follows shows that the polygon configuration cannot occur in F.

THEOREM 3. 2-1. *A branched regular curve family F filling π can contain neither a closed curve nor a simple polygon of branched curves.*

Proof.[5] Suppose F contains a closed curve C. Kaplan has shown [IV] that if a regular curve family contains a closed curve, the interior of the curve must contain a boundary point, i. e., in this case an element of B. Thus it follows from Theorem 2. 2-2 that the sum of the indices of the branch points interior to the closed curve is a negative integer, since by that theorem the index of a branch point of multiplicity n is $1-n$. However, the index of the closed curve itself is one, which is a contradiction to Theorem 2. 2-1. Thus F contains no closed curve.

Next we suppose that F contains a polygon K of branched curves. We construct a curve family filling the circular region $R=\{(x,y) \mid x^2+y^2 \leqq 4\}$ of the xy-plane as follows: fill the annular region $R_1=\{(x,y) \mid 1<x^2+y^2 \leqq 4\}$ with circles $x^2+y^2=c^2$, $1<c \leqq 2$, then map K with its interior onto R_2, the unit circular disc, K mapping onto the circumference. Then $R=R_1 \bigcup R_2$, except for the images of the singular points interior to K plus certain of the images of those on K, is filled by a curve family F'. $F'[R_1-S]$ and $F'[R_2]$ (where $S=$ singular points mentioned above) are regular and it follows from Theorem 1. 2-3 that F' is regular in $R-S$. We now examine the singularities more closely. Let b be a singular point on K, i. e., the common endpoint of two curves of the polygon; b is a branch point of multiplicity n and thus $St(b)$ contains more than the two curves of K which meet at b. If $St(b) \bigcap \operatorname{int} K=0$ it is clear that the image b' of b will not be a singular point of F'; however, if this is not the case, b' will be, and there will be at least three curves of F' which have the image b' of b as endpoint. Thus by Theorem 2. 2-2 the index $\rho(b')$ of this singularity of F' is $\leqq -\frac{1}{2}$. Hence if we consider the curve $x^2+y^2=4$, whose index is one, we again reach a contradiction, for this curve contains singularities, all of whose indices are negative and thus the requirements of Theorem 2. 2-1 cannot be satisfied. Should the image K' of K contain no singularities of F', we are again reduced to the case proved earlier, since F' will in fact be a branched regular curve family. Whence the theorem is proved.

We shall now describe some other simple configurations which cannot occur in a branched regular curve family. Let us suppose that a cross-section pq crosses a curve C of F at exactly two points t and u between p and q. Denote by $(tu)_1$ the arc of pq between t and u, and by $(tu)_2$ the arc on C. Then $K=(tu)_1 \bigcup (tu)_2$ is a simple closed curve. We distinguish three possible cases: (i) if p and q are exterior to K, we call this pattern a *bay*, (ii) if one of the points is interior and the other exterior, we call it a *spiral*,

[5] It is suggested that many of the proofs can be followed much more easily by the reader if he makes his own diagrams where none is provided.

and (iii) if both p and q are interior to K, we call it a *harbor* (see Figure 2). From the definition it is seen that the index $\rho(K)$ is $\frac{1}{2}$, 1 and $1\frac{1}{2}$ respectively in the three cases. Since the sum of the indices of the branch points interior to K must be 0 (if there are none) or a negative integer, we have at once:

THEOREM 3. 2-2. *A branched regular curve family filling π can contain neither a bay, nor a spiral, nor a harbor.*

THEOREM 3. 2-3. *The set $L(C+)$ is either empty or contains a single branch point. Similarly $L(C-)$.*

Proof. Suppose $L(C+)$ contained a branch point b and in addition contained points other than b. Let p_n be a sequence of points on C which

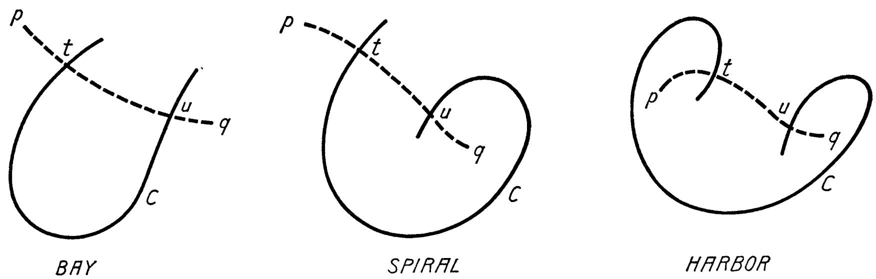

FIGURE 2

approach b, which we assume to have multiplicity n. We consider an admissible neighborhood $U(b)$ with its homeomorphism f carrying U onto the interior of the unit circle. Let F' denote the level curves of $\Re(z^n)$ onto which $F[U]$ is carried homeomorphically by f. Since $L(C+)$ contains points other than b it is clear that $f(C)$ is not one of the radial curves, $\theta = (k/n)\pi$, in F' but must have its image in one or more of the sectors formed by these radial curves, and in fact since $f(p_n) \to f(b)$ the origin, $f(C)$ must cross one of the sectors an infinite number of times. Hence it must be possible to choose points q_n on C near p_n so that q_n approach a regular point q whose image lies on one of the radial lines. Thus we are forced under our assumption to conclude that $L(C+)$ contains at least one regular point q in $F[U(b)]$. Finally, it is clear from the structure of $f(U)$ that this implies that a cross-section through q crosses C an infinite number of times and hence must form at least one bay, spiral, or harbor in so doing. This contradicts the previous theorem; whence it follows that if b is in $L(C+)$, it is the only point in this set.

We remark that this proves in particular that if a curve C is bounded in the positive direction, then $L(C+)$ must be a branch point.

THEOREM 3.2-4. *An arc pq of $R = \pi - B$ is a cross-section if and only if it meets each chain (or curve) at most once.*

Proof. If pq satisfies the conditions, it is a cross-section by definition. Now suppose pq is a cross-section and that it crosses some chain twice. Let t and u be successive points of crossing and denote by $(tu)_1$ the arc from t to u on pq, and by $(tu)_2$ the arc on the chain. We then have a simple closed curve $K = (tu)_1 \cup (tu)_2$. Interior to K there are only a finite number of branch points, hence by virtue of the preceding theorem we can surely find a curve C through some point r interior to K which leaves K in both directions. Letting r_1 be the first point in the positive direction from r at which C intersects $(tu)_1$, and r_2 the first in the negative direction, then the arcs from r_1 to r_2 on $(tu)_1$ and C must form a bay, a spiral, or a harbor. Since this is a contradiction to Theorem 3.2-2 the theorem follows.

3.3. Trees. In this section we define an equivalence relation which decomposes the oriented plane, π, into a collection of disjoint closed sets, each of which is a sum of curves of F and points of B, and each of which is a topological tree of a certain type which we define below:

Definition. Let the closed set T of the oriented plane be decomposable into the sum of an at most denumerable collection of subsets C_i, each closed in π, and satisfying the following four conditions:

(1) Each set C_i is the homeomorphic image of either a closed, half-open, or open line segment (whence we will refer to it as a curve).

(2) Each set C_i has at most an endpoint in common with any C_j, $i \neq j$; and if we denote by $St(b)$ the collection of all curves with b as endpoint, then $St(b)$ consists of a finite even number of curves $\geqq 4$.

(3) There is a unique finite chain $c(C_i, C_j) = (C_i, C_{i_2}, \cdots, C_{i_k}, C_j)$ from C_i to C_j for every i, j; i. e., each curve of the chain having an endpoint in common with the preceding curve as in the definitions of **3.2**.

(4) The sets *open* $St(b)$, consisting of the curves of $St(b)$ without their endpoints opposite b, and *open* C_i, consisting of C_i without its endpoints, are both open sets in T (as a subspace of π).

Then we say that T is a *tree*. (See Figure 3.) Our use of this term is much less general than is usual, but since we consider only this specialized type of tree throughout, there should be no confusion in the use of the term.

The decomposition of any tree T into sets C_i is *unique* quite clearly, except for the numbering, and therefore we may speak without ambiguity of *the* curves of T and *the* endpoints of curves (or, i. e., branch points) of T. Note that a tree is connected and, in fact, arcwise connected by (3) and that by the uniqueness of the chains of (3) there can be no closed curve in T. Condition (4) plus the fact that T is closed in π is equivalent to the following statement: If (p_n) is any sequence of points of T and $p_n \to p \varepsilon \pi$, then $p \varepsilon T$ and all the points of p_n after some N will lie either on $St(p)$ or on a single curve C_i of T depending on whether p is or is not an endpoint of some curve of T. In the language of combinatorial topology each tree, as described above, is a locally-finite, connected, one-dimensional complex containing no one-cycles. In order to exhibit this, it would be necessary to introduce arbitrarily an infinite number of vertices tending to infinity on each curve of the tree which is homeomorphic to a half-open line segment. Once this is done, the statement is clearly true.

