Functions

A function is a set of ordered pairs where no two ordered pairs are "first same and second different"

ex $\{(3,7),(5,2),(4,1)\}$

This is a function

$\{(3,7),(5,2),(3,1)\}$

This is NOT a function.

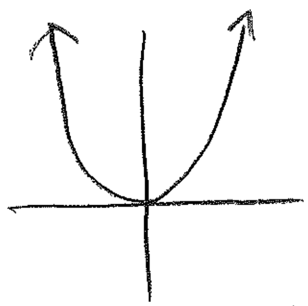
$\{(3,7),(5,2),(4,7)\}$

This is a function.

The Vertical Line Test

A set of ordered pairs is a function exactly when no vertical line hits the graph more than once.

ex

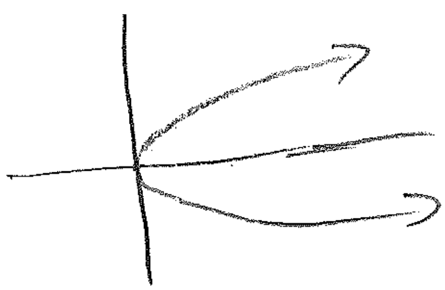


this is a picture of the set

$\{(0,0), (1,1), (2,4), (3,9), (-1,1), (-2,4), (-3,9), (\frac{1}{2}, \frac{1}{4}), \ldots\}$

this is a function.

ex



this is a picture of the set

$\{(0,0), (1,1), (4,2), (9,3), (1,-1), (4,-2), (9,-3), \ldots\}$

this is NOT a function.

We write

$f(x) = y$ if (x,y) is in f.

ex

If $f = \{(3,7), (5,2), (4,1)\}$

then $f(3) = 7$

$f(5) = 2$

$f(4) = 1$

$f(9)$ is undefined

$f(7)$ " " .

We say that

" f is given by

$f(x) = x^2$ "

to mean

that $f = \{(x, x^2) : x \text{ is a real \#}\}$

i.e. $f = \{(0,0), (1,1), (2,4), (3,9),$

$(-1,1), (-2,4), \ldots\}$

One to One Functions

We say that
a function, f, is
onetoone (1-1) if
there are no two
ordered pairs in f
that are "second same".

ex $f = \{(3,7), (5,2), (4,7)\}$

this is a function but is
NOT onetoone.

ex $f = \{(3,7), (5,2), (4,1)\}$

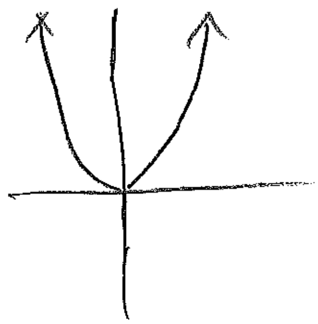
this is a function and
it is onetoone.

Horizontal Line Test

Let f be a function.
Then f is onetoone
exactly when

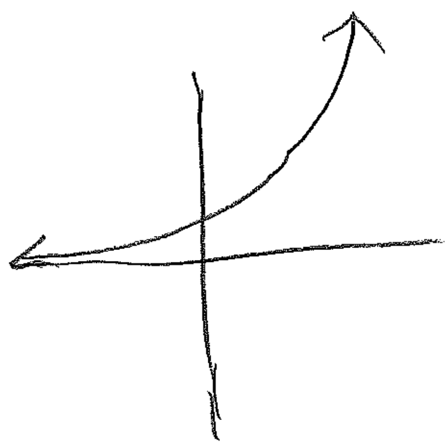
no horizontal line hits the graph more than once.

ex



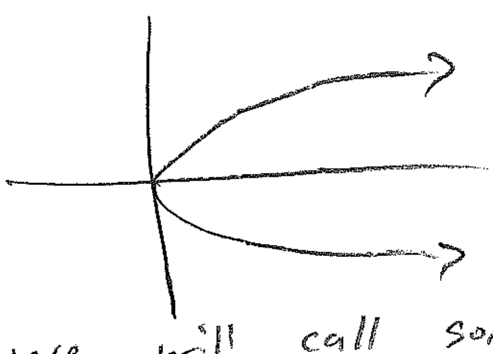
NOT 1-1

ex



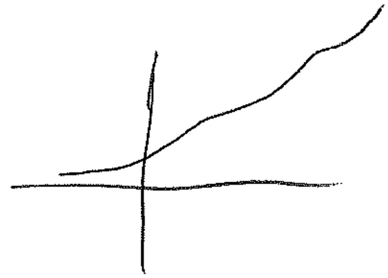
IS 1-1.

later

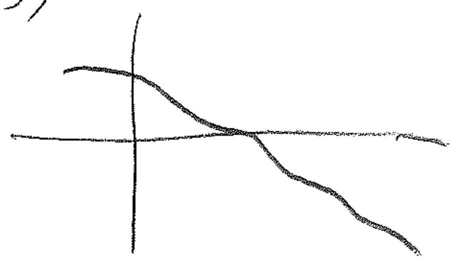


we will call something that passes HLT an antifunction.

If f is increasing,
then f is 1-1



If f is decreasing,
then f is 1-1.



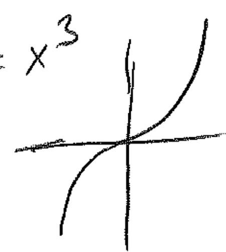
Note:

f is increasing
when f' is positive

f is decreasing
when f' is negative.

So to see if f is 1-1,
you could verify that
$f'(x)$ is always positive
OR $f'(x)$ is always negative.

ex if $f(x) = x^3$, then $f'(x) = 3x^2$.
$3x^2$ is always positive,
So f is increasing.



Inverse Functions

If f is a function,

then we define

$$f^{\leftarrow} := \{(y,x) : (x,y) \text{ in } f\}$$

ex If $f = \{(3,7),(5,2),(4,7)\}$

then $f^{\leftarrow} = \{(7,3),(2,5),(7,4)\}$.

Note that $f^{\leftarrow}$ is a function exactly when f is 1-1.

If f is 1-1,

then we call $f^{\leftarrow}$

the inverse function of f

and we define $f^{-1} := f^{\leftarrow}$

ex if $f = \{(3,7),(5,2),(4,1)\}$

then $f^{-1} = \{(7,3),(2,5),(1,4)\}$

$\text{Domain}(f^{-1}) = \text{Range}(f)$

$\text{Range}(f^{-1}) = \text{Domain}(f)$

say

$f = \{(x,y), \dots\}$

$f^{-1} = \{(y,x), \dots\}$

$f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x$

If $f(x_0) = y_0$,

then $f^{-1}(y_0) = x_0$.

If f is increasing,

then f^{-1} is increasing,

If f is decreasing,

then f^{-1} is decreasing.

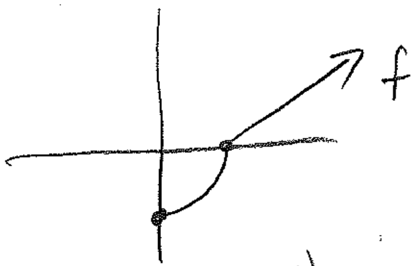
The graph of f^{-1}

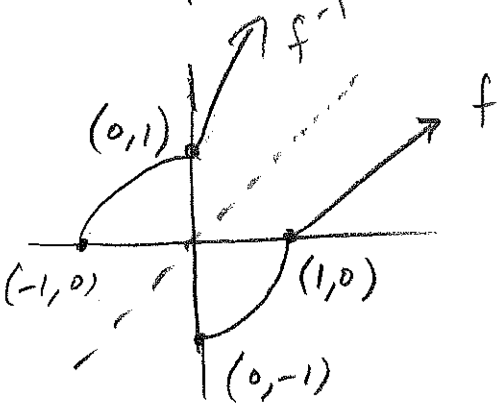
is the graph of f

reflected across the

line $y = x$.

ex





ex If $f(x) = x^3$.

Find $f^{-1}(x)$.

Write $y = x^3$

Now write $x = y^3$

Now solve for y

$$x = y^3$$

$$\longrightarrow \quad y = \sqrt[3]{x}$$

so $f^{-1}(x) = \sqrt[3]{x}$

ex if $f(x) = \dfrac{3x+4}{5x+6}$

then find $f^{-1}(x)$.

$$y = \frac{3x+4}{5x+6}$$

$$x = \frac{3y+4}{5y+6}$$

$\rightarrow \ x(5y+6) = 3y+4$

$\rightarrow \ 5xy + 6x = 3y + 4$

$\rightarrow \ 5xy - 3y = -6x + 4$

$\rightarrow \ y(5x-3) = -6x + 4$

$\rightarrow \ y = \dfrac{-6x+4}{5x-3}$

so $f^{-1}(x) = \dfrac{-6x+4}{5x-3}$

The Number e

We define

$$n! := n(n-1)(n-2)\cdots(2)(1)$$

$$0! := 1$$

We call $n!$ "n factorial"

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$$

$$8! = 8\cdot 7\cdot 6\cdot 5\cdot 3\cdot 4\cdot 2\cdot 1 = 40{,}320$$

We define

$$e := \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e \approx 2.718$$

Exponential Functions

We define

$$\exp_b(x) := b^x$$

We call $\exp_b$ the exponential function with base b.

We define $\exp(x) := e^x$.

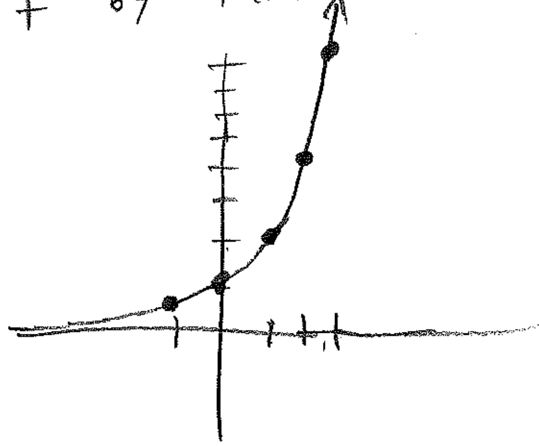
ex $\exp_2$ is the function f given by

$$f(x) = 2^x.$$

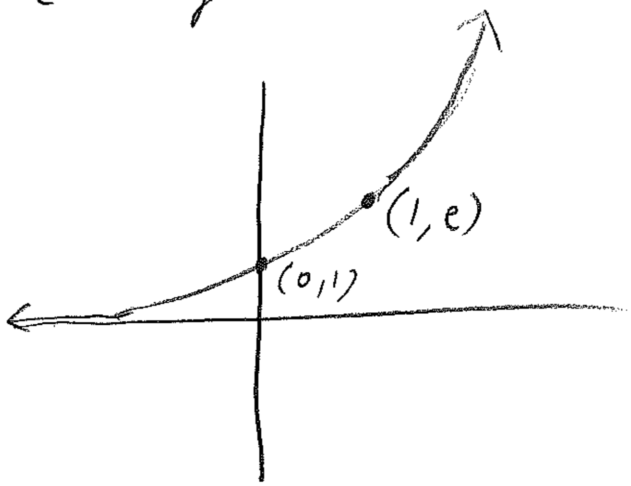
Graphs of Exponential Functions

ex Graph f by $f(x) = 2^x$

x	y
0	1
1	2
2	4
3	8
-1	$\frac{1}{2}$



ex $y = e^x$



Logarithmic Functions

Note that $\exp_b$ is increasing (when $b > 1$; if $b < 1$, then $\exp_b$ is decreasing). So $\exp_b$ is 1-1.

So $\exp_b$ has an inverse function. That inverse function is called $\log_b$.

We define

$$\log_b(x) := y \quad \text{if} \quad b^y = x.$$

ex $\log_2(8) = 3$

because $2^3 = 8$.

We define

$\ln(x) := \log_e(x)$.

ex $\ln(e^5) = \log_e(e^5) = 5$

We define

$\log(x) := \log_{10}(x)$

ex $\log(100) = \log_{10}(100) = 2$

---

ex Convert to an exponential equation.

$\log_b(x) = y \rightarrow b^y = x$

$\log_r(w) = k \rightarrow r^k = w$

ex Convert to a log equation.

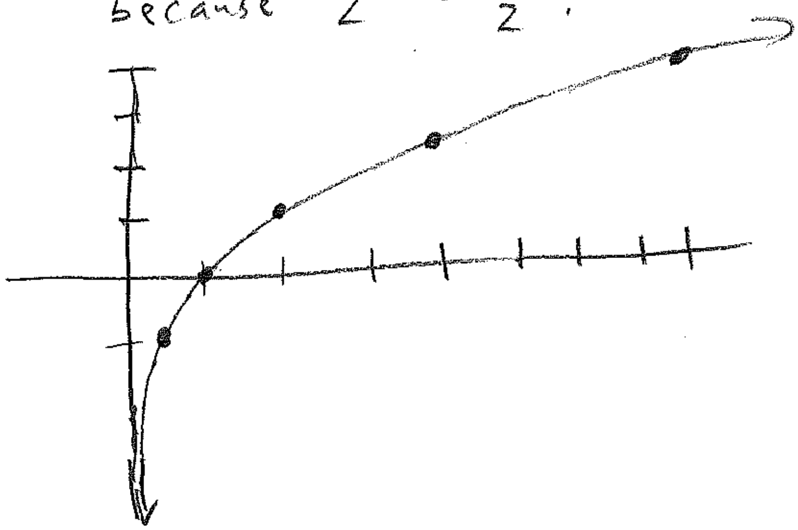
$b^y = x \longrightarrow \log_b(x) = y$

$u^h = l \rightarrow \log_u(l) = h$

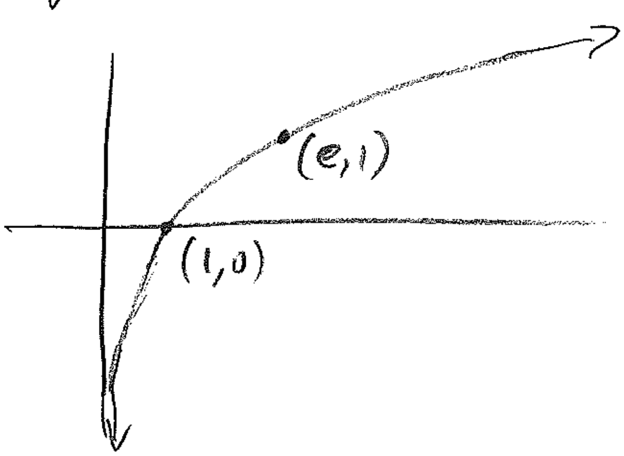
Graphs of Log Functions

ex Graph $y = \log_2(x)$

x	y	
1	0	because $2^0 = 1$
2	1	because $2^1 = 2$
4	2	because $2^2 = 4$
8	3	because $2^3 = 8$
$\frac{1}{2}$	-1	because $2^{-1} = \frac{1}{2}$.



ex $y = \ln(x)$



Properties of Log Functions

$\log_b(1) = 0$

ex $\log_2(1) = 0$

$\log_b(b) = 1$

ex $\log_2(2) = 1$

$\log_b(b^x) = x$

ex $\log_2(2^5) = 5$

$b^{\log_b(x)} = x$

ex $2^{\log_2(8)} = 8$

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

ex $\log_2(4 \cdot 8) = \log_2(4) + \log_2(8)$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

ex $\log_2\left(\frac{8}{4}\right) = \log_2(8) - \log_2(4)$

$$\log_b(x^p) = p\log_b(x)$$

ex $\log_2(8^5) = 5\log_2(8)$

$$\log_b(x) = \frac{\log_B(x)}{\log_B(b)}$$

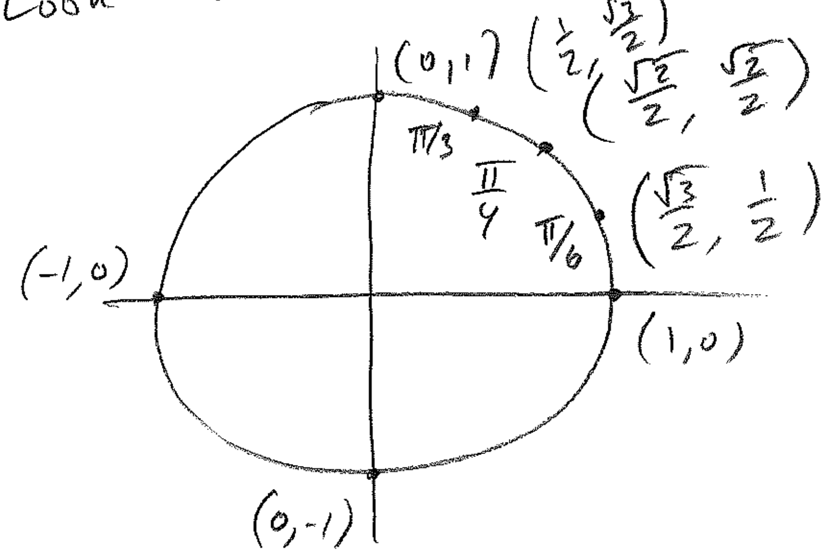
ex $\log_2(8) = \dfrac{\log_e(8)}{\log_e(2)} = \dfrac{\ln(8)}{\ln(2)} = 3$

$$b^x = B^{\log_B(b)x}$$

ex $2^x = e^{\log_e(2)x} = e^{\ln(2)x}$

Trigonometric Functions

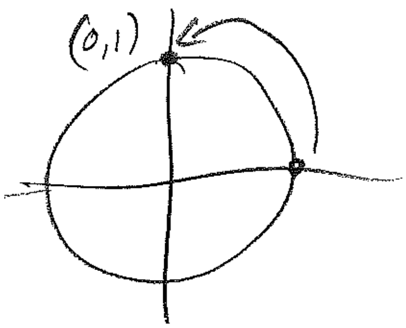
Look at a unit circle,



We define $\sin(\theta)$ to be the second coordinate of the point you land at after starting at $(1,0)$ and traveling counterclockwise by an arclength of θ.

$\cos(\theta)$ is defined as the first coordinate.

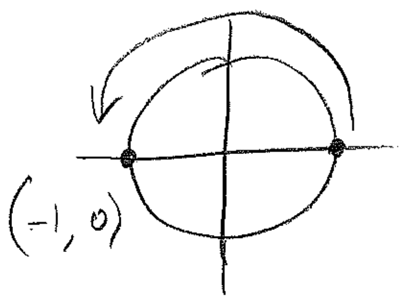
ex find $\sin\left(\frac{\pi}{2}\right)$



so $\sin\left(\frac{\pi}{2}\right) = 1.$

and $\cos\left(\frac{\pi}{2}\right) = 0$

ex find $\cos(\pi)$.



so $\cos(\pi) = -1$

and $\sin(\pi) = 0$

We define

$$\tan(\theta) := \frac{\sin(\theta)}{\cos(\theta)}$$

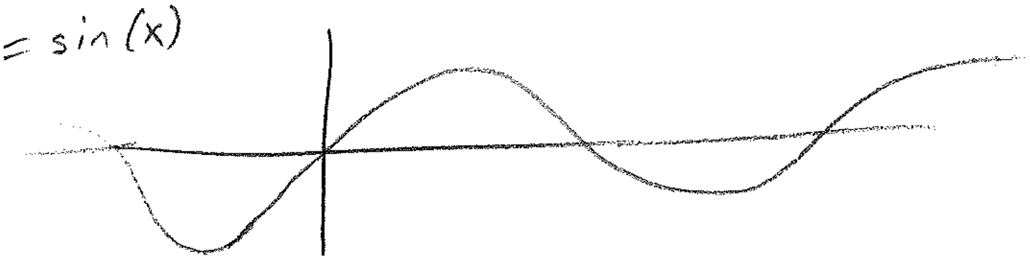
$$\csc(\theta) := \frac{1}{\sin(\theta)}$$

$$\sec(\theta) := \frac{1}{\cos(\theta)}$$

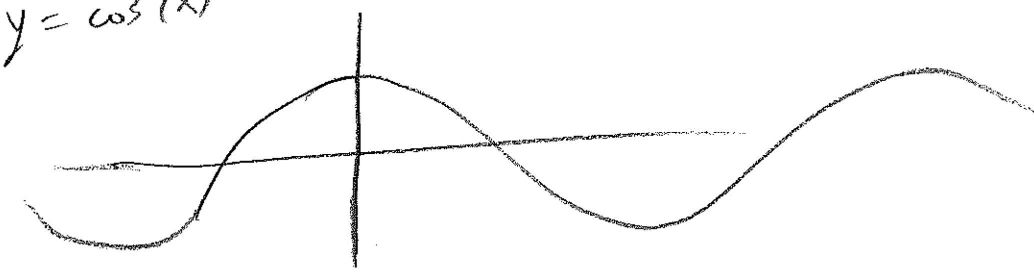
$$\cot(\theta) := \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

Graphs of Trig Functions

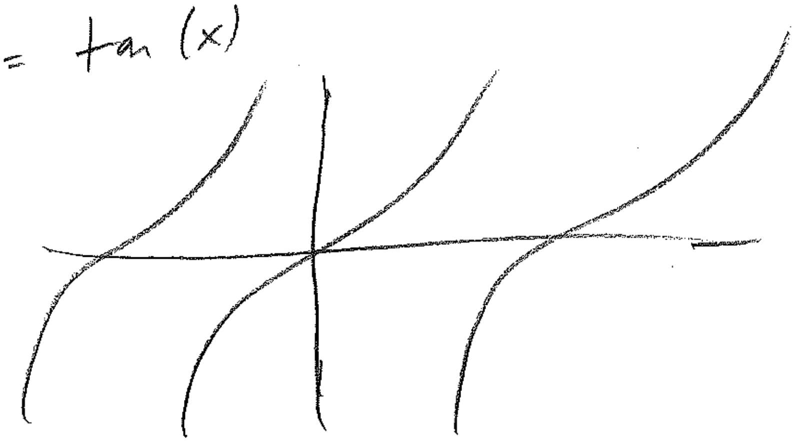
$y = \sin(x)$



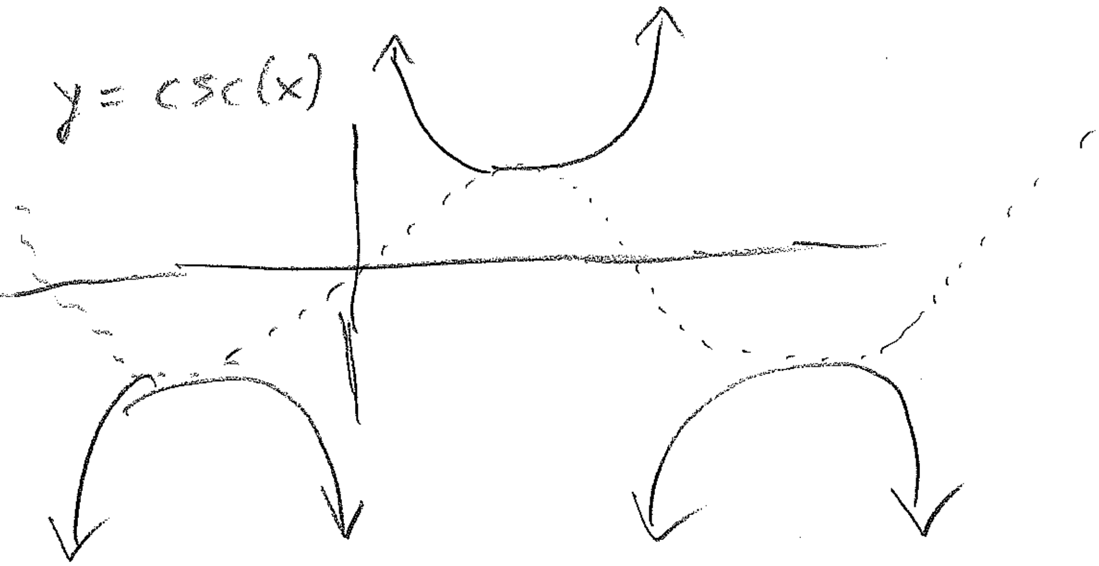
$y = \cos(x)$



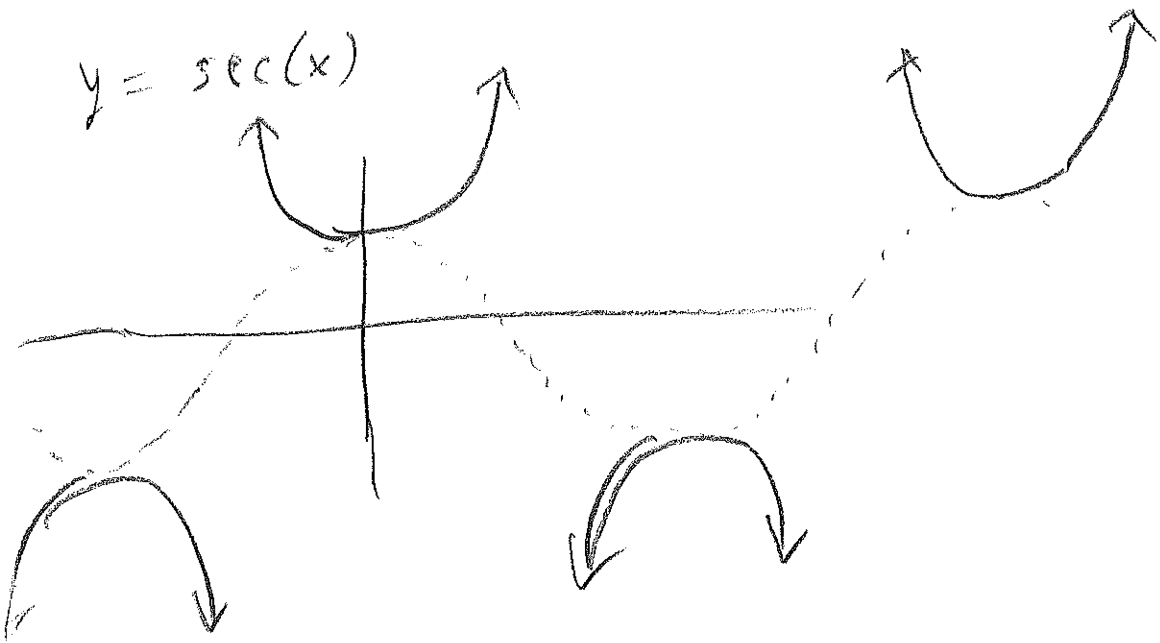
$y = \tan(x)$



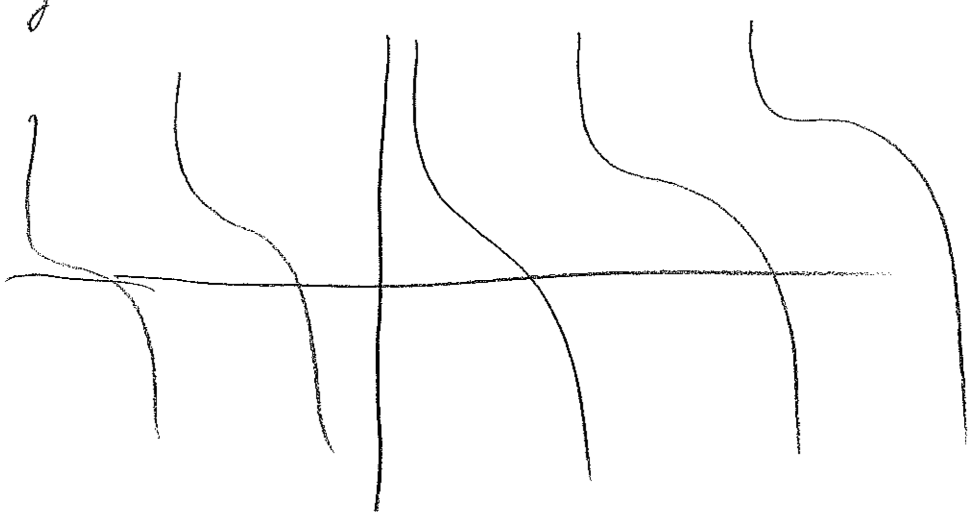
$y = \csc(x)$



$y = \sec(x)$

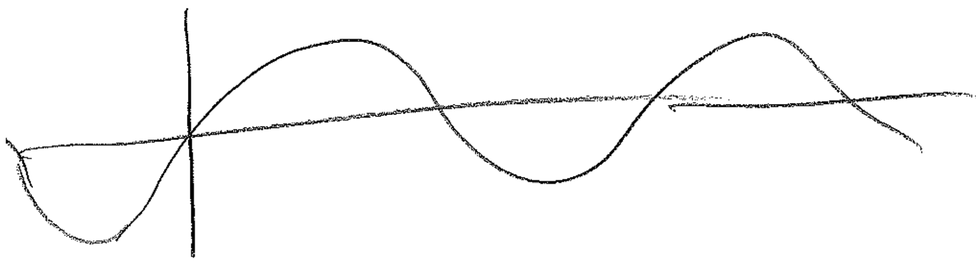


$y = \cot(x)$



Inverse Trigonometric Functions

$y = \sin(x)$

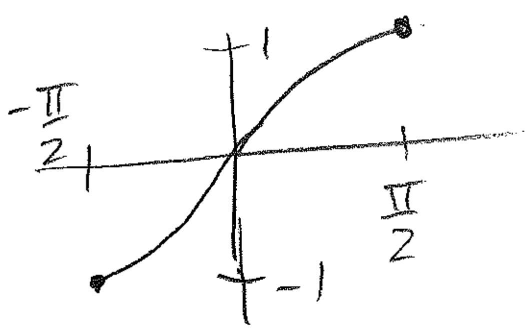


sin is NOT 1-1,

so sin has no inverse function.

So we take a piece of this graph that IS 1-1.

We take this piece.



Call this function $\sin^*$

So $\sin^*$ is 1-1

So $\sin^*$ has an inverse function.

We call that inverse function arcsin.

We define

$\arcsin(x) := y$ if

$\sin^*(y) = x$.

i.e.

$\arcsin(x) := y$ if

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ AND $\sin(y) = x$.

ex $\arcsin(1) = \frac{\pi}{2}$

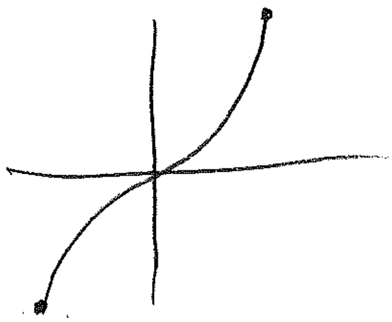
because $\frac{\pi}{2}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

AND $\sin(\frac{\pi}{2}) = 1$.

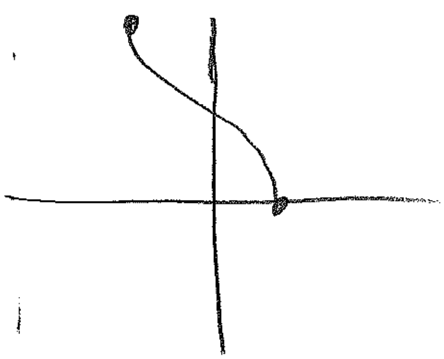
You can do a similar thing with cos, tan, csc, sec, cot, to define arccos, arctan, arccsc, arcsec, arccot.

Graphs of Inverse Trig Functions

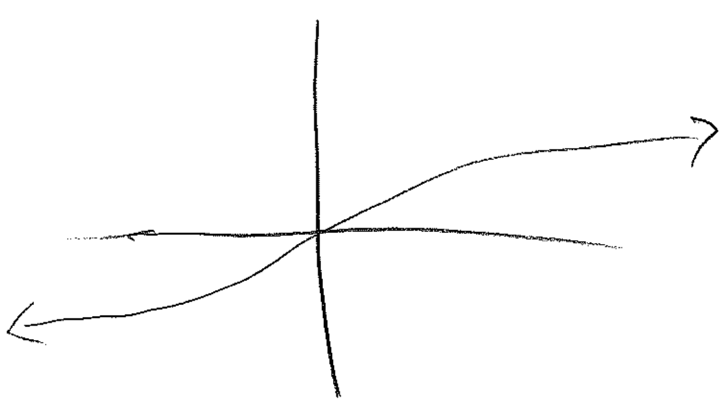
$y = \arcsin(x)$



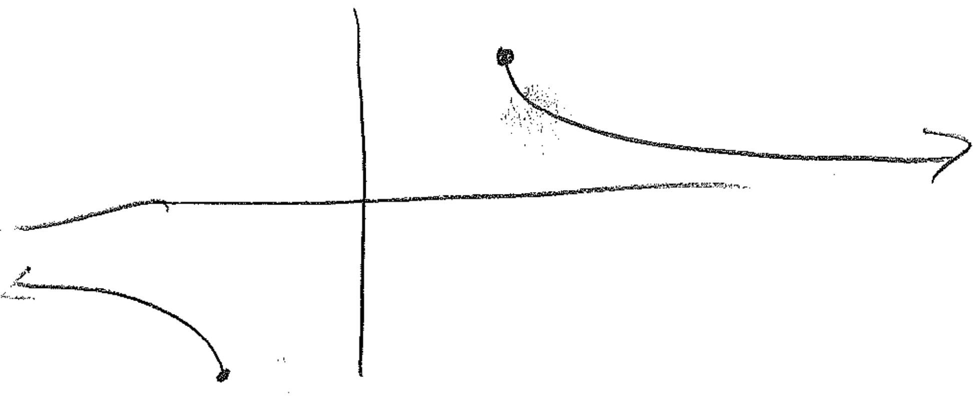
$y = \arccos(x)$



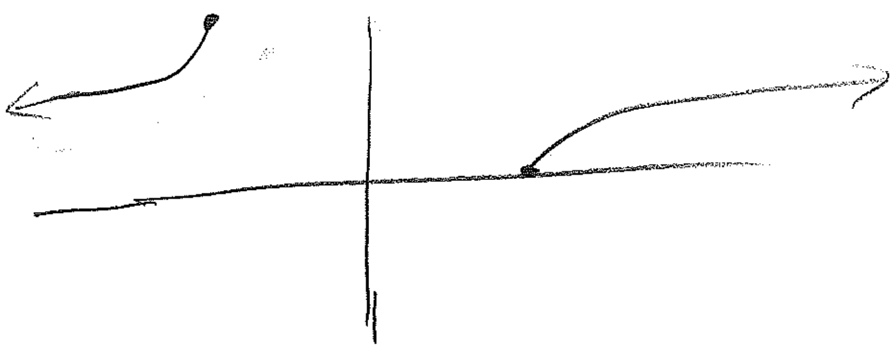
$y = \arctan(x)$



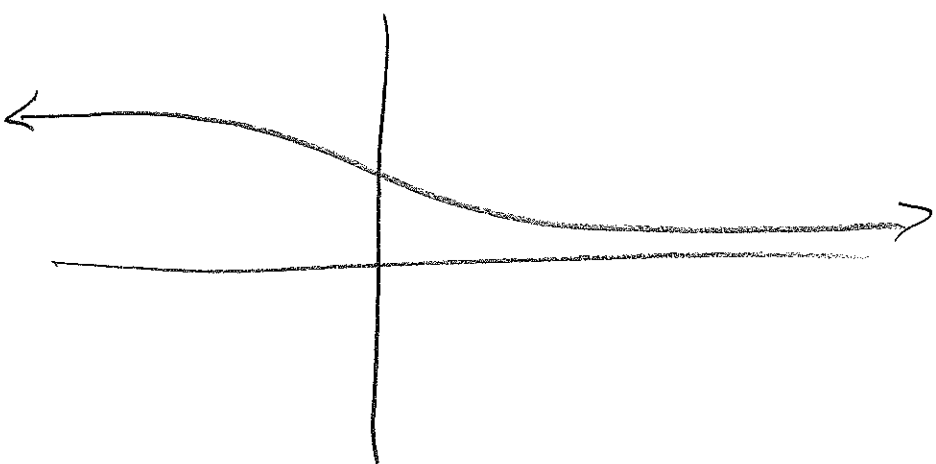
$y = \operatorname{arccsc}(x)$



$y = \operatorname{arcsec}(x)$



$y = \operatorname{arccot}(x)$



We also write $\sin^{-1}(x)$ instead of $\arcsin(x)$.

Hyberbolic Functions

We define

$$\sinh(x) := \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh(x) := \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\tanh(x) := \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{csch}(x) := \frac{1}{\sinh(x)}$$

$$\operatorname{sech}(x) := \frac{1}{\cosh(x)}$$

$$\coth(x) := \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)}$$

Compositions of Trig Functions with Inverse Trig Functions

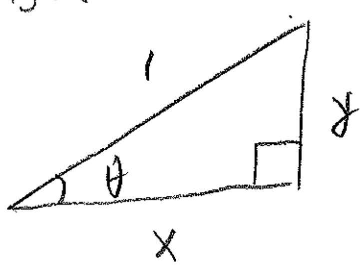
$$\sin(\arcsin(x)) = x$$

$$\sin(\arccos(x)) = ?$$

Say $\theta := \arccos(x)$.

So $\cos(\theta) = x$

Now make a triangle

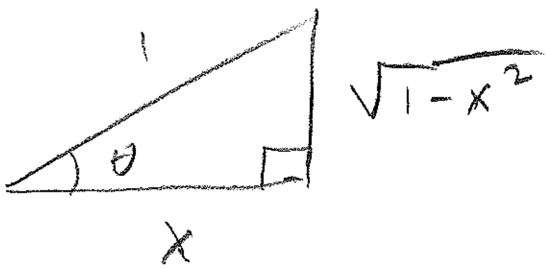


use the pythagorean theorem

$$x^2 + y^2 = 1$$

$$\rightarrow \quad y^2 = 1 - x^2$$

$$\rightarrow \quad y = \sqrt{1 - x^2}$$



so $\sin(\arccos(x))$

$= \sin(\theta)$

$= \dfrac{\sqrt{1-x^2}}{1}$

$= \sqrt{1-x^2}$

The Inverse Function Rule

If you have an original function, Ω, and Ω has an inverse function, I,

then $$I'(x) = \frac{1}{\Omega'(I(x))}$$

ex If $\Omega(x) = x^3$,

then $I(x) = \sqrt[3]{x}$

and $\Omega'(x) = 3x^2$

so $$I'(x) = \frac{1}{\Omega'(I(x))}$$

$$= \frac{1}{\Omega'(\sqrt[3]{x})}$$

$$= \frac{1}{3(\sqrt[3]{x})^2}$$

$$= \frac{1}{3x^{2/3}}$$

also $I(x) = x^{1/3}$

so $I'(x) = \frac{1}{3} x^{-2/3}$

ex Let $\Omega(x) = \dfrac{3x+4}{5x-1}$.

Let $I(x) = \Omega^{-1}(x)$.

Find $I'(x)$.

$$\Omega'(x) = \frac{(3)(5x-1) - (3x+4)(5)}{(5x-1)^2}$$

$$I'(x) = \frac{1}{\Omega'(I(x))}$$

$$= \frac{1}{\dfrac{(3)(5I(x)-1) - (3I(x)+4)(5)}{(5I(x)-1)^2}}$$

ex Let

$\Omega(x) = 2x^3 + 3x^2 + 7x + 4.$

Let $I(x) = \Omega^{-1}(x).$

Note that $\Omega(0) = 4.$

Find $I'(4).$

$\Omega'(x) = 6x^2 + 6x + 7$

$I'(x)$

$$= \frac{1}{\Omega'(I(x))}$$

so $I'(4)$

$$= \frac{1}{\Omega'(I(4))}$$

$\Omega(0) = 4$

so $I(4) = 0$

$$= \frac{1}{\Omega'(0)}$$

$$= \boxed{\frac{1}{7}}$$

Exponential Function Rules

If $f(x) = e^x$,

then $f'(x) = e^x$.

If $f(x) = e^{A(x)}$,

then $f'(x) = A'(x)\, e^{A(x)}$.

ex If $f(x) = e^{(x^3)}$,

then $f'(x) = 3x^2 e^{(x^3)}$.

If $f(x) = b^x$,

then $f'(x) = \ln(b)\, b^x$.

ex If $f(x) = 2^x$,

then $f'(x) = \ln(2)\, 2^x$.

If $f(x) = b^{A(x)}$,

then $f'(x) = A'(x)\, b^{A(x)} \ln(b)$

ex If $f(x) = 3^{(x^5)}$,

then $f'(x) = \ln(3)\, 5x^4\, 3^{(x^5)}$.

e^x

$$f(x) = e^{x^2-5x+3}$$

$$f'(x) = (2x-5)e^{x^2-5x+3}$$

$$f(x) = \underbrace{xe^x}$$

$$f'(x) = (1)(e^x) + (x)(e^x)$$

$$f(x) = \frac{e^{-3x}}{x^2+1}$$

$$f'(x) = \frac{(-3e^{-3x})(x^2+1) - (e^{-3x})(2x)}{(x^2+1)^2}$$

Logarithmic Function Rules

If $f(x) = \ln(x)$,

then $f'(x) = \dfrac{1}{x}$.

If $f(x) = \ln(A(x))$,

then $f'(x) = \dfrac{A'(x)}{A(x)}$.

ex If $f(x) = \ln(5x^2 + x)$

then $f'(x) = \dfrac{10x+1}{5x^2+x}$

If $f(x) = \log_b(x)$,

then $f'(x) = \dfrac{1}{\ln(b)x}$

ex If $f(x) = \log_2(x)$,

then $f'(x) = \dfrac{1}{\ln(2)x}$.

If $f(x) = \log_b(A(x))$,

then $f'(x) = \dfrac{A'(x)}{\ln(b)A(x)}$.

ex If $f(x) = \log_3(4x^5 - x)$,

then $f'(x) = \dfrac{20x^4 - 1}{\ln(3)(4x^5-x)}$

ex

$$f(x) = \ln\left(7x^3 - 5x + 1\right)$$

$$f'(x) = \frac{21x^2 - 5}{7x^3 - 5x + 1}$$

$$f(x) = \ln(x+1)$$

$$f'(x) = \frac{1}{x+1}$$

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x}$$

$$f(x) = \ln\left((x^3+3)(x^2-1)\right)$$

$$f'(x) = \frac{(3x^2)(x^2-1) + (x^3+3)(2x)}{(x^3+3)(x^2-1)}$$

$$f(x) = \ln\left(\frac{\sqrt{x^2+4}}{x}\right)$$

$$f'(x) = \frac{\dfrac{\left(\frac{1}{2}(x^2+4)^{-1/2}(2x)\right)(x) - \left(\sqrt{x^2+4}\right)(1)}{x^2}}{\dfrac{\sqrt{x^2+4}}{x}}$$

Trigonometric Function Rules

If $f(x) = \sin(x)$,
then $f'(x) = \cos(x)$.

If $f(x) = \cos(x)$,
then $f'(x) = -\sin(x)$.

If $f(x) = \tan(x)$,
then $f'(x) = \sec^2(x)$.

If $f(x) = \cot(x)$,
then $f'(x) = -\csc^2(x)$.

If $f(x) = \sec(x)$,
then $f'(x) = \sec(x)\tan(x)$

If $f(x) = \csc(x)$,
then $f'(x) = -\csc(x)\cot(x)$.

Inverse Trig Function Rules

If $f(x) = \arcsin(x)$,

then $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

If $f(x) = \arccos(x)$,

then $f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$

If $f(x) = \arctan(x)$,

then $f'(x) = \dfrac{1}{x^2+1}$

If $f(x) = \operatorname{arccot}(x)$,

then $f'(x) = \dfrac{-1}{x^2+1}$.

