THE TOPOLOGY OF THE LEVEL CURVES OF HARMONIC FUNCTIONS WITH CRITICAL POINTS.* [1]

By WILLIAM M. BOOTHBY.

Introduction. In a previous paper,[2] of which this is a continuation, topological properties of curve families which filled the Euclidean plane π, or a simply connected domain in π, were investigated. The families were assumed regular (i. e. locally homeomorphic to parallel lines) except at a possibly infinite collection of isolated singularities at each of which the family had the structure of a multiple saddle point; such families were called *branched regular curve families.* Further investigation of these families, in particular their relation to harmonic functions, is the aim of this paper. In what follows the definitions and theorems in [I] will be assumed, and the same notation will be used. In particular F, G will denote branched regular curve families filling the plane π, B will denote the set of singular points, R the domain $\pi - B$ in which F is regular, and so on. The Euclidean plane will be taken as a model for all simply connected domains.

The principal result of [I] was to prove that any branched regular curve family F filling π can be given as the family of level curves of a function $f(p)$ which is continuous on all of π and has no relative extrema. This generalizes a portion of [II] in which the same theorem is proved for a curve family without singularities in π. In this paper there are two main results: the first, proved in Section **1**, is that F is actually homeomorphic to the level curves of a harmonic function; the second, proved in Section **2**, asserts the existence of a decomposition of F into a countable collection of subfamilies of curves, each of which has the structure of the parallel lines $y = \text{constant}$ of the upper half-plane. Such subfamilies will be called half-parallel, and this decomposition has consequences for the study of harmonic functions and analytic functions which will be mentioned below. These two results generalize

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[2] The Topology of Regular Curve Families with Multiple Saddle Points (pp. 405-438 of this volume). Theorems or section numbers preceded by I refer to this paper.

those of Kaplan [III] to curve families with singularities of the saddle point type.

The methods of proof used in [I] will also be of value here, i. e., it was first noted that the family F decomposes into single curves extending to infinity in each direction and collections of curves ending at branch points, where each collection forms a tree. Then it was shown that by removing enough curves we are left with a simply connected domain R^* in which the remaining curves F^* of F form a regular family. This allows us to use the theorems of Kaplan. By this method we are able to show in **1. 2** that for any branched regular curve family F, there exists a complementary family G, i. e. there is a branched regular curve family G with the same singularities as F and such that no pair of curves of F and G intersect more than once. For, from Kaplan [IV] it follows that for the family F^* filling R^* there is such a family G^*. Thus it remains only to modify G^* (along the curves removed from π to give us R^*) in such a fashion that the modified family $\tilde{G}^*$ becomes complementary to F when we replace the curves again. The details of this procedure are elaborated in **1. 2**.

Now given F, G complementary, by [I] there exist two functions f, g defined on π which have F, G as level curves. Using f and g, we may define a map $T:\pi \rightarrow uv$-plane by $T(p) = [f(p), g(p)]$ and since f, g have no extrema and have regular curve families as level curves it may easily be shown that T is light and interior. It follows from well known theorems that there is a homeomorphism $h: D \rightarrow \pi$, D a simply connected domain in the xy-plane, such that $w(x,y) = T[h(x,y)]$ is harmonic; but h maps the level curves of w onto F. This concludes in outline the proof of the first principal theorem. Since the converse is well known, this theorem give a characterization by *local* topological properties of the level curves of a function harmonic in a simply connected domain. We note an important but immediate corollary: F is homeomorphic to the family of solutions of a system of differential equations: $dy/dt = p(x,y)$, $dx/dt = q(x,y)$. Section **1. 3** gives us in detail the proof outlined above.

In Section **2** the decomposition theorem mentioned is proved, again by a reduction to theorems of Kaplan for families without singularities in a simply connected domain, in this case, as above, applied to the family F^* filling, and regular everywhere, in the domain R^* obtained by removing curves from F. The importance of this theorem lies in its possible applications to functions analytic in a simply connected domain, as follows. For example, let $u(x,y)$ be the real part of an entire function. The family F of its level curves will then be a branched regular curve family. Now for F by the decomposition theorem there exists a countable set A and a decomposition of F into sets D_α, $\alpha \varepsilon A$, which are simply connected, overlap at most on their boundaries (and then along curves of F) and in each of which our entire function is 1-1. Thus the sets D_α furnish a decomposition of the Riemann surface of the inverse function into sheets. This is again a generalization of Kaplan [III] where these results were obtained for analytic functions, with non-vanishing derivative, defined in a simply connected domain.

1. The Branched Regular Curve Family as the Level Curves of a Harmonic Function.

1.1. Preliminary properties and definitions. In this paper a slightly more general definition of cross-section will be needed than in [I], as follows: a curve in R is a cross-section if every arc on it is a cross-section. This removes the restriction that a cross-section be an arc, i. e. it may extend to infinity in one or both directions or even be a bounded open or half-open curve.

Whitney [VI] has shown that in an orientable regular curve family filling a region S there is a function $f(p,t)$ with the properties: for each p in S and any t in $-\infty < t < \infty$, there is a unique point $q = f(p,t)$ lying on the curve C through p; $f(p,t)$ is continuous in both variables; $f(p,0) = p$ and as t increases (decreases) $f(p,t)$ moves continuously in the positive (negative) direction on C. Just as in [II] we have as an immediate corollary to this theorem the following:

THEOREM 1.1-1. *Let γ be a cross-section in F and $S(\gamma)$ the set of curves of F crossing γ. Then $S(\gamma)$ forms an open, simply connected set, F is regular in this set and, in fact there is a homeomorphism of $S(\gamma)$ onto* (i) *a strip $0 \leqq y \leqq 1$,* (ii) *a half-plane $y \geqq 0$ or* (iii) *the xy-plane, carrying the curves of F in $S(\gamma)$ onto the lines $y =$ constant, the case depending on whether* (i) *γ is an arc,* (ii) *γ is a half-open curve, or* (iii) *γ is an open curve, respectively.*

Proof. We will prove only case (ii), the proofs of the other cases being similar. It is a consequence of Theorem I 1.2-2 that $S(\gamma)$ is open and F is regular in $S(\gamma)$. From what follows it is clear that the domain filled by $S(\gamma)$ is simply connected, since it is homeomorphic to a half-plane. Let γ be parametrized by τ, i. e. $\phi(\tau)$ maps $0 \leqq \tau < \infty$ homeomorphically onto γ. Now $S(\gamma) - \gamma$ consists of two disjoint domains A and B and each curve, since it crosses γ exactly once, has an arc in A and an arc in B. Hence the curve family $F[S(\gamma)]$ is orientable, since we may assign to each C in this set the direction as positive along which we pass from A into B. Then by the above mentioned theorem of Whitney we have a function $f(p, t)$ defined on S. Let $p = \phi(\tau)$, then the function $f(\phi(\tau), t)$ at once gives us, if we set $y = \tau$, $x = t$, a homeomorphism of $S(\gamma)$ onto the upper half plane with $x = 0$ the image of γ. This is essentially the same as the situation in [II, p. 174, Theorem 30].

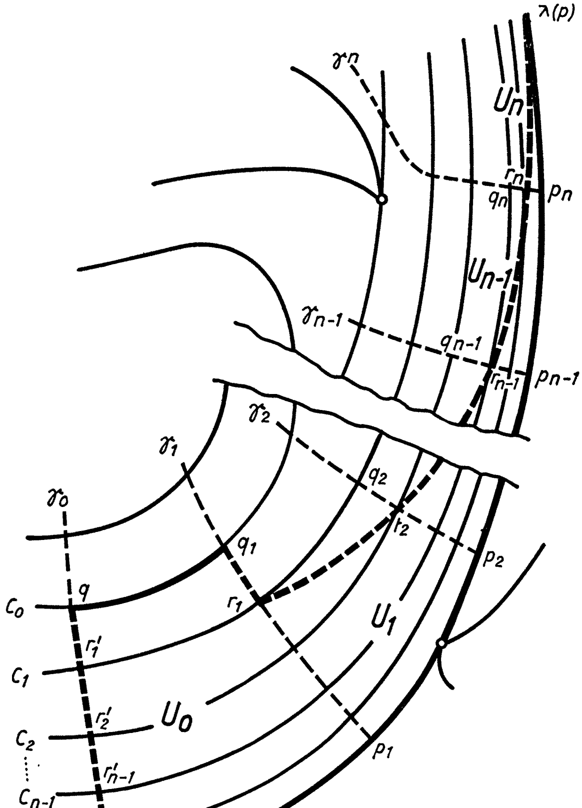

FIGURE 1
CONSTRUCTION OF EXTENDED R-SET

In [I] a useful "half-neighborhood" called an r-set was defined for any arc pq lying on a "maximal curve" C^*. The set was, briefly, a closed set in $\mathfrak{D}^*(C^*)$ which abutted on pq and in which the curve family had the structure of the family F_1 of parallel lines, $y = k$, in a closed rectangle $R_1 = \{(x, y) \mid |x| \leqq 1, 0 \leqq y \leqq 1\}$, the only singular points in this set being those on pq itself. Below we shall need a similar half-neighborhood for a half-open curve $\lambda(p)$ extending from a point p to infinity along a maximal curve C^*. Of course, the most important case of such a situation is the "cut," $\lambda(b)$, from a branch point b. An *extended r-set*, $U(\lambda(p))$, of $\lambda(p)$ on C^* will then be a closed set contained in $\mathfrak{D}^*(C^*)$ together with a definite homeomorphism k of this closed set onto $\tilde{R}_1 = R_1 - \{(1,0)\}$, i. e. R_1 without its lower, left-hand corner point, where the following conditions are satisfied by U and k: (1) $F[U]$ is homeomorphic to F_1; (2) the inverse image under k of $x = -1$ is a cross-section, which we will denote by γ, joining p to a point q, $k(p) = (-1, 0)$ and $k(q) = (-1, -1)$; (3) the inverse image of $y = 1$ is an arc from q to $q_1 = k^{-1}[(1,1)]$ on a curve of F; and (4) the inverse image of $x = +1$ is a half-open cross-section Γ from q_1 to infinity, asymptotic to $\lambda(p)$; finally (5) k maps $\lambda(p)$ on the portion of $y = 0$ in $\tilde{R}_1$. Of course, the missing corner-point $(1, 0)$ of $\tilde{R}_1$ corresponds to the point at infinity. (See Figure 1.)

THEOREM 1. 1-2. *Let $\lambda(p)$ be a half-open arc on C^* as defined above and let $V[\lambda(p)]$ be any open set containing $\lambda(p)$, then there is an extended r-set, $U(\lambda(p))$ interior to V.*

Proof. The proof will consist of two parts: the first part (A) being the description of the set U which is to be the extended r-set, and the second part (B) being the description of the homeomorphism k from U to $\tilde{R}_1$. (Fig. 1.)