It is clear that in particular any regular curve C of a curve family F is a tree with C being itself the only curve of the decomposition. Now, among the elements of our family F we define the relation *joins* as follows: C is said to *join* C' if and only if there is a finite chain $c(C, C')$ of curves of F from C to C'. If we add to this definition that every curve joins itself, then this is easily shown to be an equivalence relation on the curves of F. We denote by T_C the equivalence class of C, including with each curve its endpoint, i. e., T_C is the set of all curves of F, together with their endpoints, which join C. These equivalence classes are disjoint sets and will be shown below to be trees in the sense of our definition. Thus it will be shown that the curves of a branched regular curve family F fall into classes, each of which is a tree.

THEOREM 3. 3-1. *An arc pq on π is a cross-section of F if and only if it lies entirely in $R = \pi - B$ and has at most one point of intersection with each set T_C.*

Proof. If pq has at most one point in common with each set T_C, since T_C is itself a sum of curves of F with their endpoints, then it will have at most one point in common with each curve of F and hence be a cross-section by definition.

On the other hand, by Theorem 3. 2-4, it is necessary that pq meet any set T_C at most once if it is a cross-section, since if pq met T_C at points r, s, then either C_r, C_s are the same curve or else there is a chain $c(C_r, C_s)$ either of which is impossible by that theorem.

THEOREM 3. 3-2. *Each set T_C of a branched regular curve family F is a tree in the sense of our definition.*

Proof. In the event that C is a regular curve the theorem is trivial since $T_C = C$, as already noted. Now let T_C contain a singular curve, then it follows that it contains only such and at most a countable number, since there are at most a countable number of singular curves in F. Each curve of F in T_C, together with its endpoint(s), will constitute a curve C_i of the decomposition of T_C. Each such set is closed in π, since we include endpoints, and is homeomorphic to either a closed or half-open segment, the latter if the curve extends to infinity in one direction. Thus (1) is satisfied. Condition (2) is, however, also satisfied since each set C_i, being the point set closure of a curve of F, has at most an endpoint in common with any set C_j, $i \neq j$, and, if b is any endpoint, then $St(b)$ contains at least four curves and always an even number, $2n =$ twice the multiplicity of b as a branch point. Likewise (3) is satisfied, i. e., the existence of a chain $c(C, C')$ from $C \subset T_C$ to $C' \subset T_C$ is part of the definition of T_C; and the uniqueness is due to the fact that there can be no polygons of branched curves of F by Theorem 3. 2-1. Finally, we prove simultaneously that condition (4) is satisfied and that T_C is closed as a subset of π. Let p_n be any sequence of points of T_C with a point p of π as limit point. Now if p is a regular point of F, then we take an r-neighborhood $U(p)$ and note that unless every p_n lies on the same curve of F, which is necessarily C_p itself, then we have more than one curve of T_C intersecting $U(p)$ and hence a cross-section through p which must cross T_C more than once, contrary to Theorem 3. 3-1. If p is a branch point, then taking an admissible neighborhood of p, we observe that unless we assume all the points p_n, $n > N$, to lie on $St(p)$, we arrive at the same contradictory conclusion by considering a cross-section from p into one of the sectors of the admissible neighborhood. Thus we conclude that the theorem must be true.

We return to a discussion of a tree T, in the sense of our definition, to study its general structure and its relation to its complementary domain in π. Whether it is or is not a set T_C, i. e. composed of curves of a branched regular curve family or merely abstractly given, is at the moment immaterial. As previously noted, the decomposition of T into curves is unique, and hence we may refer without ambiguity to *the* curves and *the* branch points (or endpoints) of T. Since T is assumed to be imbedded in an oriented plane, a cyclic order is induced on the curves of $St(b)$ [see I]; this allows us to make the following definitions.

We shall call curves C, C' *clockwise adjacent* if they have a common endpoint b and, in the order induced on $St(b)$ by the orientation of π, moving clockwise the curve C' follows C. In the case of curves of F this means that in each (orientation preserving) homeomorphism of $F[U]$, U an admissible neighborhood of b, onto the level curves of $\Re(z^n)$ in $|z| < 1$, the images of $C \cap U$ and $C' \cap U$ will correspond to radial lines $\theta = (k/n)\pi$, $\theta' = ((k-1)/n)\pi$ respectively. Clearly, given a curve C with endpoint b, there is exactly one curve C' such that C, C' form a clockwise adjacent pair. We call C', C *counterclockwise* adjacent if C, C' are clockwise adjacent. A chain $C_1, \cdots, C_n$ is called an *adjacent chain* if C_i, C_{i+1} are clockwise adjacent for each $i = 1, \cdots, n-1$ or if, for each i, they are counterclockwise adjacent. We shall also consider infinite collections $\{\cdots, C_{-2}, C_{-1}, C_0, C_1, \cdots\}$ of branched curves with the property that for every $k < m$ the subcollection $C_k, \cdots, C_m$ is a chain. Such a collection will be called an *infinite chain* or *half-infinite chain* depending on whether the indices run through both all positive and all negative integers, or whether there is a least negative or a greatest positive integer. An infinite chain will be called adjacent if every subchain is adjacent.

It is convenient at this point to give some attention to a theorem due to Adkisson and MacLane [I] which states that if $\bar{T}$, $\bar{T}'$ are two homeomorphic Peano continua lying on spheres S, S' respectively, then a homeomorphism from $\bar{T}$ to $\bar{T}'$ can be extended to a homeomorphism of S to S' if and only if it preserves the relative sense of every pair of triods of $\bar{T}$. By a triod, $t = [\alpha, \beta, \gamma]$, of $\bar{T}$ is meant any set of three arcs α, β, γ in $\bar{T}$ which have only a single point, called the vertex, in common. A homeomorphism is said to preserve the relative sense of triods of $\bar{T}$ if every two triods t_1, t_2 which have the same sense (i. e., both clockwise or both counterclockwise) on S are carried into two triods t_1', t_2' of $\bar{T}'$ which have the same sense on S'. Let us denote by $\bar{\pi}$ the plane π plus the point ∞ and by $\bar{T}$ the tree T plus the point ∞. Assuming for the moment that the set $\bar{T}$ is a Peano continuum, the theorem above is applicable to our situation. It is a direct consequence of this theorem that, if T, T' are two homeomorphic trees on π and the xy-plane respectively, then any homeomorphism between them may be extended to a homeomorphism of the planes if and only if the relative sense of the curves of $St(b_1)$, $St(b_2)$ is preserved for every pair of branch points b_1, b_2 of T, the only possible vertices for triods being branch points of T and ∞. In order to show that this is a consequence of the theorem, it must only be shown that the relative sense of every pair of triods of $\bar{T}$ is preserved if this is true for every triod of T, i. e. we must consider triods with vertex at ∞. This will be seen to follow from Theorem 6 of the same paper which states that two non-intersecting triods $t_1 = [\alpha_1, \beta_1, \gamma_1]$, $t_2 = [\alpha_2, \beta_2, \gamma_2]$ have opposite sense on a sphere S if and only if there exists on S a θ-graph whose vertices are the vertices of t_1 and t_2 and whose three (non-intersecting) arcs contain respectively the legs α_1 and α_2, β_1 and β_2, γ_1 and γ_2. Now it is clear from condition (3) in the definition of a tree (the arcwise connectedness) that given any triod with vertex at ∞, it is possible to find at least one triod with vertex at a branch point of T which is, with it, part of a θ-graph, i. e. a triod with sense opposite that of the given one and with vertex at a branch point. Thus if the sense of triods whose vertices are branch points is always preserved, so also for those whose vertex is at ∞. The conclusion is immediate that we may restate the theorem of Adkisson and MacLane, as we have above, for our own purpose here, if it is true that $\bar{T}$ is a Peano space. But this is clearly true: it is connected by condition (3) of the definition; it is locally connected and locally compact since by condition (4) open C and open $St(b)$ are open sets in T, however these sets have these properties and they cover T. Now T is closed in π by definition, and, on $\bar{\pi}$, ∞ is a limit point of T. By adding this point to T to obtain $\bar{T}$ we clearly obtain a closed set, i. e. $\bar{T}$ is in fact the closure of T in $\bar{\pi}$. Hence $\bar{T}$ is a closed, connected subset of the 2-sphere $\bar{\pi}$. It follows that $\bar{T}$ is compact and hence locally compact. Finally $\bar{T}$ is locally connected since a compact continuum cannot fail to be locally connected at just one point [XI] and, of course, the addition of a single point, ∞, to T does not disturb the local connectedness of T. Thus $\bar{T}$ is a Peano space.