If $f(x) = \operatorname{arcsec}(x)$,

then $f'(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

If $f(x) = \operatorname{arccsc}(x)$,

then $f'(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$

why is

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \ ?$$

use the inverse function rule.

$\Omega(x) = \sin(x)$.

$I(x) = \arcsin(x)$.

so

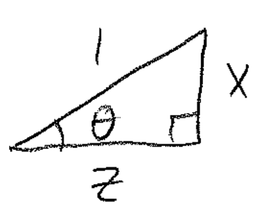
$$I'(x) = \frac{1}{\Omega'(I(x))}$$

$$= \frac{1}{\cos(\arcsin(x))}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

because to find $\cos(\arcsin(x))$, say $\theta := \arcsin(x)$.

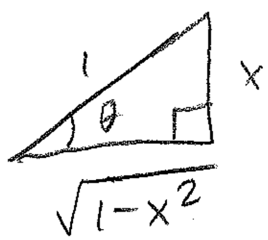
so $\sin(\theta) = x$



so $z^2 + x^2 = 1^2$

$\rightarrow \quad z^2 = 1 - x^2$

$\rightarrow \quad z = \sqrt{1-x^2}$



so $\cos(\arcsin(x))$

$= \cos(\theta)$

$= \sqrt{1-x^2}$

Hyperbolic Function Rules

If $f(x) = \sinh(x)$,

then $f'(x) = \cosh(x)$.

If $f(x) = \cosh(x)$,

then $f'(x) = \sinh(x)$.

If $f(x) = \tanh(x)$,

then $f'(x) = \operatorname{sech}^2(x)$.

If $f(x) = \coth(x)$,

then $f'(x) = -\operatorname{csch}^2(x)$.

If $f(x) = \operatorname{sech}(x)$,

then $f'(x) = -\operatorname{sech}(x)\tanh(x)$.

If $f(x) = \operatorname{csch}(x)$,

then $f'(x) = -\operatorname{csch}(x)\coth(x)$.

Logarithmic Differentiation

ex Let $f(x) = x^x$.

Find $f'(x)$.

Define a new function, F, by $F(x) := \ln(f(x))$.

So $F(x) = \ln(x^x)$
$= x \ln(x)$

So $F'(x) = (1)(\ln(x)) + (x)\left(\frac{1}{x}\right)$
$= \ln(x) + 1$

Also $F(x) = \ln(f(x))$,

so $F'(x) = \dfrac{f'(x)}{f(x)}$

so $f'(x) = F'(x) f(x)$

$$\boxed{\text{So } f'(x) = (\ln(x) + 1) x^x}$$

ex Let $f(x) = \dfrac{(x-2)^3}{\sqrt{x^2+1}}$

Find $f'(x)$.

Define $F(x) := \ln(f(x))$

$$= \ln\left(\frac{(x-2)^3}{(x^2+1)^{1/2}}\right)$$

$$= \ln\left((x-2)^3\right) - \ln\left((x^2+1)^{1/2}\right)$$

$$= 3\ln(x-2) - \frac{1}{2}\ln(x^2+1)$$

So $F'(x) = \dfrac{3}{x-2} - \dfrac{1}{2}\cdot\dfrac{2x}{x^2+1}$

$$= \frac{3}{x-2} - \frac{x}{x^2+1}$$

Also $F(x) = \ln(f(x))$

so $F'(x) = \dfrac{f'(x)}{f(x)}$

So $f'(x) = F'(x) f(x)$.

So $\boxed{f'(x) = \left(\dfrac{3}{x-2} - \dfrac{x}{x^2+1}\right)\dfrac{(x-2)^3}{\sqrt{x^2+1}}}$

L'Hopital's Rule

We say that

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has an indeterminate form of $\frac{0}{0}$ if

$$\lim_{x \to a} f(x) = 0$$

AND $\lim_{x \to a} g(x) = 0.$

We say that

$\lim_{x \to a} \frac{f(x)}{g(x)}$ has

an indeterminate form of $\frac{\infty}{\infty}$ if

$$\lim_{x \to a} f(x) \in \{-\infty, \infty\}$$

AND

$$\lim_{x \to a} g(x) \in \{-\infty, \infty\}$$

The Rule:

If $\lim_{x \to a} \frac{f(x)}{g(x)}$

has an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

---

ex Find $\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$

I.F. of $\frac{0}{0}$

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{2x - 1}{1}$$

$$= 2(2) - 1$$

$$= \boxed{3}$$

ex Find

$$\lim_{x \to \infty} \frac{3x^2 - 2x + 1}{2x^2 + 3}$$

I.F. of $\frac{\infty}{\infty}$

so
$$\lim_{x \to \infty} \frac{3x^2 - 2x + 1}{2x^2 + 3}$$

$$= \lim_{x \to \infty} \frac{6x - 2}{4x}$$

$$= \lim_{x \to \infty} \frac{6}{4}$$

$$= \frac{6}{4}$$

$$= \boxed{\frac{3}{2}}$$

If $Deg(T) > Deg(B)$,

then $$\lim_{x \to \infty} \frac{T(x)}{B(x)} = \frac{LC(T)}{LC(B)} (\infty)^{Deg(T) - Deg(B)}$$

ex
$$\lim_{x \to \infty} \frac{9x^5 - x}{4x^2 + 7} = \frac{9}{4}(\infty)^{5-2}$$

$$= \frac{9}{4}(\infty)^3$$

$$= \frac{9}{4}(\infty)$$

$$= \boxed{\infty}$$

$0 \cdot \infty$

to find $\lim_{x \to a} f(x)g(x)$

when it has an I.F.

of $0 \cdot \infty$,

rewrite $f(x)g(x)$

as $\dfrac{f(x)}{\frac{1}{g(x)}}$ or $\dfrac{g(x)}{\frac{1}{f(x)}}$.

$\infty - \infty$

to find $\lim_{x \to a} \left(f(x) - g(x)\right)$

when it has an I.F.

of $\infty - \infty$

rewrite $f(x) - g(x)$

as $$\frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$

0^0, ∞^0, 1^∞

to find $\lim_{x \to a} f(x)^{g(x)}$

when it has an I.F.

of 0^0, ∞^0, 1^∞,

(i) Let $y := \lim_{x \to a} f(x)^{g(x)}$.

(ii) Consider

$$\begin{aligned} &\ln(y) \\ &= \ln\left(\lim_{x \to a} f(x)^{g(x)}\right) \\ &= \lim_{x \to a} \ln\left(f(x)^{g(x)}\right) \quad \text{because } \ln \text{ is continuous} \\ &= \lim_{x \to a} g(x) \ln(f(x)) \end{aligned}$$

this should have an I.F.

of $0 \cdot \infty$.

If $\ln(y) = L$,

then $y = e^L$.

ex Find

$$\lim_{x \to \infty} e^{-x}\sqrt{x}.$$

I.F. of $0 \cdot \infty$

$$\lim_{x \to \infty} e^{-x}\sqrt{x}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}}{e^x} \qquad \text{I.F. of } \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{2}x^{-1/2}}{e^x}$$

$$= \lim_{x \to \infty} \frac{1}{2} \frac{1}{\sqrt{x}} \frac{1}{e^x}$$

$$= \frac{1}{2}(0)(0)$$

$$= \boxed{0}$$

ex Find

$$\lim_{x \to 1+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1} \right)$$

I.F. of $\infty - \infty$

$$\lim_{x \to 1+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1} \right)$$

$$= \lim_{x \to 1+} \frac{(x-1) - \ln(x)}{\ln(x)(x-1)} \quad \text{I.F. of } \frac{0}{0}$$

$$= \lim_{x \to 1+} \frac{1 - \frac{1}{x}}{\left(\frac{1}{x}\right)(x-1) + (\ln(x))(1)}$$

$$= \lim_{x \to 1+} \frac{1 - \frac{1}{x}}{1 - \frac{1}{x} + \ln(x)} \quad \frac{0}{0} \text{ again}$$

$$= \lim_{x \to 1+} \frac{\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}}$$

$$= \frac{1}{1+1}$$

$$= \boxed{\frac{1}{2}}$$

ex Find

$$\lim_{x \to 0^+} x^x$$

I.F. of 0^0.

let $y := \lim_{x \to 0^+} x^x$

so $\ln(y)$

$$= \ln\left(\lim_{x \to 0^+} x^x\right)$$

$$= \lim_{x \to 0^+} \ln(x^x)$$

$$= \lim_{x \to 0^+} x \ln(x) \quad \text{I.F. of } 0 \cdot \infty$$

$$= \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \quad \text{I.F. of } \frac{\infty}{\infty}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0^+} -\frac{1}{x} \cdot \frac{x^2}{1}$$

$$= \lim_{x \to 0^+} -x$$

$$= 0$$

so $\ln(y) = 0$

so $y = e^0$

so $y = 1$

so $\boxed{\lim_{x \to 0^+} x^x = 1}$

ex Let k be a real #.

Find $\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x$.

I.F. of 1^∞.

let $y := \lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x$.

so $\ln(y)$

$= \ln\left(\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x\right)$

$= \lim_{x \to \infty} \ln\left(\left(1 + \frac{k}{x}\right)^x\right)$

$= \lim_{x \to \infty} x \ln\left(1 + \frac{k}{x}\right)$ I.F. of $0 \cdot \infty$

$= \lim_{x \to \infty} \dfrac{\ln\left(1 + \frac{k}{x}\right)}{\frac{1}{x}}$ I.F. of $\frac{0}{0}$

$= \lim_{x \to \infty} \dfrac{-kx^{-2}}{-x^{-2}}$

$= \lim_{x \to \infty} k$

$= k$

so $\ln(y) = k$

so $y = e^k$

$\boxed{\text{so } \lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x = e^k}$

Antiderivatives

If we reverse our rules for derivatives,

then we get some rules for antiderivatives.

$$\int 0\,dx = C_1$$

$$\int 1\,dx = x + C_1$$

$$\int k\,dx = kx + C_1$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C_1 \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln(x) + C_1$$

$$\int (A(x) + B(x))\,dx = \int A(x)\,dx + \int B(x)\,dx$$

$$\int k\,A(x)\,dx = k\int A(x)\,dx$$

$$\int e^x \, dx = e^x + C_1$$

$$\int \ln(x) \, dx = x\ln(x) - x + C_1$$

$$\int b^x \, dx = \frac{1}{\ln(b)} b^x + C_1$$

$$\int \log_b(x) \, dx = \frac{1}{\ln(b)} \left(x\log_b(x) - x \right) + C_1$$

$$\int \sin(x) \, dx = -\cos(x) + C_1$$

$$\int \cos(x) \, dx = \sin(x) + C_1$$

$$\int \tan(x) \, dx = -\ln(\cos(x)) + C_1$$

$$\int \cot(x) \, dx = \ln(\sin(x)) + C_1$$

$$\int \sec(x) \, dx = \ln\left(\sec(x) + \tan(x) \right) + C_1$$

$$\int \csc(x) \, dx = -\ln\left(\csc(x) + \cot(x) \right) + C_1$$

$$\int \sec^2(x) \, dx = \tan(x) + C_1$$

$$\int \csc^2(x) \, dx = -\cot(x) + C_1$$

$$\int \sec(x)\tan(x) \, dx = \sec(x) + C_1$$

$$\int \csc(x)\cot(x) \, dx = -\csc(x) + C_1$$

$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C$$

$$\int \frac{1}{a^2x^2+b^2}\,dx = \frac{1}{ab}\arctan\left(\frac{a}{b}x\right) + C$$

$$\int \frac{1}{\sqrt{b^2-a^2x^2}}\,dx = \frac{1}{a}\arcsin\left(\frac{a}{b}x\right) + C$$

$$\int \frac{1}{x\sqrt{a^2x^2-b^2}}\,dx = \frac{1}{b}\operatorname{arcsec}\left(\frac{a}{b}x\right) + C$$

Ex Find $\displaystyle\int \frac{1}{4x^2+9}\,dx$

$$\int \frac{1}{4x^2+9}\,dx$$

$$= \int \frac{1}{2^2x^2+3^2}\,dx$$

$$= \frac{1}{(2)(3)}\arctan\left(\frac{2}{3}x\right) + C$$

$$= \frac{1}{6}\arctan\left(\frac{2}{3}x\right) + C$$

Integration by Substitution

ex Find $\int 8x(x^2+1)^3 dx$

let $u := x^2+1$

so $\frac{du}{dx} = 2x$

so $du = 2x\,dx$

so $dx = \frac{du}{2x}$

now $\int 8x\,(x^2+1)^3 dx$

$= \int 8x\,u^3 \frac{du}{2x}$

$= \int 4u^3 du$

$= u^4 + C_1$

$= (x^2+1)^4 + C_1$

ex Complete the square

$$3x^2 + 18x + 22$$

---

this means write it
in the form $a(x+I)^2 + \Omega$
where I is some number,
Ω is some number,
a is some number.

$$3x^2 + 18x + 22$$

$$= 3\left(x^2 + 6x + \frac{22}{3}\right) \qquad \frac{6}{2} = 3,\ 3^2 = 9$$

$$= 3\left(\underbrace{x^2 + 6x + 9}\, - 9 + \frac{22}{3}\right)$$

$$= 3\left((x+3)^2 - 9 + \frac{22}{3}\right)$$

$$= 3(x+3)^2 + 3(-9) + 3\left(\frac{22}{3}\right)$$

$$= 3(x+3)^2 - 27 + 22$$

$$= \boxed{3(x+3)^2 - 5}$$

ex Find

$$\int \frac{1}{x^2+8x+17}\,dx$$

---

complete the square

$$x^2+8x+17 \qquad \frac{8}{2}=4,\ 4^2=16$$

$$= \underbrace{x^2+8x+16}-16+17$$

$$= (x+4)^2+1$$

so $$\int \frac{1}{x^2+8x+17}\,dx$$

$$= \int \frac{1}{(x+4)^2+1}\,dx$$

let $u := x+4$

so $\frac{du}{dx} = 1$

so $du = dx$

so $dx = du$

now $\int \frac{1}{(x+4)^2+1}\,dx$

$$= \int \frac{1}{u^2+1}\,du$$

$$= \arctan(u) + C_1$$

$$= \arctan(x+4) + C_1$$

Note that to find

$\int \tan(x)\,dx$, write as

$$\int \frac{\sin(x)}{\cos(x)}\,dx$$

let $u := \cos(x)$

then $\frac{du}{dx} = -\sin(x)$

so $dx = \frac{du}{-\sin(x)}$

now $\int \frac{\sin(x)}{\cos(x)}\,dx$

$$= \int \frac{\sin(x)}{u} \frac{du}{-\sin(x)}$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln(u) + C_1$$

$$= -\ln(\cos(x)) + C_1$$

Integration by Parts

$$\int A(x)B'(x)\,dx$$

$$= A(x)B(x) - \int A'(x)B(x)\,dx$$

If we let $u := A(x)$

and $v := B(x)$,

then $\frac{du}{dx} = A'(x)$

and $\frac{dv}{dx} = B'(x)$

so $du = A'(x)\,dx$

and $dv = B'(x)\,dx$

so $\int A(x)B'(x)\,dx = \int u\,dv$

and $A(x)B(x) - \int A'(x)B(x)\,dx = uv - \int v\,du$

so $\int u\,dv = uv - \int v\,du$

ex Find $\int x e^x \, dx$

let $u := x$

and $dv := e^x dx$

so $\frac{du}{dx} = 1$, $du = dx$

and $v = e^x$ (because $\frac{dv}{dx} = e^x \Longleftrightarrow dv = e^x dx$)

so $\int x e^x \, dx$

$= \int u \, dv$

$= uv - \int v \, du$

$= x e^x - \int e^x \, dx$

$= \boxed{x e^x - e^x + C}$

ex Find $\int x^2 e^x \, dx$

let $u := x^2$

and $dv := e^x \, dx$

so $\frac{du}{dx} = 2x$ so $du = 2x\,dx$

and $\frac{dv}{dx} = e^x$ so $v = e^x$

now $\int x^2 e^x \, dx$

$= \int u \, dv$

$= uv - \int v \, du$

$= x^2 e^x - \int e^x 2x \, dx$

$= x^2 e^x - 2 \int x e^x \, dx$

now find $\int x e^x dx$

with an IBP

$= \boxed{x^2 e^x - 2\left(x e^x - e^x\right) + C_1}$

ex Find

$$\int \ln(x)\,dx$$

---

let $u := \ln(x)$

let $dv := dx$

so $\frac{du}{dx} = \frac{1}{x}$ so $du = \frac{1}{x}dx$

and $\frac{dv}{dx} = 1$ so $v = x$

now $\int \ln(x)\,dx$

$= \int u\,dv$

$= uv - \int v\,du$

$= \ln(x)x - \int x \frac{1}{x}\,dx$

$= x\ln(x) - \int 1\,dx$

$= \boxed{x\ln(x) - x + C}$

ex Find $\int x \ln(x)\,dx$

let $u := \ln(x)$

let $dv := x\,dx$

so $\frac{du}{dx} = \frac{1}{x}$ so $du = \frac{1}{x}dx$

and $\frac{dv}{dx} = x$ so $v = \frac{1}{2}x^2$

now $\int x \ln(x)\,dx$

$= \int u\,dv$

$= uv - \int v\,du$

$= \ln(x)\frac{1}{2}x^2 - \int \frac{1}{2}x^2 \frac{1}{x}\,dx$

$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\,dx$

$= \boxed{\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C_1}$

ex Find $\int x\sin(x)\,dx$

let $u := x$

and $dv := \sin(x)\,dx$

so $\frac{du}{dx} = 1$ so $du = dx$

and $\frac{dv}{dx} = \sin(x)$ so $v = -\cos(x)$

now $\int x\sin(x)\,dx$

$= \int u\,dv$

$= uv - \int v\,du$

$= x(-\cos(x)) - \int -\cos(x)\,dx$

$= -x\cos(x) + \int \cos(x)\,dx$

$= \boxed{-x\cos(x) + \sin(x) + C}$

ex Find $\int e^x \sin(x)\,dx$

let $u := e^x$

and $dv := \sin(x)\,dx$

so $\frac{du}{dx} = e^x$ so $du = e^x\,dx$

and $\frac{dv}{dx} = \sin(x)$ so $v = -\cos(x)$

now $\int e^x \sin(x)\,dx$

$= \int u\,dv$

$= uv - \int v\,du$

$= e^x(-\cos(x)) - \int -\cos(x)\,e^x\,dx$

$= -e^x\cos(x) + \int e^x \cos(x)\,dx$

we need $\int e^x \cos(x)\,dx$

$\int e^x \cos(x)\,dx$

let $u := e^x$

and $dv := \cos(x)\,dx$

so $\frac{du}{dx} = e^x$ so $du = e^x\,dx$

and $\frac{dv}{dx} = \cos(x)$ so $v = \sin(x)$

now $\int e^x \cos(x)\,dx$

$= \int u\,dv$

$= uv - \int v\,du$

$= e^x \sin(x) - \int \sin(x) e^x\,dx$

$= e^x \sin(x) - \int e^x \sin(x)\,dx$

So

$$\int e^x \sin(x) = -e^x \cos(x) + \int e^x \cos(x)\,dx$$

$$\rightarrow \int e^x \sin(x) = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx$$

$$\rightarrow 2\int e^x \sin(x) = -e^x \cos(x) + e^x \sin(x)$$

$$\rightarrow \boxed{\int e^x \sin(x) = \frac{1}{2}\left(e^x \sin(x) - e^x \cos(x)\right) + C_1}$$

Trigonometric Integrals

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

To find

$$\int \sin^m(x)\cos^n(x)\,dx$$

(i) If m is odd

and $m = 2k+1$,

then $\int \sin^m(x)\cos^n(x)\,dx$

$$= \int \sin^{2k+1}(x)\cos^n(x)\,dx$$

$$= \int \sin^{2k}(x)\sin(x)\cos^n(x)\,dx$$

$$= \int \left(\sin^2(x)\right)^k \sin(x)\cos^n(x)\,dx$$

$$= \int \left(1-\cos^2(x)\right)^k \sin(x)\cos^n(x)\,dx$$

now do int. by sub.

with $u := \cos(x)$.

(ii) If n is odd,

and $n = 2k+1$,

then $\int \sin^m(x)\cos^n(x)\,dx$

$= \int \sin^m(x)\cos^{2k+1}(x)\,dx$

$= \int \sin^m(x)\cos^{2k}(x)\cos(x)\,dx$

$= \int \sin^m(x)\left(\cos^2(x)\right)^k \cos(x)\,dx$

$= \int \sin^m(x)\left(1-\sin^2(x)\right)^k \cos(x)\,dx$

now do an int. by sub.

with $u := \sin(x)$.

(iii) If m and n

are both even,

use $\sin^2(x) = \frac{1}{2}\left(1-\cos(2x)\right)$

$\cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$

$\boxed{\text{Evaluating } \int \tan^m(x)\sec^n(x)dx}$

Let $m, n \in \mathbb{N}$.
To evaluate $\int \tan^m(x)\sec^n(x)dx$, use these rules.

(iv) If m is odd (so $m = 2k+1$ for some k), then write
$\int \tan^m(x)\sec^n(x)dx$
$= \int \tan^{2k+1}(x)\sec^n(x)dx$
$= \int \tan^{2k}(x)\tan(x)\sec^n(x)dx$
$= \int [\tan^2(x)]^k \tan(x)\sec^n(x)dx$
$= \int [\tan^2(x)]^k \tan(x)\sec^{n-1}(x)\sec(x)dx$
$= \int [\sec^2(x)-1]^k \tan(x)\sec^{n-1}(x)\sec(x)dx$ (since $\tan^2(x) = \sec^2(x) - 1$, since $\tan^2(x) + 1 = \sec^2(x)$)
Then do integration by substitution with $u := \sec(x)$.

(v) If n is even (so $n = 2k$ for some k), write
$\int \tan^m(x)\sec^n(x)dx$
$= \int \tan^m(x)\sec^{2k}(x)dx$
$= \int \tan^m(x)[\sec^2(x)]^k dx$
$= \int \tan^m(x)[\sec^2(x)]^{k-1}\sec^2(x)dx$
$= \int \tan^m(x)[1+\tan^2(x)]^{k-1}(x)\sec^2(x)dx$ (since $\sec^2(x) = 1 + \tan^2(x)$)
Then do integration by substitution with $u := \tan(x)$.

(vi)
If $m = 0$ and n is odd, then use the formula
$\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$
This formula can be obtained with integration by parts;
let $u := \sec^{n-2}(x)$ and $dv := \sec^2(x)dx$
then $\frac{du}{dx} = (n-2)\sec^{n-3}(x)\sec(x)\tan(x)$ and $v = \int \sec^2(x)dx$
so $du = (n-2)\sec^{n-3}(x)\sec(x)\tan(x)dx$ and $v = \tan(x)$.
$\int \sec^n(x)dx$
$= \int udv$
$= uv - \int vdu$
$= \sec^{n-2}(x)\tan(x) - \int \tan(x)(n-2)\sec^{n-3}(x)\sec(x)\tan(x)dx$
$= \sec^{n-2}(x)\tan(x) - (n-2)\int \tan^2(x)\sec^{n-2}(x)dx$
$= \sec^{n-2}(x)\tan(x) - (n-2)\int [\sec^2(x)-1]\sec^{n-2}(x)dx$
$= \sec^{n-2}(x)\tan(x) - (n-2)\int [\sec^n(x) - \sec^{n-2}(x)]dx$
$= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)dx + (n-2)\int \sec^{n-2}(x)dx$

So $\int \sec^n(x)dx = \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)dx + (n-2)\int \sec^{n-2}(x)dx$
So $\int \sec^n(x)dx + (n-2)\int \sec^n(x)dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)dx$
So $(n-1)\int \sec^n(x)dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)dx$
So $\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$

(vii)
If m is even and $n = 0$, then use the formula
$\int \tan^m(x)dx = \frac{\tan^{m-1}(x)}{m-1} - \int \tan^{m-2}(x)dx$
This formula can be obtained like this.
$\int \tan^m(x)dx$
$= \int \tan^{m-2}(x)\tan^2(x)dx$
$= \int \tan^{m-2}(x)[\sec^2(x) - 1]dx$
$= \int [\tan^{m-2}(x)\sec^2(x) - \tan^{m-2}(x)]dx$
$= \int \tan^{m-2}(x)\sec^2(x)dx - \int \tan^{m-2}(x)dx$
let $u := \tan(x)$
then $\frac{du}{dx} = \sec^2(x)$
so $dx = \frac{du}{\sec^2(x)}$
$= \int u^{m-2}\sec^2(x)\frac{du}{\sec^2(x)} - \int \tan^{m-2}(x)dx$
$= \int u^{m-2}du - \int \tan^{m-2}(x)dx$
$= \frac{1}{m-1}u^{m-1} - \int \tan^{m-2}(x)dx$

(viii)
You can try converting sec tan integrals to sin and cos.

To evaluate $\int f(ax)dx$ you can just evaluate $\int f(x)dx$ to get $\int f(x)dx = F(x) + C$;
then $\int f(ax)dx = \frac{1}{a}F(ax) + C$.

For example, to evaluate $\int \sin^3(6x)\cos^2(6x)dx$, you can just evaluate $\int \sin^3(x)\cos^2(x)dx$ to get
$\int \sin^3(x)\cos^2(x)dx = -\frac{1}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C$, then
$\int \sin^3(6x)\cos^2(6x)dx = \frac{1}{6}[-\frac{1}{3}\cos^3(6x) + \frac{1}{5}\cos^5(6x)] + C$.

ex Find

$$\int \sin^3(x)\cos^2(x)\,dx$$

$$= \int \sin^2(x)\sin(x)\cos^2(x)\,dx$$

$$= \int \left(1-\cos^2(x)\right)\sin(x)\cos^2(x)\,dx$$

let $u := \cos(x)$

so $\dfrac{du}{dx} = -\sin(x)$

so $dx = \dfrac{du}{-\sin(x)}$

$$= \int (1-u^2)\sin(x)\,u^2\,\frac{du}{-\sin(x)}$$

$$= -\int (1-u^2)u^2\,du$$

$$= -\int (u^2-u^4)\,du$$

$$= -\left(\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right) + C$$

$$= -\tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + C$$

$$= \boxed{-\tfrac{1}{3}\cos^3(x) + \tfrac{1}{5}\cos^5(x) + C}$$

ex Find

$$\int \sin^2(x)\cos^5(x)\,dx$$

$$= \int \sin^2(x)\cos^4(x)\cos(x)\,dx$$

$$= \int \sin^2(x)\left(\cos^2(x)\right)^2\cos(x)\,dx$$

$$= \int \sin^2(x)\left(1-\sin^2(x)\right)^2\cos(x)\,dx$$

let $u := \sin(x)$

so $\dfrac{du}{dx} = \cos(x)$

so $dx = \dfrac{du}{\cos(x)}$

$$= \int u^2\left(1-u^2\right)^2\cos(x)\,\frac{du}{\cos(x)}$$

$$= \int u^2\left(u^4-2u^2+1\right)du$$

$$= \int \left(u^6-2u^4+u^2\right)du$$

$$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C_1$$

$$= \boxed{\frac{1}{7}\sin^7(x) - \frac{2}{5}\sin^5(x) + \frac{1}{3}\sin^3(x) + C_1}$$

ex Find

$$\int \sin^2(x)\cos^4(x)\,dx$$

$$= \int \frac{1}{2}\left(1-\cos(2x)\right)\left(\frac{1}{2}\left(1+\cos(2x)\right)\right)^2 dx$$

$$= \int \frac{1}{2}\left(1-\cos(2x)\right)\frac{1}{4}\left(1+\cos(2x)\right)^2 dx$$

$$= \int \frac{1}{8}\left(1-\cos(2x)\right)\left(\cos^2(2x)+2\cos(2x)+1\right)dx$$

$$= \frac{1}{8}\int \cos^2(2x)+2\cos(2x)+1-\cos^3(2x)-2\cos^2(2x)-\cos(2x)\Big)dx$$

$$= \frac{1}{8}\int\left(-\cos^3(2x)-\cos^2(2x)+\cos(2x)+1\right)dx$$

ex Find

$$\int \sin^2(x)\cos^4(x)\,dx$$

$$= \int \frac{1}{2}\left(1-\cos(2x)\right)\left(\frac{1}{2}\left(1+\cos(2x)\right)\right)^2 dx$$

$$= \int \frac{1}{2}\left(1-\cos(2x)\right)\frac{1}{4}\left(1+\cos(2x)\right)^2 dx$$

$$= \int \frac{1}{8}\left(1-\cos(2x)\right)\left(\cos^2(2x)+2\cos(2x)+1\right)dx$$

$$= \frac{1}{8}\int \cos^2(2x)+2\cos(2x)+1-\cos^3(2x)-2\cos^2(2x)-\cos(2x)\Big)dx$$

$$= \frac{1}{8}\int\left(-\cos^3(2x)-\cos^2(2x)+\cos(2x)+1\right)dx$$

$$\int -\cos^3(2x)\,dx$$

$$= \int -\cos^2(2x)\cos(2x)\,dx$$

$$= \int -\left(1-\sin^2(2x)\right)\cos(2x)\,dx$$

let $u := \sin(2x)$

so $\frac{du}{dx} = \cos(2x)2$

so $dx = \frac{du}{2\cos(2x)}$

now $\int -\left(1-\sin^2(2x)\right)\cos(2x)\,dx$

$$= \int -(1-u^2)\cos(2x)\,\frac{du}{2\cos(2x)}$$

$$= -\frac{1}{2}\int (1-u^2)\,du$$

$$= -\frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C_1$$

$$= -\frac{1}{2}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C_1$$

$$-\int \cos^2(2x)\,dx$$

$$= -\int \frac{1}{2}\left(1+\cos(4x)\right)dx$$

$$= -\int\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)dx$$

$$= \int\left(-\frac{1}{2} - \frac{1}{2}\cos(4x)\right)dx$$

$$= -\frac{1}{2}x - \frac{1}{2}\int \cos(4x)\,dx$$

let $u := 4x$

so $\frac{du}{dx} = 4$

so $dx = \frac{du}{4}$

now $-\frac{1}{2}x - \frac{1}{2}\int \cos(4x)dx$

$= -\frac{1}{2}x - \frac{1}{2}\int \cos(u)\frac{du}{4}$

$= -\frac{1}{2}x - \frac{1}{8}\sin(u) + C_1$

$= -\frac{1}{2}x - \frac{1}{8}\sin(4x) + C_1$

$\int \cos(2x)dx$

let $u := 2x$

so $\frac{du}{dx} = 2$

so $dx = \frac{du}{2}$

now $\int \cos(2x)\,dx$

$= \int \cos(u)\frac{du}{2}$

$= \frac{1}{2}\sin(u) + C$

$= \frac{1}{2}\sin(2x) + C_1$

$\int 1\,dx$

$= x + C_1$

Trigonometric Substitution

If you have an integral that involves

$\sqrt{a^2-u^2}$,

$\sqrt{a^2+u^2}$,

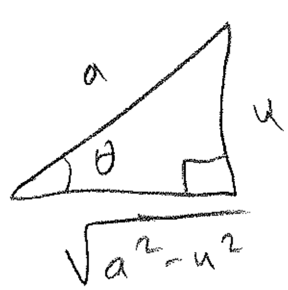
$\sqrt{u^2-a^2}$,

then you can try evaluating by trig substitution.

(i) for integrals involving $\sqrt{a^2-u^2}$,

let $u = a\sin(\theta)$

then $\sqrt{a^2-u^2} = a\cos(\theta)$

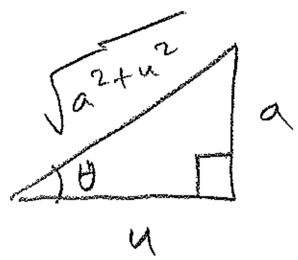


(ii) for integrals

involving $\sqrt{a^2+u^2}$,

let $u = a\tan(\theta)$

then $\sqrt{a^2+u^2} = a\sec(\theta)$

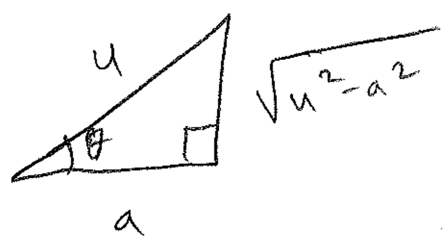


(iii) for integrals

involving $\sqrt{u^2-a^2}$,

let $u = a\sec(\theta)$,

then $\sqrt{u^2-a^2} = \pm a\tan(\theta)$



ex Find

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx$$

let $x = 4\sin(\theta)$

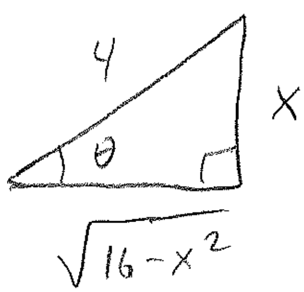
so $\frac{dx}{d\theta} = 4\cos(\theta)$

so $dx = 4\cos(\theta)\,d\theta$

we also have

$\sqrt{16-x^2} = 4\cos(\theta)$

from the diagram



now $$\int \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$$= \int \frac{(4\sin(\theta))^3}{4\cos(\theta)}\,4\cos(\theta)\,d\theta$$

$$= \int (4\sin(\theta))^3 \, d\theta$$

$$= \int 64 \sin^3(\theta) d\theta$$

$$= 64 \int \sin^3(\theta) d\theta$$

$$= 64 \int \sin^2(\theta) \sin(\theta) \, d\theta$$

$$= 64 \int \left(1 - \cos^2(\theta)\right) \sin(\theta) \, d\theta$$

let $u := \cos(\theta)$

so $\frac{du}{d\theta} = -\sin(\theta)$

so $d\theta = \frac{du}{-\sin(\theta)}$

now

$$= 64 \int (1-u^2) \sin(\theta) \frac{du}{-\sin(\theta)}$$

$$= -64 \int (1-u^2) \, du$$

$$= 64 \int (u^2 - 1) \, du$$

$$= 64 \left(\frac{1}{3} u^3 - u \right) + C$$

$$= 64\left(\frac{1}{3}\cos^3(\theta) - \cos(\theta)\right) + C$$

$$\boxed{= 64\left(\frac{1}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3 - \left(\frac{\sqrt{16-x^2}}{4}\right)\right) + C}$$

---

ex Find $\displaystyle\int \frac{x^2}{\sqrt{9-25x^2}}\,dx$

let $u = 5x$

let $u = 3\sin(\theta)$

so $5x = 3\sin(\theta)$

so $x = \frac{3}{5}\sin(\theta)$

so $\frac{dx}{d\theta} = \frac{3}{5}\cos(\theta)$

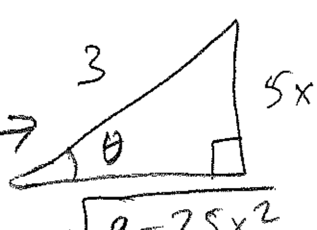
So $dx = \frac{3}{5}\cos(\theta)\,d\theta$

We also have $\sqrt{9-25x^2} = 3\cos(\theta)$

$5x = 3\sin(\theta)$

so

$\sin(\theta) = \frac{5x}{3}$



now

$$\int \frac{x^2}{\sqrt{9-25x^2}}\,dx$$

$$= \int \frac{\left(\frac{3}{5}\sin(\theta)\right)^2}{3\cos(\theta)} \frac{3}{5}\cos(\theta)\,d\theta$$

$$= \int \frac{3^2}{5^2}\sin^2(\theta)\frac{1}{5}\,d\theta$$

$$= \frac{9}{125}\int \sin^2(\theta)\,d\theta$$

$$= \frac{9}{125}\int \frac{1}{2}\left(1-\cos(2\theta)\right)d\theta$$

$$= \frac{9}{250}\int \left(1-\cos(2\theta)\right)d\theta$$

$$= \frac{9}{250}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C_1$$

note that $x = \frac{3}{5}\sin(\theta)$

$\longrightarrow \frac{5}{3}x = \sin(\theta)$

$\longrightarrow \arcsin\left(\frac{5x}{3}\right) = \theta$, so

$$= \boxed{\frac{9}{250}\left(\arcsin\left(\frac{5x}{3}\right) - \frac{1}{2}\sin\left(2\arcsin\left(\frac{5x}{3}\right)\right)\right) + C_1}$$

ex Find $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

complete the square

$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

let $u = x+1$

let $u = 2\sin(\theta)$

so $x+1 = 2\sin(\theta)$

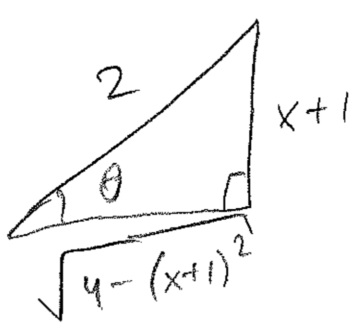
so $x = 2\sin(\theta) - 1$

so $\frac{dx}{d\theta} = 2\cos(\theta)$

so $dx = 2\cos(\theta)\,d\theta$

we also have

$$\sqrt{4-(x+1)^2} = 2\cos(\theta)$$



now $\displaystyle\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$

$$= \int \frac{2\sin(\theta)-1}{2\cos(\theta)}\, 2\cos(\theta)\,d\theta$$

$$= \int \left(2\sin(\theta)-1\right)d\theta$$

$$= -2\cos(\theta) - \theta + C$$

$$= -2\,\frac{\sqrt{4-(x+1)^2}}{2} - \arcsin\left(\frac{x+1}{2}\right) + C$$

note $\quad x+1 = 2\sin(\theta)$

$\longrightarrow \quad \frac{x+1}{2} = \sin(\theta)$

$\longrightarrow \quad \arcsin\left(\frac{x+1}{2}\right) = \theta$

$$= -\sqrt{4-(x+1)^2} - \arcsin\left(\frac{x+1}{2}\right) + C$$

ex Find

$$\int \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int \frac{x^3}{\left(\sqrt{4x^2+9}\right)^3}\,dx$$

let $u = 2x$

let $u = 3\tan(\theta)$

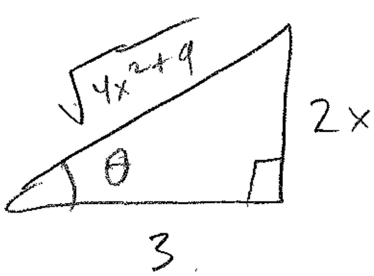
so $2x = 3\tan(\theta)$

so $x = \frac{3}{2}\tan(\theta)$

so $\frac{dx}{d\theta} = \frac{3}{2}\sec^2(\theta)$

so $dx = \frac{3}{2}\sec^2(\theta)\,d\theta$

also $\sqrt{4x^2+9} = 3\sec(\theta)$



now $$\int \frac{x^3}{\left(\sqrt{4x^2+9}\right)^3}\,dx$$

$$= \int \frac{\left(\frac{3}{2}\tan(\theta)\right)^3}{\left(3\sec(\theta)\right)^3}\,\frac{3}{2}\sec^2(\theta)\,d\theta$$

$$= \int \frac{\frac{3^3}{2^3}\tan^3(\theta)}{3^3\sec^3(\theta)}\,\frac{3}{2}\sec^2(\theta)\,d\theta$$

$$= \frac{3^3}{2^3}\,\frac{1}{3^3}\,\frac{3}{2}\int \frac{\tan^3(\theta)}{\sec(\theta)}\,d\theta$$

$$= \frac{3}{2^4}\int \frac{\sin^3(\theta)}{\cos^3(\theta)}\cos(\theta)\,d\theta$$

$$= \frac{3}{2^4}\int \frac{\sin^3(\theta)}{\cos^2(\theta)}\,d\theta$$

$$= \frac{3}{2^4}\int \frac{\sin^2(\theta)\sin(\theta)}{\cos^2(\theta)}\,d\theta$$

$$= \frac{3}{2^4}\int \frac{\left(1-\cos^2(\theta)\right)\sin(\theta)}{\cos^2(\theta)}\,d\theta$$

let $u := \cos(\theta)$

so $\dfrac{du}{d\theta} = -\sin(\theta)$

so $du = -\sin(\theta)\, d\theta$

so $d\theta = \dfrac{du}{-\sin(\theta)}$

now $\dfrac{3}{16}\displaystyle\int \frac{(1-u^2)\sin(\theta)}{u^2}\,\frac{du}{-\sin(\theta)}$

$$= -\frac{3}{16}\int \frac{1-u^2}{u^2}\,du$$

$$= -\frac{3}{16}\int\left(\frac{1}{u^2} - \frac{u^2}{u^2}\right)du$$

$$= -\frac{3}{16}\int\left(u^{-2} - 1\right)du$$

$$= -\frac{3}{16}\left(-u^{-1} - u\right) + C$$

$$= \frac{3}{16}\left(\frac{1}{u} + u\right) + C$$

$$= \frac{3}{16}\left(\frac{1}{\cos(\theta)} + \cos(\theta)\right) + C$$

$$= \boxed{\frac{3}{16}\left(\frac{\sqrt{4x^2+9}}{3} + \frac{3}{\sqrt{4x^2+9}}\right) + C}$$

$ Partial Fractions

Note that
$\frac{1}{x+1} + \frac{1}{x+2}$
$= \frac{x+2}{(x+1)(x+2)} + \frac{x+1}{(x+1)(x+2)}$
$= \frac{x+2+x+1}{(x+1)(x+2)}$
$= \frac{2x+3}{(x+1)(x+2)}$.

If we start with $\frac{2x+3}{(x+1)(x+2)}$
and we write it as $\frac{1}{x+1} + \frac{1}{x+2}$,
then we say that we have expanded $\frac{2x+3}{(x+1)(x+2)}$
into partial fractions.

Definition
Let $f(x)$ be a polynomial expression.
We say that $f(x)$ is linear iff $f(x) = ax + b$.
DONE

Example
$3x + 4$ is linear.
DONE

Definition
Let $f(x)$ be a polynomial expression.
We say that $f(x)$ is quadratic iff $f(x) = ax^2 + bx + c$.
DONE

Example
$3x^2 + 4x + 5$ is quadratic.
DONE

Definition
Let $f(x)$ be a quadratic expression.
We say that $f(x)$ is irreducible iff
$f(x)$ has no real number zeros.
DONE

Example
$x^2 + 1$ is irreducible
because there is no real number c s.t. $c^2 + 1 = 0$.
DONE

Theorem
$ax^2 + bx + c$ is irreducible
exactly when $b^2 - 4ac < 0$.
DONE

Example
$3x^2 + 4x + 5$ is irreducible
because $4^2 - 4(3)(5) < 0$.
DONE

Reminder:
If $f(x)$ is a polynomial expression
with real number coefficients,
then $f(x)$ can be factored
into linear factors.
and irreducible quadratic factors.