(A) We begin by choosing a sequence $p = p_0, p_1, p_2, \cdots$ of regular points on $\lambda(p)$ which recede monotonely to infinity along that half-open curve. We shall denote the arc joining p_n to p_{n+1} on $\lambda(p)$ merely by p_np_{n+1}, and as first step we shall choose rather carefully an r-set U_n abutting on p_np_{n+1} for each n. From an examination of the discussion of r-sets in [I], it is not difficult to see that given an arc on a maximal chain whose two endpoints are regular points, if we take arbitrary cross-sections through these endpoints, then it is always possible to find an r-set abutting on this arc whose cross-sectional sides are on these chosen cross-sections. Now let us choose for each n a cross-section γ_n extending from p_n into $\mathfrak{D}^*(C^*)$. For each n we shall choose the r-set U_n interior to V and complying with the following conditions: first, so that its cross-sectional sides are on γ_n and γ_{n+1}, and so that it lies within an ϵ_n-neighborhood of p_np_{n+1}, $\epsilon_n \to 0$. And second, having chosen U_{n-1}, and denoting by q_np_n the arc on γ_n which forms that one of the cross-sectional sides of U_{n-1} on γ_n, we choose U_n in such a manner that its cross-sectional side r_np_n along γ_n is contained in q_np_n, i. e. r_n lies between q_n and p_n on γ_n. Then $U_{n-1} \cap U_n = r_np_n$ by Theorem I 3. 2-2, and C_{r_n} (denoted below by C_n) intersects γ_0 at some point r'_n. Of course, each U_n is in $\mathfrak{D}^*(C^*)$ since each γ_n is. Thirdly, we require the U_n so chosen that r_n is contained in a

δ_n-neighborhood of $p = p_0$ for a definite sequence $\delta_n \to 0$. The properties of r-sets as developed in [I] make it obvious that these conditions on U_n can be complied with. (Note: below we denote γ_0 simply by γ.)

Let k_n denote the homeomorphism of U_n onto R_1; we shall always assume k_n so chosen that as we move from p towards infinity on $\lambda(p)$ the image point moves from left to right along the x-axis. Then if we consider the image of U_n in R_1, we have $k_n(p_n) = (-1, 0)$, $k_n(r_n) = (-1, -1)$, $k_n(p_n r_n) = (\text{line } x = -1)$; $k_n(p_{n+1}) = (1, 0)$, $k_n(q_{n+1}) = (1, 1)$ and $k_n(r_{n+1}) = (1, a)$ where $0 < a < 1$. Now it is clear, as noted above, from the description of the sets U_n, that the curves C_n determined by r_n and q_{n+1}, $n = 1, 2, \cdots$, i. e. those curves of F on which lie the arcs $r_n q_{n+1}$, carried by k_n onto the line $y = +1$ in R_1, will cross the cross-section pq on γ at points r'_n which form a monotone sequence and which by our third requirement approach p. Also the points qq_1 determine an arc on a curve C_0, which maps under k_0 on $y = +1$. It follows that $S(\gamma)$, the set of all curves crossing γ, will contain all the sets U_n. Now for each n let a cross-section $r_n r_{n+1}$ be determined in U_n as the inverse image of the straight line in R_1 joining $k_n(r_n)$ and $k_n(r_{n+1})$. If we direct each curve C_n so that $\mathfrak{D}^\#(C_n)$ contains $\lambda(p)$ then $r_{n-1}r_n$ will lie in $\mathfrak{D}^*(C_n)$, except for the endpoint r_n which is on C_n. Hence the arcs $q_1 r_1$, $q_1 r_1 r_2$, $q_1 r_1 r_2 r_3, \cdots$ are each cross-sections by Theorem I 3.5-3, and they approach as limit an arc Γ from q_γ to infinity, which will be then a cross-section. The set U bounded by (1) $\lambda(p)$, (2) the cross-sectional arc pq on γ, (3) the arc qq_1 on C_0 and (4) the cross-section Γ will, as will be shown below, be an r-set interior to $V(\lambda(p))$, i. e. we shall exhibit the homeomorphism k of the definition.

(B) Now change the meaning of γ slightly to let γ denote only the arc pq on γ and, as above, let $S(\gamma)$ denote the set of curves of F crossing γ, and $S(\Gamma)$ the set of curves crossing Γ. We see from above that $S(\Gamma) \cup C^* = S(\gamma)$. Each of these sets is split into two domains if we remove γ, and we shall let $S'(\Gamma)$, $S'(\gamma)$ denote respectively the domains containing $\lambda(p)$. By Theorem 1. 1-1 there is a homeomorphism $k_1 : S'(\gamma) \to R_1''$, $R_1'' = \{(x, y) \mid -1 \leqq x < \infty, 0 \leqq y \leqq 1\}$ and $k_1(\gamma)$ is the line $x = -1$, $k_1(\lambda(p))$ is the x-axis for $x \geqq -1$. The curve C_0 on which q, q_1 lie will map onto the line $y = 1$ and Γ'', the image of Γ, will be an arc given by $x = \phi_2(y)$, $0 \leqq y < 1$, $k_1(q_1) = (\phi_2(1), 1)$ and $\lim_{y \to 0} \phi_2(y) = \infty$.

Now we let $R_1' = \{(x, y) \mid -1 \leqq x < 1, 0 \leqq y \leqq 1\}$, i. e. a rectangle with the right side missing. We shrink $S'(\gamma)$ into R_1' along the lines $y = \text{constant}$ by the homeomorphism $k_2 : S'(\gamma) \to R_1'$ where $k_2 : (x, y) \to (x, y)$

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for $-1 \leqq x \leqq 0$ and $k_2: (x, y) \rightarrow (x/(x+1), y)$ for $0 \leqq x < +\infty$. Let $x = \phi_1(y)$ be the function whose graph is Γ', the image of Γ'' under k_2. Γ' will be a half-open arc from the point $(\phi_1(1), 1)$ on the line $y = 1$ to the point $(1, 0)$ as limit point, i. e. as $y \rightarrow 0$ the point $(\phi_1(y), y)$ approaches $(1, 0)$ along Γ'. The side $x = -1$, the side $y = 0$, the segment from $(-1, 1)$ to $(\phi_1(1), 1)$ on $y = 1$, and finally Γ' will bound the image of U under $k_2 k_1$. Denote this portion of R_1' by U'. Then we perform a final homeomorphism $k_3: U' \rightarrow \tilde{R}_1$ where $k_3: (x, y) \rightarrow \{(2 + 2x)/(1 + \phi_1(y)) - 1, y\}$. The combined homeomorphism $k = k_3 k_2 k_1: U \rightarrow R_1$ is easily seen to be the desired homeomorphism as required in the definition of an extended r-set. Hence the proof of the theorem is complete. With these preliminaries completed we can prove the existence of the family G, complementary to F.

1.2. Complementary curve families. Given a branched regular curve family F filling π, we shall call another such family, G, filling π *complementary* to F if (1) the singularities of G are exactly those of F and each is of the same type, i. e., a point b is an n-th order branch point of G if and only if it is an n-th order branch point of F; and (2) every curve of G is a cross-section of F. It follows at once from this definition and Theorem I 3.2-4 that if G is complementary to F, then F is complementary to G. Hence we may speak of two complementary families, F and G, filling π. They will have a common set of singular points, B.

The major result of this section is to establish that every branched regular curve family F has a complementary family G. In [IV] it is shown that this is true when the set of singular points is empty, i. e. for a family F^* regular throughout a simply connected domain R^*, for we may by [IV] map F^* onto a family F' filling the xy-plane and defined by differential equations, $dx/dt = f(x, y)$; $dy/dt = g(x, y)$. The orthogonal trajectories define a family G' complementary to F' and the inverse image G^* of G' is then the desired complementary family to F^*. This result immediately gives us a family G^* complementary to F^* in $R^* = \pi - \tilde{\mathfrak{J}}$, $\tilde{\mathfrak{J}}$ being the cuts removed from π to make R simply connected [I]. The method we shall use to establish the existence of a family G complementary to F will be to consider first F^* and its complementary family G^*, both defined in R^* and then to modify G^* slightly near the boundary of R^*, i. e., near the cuts $\lambda(b)$, obtaining a family $\tilde{G}^*$ which becomes a family $\tilde{G}$ of the desired type when $\tilde{\mathfrak{J}}$, the set removed from π to give R^*, is replaced again. Theorem I 4.1-3 tells us that we may cover $\tilde{\mathfrak{J}}$ with a collection $\{V[\lambda(b)]\}$ of disjoint open sets; we shall assume such a covering, to be fixed throughout what follows, and moreover, assume that each $V \subset U_\epsilon[\lambda(b)]$ an ϵ-neighborhood of $\lambda(b)$ where $\epsilon > 0$ is fixed. Any modification in G^* will actually take place deep inside V, i. e., in an open set whose closure lies in V. We shall actually discuss the modification for one such V and, assuming similar modifications have taken place in each V, we will denote by $\tilde{G}^*$ the modified G^*. $\tilde{G}^*$ will then be shown to be such that when $\tilde{\mathfrak{J}}$ is replaced, $\tilde{G}^*$ becomes a set $\tilde{G}$ complementary to F.