In Section **3.4** a numbering system for the curves of a tree is described and it is made clear that given any tree it is possible to construct a homeomorphic rectilinear model of it in the xy-plane, i. e. a model consisting of closed and half-open line segments. Moreover, this may even be accomplished so that any one given chain, either finite or infinite, will have its image in the model on a straight line, e. g. on the x-axis. Finally all this may be done preserving the sense in $St(b)$ for every $b \varepsilon B$ and thus the homeomorphism of our tree onto its model may be extended to a homeomorphism of π onto the xy-plane. Thus any tree, no matter how badly "twisted," may be straightened out, by a homeomorphism of the entire plane, into a rectilinear tree. This result is established in **3.4** entirely independently of the remainder of this section, the proof being deferred for convenience only. In order to avoid long and cumbersome proofs the theorems of this section will be established by free use of this fact plus some slight appeal to intuition as far as rectilinear graphs in the plane are concerned.

We now consider relations between a tree T and its complementary domains. In this connection it is convenient to consider a special class of adjacent chains (of curves of T) which we shall call maximal chains. An adjacent chain of curves of a tree is said to be *maximal* if it is not a subchain of any adjacent chain. It is an immediate consequence of our definitions that a chain of adjacent curves is maximal if and only if (1) it is infinite, or (2) it is half-infinite and its initial (or terminal) curve has only one endpoint, or (3) it is a finite chain and both its initial and terminal curves

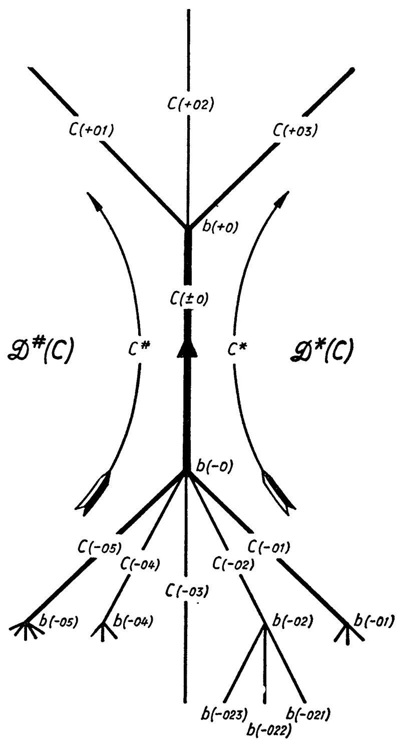

FIGURE 3
RECTILINEAR TREE

have each only one endpoint (a curve of a tree with only one endpoint extends to infinity in the direction opposite to that with the endpoint). Moreover, since a maximal chain is closed as a subset of the tree and a tree is a closed subset of π, so also is every maximal chain a closed subset, and thus is in fact an open curve extending to infinity in each direction, thus dividing the plane into two Jordan domains. (Reference to Figure 3 will clarify this section as well as the following one.)

THEOREM 3. 3-3. *If T is itself a single curve, then it is its only maximal chain. When T contains more than one curve, then* (1) *each curve of T is contained in exactly two maximal chains which intersect only on this curve and* (2) *every branch point of order n is contained in exactly $2n$ maximal chains whose only common point is the branch point itself. Conversely, the intersection of any two maximal chains can be empty, be a single branch point, or, at most a curve of the tree.*

Proof. Let $C_1, \cdots, C_k$ be any clockwise adjacent chain of two or more curves. Now if C_1 has only one endpoint, then there is no curve C' adjacent to C_1 such that $C', C_1, \cdots, C_k$ is a clockwise adjacent chain, since adjacent curves must have an endpoint in common and at a given endpoint there is exactly one curve clockwise adjacent to a given curve. But, if C_1 has two endpoints, then there is exactly one curve C' such that $C', C_1, \cdots, C_k$ is a clockwise adjacent chain. Similar remarks apply to C_k. If neither C_1 nor C_k has more than a single endpoint, then the chain is maximal; in any other case we may extend the chain, one curve at a time added to the initial or final curve, until we arrive at endcurves which have only one endpoint; or, if we do not come to a curve with one endpoint, indefinitely. In any of these cases, the resulting adjacent chain is maximal since it is an open curve extending to ∞ in both directions. Thus every finite adjacent chain which is not alerady maximal can be extended to a unique maximal chain.

If we begin with a single curve C, with at least one endpoint b, then there is one curve clockwise adjacent to C in $St(b)$ and one counterclockwise adjacent. Thus in $St(b)$ we have C, C' and C, C'', unique adjacent chains containing C, one clockwise and one counterclockwise. Hence, C is contained in just exactly two maximal chains, one of which contains C, C', the other C, C''. Similarly, if b is a branch point, there are just $2n$ pairs of adjacent curves in $St(b)$, whence b is contained in $2n$ maximal chains.

Now consider the converse. If two chains intersect, they surely must have a branch point b in common. If this is their only point of intersection in open $St(b)$, then they can intersect at no other point, since the tree is arcwise connected and can contain no closed polygon. If they intersect along two curves of $St(b)$, they must be adjacent curves since the chains are adjacent chains; hence by the preceding remarks on uniqueness they must coincide. This leaves only the possibility that they intersect along a single curve of $St(b)$, and in this case again, since there are no closed curves in the tree, they either have no other intersection or they coincide.

THEOREM 3. 3-4. *Every maximal chain of a tree T divides the plane* *into two domains, whose complete boundary it is; and one of these domains contains no points of T.*

Proof. The first part of this theorem is just the Jordan curve theorem. The second part becomes clear when we consider a rectilinear model of the tree. Using the Theorem of Adkisson and MacLane and its consequences as mentioned above, we first map π onto the xy-plane so that the maximal chain becomes the x-axis, every curve of T a chain of line segments, and moreover, so that the orientation is preserved, i. e., so that every clockwise adjacent pair of π will still be clockwise adjacent on the xy-plane, and conversely. Now the contention is that all of the image of T, except what is on the x-axis, will lie in one half-plane, say the upper half-plane. If this is not the case, then there will be a point (u, v) of the upper half-plane and a point (x, y) of the lower half-plane, each in the image T' of T. Then, from $C'(u, v)$, the image-curve containing (u, v), there is a chain to any image-curve on the x-axis, i. e., in the given maximal chain. Let C be the last line segment on the last curve of some such chain to lie in the upper half-plane (more precisely, the endpoint p of C lies on the x-axis, but the rest of the curve lies in the upper half-plane). Similarly, we may choose a line segment C' of T' which lies in the lower half-plane except for one endpoint q. For convenience assume p is to the left of q. Clearly, p and q are branch points on that maximal chain of T' which is the image of our maximal chain in π. Now let C_1, C_2, C_3, C_4 be curves of the maximal chain, i. e., line segments on the x-axis, numbered from left to right, such that p is the common endpoint of the first pair, q of the second. Then it is clear that $[C_1, C, C_2]$ is a triod, with vertex p, of curves in clockwise order, and $[C_3, C, C_4]$ is a triod, with vertex q, of curves in counterclockwise order. But this is impossible since C_1, C_2 and C_3, C_4 are both presumed to be adjacent in the same sense, lying as they do on an adjacent chain. Thus all of T' must lie in either the closed upper half-plane or the closed lower half-plane; whence, the theorem is immediate.

Now let C be a *directed* curve of T, a tree consisting of more than one curve. Then we have seen that C determines exactly two maximal curves which we shall denote by C^* and $C^\#$ with the following convention: as we move along C^* in the direction corresponding to the positive direction on C, then the complementary domain of C^* "to the right" (this can clearly be defined in a topologically invariant manner, by the method used in the proof above) will contain no points of T, and as we move along $C^\#$ in the direction corresponding to the positive direction on C, the complementary domain "to the left" will contain no points of T. These domains will be denoted $\mathfrak{D}^*(C^*)$ and $\mathfrak{D}^\#(C^\#)$ respectively, or more simply by $\mathfrak{D}^*(C)$ and $\mathfrak{D}^\#(C)$ respectively. Now as proved above, C^* is the common boundary of two Jordan domains, and the notation for one of them was given as $\mathfrak{D}^*(C^*)$, the other will be denoted by $\mathfrak{D}^\#(C^*)$. Similarly, $C^\#$ divides the plane into the domains $\mathfrak{D}^\#(C^\#)$ and $\mathfrak{D}^*(C^\#)$. When T is just a single curve then C, C^* and $C^\#$ are all the same curve, and $\mathfrak{D}^*(C^\#) = 0 = \mathfrak{D}^\#(C^*)$. If we reverse the direction on C, we must replace $^\#$ by * throughout.