If $\frac{T(x)}{B(x)}$ is a rational expression
and $\text{Deg}(T) < \text{Deg}(B)$
and you want to expand $\frac{T(x)}{B(x)}$ into partial fractions,
then you could do this.
(i)
Factor $B(x)$ into linear factors
and irreducible quadratic factors.
So factor $B(x)$ as much as possible.
Say you get
$B(x) = (3x+4)^2(6x-5)(7x-1)^4$
$(x^2+1)^3(3x^2+4x+5)^2$.
(ii)
If $(ax+b)^m$ appears
in the complete factorization of $B(x)$,
then write $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_m}{(ax+b)^m}$.
If $(ax^2+bx+c)^m$ appears
in the complete factorization of $B(x)$,
then write $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ... + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$.
So if $B(x) = (3x+4)^2(6x-5)(7x-1)^4$
$(x^2+1)^3(3x^2+4x+5)^2$,
then we would write
$\frac{T(x)}{B(x)}$
$= \frac{A}{3x+4} + \frac{B}{(3x+4)^2}$
$+ \frac{C}{6x-5}$
$+ \frac{D}{7x-1} + \frac{E}{(7x-1)^2} + \frac{F}{(7x-1)^3} + \frac{G}{(7x-1)^4}$
$+ \frac{Hx+I}{x^2+1} + \frac{Jx+K}{(x^2+1)^2} + \frac{Lx+M}{(x^2+1)^3}$
$+ \frac{Nx+\Omega}{3x^2+4x+5} + \frac{Px+Q}{(3x^2+4x+5)^2}$.
(iii) Determine the top coefficients of the partial fractions.
To do this, if we wrote $\frac{T(x)}{B(x)} = P_1(x) + P_2(x) + ... + P_k(x)$
then $T(x) = B(x)P_1(x) + B(x)P_2(x) + ... + B(x)P_k(x)$.
Compare the coefficients of the left and right side
to create a system of equations
for the partial fraction coefficients.
Solve this system of equations.

Practice 1
Write out the form
of the partial fraction expansion.

Procedure:

$$\frac{...}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}$$
$$+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$$

$$\frac{...}{x^2(x-1)^3(4x+5)(x^2+1)^2(x^2+x+1)^2}$$

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}+\frac{F}{4x+5}$$
$$+\frac{Hx+I}{x^2+1}+\frac{Jx+K}{(x^2+1)^2}+\frac{Lx+M}{x^2+x+1}+\frac{Nx+P}{(x^2+x+1)^2}$$

$$\frac{...}{(x+3)(3x+1)}$$

$$\frac{A}{x+3}+\frac{B}{3x+1}$$

$$\frac{...}{x(x+1)^2}$$

$$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$\frac{...}{x^2(x+1)}$$

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$\frac{...}{x(x^2+1)}$$

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$\frac{...}{(x+4)(x+1)}$$

$$\frac{A}{x+4}+\frac{B}{x+1}$$

$$\frac{...}{(x-1)(x^2+x+1)}$$

$$\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$\frac{...}{(x+1)(x+3)}$$
$$\frac{A}{x+1}+\frac{B}{x+3}$$

$$\frac{...}{(x+1)^3(x^2+4)^2}$$
$$\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{Dx+E}{x^2+4}+\frac{Fx+G}{(x^2+4)^2}$$

$$\frac{...}{(x-1)(x+1)(x^2+1)}$$
$$\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$\frac{...}{(x^2+1)(x^2+4)^2}$$
$$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}+\frac{Ex+F}{(x^2+4)^2}$$

$$\frac{...}{x(x^2+1)(x^2-x+3)}$$
$$\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2-x+3}$$

$$\frac{...}{x^3(x-1)(x^2+x+1)}$$
$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}+\frac{Ex+F}{x^2+x+1}$$

DONE

Practice 2

Find a partial fraction expansion for $\dfrac{5x-4}{(x-2)(x+1)}$.

Procedure:

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \text{ for all } x$$

$\longrightarrow 5x-4 = A(x+1) + B(x-2)$
$\longrightarrow 5x-4 = Ax + A + Bx - 2B$
$\longrightarrow 5x-4 = [A+B]x + [A-2B]$
Equating coefficients, we get
$A + B = 5,$
$A - 2B = -4.$
Solving this system of equations gives
$A = 2, B = 3.$

So
$$\frac{5x-4}{(x-2)(x+1)} = \frac{2}{x-2} + \frac{3}{x+1}.$$

DONE

$ Integration by Partial Fractions

Reminder:
If we have a rational expression
$\frac{T(x)}{B(x)}$ and $\text{Deg}(T) > \text{Deg}(B)$,
then there is a polynomial expression $Q(x)$
and a polynomial expression $R(x)$ such that
$\frac{T(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}$ and $\text{Deg}(R) < \text{Deg}(B)$;
to find $Q(x)$ and $R(x)$, do a polynomial division.

Let $\frac{T(x)}{B(x)}$ be a rational expression.
If you want to find $\int \frac{T(x)}{B(x)} dx$
and $\text{Deg}(T) > \text{Deg}(B)$,
then do a polynomial division
and note that
$\int \frac{T(x)}{B(x)} dx$
$= \int (Q(x) + \frac{R(x)}{B(x)}) dx$.
Now $Q(x)$ is a polynomial and can easily be integrated.
We only need to worry about integrating $\frac{R(x)}{B(x)}$
where $\text{Deg}(R) < \text{Deg}(B)$.

If $\frac{T(x)}{B(x)}$ is a rational expression.
with $\text{Deg}(T) < \text{Deg}(B)$.
and you want to find $\int \frac{T(x)}{B(x)} dx$,
then expand $\frac{T(x)}{B(x)}$ into partial fractions
and note that
$\int \frac{T(x)}{B(x)} dx$
$= \int$ "stuff like $\frac{A}{(ax+b)^k}$ and $\frac{Ax+B}{(ax^2+bx+c)^k}$" dx
So we only need to be able to integrate
$\frac{A}{(ax+b)^k}$ and $\frac{Ax+B}{(ax^2+bx+c)^k}$.

If we can integrate
$\frac{A}{(ax+b)^k}$ when $k = 1$
$\frac{A}{(ax+b)^k}$ when $k > 1$
$\frac{Ax+B}{(ax^2+bx+c)^k}$ when $k = 1$
$\frac{Ax+B}{(ax^2+bx+c)^k}$ when $k > 1$,
then we can integrate any rational expression.

Example

Find $\int \frac{2}{3x+4}dx$

Procedure:

$\int \frac{2}{3x+4}dx$
$= 2\int \frac{1}{3x+4}dx$
let $u := 3x + 4$
then $\frac{du}{dx} = 3$
so $du = 3dx$
so $dx = \frac{du}{3}$
now
$2\int \frac{1}{3x+4}dx$
$= 2\int \frac{1}{u}\frac{du}{3}$
$= \frac{2}{3}\int \frac{1}{u}du$
$= \frac{2}{3}\ln(u) + C$
$= \frac{2}{3}\ln(3x + 4) + C$

DONE

Example

Find $\int \frac{2}{(3x+4)^5}dx$

$\int \frac{2}{(3x+4)^5}dx$
$= 2\int (3x + 4)^{-5}dx$
let $u := 3x + 4$
then $\frac{du}{dx} = 3$
so $du = 3dx$
so $dx = \frac{du}{3}$
now
$2\int (3x + 4)^{-5}dx$
$= 2\int u^{-5}\frac{du}{3}$
$= \frac{2}{3}\int u^{-5}du$
$= \frac{2}{3}(\frac{1}{-4}u^{-4}) + C$
$= -\frac{1}{6}(3x + 4)^{-4} + C$

DONE

Example

Find $\int \frac{7x+8}{3x^2+4x+5}dx$

Procedure:

$\int \frac{7x+8}{3x^2+4x+5}dx$
$= \int \frac{7x}{3x^2+4x+5}dx + \int \frac{8}{3x^2+4x+5}dx$
note that $(3x^2+4x+5)' = 6x+4$
$= 7\int \frac{x}{3x^2+4x+5}dx + 8\int \frac{1}{3x^2+4x+5}dx$
$= \frac{7}{6}\int \frac{6x}{3x^2+4x+5}dx + 8\int \frac{1}{3x^2+4x+5}dx$
$= \frac{7}{6}\int \frac{6x+4-4}{3x^2+4x+5}dx + 8\int \frac{1}{3x^2+4x+5}dx$
$= \frac{7}{6}\int (\frac{6x+4}{3x^2+4x+5} - \frac{4}{3x^2+4x+5})dx + 8\int \frac{1}{3x^2+4x+5}dx$
$= \frac{7}{6}(\int (\frac{6x+4}{3x^2+4x+5}dx - \int \frac{4}{3x^2+4x+5})dx) + 8\int \frac{1}{3x^2+4x+5}dx$
$= \frac{7}{6}\int \frac{6x+4}{3x^2+4x+5}dx - \frac{7}{6}\int \frac{4}{3x^2+4x+5}dx + 8\int \frac{1}{3x^2+4x+5}dx$
$= \frac{7}{6}\int \frac{6x+4}{3x^2+4x+5}dx - \frac{7*4}{6}\int \frac{1}{3x^2+4x+5}dx + \int \frac{8}{3x^2+4x+5}dx$
$= \frac{7}{6}\int \frac{6x+4}{3x^2+4x+5}dx + (8 - \frac{7*4}{6})\int \frac{1}{3x^2+4x+5}dx$
$= \frac{7}{6}\ln(3x^2+4x+5) + (8 - \frac{7*4}{6})\int \frac{1}{3x^2+4x+5}dx$
complete the square
$= \frac{7}{6}\ln(3x^2+4x+5) + (8 - \frac{7*4}{6})\int \frac{1}{3(x+\frac{2}{3})^2+\frac{11}{3}}dx$
use the formula $\int \frac{1}{\alpha^2x^2+\beta^2}dx = \frac{1}{\alpha\beta}\arctan(\frac{\alpha}{\beta}x) + C$.
$= \frac{7}{6}\ln(3x^2+4x+5) + (8 - \frac{7*4}{6})\frac{1}{\sqrt{3}\sqrt{\frac{11}{3}}}\arctan(\frac{\sqrt{3}}{\sqrt{\frac{11}{3}}}(x+\frac{2}{3})) + C$
$= \frac{7}{6}\ln(3x^2+4x+5) + (\frac{48}{6} - \frac{28}{6})\frac{1}{\sqrt{11}}\arctan(\frac{3}{\sqrt{11}}(x+\frac{2}{3})) + C$
$= \frac{7}{6}\ln(3x^2+4x+5) + (\frac{20}{6})\frac{1}{\sqrt{11}}\arctan(\frac{3x+2}{\sqrt{11}}) + C$
$= \frac{7}{6}\ln(3x^2+4x+5) + \frac{10}{3\sqrt{11}}\arctan(\frac{3x+2}{\sqrt{11}}) + C$

DONE

We can also use these formulas.

Theorem

$\int \frac{A}{ax+b} dx = \frac{A}{a} \ln(ax+b) + C$

DONE

Example

$\int \frac{2}{3x+4} dx$
$= \frac{2}{3} \ln(3x+4) + C$

DONE

Theorem

If $k > 1$, then $\int \frac{A}{(ax+b)^k} dx = \frac{A}{a} \frac{1}{1-k}(ax+b)^{1-k} + C$.

DONE

Example

$\int \frac{2}{(3x+4)^5} dx$
$= \frac{2}{3} \frac{1}{1-5}(3x+4)^{1-5} + C$
$= \frac{2}{3} \frac{1}{-4}(3x+4)^{-4} + C$
$= -\frac{1}{6}(3x+4)^{-4} + C$

DONE

Theorem

If $a \neq 0$ and $ax^2 + bx + c$ has no real roots,
then $\int \frac{Ax+B}{ax^2+bx+c} dx$
$= \frac{A}{2a} \ln(ax^2 + bx + c) + \frac{2aB-bA}{a\sqrt{4ac-b^2}} \arctan(\frac{2ax+b}{\sqrt{4ac-b^2}}) + C$.

DONE

Example

Find $\int \frac{7x+8}{3x^2+4x+5} dx$

Procedure:
$\int \frac{7x+8}{3x^2+4x+5} dx$
$= \frac{A}{2a} \ln(ax^2 + bx + c) + \frac{2aB-bA}{a\sqrt{4ac-b^2}} \arctan(\frac{2ax+b}{\sqrt{4ac-b^2}}) + C$
$= \frac{(7)}{2(3)} \ln(3x^2 + 4x + 5) + \frac{2(3)(8)-(4)(7)}{(3)\sqrt{4(3)(5)-(4)^2}} \arctan(\frac{2(3)x+(4)}{\sqrt{4(3)(5)-(4)^2}}) + C$
$= \frac{7}{6} \ln(3x^2 + 4x + 5) + \frac{48-28}{3\sqrt{60-16}} \arctan(\frac{6x+4}{\sqrt{60-16}}) + C$
$= \frac{7}{6} \ln(3x^2 + 4x + 5) + \frac{20}{3\sqrt{44}} \arctan(\frac{6x+4}{\sqrt{44}}) + C$
$= \frac{7}{6} \ln(3x^2 + 4x + 5) + \frac{20}{3*2\sqrt{11}} \arctan(\frac{6x+4}{2\sqrt{11}}) + C$
$= \frac{7}{6} \ln(3x^2 + 4x + 5) + \frac{10}{3\sqrt{11}} \arctan(\frac{3x+2}{\sqrt{11}}) + C$

DONE

Improper Integrals

If $A \subseteq \mathbb{R}$
(A is a subset of real #s),

then we say that A
is bounded if there
is an interval (l, r)
such that $A \subseteq (l, r)$ and $l, r \in \mathbb{R}$.

ex. Let $A := \left\{ \frac{1}{p} : p \text{ is prime} \right\}$

$$= \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \ldots \right\}$$

Note that $A \subseteq (0, 1)$

So A is bounded.

ex. Let $A = \{ p : p \text{ is prime} \}$

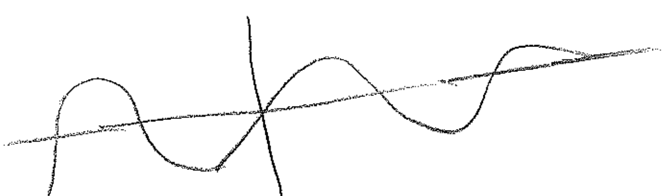
$$= \{ 2, 3, 5, 7, 11, \ldots \}$$

A is NOT bounded,
there are infinitely many
prime #'s, and they are
all natural #'s.

If f is a function, then we say that f is bounded if $\text{Range}(f)$ is bounded.

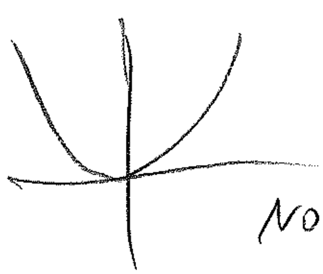
We say that f is bounded on X if $\{f(x) : x \in X\}$ is bounded.

ex Let $f(x) = \sin(x)$. is f bounded?



yes

ex Let $f(x) = x^2$ is f bounded

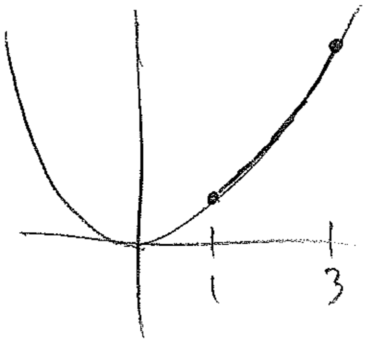


No.

ex Let $f(x) = x^2$

is f bounded

on $[1, 3]$?



Yes.

We have defined

$\int_a^b f(x)\,dx$ when

both

(i) (a,b) is bounded

(ii) f is bounded on (a,b).

We want to define

$\int_a^b f(x)\,dx$ when

one of (i), (ii)

is false.

ex $\int_1^\infty \frac{1}{x^2}\,dx$ SE

we haven't defined this yet because $(1,\infty)$ is unbounded.

but $\int_1^b \frac{1}{x^2}\,dx$

$$\int \frac{1}{x^2}dx = \int x^{-2}dx = -x^{-1} + C$$

$$= \left[-\frac{1}{x}\right]_1^b$$

$$= \left[-\frac{1}{b}\right] - \left[-\frac{1}{1}\right]$$

$$= -\frac{1}{b} + 1$$

$$= 1 - \frac{1}{b}$$

we define

$$\int_1^\infty \frac{1}{x^2}dx := \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$$

so

$$\int_1^\infty \frac{1}{x^2}dx$$

$$= \lim_{b\to\infty}\left(1-\frac{1}{b}\right)$$

$$= 1$$

ex $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

we have defined this yet because $y = \frac{1}{\sqrt{x}}$ is NOT bounded on $(0,1)$.

but $\int_a^1 \frac{1}{\sqrt{x}}\,dx$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = 2x^{1/2} + C$$

$$= \left[2x^{1/2}\right]_a^1$$

$$= \left[2(1)^{1/2}\right] - \left[2(a)^{1/2}\right]$$

$$= 2 - 2\sqrt{a}$$

we define

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx := \lim_{a \to 0} \int_a^1 \frac{1}{\sqrt{x}}\,dx$$

so $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$= \lim_{a \to 0} \left(2 - 2\sqrt{a}\right)$$

$$= 2$$

We define

$$\int_a^\infty f(x)dx := \lim_{b \to \infty} \int_a^b f(x)dx$$

$$\int_{-\infty}^b f(x)dx := \lim_{a \to -\infty} \int_a^b f(x)dx$$

$$\int_{-\infty}^\infty f(x)dx := \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$$

If f has a vertical asymptote at $x_0 \in [a,b]$
(x_0 in $[a,b]$)

then we define

$$\int_a^{x_0} f(x)dx := \lim_{c \to x_0-} \int_a^c f(x)dx$$

$$\int_{x_0}^b f(x)dx := \lim_{c \to x_0+} \int_c^b f(x)dx$$

$$\int_a^b f(x)dx := \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx$$

ex Find

$$\int_0^\infty \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}\,dx$$

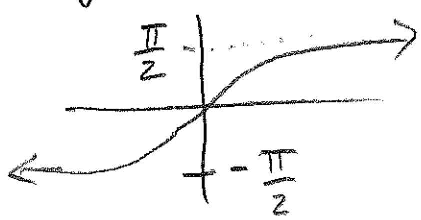
$$= \lim_{b\to\infty} \Big[\arctan(x)\Big]_0^b$$

$$= \lim_{b\to\infty} \Big(\big[\arctan(b)\big] - \big[\arctan(0)\big]\Big)$$

$$= \lim_{b\to\infty} \arctan(b)$$

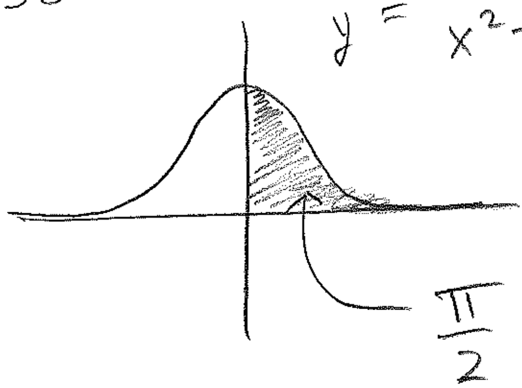
$$= \boxed{\frac{\pi}{2}}$$

$y = \arctan(x)$



So

$y = \dfrac{1}{x^2+1}$



ex Find

$$\int_1^\infty \frac{1}{(3x+1)^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{(3x+1)^2}\,dx$$

$$\int \frac{1}{(3x+1)^2}\,dx$$

let $u := 3x+1$

$$\frac{du}{dx} = 3$$

so $dx = \frac{du}{3}$

$$\int \frac{1}{(3x+1)^2}\,dx$$

$$= \int \frac{1}{u^2}\,\frac{du}{3}$$

$$= \frac{1}{3}\int \frac{1}{u^2}\,du$$

$$= \frac{1}{3}\int u^{-2}\,du$$

$$= -\frac{1}{3}u^{-1} + C_1$$

$$= -\frac{1}{3}\frac{1}{(3x+1)} + C_1$$

now $\lim_{b \to \infty} \int_1^b \frac{1}{(3x+1)^2}\,dx$

$$= \lim_{b \to \infty} \left[-\frac{1}{3(3x+1)} \right]_1^b$$

$$= \lim_{b \to \infty} \left(\left[-\frac{1}{3(3b+1)} \right] - \left[-\frac{1}{3(3+1)} \right] \right)$$

$$= \lim_{b \to \infty} \left(\left[-\frac{1}{3(3b+1)} \right] - \left[-\frac{1}{12} \right] \right)$$

$$= 0 - \left(-\frac{1}{12} \right)$$

$$= \boxed{\frac{1}{12}}$$

ex Find

$$\int_0^\infty e^{-x}\,dx$$

$$= \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[-e^{-x}\right]_0^b$$

$$= \lim_{b\to\infty} \left(\left[-e^{-b}\right] - \left[-e^{-0}\right]\right)$$

$$= \lim_{b\to\infty} \left(\left[-\frac{1}{e^b}\right] - \left[-1\right]\right)$$

$$= 0 + 1$$

$$= \boxed{1}$$

ex Find

$$\int_1^\infty \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

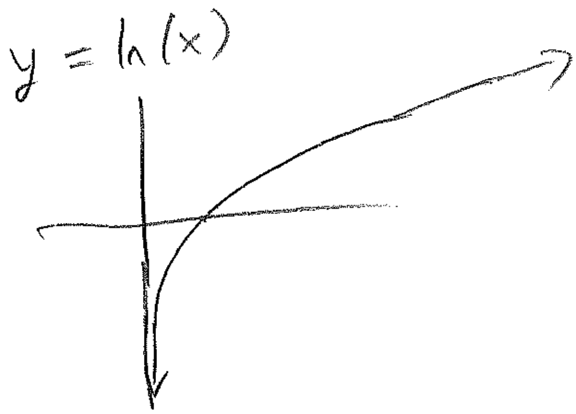
$$= \lim_{b\to\infty} \left[\ln(x)\right]_1^b$$

$$= \lim_{b\to\infty} \left(\left[\ln(b)\right] - \left[\ln(1)\right]\right)$$

$$= \lim_{b\to\infty} \ln(b) - 0$$

$$= \lim_{b\to\infty} \ln(b)$$

$$= \infty$$



so $\lim_{b\to\infty} \ln(b)$ DNE.

If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ DNE,

then we say that

$\int_a^\infty f(x)\,dx$ diverges.

otherwise we say

$\int_a^\infty f(x)\,dx$ converges.

ex Find

$$\int_0^\infty \cos(x)\,dx$$

$$= \lim_{b \to \infty} \int_0^b \cos(x)\,dx$$

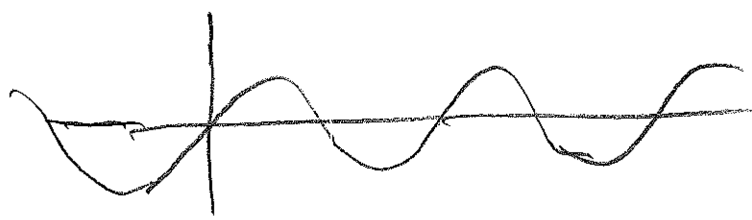
$$= \lim_{b \to \infty} \left[\sin(x)\right]_0^b$$

$$= \lim_{b \to \infty} \left(\left[\sin(b)\right] - \left[\sin(0)\right]\right)$$

$$= \lim_{b \to \infty} \sin(b)$$

DNE

$y = \sin(x)$



So $\int_0^\infty \cos(x)\,dx$

diverges.

ex Find

$$\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx.$$

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt[3]{x}}\,dx \qquad \int \frac{1}{\sqrt[3]{x}}\,dx$$

$$= \lim_{a\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_a^1 \qquad = \int \frac{1}{x^{1/3}}\,dx$$

$$= \lim_{a\to 0^+} \left(\left[\frac{3}{2}(1)^{2/3}\right] - \left[\frac{3}{2}(a)^{2/3}\right]\right) = \int x^{-1/3}\,dx$$

$$= \lim_{a\to 0^+} \left(\left[\frac{3}{2}\right] - \left[\frac{3}{2}a^{2/3}\right]\right) \qquad = \frac{3}{2}x^{2/3} + C$$

$$= \frac{3}{2} - 0$$

$$= \boxed{\frac{3}{2}}$$

ex Find

$$\int_0^1 \frac{1}{x^2}\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx$$

$$= \lim_{a\to 0^+} \left[-\frac{1}{x}\right]_a^1$$

$$= \lim_{a\to 0^+} \left(\left[-\frac{1}{1}\right] - \left[-\frac{1}{a}\right]\right)$$

$$= \lim_{a\to 0^+} \left(-1 + \frac{1}{a}\right)$$

$$= \lim_{a\to 0^+} \left(\frac{1}{a} - 1\right)$$

$$= \infty - 1$$

$$= \infty$$

DNE

so $\int_0^1 \frac{1}{x^2}\,dx$

diverges.

$$\int \frac{1}{x^2}\,dx$$

$$= \int x^{-2}\,dx$$

$$= -x^{-1} + C$$

p Integrals

$$\int_1^\infty \frac{1}{x^p}\,dx$$ converges

exactly when $p > 1$.

If $\int_1^\infty \frac{1}{x^p}\,dx$ converges,

then $$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}.$$

$$\int_0^1 \frac{1}{x^p}\,dx$$ converges

exactly when $p < 1$.

If $\int_0^1 \frac{1}{x^p}\,dx$ converges,

then $$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}.$$

ex

$$\int_1^\infty \frac{1}{x^5}\,dx \quad \text{converges} = \frac{1}{5-1} = \frac{1}{4}$$

$$\int_1^\infty \frac{1}{x^{1/2}}\,dx \quad \text{diverges}$$

$$\int_1^\infty \frac{1}{x}\,dx \quad \text{diverges}$$

$$\int_0^1 \frac{1}{x^2}\,dx \quad \text{diverges}$$

$$\int_0^1 \frac{1}{x^{1/7}}\,dx \quad \text{converges} = \frac{1}{1-\frac{1}{7}} = \frac{1}{\frac{7}{7}-\frac{1}{7}} = \frac{1}{\frac{6}{7}} = \boxed{\frac{7}{6}}$$

$$\int_0^1 \frac{1}{x}\,dx \quad \text{diverges}$$

(more improper integrals)

ex Find

$$\int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx$$

$$\int \frac{e^x}{1+e^{2x}}\,dx$$

let $u := e^x$

$$\frac{du}{dx} = e^x$$

so $dx = \dfrac{du}{e^x}$

$$\int \frac{e^x}{1+e^{2x}}\,dx$$

$$= \int \frac{e^x}{1+u^2}\,\frac{du}{e^x}$$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \arctan(u) + C$$

$$= \arctan(e^x) + C$$

now $\int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx$

$$= \lim_{a \to -\infty} \int_a^0 \frac{e^x}{1+e^{2x}}\,dx$$

$$= \lim_{a \to -\infty} \Big[\arctan(e^x)\Big]_a^0$$

$$= \lim_{a \to -\infty} \Big(\big[\arctan(e^0)\big] - \big[\arctan(e^a)\big]\Big)$$

$$= \lim_{a \to -\infty} \Big(\big[\arctan(1)\big] - \big[\arctan(e^a)\big]\Big)$$

$$= \lim_{a \to -\infty} \left(\frac{\pi}{4} - \arctan(e^a)\right)$$

$$= \frac{\pi}{4} - \arctan(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \boxed{\frac{\pi}{4}}$$

ex. Find

$$\int_3^5 \frac{1}{\sqrt{x-3}}\, dx$$

$$\int \frac{1}{\sqrt{x-3}}\, dx$$

$$= \int (x-3)^{-1/2}\, dx$$

let $u := x-3$

so $\frac{du}{dx} = 1$

so $dx = du$

$$= \int u^{-1/2}\, du$$

$$= 2u^{1/2} + c_1$$

$$= 2\sqrt{x-3} + c_1$$

4F

now $\int_3^5 \frac{1}{\sqrt{x-3}}\,dx$

$$= \lim_{a \to 3^+} \int_a^5 \frac{1}{\sqrt{x-3}}\,dx$$

$$= \lim_{a \to 3^+} \left[2\sqrt{x-3} \right]_a^5$$

$$= \lim_{a \to 3^+} \left(\left[2\sqrt{5-3} \right] - \left[2\sqrt{a-3} \right] \right)$$

$$= 2\sqrt{2} - 0$$

$$= \boxed{2\sqrt{2}}$$

ex Find .5F

$$\int_1^\infty \frac{1}{\sqrt{x}\,(x+1)}\,dx$$

$$\int \frac{1}{\sqrt{x}\,(x+1)}\,dx$$

let $u := \sqrt{x} = x^{1/2}$

so $\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

so $dx = 2x^{1/2}\,du$

$$\int \frac{1}{x^{1/2}(u^2+1)}\,2x^{1/2}\,du$$

$$= 2\int \frac{1}{u^2+1}\,du$$

$= 2\arctan(u) + C$

$= 2\arctan(\sqrt{x}) + C_1$

6F

now $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}\,(x+1)}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}\,(x+1)}\,dx$$

$$= \lim_{b\to\infty} \Big[2\arctan(\sqrt{x})\Big]_1^b$$

$$= \lim_{b\to\infty} \Big(\big[2\arctan(\sqrt{b})\big] - \big[2\arctan(\sqrt{1})\big]\Big)$$

$$= \left[2\,\frac{\pi}{2}\right] - \left[2\,\frac{\pi}{4}\right]$$

$$= \pi - \frac{\pi}{2}$$

$$= \boxed{\frac{\pi}{2}}$$

Comparison Theorem for Impoper Integrals

If $0 \le f(x) \le F(x)$

on $[a, \infty)$,

then

(i) If $\int_a^\infty F(x)\,dx$ is convergent,
then $\int_a^\infty f(x)\,dx$ is convergent.

(ii) If $\int_a^\infty f(x)\,dx$ is divergent,
then $\int_a^\infty F(x)\,dx$ is divergent.

ex Does

$$\int_1^\infty \frac{\cos^2(x)}{x^2+1}\,dx$$

converge?

note that

$$\int_1^\infty \frac{\cos^2(x)}{x^2+1}\,dx$$

$$\leq \int_1^\infty \frac{1}{x^2+1}\,dx$$

$$\leq \int_1^\infty \frac{1}{x^2}\,dx$$

converges because $p = 2 > 1$.

So

$$\int_1^\infty \frac{\cos^2(x)}{x^2+1}\,dx$$

converges.

Numerical Integration

If $F'(x) = f(x)$,

then $\int_a^b f(x)\,dx = F(b) - F(a)$.

So if we can find

an antiderivative of f,

then we can find $\int_a^b f(x)\,dx$.

If we can't find

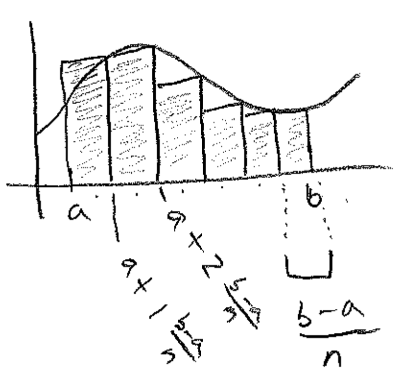
an antiderivative of f,

then we can still

<u>approximate</u> $\int_a^b f(x)\,dx$.

We defined

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\frac{b-a}{n}$$



$n = 6$

We define $x_k^R := a + k\frac{b-a}{n}$.

Note that $x_1^R, x_2^R, x_3^R, \ldots$ are the right endpoints.

Right Rule

Given $a, b, n,$

define $x_k^R := a + k\frac{b-a}{n}$

$\Delta x := \frac{b-a}{n}$.

If n is large,

then $\int_a^b f(x)dx \simeq \sum_{k=1}^{n} f(x_k^R)\Delta x$

ex Let $f(x) = e^{(x^2)}$,

$a=0, \quad b=1, \quad n=10.$

Approximate $\int_a^b f(x)dx$

by $\sum_{k=1}^{n} f(x_k^R)\Delta x$.

$$\int_a^b f(x)dx$$

$$= \int_0^1 e^{x^2} dx$$

$$\simeq \sum_{K=1}^{10} e^{\left(\frac{K}{n}\right)^2}\frac{1}{10}$$

$$= \frac{1}{10} \sum_{k=1}^{10} e^{\left(\frac{k^2}{100}\right)}$$

$$= \frac{1}{10} \left(e^{\frac{1}{100}} + e^{\frac{4}{100}} + e^{9/100} + e^{16/100} + \dots + e^{\frac{100}{100}} \right)$$

$$\approx 1.55$$

Left Rule

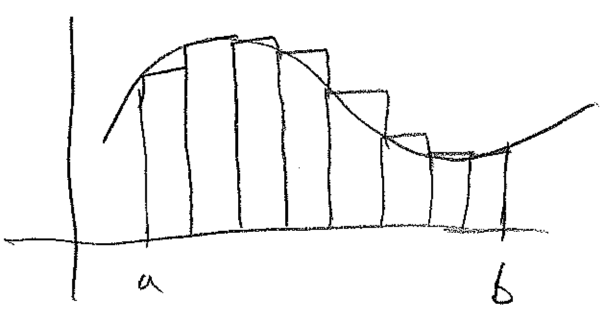
Given a, b, n

define $x_k^L := a + (k-1)\frac{b-a}{n}$.

$\Delta x := \frac{b-a}{n}$.

If n is large,

then $\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(x_k^L)\Delta x$.



Midpoint Rule

Given a, b, n,

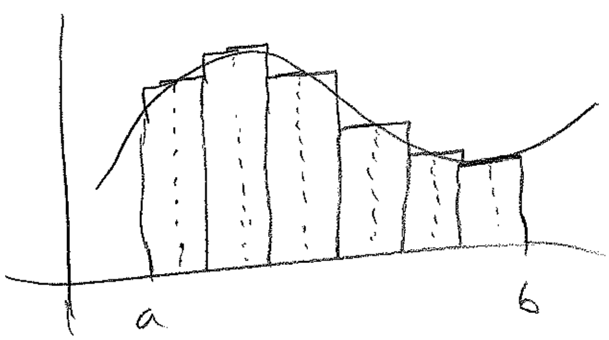
define $\overline{x_k} := a + k\left(\frac{b-a}{n}\right) - \frac{b-a}{2n}$

(so $\overline{x_k}$ is the midpoint of the kth interval)

$$\Delta x := \frac{b-a}{n}$$

If n is large,

then $\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(\overline{x_k})\Delta x$



ex Let $f(x) = \frac{1}{x}$

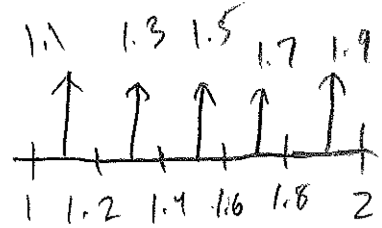
$a = 1,\ b = 2,\ n = 5.$

Approximate $\int_a^b f(x)\,dx$

by $\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x.$

$$\int_a^b f(x)\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx$$



$$\approx \sum_{k=1}^{5} \frac{1}{\bar{x}_k} \frac{2-1}{5}$$

$$= \frac{1}{\bar{x}_1}\frac{1}{5} + \frac{1}{\bar{x}_2}\frac{1}{5} + \frac{1}{\bar{x}_3}\frac{1}{5} + \frac{1}{\bar{x}_4}\frac{1}{5} + \frac{1}{\bar{x}_5}\frac{1}{5}$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\simeq .695$$

Trapezoid Rule

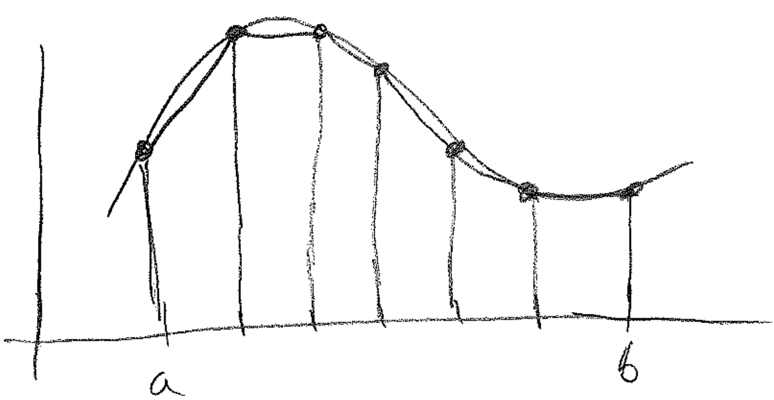
Given a, b, n,

define $x_k^R := a + k\frac{b-a}{n}$

$\Delta x := \frac{b-a}{n}$.

If n is large,

then $\int_a^b f(x)\,dx \approx \sum_{k=1}^{n-1} f(x_k^R)\Delta x + \frac{f(a)+f(b)}{2}\Delta x$



ex Let $f(x) = \frac{1}{x}$

$a = 1,\ b = 2,\ n = 5.$

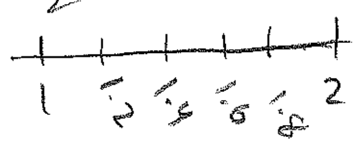
Approximate $\int_a^b f(x)\,dx$

by $\sum_{k=1}^{n-1} f(x_k^R)\,\Delta x + \frac{f(a)+f(b)}{2}\,\Delta x.$

$\int_a^b f(x)\,dx$

$= \int_1^2 \frac{1}{x}\,dx$

$\approx \sum_{k=1}^{5} \frac{1}{x_k^R}\,\frac{1}{5} + \frac{f(a)+f(b)}{2}\,\Delta x$



$= \frac{1}{5}\left(\frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \frac{1}{1.8} + \frac{1}{2}\right) + \frac{1+\frac{1}{2}}{2}\,\frac{1}{5}$

$\approx .691$

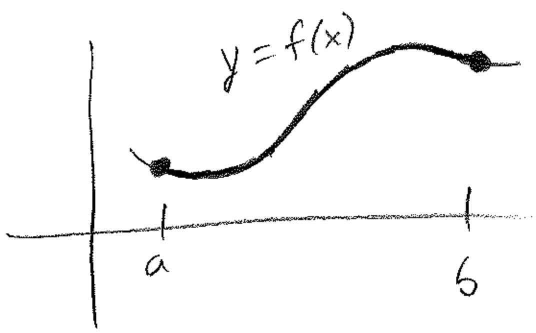
Note $\int_1^2 \frac{1}{x}\,dx = \ln(2)$

and $\ln(2) = .6931\ldots$

Arclength

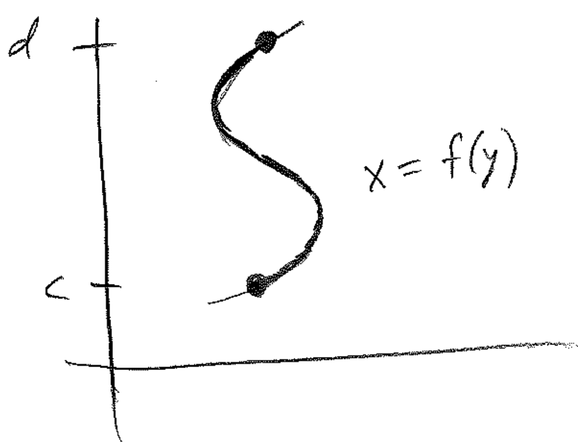
If you have a curve, C, by $y=f(x)$ for $a \le x \le b$, then we define

$$\text{Arclength}(C) := \int_a^b \sqrt{1+(f'(x))^2}\,dx.$$



If you have a curve, C, by $x=f(y)$, for $c \le y \le d$, then we define

$$\text{Arclength}(C) := \int_c^d \sqrt{1+(f'(y))^2}\,dy$$



ex Let C be the curve given by
$f(x) = x^{3/2}$ for $0 \le x \le 8$.
Find Arclength(C).

$$\text{Arclength}(C) = \int_0^8 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx$$

$$f'(x) = \frac{3}{2}x^{1/2}$$

$$\int \sqrt{1+\left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= \int \sqrt{1 + \frac{9}{4}x}\, dx$$

let $u := 1 + \frac{9}{4}x$

so $\frac{du}{dx} = \frac{9}{4}$

so $dx = \frac{4}{9}du$

$$= \int u^{1/2} \frac{4}{9} du$$

$$= \frac{4}{9}\int u^{1/2} du$$

$$= \frac{4}{9}\frac{2}{3}u^{3/2} + C_1$$

$$= \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2} + C_1$$

So Arclength(C)

$$= \int_0^8 \sqrt{1+\left(\tfrac{3}{2}x^{1/2}\right)^2}\,dx$$

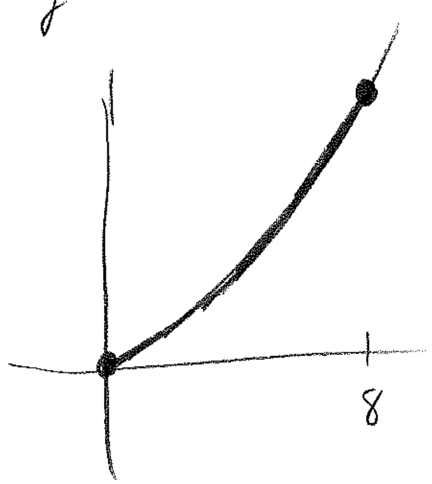
$$= \left[\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\right]_0^8$$

$$= \left[\frac{8}{27}\left(1+\frac{9}{4}(8)\right)^{3/2}\right] - \left[\frac{8}{27}\left(1+\frac{9}{4}(0)\right)^{3/2}\right]$$

$$= \boxed{\frac{8}{27}(19)^{3/2} - \frac{8}{27}}$$

$$= \boxed{\frac{8}{27}\left(19^{3/2} - 1\right)}$$

$y = x^{3/2}$



(more arclength)

ex Let $f(x) = \ln(\cos(x))$.

Find the arclength of f from 0 to $\frac{\pi}{4}$.

$$L = \int_0^{\pi/4} \sqrt{1 + \left(-\tan(x)\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\, dx \qquad \text{use } \tan^2(x) + 1 = \sec^2(x)$$

$$= \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx$$

$$= \int_0^{\pi/4} \sec(x)\, dx$$

$$= \Big[\ln\left(\sec(x) + \tan(x)\right)\Big]_0^{\pi/4}$$

$$= \left[\ln\left(\sec\left(\tfrac{\pi}{4}\right) + \tan\left(\tfrac{\pi}{4}\right)\right)\right] - \left[\ln\left(\sec(0) + \tan(0)\right)\right]$$

$$= \left[\ln(\sqrt{2} + 1)\right] - \left[\ln(1 + 0)\right]$$

$$= \ln(\sqrt{2}+1) - \ln(1)$$

$$= \boxed{\ln(\sqrt{2}+1)}$$

ex Find the arclength of the curve given by $x = y^3 + y$ for $1 \le y \le 4$.

Just set up the integral.

Let $f(y) := y^3 + y$

Then

$$L = \int_1^4 \sqrt{1 + (f'(y))^2}\, dy$$

$$= \int_1^4 \sqrt{1 + (3y^2+1)^2}\, dy$$

$$= \int_1^4 \sqrt{1 + 9y^4 + 6y^2 + 1}\, dy$$

$$= \boxed{\int_1^4 \sqrt{9y^4 + 6y^2 + 2}\, dy}$$

Surface Areas of Surfaces of Revolution.

Rotate $y = f(x)$

for $a \leq x \leq b$

around the x axis.

Let S be the

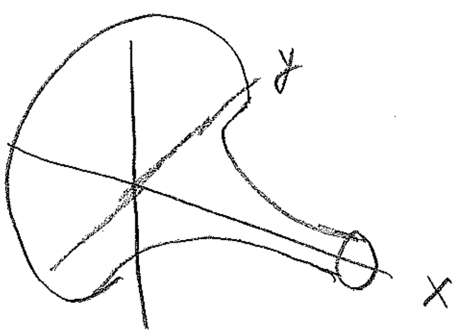
resulting surface.