Restricting ourselves then to a definite $\lambda(b)$, and its neighborhood $V[\lambda(b)]$, as model, we proceed to define for $\lambda(b)$ a certain possibly infinite collection of closed sets $W_0, W_1, \cdots$ all contained inside $V[\lambda(b)]$ and surrounding $\lambda(b)$. These are the sets in which G^* will be modified. W_0 is an extended r-set, and if the number of curves in $\lambda(b)$ is finite, and only in this case, there will be a last set W_N of this collection which is also an extended r-set. All the other sets W_i will be r-sets in the sense of [I], which we shall hereafter call merely r-sets. These sets will be chosen as follows: *First,* let $b_0 = b, b_1, b_2, \cdots$ be the branch points on $\lambda(b)$, numbered so as to recede monotonely to infinity, and let the curves in R^* of each $St(b_i)$ be numbered with two indices, the first being that of b_i, the second being given by a counterclockwise numbering of the $St(b_i)$ proceeding from the first curve which follows counterclockwise after a curve of $St(b_i)$ not on C^* $(C^* \supset \lambda(b))$ to the last curve of $St(b_i)$ not on C^*: $C_{01}, \cdots, C_{0n_1}; C_{11}, C_{12}, \cdots, C_{1n_2}; \cdots$ etc.; let C_{00} denote C. (See Figure 2.) Then all numbered curves are in R^*. *Second,* choose regular points s_{ij} on each C_{ij} and short cross-sections γ_{ij} through s_{ij}, the γ_{ij} being in each case an arc on a curve of G^* and both s_{ij} and γ_{ij} being chosen so as to lie in $V(\lambda)$. Now we choose our sets W_n as follows: $W_0 \subset V(\lambda)$ is an extended r-set bounded on one side by an arc $r_{00}s_{00}$ on γ_{00} and on one side, of course, by $(s_{00}b) \bigcup \lambda(b)$. Next, in the domain bounded by the maximal chain determined by the adjacent curves C_{00}, C_{01} we choose an r-set $W_1 \subset V(\lambda)$ of the arc $s_{00}s_{01}$ on these curves, with W_1 bounded by the arcs $s_{00}t_{00}$ on γ_{00} and $r_{01}s_{01}$ on γ_{01}. Similarly, we choose $W_2, \cdots, W_{n_1}$, each an r-set contained in $V(\lambda)$ and bounded by arcs on two of the γ_{0i}'s. It may be that b_0 is the only branch point of $\lambda(b)$, in which case the next set W_{n_1+1} is the last and must be an extended r-set, bounded on one side by an arc $s_{0n_1}t_{0n_1}$ on γ_{0n_1}. Otherwise, we choose for W_{n_1} an r-set of $s_{0n_1}b_0b_1s_{11}$, an arc on the adjacent chain C_{1n_1}, C', C_{11}; C' being the curve of $\lambda(b)$ with endpoints b_0, b_1. The r-set W_{n_1} is so chosen that its cross-sectional ends are arcs $s_{0n_1}t_{0n_1}$ and $r_{11}s_{11}$ on γ_{0n_1} and γ_{11} respectively, and that it lies in $V(\lambda)$. This process is continued until we have chosen r-sets (or extended r-sets) on both sides of every curve of $St(b_i)$ for all b_i and hence, in particular, on both sides of each curve of $\lambda(b)$. Then $\lambda(b)$ will be contained in the interior of the set $W_\lambda = \bigcup_i W_i$. W_λ is bounded by an open arc Γ extending to infinity in each direction; and Γ consists in one case of *one* infinite cross-section of F^*, not in general a curve of G^*, plus an infinite number of arcs alternately on curves of F^* and on curves of G^* (the latter of the form $r_{ij}s_{ij}t_{ij} \subset \gamma_{ij}$); or else in

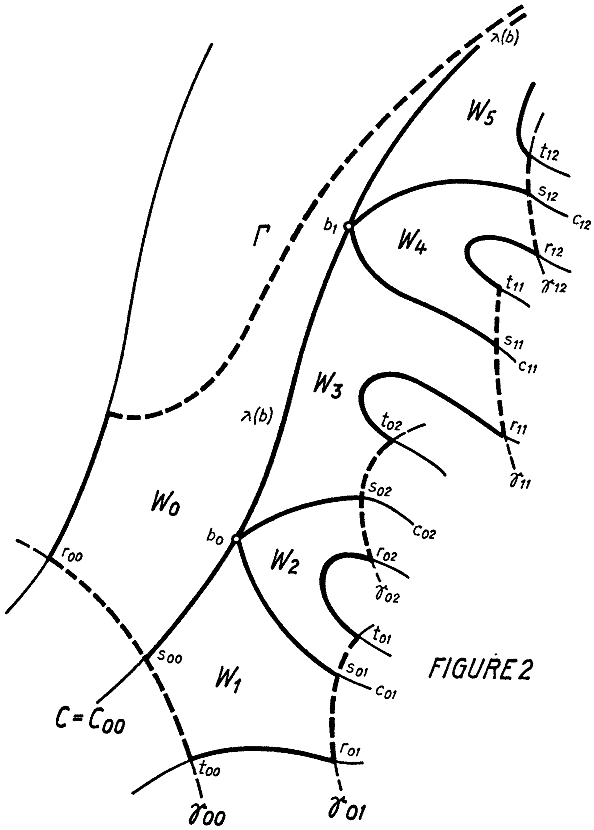

FIGURE 2

the other case, Γ consists of a finite number of such alternate arcs on F^* and G^* plus *two* half-open cross-sections of F^* extending to infinity. The first case occurs when there is one extended r-set and the number of sets W_i is infinite, the second when there are two extended r-sets and a finite collection of sets W_i comprising W_λ. Γ lies entirely inside $V(\lambda)$, and W_λ, which consists of Γ plus that one of its complementary domains inside $V(\lambda)$, is a closed set. The W_i's clearly intersect only on curves of F, namely on $\lambda(b)$ and on the arcs b_is_{ij} on each curve C_{ij} of $R^* \cap St(b_i)$ for b_i in $\lambda(b)$. We denote by $\tilde{\lambda}$ the set of all points which lie on the common boundary of two or more W_i's. *A* point of $\tilde{\lambda}$ which is a regular point clearly lies on the intersection of just two such sets, whereas each branch point b_i lies on the intersection of $2m$, where m is the multiplicity of b_i. We denote by W_i^* the set $W_i - \tilde{\lambda}$, and by W_λ^* the set $W_\lambda - \tilde{\lambda}$, and finally by V_λ^* the set $V(\lambda) - \tilde{\lambda}$. Then let $\bar{G}^* = G^*[V_\lambda^*]$ and $\bar{F}^* = F^*[V_\lambda^*]$, i. e. we remove from $V(\lambda)$ all points on two or more sets W_i.

Now each W_n has associated with it a homeomorphism k_n, of W_n onto R_1 or, if it is an extended r-set, onto $\tilde{R}_1$. In order that the modification of G^* to $\hat{G}^*$ which we are going to make will not destroy the relationship between G^* and F^* we will actually achieve it by a homeomorphism h of $\tilde{R}^*$ $(\tilde{R}^* = R^* - \bigcup_{\lambda \varepsilon \tilde{T}} \tilde{\lambda})$ onto itself, which is the identity outside of each set W, but which inside such a set carries each curve of $\bar{F}^*$ onto itself. The need for this modification arises from the fact that although $\bar{F}^*$, $\bar{G}^*$ are complementary in $\tilde{R}^*$, and hence in $V^*(\lambda) = V(\lambda) - \tilde{\lambda}$, they will not in general be complementary in $V(\lambda)$ along points of $\tilde{\lambda}$. In fact replacing $\tilde{\lambda}$ in V^* will not in general transform $\bar{G}^*$ into a regular curve family in $V(\lambda)$, since after all those points of $\tilde{\lambda}$ on λ are boundary points of the region R^* filled by G^*, and curves of G^* may have common endpoints on the boundary of the domain of G^*, or no endpoints (i. e. may extend to ∞ which is also a boundary point of R^*). Our procedure is to cut the plane along each $\tilde{\lambda}$, cutting along curves of F, i. e. whenever we cut a curve of G^*, we cut across it: in particular in cutting along $\tilde{\lambda} - \lambda$, since this is in R^*, we cut across curves of G^*. Then keeping the curves of F fixed (not pointwise) we move the "cut ends" of curves of G^*, with their individual points "sliding" along curves of F, into such positions that each regular point on $\tilde{\lambda}(b)$ becomes the endpoint of exactly one curve of G^* from each side of $\tilde{\lambda}$, and the branch points of multiplicity m the endpoint of $2m$ curves, one from each sector. Then replacing $\tilde{\lambda}$, $\tilde{G}^*$ the modified G^* becomes a regular family at every regular point of F and is in fact complementary to F. We shall describe this operation piecewise, for each W_n^* and, in fact, at first as a homeomorphism on the image of W_n^* in R_1 (or $\tilde{R}_1$ as the case may be), (I) for r-sets, (II) for extended r-sets.

(I) We begin by defining a typical homeomorphism f_I in R_1 on the image under k_i of $\bar{F}^*[W_i^*]$, $\bar{G}^*[W_i^*]$, W_i an r-set. The image of W_i^* will be $R_1^* = R_1 - (x\text{-axis})$, and we will denote the images of the curve families as F_1^*, G_1^*, respectively. The former will, of course, be just the lines $y = a$, $0 < a \leqq 1$, and the latter will be a regular curve family filling R_1^*, complementary to F_1^*, and having among its curves the two lines $x = \pm 1$, images of arcs γ_{ij}, which lie on curves of G^*. It will be seen that G_1^* consists exactly of the curves whose inverse images cross C', the inverse image of $y=1$ in R_1^*, for, if we consider any curve of $\tilde{G}^*$ with a point inside W_i, it is clear that it must leave W_i in each direction, there being no boundary points of $\tilde{R}^*$ interior to W_i; and hence, it must either cross C' or have two endpoints on $\tilde{\lambda}(b)$. It could scarcely have both endpoints on $\tilde{\lambda}(b)$, however, without crossing some curve of F^* twice inside W_i, which is impossible since the curves of G^* are cross-sections of F^*. Moreover, no curve of G^* will cross C' more than once, since C' is a cross-section of G^*. Thus we may define a function f_I mapping R_1^* onto itself as follows: Let $\bar{x}=f(x,y)$ be defined by $f(x,1)\equiv x$ and $f(x,y)=$ constant on each curve of G_1^*, and let $\bar{y}=g(x,y)$ be defined by $g(x,y)\equiv y$. Then it follows from the above remarks and the work of Kaplan [II] and [III] that $f_I\colon (x,y)\to(\bar{x},\bar{y})$ is a homeomorphism of R_1^* onto itself which takes each curve of F_1^* onto itself, and each curve of G_1^* onto a line $x=b$, $-1\leqq b\leqq 1$, the lines $x=\pm 1$ being held pointwise fixed, as is the line $y=1$, i. e., all of the boundary of R_1^* on which f_I is defined is held pointwise fixed. $h\,|\,W_i^*$ is then defined by $k_i^{-1}f_Ik_i$, and if thus defined h maps $\tilde{F}^*[W_i^*]$ onto itself, takes $\tilde{G}^*[W_i^*]$ homeomorphically onto a new family $\tilde{G}^*[W_i^*]$ which is still complementary to F^* and which is identical to $\tilde{G}^*$ on the boundary of W_i^*. Since k_i is actually a homeomorphism of all of W_i onto R_1, it will now map $F[W_i]$ and $\tilde{G}^*[W_i]$ so that the curves $F^*[W_i]$, $\tilde{G}^*[W_i]$ will map onto the lines $y=a$ and $x=b$, respectively. We re-denote k_i by $\tilde{k}_i$ to emphasize that it acts on $\tilde{G}^*$. Thus it is clear that every curve of $\tilde{G}^*[W_i]$ has exactly one endpoint, unique to it, on $\tilde{\lambda}$ and exactly one endpoint unique to it on the curve of F^* forming the opposite side of W_i. The regularity of $\tilde{G}^*$ which we have achieved at $\tilde{\lambda}$ is precisely what is needed. We assume a similar homeomorphism defined for every index i such that W_i is an r-set; then h will be defined on every set of W except the one (or two) extended r-set(s).