If we remove *open* C from $C^* \cup C^\#$ we get either two or four half-open arcs extending to infinity from the endpoint(s) of C: two if C has one endpoint, four if it has two. We let $\delta^*(C+)$ denote the arc from the positive endpoint of C lying on C^*, and $\delta^\#(C+)$ the arc from the positive endpoint of C lying on $C^\#$. Similarly, we use the notation $\delta^*(C-)$ and $\delta^\#(C-)$ for the arcs at the other endpoint. We also let $\delta(C+)$ stand for $\delta^*(C+) \cup \delta^\#(C+)$, and $\delta(C-)$ for $\delta^\#(C-) \cup \delta^*(C-)$, and finally, $\delta(C)$ for $\delta(C+) \cup \delta(C-)$. These symbols are clarified by reference to Figure 3.

The collection of all curves C^* and $C^\#$ are then just the maximal chains of T. As already noted above each of these maximal chains bounds two domains, one of which contains no points of T. A converse to this also holds, and, denoting by $\bar{\pi}$ the extended plane and $\bar{T}$ the points of T plus the point at infinity, we have:

THEOREM 3. 3-5. *If T is a tree of π, then $\bar{\pi} - \bar{T}$ consists of an at most countable collection of Jordan domains, each bounded by a simple closed curve in $\bar{T}$ containing the point at infinity. The necessary and sufficient condition that a curve of $\bar{T}$ bound one of those domains is that it be a maximal chain of curves of T.*

COROLLARY 1. *If T is a tree of a regular curve family F, then each complementary domain decomposes into a sum of sets T_C of F.*

COROLLARY 2. *The complementary domains D_n, if infinite in number, tend uniformly to infinity with any sequence p_n, $p_n \,\varepsilon\, b\,dry\, D_n$, of their boundary points.*

Proof. As we have already noted, π may be mapped on the xy-plane so that the image of T is rectilinear and even so that a given arc (or chain) of T goes onto the x-axis. It is clear then that the complementary domains are Jordan domains. The plane has a countable basis of open sets, whence it follows trivially that the number of complementary domains must be countable also. A boundary curve of a complementary domain must contain the point at infinity, since T contains no closed curves. Finally, it is clear that π may be mapped onto the xy-plane so that a given complementary domain maps onto the upper half-plane and its boundary onto the x-axis. Thus each such boundary must be a maximal chain.

Corollary 1 follows from the fact that each set T_C is connected and disjoint from every other such set, and hence must lie in a single complementary domain. If Corollary 2 were not true we would obtain a contradiction to either property (4) of a tree or the fact that T is a closed subset of π.

3.4. A numbering system for the curves of a tree. To facilitate further proofs it will be convenient to establish a system for numbering the curves of a tree of π. The numbering proceeds from an arbitrarily chosen, directed curve C of T, which we shall call the *base curve* of the tree. Using the orientation of the plane together with the existence of a unique chain from the base curve C to each curve of T, we set up a 1-1 correspondence between curves of T and a collection of signed finite sequences, the particular collection depending on both T and C, the sign of the sequences depending only on the direction of C. (See Figure 3.)

To the curve C itself we assign, ambiguously, the sequences ± 0, and we write $C = C(\pm 0)$. If C has a positive endpoint, we denote it by $b(+0)$ and number in *clockwise* order, the curves of $St(b(+0))$ by $[C(+0)]$, $C(+01)$, $C(+02), \cdots, C(0u(+0))$, where $u(+0)$ is defined as the number of curves in $St(b(+0))$ less one, i. e., as twice the multiplicity of that branch point less one. We then denote, if it exists, the endpoint of $C(+0k)$ opposite $b(+0)$ by $b(+0k)$. We follow exactly the same procedure at the other endpoint, if there is one, of $C(\pm 0)$. This endpoint is denoted by $b(-0)$ and the curves of $St(b(-0))$ are numbered, again in *clockwise* order, $[C(-0)]$, $C(-01)$, $C(-02), \cdots, C(-0u(-0))$. If the chain $c(C, C')$ from the base curve C to another curve C' of T contains n curves, we shall say that C' is of *order* n with respect to C. The process above then has numbered every curve of T of order 1 or 2 by *exactly one* finite sequence of one or two elements respectively (except for the ambiguity in the numbering of C itself). Moreover, it assigns a unique sequence to the endpoints of the curve, with the endpoint being numbered with the same number as the curve of lowest order having it as endpoint. Two curves C, C' of the same order will be clockwise adjacent if the final integer of the sequence of C is one less than that of the sequence for C'; and two curves C, C' with C of lower order than C' will be clockwise adjacent if the sequence of C' is that of C with a final integer 1 added to it. Finally, the chain from the base curve to a curve C' consists of the curves whose numbering sequences are successive "*lower segments*" of the sequence numbering C', i. e., if $\alpha = 0p_2, \cdots, p_n$ numbers C', then $\alpha_1 = 0$, $\alpha_2 = 0p_2$, $\cdots, \alpha_{n-1} = 0p_2 \cdots p_{n-1}$, $\alpha = \alpha_n = 0p_2 \cdots p_{n-1}p_n$ number the curves of the chain from the base curve to C'.

Now if we assume that everything said above is true for every curve of order n, it is very simple to show that it may be extended *in toto* to the curves of order $n+1$; i. e., let C' be any curve of T of order $n+1$. Then C' is the terminal curve of the chain $c(C, C')$ of $n+1$ curves, all of which except C' itself have already received their unique numbering the next-to-last of them by a sequence α of n terms, which sequence also numbers the common endpoint $b(\alpha)$ of this curve and C'. As before, we number the curves in $St(b(\alpha))$ in clockwise order as $[C(\alpha)]$, $C(\alpha, 1)$, $C(\alpha, 2), \cdots, C(\alpha, u(\alpha))$. Here we let α, k denote the sequence whose first n elements are those of α, and whose $(n+1)$-st is k, i. e., we adjoin k to the sequence α. In this process C' will receive a unique numbering, and the statements above will follow through.

With the help of a little new terminology, we will express these facts in a theorem. Given two collections of sequences, A, A^*, we denote by $A \cup A^*$ the collection of all signed sequences obtained by giving those in A positive sign and those in A^* negative. Using this notation we shall call a collection A of finite sequences *allowable* if:

(1) Every sequence has 0 as first element, positive integers for the other elements, and 0 is a sequence of A.

(2) $\alpha, k \in A$ implies $\alpha, k - 1 \in A$ if $k \neq 1$, and implies $\alpha \in A$ if $k = 1$.

(3) For each $\alpha \in A$ there is defined an integer $u(\alpha) = 0$ or $\geqq 3$ and odd such that if $u(\alpha) \neq 0$ then $\alpha, 1; \alpha, 2; \cdots; \alpha, u(\alpha)$ are in A but not $\alpha, u(\alpha) + 1$; and, if $u(\alpha) = 0$, then there is no sequence of A with α as lower segment, i. e., of the form $\alpha, p_{n+1}p_{n+2} \cdots p_{n+k}$. (Note: If $u(\alpha) = 0$ we call α a *terminal* sequence.)

THEOREM 3.4-1. *Given a tree T, a curve C of T, and a direction on C, then there exist two unique allowable collections of finite sequences, A, A^*, such that there is a* 1-1 *correspondence between the curves of T and the signed sequences $A \cup A^*$ (except for $\pm$* 0 *being assigned to C), and such that there is further a* 1-1 *correspondence between the endpoints of the curves of T and those signed sequences which are not terminal of the collection $A \cup A^*$, these correspondences being as described above and having in particular the properties:*

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(1) *If $C(\alpha)$ is any curve of T, then $C(\pm 0), C(\alpha_2), \cdots, C(\alpha_{n-1})$, $C(\alpha)$ is the chain from the base curve to $C(\alpha)$.*

(2) *$C(\alpha, k)$ has the endpoint $b(\alpha)$ in common with the lower order curve $C(\alpha)$ and, if α, k is not a terminal sequence, the endpoint $b(\alpha, k)$ at the opposite end.*

(3) *$C(\alpha)$, $C(\beta)$ of the same order n are clockwise [counterclockwise] adjacent if and only if $\alpha_{n-1} = \beta_{n-1}$ and $\alpha = \alpha_{n-1}, k$; $\beta = \beta_{n-1}, k+1$ [$\beta = \beta_{n-1}, k-1$]. $C(\alpha)$, $C(\beta)$ of different order are clockwise [counterclockwise] adjacent if and only if $\beta = \alpha, 1$ [$\beta = \alpha, u(\alpha)$].*

It is obvious but tedious to prove that maximal chains, the set $\delta^*(C+)$, $\delta^\#(C+)$ and so on are numbered by sequences with certain characteristic properties. We shall not develop this aspect, but shall state one or two important properties of the numbering below:

THEOREM 3.4-2. *Two trees T on π and T' on π' are homeomorphic under a homeomorphism which may be extended to all of π if and only if we may choose and direct a base curve from each tree and orient the image plane so that the numberings of the two trees are then identical.*

Proof. If the two trees are homeomorphic under such a homeomorphism, then it is clear that for any directed curve C of T as base curve we may choose the homeomorph C' of C as base curve in T', giving it the direction induced by C, and using the orientation in π' induced by the orientation in π, we will get precisely the same numbering for each tree.