We define the

surface area of S

to be

$$A := 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$$



ex

Rotate $y = x^3$

where $0 \le x \le 2$

around the xaxis.

Find the surface area.

$$A = 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2}\, dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\, dx$$

let $u = 1 + 9x^4$

so $\frac{du}{dx} = 36x^3$

so $dx = \frac{du}{36x^3}$

$$\int x^3 \sqrt{1 + 9x^4}\, dx$$

$$= \int x^3 \sqrt{u}\, \frac{du}{36x^3}$$

$$= \frac{1}{36} \int u^{1/2}\, du$$

$$= \frac{1}{36} \frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{54}\left(1+9x^4\right)^{3/2} + C_1$$ 56

so

$$2\pi \int_0^2 x^3 \sqrt{1+9x^4}\, dx$$

$$= 2\pi \left[\frac{1}{54}\left(1+9x^4\right)^{3/2} \right]_0^2$$

$$= \frac{2\pi}{54}\left(\left[\left(1+9(2)^4\right)^{3/2} \right] - \left[\left(1+9(0)^4\right)^{3/2} \right] \right)$$

$$= \frac{\pi}{27}\left(\left[145^{3/2} \right] - \left[1^{3/2} \right] \right)$$

$$= \boxed{\frac{\pi}{27}\left(145^{3/2} - 1\right)}$$

Rotate $y=f(x)$

for $a \le x \le b$

around the y axis.

Let $\$$ be the

resulting surface.

We define the

surface area of $\$$

to be

$$A := 2\pi \int_a^b x \sqrt{1+(f'(x))^2}\, dx$$

ex Rotate $y = 1 - x^2$

where $0 \leq x \leq 1$

around the yaxis.

Find the surface area.

$$A = 2\pi \int_0^1 x\sqrt{1 + (-2x)^2}\,dx$$

$$= 2\pi \int_0^1 x\sqrt{1 + 4x^2}\,dx$$

let $u = 1 + 4x^2$

so $\frac{du}{dx} = 8x$

so $dx = \frac{du}{8x}$

$$\int x\sqrt{1+4x^2}\,dx$$

$$= \int x\sqrt{u}\,\frac{du}{8x}$$

$$= \frac{1}{8}\int u^{1/2}\,du$$

$$= \frac{1}{8}\,\frac{2}{3}\,u^{3/2} + C_1$$

$$= \frac{1}{12}(1+4x^2)^{3/2} + C$$

so

$$2\pi \int_0^1 x\sqrt{1+4x^2}\,dx$$

$$= 2\pi \left[\frac{1}{12}(1+4x^2)^{3/2}\right]_0^1$$

$$= \frac{2\pi}{12}\left(\left[(1+4(1)^2)^{3/2}\right] - \left[(1+4(0)^2)^{3/2}\right]\right)$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1^{3/2}\right)$$

$$= \boxed{\frac{\pi}{6}\left(5^{3/2} - 1\right)}$$

Rotate $x = f(y)$

for $c \le y \le d$

around the x axis.

Let $\mathcal{S}$ be the resulting surface.

We define the surface area of $\mathcal{S}$ to be

$$A := 2\pi \int_c^d y\sqrt{1+(f'(y))^2}\, dy$$

ex Rotate $x = 1 + 2y^2$

where $1 \leq y \leq 2$

around the x axis.

Find the surface area.

$$A = 2\pi \int_1^2 y \sqrt{1 + (4y)^2}\, dy$$

$$= 2\pi \int_1^2 y \sqrt{1 + 16y^2}\, dy$$

let $u = 1 + 16y^2$

so $\frac{du}{dy} = 32y$

so $dy = \frac{du}{32y}$

$$\int y \sqrt{1 + 16y^2}\, dy$$

$$= \int y \sqrt{u}\, \frac{du}{32y}$$

$$= \frac{1}{32} \int u^{1/2}\, du$$

$$= \frac{1}{32} \frac{2}{3} u^{3/2} + C_1$$

$$= \frac{1}{48}(1+16y^2)^{3/2} + C$$

so

$$2\pi \int_1^2 y\sqrt{1+16y^2}\,dy$$

$$= 2\pi\left[\frac{1}{48}(1+16y^2)^{3/2}\right]_1^2$$

$$= \frac{2\pi}{48}\left(\left[(1+16(2)^2)^{3/2}\right] - \left[(1+16(1)^2)^{3/2}\right]\right)$$

$$= \boxed{\frac{\pi}{24}\left(65^{3/2} - 17^{3/2}\right)}$$

Rotate $x = f(y)$

for $c \leq y \leq d$

around the y axis.

Let $\$$ be the

resulting surface.

We define the

surface area of $\$$

to be

$$A := 2\pi \int_c^d f(y) \sqrt{1 + (f'(y))^2}\, dy$$

ex Rotate $x = y^3$

where $1 \le y \le 2$

around the y axis.

Find the surface area.

$$A = 2\pi \int_1^2 y^3 \sqrt{1 + (3y^2)^2}\, dy$$

$$= 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4}\, dy$$

let $u = 1 + 9y^4$

so $\frac{du}{dy} = 36y^3$

so $dy = \frac{du}{36y^3}$

$$\int y^3 \sqrt{1 + 9y^4}\, dy$$

$$= \int y^3 \sqrt{u}\, \frac{du}{36y^3}$$

$$= \frac{1}{36} \int u^{1/2}\, du$$

$$= \frac{1}{36} \frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{54}\left(1+9y^4\right)^{3/2} + C_1$$

so

$$2\pi\int_1^2 y^3\sqrt{1+9y^4}\,dy$$

$$= 2\pi\left[\frac{1}{54}\left(1+9y^4\right)^{3/2}\right]_1^2$$

$$= \frac{2\pi}{54}\left(\left[\left(1+9(2)^4\right)^{3/2}\right] - \left[\left(1+9(1)^4\right)^{3/2}\right]\right)$$

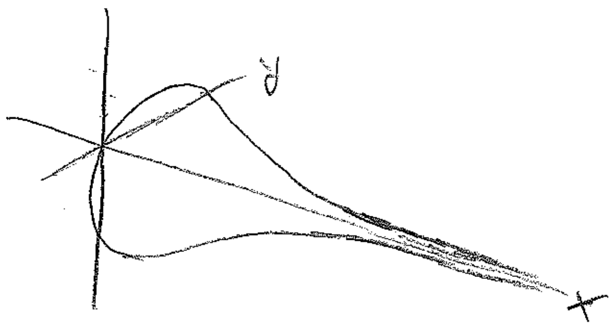
$$= \boxed{\frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)}$$

Gabriel's Horn

If you rotate

$$y = \frac{1}{x} \quad \text{where} \quad 1 \leq x < \infty$$

around the xaxis, then the resulting surface is called gabriel's horn.



We can find the volume inside by using the disk method

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx$$

$$= \pi \int_1^\infty \frac{1}{x^2} dx$$

$$= \pi (1)$$

$$= \pi$$

The surface area is

$$A = 2\pi \int_1^\infty f(x)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{\frac{1}{x^4}\left(x^4+1\right)}\, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{\frac{1}{x^4}}\sqrt{x^4+1}\, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\frac{1}{x^2}\sqrt{x^4+1}\, dx$$

$$= 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$$

$$\geq 2\pi \int_1^\infty \frac{\sqrt{x^4}}{x^3}\, dx$$

$$= 2\pi \int_1^\infty \frac{x^2}{x^3}\, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\, dx$$

this last integral diverges $(p=1)$

so $\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$ diverges.

Note $2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx = \infty$

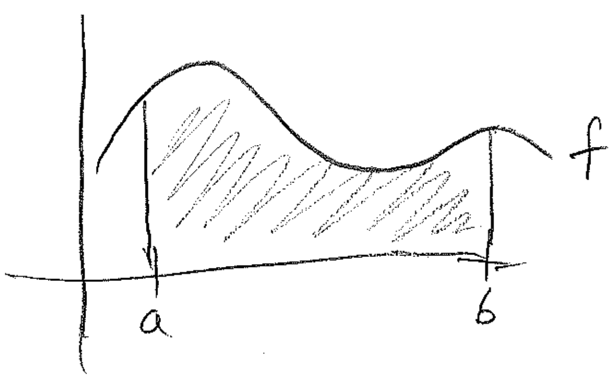
So gabriel's horn has a finite volume but an infinite surface area.

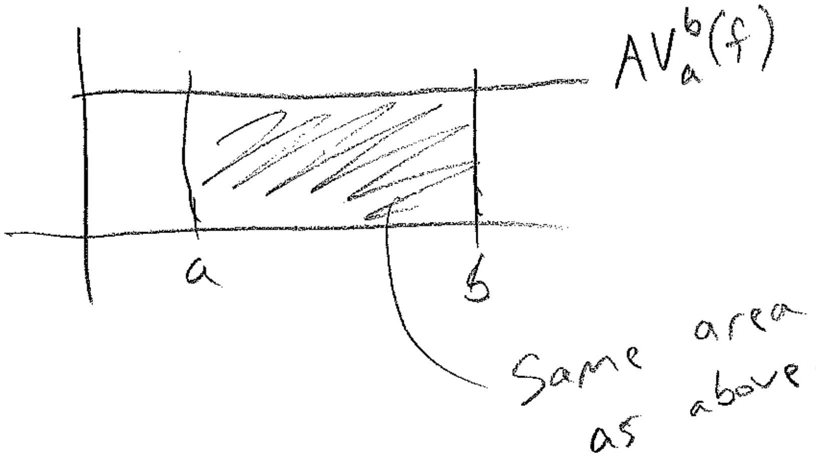
The Average Value of a Function

We define

$$AV_a^b(f) := \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We call $AV_a^b(f)$ the average value of f from a to b.





ex Let $f(x) = -x^2 + 4x$.

Find $AV_0^4(f)$.

$AV_0^4(f)$

$$= \frac{1}{4-0} \int_0^4 (-x^2 + 4x)\,dx$$

$$= \frac{1}{4}\left[-\frac{1}{3}x^3 + 2x^2\right]_0^4$$

$$= \frac{1}{4}\left(\left[-\frac{1}{3}(4)^3 + 2(4)^2\right] - \left[-\frac{1}{3}(0)^3 + 2(0)^2\right]\right)$$

$$= \frac{1}{4}\left(\left[-\frac{1}{3}(64) + 32\right] - \left[0 + 0\right]\right)$$

$$= \frac{1}{4}\left(-\frac{64}{3} + \frac{96}{3}\right)$$

$$= \frac{1}{4}\left(\frac{32}{3}\right)$$

$$= \boxed{\frac{8}{3}}$$

Work

If you apply a constant force of F to an object and the object moves along a line a distance of d, then we define the work done to be

$$W := Fd$$

ex I push an object with a force of 5N. The object has moved 2 meters. Find the work done.

$$W = (5N)(2m)$$
$$= \boxed{10\,Nm}$$

If you apply a varying force, say $F(x)$ when the object is at x, to an object and the object moves along the xaxis from a to b, then we define the work done to be

$$W := \int_a^b F(x)\,dx$$

ex An object moves along the xaxis from 1 to 3 in meters. If at x, then the object feels a force of $(x^2+2x)\,N$. Find the work done.

$$W = \int_1^3 (x^2 + 2x)\,dx$$

$$= \left[\frac{1}{3}x^3 + x^2\right]_1^3$$

$$= \left[\frac{1}{3}(3)^3 + (3)^2\right] - \left[\frac{1}{3}(1)^3 + (1)^2\right]$$

$$= \left[\frac{1}{3}(27) + 9\right] - \left[\frac{1}{3} + 1\right]$$

$$= [9 + 9] - \left[\frac{4}{3}\right]$$

$$= 18 - \frac{4}{3}$$

$$= \frac{54}{3} - \frac{4}{3}$$

$$= \frac{50}{3} \quad (Nm)$$

Mass Distributions

If a thin bar is placed on the x axis and the mass density at x is $\rho(x)$, then the total mass from a to b is

$$m = \int_a^b \rho(x)\,dx.$$

ex A bar runs

from 0 to 2

with $\rho(x) = x^2+1$ (kg/m).

Find the total mass.

$$m = \int_0^2 (x^2+1)\,dx$$

$$= \left[\tfrac{1}{3}x^3 + x\right]_0^2$$

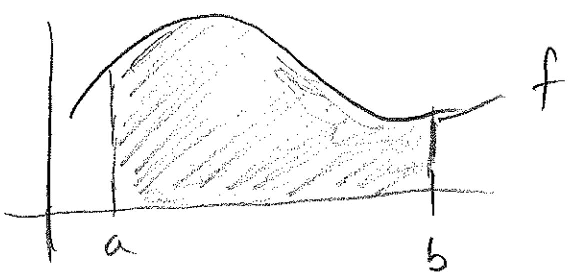
$$= \left[\tfrac{1}{3}(2)^3 + (2)\right] - \left[\tfrac{1}{3}(0)^3 + (0)\right]$$

$$= \left[\frac{8}{3} + 2\right] - \left[0 + 0\right]$$

$$= \frac{8}{3} + \frac{6}{3}$$

$$= \frac{14}{3} \text{ (kg)}$$

If f is a function and a thin flat plate $\mathcal{L}$ (a lamina) with a constant mass density k is made in the shape of the region under f from a to b,



then the total mass of $\mathcal{L}$ is

$$m = k\int_a^b f(x)\,dx.$$

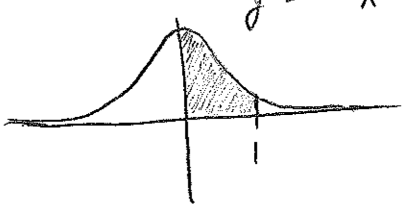
ex Let $f(x) = \frac{1}{x^2+1}$.

Let R be the region under f from 0 to 1.

Let $\mathcal{L}$ be a lamina in the shape of R with a constant mass density of $3\,kg/m^2$.

Find the total mass of $\mathcal{L}$.

$y = \frac{1}{x^2+1}$



$$m = 3\int_0^1 \frac{1}{x^2+1}\,dx$$

$$= 3\Big[\arctan(x)\Big]_0^1$$

$$= 3\Big(\big[\arctan(1)\big] - \big[\arctan(0)\big]\Big)$$

$$= 3\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{3\pi}{4}\,(kg)$$

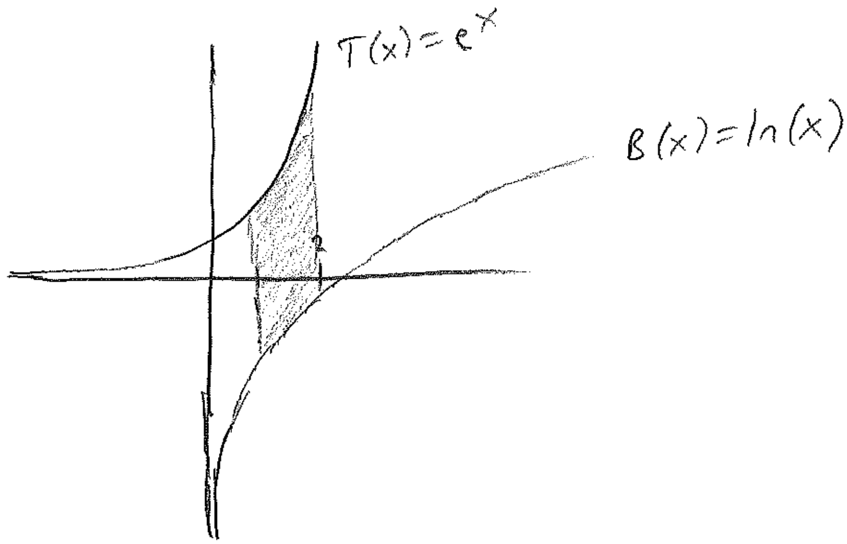
ex Let $T(x) = e^x$

$B(x) = \ln(x)$.

Let R be the region between T and B from 1 to 2.

Let $\mathcal{L}$ be a lamina in the shape of R with a constant mass density of $\frac{1}{2}$ kg/m^2

Find the total mass of $\mathcal{L}$.



$$m = \frac{1}{2}\int_1^2 \left(e^x - \ln(x)\right)dx$$

$$= \frac{1}{2}\left[e^x - \left(x\ln(x) - x\right)\right]_1^2$$

$$= \frac{1}{2}\left(\left[e^2 - (2\ln(2) - 2)\right] - \left[e^1 - (1\ln(1) - 1)\right]\right)$$

$$= \frac{1}{2}\left(\left[e^2 - 2\ln(2) + 2\right] - \left[e - (0 - 1)\right]\right)$$

$$= \frac{1}{2}\left(e^2 - 2\ln(2) + 2 - e - 1\right)$$

$$= \frac{1}{2}\left(e^2 - e - 2\ln(2) + 1\right)$$

$$\approx 2.14\ (kg)$$

(more mass distributions)

Let R be the region between T and B from a to b.

Let $\mathcal{L}$ be a lamina in the shape of R with constant density k.

We define

$$M_x := k\int_a^b x\left(T(x)-B(x)\right)dx$$

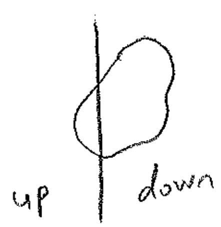
$$M_y := k\int_a^b \frac{1}{2}\left(T(x)^2-B(x)^2\right)dx$$

We call M_x the moment of $\mathcal{L}$ about the yaxis.

We call M_y the moment of $\mathcal{L}$ about the xaxis.

Note that M_x is supposed to be a measure of "how much $\mathcal{L}$ wants to rotate around the y axis".

If $M_x > 0$, then $\mathcal{L}$ wants to rotate down on the right of the y axis and up on the left of the y axis.

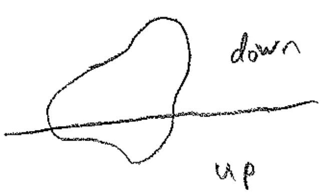


Here, $M_x > 0$.

So if $M_x > 0$, then "$\mathcal{L}$ falls where $x > 0$"
If $M_x < 0$, then "$\mathcal{L}$ falls where $x < 0$"

Note that M_y is supposed to be a measure of "how much $\mathcal{L}$ wants to rotate around the xaxis".

If $M_y > 0$, then $\mathcal{L}$ wants to rotate down above the xaxis and up the xaxis below the xaxis.



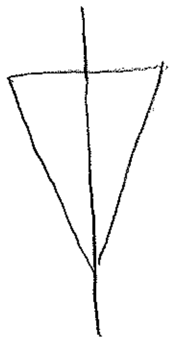
Here $M_y > 0$.

So if $M_y > 0$, then "$\mathcal{L}$ falls where $y > 0$"
If $M_y < 0$, then "$\mathcal{L}$ falls where $y < 0$"

If $M_x = 0$,

then $\mathcal{L}$ will balance

on the y axis.



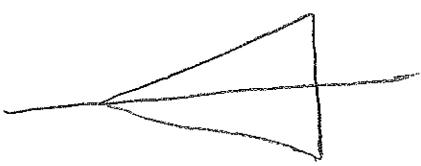
Here,

$M_x = 0$.

If $M_y = 0$,

then $\mathcal{L}$ will balance

on the x axis.



Here,

$M_y = 0$.

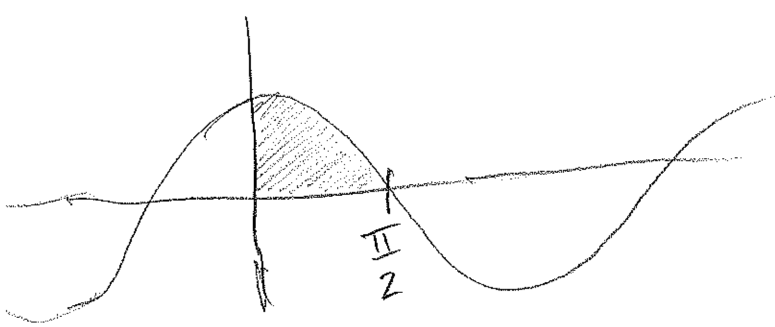
ex Let $T(x) = \cos(x)$.

$B(x) = 0$.

Let R be the region between T and B from 0 to $\frac{\pi}{2}$.

Let $\mathcal{L}$ be a lamina in the shape of R with constant mass density $5\,kg/m^2$.

Find M_x and M_y.



M_x

$$= 5\int_0^{\pi/2} x\left(\cos(x) - 0\right)dx$$

$$= 5\int_0^{\pi/2} x\cos(x)\,dx$$

$$= 5\left[x\sin(x) + \cos(x)\right]_0^{\pi/2}$$

$$= 5\left(\left[\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right] - \left[0\sin(0) + \cos(0)\right]\right)$$

$$= 5\left(\left[\frac{\pi}{2} + 0\right] - \left[0 + 1\right]\right)$$

$$= 5\left(\frac{\pi}{2} - 1\right)$$

M_y

$$= 5\int_0^{\pi/2} \frac{1}{2}\left((\cos(x))^2 - (0)^2\right) dx$$

$$= \frac{5}{2}\int_0^{\pi/2} \cos^2(x)\, dx$$

$$= \frac{5}{2}\int_0^{\pi/2} \frac{1}{2}\left(1 + \cos(2x)\right) dx$$

$$= \frac{5}{4}\int_0^{\pi/2} \left(1 + \cos(2x)\right) dx$$

$$= \frac{5}{4}\left[x + \frac{1}{2}\sin(2x)\right]_0^{\pi/2}$$

$$= \frac{5}{4}\left(\left[\frac{\pi}{2} + \frac{1}{2}\sin\left(2\frac{\pi}{2}\right)\right] - \left[0 + \frac{1}{2}\sin(2\cdot 0)\right]\right)$$

$$= \frac{5}{4}\left(\left[\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right] - \left[0 + \frac{1}{2}\sin(0)\right]\right)$$

$$= \frac{5}{4}\left(\left[\frac{\pi}{2} + 0\right] - \left[0 + 0\right]\right)$$

$$= \frac{5}{4}\left(\frac{\pi}{2}\right)$$

$$= \frac{5\pi}{8}$$

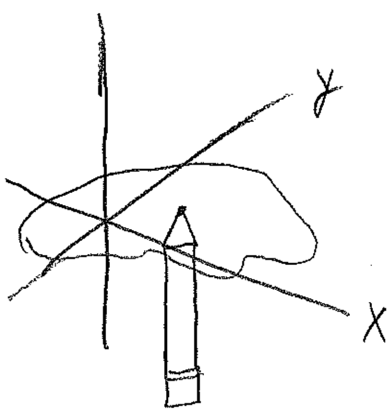
We define

$$\bar{x} := \frac{M_x}{m}$$

$$\bar{y} := \frac{M_y}{m}$$

We call $(\bar{x}, \bar{y})$ the center of mass of $\mathcal{L}$.

You could balance $\mathcal{L}$ on a pencil at $(\bar{x}, \bar{y})$.



ex Let $T(x) = \cos(x)$

$B(x) = 0$

from 0 to $\frac{\pi}{2}$.

Constant density of $5\ kg/m^2$.

Find the center of mass.

$$m = k\int_a^b (T(x) - B(x))\,dx$$

$$= 5\int_0^{\pi/2} (\cos(x) - 0)\,dx$$

$$= 5\int_0^{\pi/2} \cos(x)\,dx$$

$$= 5\left[\sin(x)\right]_0^{\pi/2}$$

$$= 5\left(\left[\sin\left(\tfrac{\pi}{2}\right)\right] - \left[\sin(0)\right]\right)$$

$$= 5([1] - [0])$$

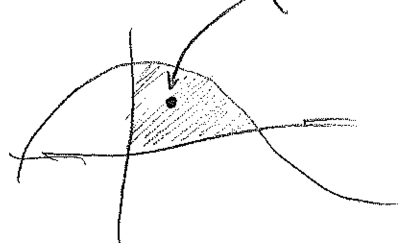
$$= 5 \quad (kg)$$

now $\bar{x}$ $\qquad$ $\bar{y}$

$$\bar{x} = \frac{M_x}{m} = \frac{5\left(\frac{\pi}{2} - 1\right)}{5} = \frac{\pi}{2} - 1 \qquad \bar{y} = \frac{M_y}{m} = \frac{\frac{5\pi}{8}}{5} = \frac{\pi}{8}$$

so $(\bar{x}, \bar{y}) = \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$

$\left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$



Probability Density Functions

If we have a set of outcomes, Ω, then a function

$$f : \Omega \to \mathbb{R}$$

is called a random variable.

ex I toss a coin.

$$\Omega = \{H, T\}$$

If we define a function, f, by

$$f(H) := 1, \ f(T) := -1,$$

then f is a random variable.

ex We measure the height of a tree.

$$\Omega = [0, \infty)$$

If we define

$$f : \Omega \to \mathbb{R}$$

by $f(w) = w$.

then f is a random variable.

Let f be a random variable.

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$.

We say that φ is a probability density function for f if this is true:

$$P(f \leq a) = \int_{-\infty}^{a} \varphi(x)\,dx$$

So φ is a probability density function for f if probabilities involving f are areas under the graph of φ.

If φ is a probability density function for f, then

$$P(f = a) = 0$$

$$P(f \le a) = \int_{-\infty}^{a} \varphi(x)\,dx$$

$$P(a \le f) = \int_{a}^{\infty} \varphi(x)\,dx$$

$$P(a \le f \le b) = \int_{a}^{b} \varphi(x)\,dx$$

$$P(f \in \mathbb{R}) = \int_{-\infty}^{\infty} \varphi(x)\,dx$$

for all x, $\varphi(x) \ge 0$

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$$

ex

Let $\varphi(x) = \begin{cases} .006x(10-x), & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$

Let f be a random variable with density function φ.

Find $P(4 \le f \le 8)$

$P(4 \le f \le 8)$

$= \int_4^8 \varphi(x)\,dx$

$= \int_4^8 .006x(10-x)\,dx$

$= \int_4^8 (.06x - .006x^2)\,dx$

$= \left[\frac{.06}{2}x^2 - \frac{.006}{3}x^3\right]_4^8$

$= \left[\frac{.06}{2}(8)^2 - \frac{.006}{3}(8)^3\right] - \left[\frac{.06}{2}(4)^2 - \frac{.006}{3}(4)^3\right]$

$\approx .544$

ex Let $\varphi(x) = \begin{cases} \frac{3}{64} x\sqrt{16-x^2}, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$

Let f be a random variable with density φ.

(a) Find $P(1 \le f \le 3)$

$P(1 \le f \le 3)$

$= \int_1^3 \varphi(x)\,dx$

$= \int_1^3 \frac{3}{64} x\sqrt{16-x^2}\,dx$

let $u = 16 - x^2$

so $\frac{du}{dx} = -2x$

so $dx = \frac{du}{-2x}$

$\int x\sqrt{16-x^2}\,dx$

$= \int x\sqrt{u}\,\frac{du}{-2x}$

$= -\frac{1}{2}\int u^{1/2}\,du$

$= -\frac{1}{2}\frac{2}{3}u^{3/2} + C$

$= -\frac{1}{3}(16-x^2)^{3/2} + C_1$

now

$$\int_1^3 \frac{3}{64} x \sqrt{16-x^2}\, dx$$

$$= \frac{3}{64}\left[-\frac{1}{3}\left(16-x^2\right)^{3/2}\right]_1^3$$

$$= \frac{3}{64}\left(\left[-\frac{1}{3}\left(16-(3)^2\right)^{3/2}\right] - \left[-\frac{1}{3}\left(16-(1)^2\right)^{3/2}\right]\right)$$

$$= \frac{3}{64}\left(-\frac{1}{3}\left(16-9\right)^{3/2} + \frac{1}{3}\left(16-1\right)^{3/2}\right)$$

$$= \frac{3}{64}\left(\frac{1}{3}15^{3/2} - \frac{1}{3}7^{3/2}\right)$$

$$= \frac{1}{64}\left(15^{3/2} - 7^{3/2}\right)$$

$$\approx .6183$$

(b) Find $P(f > 3)$.

$$P(f>3)$$

$$= \int_3^{\infty} \varphi(x)\,dx$$

$$= \int_3^4 \frac{3}{64} x \sqrt{16-x^2}\,dx$$

$$= \frac{3}{64}\left[-\frac{1}{3}(16-x^2)^{3/2}\right]_3^4$$

$$= \frac{3}{64}\left(\left[-\frac{1}{3}(16-4^2)^{3/2}\right] - \left[-\frac{1}{3}(16-3^2)^{3/2}\right]\right)$$

$$= \frac{3}{64}\left(\left[-\frac{1}{3}(0)^{3/2}\right] - \left[-\frac{1}{3}(16-9)^{3/2}\right]\right)$$

$$= \frac{3}{64}\left(\frac{1}{3} 7^{3/2}\right)$$

$$= \frac{1}{64}\left(7^{3/2}\right)$$

$$\approx .2893$$

(c) Find $P(f<1)$

$$P(f<1)$$

$$= \int_{-\infty}^{1} \varphi(x)\,dx$$

$$= \int_0^1 \frac{3}{64} x \sqrt{16-x^2}\,dx$$

$$= \frac{3}{64}\left[-\frac{1}{3}(16-x^2)^{3/2}\right]_0^1$$

$$= \frac{3}{64}\left(\left[-\frac{1}{3}(16-(1)^2)^{3/2}\right]-\left[-\frac{1}{3}(16-0^2)^{3/2}\right]\right)$$

$$= \frac{3}{64}\left(-\frac{1}{3}(15)^{3/2}+\frac{1}{3}(16)^{3/2}\right)$$

$$= \frac{1}{64}\left(16^{3/2}-15^{3/2}\right)$$

$$\simeq .0922$$

Note: $.6183 + .2893 + .0922 = .998$

$$\simeq 1$$

ex Let

$$\varphi(x) = \begin{cases} c x \sqrt{r^2 - x^2}, & 0 \le x \le r \\ 0, & \text{otherwise} \end{cases}$$

Here r is given.

Find c that makes φ a density function.

We need $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} \varphi(x)\,dx$$

$$= \int_0^r c x \sqrt{r^2 - x^2}\,dx$$

let $u = r^2 - x^2$

so $\frac{du}{dx} = -2x$

so $dx = \frac{du}{-2x}$

$$\int c x \sqrt{r^2 - x^2}\,dx$$

$$= c \int x \sqrt{u} \frac{du}{-2x}$$

$$= -\frac{c}{2} \int u^{1/2}\,du$$

$$= -\frac{c}{2}\,\frac{2}{3}u^{3/2} + C$$

$$= -\frac{c}{3}(r^2 - x^2)^{3/2} + C$$

now

$$\int_0^r cx\sqrt{r^2 - x^2}\,dx$$

$$= \left[-\frac{c}{3}(r^2 - x^2)^{3/2}\right]_0^r$$

$$= \left[-\frac{c}{3}(r^2 - r^2)^{3/2}\right] - \left[-\frac{c}{3}(r^2 - 0^2)^{3/2}\right]$$

$$= \frac{c}{3}(r^2)^{3/2}$$

$$= \frac{c}{3}r^3$$

If $\frac{c}{3}r^3 = 1$

$\rightarrow cr^3 = 3$

$\rightarrow \boxed{c = \frac{3}{r^3}}$

Note if $r = 4$,

then $c = \frac{3}{64}$.

Consumer Surplus and Producer Surplus

If x is the # of items produced, then we will write $\delta(x)$ for the price that consumers are willing to pay for a single item and $\sigma(x)$ for the price that producers are willing to charge for a single item.

We call δ the demand function and σ the supply function.

ex A company made 1,000 hats. Customers are willing to pay \$5 for a hat. The company is willing to sell each hat for \$15.

So $\delta(1{,}000) = 5$ and $\sigma(1{,}000) = 15$

Assume that

δ is decreasing

and σ is increasing.

If items are sold

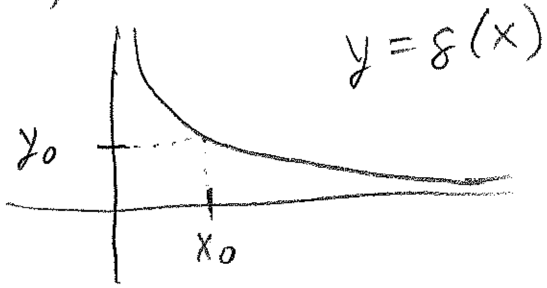
at a price of y_0

for a single item,

then there is a

# of items produced,

x_0, where $\delta(x_0) = y_0$.

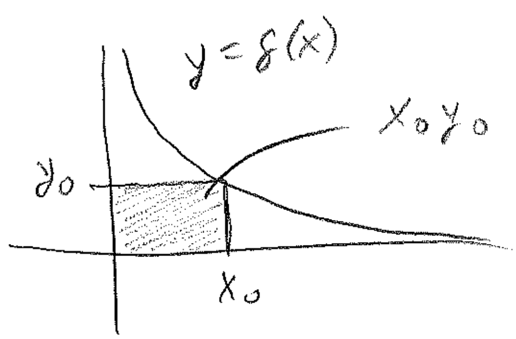


If the price is y_0,

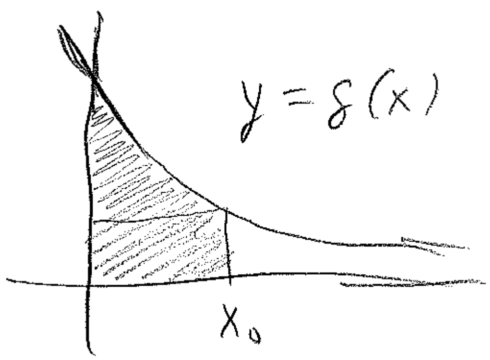
then how much money

will consumers spend?

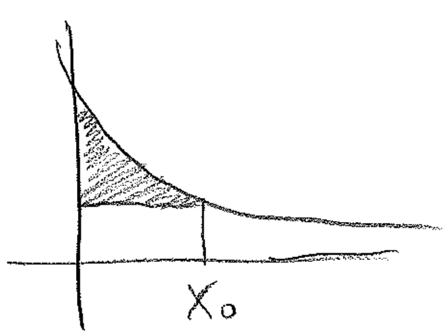
The answer is $x_0 y_0$.



The area under g from 0 to x_0 can be thought of as the amount consumers would have been willing to spend at lower production levels than x_0.



So the amount the consumers are keeping to themselves is this area



This area is called the consumer's surplus.

If $\delta(x_0) = y_0$,
then we define

$$CS(x_0) := \int_0^{x_0} \delta(x)\,dx - x_0 y_0 .$$

We call $CS(x_0)$
the consumer surplus
at production level x_0.

We can say similar
things for producers.

If $\sigma(x_0) = y_0$,
then we define

$$PS(x_0) = x_0 y_0 - \int_0^{x_0} \sigma(x)\,dx$$

We call $PS(x_0)$
the producer's surplus
at production level x_0.

ex Let

$$S(x) = \frac{500}{x+10} - 3$$

Note that $S(40) = 7$.

Find $CS(40)$.

$CS(40)$

$= \int_0^{40} \left(\frac{500}{x+10} - 3 \right) dx - (40)(7)$

$= \left[500 \ln(x+10) - 3x \right]_0^{40} - 280$

$= \left[500 \ln(40+10) - 3(40) \right] - \left[500 \ln(0+10) - 3(0) \right] - 280$

$= 500 \ln(50) - 120 - 500 \ln(10) - 280$

$= 500 \left(\ln(50) - \ln(10) \right) - 400$

$= 500 \ln\left(\frac{50}{10} \right) - 400$

$= 500 \ln(5) - 400$

$\approx \boxed{404.72}$

ex Let $\sigma(x) = 11e^{.01x} + 17$.

Note that $\sigma(7) \approx 28.80$

Find $PS(7)$.

$PS(7)$

$= (7)(28.80) - \int_0^7 (11e^{.01x} + 17)\,dx$

$= 201.6 - \left[\frac{11}{.01} e^{.01x} + 17x\right]_0^7$

$= 201.6 - \left(\left[\frac{11}{.01} e^{.01(7)} + 17(7)\right] - \left[\frac{11}{.01} e^{.01(0)} + 17(0)\right]\right)$

$= 201.6 - \left(\left[1100e^{.07} + 119\right] - \left[1100 + 0\right]\right)$

$= 201.6 - \left[1100e^{.07} + 119\right] + \left[1100\right]$

$= 201.6 - 1100e^{.07} - 119 + 1100$

$= 1182.6 - 1100e^{.07}$

$\approx \boxed{2.841}$

\$ Sequences

Infinite Sequences

Definition

Let ψ be a function.
We say that ψ is an infinite sequence
iff $\text{Domain}(\psi) = \mathbb{N}$.

DONE

Example

If $\psi : \mathbb{N} \longrightarrow \mathbb{R}$ by $\psi(n) := \sqrt{n}$,
then ψ is an (infinite) sequence.
Here $\psi = \{(1, \sqrt{1}), (2, \sqrt{2}), (3, \sqrt{3}), ...\}$.

DONE

Definition

If ψ is an infinite sequence,
then we define $\text{Term}_n(\psi) := \psi(n)$.

DONE

Example

If $\psi : \mathbb{N} \longrightarrow \mathbb{R}$ by $\psi(n) := \sqrt{n}$,
then $\text{Term}_1(\psi) = 1$, $\text{Term}_2(\psi) = \sqrt{2}$, $\text{Term}_9(\psi) = 3$.

DONE

So if ψ is an infinite sequence,
then the outputs are called the terms.

Definition

Let ψ be an infinite sequence.
Let X be a set.
We say that ψ is within X
iff $\text{Range}(\psi) \subseteq X$.

DONE

Example

If $\psi : \mathbb{N} \longrightarrow \mathbb{R}$ by $\psi(n) := \sqrt{n}$,
then ψ is within $\mathbb{R}$.

DONE

We also say that ψ is built from X
or that ψ is a sequence of members from X.

Definition

If X is a set, then we define
$\text{Sequences}_\infty(X)$
$:= \{\psi \text{ s.t. } \psi \text{ is an infinite sequence within } X\}$.
$X^\infty := \text{Sequences}_\infty(X)$.

DONE

Example

$\{0, 1\}^\infty$ is the set of
all infinite sequences within $\{0, 1\}$.
Here is one such sequence
$\{(1, 1), (2, 1), (3, 0), (4, 1), (5, 0), (6, 0), ...\}$

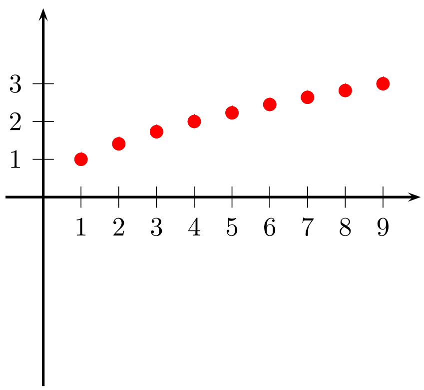
DONE

So we define X^∞
to be the set of all infinite sequences within X.

If $\psi \in \mathbb{R}^\infty$,
then we can graph ψ.

Example

If $\psi : \mathbb{N} \longrightarrow \mathbb{R}$ by $\psi(n) = \sqrt{n}$,
then we can graph ψ.

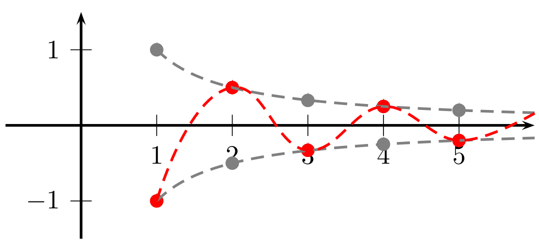


Note that ψ is only defined at 1,2,3,...
so the graph consists of isolated points.

DONE

Example

If $\psi : \mathbb{N} \longrightarrow \mathbb{R}$ by $\psi(n) = \frac{(-1)^n}{n}$,
then we can graph ψ.



DONE

Definition

Let $\psi \in \mathbb{R}^\infty$.
We say that ψ is increasing iff
the graph of ψ is going up
as you look from left to right.

DONE

Definition

Let $\psi \in \mathbb{R}^\infty$.
We say that ψ is decreasing iff
the graph of ψ is going down
as you look from left to right.

DONE

Definition

Let $\psi \in \mathbb{R}^\infty$.
We say that ψ is constant iff
the graph of ψ stays level
as you look from left to right.

DONE

So
ψ is increasing iff
if $a < b$, then $\psi(a) < \psi(b)$.
ψ is decreasing iff
if $a < b$, then $\psi(a) > \psi(b)$.
ψ is constant iff
if $a < b$, then $\psi(a) = \psi(b)$.

A subsequence of a sequence
should be a new sequence that is
"part of the original sequence".
So a subsequence should be
a new sequence you get by
"dropping some terms of the original sequence
without disturbing the order of the terms".

One way to specify which terms to drop
is to specify which terms to keep.
If we have an increasing sequence $I \in \mathbb{N}^\infty$,
then we can use I to specify which terms to keep:
if Ψ is the original sequence,
then keep $\mathrm{Term}_{I(1)}(\Psi), \mathrm{Term}_{I(2)}(\Psi), \mathrm{Term}_{I(3)}(\Psi), \ldots$.

Definition

Let ψ and Ψ be infinite sequences.
We say that ψ is a subsequence of Ψ iff
there is an increasing sequence $I \in \mathbb{N}^\infty$
s.t. $\psi(n) = \Psi(I(n))$

DONE

Example

If $\Psi(n) = \frac{1}{n}$
and $\psi(n) = \frac{1}{n^2}$,
then ψ is a subsequence of Ψ
because if $I(n) := n^2$,
then I is increasing and $\psi(n) = \Psi(I(n))$.

The terms of the original sequence Ψ are
$\mathrm{Term}_1(\Psi) = 1, \mathrm{Term}_2(\Psi) = \frac{1}{2}$,
$\mathrm{Term}_3(\Psi) = \frac{1}{3}, \mathrm{Term}_4(\Psi) = \frac{1}{4}, \ldots$
The terms of the new sequence ψ are
$\mathrm{Term}_1(\psi) = 1, \mathrm{Term}_2(\psi) = \frac{1}{4}$,
$\mathrm{Term}_3(\psi) = \frac{1}{9}, \mathrm{Term}_4(\psi) = \frac{1}{16}, \ldots$
So we can get the new sequence ψ
from the original sequence Ψ
by keeping $\mathrm{Term}_{I(1)}(\Psi), \mathrm{Term}_{I(2)}(\Psi)$,
$\mathrm{Term}_{I(3)}(\Psi), \mathrm{Term}_{I(4)}(\Psi), \ldots$

DONE

Note that a sequence
is a subsequence of itself;
just take $I(n) = n$.

Notation for Sequences

If γ is an infinite sequence, then we define

$$(\gamma(1), \gamma(2), \gamma(3), \ldots)$$

$$:= \{(1, \gamma(1)), (2, \gamma(2)), (3, \gamma(3)), \ldots\}$$

ex

$$(\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots)$$

$$= \{(1, \sqrt{1}), (2, \sqrt{2}), (3, \sqrt{3}), \ldots\}$$

so $(\gamma(1), \gamma(2), \gamma(3), \ldots) := \gamma$.