(II) Now let us suppose that we are dealing with an extended r-set say W_0, with its associated homeomorphism k_0 onto $\tilde{R}_1$. Again let F_1^*, G_1^* denote the images of the respective families of W_0 in $\tilde{R}_1^*=\tilde{R}_1-(x\text{-axis})$, F_1^* being just the lines $y=a$; the line $x=-1$ in $\tilde{R}_1^*$, but not in general the line $x=+1$, being a curve of G_1^*. f_{II} will be given as the composition of four homeomorphisms of $\tilde{R}_1^*$ onto itself. Before we can describe f_1, the first of these, we must note that there is in W_0 at least one curve ψ of G^*, distinct from the arc $r_{00}s_{00}$ on γ_{00} (the inverse image of $x=-1$), whose image ψ_1 in $\tilde{R}_1$ joins a point $(x'',0)$ to a point $(x',1)$, where $-1<x'',x'<0$,

i. e., a curve of G^* joining one side of W_0 to the other, and intersecting each side at a regular point of R^*, in particular, on C_{00}, not on $\lambda(b)$ (see Figure 3). That such a curve exists follows from the fact that in the family G^*, regular in R^*, the arc $r_{00}s_{00}$ on a curve of G^* has an r-neighborhood U (by Theorem I 1. 2-2) with $\bar{U} \subset R^*$. The curves $C_{s_{00}}$ and $C_{r_{00}}$ (see Figure 2) have small arcs entirely in this neighborhod, since they are cross-sections of G^*, and each of these will be crossed by an infinite number of curves of G^* on each side of $s_{00}t_{00}$, one of which will serve our purpose; namely, one crossing for each of these arcs that part which is the inverse image respectively of the

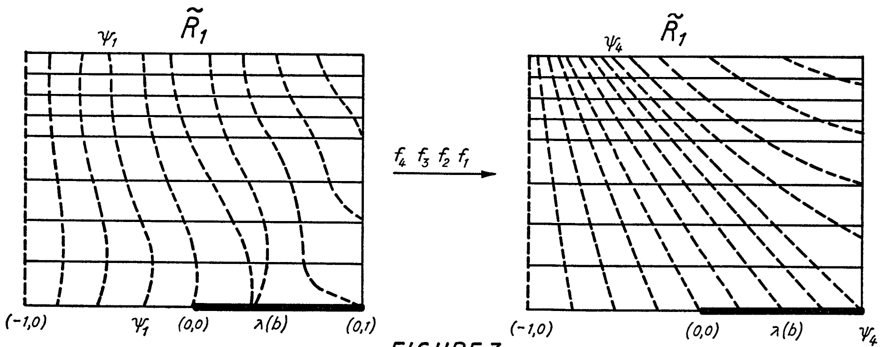

FIGURE 3

segments $(-1,1)$ to $(0,1)$ and $(-1,0)$ to $(-\epsilon,0)$, $1>\epsilon>0$. ψ_1 will be given by a continuous function $x=\psi_1(y)$, $0\leqq y\leqq 1$, and we shall use it to define $f_1: R_1^* \to R_1^*$ given by $f_1\colon (x,y)\to(\bar{x},\bar{y})$ where

$$\bar{x}=\{[1+\psi_2(y)]x-[\psi_1(y)-\psi_2(y)]\}/[1+\psi_1(y)] \text{ for } -1\leqq x\leqq \psi_1(y),$$

$$\bar{x}=\{[1-\psi_2(y)]x-[\psi_1(y)-\psi_2(y)]\}/[1-\psi_1(y)] \text{ for } \psi_1(y)\leqq x\leqq +1;$$

$$\bar{y}\equiv y$$

(where $\psi_2(y)=(x'-x'')y+x''$, this being the equation of the line joining $(x',1)$ to $(x'',0)$, the curve into which ψ_1 is mapped by f_1).

The next homeomorphism, $f_2\colon R_1^*\to R_1^*$ will carry ψ_2 into ψ_3, the line $x=x'$. f_2 is given by $f_2\colon (x,y)\to(\bar{x},\bar{y})$ where:

$$\bar{x}=\{(1+x')x+[x'-\psi_2(y)]\}/[1+\psi_2(y)] \text{ for } -1\leqq x\leqq \psi_2(y),$$

$$\bar{x}=\{(1-x')x+[x'-\psi_2(y)]\}/[1-\psi_2(y)] \text{ for } \psi_2(y)\leqq x\leqq 1;$$

$$\bar{y}\equiv y.$$

Each of these homeomorphisms holds the boundary curves $x = \pm 1$, $y = 1$ pointwise fixed. To describe f_3 we first denote by M that portion of $\tilde{R}_1{}^*$ which lies on or to the left of ψ_3, i. e., $M = \{(x, y) \mid -1 \leqq x \leqq x', 0 \leqq y \leqq 1\}$. M is bounded on each side by a line $x =$ constant, i. e. $x = -1$, and $x = x'$, each the image of a curve of G^* under the composition of the above maps, and M is bounded on top and bottom by images of curves of F^*. The image of F^* in M is the family of lines $y = a$. Hence by precisely the same argument as in the definition of f_I for the neighborhood of type I above, we may find a homeomorphism $f_3 : M \rightarrow M$ which holds the boundary of M pointwise fixed, takes each curve $y = a$ onto itself, and takes the image family of G^* onto the lines $x = b$, $-1 \leqq b \leqq x'$. We extend f_3 to all of $R_1{}^*$ by defining it as the identity on the rest of this set. Again, f_3 will be a homeomorphism leaving the boundary curves $x = \pm 1$, $y = 1$, as well as the curve ψ_3 and all of $\tilde{R}_1{}^*$ to the right of ψ_3, pointwise fixed.

Finally, we define a homeomorphism $f_4 : \tilde{R}_1{}^* \rightarrow \tilde{R}_1{}^*$, again by giving $f_4 : (x, y) \rightarrow (\bar{x}, \bar{y})$ as follows:

$$\bar{x} = (\psi_4(y) + 1)((x + 1)/x' + 1)) - 1 \text{ for } -1 \leqq x \leqq x',$$

$$\bar{x} = (1 - \psi_4(y))((x - x')/(1 - x')) + \psi_4 \text{ for } x' \leqq x \leqq +1;$$

$$\bar{y} \equiv y,$$

where ψ_4 denotes the line $x = \psi_4(y) = (x' - 1)y + 1$ joining $(x', 1)$ to $(1, 0)$, this being the image of ψ_3 under f_4. The image of M under f_4 will be denoted by M_1 and will be the trapezoid bounded by ψ_4, the x-axis, the line $x = -1$, and the segment from $(-1, 1)$ to $(x', 1)$ on the line $y = 1$. f_4 takes the lines $y = a$ onto themselves, and the lines $x = b$, $-1 \leqq b \leqq x'$ of M onto a family of nonintersecting straight lines joining the points of the top edge of M_1 to the bottom (as listed above). f_4 leaves the lines $x = \pm 1$ and $y = 1$ pointwise fixed.

Now we define $f_{II} : \tilde{R}_1{}^* \rightarrow \tilde{R}_1{}^*$ as the homeomorphism $f_4 f_3 f_2 f_1$, and we define $h \mid W_0{}^*$ as $k_0{}^{-1} f_{II} k_0$ (see Figure 3). Then $h \mid W_0{}^*$ is a homeomorphism of $W_0{}^* = W_0 - \tilde{\lambda}$ onto itself which is pointwise fixed on the boundary of $W_0{}^*$ in $\tilde{R}^*$, i. e., on $t_{00}s_{00}$, on $C_{t_{00}}$, and on the extended cross-section which bounds one side of W_0. h also takes the curves of $G^*[W_0{}^*]$ homeomorphically onto a family $\tilde{G}^*$, at the same time mapping each curve of F^* onto itself. Now, if as above for k_i, we re-denote k_0 by $\tilde{k}_0$, then we have a homeomorphism of all of W_0 onto $\tilde{R}_1$ which takes $\tilde{\lambda}$ onto the x-axis between $(-1, 0)$ and $(1, 0)$, with b_0 mapping onto $(0, 0)$, and s_{00} onto $(-1, 0)$, and which moreover, takes the curves of F onto the lines $y = a$ and takes part of $\tilde{G}^*$ onto the straight lines joining the top and bottom of M_1 as described above, the remainder of $\tilde{G}^*$ mapping onto a regular family filling the rest of R_1. The curve Ψ of $\tilde{G}^*$, image of ψ under $h|W_0^*$ divides W_0 into two domains, one of which maps onto M_1, the other onto $R_1 - M_1$. We shall denote the one which maps onto M_1, together with its boundary, by $\tilde{W}_0$, the boundary consisting of two curves of $\tilde{G}^*$, namely $r_{00}s_{00}$ and Ψ, together with $C_{r_{00}}$ and $s_{00}b_0 \cup \lambda(b)$ in F. It is obvious that $M_1 \subset \tilde{R}_1$ can be mapped onto $\tilde{R}_1$ by a homeomorphism g which holds $x = -1$ and $y = 0$ pointwise fixed, takes each line $y = a$ into itself, and finally moves the image curves of $\tilde{G}^*$ in M_1 onto the lines $x = b$, $-1 \leqq b \leqq 1$, keeping, of course, their lower endpoints fixed, thus taking the line Ψ onto $x = 1$. Then $g\tilde{k}_0 \colon \tilde{W}_0 \to \tilde{R}_1$ with F going onto the lines $y =$ constant and $\tilde{G}^*$ onto the lines $x =$ constant. W_0 is then again, like W_0, an extended r-set of $\lambda(b)$, but of a kind which is bounded by curves of two complementary families and has associated a homeomorphism $g\tilde{k}_0$ which maps the curves of the respective families onto the lines parallel to the axes in $\tilde{R}_1$. Hereafter, we shall denote $g\tilde{k}_0$ merely by $\tilde{k}_0$. Now if W_N is a second extended r-set in W, then it must be the last W_i defined for $\lambda(b)$ and on it we define, in a manner entirely parallel to the above discussion, f_{II}, $h|W_N^*$, $\tilde{W}_N$, $\tilde{k}_N$, etc.

Thus we have defined $h|W_i^*$ for all i. Now since the W_i^* are overlapping closed sets of V_λ^* with only a finite number of the sets W_i containing any given point, and since h is actually the identity along their overlapping boundaries as well as on Γ, the boundary of W_λ, we have defined a homeomorphism h of $W_\lambda^* = W_\lambda - \tilde{\lambda}$ onto itself. Assume that h is similarly defined for a set $W_\lambda^* \subset V[\lambda(b)]$ for *every* cut $\lambda(b)$ contained in $\tilde{\mathfrak{T}}$, and define h as the identity outside the W_λ's. We remark that the collection of all the sets W_λ for $\lambda(b)$ in $\tilde{\mathfrak{T}}$, together with the set $\pi - \bigcup_{\lambda \varepsilon \tilde{T}} W_\lambda$, is a collection of overlapping closed sets which has a locally finite character, i. e., every neighborhood of any point meets only a finite number of the closed sets. This is clear because the cuts, λ, recede to infinity, and each W_λ lies in an ϵ-neighborhood of the cut λ, $\varepsilon > 0$ being fixed. Then it follows that h is a homeomorphism of $\tilde{R}^*$ onto itself, where by $\tilde{R}^*$ we mean $R^* - [\bigcup_{\lambda \varepsilon \tilde{T}} \tilde{\lambda}(b)]$. h carries every curve of F^* onto itself homeomorphically, and every curve of $G^*[R^*]$ homeomorphically onto a family $\tilde{G}^*$ which is complementary to F^* in $\tilde{R}^*$ and which coincides with G^* except in the interior of the W_λ's.