On the other hand let T, T' be two trees with identical numberings. We first show that they are homeomorphic. We let $f(\alpha): C(\alpha) \to C'(\alpha)$ be any homeomorphism of $C(\alpha)$ *onto* $C'(\alpha)$ such that $b(\alpha_{n-1})$ maps onto $b'(\alpha_{n-1})$ or, if $\alpha = 0$, so that $b(0+)$ maps onto $b'(0+)$, then $f(\alpha)$ coincides with $f(\alpha_{n-1})$ at $b(\alpha_{n-1})$, the only point where their domains overlap. The map $f: T \to T'$ defined by $f(x) = f(\alpha)x$, for α such that $x \,\varepsilon\, C(\alpha)$, is 1-1, and is continuous on each of a family of closed sets covering T. Now let x_n be any sequence of points on T such that $x_n \to x \,\varepsilon\, T$, then by property (4) of trees, for $n \geqq N$, x_n will lie on C_x or $St(x)$, the latter if x is a branch point. From the continuity of f on C_x and $St(x)$ for every $x \,\varepsilon\, T$, it follows that $f(x_n) \to f(x)$. Hence, since x_n was any sequence and x any point, f is continuous on T. It follows in the same manner that f^{-1} is continuous on T'. Thus f is a homeomorphism from T to T'.

Now in view of the fact that for every branch point b of T, the sense in $St(b)$ must be preserved by f as defined above, by virtue of our earlier slight modification of the theorem of Adkisson and MacLane, it follows that f can be extended to a homeomorphism on all of π.

We now remark that this theorem makes it possible to construct a rectilinear model of any tree T on the xy-plane, and assures us that there will be a homeomorphism of π onto the xy-plane carrying T onto this model. The model is constructed by considering any numbering $A \cup A^*$ of T, and, using line segments of length $\geqq 1$ as our elements, building up the model piece by piece: We begin with a base element corresponding to the sequence ± 0, add segments corresponding to the 2nd order sequences, 3rd order, etc., each time moving further out from our base segment so that its distance from any n-th order segment approaches infinity with n. In this process it is clearly possible to construct the model so that the image of any one particular chain is a straight line, e. g., the x-axis. (See Figure 3.)

3. 5. r-sets and cross-sections. For an arc pq, lying on an adjacent chain of curves $C_1, \cdots, C_n$, it is possible to get a serviceable analog of the r-neighborhood of an arc on a regular curve (cf. Theorem 1. 2-2). By suitably directing C_1, we have both pq and $C_1, \cdots, C_n$ as arcs on $C_1{}^*$. By an *r-set of pq* will be meant any closed set $U \subset \mathfrak{D}^*(C_1{}^*)$ together with a homeomorphism g of U onto the rectangle R_1 of the xy-plane, where $R_1 = \{(x, y) \mid -1 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$, with g having the properties:

(1) g carries $F[U]$ onto the lines $y = \text{constant}$ in R_1.

(2) $g^{-1}(\lambda_i)$ are cross-sections, where λ_i, $i = 1, 2$ are, respectively, that part of the lines $x = -1$ and $x = +1$ in R_1.

(3) pq is mapped into the set $\{(x, y) \mid y = 0, -1 < x < 1\}$.

$U(pq)$ contains no branch points except those on $C_1{}^*$ itself and lies entirely in $\mathfrak{D}^*(C_1{}^*) \cup C_1{}^*$, which contains no points of T_{C_1} except those on $C_1{}^*$. (See Figure 4.) In the event that pq lies on a single curve C_1, then two r-sets which coincide along C_1 but lie on different sides of C_1 will together form an r-neighborhood of C_1. We shall find it convenient to refer to an r-set of a single point p, by which we shall mean one side of a regular neighborhood of p if p is a regular point and one sector of an admissible neighborhood if p is a branch point. We now prove an analog of part of Theorem 1. 2-2.

THEOREM 3. 5-1. *Given any arc pq on an adjacent chain of curves $C_1, \cdots, C_n$; $n > 1$ and any $\epsilon > 0$, there exists an r-set of pq which lies inside the ϵ-neighborhood of pq and in that complementary domain of T whose boundary is the maximal chain containing the given adjacent chain.*

Proof. Let C_1 be directed so that $C_1{}^*$ contains the chain $C_1, \cdots, C_n$. We leave F unchanged in $\mathfrak{D}^*(C_1{}^*)$, the complementary domain of T_{C_1} in which U is to lie, but we alter F in the remainder of the plane by mapping the lines $y = \text{constant}$ of the lower half-plane, including the x-axis onto $C_1{}^* \bigcup \mathfrak{D}^\#(C_1{}^*)$ in such a manner that the x-axis is mapped onto $C_1{}^*$. Then

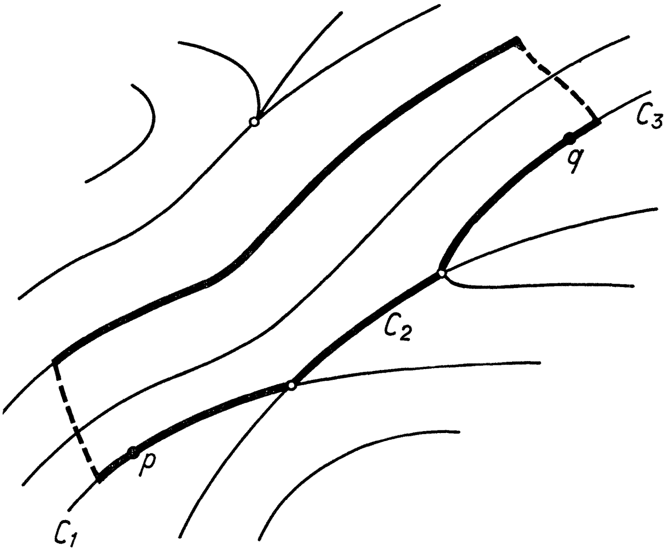

FIGURE 4 AN R-SET

by Theorem 1.2-3 this new family is regular at all points of $C_1{}^*$ and agrees with F in $C_1{}^* \bigcup \mathfrak{D}^*(C_1{}^*)$. Hence by Theorem 1.2-2, there is an arbitrarily small r-neighborhood of pq, call it V. Then $U(pq) = \overline{V \bigcap [C_1{}^* \bigcup \mathfrak{D}^*(C_1{}^*)]}$ will be our desired r-set.

THEOREM 3.5-2. *Let a sequence of points q_n on distinct curves C_n approach the point q, where q is a regular point or a branch point, then there is a curve C which may be so directed that q lies on C^* and an infinite subsequence q_m of q_n lies in $\mathfrak{D}^*(C^*)$. Moreover, if p is any point on C^*, then there is a sequence of points p_m in $\mathfrak{D}^*(C^*)$ such that $p_m \to p'$, and such that for $m > M$ the curves C_{p_m} and C_{q_m} are identical.*

Proof. Let T be the tree of F which contains q, then there are at most a finite number of complementary domains of T on whose boundary q lies, and since the q_n lie on distinct curves, there must be a subsequence q_m of these points lying in one of these complementary domains. The maximal chain which bounds this domain, can, for some suitably directed C be given as $\mathfrak{D}^*(C^*)$. Now let p be any other point on C^*. We may take an r-set $U(pq)$ of pq in $\mathfrak{D}^*(C^*)$ and, if f denotes the homeomorphism $f: U(pq) \to R_1$, then the curves C_m containing q_m map onto lines $y = k_m$ for $m \geqq M$. If $f(p) = (x, 0)$, then the points $p_m = f^{-1}(x, k_m)$ will be the desired sequence.