If f is a function,
then we define

$$\left(f(x)\right)_{x=A}^{\infty}$$

$$:= \left(f(A), f(A+1), f(A+2), \ldots\right)$$

ex $\left(x^2\right)_{x=3}^{\infty}$

$$= \left(9, 16, 25, 36, 49, \ldots\right)$$

In this notation,
the symbol x is
called the index.
Any symbol that
is NOT already
in use can be
used as the index.

ex $\left(n^2\right)_{n=3}^{\infty} = \left(9, 16, 25, 36, 49, \ldots\right)$

$$\left(\frac{(-1)^k}{k}\right)_{k=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\right)$$

$$\left(3^l - 2^l\right)_{l=-1}^{\infty} = \left(-\frac{1}{6}, 0, 1, 5, 19, \ldots\right)$$

If we define

$$x_n := f(n),$$

then $\left(f(x)\right)_{x=A}^{\infty} = \left(x_n\right)_{n=A}^{\infty}$

and $\left(x_n\right)_{n=A}^{\infty} = \left(x_A, x_{A+1}, x_{A+2}, \cdots\right)$

So $\left(x_n\right)_{n=A}^{\infty} := \Psi$

where $\Psi : \mathbb{N} \to \mathbb{X}$

by $\Psi(n) = x_{A+n-1}$.

ex if $\Psi = \left(\text{cat}, \text{dog}, \text{bird}, \text{dog}, \text{dog}, \text{bird}, \text{cat}, \ldots\right)$

then $\Psi : \mathbb{N} \to \{\text{dog}, \text{cat}, \text{bird}\}$

FACT:

$$(x_n)_{n=A}^{\infty} = \left(x_{n+(A-B)}\right)_{n=B}^{\infty}$$

ex $\quad (n^2)_{n=3}^{\infty}$

$$= \left((n-4)^2\right)_{n=7}^{\infty}$$

$$= \left((n+3)^2\right)_{n=0}^{\infty}$$

Say I have $(x_n)_{n=A}^{\infty}$

and I want to start with B.

Define a new variable, k,

such that when $n=A$, then $k=B$.

So let $k := n - A + B$

So $n = k + (A-B)$

so $(x_n)_{n=A}^{\infty}$

$$= \left(x_{k+(A-B)}\right)_{k+(A-B)=A}^{\infty}$$

$$= \left(x_{k+(A-B)}\right)_{k=B}^{\infty} = \left(x_{n+(A-B)}\right)_{n=B}^{\infty}$$

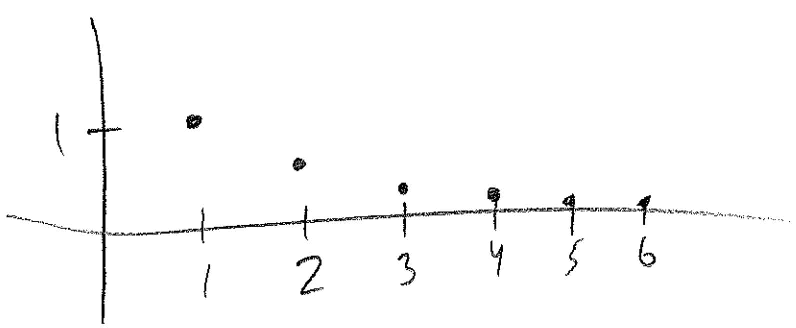
Limits of Sequences

ex Note that

$$\left(\frac{1}{n}\right)_{n=1}^{\infty}$$

$$= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots\right)$$

You can see that if n approaches ∞, then $\frac{1}{n}$ approaches 0.



We express this by writing

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$$

We say that

$$\lim_{n \to \infty} x_n = a \quad \text{iff}$$

if n approaches ∞,

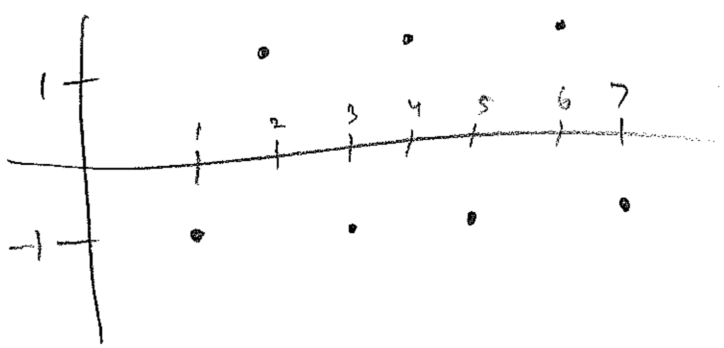
then x_n approaches a.

ex Note that

$$\left((-1)^n\right)_{n=1}^{\infty}$$

$$= (-1, 1, -1, 1, -1, 1, -1, \dots)$$

You can see that if n approaches ∞, then $(-1)^n$ does NOT approach a real number.



We express this by writing

$$\lim_{n\to\infty} (-1)^n \quad \text{DNE}$$

We say that

$\lim_{n\to\infty} x_n$ exists iff

if n approaches ∞,

then x_n approaches

a real #.

We say that

$\lim_{n\to\infty} x_n$ DNE iff

if n approaches ∞,

then x_n does NOT

approach a real #.

We say that $(x_n)_{n=1}^{\infty}$

converges iff $\lim_{n\to\infty} x_n$ exists.

We say that $(x_n)_{n=1}^{\infty}$

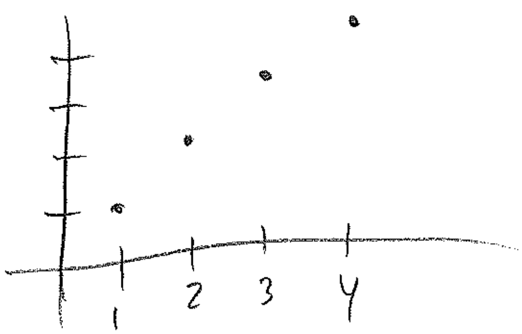
diverges iff $\lim_{n\to\infty} x_n$ DNE.

ex
Note that

$$(n)_{n=1}^{\infty}$$

$$= (1, 2, 3, 4, 5, \ldots)$$

You can see that if n approaches ∞, then x_n approaches ∞.



We express this by writing

$$\lim_{n \to \infty} n = \infty$$

also, $\lim_{n \to \infty} n$ DNE

We say that

$$\lim_{n\to\infty} x_n = \infty \quad \text{iff}$$

if n approaches ∞,

then x_n approaches ∞.

We say that

$$\lim_{n\to\infty} x_n = -\infty \quad \text{iff}$$

if n approaches ∞,

then x_n approaches $-\infty$.

Properties of Limits of Sequences

$$\lim_{n\to\infty} c = c$$

ex $\lim_{n\to\infty} 7 = 7$

$$\lim_{n\to\infty} (c\,x_n) = c \lim_{n\to\infty} (x_n)$$

ex If $\lim_{n\to\infty} x_n = 3$,

then $\lim_{n\to\infty} 2x_n$

$= 2 \lim_{n\to\infty} (x_n)$

$= 2(3)$

$= 6.$

$$\lim_{n\to\infty} (x_n + y_n)$$

$$= \lim_{n\to\infty} (x_n) + \lim_{n\to\infty} (y_n)$$

ex if $\lim_{n\to\infty} x_n = 3$

and $\lim_{n\to\infty} y_n = 5$,

then $\lim_{n\to\infty} (x_n + y_n)$

$$= \lim_{n\to\infty} (x_n) + \lim_{n\to\infty} (y_n)$$

$$= 3 + 5$$

$$= 8$$

$$\lim_{n\to\infty} (x_n y_n)$$

$$= \lim_{n\to\infty} (x_n) \lim_{n\to\infty} (y_n)$$

ex if $\lim_{n\to\infty} x_n = 3$

and $\lim_{n\to\infty} y_n = 5$,

then $\lim_{n\to\infty} (x_n y_n)$

$$= \lim_{n\to\infty} (x_n) \lim_{n\to\infty} (y_n)$$

$$= (3)(5)$$

$$= 15$$

$$\lim_{n\to\infty} \frac{x_n}{y_n}$$

$$= \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

as long as $\lim_{n\to\infty} y_n \neq 0$.

ex if $\lim_{n\to\infty} x_n = 3$

and $\lim_{n\to\infty} y_n = 5$

then $\lim_{n\to\infty} \frac{x_n}{y_n}$

$$= \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

$$= \frac{3}{5}$$

15I

if $\lim_{n \to \infty} y_n = 0$

and $\lim_{n \to \infty} x_n \neq 0$,

then $\lim_{n \to \infty} \frac{x_n}{y_n}$ DNE.

if $\lim_{n \to \infty} x_n = 3$

and $\lim_{n \to \infty} y_n = 0$,

then $\lim_{n \to \infty} \frac{x_n}{y_n}$ DNE

$$\lim_{n\to\infty} (x_n)^p = \left(\lim_{n\to\infty} x_n\right)^p$$

ex if $\lim_{n\to\infty} x_n = 3$

then $\lim_{n\to\infty} (x_n)^2$

$$= \left(\lim_{n\to\infty} x_n\right)^2$$

$$= 3^2$$

$$= 9$$

$$\lim_{n\to\infty}\left(\frac{1}{x_n}\right) = \frac{1}{\lim_{n\to\infty}(x_n)}$$ 17 I

as long as $\lim_{n\to\infty} x_n \neq 0$.

ex if $\lim_{n\to\infty} x_n = 3$,

then $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{3}$.

Limits of Rational Sequences

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} n = \infty$$

Let T and B be polynomial functions.

If $\mathrm{Deg}(T) < \mathrm{Deg}(B)$, then $\displaystyle\lim_{n\to\infty} \frac{T(n)}{B(n)} = 0$.

If $\mathrm{Deg}(T) = \mathrm{Deg}(B)$, then $\displaystyle\lim_{n\to\infty} \frac{T(n)}{B(n)} = \frac{LC(T)}{LC(B)}$.

If $\mathrm{Deg}(T) > \mathrm{Deg}(B)$, then $\displaystyle\lim_{n\to\infty} \frac{T(n)}{B(n)}$ DNE.

ex

$$\lim_{n\to\infty} \frac{3n^2+4n+5}{6n^3+7n^2+8n+9}$$

$$= 0$$

---

ex

$$\lim_{n\to\infty} \frac{3n^2+4n+5}{6n^2+7n+8}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}.$$

---

ex

$$\lim_{n\to\infty} \frac{3n^2+4n+5}{6n+7}$$

DNE

---

ex

$$\lim_{n\to\infty} \frac{n}{n+1}$$

$$= 1$$

If $b\,\mathrm{Deg}(T) < t\,\mathrm{Deg}(B)$,

then $$\lim_{n\to\infty} \frac{\sqrt[t]{T(n)}}{\sqrt[b]{B(n)}} = 0$$

If $b\,\mathrm{Deg}(T) = t\,\mathrm{Deg}(B)$,

then $$\lim_{n\to\infty} \frac{\sqrt[t]{T(n)}}{\sqrt[b]{B(n)}} = \frac{\sqrt[t]{LC(T)}}{\sqrt[b]{LC(B)}}$$

If $b\,\mathrm{Deg}(T) > t\,\mathrm{Deg}(B)$,

then $$\lim_{n\to\infty} \frac{\sqrt[t]{T(n)}}{\sqrt[b]{B(n)}}$$ DNE.

ex $$\lim_{n\to\infty} \frac{\sqrt[4]{2n^3-9n}}{\sqrt[5]{7n^8+n^3}}$$

$$= 0$$

ex $$\lim_{n\to\infty} \frac{\sqrt[5]{2n^3+1}}{\sqrt[10]{7n^6-4n}}$$

$$= \frac{\sqrt[5]{2}}{\sqrt[10]{7}}$$

ex $$\lim_{n\to\infty} \frac{\sqrt[5]{2n^8-9n}}{\sqrt[4]{7n^3+n^2}}$$

DNE

If $f; \mathbb{R} \to \mathbb{R}$

and $f(n) = x_n$,

then $\lim_{n \to \infty} x_n = \lim_{x \to \infty} f(x)$.

ex Find $\lim_{n \to \infty} \frac{\ln(n)}{n}$.

$$= \lim_{x \to \infty} \frac{\ln(x)}{x}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1} \quad \text{l'hopital}$$

$$= \lim_{x \to \infty} \frac{1}{x}$$

$$= 0$$

If $(w_n)_{n=1}^{\infty}$

is a subsequence of $(x_n)_{n=1}^{\infty}$

and $\lim\limits_{n \to \infty} x_n = a$,

then $\lim\limits_{n \to \infty} w_n = a$.

ex $\lim\limits_{n \to \infty} (-1)^n$ DNE

$((-1)^n)_{n=1}^{\infty} = (-1, 1, -1, 1, -1, 1, -1, \ldots)$

$(1, 1, 1, 1, \ldots)$ is a subseq.

$(-1, -1, -1, -1, \ldots)$ is a subseq.

$(1, 1, 1, \ldots) \to 1$

$(-1, -1, -1, \ldots) \to -1$

So $((-1)^n)_{n=1}^{\infty}$ diverges

(more limits of sequences)

If $x_n \le y_n$,

then $\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$.

If $x_n \le y_n \le z_n$

and $\lim_{n\to\infty} x_n = a$

and $\lim_{n\to\infty} z_n = a$,

then $\lim_{n\to\infty} y_n = a$.

This is called the squeeze theorem.

If $\lim_{n\to\infty} |x_n| = 0$,

then $\lim_{n\to\infty} x_n = 0$.

If $\lim_{n\to\infty} x_n = a$,

then $\lim_{n\to\infty} |x_n| = |a|$.

Converse is false.

If $|c|<1$,

then $\lim_{n \to \infty} c^n = 0$.

If $c \in \mathbb{R}$,

then $\lim_{n \to \infty} \frac{c^n}{n!} = 0$.

$$\lim_{n \to \infty} \frac{n!}{n^n} = 0$$

$$\lim_{n \to \infty} \frac{n^n}{n!} = \infty$$

We say that

$(x_n)_{n=1}^{\infty}$ is bounded

if there is an $r \in \mathbb{R}$

such that

if $n \in \mathbb{N}$, then $|x_n| < r$.

ex $\left(\frac{\sin(n)}{n}\right)_{n=1}^{\infty}$ is

bounded because

$$\left|\frac{\sin(n)}{n}\right| \leq 1.$$

$(n)_{n=1}^{\infty}$ is

unbounded.

We say that

$(x_n)_{n=1}^{\infty}$ is

monotone up if

$x_n \leq x_{n+1}$

monotone down if

$x_n \geq x_{n+1}$

monotone if

monotone up or monotone down.

ex

$\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$ is

monotone up

$\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is

monotone down

$(\sin(n))_{n=1}^{\infty}$ is

NOT monotone

If $(x_n)_{n=1}^{\infty}$ converges,

then $(x_n)_{n=1}^{\infty}$ is bounded.

ex $(n)_{n=1}^{\infty}$ is unbounded,

so $(n)_{n=1}^{\infty}$ diverges.

If $(b_n)_{n=1}^{\infty}$ is bounded

and $\lim\limits_{n\to\infty} z_n = 0$,

then $\lim\limits_{n\to\infty} b_n z_n = 0$.

ex $\lim\limits_{n\to\infty} \dfrac{\sin(n)}{n} = 0$

because $(\sin(n))_{n=1}^{\infty}$

is bounded

and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

If $(x_n)_{n=1}^{\infty}$ is bounded and monotone, then $(x_n)_{n=1}^{\infty}$ converges.

This is called the monotone convergence theorem for sequences.

If $(x_n)_{n=1}^{\infty}$ is bounded, then $(x_n)_{n=1}^{\infty}$ has a convergent subsequence.

This is called the bolzano weierstrass theorem.

Infinite Series

If $(x_n)_{n=A}^{\infty} \in \mathbb{R}^{\infty}$,

then we define

$$\sum_{n=A}^{\star} x_n := \left(\sum_{n=A}^{N} x_n \right)_{N=A}^{\infty} .$$

ex $\displaystyle\sum_{n=3}^{\star} \frac{1}{n^2}$

$$= \left(\sum_{n=3}^{N} \frac{1}{n^2} \right)_{N=3}^{\infty}$$

$$= \left(\frac{1}{3^2}, \frac{1}{3^2} + \frac{1}{4^2}, \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}, \ldots \right)$$

We call $\displaystyle\sum_{n=A}^{\star} x_n$

the sequence of partial sums of $(x_n)_{n=A}^{\infty}$.

We define

$$\sum_{n=A}^{\infty} x_n := \lim_{N\to\infty} \sum_{n=A}^{N} x_n$$

So $\sum_{n=A}^{\infty} x_n$ is the limit of the sequence $\sum_{n=A}^{\star} x_n$.

We say that

$\sum_{n=A}^{\infty} x_n$ converges

iff

$\sum_{n=A}^{\star} x_n$ converges

We say that

$\sum_{n=A}^{\infty} x_n$ diverges

if

$\sum_{n=A}^{\star} x_n$ diverges

We define $x_A + x_{A+1} + x_{A+2} + \cdots := \sum_{n=A}^{\infty} x_n$.

ex Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$+ \left[\frac{1}{3} - \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{5}\right]$$

$$+ \dots + \left[\frac{1}{N} - \frac{1}{N+1}\right]$$

$$= 1 - \frac{1}{N+1}$$

So $\lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n(n+1)}$

$$= \lim_{N \to \infty} \left(1 - \frac{1}{N+1} \right)$$

$$= 1$$

So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Telescoping Series

If $\sum_{n=A}^{N} x_n$ can be collapsed into a smaller sum, then we say that $\sum_{n=A}^{\infty} x_n$ is a telescoping series.

ex Find

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1}$$

$$\sum_{n=2}^{N} \frac{2}{n^2-1}$$

$$= \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \left[\frac{1}{1} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{3} - \frac{1}{5}\right]$$

$$+ \left[\frac{1}{4} - \frac{1}{6}\right] + \left[\frac{1}{5} - \frac{1}{7}\right] + \cdots$$

$$+ \left[\frac{1}{N-3} - \frac{1}{N-1}\right] + \left[\frac{1}{N-2} - \frac{1}{N}\right]$$

$$+ \left[\frac{1}{N-1} - \frac{1}{N+1}\right]$$

$$= \frac{1}{1} + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}$$

so $$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{N\to\infty} \sum_{n=2}^{N} \frac{2}{n^2-1} = \boxed{\frac{3}{2}}$$

The Divergence Test

If $\sum_{n=A}^{\infty} x_n$ converges,

then $\lim_{n \to \infty} x_n = 0$.

If $\lim_{n \to \infty} x_n \neq 0$,

then $\sum_{n=A}^{\infty} x_n$ diverges.

---

ex Test $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)$

(determine if it converges
or diverges)

$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)$

$= 1$

$\neq 0$

so $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)$ diverges

$$\sum_{n=A}^{\infty} c \quad \text{diverges if } c \neq 0.$$

ex $\displaystyle\sum_{n=1}^{\infty} 3$ diverges

Note that the divergence test does NOT say that

if $\displaystyle\lim_{n\to\infty} x_n = 0$,

then $\displaystyle\sum_{n=A}^{\infty} x_n$ converges.

ex. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} \ln\left(\frac{n}{n}+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)$$

$$= \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)$$

$$= \ln(1) = 0$$

3K

$$\sum_{n=1}^{N} \ln\left(\frac{n+1}{n}\right)$$

$$= \sum_{n=1}^{N} \left(\ln(n+1) - \ln(n) \right)$$

$$= \left[\ln(2) - \ln(1)\right] + \left[\ln(3) - \ln(2)\right]$$

$$+ \left[\ln(4) - \ln(3)\right] + \left[\ln(5) - \ln(4)\right]$$

$$+ \dots + \left[\ln(N+1) - \ln(N)\right]$$

$$= \ln(N+1) - \ln(1)$$

$$= \ln(N+1)$$

so

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \ln\left(\frac{n+1}{n}\right)$$

$$= \lim_{N\to\infty} \ln(N+1)$$

$$= \infty$$

so $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges.

4K

The series $\sum_{n=1}^{\infty} \frac{1}{n}$

is called the harmonic series.

FACT: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

proof:

$$\sum_{n=1}^{2^k} \frac{1}{n}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots + \frac{1}{2^k}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$+$ (next 8 terms) $+$ (next 16 terms) $+ \dots (\dots)$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$+ (\dots) + \dots + (\dots)$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots + \left(\frac{1}{2}\right)$$

$$= 1 + \frac{1}{2}k$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{converges}$$

if $\lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n}$ converges.

i.e.

$\sum_{n=1}^{\infty} \frac{1}{n}$ converges if

this sequence $\left(\sum_{n=1}^{N} \frac{1}{n} \right)_{N=1}^{\infty}$

converges

but $\left(\sum_{n=1}^{2^k} \frac{1}{n} \right)_{K=1}^{\infty}$

is a subsequence

of $\left(\sum_{n=1}^{N} \frac{1}{n} \right)_{N=1}^{\infty}$

and $\left(\sum_{n=1}^{2^k} \frac{1}{n} \right)_{K=1}^{\infty}$

diverges because

$$\lim_{k \to \infty} \sum_{n=1}^{2^k} \frac{1}{n} > \lim_{k \to \infty} \left(1 + \frac{1}{2}k \right) = \infty.$$

if $\left(\sum_{n=1}^{N} \frac{1}{n}\right)_{N=1}^{\infty}$

converged, then

every subsequence

must also converge;

but we found

a subsequence that

diverges,

so $\left(\sum_{n=1}^{N} \frac{1}{n}\right)_{N=1}^{\infty}$

diverges,

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Note that

$\lim_{n\to\infty} \frac{1}{n} = 0$

but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

The Hyperharmonic Series Test

If $p > 0$,

then we call $\sum_{n=A}^{\infty} \frac{1}{n^p}$

a hyperharmonic series.

ex $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a

hyperharmonic series

$\sum_{n=A}^{\infty} \frac{1}{n^p}$ converges when $p > 1$

$\sum_{n=A}^{\infty} \frac{1}{n^p}$ diverges when $p \leq 1$.

8K

ex Test $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges, HHT $p=2$

ex Test $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

diverges HHT $p=\frac{1}{2}$

ex Test $\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^4}}$

diverges HHT, $p=\frac{4}{7}$

ex Test $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^7}}$

converges HHT, $p=\frac{7}{4}$

Not important

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The Geometric Series Test

We say that $\sum_{n=A}^{\infty} r^n$ is a geometric series.

ex $\sum_{n=0}^{\infty} 2^n$ is a geometric series.

Reminder:

$$\sum_{n=0}^{N} r^n = \frac{r^{N+1} - 1}{r - 1}$$

$$\sum_{n=1}^{N} r^n = \frac{r^{N+1} - r}{r - 1}$$

$$\sum_{n=A}^{N} r^n = \frac{r^{N+1} - r^A}{r - 1}$$

10K

proof:

let $\$ = \sum_{n=0}^{N} r^n$

then $\$ = 1 + r + r^2 + r^3 + \dots + r^N$

and $r\$ = r + r^2 + r^3 + r^4 + \dots + r^{N+1}$

and $r\$ - \$ = (r + r^2 + \dots + r^{N+1}) - (1 + r + \dots + r^N)$

so $r\$ - \$ = r^{N+1} - 1$

so $(r-1)\$ = r^{N+1} - 1$

so $\$ = \dfrac{r^{N+1} - 1}{r - 1}$

so $\displaystyle\sum_{n=0}^{N} r^n = \frac{r^{N+1} - 1}{r - 1}$

$$\sum_{n=A}^{N} r^n$$

$$= \sum_{n=0}^{N} r^n - \sum_{n=0}^{A-1} r^n$$

$$= \frac{r^{N+1} - 1}{r - 1} - \left(r^0 + r^1 + r^2 + \dots + r^{A-1}\right)$$

$$= \frac{r^{N+1}-1}{r-1} - \left(\frac{r-1}{r-1} + \frac{r(r-1)}{r-1} + \frac{r^2(r-1)}{r-1} + \dots + \frac{r^{A-1}(r-1)}{r-1} \right)$$

$$= \frac{r^{N+1}-1}{r-1} - \left(\frac{(r-1)+(r^2-r)+(r^3-r^2)+\dots+(r^A-r^{A-1})}{r-1} \right)$$

$$= \frac{r^{N+1}-1}{r-1} - \left(\frac{r^A-1}{r-1} \right)$$

$$= \frac{r^{N+1}-1}{r-1} + \frac{1-r^A}{r-1}$$

$$= \frac{r^{N+1}-r^A}{r-1}$$

$$\sum_{n=0}^{\infty} r^n \quad \text{converges}$$

exactly when $|r| < 1$.

ex $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

$r = \frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$.

so converges

ex $\displaystyle\sum_{n=0}^{\infty} 2^n$

$r = 2$ and $|2| \geq 1$

so diverges

ex $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$

$r = -\frac{1}{3}$ and $\left|-\frac{1}{3}\right| = \frac{1}{3} < 1$

so converges

ex $\displaystyle\sum_{n=0}^{\infty} (-3)^n$

$r = -3$ and $|-3| = 3 \geq 1$

so diverges

If $|r|<1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

$$\sum_{n=A}^{\infty} r^n = \frac{r^A}{1-r}$$

ex $$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= \frac{1}{\frac{1}{2}}$$

$$= 2$$

ex $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$= 1$$

ex $\sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^n$

$$= \frac{\left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{32}}{\frac{1}{2}}$$

$$= \frac{1}{32} \cdot \frac{2}{1}$$

$$= \frac{1}{16}$$

ex Find $\sum_{n=3}^{\infty} \frac{1}{7^n}$ 15K

$$\sum_{n=3}^{\infty} \frac{1}{7^n}$$

$$= \sum_{n=3}^{\infty} \left(\frac{1}{7}\right)^n$$

$$= \frac{\left(\frac{1}{7}\right)^3}{1 - \frac{1}{7}}$$

$$= \frac{\frac{1}{7^3}}{\frac{6}{7}}$$

$$= \frac{1}{7^3} \cdot \frac{7}{6}$$

$$= \frac{1}{7^2 6}$$

$$= \frac{1}{294}$$

ex Find $\sum_{n=1}^{\infty} \frac{2}{3^n}$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{3^n}$$

$$= 2 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= 2 \frac{\frac{1}{3}}{1 - \frac{1}{3}}$$

$$= 2 \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$= 2 \frac{1}{3} \cdot \frac{3}{2}$$

$$= 1$$

Comparison Tests

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$ geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$ NOT geometric series

If we know one series converges, then maybe a similar looking series will also converge.

Direct Comparison Test:

Let $x_n \geq 0$

and $y_n \geq 0$.

Let $x_n \leq y_n$.

If $\sum_{n=A}^{\infty} y_n$ converges,

then $\sum_{n=A}^{\infty} x_n$ converges.

If $\sum_{n=A}^{\infty} x_n$ diverges,

then $\sum_{n=A}^{\infty} y_n$ diverges.

ex Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

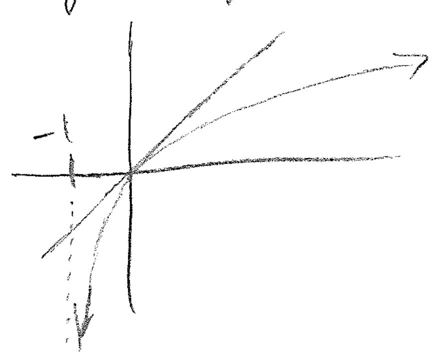
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

ex Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

$\displaystyle\frac{1}{\ln(n+1)} \geq \frac{1}{n}$ because $\ln(n+1) \leq n$

$y = x$, $y = \ln(x+1)$



$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges

Limit Comparison Test: 20K

Let $x_n > 0$

Let $y_n > 0$

Let $\lim\limits_{n\to\infty} \frac{x_n}{y_n} \in (0,\infty)$.

Then $\sum\limits_{n=A}^{\infty} x_n$ converges

exactly when

$\sum\limits_{n=A}^{\infty} y_n$ converges.

ex Test $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

If we try direct comparison with $\frac{1}{n^2}$,

then we run into a problem.

$$\frac{1}{n^2-1} > \frac{1}{n^2}$$

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, but we can't conclude that the bigger series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges.

So let's try limit comparison with $\frac{1}{n^2}$.

22K

take $x_n = \frac{1}{n^2}$

and $y_n = \frac{1}{n^2-1}$

$$\frac{x_n}{y_n}$$

$$= \frac{\frac{1}{n^2}}{\frac{1}{n^2-1}}$$

$$= \frac{n^2-1}{n^2}$$

$$= 1 - \frac{1}{n^2}$$

and $\lim_{n\to\infty} \frac{x_n}{y_n}$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right)$$

$$= 1 \in (0,\infty)$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

so $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ converges.

We call $\sum_{n=1}^{\infty} \frac{1}{an+b}$

a wide harmonic series.

ex $\sum_{n=1}^{\infty} \frac{1}{3n+4}$ is

a wide harmonic series.

FACT: $\sum_{n=1}^{\infty} \frac{1}{an+b}$ diverges

proof:

Case 1, $a > 0$.

$$x_n = \frac{1}{n}$$

$$y_n = \frac{1}{an+b}$$

$$\frac{x_n}{y_n} = \frac{an+b}{n}$$

$$\lim_{n \to \infty} \frac{x_n}{y_n} = a \in (0,\infty)$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

so $\sum_{n=1}^{\infty} \frac{1}{an+b}$ diverges

Case 2, $a=0$

$$\sum_{n=1}^{\infty} \frac{1}{b}$$ diverges by divergence test.

Case 3, $a<0$

$$\sum_{n=1}^{\infty} \frac{1}{an+b}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{-an-b}$$

by case 1,

$$\sum_{n=1}^{\infty} \frac{1}{-an-b}$$ diverges

so $$-\sum_{n=1}^{\infty} \frac{1}{-an-b}$$

diverges.

So any wide harmonic series diverges.

ex $$\sum_{n=1}^{\infty} \frac{1}{7n+2}$$ diverges.

Let T and B be polynomial functions.

Let $B(n) \neq 0$ for all $n \in \mathbb{N}$.

Then $\displaystyle\sum_{n=1}^{\infty} \frac{T(n)}{B(n)}$

converges exactly when

$\mathrm{Deg}(B) - \mathrm{Deg}(T) > 1.$

proof:

Say $\mathrm{Deg}(T) = \tau$

$\mathrm{Deg}(B) = \beta$

Say $\displaystyle\frac{T(n)}{B(n)} = \frac{a_\tau n^\tau + a_{\tau-1} n^{\tau-1} + \dots + a_0}{b_\beta n^\beta + b_{\beta-1} n^{\beta-1} + \dots + b_0}$

take $\displaystyle y_n = \frac{1}{n^{\beta-\tau}}$

$\displaystyle x_n = \frac{T(n)}{B(n)}$

26K

$$\frac{x_n}{y_n}$$

$$= n^{\beta-\tau} \frac{a_\tau n^\tau + \cdots + a_0}{b_\beta n^\beta + \cdots + b_0}$$

$$= \frac{a_\tau n^\beta + \cdots + a_0 n^{\beta-\tau}}{b_\beta n^\beta + \cdots + b_0}$$

So $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{a_\tau}{b_\beta}$

if $a_\tau > 0$ and $b_\beta > 0$,

then $\frac{a_\tau}{b_\beta} > 0$.

if $a_\tau < 0$ and $b_\beta < 0$,

then $\frac{a_\tau}{b_\beta} > 0$

if one of a_τ and b_β is negative, then redo with $y_n = \frac{-T(n)}{B(n)}$.

So we have $\lim_{n\to\infty} \frac{x_n}{y_n} \in (0,\infty)$.

So by LCT,

$$\sum_{n=1}^{\infty} \frac{T(n)}{B(n)} \text{ converges}$$

exactly when

$$\sum_{n=1}^{\infty} \frac{1}{n^{\beta-\tau}} \text{ converges}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\beta-\tau}}$ converges

exactly when $\beta - \tau > 1$
(by HHT)

So $\displaystyle\sum_{n=1}^{\infty} \frac{T(n)}{B(n)}$ converges

exactly when

$$\mathrm{Deg}(B) - \mathrm{Deg}(T) > 1.$$

ex Test $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

Deg(B) − Deg(T)

$= 3 - 1$

$= 2$

> 1

so converges

ex Test $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Deg(B) − Deg(T)

$= 2 - 1$

$= 1$

≤ 1

so diverges

ex Test $\sum_{n=1}^{\infty} \frac{7n^2-4n}{3n^5-2n^3+1}$

$\text{Deg}(B) - \text{Deg}(T)$

$= 5 - 2$

$= 3$

> 1

converges

ex Test $\sum_{n=1}^{\infty} \frac{3n^2+4n+5}{8n^2+7n+6}$

$\text{Deg}(B) - \text{Deg}(T)$

$= 2 - 2$

$= 0$

≤ 0

diverges.

Let T and B be polynomial functions. 30 K

Let $B(n) \neq 0$ for all $n \in \mathbb{N}$.

Then $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[t]{T(n)}}{\sqrt[b]{B(n)}}$ converges

exactly when

$$t\,\mathrm{Deg}(B) - b\,\mathrm{Deg}(T) > bt$$

ex.

Test $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

$t\,\mathrm{Deg}(B) - b\,\mathrm{Deg}(T)$

$= 2(3) - 1(3)$

$= 6 - 3$

$= 3$

and

bt

$= (1)(2)$

$= 2$

converges

31K

ex Test

$$\sum_{n=1}^{\infty} \frac{n^2+n+1}{\sqrt{n^6+n^2+1}}$$

t Deg (B) − b Deg (T)

$= (1)(6) - 2(2)$

$= 6 - 4$

$= 2$

and

bt

$= (2)(1)$

$= 2$

diverges

The Alternating Series Test

We say that

$\sum_{n=A}^{\infty} x_n$ is alternating

if $\text{Sign}(x_{n+1}) = -\text{Sign}(x_n)$.

ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an

alternating series

because $\left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty}$

$= \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots\right)$

if $(\gamma_n)_{n=1}^{\infty} = (2, 2, 1, 1, 2, 2, 1, 1, 2, 2, \dots)$

then $\sum_{n=1}^{\infty} \frac{(-1)^{\gamma_n}}{n}$ is NOT

an alternating series.

because $\left(\frac{(-1)^{\gamma_n}}{n}\right)_{n=1}^{\infty}$

$= \left(1, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, -\frac{1}{8}, \dots\right)$

If $\varphi_n > 0$, then

$$\sum_{n=1}^{\infty} (-1)^n \varphi_n$$ converges

exactly when $$\sum_{n=1}^{\infty} (-1)^{n+1} \varphi_n$$

converges.

Let $\varphi_n > 0$.

If φ is monotone down,

i.e. $\varphi_n \geq \varphi_{n+1}$

and $\lim_{n \to \infty} \varphi_n = 0$,

then $$\sum_{n=1}^{\infty} (-1)^n \varphi_n$$ converges.

ex Test $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Let $\varphi_n = \frac{1}{n}$.

(i) Is φ_n decreasing?

$\frac{1}{n} > \frac{1}{n+1}$. yes

(ii) Is $\lim_{n\to\infty} \varphi_n = 0$? yes

so by AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

ex Test $\sum_{n=1}^{\infty} (-1)^n \frac{3n+1}{2n+1}$.

Let $\varphi_n = \frac{3n+1}{2n+1}$.

(i) is φ_n decreasing?

NO.

(ii) Is $\lim_{n\to\infty} \varphi_n = 0$?

NO $\lim_{n\to\infty} \varphi_n = \frac{3}{2}$

Let $x_n = (-1)^n \frac{3n+1}{2n+1}$

$\lim_{n\to\infty} x_n$ DNE because

Let $u_n = x_{2n}$

$v_n = x_{2n+1}$

Then u_n and v_n are subsequences of x_n, and

$\lim_{n\to\infty} u_n = \frac{3}{2}$ and $\lim_{n\to\infty} v_n = -\frac{3}{2}$.

So $\lim_{n\to\infty} x_n$ DNE, so by div. test, $\sum_{n=1}^{\infty} x_n$ diverges.

ex Test $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ SL

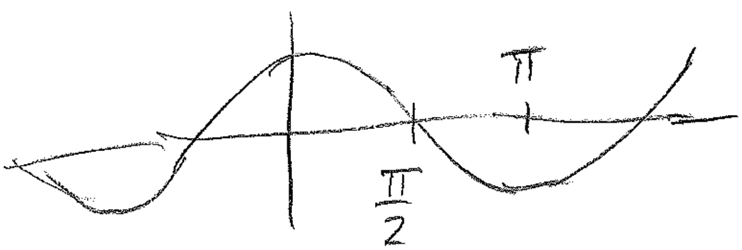
$\varphi_n = \sin\left(\frac{1}{n}\right)$

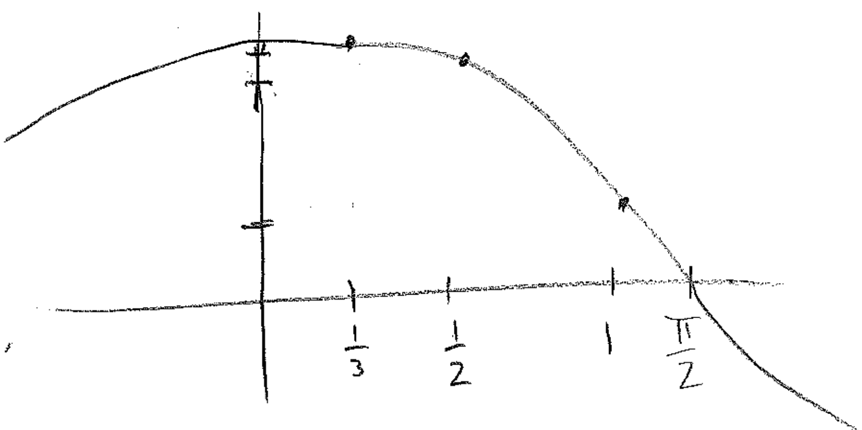
(i) Is φ_n decreasing?

$$\varphi_n' = \cos\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)$$

is $\cos\left(\frac{1}{n}\right)$ always positive?

$y = \cos(x)$





yes

so φ_n is decreasing ✓

6L

(ii) is $\lim_{n \to \infty} \varphi_n = 0$?

$$\lim_{n \to \infty} \sin\left(\frac{1}{n}\right)$$

$$= \sin\left(\lim_{n \to \infty} \frac{1}{n}\right)$$

$$= \sin(0)$$

$$= 0. \quad \checkmark$$

so $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

converges by

AST.

Absolute Convergence

We say that

$\sum_{n=1}^{\infty} x_n$ converges absolutely

if $\sum_{n=1}^{\infty} |x_n|$ converges.

ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely

because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

ex. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does NOT

converge absolutely

because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Note that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges,

but NOT absolutely.

We say that

$$\sum_{n=1}^{\infty} x_n$$ is conditionally convergent if

$$\sum_{n=1}^{\infty} x_n$$ converges

and $$\sum_{n=1}^{\infty} |x_n|$$ diverges.

ex $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ is conditionally convergent.

ex $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$ is NOT conditionally convergent

FACT :

If $\sum_{n=1}^{\infty} x_n$ converges absolutely and $\sum_{n=1}^{\infty} y_n$ is a rearrangement of $\sum_{n=}^{\infty} x_n$,

then $\sum_{n=1}^{\infty} y_n$ converges

and $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n$.

ex $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges absolutely.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots +$$

so $\sum_{n=1}^{\infty} y_n = \frac{1}{1} + \frac{1}{9} + \frac{1}{4} + \frac{1}{16} + \frac{1}{25} + \frac{1}{49} + \frac{1}{36} + \frac{1}{64} + \cdots$

converges to the same limit as $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

FACT:

If $\sum_{n=1}^{\infty} x_n$ is convergent

and not absolutely convergent

and $r \in \mathbb{R}$, then

there is a rearrangement, y_n,

such that $\sum_{n=1}^{\infty} y_n = r$.

There is also a rearrangement, w_n,

such that $\sum_{n=1}^{\infty} w_n = \infty$.

The Ratio Test

Let $(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\infty}$

with $x_n \neq 0$.

Let $L := \lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right|$.

If $L < 1$,

then $\sum_{n=1}^{\infty} x_n$ converges absolutely.

If $L > 1$,

then $\sum_{n=1}^{\infty} x_n$ diverges.

If $L = 1$,

then nothing can be concluded.

ex Test $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

let $x_n = \frac{(-1)^n}{n!}$

$$L = \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n \to \infty} \left| \frac{n(n-1)(n-2)\cdots(1)}{(n+1)(n)(n-1)(n-2)\cdots(1)} \right|$$

$$= \lim_{n \to \infty} \left| \frac{1}{n+1} \right|$$

$$= \lim_{n \to \infty} \frac{1}{n+1}$$

$$= 0$$

so $L = 0$.

so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

converges absolutely.

ex Test $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^3}$

let $x_n = (-1)^n \frac{2^n}{n^3}$

L

$= \lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right|$

$= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} \frac{2^{n+1}}{(n+1)^3}}{(-1)^n \frac{2^n}{n^3}} \right|$

$= \lim_{n \to \infty} \left| (-1) \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} \right|$

$= \lim_{n \to \infty} \left| 2 \frac{n^3}{(n+1)^3} \right|$

$= \lim_{n \to \infty} 2 \frac{n^3}{(n+1)^3}$

$= 2 \lim_{n \to \infty} \frac{n^3}{(n+1)^3}$

$$= 2 \lim_{n \to \infty} \left(\frac{n}{n+1} \right)^3$$ 15L

$$= 2 \left(\lim_{n \to \infty} \frac{n}{n+1} \right)^3$$

$$= 2 \, (1)^3$$

$$= 2$$

so $\quad L = 2.$

so $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^3}$ diverges.

Note.

$$\lim_{n \to \infty} \frac{n^3}{(n+1)^3}$$

$$= \lim_{n \to \infty} \frac{n^3}{n^3 + 3n^2 + 3n + 1}$$

$$= \frac{1}{1}$$

$$= 1.$$

ex Test $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

using the ratio test.

$$X_n = \frac{(-1)^n}{n^2+1}$$

L

$$= \lim_{n\to\infty} \left| \frac{X_{n+1}}{X_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)^2+1}}{\frac{(-1)^n}{n^2+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+1}{(n+1)^2+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+1}{n^2+2n+1+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+1}{n^2+2n+2} \right|$$

$$= \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2}$$

$$= \frac{1}{1}$$

$$= 1$$

so $L = 1$

we can't conclude anything.

We should have used AST, it converges.

The Root Test

Let $(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\infty}$.

Let $L := \lim\limits_{n\to\infty} \sqrt[n]{|x_n|}$.

If $L < 1$,
then $\sum\limits_{n=1}^{\infty} x_n$ converges absolutely.

If $L > 1$,
then $\sum\limits_{n=1}^{\infty} x_n$ diverges

If $L = 1$,

then nothing can
be concluded.

ex Test $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+4}\right)^n$

let $X_n := \left(\frac{2n+3}{3n+4}\right)^n$.