It remains to prove that by adding the boundary points of $\tilde{R}^*$, i. e., $\bigcup_{\lambda \varepsilon \tilde{T}} \tilde{\lambda}$, the curves of $\tilde{G}^*$ become curves of a family $\tilde{G}$ complementary to F in π. To prove this we must first prove that $\tilde{G}$ is regular in $R = \pi - B$. Now if v is a point of $\tilde{R}^*$, this is clear, since $\tilde{G} = \tilde{G}^*$ (which is homeomorphic to $\tilde{G}^*$)

in some neighborhod of p. In fact, it is clear (from the method used in getting the homeomorphism f_I above) that there is an arbitrarily small r-neighborhod of p whose closure maps onto $R_0 = \{(x, y) \mid \; |x| \leqq 1, |y| \leqq 1\}$ so that the lines $x =$ constant are the images of the curves of $\tilde{G}$, those lines $y =$ constant are the image curves of F.

Now, however, suppose that p is a regular point on $\tilde{\lambda}(b)$. Then p will be on the common boundary of just two of the neighborhoods W_i, since p is not a branch point. Let W_n, W_m be the two neighborhoods. Then p is interior to $W_n \cup W_m$, and it follows from Theorem I 1. 2-3 that $\tilde{G}[W_n \cup W_m]$ is regular at p, since $\tilde{G}$ is regular in W_n and in W_m separately, as may be seen from the existence of the maps $\tilde{k}_n, \tilde{k}_m$ onto R_1 (or $\tilde{R}_1$ as the case may be) with $\tilde{G}$ mapping onto the lines $x =$ constant. Thus $\tilde{G}$ is regular at every point of R, so that the singularities of $\tilde{G}$ are contained in the set B of singularities of F, and are thus isolated. Now each branch point is in a cut, and hence will be $b_i \varepsilon \lambda(b)$ for some i and some $\lambda(b)$. b_i is on the common boundary of just $2m$ sets W_n, where m is the multiplicity of b_i. Then it is clear that there are just exactly $2m$ curves of $\tilde{G}[W]$, exactly one in each of these $2m$ sets, which have b_i as a limit point in one direction. For, if W_n has b_i on its boundary, then in the homeomorphism $\tilde{k}_n : W_n \rightarrow R_1$ the point b_i will map onto a point $(a, 0)$ and the inverse image of the line $x = a$ is the single curve of $\tilde{G}[W_n]$ which has b_i as a limit point. It follows at once that b_i is a branch of multipilicity $2m$ of $\tilde{G}$. Hence we have established that $\tilde{G}$ is a branched regular curve family with the same branch points as F. Again, just as above, it is clear that it is possible to find an arbitrarily small neighborhood U of each b_i which is homeomorphic to $|z| < 1$, and moreover, with a homeomorphism k carrying $F[\bar{U}]$ onto the level curves of $\Re(z^m)$ and $\tilde{G}[\bar{U}]$ onto the level curves of $\Im(z^m)$.

Finally, to complete the proof that $\tilde{G}$ is complementary to F, we note that by Corollary 2 to Theorem I 3. 5-3 we have at once that every curve of $\tilde{G}$ is a cross-section of F. This completes the proof of the following:

THEOREM 1. 2-1. *Every branched regular curve family F has at least one complementary family G $(= \tilde{G})$ as described above.*

1. 3. The fundamental theorem. Given any branched regular curve family F on π, we have shown the existence of a complementary family G; and also, we have shown [I] that each of these families is the level curve family of a continuous function (without relative extrema) $f(p)$ and $g(p)$ respectively. This enables us to define a single-valued mapping T_1 from the plane π to the complex w-plane as follows: $T_1(p) = u + iv$ where $u = f(p)$

and $v = g(p)$. $T_1(p)$ is clearly continuous, because f and g are continuous. Moreover, T_1 is locally a homeomorphism on R and is exactly m-to-1 in the neighborhood of an m-th-order branch point. To show this, it is sufficient to consider the special neighborhoods mentioned in the proof of the previous theorem, i. e., for every regular point we consider only a neighborhood U such that there is a homeomorphism of U onto the rectangle R_1 of the xy-plane such that $F[U]$ goes onto the lines $y =$ constant and $G[U]$ onto the lines $x =$ constant. Then T_1 becomes a map of R_1 onto a rectangle in the uv-plane carrying the lines $y =$ constant onto $u =$ constant and $x =$ constant onto $v =$ constant. It is clearly a homeomorphism since it is monotone on each line $x =$ constant and each line $y =$ constant. This is exactly as in [III]. It is equally easy to show that in a neighborhood V of a branch point, where $F[V]$ and $G[V]$ map onto $\Re(z^m)$ and $\Im(z^m)$ respectively under a homeomorphism of V onto $|z| < 1$, T_1 carries V onto an open set and is at most m-to-1, where m is the multiplicity of the branch point (cf. [III]). Hence T_1 is not only interior but light (since for every point there is a neighborhood in which f and g take on the same value only a finite number of times in the neighborhood). It follows from Stoilow [V, Chapter V, part III, §5] and Whyburn [VII] that T_1 is topologically equivalent to an analytic function $W = \phi(z)$, i. e., there exists a homeomorphism $p = h(z)$ of the plane π onto either the domain $D_1 = \{z \mid |z| < 1\}$ or $D_\infty = \{z \mid |z| < \infty\}$ of the z-plane such that $\phi(z) = T_1[h(z)]$ is analytic. The family F' of level curves of the real part of $\phi(z)$ are just those curves mapping onto the lines $u =$ constant of the w-plane and hence are homeomorphic to F under h. It is thus proved that:

THEOREM 1. 3-1. *Given any branched regular curve family F there exists a function harmonic in either the finite plane or the unit circle whose level curves are homeomorphic to F.*

Since, if the function $u(x, y)$ is harmonic in a domain D, it is differentiable in D, its level curves will satisfy the differential equations $dx/dt = u_y$, $dy/dt = -u_x$, we have at once:

THEOREM 1. 3-2. *Given any branched regular curve family F, then there is a solution family of a system of differential equations to which it is homeomorphic.*

2. Decomposition of F into Half-Parallel Subfamilies.

2. 1. Extended cross-sections.

THEOREM 2. 1-1. *Let p be any regular point of π, C_p the curve of F through p, and let C be a curve containing a point q such that there is a cross-section pq. Then there will be a cross-section from p to an arbitrary point q' of T_C if and only if $q' \varepsilon C^*$, where C is directed so that $p \varepsilon \mathfrak{D}^*(C)$. Moreover, if $q' \varepsilon C^*$ and $U(qq')$ is any r-set (in $\mathfrak{D}^*(C)$) of qq', we may choose the cross-section qq' as follows: $qq' \equiv qrq'$ where qr lies on pq and rq' is in $U(qq')$.*

Proof. Suppose q' to lie on C^* and let $U(qq')$ be any r-set of qq'. Now moving along pq from p, the cross-section pq lies entirely inside $U(qq')$ from some point on, so we may choose some r on pq, with rq interior to U, letting prq now denote pq. We direct C_r so that $pr \subset \mathfrak{D}^*(C_r)$ and $rq \subset \mathfrak{D}^\#(C_r)$, which we can do by Theorem I 3. 5-3 since prq is a cross-section. We replace rq by a cross-section rq' in U which is found as follows: U by definition is homeomorphic to the rectangle R_1 in the xy-plane, and we join in R_1 the image of r to that of q' by a straight line, whose inverse image we then take for rq'. Since the straight line is a cross-section of the image of F, i. e., the lines $y = k$, rq' will be also a cross-section, and will lie in the same domain $\mathfrak{D}^\#(C_r)$ as rq, since each cross the same curves in U. Hence, by Theorem I 3. 5-3, we know that prq' is a cross-section.

It remains only to prove that if C' is any curve of T_C not on C^*, then there is no cross-section to C' from p. Now p lies in $\mathfrak{D}^*(C^*)$ and C' in $\mathfrak{D}^\#(C^*)$, hence any such cross-section, if it existed, would have to cross C^* and thus would have two points on T_C, contrary to the assumption that it is a cross-section.

THEOREM 2. 1-2. *Let the trees of F be numbered as in* [I, *Section* 4], *i. e., in a standard numbering, using the concentric circles K_n of center p and radius n; further, let the cuts $\tilde{\mathfrak{J}}$ be removed from F, leaving $F^* = F[R^*]$. Then, outside every circle K_n lies at least one curve of F^* which can be reached from p by a cross-section lying in $R^* \cap \mathfrak{D}^*(C_p)$.*

Proof. Denote by $\{C\}$ the collection of all curves in $\mathfrak{D}^*(C_p)$ which can be reached by a cross-section from p lying in $R^* \cap \mathfrak{D}^*(C_p)$. We direct each curve of $\{C\}$ so that $\mathfrak{D}^*(C) \subset \mathfrak{D}^*(C_p)$. The existence of a cross-section from p to $q \varepsilon C$ makes this possible, i. e., direct C so that $\mathfrak{D}^\#(C) \supset pq$. $\{C\}$ will certainly not be empty since we assume p to be a regular point.