THEOREM 3. 5-3. *An arc pqr is a cross-section of F if and only if* (1) *it contains no branch points,* (2) *one of the domains $\mathfrak{D}^*(C_q)$, $\mathfrak{D}^\#(C_q)$ contains p, the other r, and* (3) *pq and qr are each cross-sections.*

Proof. We first assume that the arc pqr is a cross-section through q. Then (1) and (3) follow by definition of cross-section. By the Lemma stated in [IV, p. 158] there is an r-neighborhood of q, $V(q)$, such that the image of pqr in R_0 is the y-axis. Every curve crossing $V(q)$ crosses pqr; hence, no curve has more than one line $y = \text{constant}$ as image in R_0. The point q itself maps on $(0, 0)$ and C_q on the x-axis; hence, $V - C_q$ splits into two domains, one containing p, whose image in R_0 is $(0, 1)$, and the other r, whose image in R_0 is $(0, -1)$. Moreover, one of these domains lies in $\mathfrak{D}^*(C_q)$ and the other in $\mathfrak{D}^\#(C_q)$, for q is a point on the common boundary of these two domains and hence every neighborhood of q contains points of each domain.

Now, if we assume that pqr is an arc with the properties (1), (2) and (3), we may show that it is a cross-section by showing that it intersects any set T_C at most once. This is clear at once if we remember that a set T_C cannot have points in each of the domains $\mathfrak{D}^*(C_q)$ and $\mathfrak{D}^\#(C_q)$ so that if pqr had more than one point in common with T_C, each such common point would have to lie in the same domain, i. e., both on pq, or both on qr. This is impossible, however, since both of these arcs are cross-sections. It follows that pqr is a cross-section.

The following corollary is immediate:

COROLLARY 1. *If C, C' both intersect a cross-section pq, and each is directed to cross pq in the same direction, then either $\mathfrak{D}^*(C) \supset \mathfrak{D}^*(C')$ or $\mathfrak{D}^\#(C) \subset \mathfrak{D}^\#(C')$.*

COROLLARY 2. *If an arc is such that any point on it is interior to a subarc which is a cross-section, then it is a cross-section.*

Proof. Let pq be such an arc; then we may cover pq with a finite number of r-neighborhoods which overlap. Then, applying the theorem repeatedly a finite number of times gives the result desired.

4. The Family *F* as the Level Curves of a Continuous Function.

In this section it will be shown that there is a continuous function $f(x, y)$ whose level curves are exactly the family F. The proof of this statement will depend on our ability to remove certain branched curves of F together with their endpoints so as to leave a subset R^* of the plane π, which is open, connected, and simply connected and is such that $F^* = F[R^*]$ is a regular curve family filling R^*. It will then follow from [IV] that there is a continuous function $f^*(x, y)$ defined on R^* and having the family F^* as level curves. Finally, it is shown that $f^*(x, y)$ may be extended to a continuous function on all of the plane with the curves of F as level curves. In this and the next section we will restrict the use of the term *tree* to those sets T_C containing singular curves, i. e. the term will not be used to include a regular curve.

4. 1. The numbering of the trees of *F*.

THEOREM 4. 1-1. *If K is any compact subset of π, then there are at most a finite number of distinct trees of F which intersect K on more than one curve of the tree. Moreover, no more than a finite number of curves from any one tree can intersect K.*

Proof. The second part of the theorem is an immediate consequence of the fact that any point p which is a limit of a sequence of points p_n of a tree must be a point of the tree and, in addition to this, for $n \geqq N$, must lie in $St(p)$ if p is a branch point, or on C_p if p is a regular point. If an infinite sequence of curves of a single tree intersected K, we could, by compactness of K, choose a sequence of points on *distinct* curves of this tree such that the sequence would have a limit point, and hence could not conform to the requirements above.

We prove the first part of the theorem by assuming it false and arriving at a contradiction. Let T_i $(i = 1, 2, \cdots)$ be an infinite collection of trees, each intersecting the compact set K on at least the two curves C_i, C_i'. Using the compactness of K we may choose an infinite subsequence from this collection of trees together with points $p_i \in C_i \cap K$ and $q_i \in C_i' \cap K$ on each tree T_i of the subsequence such that $p_i \to p$ and $q_i \to q$. We then throw away all trees except those of our subsequence and renumber. We may assume p is a regular point lying on a curve C and that the p_i all lie in a single complementary domain of T_C, i. e. in $\mathfrak{D}^*(C)$. If this is not the case, using Theorem 3. 5-2 we may rechoose p and p_i so that they satisfy these conditions.

This may move them out of K, but this causes no difficulty later. We still have $p_i \varepsilon C_i \subset T_i$. In this process, to get the p_i all in a single complementary domain of T_C, it may be necessary to choose a subsequence. If so we throw out the trees T_i containing the points p_i which we remove to get our subsequence, then we renumber. That is, we simultaneously pass to subsequences of p_i and of q_i which correspond, so that we still have $p_i \varepsilon C_i$, $q_i \varepsilon C_i'$ and and $C_i, C_i' \subset T_i$ after renumbering. This method is followed whenever it is necessary to select a subsequence of either p_i or q_i and renumber. Now we may similarly assume q is a regular point on a curve C' and that the q_i lie in $\mathcal{D}^*(C')$, or else use Theorem 3. 5-2 as above, and again choose a subsequence. When this process is finished we may be sure that $p \neq q$, for if this were not the case, choosing an r-neighborhood of $p = q$, we would find it crossed by curves C_i, C_i' of the same tree T_i, and this is impossible since it would imply that a cross-section crossed T_i twice.

Now a brief consideration of the image in R_0 of non-intersecting r-neighborhoods $U(p)$ and $U(q)$ and the images of p_i and q_i which lie in these neighborhoods for $i > N$, makes it obvious that we may choose three trees from our sequence of trees: T_{i_1}, T_{i_2}, T_{i_3} which are such that p_{i_1}, p_{i_2}, p_{i_3} lie in $U(p)$ and may be connected in that order by a cross-section γ running from p_{i_1} through p_{i_2} to p_{i_3}; and at the same time q_{i_1}, q_{i_2}, q_{i_3} lie in $U(q)$ and similarly may be connected by a cross-section γ' running from q_{i_1} through q_{i_2} to q_{i_3} (reference to Figure 5 will make the procedure here more easily followed). Then, denoting by c_j the chain $c(C_{i_j}, C_{i_j}')$ in T_{i_j}, $j = 1, 2, 3$, we may define the following three arcs, λ_{i_1}, λ_{i_2}, λ_{i_3} having only their endpoints p_{i_2} and q_{i_2} in common: λ_{i_2} is the arc $p_{i_2}p_{i_1}$ on γ plus the arc $p_{i_1}q_{i_1}$ on c_1 plus the arc $p_{i_1}q_{i_2}$ on γ'. λ_{i_3} is similarly defined with 1 replaced by 3 in the subscripts above, and finally λ_{i_2} is the arc $p_{i_2}q_{i_2}$ on c_2. Two of these arcs, say λ_{j_1}, λ_{j_2}, must form a simple closed curve Γ containing the third λ_{j_3} in its interior. But this is impossible, since each arc contains a branch point; in particular the arc λ_{j_3}, thus enclosed in the interior (in our example) would contain a branch point; and from this branch point issues a chain of curves of T_{j_3}, all distinct from λ_{j_3}, which must leave Γ at some point r. This point r cannot be on T_{j_1} or T_{j_2} since two trees cannot intersect, nor can it be on γ or γ' since then this cross-section would have two points on the same tree, which is ruled out by Theorem 3. 3-1. Hence, we conclude that our initial assumption is impossible and that the theorem must be true.