L

$= \lim_{n\to\infty} \sqrt[n]{|X_n|}$

$= \lim_{n\to\infty} \frac{2n+3}{3n+4}$

$= \frac{2}{3}$

so $L = \frac{2}{3}$

so $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+4}\right)^n$ converges.

ex Test $\sum_{n=1}^{\infty} \frac{n^n}{3^{2n+1}}$

let $x_n := \frac{n^n}{3^{2n+1}}$

L

$= \lim_{n\to\infty} \sqrt[n]{|x_n|}$

$= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^{2n+1}}}$

$= \lim_{n\to\infty} \left(\frac{n^n}{3^{2n+1}}\right)^{1/n}$

$= \lim_{n\to\infty} \left(\frac{(n^n)^{1/n}}{(3^{2n+1})^{1/n}}\right)$

$= \lim_{n\to\infty} \left(\frac{n}{3^{2+\frac{1}{n}}}\right)$

$= \frac{\infty}{9} = \infty.$

So $L = \infty$

So $\sum_{n=1}^{\infty} \frac{n^n}{3^{2n+1}}$ diverges.

ex Test $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

using the root test.

let $x_n := \left(1+\frac{1}{n}\right)^n$

$$L = \lim_{n\to\infty} \sqrt[n]{|x_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1.$$

So $L = 1$.

we can't conclude anything.

We should have used the divergence test, the series diverges because $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$.

The Integral Test

Let f be a decreasing function that is positive on $[A, \infty)$.

Then $\sum_{n=A}^{\infty} f(n)$ converges exactly when $\int_A^{\infty} f(x)\,dx$ converges.

Note that in general,

$$\int_A^{\infty} f(x)\,dx \neq \sum_{n=A}^{\infty} f(n)$$

ex Test $\sum_{n=1}^{\infty} e^{-n}$

$$\int_1^{\infty} e^{-x}\,dx$$

$$= \lim_{b\to\infty} \int_1^{b} e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[-e^{-x}\right]_1^{b}$$

$$= \lim_{b\to\infty} \left(\left[-e^{-b}\right] - \left[-e^{-1}\right]\right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{e^{b}} + \frac{1}{e}\right)$$

$$= \frac{1}{e}$$

So $\int_1^{\infty} e^{-x}\,dx$ converges

So $\sum_{n=1}^{\infty} e^{-n}$ converges.

ex Test $\sum_{n=1}^{\infty} ne^{-n}$

$$\int_1^\infty xe^{-x}\,dx$$

$$= \lim_{b\to\infty}\int_1^b xe^{-x}\,dx$$

$$\int xe^{-x}\,dx$$

let $u = x$, $dv = e^{-x}dx$

so $du = dx$, $v = -e^{-x}$

$$\int xe^{-x}\,dx$$

$$= \int u\,dv$$

$$= uv - \int v\,du$$

$$= x(-e^{-x}) - \int -e^{-x}\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + C$$

So $\lim_{b \to \infty} \int_1^b x e^{-x}\,dx$

$$= \lim_{b \to \infty} \left[-xe^{-x} - e^{-x} \right]_1^b$$

$$= \lim_{b \to \infty} \left(\left[-be^{-b} - e^{-b} \right] - \left[-(1)e^{-1} - e^{-1} \right] \right)$$

$$= \lim_{b \to \infty} \left(\left[\frac{-b}{e^b} - \frac{1}{e^b} \right] - \left[-\frac{1}{e} - \frac{1}{e} \right] \right)$$

$$= \left([0 - 0] - \left[-\frac{2}{e} \right] \right)$$

$$= \frac{2}{e}$$

So $\int_1^\infty x e^{-x}$ converges

So $\sum_{n=1}^{\infty} n e^{-n}$ converges.

ex Test $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{b\to\infty} \int_2^{b} \frac{1}{x\ln(x)}\,dx$$

$$\int \frac{1}{x\ln(x)}\,dx$$

let $u = \ln(x)$

so $\frac{du}{dx} = \frac{1}{x}$

so $dx = x\,du$

$$\int \frac{1}{x\ln(x)}\,dx$$

$$= \int \frac{1}{xu}\,x\,du$$

$$= \int \frac{1}{u}\,du$$

$$= \ln(u) + C$$

$$= \ln(\ln(x)) + C$$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{b\to\infty} \Big[\ln(\ln(x))\Big]_2^b$$

$$= \lim_{b\to\infty} \Big(\big[\ln(\ln(b))\big] - \big[\ln(\ln(2))\big] \Big)$$

$$= \infty - \ln(\ln(2))$$

$$= \infty$$

so $\displaystyle\int_2^\infty \frac{1}{x\ln(x)}\,dx$ diverges

so $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln(n)}$ diverges

Estimation of Series

If N is large,

then $\sum_{n=A}^{\infty} x_n \approx \sum_{n=A}^{N} x_n$.

We would like to know how good of an approximation is $\sum_{n=A}^{N} x_n$ for $\sum_{n=A}^{\infty} x_n$

Define $\$:= \sum_{n=A}^{\infty} x_n$

$$\$_N := \sum_{n=A}^{N} x_n$$

$$R_N := \$ - \$_N.$$

Note that $\$_N$ is close to $\$$ when R_N is close to 0

Alternating Series Test Remainder

ZM

If $\$ = \sum_{n=1}^{\infty} (-1)^n \varphi_n$

and $\varphi_n \geq 0$

and $\varphi_n \geq \varphi_{n+1}$

and $\lim_{n\to\infty} \varphi_n = 0$,

then $\sum_{n=1}^{\infty} (-1)^n \varphi_n$ converges

and $|R_N| \leq \varphi_{N+1}$

ex Find N such that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} - \sum_{n=1}^{N} \frac{(-1)^n}{n!} \right| < .001$$

So find N such that

$$|R_N| < .001$$

$$\varphi_n = \frac{1}{n!} \quad \text{and} \quad |R_N| \leq \varphi_{N+1}$$

so find N such that

$$\varphi_{N+1} < .001.$$

$$\varphi_{N+1} < .001$$

$$\rightarrow \frac{1}{(N+1)!} < .001$$

$$\rightarrow 1 < .001\,(N+1)!$$

$$\rightarrow \frac{1}{.001} < (N+1)!$$

$$\rightarrow 1{,}000 < (N+1)!$$

4M

If $N = 5$,

$$(N+1)! = (5+1)! = 6! = 720$$

If $N = 6$,

$$(N+1)! = (6+1)! = 7! = 5{,}040.$$

So if $N=6$, then

$$1{,}000 < (N+1)!$$

so $|R_N| < .001$

Note that

$$\sum_{n=1}^{6} \frac{(-1)^n}{n!} = -.6319\overline{4}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e} - 1 = -.63212\ldots$$

Integral Test

Remainder

If $S = \sum_{n=1}^{\infty} f(n)$

and f is positive

and f is decreasing

and $\int_1^{\infty} f(x)\,dx$ converges,

then $\sum_{n=1}^{\infty} f(n)$ converges

and $|R_N| \le \int_N^{\infty} f(x)\,dx$.

ex Find N such that

6M

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{N} \frac{1}{n^2} \right| < .0001$$

So find N such that

$$|R_N| < .0001$$

$$|R_N| \leq \int_N^{\infty} \frac{1}{x^2}\, dx$$

So find N such that

$$\int_N^{\infty} \frac{1}{x^2}\, dx < .0001.$$

$$\int_N^{\infty} \frac{1}{x^2}\, dx$$

$$= \lim_{b \to \infty} \int_N^{b} \frac{1}{x^2}\, dx$$

$$= \lim_{b \to \infty} \left[-\frac{1}{x} \right]_N^b$$

$$= \lim_{b \to \infty} \left(\left[-\frac{1}{b} \right] - \left[-\frac{1}{N} \right] \right)$$

7M

$$= \frac{1}{N}$$

$$\int_N^\infty \frac{1}{x^2}\,dx < .0001$$

$$\rightarrow \quad \frac{1}{N} < .0001$$

$$\rightarrow \quad 1 < .0001N$$

$$\rightarrow \quad \frac{1}{.0001} < N$$

$$\rightarrow \quad 10{,}000 < N$$

so if $N = 10{,}001$,

then $|R_N| < .0001$.

Note that

$$\sum_{n=1}^{10{,}001} \frac{1}{n^2} = 1.64483408\ldots$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1.64493406\ldots$$

Power Series

An expression like

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_N x^N$$

is called a polynomial expression.

If f is a function

and $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_N x^N$,

then f is called a polynomial function.

$$c_0 + c_1 (x-c) + c_2 (x-c)^2 + \ldots$$

$$+ c_N (x-c)^N \quad \text{is}$$

also a polynomial expression.

and f by

$$f(x) = \sum_{n=0}^{N} c_n (x-c)^n \quad \text{is}$$

a polynomial function.

An expression like

$c_0 + c_1(x-c) + c_2(x-c)^2 + \dots$

is called a power series expression or an analytic expression.

If f is a function and $f(x) = \sum_{n=0}^{\infty} c_n (x-c)^n$,

then we call f a power series function or an analytic function.

ex If $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}(x-3)^n$,

then f is a power series function and $f(3) = 1$

because $f(3)$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}(0)^n$$

$$= \frac{1}{2^0}(0)^0 + \frac{1}{2^1}(0)^1 + \frac{1}{2^2}(0)^2 + \frac{1}{2^3}(0)^3 + \dots$$

and $0^0 := 1$, 10M

so $f(3) = 1$.

$$f(4) = \sum_{n=0}^{\infty} \frac{1}{2^n}(4-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}(1)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= \frac{1}{\frac{1}{2}}$$

$$= 2$$

11M

$$f(2) = \sum_{n=0}^{\infty} \frac{1}{2^n}(2-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}(-1)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$$

$$= \frac{1}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{1}{\frac{2}{2}+\frac{1}{2}}$$

$$= \frac{1}{\frac{3}{2}}$$

$$= \frac{2}{3}$$

$$f(5) = \sum_{n=0}^{\infty} \frac{1}{2^n}(5-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}(2)^n$$

$$= \sum_{n=0}^{\infty} 1$$

$$= \infty$$

diverges

so $f(5)$ is undefined.

We say the expression

$$\sum_{n=0}^{\infty} c_n (x-c)^n \quad \text{is}$$

centered at c.

ex $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}(x-3)^n$

is centered at 3.

FACT:

If $f(x) = \sum_{n=0}^{\infty} c_n (x-c)^n$

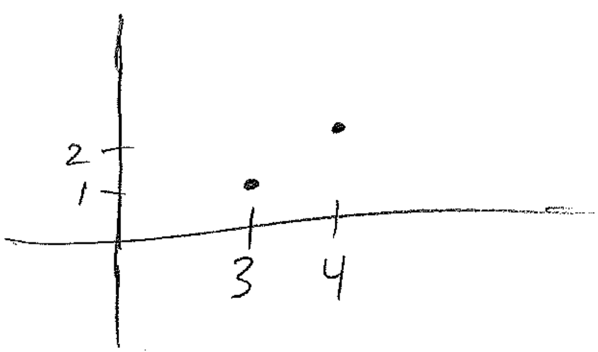
and $d(c,a) < d(c,b)$

and $f(b)$ converges,

then $f(a)$ converges.

ex Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} (x-3)^n$

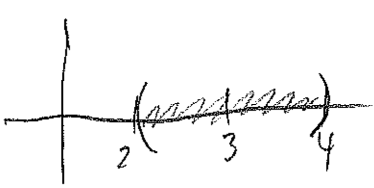
We know that $f(4)$ converges.



so if $d(3,a) < d(3,4)$

then $f(a)$ converges

so if a is in here



then $f(a)$ converges

So Domain(f) must be an interval.

FACT:

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-c)^n$

Let $D := \{x : f(x) \text{ converges}\}$,

so $D = \text{Domain}(f)$.

Then exactly one of these is true.

(i) $D = \{c\}$

(ii) there is a number, r, such that

$D = (c-r, c+r)$

OR $D = [c-r, c+r]$

OR $D = [c-r, c+r)$

OR $D = (c-r, c+r]$

(iii) $D = (-\infty, \infty)$.

Note If

$I = (a,b)$ OR $I = [a,b]$

OR $I = [a,b)$ OR $I = (a,b]$,

then $\text{Length}(I) := b - a$

ex $\text{Length}([3,7)) := 4$.

We define

$IOC(f) := D$

we call $IOC(f)$

the interval of

convergence of f.

So $IOC(f)$ is the

domain of f.

We define $ROC(f)$

$:= \frac{1}{2}\text{Length}(IOC(f))$.

We call $ROC(f)$

the radius of convergence of f.

ex. Let $f(x) = \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n 3^n}$ 16M

Find ROC(f) and IOC(f).

Find all x values where $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$ converges.

use ratio test, $a_n = \frac{(3x-2)^n}{n3^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}}{\frac{(3x-2)^n}{n3^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(3x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(3x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(3x-2)}{3} \cdot \frac{n}{n+1} \right|$$

$$= \left| \frac{3x-2}{3} \right|$$

If $L < 1$,

then $f(x)$ converges.

$L < 1$

$\rightarrow \left| \frac{3x-2}{3} \right| < 1$

$\rightarrow \frac{3x-2}{3} < 1$ AND $\frac{3x-2}{3} > -1$

$\rightarrow 3x-2 < 3$ AND $3x-2 > -3$

$\rightarrow 3x < 5$ AND $3x > -1$

$\rightarrow x < \frac{5}{3}$ AND $x > -\frac{1}{3}$

$\rightarrow x \in \left(-\frac{1}{3}, \frac{5}{3}\right)$

so $\left(-\frac{1}{3}, \frac{5}{3}\right) \subseteq D$

but we don't know yet

if $\frac{5}{3} \in D$ or

if $-\frac{1}{3} \in D$.

$$f\left(\frac{5}{3}\right)$$

$$= \sum_{n=1}^{\infty} \frac{\left(3\left(\frac{5}{3}\right)-2\right)^n}{n3^n}$$

$$= \sum_{n=1}^{\infty} \frac{3^n}{n3^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, so $\frac{5}{3} \notin D$.

$$f\left(-\frac{1}{3}\right)$$

$$= \sum_{n=1}^{\infty} \frac{\left(3\left(-\frac{1}{3}\right)-2\right)^n}{n3^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, by AST

so $-\frac{1}{3} \in D$.

So

$$\boxed{IOC(f) = \left[-\tfrac{1}{3}, \tfrac{5}{3}\right)}$$

and

$ROC(f)$

$$= \frac{1}{2}\,\text{Length}\big(IOC(f)\big)$$

$$= \frac{1}{2}\left(\frac{5}{3} - \left(-\frac{1}{3}\right)\right)$$

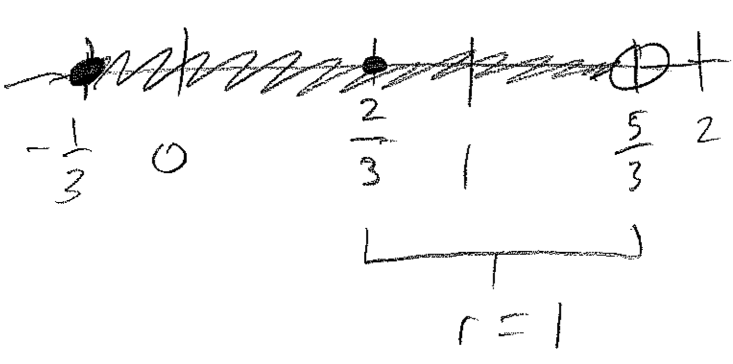
$$= \frac{1}{2}\left(\frac{5}{3} + \frac{1}{3}\right)$$

$$= \frac{1}{2}\left(\frac{6}{3}\right)$$

$$= \frac{1}{2}(2)$$

$$= 1$$

$$\boxed{\text{So} \quad ROC(f) = 1}$$



ex Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ 20M

Find IOC(f) and ROC(f).

Let $a_n = \frac{(-1)^n x^n}{n+1}$

Let $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{n+2}}{\frac{(-1)^n x^n}{n+1}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(-1)\, x}{1} \cdot \frac{n+1}{n+2} \right|$$

$$= \lim_{n \to \infty} \left| x \cdot \frac{n+1}{n+2} \right|$$

$$= |x|$$

$L < 1$

$\rightarrow \quad |x| < 1$

$\rightarrow \quad x < 1$ AND $x > -1$

$\rightarrow \quad x \in (-1, 1)$

so $(-1, 1) \subseteq D$

$f(-1)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges

LCT with $\frac{1}{n}$.

so $-1 \notin D$

$f(1)$ 22M

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (1)^n}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

converges, AST.

so $1 \in D$

so $\boxed{IOC(f) = (-1, 1]}$

and $ROC(f)$

$= \frac{1}{2} \text{Length}(IOC(f))$

$= \frac{1}{2}(1-(-1))$

$= \frac{1}{2}(1+1)$

$= \frac{1}{2}(2)$

$= 1$

$\boxed{\text{so } ROC(f) = 1.}$

ex Let $f(x) = \sum_{n=1}^{\infty} \sqrt{n}\, x^n$

Find IOC(f) and ROC(f).

---

let $a_n = \sqrt{n}\, x^n$

let $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n \to \infty} \left| \frac{\sqrt{n+1}\, x^{n+1}}{\sqrt{n}\, x^n} \right|$$

$$= \lim_{n \to \infty} \left| x \frac{\sqrt{n+1}}{\sqrt{n}} \right|$$

$$= |x|$$

$L < 1$

$\rightarrow |x| < 1$

$\rightarrow x < 1$ AND $x > -1$

$\rightarrow x \in (-1, 1)$

$f(-1)$

$$= \sum_{n=1}^{\infty} \sqrt{n}\,(-1)^n$$

diverges by divergence test,

$\lim_{n\to\infty} \sqrt{n}\,(-1)^n \neq 0.$

so $-1 \notin D$

$f(1)$

$$= \sum_{n=1}^{\infty} \sqrt{n}\,(1)^n$$

$$= \sum_{n=1}^{\infty} \sqrt{n}$$

diverges by div. test,

$\lim_{n\to\infty} \sqrt{n} \neq 0.$

So $1 \notin D$.

25M

So $\boxed{\underline{I}OC(f) = (-1,1)}$

and $ROC(f)$

$= \frac{1}{2} \text{Length}(IOC(f))$

$= \frac{1}{2}(1-(-1))$

$= \frac{1}{2}(2)$

$= 1$

So $\boxed{ROC(f) = 1}$

ex Let $f(x) = \sum_{n=1}^{\infty} \frac{2^n x^n}{n^3}$

Find IOC(f) and ROC(f).

Let $a_n = \frac{2^n x^n}{n^3}$

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\frac{2^{n+1} x^{n+1}}{(n+1)^3}}{\frac{2^n x^n}{n^3}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2x}{1} \frac{n^3}{(n+1)^3} \right|$$

$$= |2x|$$

27M

$L < 1$

$\rightarrow |2x| < 1$

$\rightarrow 2x < 1$ AND $2x > -1$

$\rightarrow x < \frac{1}{2}$ AND $x > -\frac{1}{2}$

$\rightarrow x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

$f\left(-\frac{1}{2}\right)$

$$= \sum_{n=1}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{n^3}$$

$$= \sum_{n=1}^{\infty} \frac{2^n (-1)^n \frac{1}{2^n}}{n^3}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

converges, AST

$f\left(\frac{1}{2}\right)$ 28M

$$= \sum_{n=1}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{n^3}$$

$$= \sum_{n=1}^{\infty} \frac{2^n \frac{1}{2^n}}{n^3}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges,
HHT $p = 3 > 1$.

so $IOC(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$

and $ROC(f) = \frac{1}{2}$

ex Let $f(x) = \sum_{n=0}^{\infty} n!\, x^n$

Find IOC(f) and ROC(f).

Let $a_n = n!\, x^n$

Let $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n \to \infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(n+1)n!\, x^{n+1}}{n!\, x^n} \right|$$

$$= \lim_{n \to \infty} |(n+1)x|$$

$$= \infty$$

So for any $x \neq 0$, $L = \infty$.

if $x = 0$, then $\lim_{n \to \infty} |(n+1)x|$

$$= \lim_{n \to \infty} |0|$$

$$= 0.$$

$$< 1$$

30M

so $IOC(f) = \{0\}$

and $ROC(f) = 0$

ex Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Find IOC(f) and ROC(f).

Let $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Let $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n \to \infty} \left| \frac{\dfrac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}}{\dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2}(n+1)!(n+1)!} \; \frac{2^{2n} n! n!}{(-1)^n x^{2n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^2}{2^2(n+1)(n+1)} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^2}{4(n+1)^2} \right| = 0 < 1$$

so $IOC(f) = (-\infty, \infty)$

and $ROC(f) = \infty$.

Pointwise Convergence

Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions.

We say that $(f_n)_{n=1}^{\infty}$ converges pointwise to f on D

iff

if $x \in D$, then $\lim_{n\to\infty} f_n(x) = f(x)$.

ex If $f_n(x) = \frac{1}{n}x$

and $f(x) = 0$,

then $(f_n)_{n=1}^{\infty}$ converges pointwise to f, on $\mathbb{R}$

because if $x \in \mathbb{R}$

then $\lim_{n\to\infty} \frac{1}{n}x = 0$.

If all the functions in the sequence $(f_n)_{n=1}^{\infty}$ are continuous, and $(f_n)_{n=1}^{\infty}$ converges pointwise to f, then f might NOT be continuous.

A technical definition of pointwise convergence:

we say that $(f_n)_{n=1}^{\infty}$ converges pointwise to f on D

iff

if $x \in D$ and $\varepsilon > 0$,

then there is a number N

such that

if $n > N$, then $|f_n(x) - f(x)| < \varepsilon$.

Uniform Convergence

Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions.

We say that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on D

iff

if $\varepsilon > 0$, then there is a number N such that

if $x \in D$ and $n > N$, then $|f_n(x) - f(x)| < \varepsilon$.

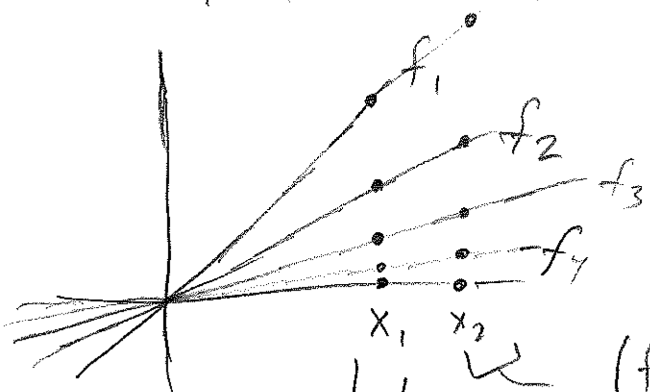
So here, our N must work for every $x \in D$.

ex if $f_n(x) = \frac{1}{n} x$ 4N

and $f(x) = 0$.

then $(f_n)_{n=1}^{\infty}$ does

NOT converge uniformly to f.



$(f_n(x_2))_{n=1}^{\infty}$

gives a sequence

$(f_n(x_1))_{n=1}^{\infty}$

$(f_n(x_1))_{n=1}^{\infty}$ converges to 0

$(f_n(x_2))_{n=1}^{\infty}$ converges to 0.

but you might have to go further out in this sequence $(f_n(x_2))_{n=1}^{\infty}$ for the terms to be less than .01 than you must go out in this sequence $(f_n(x_1))_{n=1}^{\infty}$ for the terms to be less than .01.

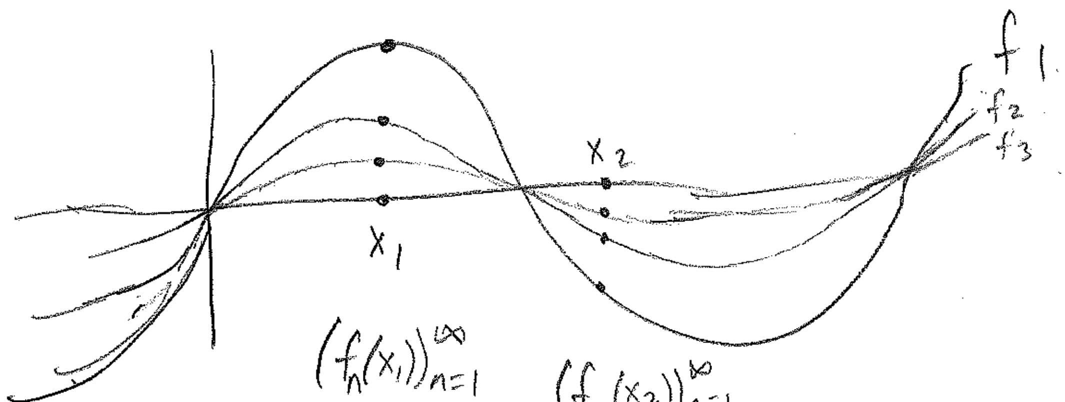
ex if $f_n(x) = \frac{1}{n}\sin(x)$

and $f(x) = 0$,

then $(f_n)_{n=1}^{\infty}$ is

converging uniformly to f

on $\mathbb{R}$.



$(f_n(x_1))_{n=1}^{\infty}$ $(f_n(x_2))_{n=1}^{\infty}$

The sequence $(f_n(x_1))_{n=1}^{\infty}$ is the "slowest" of all the sequences

$(f_n(x_1))_{n=1}^{\infty}, (f_n(x_2))_{n=1}^{\infty}, (f_n(x_3))_{n=1}^{\infty}, \ldots$

So if we pick an $\varepsilon > 0$ and going beyond the Nth term of the sequence $(f_n(x_1))_{n=1}^{\infty}$ makes the terms less than ε, then going beyond the Nth term in ANY sequence $(f_n(x))_{n=1}^{\infty}$ will make the terms less than ε.

6N

If $(f_n)_{n=1}^{\infty}$ converges to f uniformly and all $f_1, f_2, f_3, \ldots$ are continuous, then f is continuous.

If $(f_n)_{n=1}^{\infty}$ converges to f uniformly and $(f_n')_{n=1}^{\infty}$ converges uniformly to g, then* $f' = g$.

* (there are more conditions needed)

If $(f_n)_{n=1}^{\infty}$ converges to f uniformly and $f_1, f_2, f_3, \ldots$ are integrable, then

$$\int f(x)\,dx = \lim_{n \to \infty} \int f_n(x)\,dx$$

If you have a power series function, f, this means that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n,$$

then there is a sequence of functions

$$f_0(x) = \sum_{n=0}^{0} a_n (x-a)^n = a_0$$

$$f_1(x) = \sum_{n=0}^{1} a_n (x-a)^n = a_0 + a_1(x-a)$$

$$f_2(x) = \sum_{n=0}^{2} a_n (x-a)^n$$

$$\vdots$$

$$f_N(x) = \sum_{n=0}^{N} a_n (x-a)^n$$

$$\vdots$$

So we have a sequence of functions $(f_N)_{N=0}^{\infty}$. Note that each function in this sequence is a polynomial function.

And this sequence $(f_N)_{N=0}^{\infty}$

converges uniformly to f

where $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$.

So we can say a few

things about f:

(i) all $f_1, f_2, f_3, \ldots$ are continuous,

so f is continuous.

(ii) all f_1, f_2, f_3 are

differentiable, so

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1}$$

(iii) all f_1, f_2, f_3 are

integrable, so

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \int a_n (x-a)^n\,dx.$$

ex If $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}(x-3)^n$,

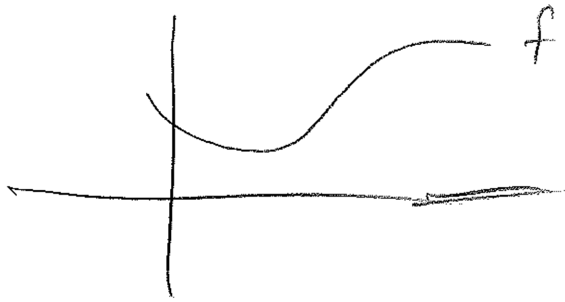
then $f'(x) = \sum_{n=0}^{\infty} \frac{n}{2^n}(x-3)^{n-1}$

and $\int f(x)dx = \sum_{n=0}^{\infty} \frac{1}{2^n(n+1)}(x-3)^{n+1} + C$

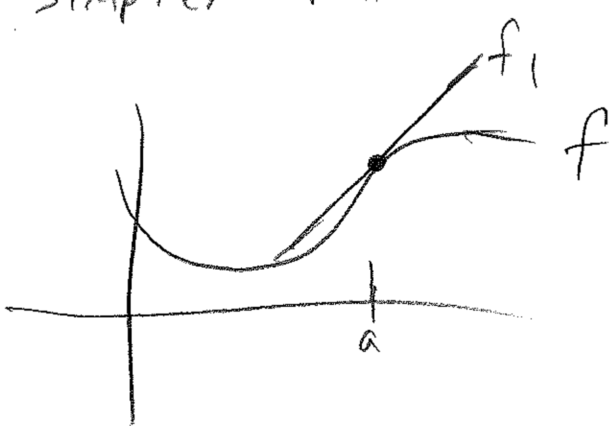
In general, the radius of convergence of $f(x)$, $f'(x)$, $\int f(x)dx$ are all the same, but endpoint convergence might change.

Taylor Functions

If we have a function, f,



then there might be another function that looks like f near a that is a simpler function.



Here f_1 is simpler than f and looks like f near a.

What is the formula for f_1?

$f_1(x) = mx + b,$

$m = f'(a)$ and $(a, f(a))$
is on f_1 so

$$f(a) = ma + b \rightarrow b = f(a) - ma$$

So $f_1(x) = f'(a)x + f(a) - ma$

$$= f'(a)x + f(a) - f'(a)a$$

$$= f'(a)(x-a) + f(a)$$

So $f_1(x) = f'(a)(x-a) + f(a).$

So basically we just pretend
that $f(x) = a_1(x-a) + a_0$
and note that $f(a) = a_0$
and $f'(x) = a_1$ so $f'(a) = a_1$

Now pretend
$$f(x) = a_2(x-a)^2 + a_1(x-a) + a_0$$
and note that $f(a) = a_0$
$$f'(a) = a_1$$
$$f''(a) = 2a_2$$

<u>if</u> $f(x) = a_3(x-a)^3 + a_2(x-a)^2 + a_1(x-a) + a_0$,

then $f(a) = a_0$

$f'(a) = a_1$

$f''(a) = 2a_2$

$f'''(a) = 3!\,a_3$

FACT: <u>If</u> $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$,

then $a_n = \dfrac{f^{(n)}(a)}{n!}$.

If f is a function, then we define a new function $f^{\tau @ a}$ by

$$f^{\tau @ a}(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We call $f^{\tau @ a}$ the taylor function of f at a.

We call the expression $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ the taylor series of f at a.

VERY IMPORTANT:

$f^{\tau @ a}$ is NOT always equal to f.

We define f^{μ} by
$f^{\mu}(x) := f^{\tau @ 0}(x)$.

We call f^{μ} the maclaurin function of f.

So $f^{\mu}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

ex Let $f(x) = e^x$.

Find $f^{\mu}(x)$.

We need to find

$$\frac{f^{(0)}(0)}{0!}, \quad \frac{f^{(1)}(0)}{1!}, \quad \frac{f^{(2)}(0)}{2!},$$

$$\frac{f^{(3)}(0)}{3!}, \ldots$$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x$

So $f^{(n)}(0) = e^0 = 1$

So $$\frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

So $$\boxed{f^{\mu}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$$

We do NOT yet know if $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

i.e. we don't know

$f(x) = f^M(x)$ yet.

All we have right now is two functions,

$f(x) = e^x$ and $f^M(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

ex Let $f(x) = \sin(x)$.

Find $f^{M}(x)$.

So find $\frac{f^{(0)}(0)}{0!}, \frac{f^{(1)}(0)}{1!}, \frac{f^{(2)}(0)}{2!}, \ldots$

$f(x) = \sin(x)$	$\frac{f(0)}{0!} = \frac{0}{0!}$
$f'(x) = \cos(x)$	$\frac{f^{(1)}(0)}{1!} = \frac{1}{1!}$
$f''(x) = -\sin(x)$	$\frac{f^{(2)}(0)}{2!} = \frac{0}{2!}$
$f^{(3)}(x) = -\cos(x)$	$\frac{f^{(3)}(0)}{3!} = \frac{-1}{3!}$
$f^{(4)}(x) = \sin(x)$	$\frac{f^{(4)}(0)}{4!} = \frac{0}{4!}$
$f^{(5)}(x) = \cos(x)$	$\frac{f^{(5)}(0)}{5!} = \frac{1}{5!}$

so if n is odd, $\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n!}$

if n even, then $\frac{f^{(n)}(0)}{n!} = 0$

So $f^M(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

ex Let $f(x) = \cos(x)$.

Find $f^M(x)$.

$f(x) = \cos(x)$	$\frac{f^{(0)}(0)}{0!} = \frac{1}{0!}$
$f'(x) = -\sin(x)$	$\frac{f^{(1)}(0)}{1!} = \frac{0}{1!}$
$f''(x) = -\cos(x)$	$\frac{f^{(2)}(0)}{2!} = \frac{-1}{2!}$
$f^{(3)}(x) = \sin(x)$	$\frac{f^{(3)}(0)}{3!} = \frac{0}{3!}$
$f^{(4)}(x) = \cos(x)$	$\frac{f^{(4)}(0)}{4!} = \frac{1}{4!}$
$f^{(5)}(x) = -\sin(x)$	

So $f^M(x)$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

ex Let $f(x) = \dfrac{1}{1-x}$.

Find $f^M(x)$.

$f(x) = ((1-x)^{-1}$	$\dfrac{f^{(0)}(0)}{0!} = 1$
$f'(x) = -(1-x)^{-2}(-1)$	$\dfrac{f^{(1)}(0)}{1!} = 1$
$f''(x) = -2(1-x)^{-3}(-1)$	$\dfrac{f^{(2)}(0)}{2!} = -1$
$f'''(x) = -3!(1-x)^{-4}(-1)$	$\dfrac{f^{(3)}(0)}{3!} = 1$
$f^{(4)}(x) = -4!(1-x)^{-5}(-1)$	$\dfrac{f^{(4)}(0)}{4!} = 1$

So $f^M(x) = \displaystyle\sum_{n=0}^{\infty} x^n$

Shortcuts for Finding

Taylor Functions

Reminder:

If $f(x) = e^x$,

then $f^{M}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

If $f(x) = \sin(x)$,

then $f^{M}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

If $f(x) = \cos(x)$,

then $f^{M}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

If $f(x) = \frac{1}{1-x}$,

then $f^{M}(x) = \sum_{n=0}^{\infty} x^n$

If $f(x) = \ln(x)$,

then $f^{T@1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$

If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$,

then $f^{T@a}(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$.

ex We know

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x|<1.$$

so if $f(x) = \frac{1}{1-x}$, then

$$f^{M}(x) = \sum_{n=0}^{\infty} x^n.$$

So if $f(x)$ is equal to a power series centered at a, then $f^{T@a}(x)$ is that power series.

The "converse" is false.

If you only have $f^{T@a}(x)$, then it might not be true that $f(x) = f^{T@a}(x)$.

If $f(x) = \Omega(I(x))$,

then $f^{\tau @ a}(x) = \Omega^{\tau @ a}(I(x))$

as long as this gives you

a <u>power</u> series.

ex Let $f(x) = e^{-x}$.

Let $\Omega(x) = e^{x}$

Let $I(x) = -x$.

So $f^{\mu}(x)$

$$= \Omega^{\mu}(I(x))$$

$$= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

If φ is a polynomial function and $f(x) = \varphi(x)A(x)$, then $f^{\tau@a}(x) = \varphi(x)\,A^{\tau@a}(x)$.

ex If $f(x) = xe^x$,

then $f^{M}(x)$

$$= x\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}.$$

If $f(x) = A(x) + B(x)$,

then $f^{\tau @ a}(x) = A^{\tau @ a}(x) + B^{\tau @ a}(x)$.

ex If $f(x) = e^x + e^{-x}$.

then $f^{M}(x)$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} x^n$$

If $T(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$,

then $T'(x) = \sum_{n=0}^{\infty} n\, a_n (x-a)^{n-1}$

and $\int T(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1} + C_1$

ex say that $f(x) = \arctan(x)$.

find $f^{M}(x)$.

$$f'(x) = \frac{1}{1+x^2}$$

let $w = -x^2$

$$= \frac{1}{1-(-x^2)} = \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

so $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C_1$

and $\arctan(0) = 0$, so $C = 0$,

so $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, so $f^{M}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

7,Ω

ex Let $f(x) = \dfrac{1}{3x+4}$.

Find $f^{\tau @ 2}(x)$.

$f(x)$

$= \dfrac{1}{3x+4}$

$= \dfrac{1}{3\left(x+\frac{4}{3}\right)}$

$= \dfrac{1}{3} \dfrac{1}{x+\frac{4}{3}}$

$= \dfrac{1}{3} \dfrac{1}{x-2+2+\frac{4}{3}}$

let $k := 2 + \frac{4}{3}$

$= \dfrac{1}{3} \dfrac{\frac{1}{-k}}{\frac{x-2+k}{-k}}$

$= \dfrac{1}{3} \dfrac{\frac{1}{-k}}{\frac{x-2}{-k} + \frac{k}{-k}}$

$$= \frac{1}{3} \frac{1}{-k} \frac{1}{\frac{x-2}{-k} - 1}$$

$$= \frac{1}{3} \frac{1}{-k} \frac{1}{-1} \frac{1}{1 - \frac{x-2}{-k}}$$

$$= \frac{1}{3} \frac{1}{k} \frac{1}{1 + \frac{x-2}{k}}$$

$$= \frac{1}{3} \frac{1}{k} \frac{1}{1 - \left(-\frac{x-2}{k}\right)}$$

$$= \frac{1}{3} \frac{1}{2 + \frac{4}{3}} \frac{1}{1 - \left(-\frac{x-2}{2+\frac{4}{3}}\right)}$$

$$= \frac{1}{3 \cdot 2 + 4} \frac{1}{1 - \left(-\frac{x-2}{2+\frac{4}{3}}\right)}$$

let $w := -\frac{x-2}{2+\frac{4}{3}}$

$$= \frac{1}{3 \cdot 2 + 4} \frac{1}{1-w}$$

$$= \frac{1}{3 \cdot 2 + 4} \sum_{n=0}^{\infty} w^n$$

$$= \frac{1}{3\cdot 2+4} \sum_{n=0}^{\infty} \left(-\frac{x-2}{2+\frac{4}{3}} \right)^n$$

$$= \frac{1}{3\cdot 2+4} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(2+\frac{4}{3}\right)^n} (x-2)^n$$

So $f^{\tau @ 2}(x) = \frac{1}{10} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(2+\frac{4}{3}\right)^n} (x-2)^n$

In general,

If $f(x) = \frac{1}{\alpha x + \beta}$ and $a \neq -\frac{\beta}{\alpha}$,

then $f^{\tau @ a}(x)$

$$= \frac{1}{\alpha a + \beta} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(a+\frac{\beta}{\alpha}\right)^n} (x-a)^n$$

ex Let $f(x) = \frac{1}{7x-2}$.

Find $f^{\tau @ 3}(x)$.

$f^{\tau @ 3}(x)$

$$= \frac{1}{7(3)+(-2)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(3+\frac{-2}{7}\right)^n} (x-3)^n$$

$$= \frac{1}{19} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{21}{7} - \frac{2}{7}\right)^n} (x-3)^n$$

$$= \frac{1}{19} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{19}{7}\right)^n} (x-3)^n$$

$$= \frac{1}{19} \sum_{n=0}^{\infty} \frac{(-1)^n}{\frac{19^n}{7^n}} (x-3)^n$$

$$= \frac{1}{19} \sum_{n=0}^{\infty} \frac{(-1)^n 7^n}{19^n} (x-3)^n$$

$$\boxed{\text{so} \quad f^{T@3}(x) = \frac{1}{19} \sum_{n=0}^{\infty} \frac{(-1)^n 7^n}{19^n} (x-3)^n}$$

If $f(x) = \dfrac{1}{(\alpha x + \beta)^k}$

and $a \neq \dfrac{-\beta}{\alpha}$,

then to find $f^{\tau @ a}(x)$,

you can do this.

Note that

$$\int f(x)\,dx$$

$$= \int (\alpha x + \beta)^{-k}\,dx$$

$$= \frac{1}{\alpha}\,\frac{1}{-k+1}(\alpha x + \beta)^{-k+1} + C_1$$

now: $\int\int f(x)\,dx\,dx$

$$= \int\left(\frac{1}{\alpha}\,\frac{1}{-k+1}(\alpha x + \beta)^{-k+1} + C_1\right)dx$$

$$= \frac{1}{\alpha^2}\,\frac{1}{(-k+1)}\,\frac{1}{(-k+2)}(\alpha x + \beta)^{-k+2} + C_1 x + C_2$$

Now

$$\underbrace{\int\int\cdots\int}_{k-1 \text{ times}} f(x) \underbrace{dx\cdots dxdx}_{k-1}$$

$$= \frac{1}{\alpha^{k-1}(-k+1)(-k+2)\cdots(-1)} \frac{1}{\alpha x+\beta} + p(x)$$

where $\mathrm{Deg}(p) = k-2$

$$= \frac{1}{\alpha^{k-1}(-1)^{k-1}(k-1)(k-2)\cdots(1)} \frac{1}{\alpha x+\beta} + p(x)$$

$$= \frac{(-1)^{k-1}}{\alpha^{k-1}(k-1)!} \frac{1}{\alpha x+\beta} + p(x)$$

$$= \frac{(-1)^{k-1}}{\alpha^{k-1}(k-1)!} \frac{1}{\alpha a+\beta} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(a+\frac{\beta}{\alpha}\right)^n}(x-a)^n + p(x)$$

let $F(x) := \frac{(-1)^{k-1}}{\alpha^{k-1}(k-1)!} \frac{1}{\alpha a+\beta} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(a+\frac{\beta}{\alpha}\right)^n}(x-a)^n + p(x)$

then $f(x)$

$$= F^{(k-1)}(x)$$

$$= \frac{(-1)^{k-1}}{\alpha^{k-1}(k-1)!} \frac{1}{\alpha a+\beta} \sum_{n=k-1}^{\infty} \frac{(-1)^n n!}{\left(a+\frac{\beta}{\alpha}\right)^n (n-k+1)!}(x-a)^{n-k+1}$$

So if $f(x) = \dfrac{1}{(\alpha x + \beta)^k}$,

then $f^{\tau @ a}(x)$

$$= \frac{(-1)^{k-1}}{\alpha^{k-1}(k-1)!} \frac{1}{\alpha a + \beta} \sum_{n=k-1}^{\infty} \frac{(-1)^n n!}{\left(a + \frac{\beta}{\alpha}\right)^n (n-k+1)!} (x-a)^{n-k+1}$$

ex Let $f(x) = \dfrac{1}{(3x+5)^7}$

Find $f^{T@2}(x)$.

$f^{T@2}(x)$

$$= \frac{(-1)^6}{3^6\, 6!} \frac{1}{3(2)+5} \sum_{n=6}^{\infty} \frac{(-1)^n n!}{\left(2+\frac{5}{3}\right)^n (n-6)!} (x-2)^{n-6}$$

$$= \frac{1}{3^6 6!} \frac{1}{11} \sum_{n=0}^{\infty} \frac{(-1)^{n+6} (n+6)!}{\left(\frac{6}{3}+\frac{5}{3}\right)^{n+6} n!} (x-2)^n$$

$$= \frac{1}{3^6 6! 11} \sum_{n=0}^{\infty} \frac{(-1)^n (n+6)!}{\left(\frac{11}{3}\right)^{n+6} n!} (x-2)^n$$

$$= \frac{1}{3^6 6! 11} \sum_{n=0}^{\infty} \frac{(-1)^n (n+6)!}{\frac{11^{n+6}}{3^{n+6}}\, n!} (x-2)^n$$

$$= \frac{1}{3^6 6! 11} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+6} (n+6)!}{11^{n+6}\, n!} (x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^n (n+6)!}{6!\, 11^{n+7}\, n!} (x-2)^n$$

ex Let $f(x) = \dfrac{4x}{x^2+2x-3}$.