Now define on the curves of $\{C\}$ the positive real-valued function $d(C) = \operatorname*{g.l.b.}_{x \varepsilon C}$ {distance from x to p}. We have at once that C is outside K_n if and only if $d(C) > n$. Also it is clear that $\mathfrak{D}^*(C) \supset \mathfrak{D}^*(C')$ implies that $d(C) < d(C')$. To prove the theorem we must show that the numbers $d(C)$ are unbounded. We asume that this is not so; then there is a least upper bound d' of $d(C)$ for C in $\{C\}$. To show that this is impossible we shall choose $N > d'$ and consider intersections of curves of $\{C\}$ with K_N. Since we have assumed $d(C) \leqq d' < N$, every curve of $\{C\}$ will intersect K_N, although by Theorem I 4. 1-1 only a finite number of these curves lie completely inside K_N. All but a finite number of curves of $\{C\}$ in fact, not only have both endpoints outside K_N, but are themselves the only curve of T_C intersecting K_N. Hence, we may choose an infinite sequence of curves C_m of $\{C\}$ such that $d(C_m) \rightarrow d'$, $T_{C_m} \cap K_N \equiv C_m \cap K_N$, and $C_m \cap K_N$ contains neither endpoint of C_m. Having chosen such a sequence we find a subsequence q'_n of points $q'_n \varepsilon C_n$ which approach a regular point q as a limit and all lie on one side of the image of C_q in an r-neighborhood $U(q)$ (i. e., in the upper or lower half of R_0, the image of $U(q)$). This may be done as follows: first, by compactness of K_N we may find $q_{m_i} \varepsilon C_{m_i} \cap K_N$ (a subsequence of the m's) which converges to some point q'. Second, if q' is a regular point, we let $q = q'$ and choose a subsequence q'_n of the q_{m_i}'s, all of whose points lie in one side only of $U(q)$. Or, if q' is a branch point, let $V(q')$ be any admissible neighborhood of q'; then an infinite subsequence of the q_{m_i}'s will lie in one sector of V. If q is any regular point on either of the adjacent curves bounding this sector of V, there will be a corresponding sequence of points q'_n on the same curves C_n which contain the points q_n and such that $q'_n \rightarrow q$. The q'_n will lie on the same side of C_q in any r-neighborhood of q and is thus the desired sequence. Finally, we may choose a subsequence of q'_n which we will denote by r_n such that if qs is a cross-section from q to s in $U(q)$, where s lies on the same side of $U(q)$ as the q'_n, then the intersections $C_n \cap qs$ tend monotonely to q on qs (C_n denoting the curve on which r_n lies). Thus we have $d(C_n) \rightarrow d'$ monotonely since $\mathfrak{D}^*(C_n) \supset \mathfrak{D}^*(C_{n+1}) \supset C_q$ for all n. We direct C_q so that $\mathfrak{D}^*(C_n) \supset \mathfrak{D}^*(C_q)$.

Now choose in $\mathfrak{D}^\#(C_q)$ an r-set W of qq'' where q'' is any point of $C^\#_q$ which is in R^*. W is chosen so that its interior lies in R^*, which is possible by Theorem I 4. 1-4. Now for $n \geqq n_0$, r_n will lie in W, and since we have $\mathfrak{D}^*(C_{n_0}) \supset C^\#_q$ and $\mathfrak{D}^\#(C_{n_0}) \supset C_p$, we may extend the cross-section $pr_{n_0} \subset R^* \cap \mathfrak{D}^*(C_p)$ to a cross-section $pr_{n_0}q'' \subset R^* \cap \mathfrak{D}^*(C_p)$ by merely adding to it that cross-section $r_{n_0}q''$ in $W \cap \mathfrak{D}^*(C_{n_0})$ which is the inverse image in W of the straight line joining the images of r_{n_0} and q'' in R_1, the image of W. This will be a cross-section by Theorem I 3. 5-3. Now since q'' is a regular point of a curve $C_{q''}$, if we take its direction such that $C^{\#}_{q''} \equiv C^{\#}_{q}$, we have $\mathfrak{D}^{\#}(C_{q''}) \supset C_p, C_n$; and $\mathfrak{D}^*(C_{q''}) \subset \mathfrak{D}^*(C_n)$ for all n, whence $d(C_{q''}) \geqq d'$. Now it is easy, however, by taking an r-neighborhood of q'' (which will lie in R^*) to extend $pr_{n_0}q''$ to a slightly larger cross-section $pr_{n_0}q''s$, and since $C_s \subset \mathfrak{D}^*(C_{q''})$, we have at once that $\mathfrak{D}^*(C_{q''}) \supset \mathfrak{D}^*(C_s)$, where C_s is directed as a curve of $\{C\}$, i. e. so that $\mathfrak{D}^*(C_s) \subset \mathfrak{D}^*(C_r)$. Hence $d(C_s) > d(C_{q''}) \geqq d'$. This is contrary to the assumption that d' is a bound of $d(C)$. Hence $d(C)$ is unbounded, which is what was to be proved.

By an *extended cross-section,* we shall mean any curve in $R = \pi - B$ which meets each curve of F at most once and tends to infinity in one or both directions. An extended cross-section is said to *tend properly to infinity* in R in a given direction on it, if it tends to infinity in that direction in such a way that the curves meeting it tend uniformly to infinity with their intersection points with the cross-section. We shall also speak of an *extended cross-section* in R^* which will be an extended cross-section as above, and lie entirely in $R^* = \pi - \tilde{\mathfrak{J}}$, i. e., it meets only curves of F^*.

THEOREM 2. 1-3. *If p is any regular point on a curve C of F^*, then there is an extended cross-section in R^* from p, which lies in $\mathfrak{D}^*(C_p)$ and tends properly to infinity.*

Proof. We consider a curve C in F^* and p any point on it. As before K_n will denote a circle with center at p and radius n; and for any point s we shall let $Q_n(s)$ denote a circle with center at s and radius so chosen that $Q_n(s)$ contains K_n. Now we choose a regular curve C_1 in $\mathfrak{D}^*(C_p) \cap R^*$ for which there is a cross-section pq_1 in $\mathfrak{D}^*(C_p) \cap R^*$ from p to q_1 on C_1. Direct C_1 so that $\mathfrak{D}^*(C_p) \supset \mathfrak{D}^*(C_1)$ and choose in $\mathfrak{D}^*(C_1) \cap R^*$ a curve C_2 outside of $Q_1(q_1)$ and such that a cross-section q_1q_2 in $\mathfrak{D}^*(C_1) \cap R^*$ exists with q_2 on C_2. Having chosen C_n and $q_n \varepsilon C_n$ in this manner, we choose for C_{n+1} any regular curve outside of $Q_n(q_n)$ for which there is a cross-section q_nq_{n+1} in $\mathfrak{D}^*(C_n) \cap R^*$ to q_{n+1} on C_{n+1}. We direct C_{n+1} so that $\mathfrak{D}^*(C_n) \supset \mathfrak{D}^*(C_{n+1})$. We continue this process indefinitely by Theorem 2. 1-2. Then the curves $pq_1, pq_1q_2, pq_1q_2q_3 \cdots$ will all be cross-sections by Theorem I 3. 4-5. They approach a curve Γ extending from p to infinity in $\mathfrak{D}^*(C_p) \cap R^*$ which is an extended cross-section extending from p to infinity in R^*. The curves intersecting Γ tend uniformly to infinity with any sequence of their points of intersection tending to infinity on Γ; since if r on Γ is beyond q_n, then C_r lies outside K_n. Thus Γ is an extended cross-section tending properly to infinity in R^*.

2. 2. Half-parallel subfamilies of *F*. We mean by a *half-parallel subfamily of F* the collection of all curves of F which intersect an extended cross-section Γ tending from a point p on a curve C_p properly to infinity. And we shall mean by a *complete* half-parallel subfamily of F the curve $C_p{}^*$ together with all curves of F crossing Γ (C_p being so directed that $\mathfrak{D}^*(C_p) \supset \Gamma$). The first of these sets is homeomorphic to the lines $y = k$, $k \geqq 0$ of the half-plane by Theorem 1. 1-1 and the same reasoning as used in the proof of that theorem will establish this homeomorphism for the second case, the complete half-parallel subfamily also. The first will be denoted by S and the second by S^*. Clearly $S^* \supset S$ and when C_p is a regular curve they are identical. C_p is called the *initial curve* of S, $C_p{}^*$ the initial curve of S^*.

If $\Gamma(q)$ is any half-open cross-section of F tending from a regular point q properly to infinity, then the boundary of $S(\Gamma)$, $S(\Gamma)$ being the collection of curves intersecting Γ, is best described in terms of maximal chains C^*, $C^\#$ and the sets $\delta(C+)$, $\delta(C-)$ defined in [I, Section 3]. We shall refer to these latter two sets as *mixed maximal chains*, since they consist of two subchains of maximal chains, one clockwise adjacent, the other counterclockwise adjacent, e. g., $\delta(C+) = \delta^*(C+) \cup \delta^\#(C+)$ (which may be empty). $\delta(C)$ will denote $\delta(C+) \cup \delta(C-)$; it is empty if and only if C is a regular curve.

THEOREM 2. 2-1. *The boundary of $S(\Gamma)$ is a collection of maximal chains C^*, $C^\#$ and mixed maximal chains $\delta(C)$, where $\delta(C)$ is on the boundary if and only if C is in $S(\Gamma)$. From each set T_C of F there is either* (1) *no point,* (2) *exactly one maximal chain, or* (3) *a set $\delta(C)$ of T_C on the boundary of $S(\Gamma)$.* (1), (2) *and* (3) *are mutually exclusive.*

Proof. Suppose $C \varepsilon S(\Gamma)$ is a singular curve, then $\delta(C)$ is in the boundary of $S(\Gamma)$, for if we consider any point q on $\delta(C)$ there exists an r-set $U(pq)$ containing q and $p = C \cap \Gamma$ (since C lies on an adjacent chain with C_q); choosing a sequence of points $p_n \to p$, $p_n \varepsilon U \cap \Gamma$, we can find by Theorem I 3. 5-2 a sequence $q_n \varepsilon U$ such that $q_n \varepsilon C_{p_n}$ for all n and $q_n \to q$. Whence q is a limit point of points of $S(\Gamma)$. But, if q is in $\delta(C)$, it is on a curve of T_C other than C; and C_q therefore cannot intersect Γ. Hence C_q is not in $S(\Gamma)$, and thus q is on the boundary of $S(\Gamma)$. Moreover, no other curves of T_C can in this case be on the boundary of $S(\Gamma)$, for $S(\Gamma)$ is clearly contained in $\mathfrak{D}^*(C) \cup C \cup \mathfrak{D}^\#(C)$, a complementary domain of $\delta(C)$, whereas every other curve of T_C lies in one or two other complementary domains of $\delta(C)$. (Note: $\delta(C)$ divides π into at most three Jordan domains.

On the other hand, suppose that C is a curve of F on the boundary of $S(\Gamma)$. (Note: from what follows it is clear that the boundary is indeed a union of curves of F.) Then, directing C so that $\mathfrak{D}^*(C)$ contains the initial point of Γ, we note that if p is a point on C, limit point of a sequence p_n of $S(\Gamma)$, then there is an r-set $U(pq)$ of any arc pq on C^* and a sequence $q_n \to q$ with $q_n \varepsilon C_{p_n}$, $C_{p_n} \subset S(\Gamma)$, from which we conclude that q is either in $S(\Gamma)$ or on its boundary. If C^* does not cross Γ, then q will be on the boundary and C^* is a boundary curve of $S(\Gamma)$. When this is the case, C^* divides π into two domains $\mathfrak{D}^*(C^*) \supset S(\Gamma)$ and $\mathfrak{D}^\#(C^*) \supset T_C - C^*$, whence no other points of T_C other than those of C^* are on the boundary of $S(\Gamma)$. But, if C^* crosses Γ at a point p on a curve C', then we are back in the previous case and $\delta(C') = [C^* \bigcup C^\#] - C'$ is the boundary in T_C of $S(\Gamma)$.