This theorem will be used to give a method of numbering all non-trivial trees, i. e., trees containing singular curves. We choose any regular point p on π and let K_n designate the circle (with its interior) of center p and radius n. Now the number of trees cutting K_1 is, of course, denumerable and we number them in any order as $T_{11}, T_{12}, T_{13}, \cdots$ and choose from each a curve C_{11}, C_{12}, C_{13}, respectively, which itself intersects K_1. By the above theorem these choices of curves will be unique for all except a finite number of the trees, and for these C_{1j} is chosen at random from any one of the finite number of curves of the tree cutting K_1. Next we number the trees which

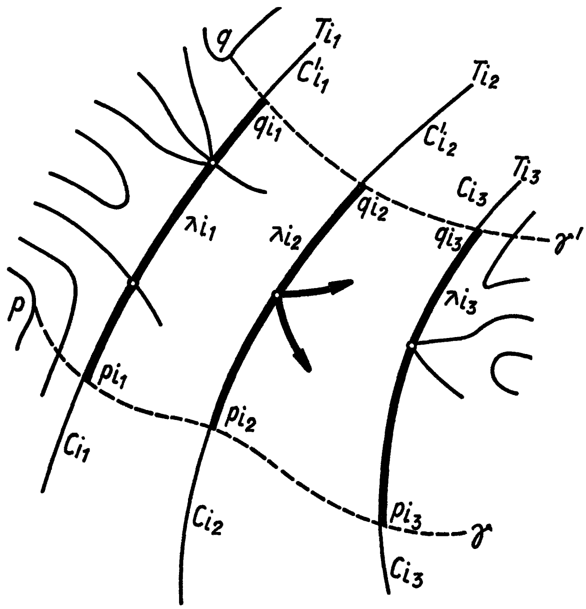

FIGURE 5

intersect K_2 but *not* K_1 as $T_{21}, T_{22}, T_{23}, \cdots$ etc., and let $C_{21}, C_{22}, C_{23}, \cdots$ respectively be curves of these trees which themselves intersect K_2. Proceeding with this process we number the trees cutting K_n but not K_{n-1} as $T_{n1}, T_{n2}, T_{n3}, \cdots$ etc., and choose from each curves $C_{n1}, C_{n2}, \cdots$ cutting K_n. This process will clearly number all the trees of F, and we choose the curves C_{ij} as base curves of the trees, hence determining within each tree T_{ij} a numbering of its curves by sets of finite sequences $A_{ij} \cup A_{ij}^*$ as described in **3. 4.** Our method of numbering the trees guarantees that for $m > n$ no tree T_{mj} intersects K_n and, moreover, for each n, there are at most a finite number of curves of the set $\bigcup_{m,j;\, m \leqq n} (T_{mj} - C_{mj})$ which intersect K_n. For future reference we shall call the above method of numbering trees a *standard* numbering of the trees of F.

With these preliminaries we are able to define the curves which we are going to remove from each tree in order to make the multiply connected region $R = \pi - B$ simply connected. Let T be any tree, then it is numbered say as T_{ij} in a standard numbering as described above, and the choice of base curve C_{ij} included in the standard numbering determines a numbering of the curves of $T = T_{ij}$. Let $b(\alpha)$ be any branch point of this tree with sequence $\alpha = e0k_2 \cdots k_{n-1}k_n$ with $k_n \neq 1$ (where e denotes the sign $+$ or $-$). Any such $b(\alpha)$ will be called the *initial point* of a cut, and the *cut*, $\lambda(b)$, will consist of all curves $C(\alpha, 1), C(\alpha, 1, 1), C(\alpha, 1, 1, 1), \cdots$ and so on *ad infinitum*, or until a terminal sequence $\alpha, 1, 1, \cdots, 1$ is reached, i. e., each cut is a chain of adjacent curves extending from $b(\alpha)$ to infinity. We assume endpoints of the curves included, of course, as part of the cut; thus each cut is of the form $\delta^*[C(\alpha)+]$ or $\delta^\#[C(\alpha)-]$, the former if $b(\alpha)$ is the positive endpoint of $C(\alpha)$ and the latter if it is the negative (see heavy lines in Figure 3). Each $\lambda(b)$ is, again, an arc from $b(\alpha)$ to infinity and includes all branch points numbered by sequences of the form $\alpha, 1, 1, \cdots, 1$. It is clear from the definition of the cuts and the properties of the numbering that every branch point of the tree is on one and only one cut $\lambda(b)$ and that no two cuts intersect at any point. We denote the collection of all the half-open arcs $\lambda(b)$ on T by $\tilde{T}$, and by $\tilde{\mathfrak{J}}$ the sum of the sets $\tilde{T}$ over all the trees of F. The set $R^* = R - \tilde{\mathfrak{J}}$ contains no branch points and is a union of curves of F. Let F^* denote the family $F[R^*]$ filling R^*.

THEOREM 4.1-2. *R^* is an open, arcwise connected, and simply connected domain, and F^* is regular in R^*.*

Proof. Let q be any point of R^* and let K_n be the first circle with center at p (in the standard numbering scheme) which contains q in its interior. Now consider how much of K_n is removed when $\tilde{\mathfrak{J}}$ is subtracted from π. None of the base curves C_{ij} is in $\tilde{\mathfrak{J}}$ since none of them is in a set $\lambda(b)$ for these curves are assigned the sequence ± 0 in the numbering, which sequence is not of the form $\alpha, 1, \cdots, 1$. And by Theorem 4.1-1 there can then be at most a finite number of other curves (than base curves) of any T_{ij} in K_n. Hence there is surely an r-neighborhood of q in $K_n - \tilde{\mathfrak{J}}$ and R^* is therefore open and F^* regular.

We wish to show that R^* is arcwise connected. Since every point has an r-neighborhood in R^*, it is clear that R^* is locally-connected. Hence, if it is connected, it is arcwise connected. Now since each cut $\lambda(b)$ is an arc, extending from a point b to infinity, the set $\tilde{\mathfrak{J}} \cup \infty$ on the extended plane $\bar{\pi}$ can clearly be deformed continuously along itself to a single point, the point at infinity. It follows from Eilenberg [III, Theorem 6, p. 77], that R^* is connected.

Finally, if K is any closed curve in R^* containing a point q of $\tilde{\mathfrak{J}}$, then q lies on a cut $\lambda(b)$ which extends to infinity from b and hence must intersect K, contrary to the assumption that K is in R^*. Thus R^* is simply connected.

THEOREM 4.1-3. *Let $B' \subset B$ be the set of all initial points of cuts $\lambda(b)$, then we may define a collection of disjoint, open sets $\{V_b\}$, one for each $b \varepsilon B'$, each containing the corresponding $\lambda(b)$.*

Proof. Referring to the closed circular discs K_n of our standard numbering of the trees of F, we have noted already that only a finite number of the cuts $\lambda(b)$ intersect any K_n. We denote by $B_n' = \{b_j^n\}$ the finite subset of B' whose elements b_j^n are for $j = 1, \cdots, j_n$ those initial points of cuts which intersect K_n but not K_{n-1}. Now, using the normality of π we are able to find disjoint open sets covering the disjoint closed sets $\lambda_1(b_j^1) = \lambda(b_j^1) \cap K_1$. We define $V_1(b_j^1)$ as the intersection of the so chosen open sets covering $\lambda_1(b_j^1)$ with the *interior* of K_1. Then we find disjoint open sets covering each of the closed sets $\lambda_2(b_j^i) = \lambda(b_j^i) \cap [K_2 - \operatorname{int} K_1]$, $i = 1, 2$ and such, moreover, that the open sets covering $\lambda_2(b_j^2)$ do not intersect K_1. Finally, we define $V_2(b_j^i)$, $i = 1, 2$ as the intersections of these open sets with the interior of K_2. Then the sets $V_2(b_j^2)$ and $V_1(b_k^1) \cup V_2(b_k^1)$ are non-intersecting open sets lying in the interior of K_2 and covering $\lambda(b_j^i) \cap int(K_2)$, for all b_j^i's in B_1' or B_2'. This process is continued indefinitely, covering, for each $\lambda(b)$, its intersection with every K_n and in such a manner that none of the open sets covering one cut has a common point with those covering another cut. Then, given any $b \varepsilon B$ it will be in B_n' for some n, hence will be of the form b_i^n, and the cut $\lambda(b_i^n)$ with it as initial point is covered by $V(b_i^n) = \bigcup_{j=n}^{\infty} V_j(b_i^n)$.

THEOREM 4.1-4. *Let F be a branched regular curve family filling the plane π. Then there exists a function $f(p)$ such that:*

(1) *$f(p)$ is defined and continuous for all p in π.*

(2) *for every real number k the locus $f(p) = k$, if not empty, consists of a finite or an at most countably infinite collection of trees (including regular curves) of F.*

(3) *in every neighborhood of any point p in π there are points q for which $f(q) > f(p)$ and points r for which $f(r) < f(p)$.*

Proof. We assume a standard numbering of the non-trivial trees of F and that thus the cuts $\lambda(b)$ and the sets $\mathfrak{J}$ and R^*, etc., are determined. Now this theorem was proved in [IV] by W. Kaplan for curve families regular throughout an open, simply connected domain; thus we may assume that there is a function $f^*(p)$ defined and continuous in R^* and with the properties above. We must show that this function can be extended to a function $f(p)$ with properties (1)-(3) above. The proof, which follows, is divided into three sections, A, B, and C.