Find $f^M(x)$.

Note $f(x)$

$$= \frac{4x}{(x+3)(x-1)}$$

$$= \frac{1}{x-1} + \frac{3}{x+3} \quad \text{(partial fractions)}$$

$$= \frac{1}{1(0)-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(0+\frac{-1}{1}\right)^n} (x-0)^n$$

$$+ 3 \frac{1}{1(0)+3} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(0+\frac{3}{1}\right)^n} (x-0)^n$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(-1)^n} x^n + \frac{3}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^n$$

$$= -\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^n$$

$$= \sum_{n=0}^{\infty} -x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^n$$

$$= \sum_{n=0}^{\infty} \left(-1 + \frac{(-1)^n}{3^n}\right) x^n$$

so $f^M(x)$

$$= \sum_{n=0}^{\infty} \left(-1 + \frac{(-1)^n}{3^n}\right) x^n.$$

Equitayloric Functions

We say that f is equitayloric iff
$$f(x) = f^{\tau @ a}(x).$$

We define
$$f_N^{\tau @ a}(x) := \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We call $f_N^{\tau @ a}$ the taylor polynomial of f centered at a with degree N.

ex If $f(x) = e^x$,

then $f_2^{\tau @ 0}(x)$
$$= \sum_{n=0}^{2} \frac{f^{(n)}(0)}{n!}(x-0)^n$$
$$= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2$$
$$= 1 + x + \frac{1}{2}x^2$$

We define

$$f_N^{r@a}(x) := f(x) - f_N^{T@a}(x).$$

We call $f_N^{r@a}(x)$

the Nth remainder of f

at a.

FACT: $f(x) = f^{T@a}(x)$

exactly when

$$\lim_{N \to \infty} f_N^{r@a}(x) = 0.$$

FACT:

If $\left| f^{(N+1)}(x) \right| \le M$

for $|x-a| \le r$,

then $$\left| f_N^{r@a}(x) \right| \le \frac{M}{(N+1)!} |x-a|^{N+1}$$

for $|x-a| \le r$.

FACT: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

i.e. if $f(x) = e^x$,

then $f(x) = f^{\mu}(x)$.

---

proof:

If $f(x) = e^x$,

then $f^{(N+1)}(x) = e^x$.

So for $|x| \leq r$

$|f^{(N+1)}(x)|$

$= |e^x|$

$\leq e^r$

so $M = e^r$.

So $|f_N^{r@0}(x)|$

$\leq \frac{e^r}{(N+1)!} |x|^{N+1}$

and $\lim_{N \to \infty} \frac{e^r}{(N+1)!} |x|^{N+1}$

$= 0$

so $\lim_{N \to \infty} \left| f_N^{r@0}(x) \right| = 0$

so $\lim_{N \to \infty} f_N^{r@0}(x) = 0$

so $f(x) = f^{\tau@0}(x)$

so $f(x) = f^{\mu}(x)$.

so $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

---

also, $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

and $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Uses of Taylor Functions

ex Calculate $\sin(2)$ correctly to 4 decimal places.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$\sin(2)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+1)!}$$

by alternating series remainder we need $N > 5$ if

we want $\left| \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+1)!} - \sum_{n=0}^{N} \frac{(-1)^n (2)^{2n+1}}{(2n+1)!} \right|$

$$< .00001$$

so $\sin(2) \approx \sum_{n=0}^{6} \frac{(-1)(2)^{2n+1}}{(2n+1)!}$

$= .9092\ldots$

ex Find $\int_0^1 e^{-x^2}\,dx$

$$\int_0^1 e^{-x^2}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{n!(2n+1)} - \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+1}}{n!(2n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}$$

$$\simeq .747$$

ex Find

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$$

$$= \lim_{x \to 0} \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x}{x^2}$$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - x}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{1}{2!} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \frac{1}{5!}x^3 + \cdots\right)$$

$$= \frac{1}{2!}$$

$$= \boxed{\frac{1}{2}}$$

ex Find

$$\lim_{x \to 0} \frac{x - \arctan(x)}{x^3}$$

$$= \lim_{x \to 0} \frac{x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}{x^3}$$

$$= \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)}{x^3}$$

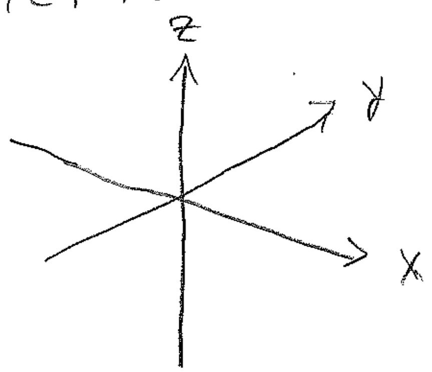
$$= \lim_{x \to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} + \cdots}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 + \cdots\right)$$

$$= \boxed{\frac{1}{3}}$$

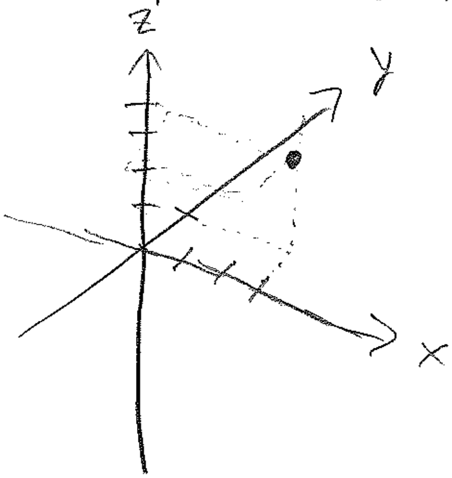
Graphing

If you want to visualize an ordered triple (x, y, z), then you could draw a picture like this.



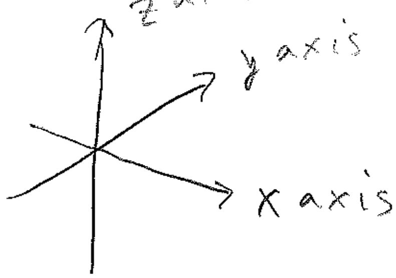
start at center,
move x units right,
move y units forward,
move z units up,
draw a dot.
This dot represents (x, y, z).

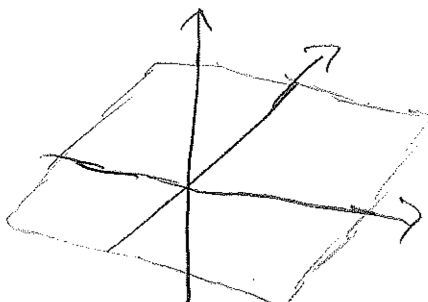
ex Graph $(3,1,4)$



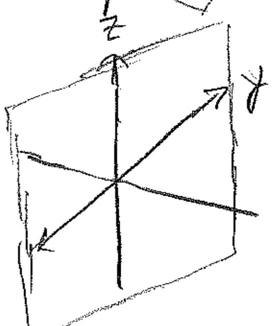
In this xyz space

we have

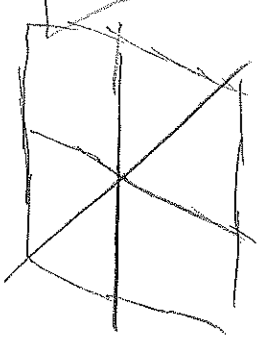




xy plane



yz plane

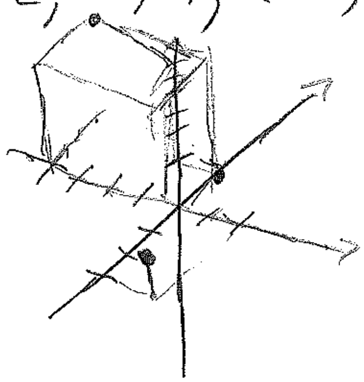


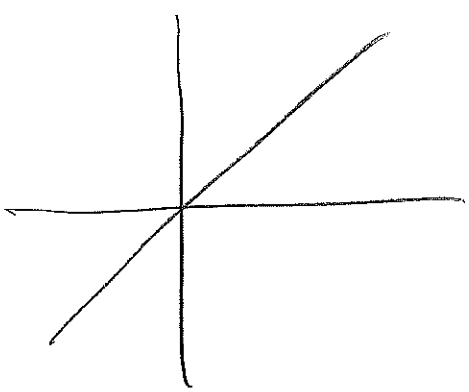
xz plane

To graph a set of ordered triples is to graph all the ordered triples in the set.

ex Graph

$\{(2,-3,1), (-4,1,5), (1,0,1)\}$

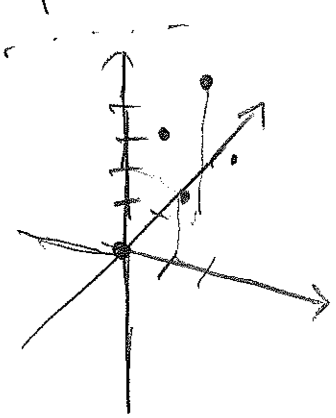




To graph an equation is to graph all the ordered triples that make the equation true.

ex Graph
$z = 2x + 3y$

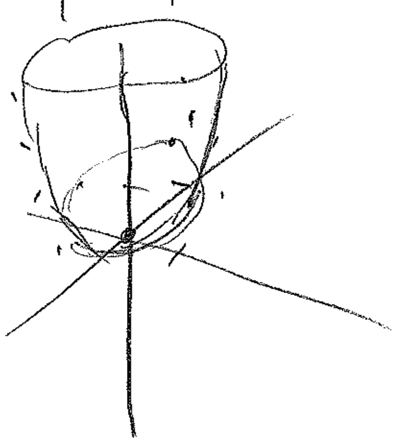
x	y	z	
0	0	0	$\to (0,0,0)$
1	0	2	$\to (1,0,2)$
0	1	3	$\to (0,1,3)$
1	1	5	$\to (1,1,5)$
2	0	4	$\to (2,0,4)$



Ex Graph

$$z = x^2 + y^2$$

x	y	z
0	0	0
1	0	1
0	1	1
1	1	2
2	0	4
0	2	4
2	2	8
-1	-1	2



Multivalued Functions

We define

$$\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$$

So $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

and $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$

so $\mathbb{R}^2 = \{(0,0), (1,0), (3,5), (-1,4), (7,\bar{2}), (\frac{1}{2}, \sqrt{3}), \ldots\}$

and $\mathbb{R}^3 = \{(0,0,0), (3,1,4), (2,1,7), (-6, \frac{4}{5}, \sqrt{7}), \ldots\}$

I write

$f ; X \longrightarrow Y$ to mean

(i) f is a function

(ii) $\text{Domain}(f) \subseteq X$

(iii) $\text{Range}(f) \subseteq Y$.

I will write

$f : X \to Y$ to mean

(i) f is a function

(ii) $\text{Domain}(f) = X$.

(iii) $\text{Range}(f) \subseteq Y$.

ex

$f(x) = x^2$

then $f : \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{1}{x}$

$f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$

$f(x) = \frac{1}{x}$

$f ; \mathbb{R} \to \mathbb{R}$

So if $f ; X \to Y$,

then the TYPES of objects you can plug into f are members of X and the TYPES of outputs are members of Y.

We say that f is a multivalued function if

$$\text{Range}(f) \subseteq \mathbb{R}^m$$

and $m > 1$.

if $m = 1$, then singlevalued.

ex Let $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (t^2 - 2t, t+1).$$

Then f is a multivalued function. Also called vectorvalued functions.

Note that $f(0) = (0, 1)$

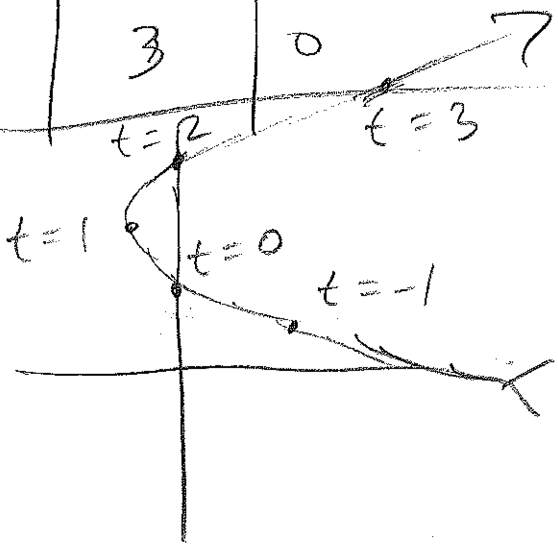
$f(1) = (-1, 2)$

$f(2) = (0, 3)$

To visualize f, we graph the set

$$\{(t^2 - 2t, t+1) : t \in \mathbb{R}\}$$

t	x	y
0	0	1
1	-1	2
2	0	3
3	3	4
-1	3	0



The equations

$x = t^2 - 2t$

$y = t + 1$

are called the parametric equations of f.

The functions f_1 by

$f_1(t) = t^2 - 2t$

and f_2 by $f_2(t) = t + 1$

are called the component functions of f.

ex

$f : \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$f(t) = (t\cos(t), t\sin(t), t)$$

Then f is a multivalued function.

Note that

$$f(0) = (0,0,0)$$

$$f\left(\tfrac{\pi}{2}\right) = \left(0, \tfrac{\pi}{2}, \tfrac{\pi}{2}\right),$$

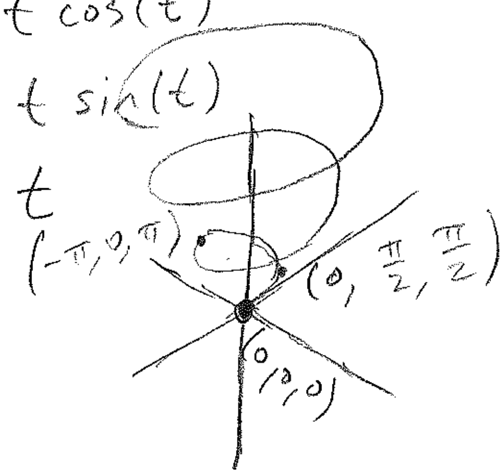
$$f(\pi) = (-\pi, 0, \pi)$$

parametric equations for f are

$$x = t\cos(t)$$

$$y = t\sin(t)$$

$$z = t$$



Parametric Equations

If $\varphi ; \mathbb{R} \to \mathbb{R}^2$

and $\varphi(t) = (\varphi_1(t), \varphi_2(t))$,

then we call

$x = \varphi_1(t)$ and $y = \varphi_2(t)$

the parametric equations for φ.

We visualize φ by graphing the set $\{\varphi(t) : t \in \text{Domain}(\varphi)\}$.

We say that φ parametrizes

$\{\varphi(t) : t \in \text{Domain}(\varphi)\}$.

ex If $\varphi(t) = (t^2 + t, t^3 + 1)$

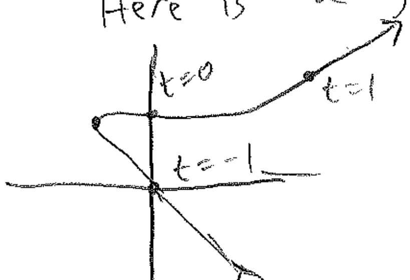
then $\varphi(0) = (0,1)$

$\varphi(1) = (2,2)$

$\varphi(2) = (6,9)$

$\varphi(-1) = (0,0)$

Here is a graph of φ.



We say that φ parametrizes this curve.

We say that φ has parametric equations

$x = t^2 + t$

$y = t^3 + 1$.

1P

ex. Let $\varphi : \mathbb{R} \to \mathbb{R}^2$

by $\varphi(t) = (-t^3 + t, 1 - t^2)$

Graph φ.

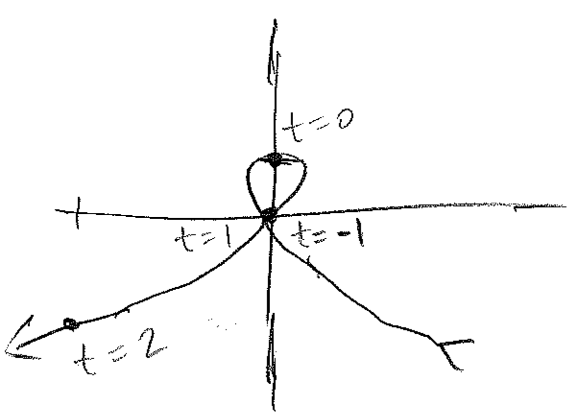
---

the parametric equations for φ are

$x = -t^3 + t$

$y = 1 - t^2$

t	x	y
0	0	1
1	0	0
-1	0	0
2	-6	-3

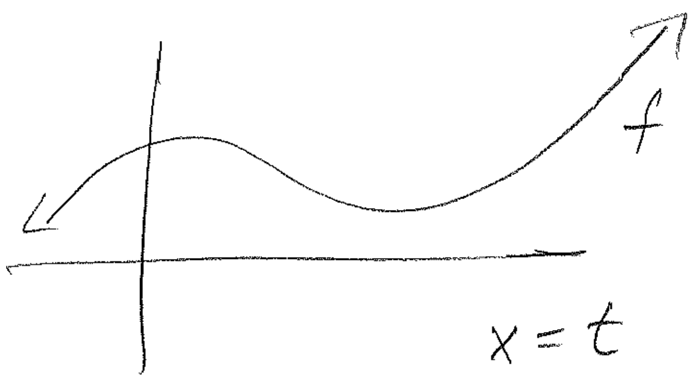


If $f: \mathbb{R} \to \mathbb{R}$

then f is parametrized

by $\varphi: \mathbb{R} \to \mathbb{R}^2$

$\varphi(t) = (t, f(t))$.



$x = t$

$y = f(t)$

ex If $f(x) = x^2$,

then f is parametrized by

$\varphi(t) = (t, t^2)$

so $x = t$

$y = t^2$

If $p, q \in \mathbb{R}^2$,

then $\text{Line}(p,q)$ is

parametrized by

$\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$

by $\varphi(t) = dt + p$

where $d = q - p$.

Note that

$\varphi(0) = p$

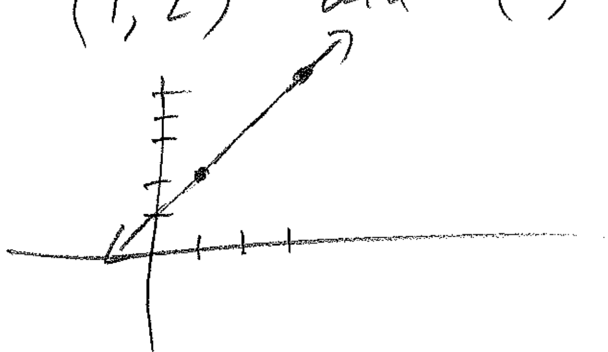
and $\varphi(1) = q$.

ex Find

parametric equations

for the line through

$(1,2)$ and $(3,5)$



d

$= q - p$

$= (3,5) - (1,2)$

$= (2,3)$

so

$\varphi(t)$

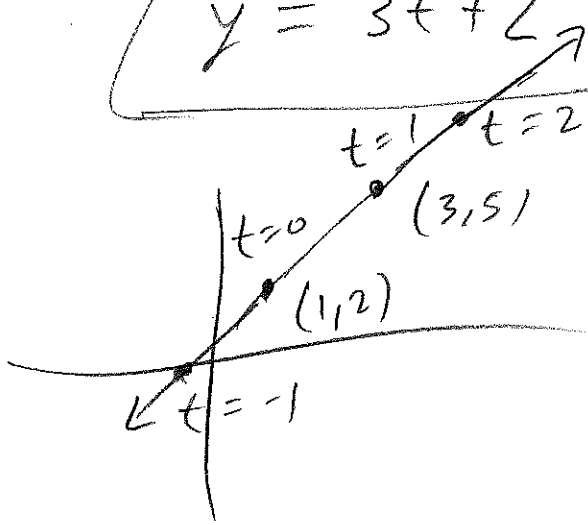
$= dt + p$

$= (2,3)t + (1,2)$

$= (2t, 3t) + (1,2)$

$= (2t+1, 3t+2)$

so $\boxed{\begin{aligned} x &= 2t+1 \\ y &= 3t+2 \end{aligned}}$

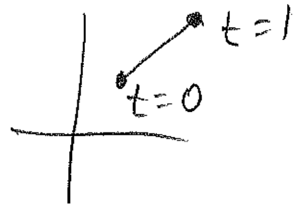


If $p, q \in \mathbb{R}^2$,

then $\text{Segment}(p, q)$

is parametrized by

$\varphi : [0,1] \longrightarrow \mathbb{R}^2$



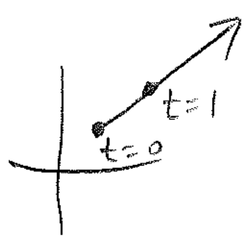
by $\varphi(t) = dt + p$.

If $p, q \in \mathbb{R}^2$,

then $\text{Ray}(p, q)$

is parametrized by

$\varphi : [0, \infty) \longrightarrow \mathbb{R}^2$



by $\varphi(t) = dt + p$

The circle with equation

$$(x-h)^2 + (y-k)^2 = r^2$$

is parametrized by

$$\varphi : [0, 2\pi] \longrightarrow \mathbb{R}^2$$

by $\varphi(t) = (r\cos(t) + h, r\sin(t) + k)$

ex the circle with center $(1,2)$ and radius 3 is parametrized

by $\varphi(t) = (3\cos(t) + 1, 3\sin(t) + 2)$

so $x = 3\cos(t) + 1$
$y = 3\sin(t) + 2.$

unit circle,

$x = \cos(t)$
$y = \sin(t)$

the ellipse with equation

$$\frac{(x-h)^2}{u} + \frac{(y-k)^2}{v} = 1$$

is parametrized by

$$\varphi : [0, 2\pi] \to \mathbb{R}^2$$

by $\varphi(t) = (u\cos(t) + h, v\sin(t) + k)$

(more parametric equations)

Eliminating the Parameter

Eliminating the parameter means taking the parametric equations

$$x = \varphi_1(t)$$

$$y = \varphi_2(t)$$

and converting into an equation involving only x and y and NOT the "parameter" t.

ex A curve C

is parametrized by

$\varphi : (-1, \infty) \longrightarrow \mathbb{R}^2$

by $\varphi(t) = \left(\dfrac{1}{\sqrt{t+1}}, \dfrac{t}{\sqrt{t+1}} \right)$.

So $$x = \frac{1}{\sqrt{t+1}}$$

$$y = \frac{t}{\sqrt{t+1}}$$

Eliminate the parameter.

$$x = \frac{1}{\sqrt{t+1}}$$

$$\longrightarrow \sqrt{t+1} = \frac{1}{x}$$

$$\longrightarrow t+1 = \frac{1}{x^2}$$

$$\longrightarrow t = \frac{1}{x^2} - 1$$

and $y = \dfrac{t}{\sqrt{t+1}}$

$$= \frac{\frac{1}{x^2} - 1}{\sqrt{\frac{1}{x^2} - 1 + 1}}$$

$$= \frac{\frac{1}{x^2} - 1}{\sqrt{\frac{1}{x^2}}}$$

$$= \frac{\frac{1}{x^2} - 1}{\frac{1}{x}}$$

$$= \left(\frac{1}{x^2} - 1\right) x$$

$$= \frac{x}{x^2} - x$$

$$= \frac{1}{x} - x$$

So $\boxed{y = \frac{1}{x} - x}$

where $x > 0$

ex A curve C

is parametrized by

$\varphi : (-\infty, \infty) \to \mathbb{R}^2$

by $\varphi(t) = (3\cos(t), 4\sin(t))$.

So $x = 3\cos(t)$

$y = 4\sin(t)$

Eliminate the parameter.

$x = 3\cos(t) \to \cos(t) = \frac{x}{3}$

$y = 4\sin(t) \to \sin(t) = \frac{y}{4}$

so $\cos^2(t) = \frac{x^2}{9}$

and $\sin^2(t) = \frac{y^2}{16}$

We know that

$\sin^2(t) + \cos^2(t) = 1$

so $\boxed{\frac{x^2}{9} + \frac{y^2}{16} = 1}$

Calculus with Parametric Equations

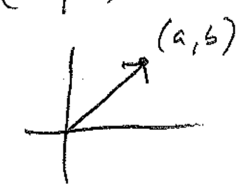
If $\varphi(t) = (\varphi_1(t), \varphi_2(t))$

then define $\varphi'(t) := (\varphi_1'(t), \varphi_2'(t))$.

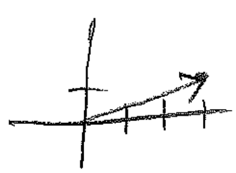
ex if $\varphi(t) = (7t^3 - 5t^2, 3t^2 + 2t)$

then $\varphi'(t) = (21t^2 - 10t, 6t + 2)$

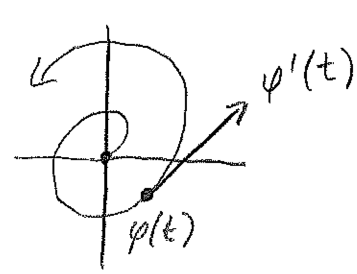
We can think of an ordered pair (a,b) as an arrow.



ex We can visualize $(3,1)$ as this arrow

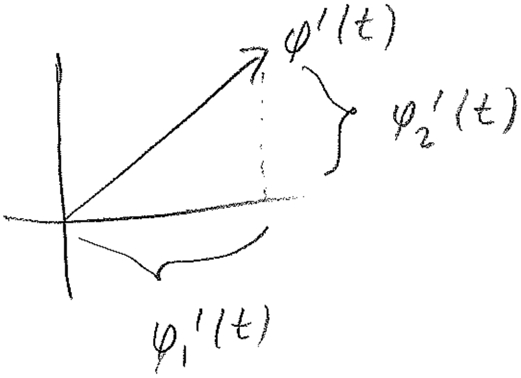


If C is a curve parametrized by φ and we place the arrow $\varphi'(t)$ at $\varphi(t)$, then the arrow $\varphi'(t)$ is tangent to C.



Note that

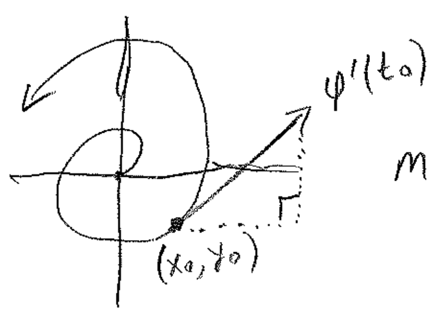
$$\varphi'(t) = (\varphi_1'(t), \varphi_2'(t))$$



Tangent Lines:

Let C be a curve parametrized by φ where $\varphi: \mathbb{R} \to \mathbb{R}^2$ by $\varphi(t) = (\varphi_1(t), \varphi_2(t))$.

If $(x_0, y_0) \in C$ and $\varphi(t_0) = (x_0, y_0)$, then we define the tangent line to C at (x_0, y_0) i.e. at t_0 to be the line through (x_0, y_0) with slope $\dfrac{\varphi_2'(t_0)}{\varphi_1'(t_0)}$.



$$m = \frac{\varphi_2'(t_0)}{\varphi_1'(t_0)}$$

We define

$$\mu(t) := \frac{\varphi_2'(t)}{\varphi_1'(t)}.$$

Note that $\mu(t)$ gives you the slope of the tangent line at t.

If $x = \varphi_1(t)$ and $y = \varphi_2(t)$, then $\mu(t) = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

so $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

ex Let C be parametrized by

$\varphi : \mathbb{R} \to \mathbb{R}^2$

by $\varphi(t) = (t+1, t^2+3t)$.

Find the equation for the tangent line to C at $t = -1$.

$$\mu(t) = \frac{\varphi_2'(t)}{\varphi_1'(t)}$$

$$= \frac{2t+3}{1}$$

$$\mu(-1) = 2(-1)+3$$

$$= -2+3$$

$$= 1.$$

So the tangent looks like

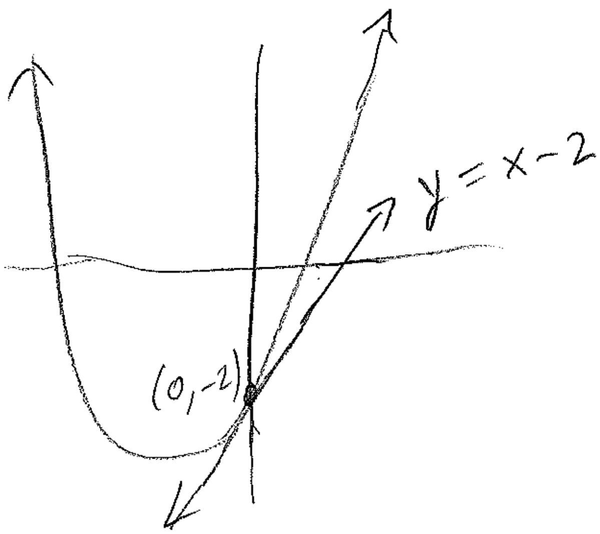
$$y = x + b$$

and $\varphi(-1) = (0, -2)$

so $-2 = (0) + b$

$\rightarrow \quad b = -2$

so $\boxed{y = x - 2}$



ex Let C be the curve parametrized by

$\varphi : \mathbb{R} \to \mathbb{R}^2$

by $\varphi(t) = (t^3 - t, t^2 - 1)$.

Find the equation of the tangent line to C at $t = 2$.

10 Q

$$\mu(t) = \frac{\varphi_2'(t)}{\varphi_1'(t)} = \frac{2t}{3t^2-1}$$

$$\mu(2) = \frac{2(2)}{3(2)^2-1} = \frac{4}{12-1} = \frac{4}{11}$$

So $y = \frac{4}{11}x + B$

and $\varphi(2) = (2^3 - 2, 2^2 - 1)$

$= (8 - 2, 4 - 1)$

$= (6, 3)$

So $3 = \frac{4}{11}(6) + B$

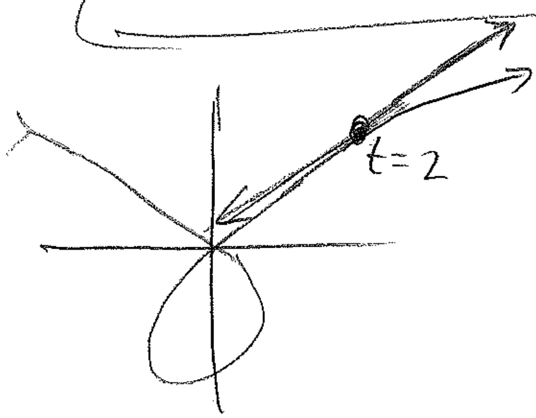
$\rightarrow 3 = \frac{24}{11} + B$

$\rightarrow B = 3 - \frac{24}{11}$

$\rightarrow B = \frac{33}{11} - \frac{24}{11}$

$\rightarrow B = \frac{9}{11}$

So $\boxed{y = \frac{4}{11}x + \frac{9}{11}}$



ex Let C be the curve parametrized by $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $\varphi(t) = (\cos(t), \sin(t))$.

Find the equation of the tangent line to C at $t = \frac{\pi}{4}$.

$$\mu(t) = \frac{\varphi_2'(t)}{\varphi_1'(t)} = \frac{\cos(t)}{-\sin(t)}$$

$$\mu\left(\frac{\pi}{4}\right) = \frac{\cos\left(\frac{\pi}{4}\right)}{-\sin\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$$

so $y = -x + b$

$$\varphi\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

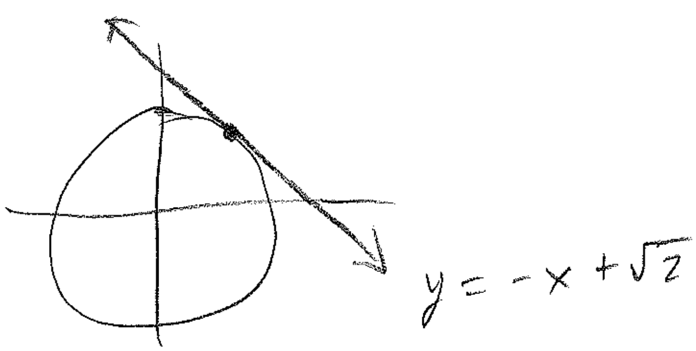
so $\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} + b$

$\rightarrow \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = b$

$\rightarrow \frac{2\sqrt{2}}{2} = b$

$\rightarrow b = \sqrt{2}$

So $\boxed{y = -x + \sqrt{2}}$



ex Let C be the curve parametrized by $\varphi : \mathbb{R} \to \mathbb{R}^2$ by $\varphi(t) = \left(2t - \pi \sin(t), 2 - \pi\cos(t)\right)$.

(a) Find the equation of the tangent line to C at $t = \pi/2$.

(b) Find the equation of the tangent line to C at $t = -\pi/2$.

$\mu(t)$

$$= \frac{\varphi_2'(t)}{\varphi_1'(t)}$$

$$= \frac{\pi \sin(t)}{2 - \pi\cos(t)}$$

(a) $\mu\left(\frac{\pi}{2}\right)$ 15Q

$$= \frac{\pi \sin\left(\frac{\pi}{2}\right)}{2 - \pi \cos\left(\frac{\pi}{2}\right)}$$

$$= \frac{\pi}{2}$$

$y = \frac{\pi}{2} x + b$

$$\varphi\left(\frac{\pi}{2}\right) = \left(2\,\frac{\pi}{2} - \pi \sin\left(\frac{\pi}{2}\right),\ 2 - \pi \cos\left(\frac{\pi}{2}\right)\right)$$

$$= (\pi - \pi,\ 2 - 0)$$

$$= (0, 2)$$

so $2 = \frac{\pi}{2}(0) + b$

$\longrightarrow$ $b = 2$

so $\boxed{y = \frac{\pi}{2} x + 2}$

(b) $\mu\left(-\frac{\pi}{2}\right)$ 16Q

$$= \frac{\pi \sin\left(-\frac{\pi}{2}\right)}{2 - \pi\cos\left(-\frac{\pi}{2}\right)}$$

$$= \frac{\pi(-1)}{2 - \pi(0)}$$

$$= \frac{-\pi}{2}$$

so $y = -\frac{\pi}{2}x + b$

$$\varphi\left(-\frac{\pi}{2}\right) = \left(2\left(-\frac{\pi}{2}\right) - \pi\sin\left(-\frac{\pi}{2}\right),\ 2 - \pi\cos\left(-\frac{\pi}{2}\right)\right)$$

$$= \left(-\pi - \pi(-1),\ 2 - 0\right)$$

$$= \left(-\pi + \pi,\ 2\right)$$

$$= (0, 2)$$

so $2 = -\frac{\pi}{2}(0) + b$

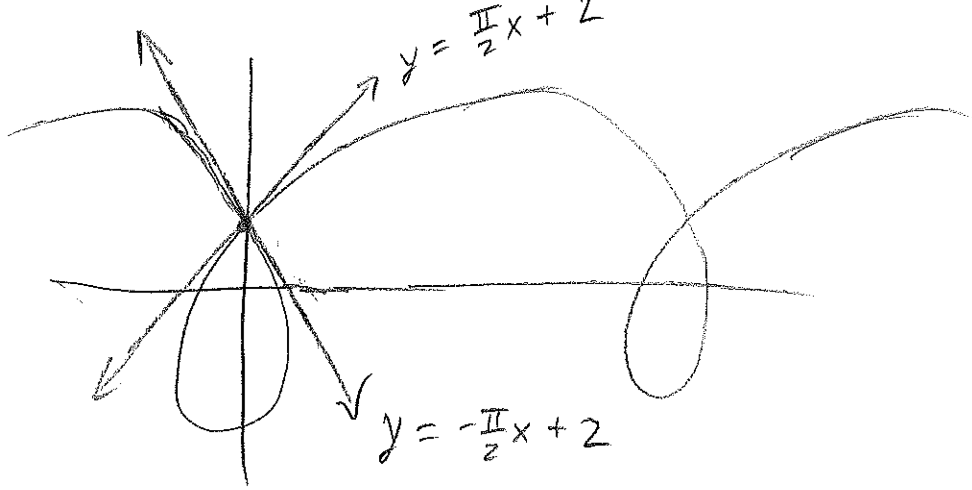
$\rightarrow$ $b = 2$

So $\boxed{y = -\frac{\pi}{2}x + 2}$

17Q

$x = 2t - \pi \sin(t)$

$y = 2 - \pi \cos(t)$



Concavity:

We define

$$\lambda(t) := \frac{\mu'(t)}{\varphi_1'(t)}.$$

If $\lambda(t) > 0$,

then C is concave up.

If $\lambda(t) < 0$,

then C is concave down.

ex Let C be the curve parametrized by

$\varphi : \mathbb{R} \to \mathbb{R}^2$

by $\varphi(t) = (t^3 - t, t^2 - 1)$.

(a) Is C concave up or concave down when $t = 0$?

(b) Find all values of t where C is concave up.

$\mu(t)$

$$= \frac{\varphi_2'(t)}{\varphi_1'(t)}$$

$$= \frac{2t}{3t^2-1}$$

$\lambda(t)$

$$= \frac{\mu'(t)}{\varphi_1'(t)}$$

$$= \frac{\dfrac{(2)(3t^2-1)-(2t)(6t)}{(3t^2-1)^2}}{3t^2-1}$$

$$= \frac{6t^2-2-12t^2}{(3t^2-1)^3}$$

$$= \frac{-6t^2-2}{(3t^2-1)^3}$$

$$= \frac{-2(3t^2+1)}{(3t^2-1)^3}$$

20Q

(a)

$\lambda(0)$

$$= \frac{-2(3(0)^2+1)}{(3(0)^2-1)^3}$$

$$= \frac{-2(1)}{(-1)^3}$$

$$= \frac{-2}{-1}$$

$$= 2$$

So G is concave up at $t=0$.

(b) Find all values of t that make $\lambda(t) > 0$.

So solve

$$\frac{-2(3t^2+1)}{(3t^2-1)^3} > 0$$

$$-2(3t^2+1) = 0$$

$$\rightarrow 3t^2+1 = 0$$

$$\rightarrow 3t^2 = -1$$

$$\rightarrow t^2 = -\frac{1}{3}$$

no (real) solutions.

$$(3t^2-1)^3 = 0$$

$$\rightarrow 3t^2-1 = 0$$

$$\rightarrow 3t^2 = 1$$

$$\rightarrow t^2 = \frac{1}{3}$$

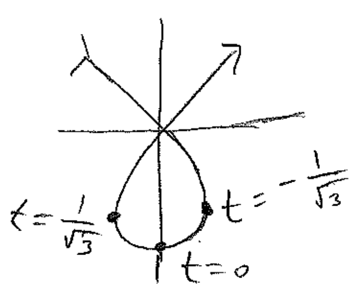
$$\rightarrow t = \sqrt{\frac{1}{3}} \text{ OR } t = -\sqrt{\frac{1}{3}}$$

$$\rightarrow t = \frac{1}{\sqrt{3}} \text{ OR } t = -\frac{1}{\sqrt{3}}$$

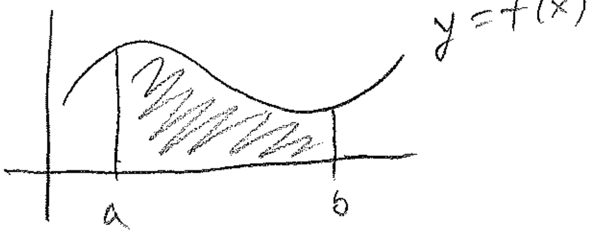
$$\lambda(t) = \frac{-2(3t^2+1)}{(3t^2-1)^3}$$

	$-\infty$ to $-1/\sqrt{3}$	$-1/\sqrt{3}$ to $1/\sqrt{3}$	$1/\sqrt{3}$ to ∞
$\lambda(t)$	$-$	$+$	$-$

C is concave up when $t \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



Areas:



$$\text{Area} = \int_a^b f(x)\,dx$$

$$= \int_a^b y\,dx$$

if $x = \varphi_1(t)$

and $y = \varphi_2(t)$

and $\varphi_1(\alpha) = a$

and $\varphi_1(\beta) = b$

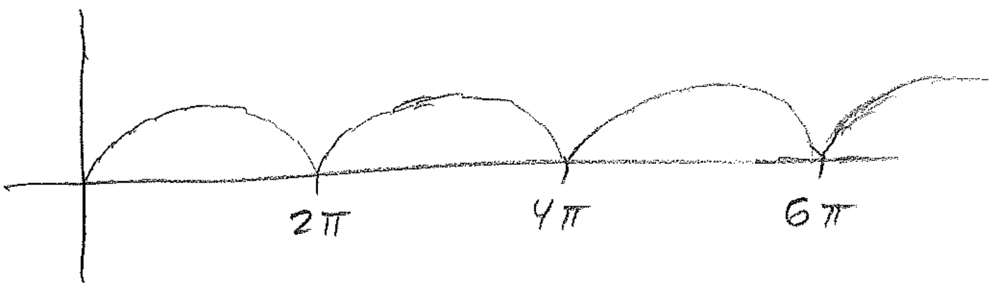
then $\frac{dx}{dt} = \varphi_1'(t)$

so $dx = \varphi_1'(t)\,dt$

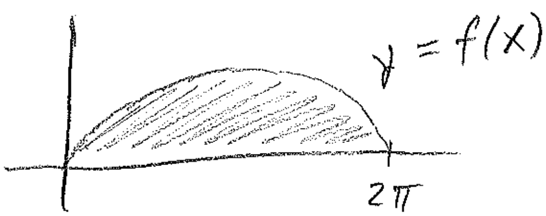
and $\int_a^b y\,dx$

$$= \int_\alpha^\beta \varphi_2(t)\,\varphi_1'(t)\,dt$$

ex Let C be the curve parametrized by $\varphi: \mathbb{R} \to \mathbb{R}^2$ by $\varphi(t) = (t - \sin(t), 1 - \cos(t))$.



Find the area under this curve from 0 to 2π.



$x = 0$

$\to \varphi_1(t) = 0$

$\to t - \sin(t) = 0$

$\to t = 0$

$x = 2\pi$

$\to \varphi_1(t) = 2\pi$

$\to t - \sin(t) = 2\pi$

$\to t = 2\pi$ So $\alpha = 0$, $\beta = 2\pi$

24Q

area

$$= \int_0^{2\pi} y \, dx$$

$$= \int_\alpha^\beta \varphi_2(t)\,\varphi_1'(t)\,dt$$

$$= \int_0^{2\pi} (1-\cos(t))(1-\cos(t))\,dt$$

$$= \int_0^{2\pi} (1 - 2\cos(t) + \cos^2(t))\,dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos(t) + \frac{1}{2}(1+\cos(2t))\right) dt$$

$$= \left[t - 2\sin(t) + \frac{1}{2}t + \frac{1}{4}\sin(2t)\right]_0^{2\pi}$$

$$= \left[2\pi - 2\sin(2\pi) + \frac{1}{2}2\pi + \frac{1}{4}\sin(2\cdot 2\pi)\right]$$
$$- \left[0 - 2\sin(0) + \frac{1}{2}0 + \frac{1}{4}\sin(2(0))\right]$$

$$= [2\pi - 0 + \pi + 0] - [0 - 0 + 0 + 0]$$

$$= \boxed{3\pi}$$

(more calculus with parametric equations)

Arclength:

If C is a curve parametrized by φ with $\varphi(t) = (\varphi_1(t), \varphi_2(t))$, then we define the arclength of C from $t=a$ to $t=b$ to be

$$L := \int_a^b \sqrt{(\varphi_1'(t))^2 + (\varphi_2'(t))^2}\, dt$$

So if $x = \varphi_1(t)$ and $y = \varphi_2(t)$, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

ex Let C be parametrized by

$\varphi(t) = (3t^2+1, 2t^3+4)$

Find the arclength of C from $t=0$ to $t=1$.