THEOREM 2.2-2. *Let q be a point on a curve C_q of $F^* = F[R^*]$ and let $\Gamma(q)$ be a cross-section tending properly to infinity in R^* in each direction. Further, let h be any homeomorphism of R^* onto the xy-plane, then $h[\Gamma(q)]$ is a cross-section of the family $h[F^*]$ (filling the xy-plane) which tends properly to infinity in both directions on the xy-plane.*

Proof. On the xy-plane we let K_n denote a circle of radius n, center $h(q)$. We must show that for every n there are points q'_n, r'_n on $\Gamma' = h[\Gamma(q)]$ such that every curve of $h(F^*)$ intersecting Γ' at points outside the arc $q'_n r'_n$ will lie outside K_n. If this is not the case, as we shall assume, we will be able to find a sequence of points t'_n receding to infinity on Γ' such that each $C_{t'_n}$ intersects a fixed circle K_N of the circles K_n. Now the inverse image K of K_N is a simple closed curve in R^* containing q in its interior. We will denote by C_n the inverse image of $C_{t'_n}$ and by t_n the inverse image of t'_n. Every C_n must then intersect K and hence intersect some circle with center at q which contains K. But this contradicts the assumption that $\Gamma(q)$ tended properly to infinity in R^*, since we have a sequence t_n approaching infinity on $\Gamma(q)$, but the curves C_{t_n} do not approach infinity. Hence the theorem must be true.

W. Kaplan introduced the notion of admissible collections of finite sequences in order to number the half-parallel subsets of a regular curve family filling an open simply connected domain. The concept is so similar to that already considered in the numbering of curves of a tree that we shall be able to use the same notation as in that section. As in Kaplan [II], we shall call a collection A of finite sequences *admissible* if

(1) A contains the one-element sequence 1 and no other one-element sequences, and

(2) $\alpha, k \varepsilon A$ implies $\alpha, k - 1 \varepsilon A$ if $k > 1$ and implies $\alpha \varepsilon A$ if $k = 1$.

Now, if we have a regular curve family F' filling the xy-plane, and if we have assigned to each point (x, y) an extended cross-section $\Gamma(x, y)$ tending properly to infinity in both directions, then for any fixed curve C_1 it was shown in [II] that we can decompose $F'[C_1 \cup \mathfrak{D}^*(C_1)]$ into a collection of non-overlapping, half-parallel subfamilies $S(\alpha)$ which will be numbered by the finite sequences $\{\alpha\}$ of an allowable collection A. Each half-parallel family $S(\alpha)$ will be the set of all curves intersecting a cross-section $\Gamma(\alpha)$ tending from an initial curve C_a to infinity and lying on some $\Gamma(x, y)$ as chosen above; C_a will be the only curve of $S(\alpha)$ mapped onto the x-axis in the homeomorphism of $S(\alpha)$ onto the lines $y = k \geqq 0$ and the complete boundary of $S(\alpha)$ will be, in addition to C_a, just exactly the curves $C_{a,k}$. Note that when we write C_a we mean to indicate that C_a is an initial curve of some $S(\alpha)$ in the decomposition of F', whereas $C(\alpha)$ will, as in [I], indicate that C is the curve of a numbered tree which has been assigned the signed sequence α in the numbering of the tree.

As a corollary to the preceding Theorem 2.2-2 plus the proof of the facts mentioned in the preceding paragraph from [II], we can immediately state the following theorem:

THEOREM 2.2-3. *Given the family $F^* = F[R^*]$ and an arbitrary regular curve C_1 of F^*, we can decompose $F^*[C_1 \cup \mathfrak{D}^*(C_1)]$ (which is the same as $F[C_1 \cup \mathfrak{D}^*(C_1) \cap R^*]$) into a collection of non-overlapping half-parallel subsets $S(\alpha)$, each $S(\alpha)$ being all curves intersecting a cross-section $\Gamma(\alpha)$ tending from a curve C_a in F^* properly to infinity in R^*.*

In order to study the relation between an arbitrary tree T of F and a given decomposition of F^* into sets $S(\alpha)$ ($\alpha \in A$, as described above), it is convenient to adopt some new notation. $A(T)$ will denote the subset of A containing all sequences α such that $S(\alpha) \cap T \neq 0$; and $A_n(T)$ the subset of all sequences of $A(T)$ of order n. We denote by $N(T)$ the smallest integer n such that $A_n(T)$ is not empty. It is clear that $\Gamma(\alpha)$ can have at most one point on T, and hence $S(\alpha) \cap T$ is a curve of F^* or is empty. If $\Gamma(\alpha) \cap T$ is the initial point of $\Gamma(\alpha)$ we say that $\Gamma(\alpha)$, or $S(\alpha)$, *begins at* T; in this case $C_a = S(\alpha) \cap T$. When $\Gamma(\alpha) \cap T$ is a point of $\Gamma(\alpha)$ other than the initial point, then $\Gamma(\alpha)$, or $S(\alpha)$ is said to *straddle* T. In the former case $S(\alpha)$ lies in one complementary domain of T, in the latter in two. Using these notations, we may state the following properties:

(1) If α, β are *distinct* elements of A with $\beta \in A(T)$, and α either an element of $A(T)$ or such that points of T lie on the boundary of $S(\alpha)$; then $S(\alpha)$, $S(\beta)$ cannot each intersect the same complementary domain of T.

(2a) If $A_N(T)$, $N = N(T)$, has one element α, then either $S(\alpha)$ straddles T, or if $S(\alpha)$ begins at T, then $C_\alpha{}^* \cap R^* = S(\alpha) \cap T$, i. e., $C_\alpha{}^*$ has just *one* curve in R^*.

(2b) If $A_N(T)$ has more than one element, then every element of $A_N(T)$ is of the form β, k for fixed β of order $N - 1$ and $C_{\beta,k}$ for $\beta, k \,\varepsilon\, A_N(T)$ are just those curves of a maximal chain C^* which are in R^*.

(3) Let γ be an element of $A_{N+k}(T)$, then every lower segment of γ of order $\geqq N(T)$ is in $A(T)$, i. e., for $0 \leqq j \leqq k$ we have $\gamma_{N+j} \,\varepsilon\, A_{N+j}(T)$, where, as previously, γ_{N+j} is the sequence consisting of the first $N + j$ elements of the sequence γ.

(4) A necessary condition that $S(\alpha)$ straddle T is that $\alpha \,\varepsilon\, A_N(T)$ and is the only element of $A_N(T)$.

First we prove (1). Let $\mathfrak{D}^*(C)$ be a complementary domain of T, bounded by C^* on T. Suppose that $S(\alpha)$ and $S(\beta)$ both have points in $\mathfrak{D}^*(C)$. Then there is a point p_1 on $\Gamma(\alpha)$, p_2 on $\Gamma(\beta)$, each in $\mathfrak{D}^*(C)$. Now since $\beta \,\varepsilon\, A(T)$, $\Gamma(\beta)$ has a point q_2 on C^* and p_2q_2, an arc on $\Gamma(\beta)$, lies in $\mathfrak{D}^*(C) \cup C^*$. In either of the possibilities for α mentioned above, there would be a point q_1 on C^* which was a limit point of points q'_n in $S(\alpha)$. If $\alpha \,\varepsilon\, A(T)$ then q'_1 and q'_n may be taken on $\Gamma(\alpha)$, otherwise q'_n will consist of points in $\mathfrak{D}^*(C)$ which are in $S(\alpha)$ but not on $\Gamma(\alpha)$. It follows by arguments used many times above, i. e. considering an r-set of q_1q_2, etc., that there is a cross-section from q_1 on C^* into $\mathfrak{D}^*(C)$, which may be shown to cross a curve also crossed by p_2q_2. This curve would have to be in both $S(\alpha)$ and $S(\beta)$ which is impossible since α, β were assumed distinct. The following lemma will be useful in proving properties 2-4.

LEMMA. *If $\alpha \,\varepsilon\, A(T)$ and $\alpha, k \not\varepsilon\, A(T)$, then no sequence γ of $A(T)$ can have α, k as a lower segment.*

Proof. $C_{\alpha,k}$ lies on the boundary of $S(\alpha)$ but is not in T, nor is any curve of $S(\alpha, k)$ in T by hypothesis. Assuming $C_{\alpha,k}$ directed so that $S(\alpha, k) \subset \mathfrak{D}^*(C_{\alpha,k})$, we have by Theorem 2. 2-1 two possibilities: (a) the entire curve $C^\#_{\alpha,k}$ is on the boundary of $S(\alpha)$ and is all of this boundary in the tree $T' = T_{C_{\alpha,k}}$, or (b) T' intersects $S(\alpha)$ on a curve C of T', and then $C_{\alpha,k} \subset \delta(C)$ where $\delta(C)$ is on the boundary of $S(\alpha)$ and is all of this boundary in T'. In case (a) every curve of $C^\#_{\alpha,k} \cap R^*$, being on the boundary of $S(\alpha)$, is a curve $C_{\alpha,k'}$ for some k'. We have $\mathfrak{D}^\#(C^\#_{\alpha,k}) \supset T$, since it contains $S(\alpha)$ which intersects T. In case (b) $\delta(C) = \delta(C+) \cup \delta(C-)$ divides π into three domains (or two if one of the sets $\delta(C \pm)$ is empty);

one of these which we denote D_1 contains C and hence $S(\alpha)$ and T. The others, D_2 and D_3, contain all other curves of T'. $\delta(C)$ is the complete boundary in T' of $S(\alpha)$, hence every curve of $\delta(C) \cap R^*$ is a curve $C_{a,k'}$ for some k'.

The remainder of the proof depends on the fact that $S(\beta) \cup S(\beta, k)$ is always a connected set. If there exists any sequence $\gamma = \alpha, k, n_1, \cdots, n_r$ such that γ is in $A(T)$ then, $S(\gamma)$ must clearly have points in $\mathfrak{D}^{\#}(C^{\#}_{a,k})$ above in case (a) or in D_1 in case (b), these being the domains of T' in which T lies, and hence, so also has the set $\Sigma = \bigcup_{j=0}^{r} S(\alpha, k, n_1, n_2, \cdots, n_j)$. Moreover, the set Σ is connected, and $S(\alpha, k)$ which is in this set lies in $\mathfrak{D}^*(C_{a,k})$ in case (a), and in D_2 or D_3 in case (b). Thus Σ has points on either $C^{\#}_{a,k}$ or $\delta(C)$, the boundary curve, i. e., for a $j \neq 0$ there is a curve of $C^{\#}_{a,k}$ or $\delta(C)$ as the case may be in $S(\alpha, k, n_1, \cdots, n_j)$. But each such curve as already pointed out is a curve $C_{a,k'}$, which is a contradiction.

The lemma implies in particular, that if α and $\alpha, n_1, \cdots, n_r \,\varepsilon\, A(T)$ then $\alpha, n_1, \cdots, n_j \,\varepsilon\, A(T)$, $j \leqq r$. Hence (3) will follow if we prove that every sequence of $A(T)$ contains a lower segment in $A_N(T)$.