(A) First it is necessary to prove that, given any tree T of F, the value of f^* is the same on each curve of $T[R^*]$, i. e., on all curves of T which lie in R^*. Let $C(\pm 0)$ be the base curve of T in the numbering; we shall proceed by induction on the order of the curves of T. If $C(\pm 0)$ has no endpoint, then it is a regular curve, lies entirely in R^* and the result is trivial. Assume it has a positive endpoint $b(+0)$. Then $C(+01)$ is in $\lambda(b(+0))$ and hence not in R^* or $T[R^*]$, but the other curves of $St(b(+0))$ are all in $T[R^*]$. To prove that $f^*(p)$ has the same value on each of these it is only necessary to prove that it has the same value on each pair of adjacent curves among them, for then the value of f^* on $C(+02)$ is the same as that on $C(+03)$ and so on until finally we have the value of $C(+0u(+0))$ the same as that on $C(+0)$. It is quite obvious that this must be so, however, for if C, C' are adjacent curves of $T[R^*]$ and $p \varepsilon C$, $q \varepsilon C'$, then there is an r-set $U(pq)$ in R^*; and, if $p_n \varepsilon U$ is a sequence of points approaching p, then there is a sequence $q_n \varepsilon C_{p_n}$ with $q_n \varepsilon U$ and q_n approaching q. But, since $q_n \varepsilon C_{p_n}$ we have $f^*(q_n) = f^*(p_n)$ and hence $f^*(q) = \lim_{n\to\infty} f^*(q_n) = \lim_{n\to\infty} f^*(p_n) = f^*(p)$. This same procedure actually tells us even more, i. e., that if $C_1, \cdots, C_n$ is any chain of adjacent curves with both $C_1, C_n \subset R^*$, then f^* must have the same value on C_1, C_n.

Now let $C(\alpha, k)$ be a curve of $T[R^*]$ whose sequence is positive and of order $n+1$, and assume that f^* has the same value on each curve of $T[R^*]$ numbered by a positive sequence of order n or less. The sequence α is of the form $\alpha = 0k_2 \cdots k_{n-1}k_n$; and we consider two cases: (1) $k_n \neq 1$ and (2) $k_n = 1$; in either event $k \neq 1$ since $C(\alpha, k)$ is in R^*. In case (1) $b(\alpha)$ is the initial point of a cut, hence $C(\alpha, 1)$ is the only curve of $St(b(\alpha))$ in the cut, and thus the only curve of $St(b(\alpha))$ not in R^*. Moreover, $St(b(\alpha))$ contains the curve $C(\alpha)$ of order n. It follows by precisely the same argument as above that f^* has the same value on each of the curves of $St(b(\alpha))$ in R^* and, in particular on $C(\alpha, k)$, as it has on the n-th order curve $C(\alpha)$ and hence that it has on $C(\pm 0)$. In case (2) both the curves $C(\alpha)$ and

$C(\alpha, 1)$ of $St(b(\alpha))$ are in a cut, i. e. not in R^*. But the curves $C(\alpha_{n-1}, 2)$, $C(\alpha_{n-1}, 1) = C(\alpha)$, and $C(\alpha, u(\alpha))$ form an adjacent chain with the first and last curves in R^*. On the first curve f^* has the same value as on $C(\pm 0)$ since it is of order n, hence it has this value also on the last, $C(\alpha, w(\alpha))$. Now, by going from adjacent curve to adjacent curve, we see that this must be the value of f^* on each curve of $St(b(\alpha))$ in R^* and in particular on $C(\alpha, k)$. This completes the first step in the proof.

(B) Next we define $f(p)$ at every point of π as follows: $f(p) = f^*(p)$ for $p \varepsilon R^*$, and $f(p) =$ value of f^* on $T[R^*]$ for $p \varepsilon T$. The single-valued function $f(p)$ will then be continuous at each point of R^* (since R^* is an open subset of π and thus the extension cannot affect the continuity of f in that domain), and will have the same value at every point of any tree. Now every point of $\tilde{\mathfrak{J}} = \pi - R^*$ lies on a cut $\lambda(b)$, which in turn lies in a neighborhood $V(\lambda(b))$ not containing points of any other cut. What must be shown is that $f(p)$ is continuous at an arbitrary point q of an arbitrary cut $\lambda(b)$. Now let q_n be any sequence of points approaching the point q of $\lambda(b)$; it must be shown that $f(q_n) \to f(q)$. We shall denote by T the tree containing q; then since $f(p)$ is constant on T, we shall assume that each q_n lies on a distinct curve and none of them is in T. This involves no loss of generality since the result is trivial otherwise. Moreover, we may restrict ourselves to sequences lying in a single complementary domain of T, the reason being that since the number of complementary domains of T with q as a boundary point is finite, any such sequence q_n, with $q_n \notin T$, can be decomposed into a *finite* number of distinct subsequences, containing all the terms of q_n, but no two having a term in common, and each consisting of points from only one complementary domain; and, finally, if for each of these subsequences we have $f(q_{n_i}) \to f(q)$, then $f(q_n) \to f(q)$. Thus we need now to consider only a sequence $q_n \to q$ such that for some $C^* \supset q$, $q_n \varepsilon \mathfrak{D}^*(C^*)$ for all n. C^* is then in T, and since no cut separates π, there is a curve C' on C^* which is in R^*. Let p be any point of C' and $U(qp)$ an r-set of qp in $\mathfrak{D}^*(C^*)$. Then, by Theorem 3. 5-2 there is in U a sequence $p_n \to p$ with $C_{p_n} \equiv C_{q_n}$ and hence $f(p_n) = f(q_n)$. But p is in R^* and $f(p)$ is continuous in R^*, therefore $\lim f(q_n) = \lim f(p_n) = f(p)$. But this is exactly what is needed, for p, q are both on T and hence $f(p) = f(q)$, so f is continuous at q. This establishes property (1) of the theorem.

Property (2) of the theorem is trivial for $f(p)$, since it is satisfied by f^* in R^* and we have added only a denumerable number of curves to the domain of f^* to get the domain of f.

(C) Finally we must prove property (3), i. e., that $f(p)$ has no weak relative extrema. This is clearly equivalent to the following, at least for regular points: if p is a regular point, then f takes a different value on every curve of every r-neighborhod of p, or again equivalently, is monotone on every cross-section. Then, since any arc pq on a curve C has an r-neighborhood, a function satisfies property (3) at every point of a curve or no point of a curve. As to branch points, we can show at once that the condition is satisfied there, for there is always a curve of $St(b)$ in R^*, hence in any neighborhood of b we may find a point q of a curve of R^* and a neighborhood of this point q inside that of b. Now $f(q) = f(b)$ and in this neighborhood of q there will be points q_1, q_2 such that $f(q_1) < f(q) < f(q_2)$, since we are in R^*, where we know f to have property (3). Since q_1, q_2 are in the given neighborhood of b, we have proved our contention for the case of branch points.

Now we wish to show that if f has property (3) on every curve of $St(b)$ except one, C, where b is any branch point, then f has property (3) on C also. Let the curves of $St(b)$ be numbered counterclockwise $C = C_1, C_2, \cdots, C_{2m}$, m being the order of the branch point b. In $U(b)$, an admissible neighborhood, we shall let s_i denote any arc into the sector bounded by C_i, C_{i+1}, such that s_i without b, its endpoint, is a cross-section, e. g., in the image of U on $|z| < 1$ we could take for s_i radii into the respective sectors. Then we indicate by s_i^+ that f increases as we move from b on s_i, by s_i^- that f decreases. Clearly C_i has property (3) if and only if s_{i-1}^+ implies s_i^- and s_{i-1}^- implies s_i^+. Hence if we have s_1^-, then we have by induction s_j^+ for even j, and in particular s_{2m}^+, whence C_1 has property (3), similarly for s_1^+.

Now let the curves of any cut $\lambda(b)$ be numbered $C_1, C_2, \cdots$ beginning with the initial curve and proceeding out from b. C_1 is the only curve of $St(b)$ not in R^*, and hence it must have property (3). If the n-th curve C_n has property (3) then C_{n+1} is the only curve of $St(C_n \cap C_{n+1})$ not having this property since the other curves (than C_n) are in R^*, thus C_{n+1} also must have the desired property. This proves by induction that every curve of every cut has property (3) and hence so has $f(p)$ for all points of π.

A regular curve family F will be said to be *orientable* if its curves may be directed so that every regular point has an r-neighborhood in which the image of each curve in R_0 is directed toward increasing x. Then we have at once:

COROLLARY. *The branched regular curve family F is orientable as a regular curve family in $R = \pi - B$.*

Proof. Exactly as in W. Kaplan [IV, Remark 2, p. 184-5].

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