$x = 3t^2+1 \qquad \frac{dx}{dt} = 6t$

$y = 2t^3+4 \qquad \frac{dy}{dt} = 6t^2$

L

$= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$

$= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$

$= \int_0^1 \sqrt{36t^2(1+t^2)}\, dt$

$= \int_0^1 \sqrt{36t^2}\sqrt{1+t^2}\, dt$

$= \int_0^1 6t\sqrt{t^2+1}\, dt$

let $u := t^2+1$

$$\frac{du}{dt} = 2t$$

so $dt = \dfrac{du}{2t}$

so $$\int 6t\sqrt{t^2+1}\,dt$$

$$= \int 6t\, u^{1/2} \frac{du}{2t}$$

$$= \int 3u^{1/2}\,du$$

$$= 3\,\tfrac{2}{3}u^{3/2} + C_1$$

$$= 2\left(t^2+1\right)^{3/2} + C_1$$

So $$\int_0^1 6t\sqrt{t^2+1}\,dt$$

$$= \left[2(t^2+1)^{3/2}\right]_0^1$$

$$= \left[2(1^2+1)^{3/2}\right] - \left[2(0^2+1)^{3/2}\right]$$

$$= 2(2)^{3/2} - 2(1)$$

$$= \boxed{2\left(2^{3/2}-1\right)}$$

ex Let C be parametrized by

$\varphi(t) = \left(e^t \cos(t),\ e^t \sin(t)\right)$.

Find the arclength of C from $t=0$ to $t=\pi$.

$x = e^t \cos(t)$ $\qquad \frac{dx}{dt} = e^t \cos(t) + e^t(-\sin(t))$

$y = e^t \sin(t)$ $\qquad \frac{dy}{dt} = e^t \sin(t) + e^t \cos(t)$

L

$$= \int_0^{\pi} \sqrt{\left(e^t \cos(t) - e^t \sin(t)\right)^2 + \left(e^t \sin(t) + e^t \cos(t)\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{(e^t)^2 \cos^2(t) - 2(e^t)^2 \cos(t)\sin(t) + (e^t)^2 \sin^2(t) + (e^t)^2 \sin^2(t) + 2(e^t)^2 \cos(t)\sin(t) + (e^t)^2 \cos^2(t)}\, dt$$

$$= \int_0^{\pi} \sqrt{2(e^t)^2 \sin^2(t) + 2(e^t)^2 \cos^2(t)}\, dt$$

$$= \int_0^{\pi} \sqrt{2(e^t)^2 \left(\sin^2(t) + \cos^2(t)\right)}\, dt$$

5R

$$= \int_0^{\pi} \sqrt{2(e^t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2}\sqrt{(e^t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2}\, e^t\, dt$$

$$= \sqrt{2}\int_0^{\pi} e^t\, dt$$

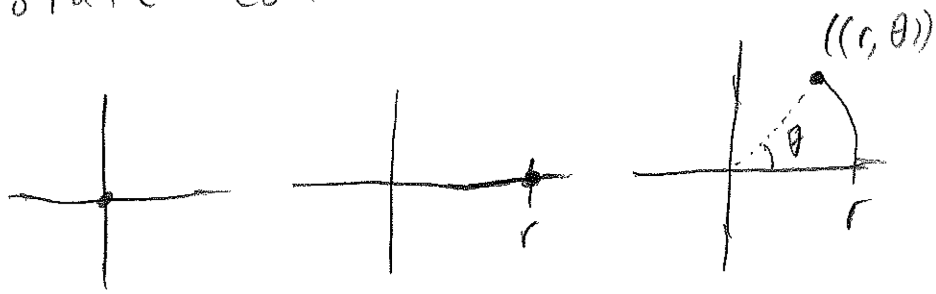
$$= \sqrt{2}\,\left[e^t\right]_0^{\pi}$$

$$= \sqrt{2}\left(\left[e^{\pi}\right] - \left[e^0\right]\right)$$

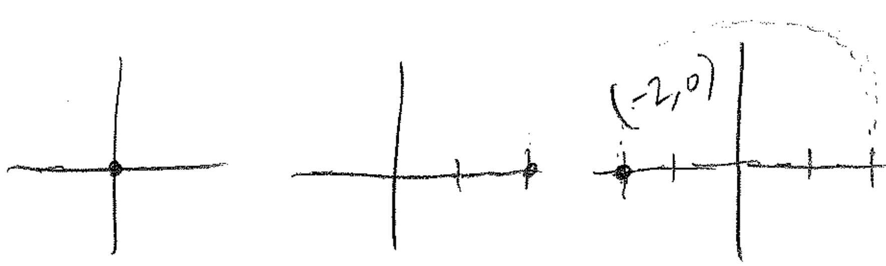
$$= \boxed{\sqrt{2}\,\left(e^{\pi} - 1\right)}$$

Polar Coordinates

We define $((r,\theta))$
to be the point
you arrive at if you
start at the origin,
travel r units to the right,
rotate counterclockwise by θ.



ex Find $((2,\pi))$



so $((2,\pi)) = (-2,0)$

If $((r,\theta)) = (x,y)$

then we call $((r,\theta))$

polar coordinates for (x,y).

We call (x,y) rectangular

or cartesian coordinates.

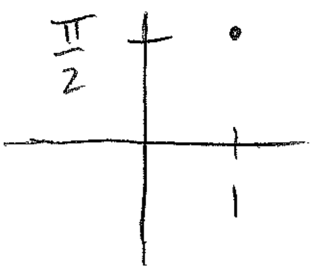
WARNING:

People usually just

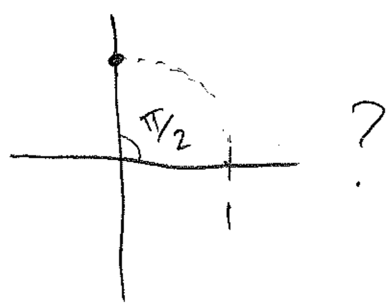
write (r,θ)

instead of $((r,\theta))$.

ex what is $(1,\frac{\pi}{2})$?



OR

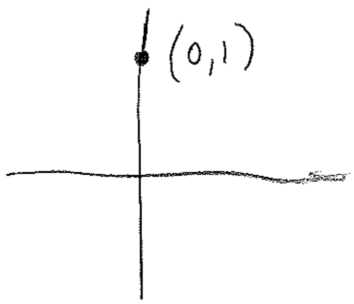


I say this is $(1,\frac{\pi}{2})$

I say this is $((1,\frac{\pi}{2}))$

Note that a point (x, y) can have many polar coordinate representations.

ex



$$(0,1) = \left(\left(1, \tfrac{\pi}{2}\right)\right)$$

$$= \left(\left(1, \tfrac{5\pi}{2}\right)\right)$$

$$= \left(\left(-1, \tfrac{3\pi}{2}\right)\right)$$

$$= \left(\left(1, -\tfrac{3\pi}{2}\right)\right)$$

$$= \left(\left(-1, -\tfrac{\pi}{2}\right)\right)$$

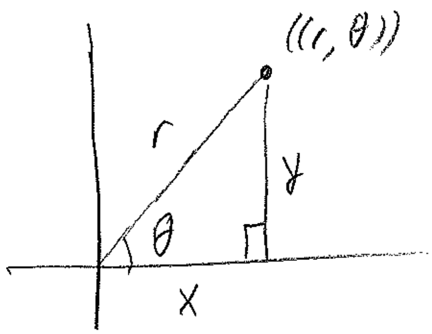
If $(x, y) = ((r, \theta))$,

then $x = r\cos(\theta)$

$y = r\sin(\theta)$

$\frac{y}{x} = \tan(\theta)$

$x^2 + y^2 = r^2$



$\sin(\theta) = \frac{y}{r}$

$\cos(\theta) = \frac{x}{r}$

$x^2 + y^2 = r^2$

So $((r, \theta)) := (\underbrace{r\cos(\theta)}_{x}, \underbrace{r\sin(\theta)}_{y})$

ex Convert to rectangular coordinates

$$\left(\left(3, \frac{\pi}{4}\right)\right)$$

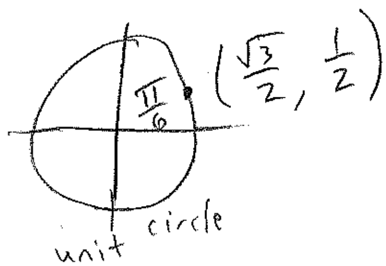
$$\left(\left(3, \frac{\pi}{4}\right)\right)$$

$$= \left(3\cos\left(\frac{\pi}{4}\right), 3\sin\left(\frac{\pi}{4}\right)\right)$$

$$= \left(3\,\frac{\sqrt{2}}{2}, 3\,\frac{\sqrt{2}}{2}\right)$$

$$= \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

$$\left(\left(2, \frac{\pi}{6}\right)\right)$$



$$= \left(2\cos\left(\frac{\pi}{6}\right), 2\sin\left(\frac{\pi}{6}\right)\right)$$

$$= \left(2\,\frac{\sqrt{3}}{2}, 2\,\frac{1}{2}\right)$$

$$= \left(\sqrt{3}, 1\right)$$

$((-1, \pi))$

$= \left(-\cos(\pi), -\sin(\pi)\right)$

$= (1, 0)$

$\left(\left(5, -\frac{\pi}{3}\right)\right)$

$= \left(5\cos\left(-\frac{\pi}{3}\right), 5\sin\left(-\frac{\pi}{3}\right)\right)$

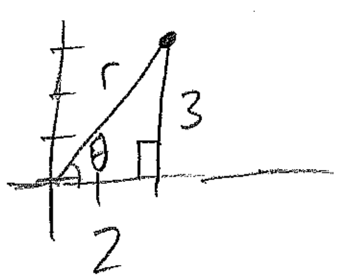
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$= \left(\frac{5}{2}, -\frac{5\sqrt{3}}{2}\right)$

---

ex Convert to polar coordinates

$(2, 3)$



$r^2 = 2^2 + 3^2$

$\rightarrow r = \sqrt{4+9}$

$\rightarrow r = \sqrt{13}$

$$\sin(\theta) = \frac{3}{\sqrt{13}}$$

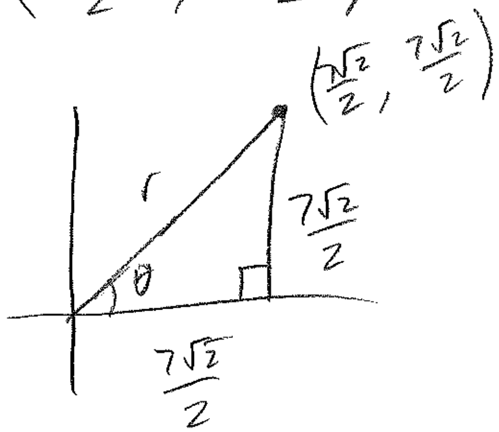
12R

$$\rightarrow \theta = \arcsin\left(\frac{3}{\sqrt{13}}\right)$$

so $(2, 3)$

$$= \left(\left(\sqrt{13}, \arcsin\left(\frac{3}{\sqrt{13}}\right)\right)\right)$$

$$\left(\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$$



$$r^2 = \left(\frac{7\sqrt{2}}{2}\right)^2 + \left(\frac{7\sqrt{2}}{2}\right)^2$$

$$\rightarrow r = \sqrt{\frac{49}{1}\frac{2}{4} + \frac{49}{1}\frac{2}{4}}$$

$$\rightarrow r = \sqrt{\frac{49}{2} + \frac{49}{2}}$$

$$\rightarrow r = \sqrt{49}$$

$$\rightarrow r = 7$$

$$\sin(\theta) = \frac{\frac{7\sqrt{2}}{2}}{7}$$

13 R

$$\rightarrow \sin(\theta) = \frac{\sqrt{2}}{2}$$

$$\rightarrow \theta = \frac{\pi}{4}.$$

$$\left(\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$$

$$= \left(\left(7, \frac{\pi}{4}\right)\right)$$

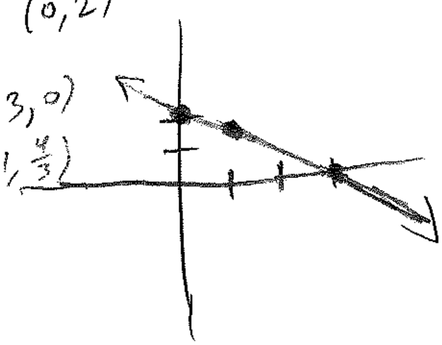
Polar Graphs

If we have an equation that involves x and y, then the graph of this equation is the graph of the set of all ordered pairs (x, y) that make the equation true.

ex $2x + 3y = 6$

Graph this equation.

x	y	
0	2	$\to (0,2)$
3	0	$\to (3,0)$
1	$\frac{4}{3}$	$\to (1, \frac{4}{3})$
$\vdots$	$\vdots$	



If we have an equation that involves r and θ, then the polar graph of this equation is the graph of the set of all polar coordinates $((r,\theta))$ that make the equation true.

ex $\quad 2r + 3\theta = 6$

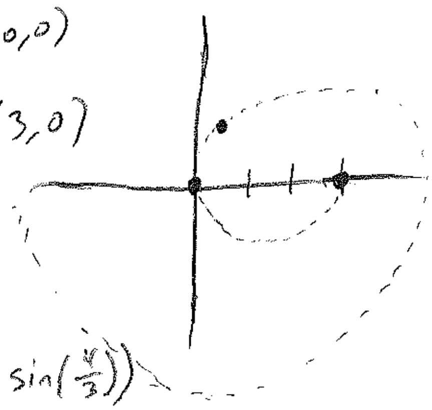
Make a polar graph for this equation

r	θ
0	2
3	0
1	$\frac{4}{3}$
$\vdots$	

$\rightarrow ((0,2)) = (0,0)$

$\rightarrow ((3,0)) = (3,0)$

$\rightarrow ((1,\tfrac{4}{3}))$
$= \left(\cos(\tfrac{4}{3}), \sin(\tfrac{4}{3})\right)$
$= (.23\ldots, .97\ldots)$

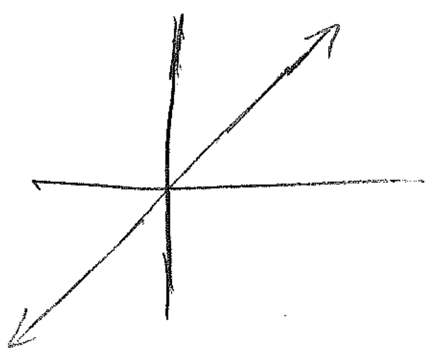


Standard Polar Graphs

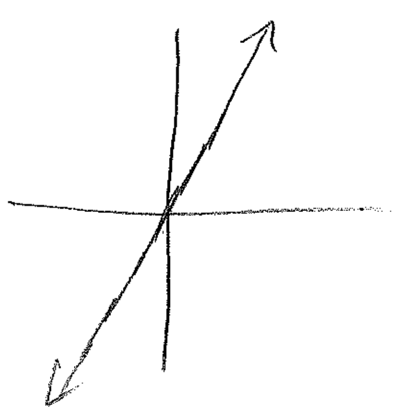
Line:

The polar graph of $\theta = c$ is the line through $(0,0)$ rotated counterclockwise by an angle of c.

ex Make a polar graph of $\theta = \frac{\pi}{4}$



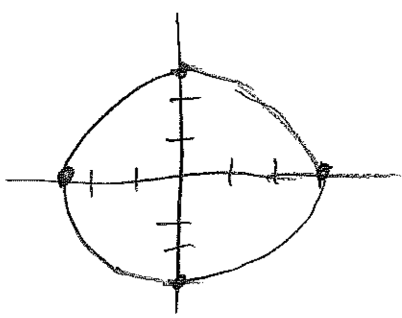
ex Make a polar graph of $\theta = \frac{\pi}{3}$



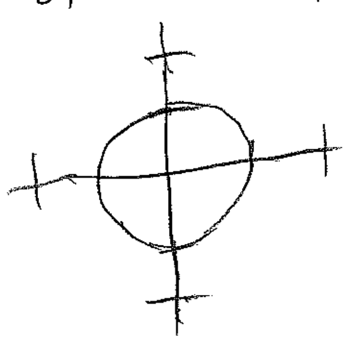
Circles:

The polar graph of $r = c$ is the circle with center $(0,0)$ and radius c.

ex Make a polar graph of $r = 3$



ex Make a polar graph of $r = 1$

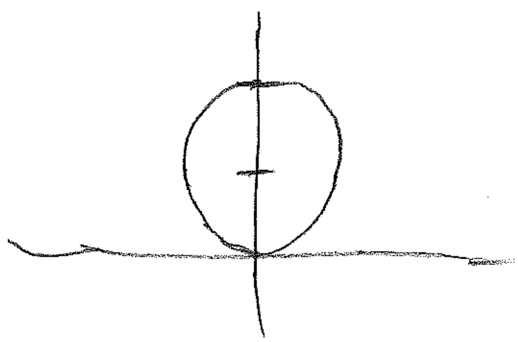


The polar graph of

$r = c\sin(\theta)$ is

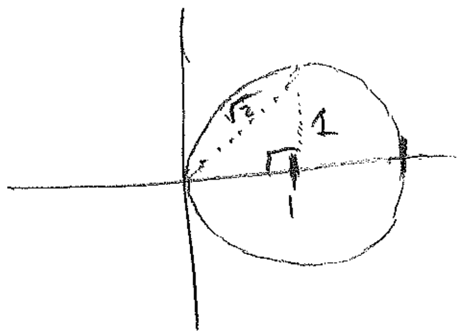
a circle

ex Make a polar graph of $r = 2\sin(\theta)$



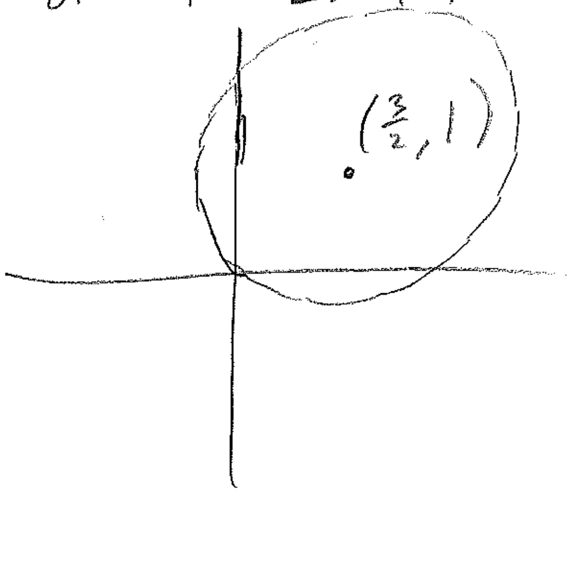
ex Make a polar graph

of $r = 2\cos(\theta)$



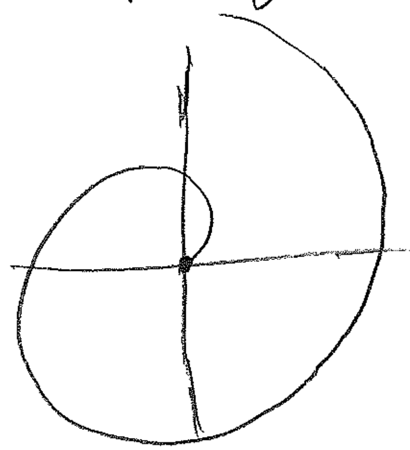
ex Make a polar graph

of $r = 2\sin(\theta) + 3\cos(\theta)$



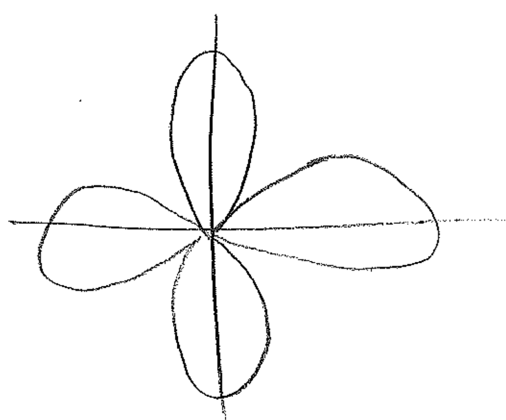
ex Make a polar graph 20R

of $r = \theta$.



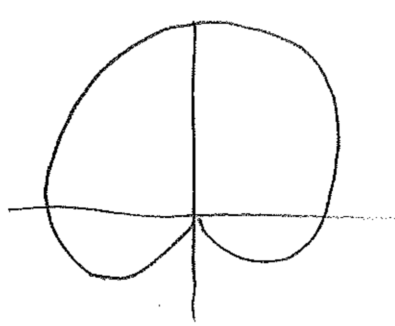
ex Make a polar graph

of $r = 3\cos(2\theta)$



ex Make a polar graph

of $r = \sin(\theta) + 1$



called

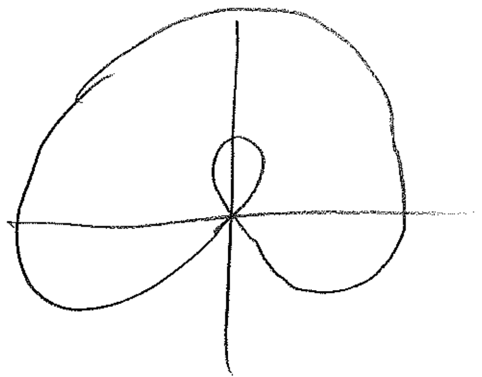
a cardioid.

can be seen

in a coffee mug.

ex Make a polar graph

of $r = 2\sin(\theta) + 1$



called a

limacon

ç french c with accent

(more polar coordinates)

If we have a rectangular equation, like $2x + 3y = 5$, then we can write down an equation involving r and θ whose polar graph is the same as the rectangular graph of the original equation. This new equation is called the polar equation.

To convert a rectangular equation into a polar equation, use

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$x^2 + y^2 = r^2$$

ex Convert

$$y = x$$

into a polar equation.

So write down an equation whose polar graph is the same as the rectangular graph of $y = x$.

$$y = x$$

$$\rightarrow \quad r\sin(\theta) = r\cos(\theta)$$

$$\rightarrow \quad \frac{r\sin(\theta)}{r\cos(\theta)} = 1$$

$$\rightarrow \quad \tan(\theta) = 1$$

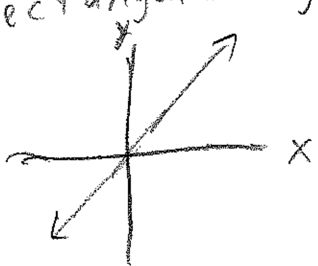
$$\rightarrow \quad \theta = \arctan(1)$$

$$\rightarrow \quad \boxed{\theta = \frac{\pi}{4}}$$

rectangular equation

$$y = x$$

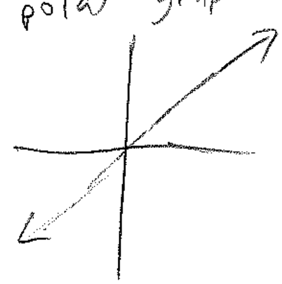
rectangular graph



polar equation

$$\theta = \frac{\pi}{4}$$

polar graph



ex Convert

$$y = x^2$$

into a polar equation.

$$y = x^2$$

$$\rightarrow r\sin(\theta) = \left(r\cos(\theta)\right)^2$$

$$\rightarrow \boxed{r\sin(\theta) = r^2\cos^2(\theta)}$$

ex Convert

$$x^2 + y^2 = 4$$

into a polar equation.

$$x^2 + y^2 = 4$$

$$\rightarrow (r\cos(\theta))^2 + (r\sin(\theta))^2 = 4$$

$$\rightarrow r^2\cos^2(\theta) + r^2\sin^2(\theta) = 4$$

$$\rightarrow r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) = 4$$

$$\rightarrow r^2 = 4$$

$$\rightarrow \boxed{r = 2}$$

Calculus with Polar Coordinates

Tangent Lines:

If C is a curve given by the polar graph of $r = f(\theta)$, then $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

So $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$.

If $t := \theta$, then C is parametrized by

$$x = f(t)\cos(t)$$
$$y = f(t)\sin(t).$$

Let $\varphi(t) = \Big(f(t)\cos(t), f(t)\sin(t)\Big)$.

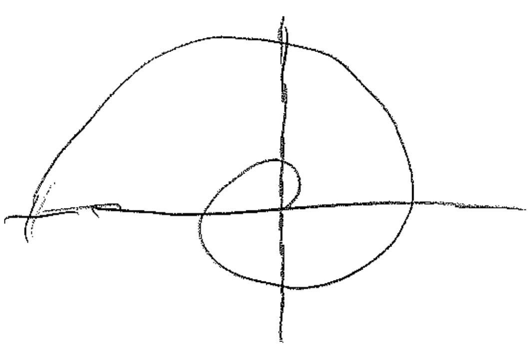
We define $\mu(t) := \dfrac{\varphi_2'(t)}{\varphi_1'(t)}$.

and $\lambda(t) := \dfrac{\mu'(t)}{\varphi_1'(t)}$.

We define the tangent line to C at θ to be the line through $\varphi(\theta)$ with slope of $\mu(\theta)$.

---

ex Let C be the curve given by the polar graph of $r = \theta$.



Find the equation of the tangent line to C when $\theta = \dfrac{3\pi}{4}$

6S

Let $\varphi(t) = \big(t\cos(t),\ t\sin(t)\big)$

(note $f(\theta) = \theta$, so $f(t) = t$)

So $\mu(t)$

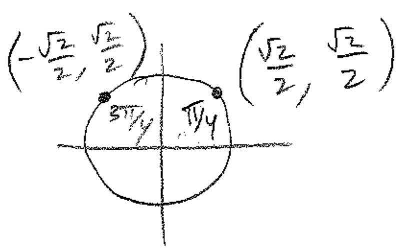
$$= \frac{\varphi_2'(t)}{\varphi_1'(t)}$$

$$= \frac{(1)(\sin(t)) + (t)(\cos(t))}{(1)(\cos(t)) + (t)(-\sin(t))}$$

$$= \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$$

and $\mu\left(\frac{3\pi}{4}\right)$

$$= \frac{\sin\left(\frac{3\pi}{4}\right) + \left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right) - \left(\frac{3\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)}$$



$$= \frac{\frac{\sqrt{2}}{2} + \frac{3\pi}{4}\left(-\frac{\sqrt{2}}{2}\right)}{-\frac{\sqrt{2}}{2} - \frac{3\pi}{4}\left(\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{\frac{\sqrt{2}}{2}\left(1 - \frac{3\pi}{4}\right)}{-\frac{\sqrt{2}}{2}\left(1 + \frac{3\pi}{4}\right)}$$

$$= (-1)\frac{1 - \frac{3\pi}{4}}{1 + \frac{3\pi}{4}}$$

$$= (-1)\frac{\frac{4}{4} - \frac{3\pi}{4}}{\frac{4}{4} + \frac{3\pi}{4}}$$

$$= (-1)\frac{\frac{4-3\pi}{4}}{\frac{4+3\pi}{4}}$$

$$= (-1)\,\frac{4-3\pi}{4}\cdot\frac{4}{4+3\pi}$$

$$= (-1)\,\frac{4-3\pi}{4+3\pi}$$

$$= \frac{3\pi - 4}{3\pi + 4}$$

So $\quad y = \dfrac{3\pi - 4}{3\pi + 4}x + b$

and $\varphi\left(\frac{3\pi}{4}\right) = \left(\frac{3\pi}{4}\cos\left(\frac{3\pi}{4}\right), \frac{3\pi}{4}\sin\left(\frac{3\pi}{4}\right)\right)$

$$= \left(\frac{3\pi}{4}\left(-\frac{\sqrt{2}}{2}\right), \frac{3\pi}{4}\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$= \left(-\frac{3\pi\sqrt{2}}{8}, \frac{3\pi\sqrt{2}}{8}\right)$$

So

$$\frac{3\pi\sqrt{2}}{8} = \frac{3\pi-4}{3\pi+4}\left(\frac{-3\pi\sqrt{2}}{8}\right) + b$$

$$\rightarrow \quad b = \frac{3\pi\sqrt{2}}{8} + \frac{3\pi-4}{3\pi+4}\left(\frac{3\pi\sqrt{2}}{8}\right).$$

So

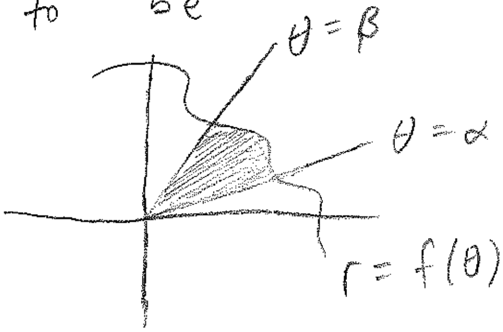
$$\boxed{y = \frac{3\pi-4}{3\pi+4}x + \left(\frac{3\pi\sqrt{2}}{8} + \frac{3\pi-4}{3\pi+4}\left(\frac{3\pi\sqrt{2}}{8}\right)\right)}$$

Areas:

If we have a curve C given by the polar graph of

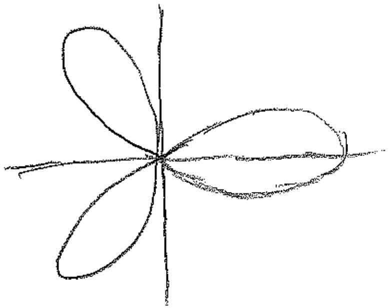
$$r = f(\theta),$$

then we define the area bounded by C as θ runs from α to β to be

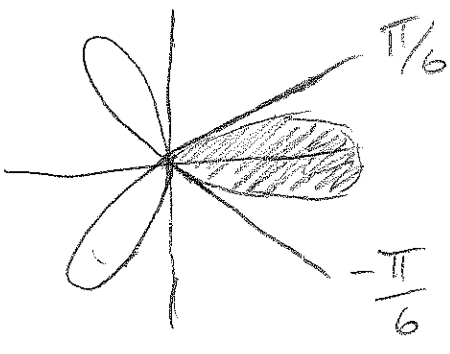


$$A := \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta.$$

ex Let C be the polar graph of $r = 3\cos(3\theta)$.



Find the area inside the petal that gets traced as θ runs from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$.



11 S

A

$$= \int_{-\frac{\pi}{6}}^{\pi/6} \frac{1}{2}\left(3\cos(3\theta)\right)^2 d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\pi/6} \frac{1}{2}\, 9\cos^2(3\theta)\, d\theta$$

$$= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$$

$$\cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$$

$$= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1}{2}\left(1+\cos(6\theta)\right) d\theta$$

$$= \frac{9}{4}\int_{-\pi/6}^{\pi/6} \left(1+\cos(6\theta)\right) d\theta$$

$$= \frac{9}{4}\left[\theta + \frac{1}{6}\sin(6\theta)\right]_{-\pi/6}^{\pi/6}$$

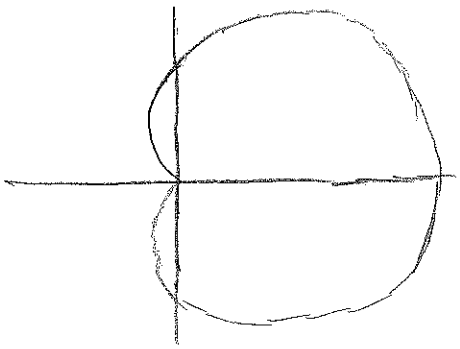
$$= \frac{9}{4}\left(\left[\frac{\pi}{6} + \frac{1}{6}\sin\left(6\frac{\pi}{6}\right)\right] - \left[-\frac{\pi}{6} + \frac{1}{6}\sin\left(6\left(-\frac{\pi}{6}\right)\right)\right]\right)$$

$$= \frac{9}{4}\left(\left[\frac{\pi}{6}\right] - \left[-\frac{\pi}{6}\right]\right)$$

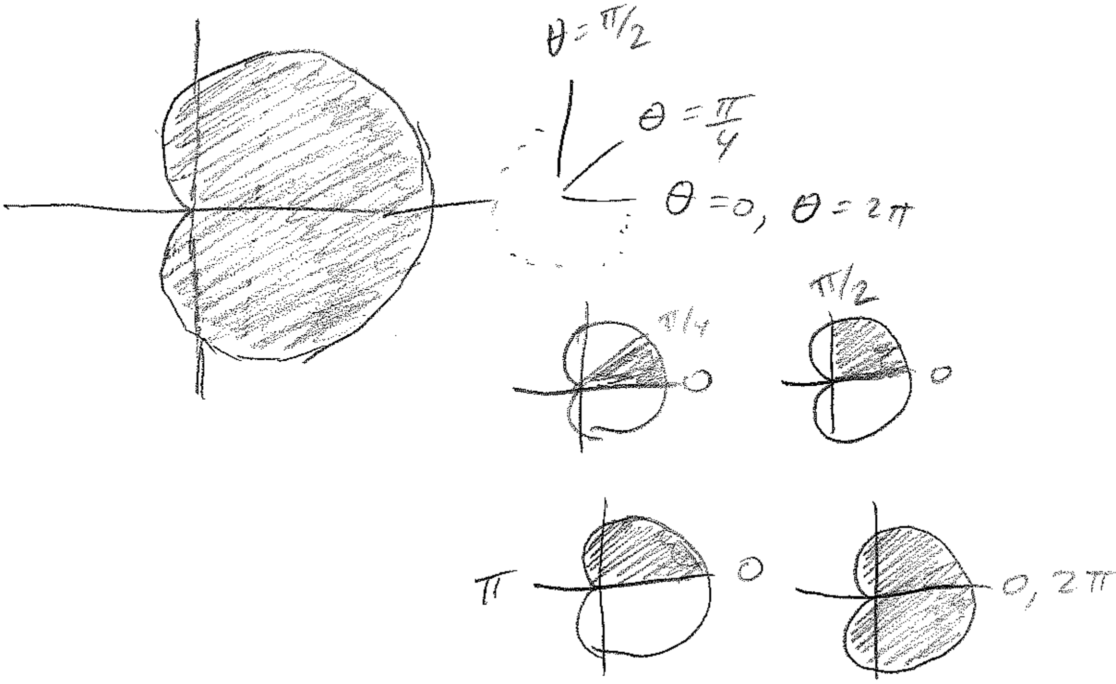
$$= \frac{9}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \frac{9}{4}\left(\frac{2\pi}{6}\right) = \frac{18\pi}{24} = \boxed{\frac{3\pi}{4}}$$

ex Let C be the polar graph of $r = 3(1 + \cos(\theta))$



Find the area inside of this curve as θ runs from 0 to 2π.



A

135

$$= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left(3(1+\cos(\theta))\right)^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} 9 \left(1+\cos(\theta)\right)^2 d\theta$$

$$= \frac{9}{2} \int_0^{2\pi} \left(1 + 2\cos(\theta) + \cos^2(\theta)\right) d\theta$$

$$= \frac{9}{2} \int_0^{2\pi} \left(1 + 2\cos(\theta) + \frac{1}{2}\left(1+\cos(2\theta)\right)\right) d\theta$$

$$= \frac{9}{2} \int_0^{2\pi} \left(1 + 2\cos(\theta) + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \frac{9}{2} \int_0^{2\pi} \left(\frac{3}{2} + 2\cos(\theta) + \frac{1}{2}\cos(2\theta)\right) d\theta$$

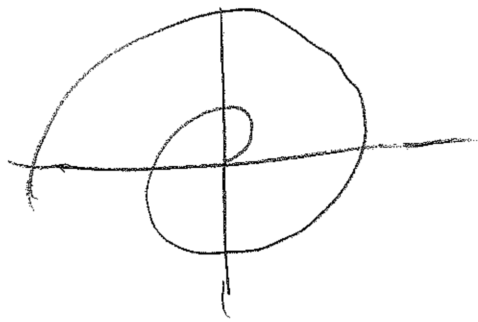
$$= \frac{9}{2} \left[\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}$$

$$= \frac{9}{2} \Big(\left[\frac{3}{2}(2\pi) + 2\sin(2\pi) + \frac{1}{4}\sin(2(2\pi))\right] - \left[\frac{3}{2}(0) + 2\sin(0) + \frac{1}{4}\sin(2\cdot 0)\right] \Big)$$

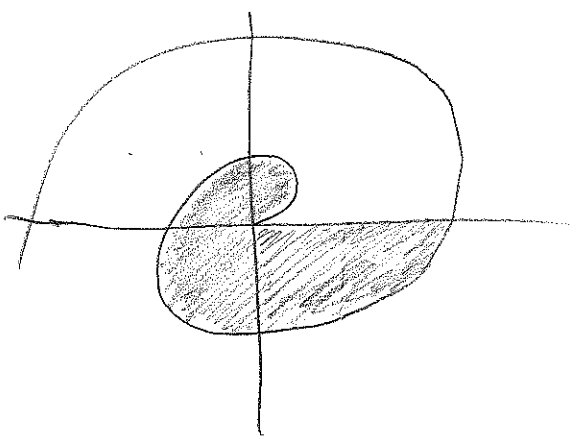
$$= \frac{9}{2} \left([3\pi + 0 + 0] - [0 + 0 - 0] \right)$$

$$= \frac{9}{2}(3\pi) = \boxed{\frac{27\pi}{2}}$$

ex Let C be the polar graph of $r = \theta$



Find the area inside of this curve as θ runs from 0 to 2π.



A 15.5

$$= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{0}^{2\pi} \frac{1}{2} \theta^2 \, d\theta$$

$$= \frac{1}{2} \int_{0}^{2\pi} \theta^2 \, d\theta$$

$$= \frac{1}{2} \left[\frac{1}{3} \theta^3 \right]_0^{2\pi}$$

$$= \frac{1}{2} \left(\left[\frac{1}{3} (2\pi)^3 \right] - \left[\frac{1}{3} (0)^3 \right] \right)$$

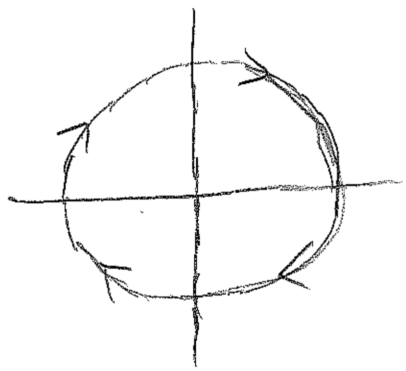
$$= \frac{1}{2} \left(\frac{1}{3} \left(8\pi^3 \right) \right)$$

$$= \frac{8\pi^3}{6}$$

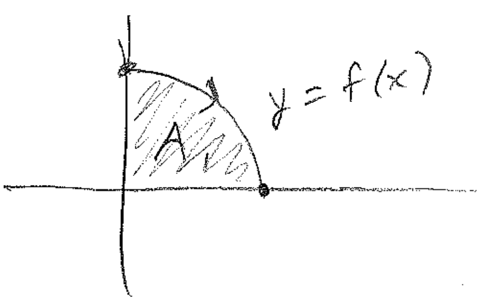
$$= \boxed{\frac{4\pi^3}{3}}$$

(more parametric areas) 25Q

ex Let C be the curve parametrized by $x = r\cos(t)$ and $y = -r\sin(t)$.



Find the area inside of this circle.



A_1

$= \int_0^1 y\,dx$

and $x = r\cos(t)$, $y = -r\sin(t)$

so $\frac{dx}{dt} = -r\sin(t)$

so $dx = -r\sin(t)dt$

$= \int_{3\pi/2}^{2\pi} -r\sin(t)(-r\sin(t)dt)$

260

$$= r^2 \int_{3\pi/2}^{2\pi} \sin^2(t)\,dt$$

$$= r^2 \int_{3\pi/2}^{2\pi} \frac{1}{2}\Big(1 - \cos(2t)\Big)\,dt$$

$$= \frac{r^2}{2} \int_{3\pi/2}^{2\pi} \Big(1 - \cos(2t)\Big)\,dt$$

$$= \frac{r^2}{2} \Big[t - \frac{1}{2}\sin(2t) \Big]_{3\pi/2}^{2\pi}$$

$$= \frac{r^2}{2} \left(\left[2\pi - \frac{1}{2}\sin(2 \cdot 2\pi) \right] - \left[\frac{3\pi}{2} - \frac{1}{2}\sin(3\pi) \right] \right)$$

$$= \frac{r^2}{2} \left(\left[2\pi - 0 \right] - \left[\frac{3\pi}{2} - 0 \right] \right)$$

$$= \frac{r^2}{2} \left(2\pi - \frac{3\pi}{2} \right)$$

$$= \frac{r^2}{2} \left(\frac{\pi}{2} \right)$$

$$= \frac{\pi r^2}{4}$$

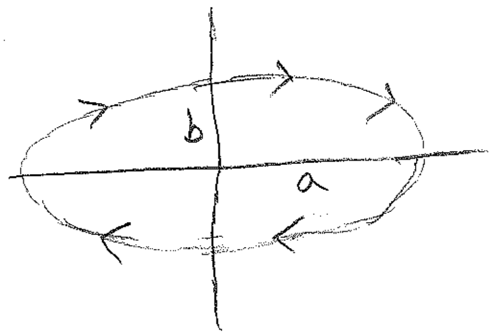
So $A = 4A_1$

$$= 4\left(\frac{\pi r^2}{4}\right)$$

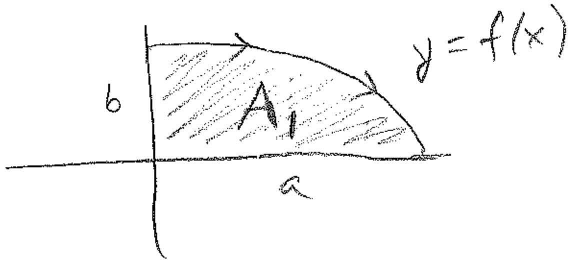
$$= \boxed{\pi r^2}$$

ex Let C be the curve parametrized by

$x = a\cos(t)$, $y = -b\sin(t)$



Find the area inside of this ellipse.



A_1

$= \int_0^1 y\,dx$

and $x = a\cos(t)$, $y = -b\sin(t)$

so $\frac{dx}{dt} = -a\sin(t)$

so $dx = -a\sin(t)\,dt$

$= \int_{3\pi/2}^{2\pi} -b\sin(t)\left(-a\sin(t)\,dt\right)$

$$= ab \int_{3\pi/2}^{2\pi} \sin^2(t)\,dt$$

$$= ab\left(\frac{\pi}{4}\right)$$

$$= \frac{ab\pi}{4}$$

So

$$A$$

$$= 4A_1$$

$$= 4\left(\frac{ab\pi}{4}\right)$$

$$= \boxed{ab\pi}$$