Now we turn to an examination of the possibilities for $A_N(T)$ and completion of the proof of (3). Suppose that α is an element of $A_N(T)$. Then either $S(\alpha)$: (i) straddles T, or (ii) begins at T. In the former case let $C = S(\alpha) \cap T$, then $\delta(C)$ is the complete boundary of $S(\alpha)$ in T, and we know that every curve in R^* of $\delta(C)$ is in the collection $\{C_{a,k}\}$. Moreover, $\delta(C)$ divides π into three (or two) domains $D_1, D_2, (D_3)$ of which the first contains C_1, and of T, *only* the curve C. Now let γ be any sequence of $A(T)$. $S(\gamma)$ must, by (1), lie in D_2 or D_3. But $\bigcup_{i=1}^{n} S(\gamma_i)$ is a connected set containing C_1 (i. e., C_a, $\alpha = 1$), hence points of D_1 and also points of D_2 or D_3. It must then contain a curve $C_{a,k}$ of $\delta(C)$, and therefore $S(\alpha)$, i. e., α is a lower segment of γ. Since this is only possible if γ is of order $> N$ we conclude α is the only element of $A_N(T)$.

In the case (ii) where $S(\alpha)$ begins at T, we have C_a on the boundary of $S(\beta)$, where β is of order $N - 1$ and $\alpha = \beta, k$. In fact, $C^{\#}_a$ is the complete boundary on T of $S(\beta)$ (the curves of $C^{\#}_a \cap R^*$ are all in the set $\{C_{\beta,k'}\}$ for some k' and therefore are in $A_N(T)$); and we have $\mathfrak{D}^{\#}(C^{\#}_a) \supset S(\beta)$, $\mathfrak{D}^*(C^*_a) \supset S(\alpha)$. Now let us suppose that $\gamma \,\varepsilon\, A(T)$, then $S(\gamma)$ by (1) cannot lie in $\mathfrak{D}^{\#}(C^{\#}_a)$, hence must lie in $\mathfrak{D}^*(C^{\#}_a)$. But $\bigcup_{i=1}^{n} S(\gamma_i)$ is connected and has a point in $\mathfrak{D}^{\#}(C^{\#}_a)$, namely, any point of C_1. Thus this set has a point on $C^{\#}_{a} \cap R^*$ and hence contains a curve $C_{a,k'}$. It follows that every sequence of $A(T)$ has a lower segment in $A_N(T)$. This proves (1) and completes the proof of (3).

To prove (4) we need show only that if $S(\alpha)$ straddles T, then no lower segment of α is in $A(T)$. If $\alpha = \beta, k$, so that β is the lower segment α_{n-1}, then if any lower segment of α is in $A(T)$, β is also by our lemma. Then C_α, being on the boundary of $S(\beta)$, we necessarily have $S(\beta)$, $S(\alpha)$ in different domains of T_{C_α}. This is impossible unless $T = T_{C_\alpha}$, for we would otherwise have points of T in two different domains of T_{C_α}. But then $S(\alpha)$ does not straddle T, for on the contrary $C_\alpha \subset T$.

As above we consider the branched regular curve family F with a regular curve C_1 of F and a decomposition of the corresponding $F^*[C_1 \cup \mathfrak{D}^*(C_1)]$ into sets $S(\alpha)$ with initial curves $C(\alpha)$. Then using properties 1-4, we have the following:

THEOREM 2. 2-4. *The complete half-parallel subfamilies* $S^*(\alpha) = S(\alpha) \cup C^*_\alpha$ *decompose* $F[\mathfrak{D}^*(C_1) \cup C_1]$ *into a family of half-parallel subsets which intersect only on a curve of their initial curves, i. e.,* $S^*(\alpha) \cap S^*(\beta) = 0$ *or* C *where* $C^* = C^*_\alpha$ *and* $C^{\#} = C^{\#}_\beta$. (See Figure 4.)

Proof. First to prove that every curve of $F[C_1 \cup \mathfrak{D}^*(C_1)]$ is included in this decomposition we note that every curve of $F^*[C_1 \cup \mathfrak{D}^*(C_1)]$ is automatically included, being already in a set $S(\alpha)$ of the decomposition of that part of the simply connected region R^* included in $\mathfrak{D}^*(C_1)$. Thus we have only to consider curves of $\tilde{\mathfrak{J}}$; let C be a curve of $F[C_1 \cup \mathfrak{D}^*(C_1)]$ which is not in R^* and let T denote the tree which contains it. Then no cross-section $\Gamma(\alpha)$ has a point on C. C will be on the boundary of two distinct sets $S(\alpha)$ and $S(\beta)$ in $\mathfrak{D}^*(C)$ and $\mathfrak{D}^{\#}(C)$ respectively. They cannot coincide since if they did, then it would mean that $S(\alpha) = S(\beta)$ would straddle T, for otherwise the set $S(\alpha)$ lies in a single domain of T. But then, in this case, since the cross-section $\Gamma(\alpha)$ would have points in two domains both having C and only C as common boundary, it would have to contain a point of C, which is clearly impossible if C is not in R^*.

Now, if either α or β, say α, is of order $> N(T)$ then, since C, a curve of T is on the boundary of $S(\alpha)$, α, k for some k is in $A(T)$. Hence by (3), α must also be in $A(T)$. Then by (4), C_α must lie on T, whence we have at once that $C^*_\alpha = C^* \supset C$ and hence C is in $S^*(\alpha)$. Thus it remains to show that either α or β must be of order $> N$. Clearly each is of order $\geqq N - 1$ since, for example, curves of T are on the boundary of $S(\alpha)$ hence either $\alpha \varepsilon A(T)$ or $\alpha, k \varepsilon A(T)$ for some k. Assume α is of order $N - 1$, then $\Gamma(\alpha)$ does not have a point on T and by Theorem 2. 2-1 all of C^* is on the boundary of $S(\alpha)$, and every curve of $C^* \cap R^*$ is in the set $\{C_{\alpha,k}\}$, and for each of these $\alpha, k \varepsilon A_N(T)$. Now, since β, k' for some k' is in $A(T)$, β is of order $\geqq N-1$. If now β were of order $N-1$, then there must be a $\beta, k' \varepsilon A_N(T)$ and β, k' would for some k equal α, k by property (2b), whence $\beta = \alpha$ by (1); or if β were of order N, then it is of the form α, k for some k. This latter would mean that the common boundary of the

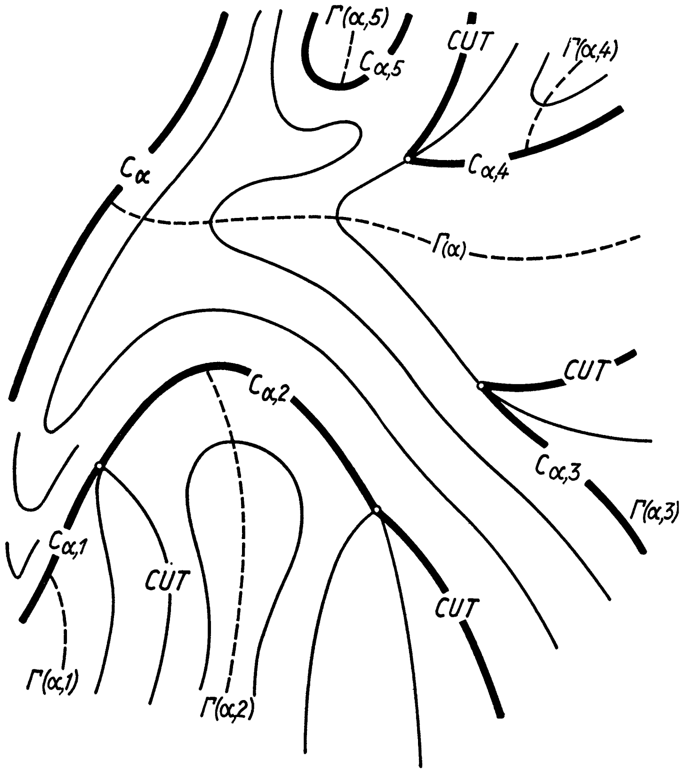

FIGURE 4

domains containing $S(\alpha)$ and $S(\beta)$ would be the curve $C_{\alpha,k}$ which must then coincide with the curve C, contrary to assumption that C is not in R^*. Hence β in this case must be of order $> N$. On the other hand, if α is of order N, then either β is of order $> N$ or C_α and C_β lie on the same maximal curve $C^\#_\alpha = C^\#_\beta$, and by (2b) in this case quite clearly, C^*_α and C^*_β could not have a boundary curve C in common. Hence either α or β is of order $> N$ and we have already shown that in this case C is in the initial curve C^*_α or C^*_β of either $S^*(\alpha)$ or $S^*(\beta)$ respectively.

Next it must be shown that if C_α is the initial curve of a set $S(\alpha)$, then for any $S(\beta)$ which intersects C^*_α, the intersection must be C_β. Let C be the curve of intersection, i. e., $C = C^*_\alpha \cap S(\beta)$. Thus, $\alpha, \beta \in A(T)$ where T is the tree containing C_α. Now $S(\alpha)$ and $S(\beta)$ cannot have points in the same complementary domain of T by (1), which means in particular that $S(\beta)$ cannot straddle T, i. e., $\Gamma(\beta) \cap C =$ initial point of $\Gamma(\beta)$, since one complementary domain of C is $\mathfrak{D}^*(C^*_\alpha)$, C lying as it does on C^*_α, and $S(\alpha)$ lies in this complementary domain. Hence $C_\beta = C$ which was to be proved. From this it follows that two complete half-parallel subfamilies intersect only along a curve of their initial curves, and when they do intersect on a curve C, one lies in $\mathfrak{D}^*(C)$ and the other in $\mathfrak{D}^\#(C)$.

THEOREM 2. 2-5. *The family F can be decomposed into complete half-parallel subfamilies which overlap at most along their initial curves.*

Proof. We merely begin with any regular curve C_1 and decompose both $C_1 \cup \mathfrak{D}^*(C_1)$ and $C_1 \cup \mathfrak{D}^\#(C_1)$ as above.

We remark in conclusion that if $f(p)$ is a function with the family F as level curves, and if $S^*(\alpha)$ is a half-parallel subfamily of the decomposition, then clearly $f(p)$ is strictly monotone on $\Gamma(\alpha)$ and hence cannot assume the same value on two curves of $S^*(\alpha)$. If F is the level curve family of the real part of a single-valued analytic function, then clearly this function is 1-1 in $S(\alpha)$, i. e. as noted earlier the $\{S^*(\alpha), \alpha \in A\}$ give a decomposition of the domain of the analytic function into domains in which it is schlicht.

NORTHWESTERN UNIVERSITY.

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