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Tractable Probabilistic Description Logic

Algorithms and Implementation

Andrew Ijano Lopes

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Tractable Probabilistic Description Logic

Algorithms and Implementation

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Resumo

Andrew Ijano Lopes. **Lógica de Descrição Probabilística Tratável: *Algoritmos e Implementação***. Monografia (Bacharelado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2020.

Elemento obrigatório, constituído de uma sequência de frases concisas e objetivas, em forma de texto. Deve apresentar os objetivos, métodos empregados, resultados e conclusões. O resumo deve ser redigido em parágrafo único, conter no máximo 500 palavras e ser seguido dos termos representativos do conteúdo do trabalho (palavras-chave). Deve ser precedido da referência do documento.

Palavras-chave: Lógicas de descrição. Palavra-chave2. Palavra-chave3.

Abstract

Andrew Ijano Lopes. **Tractable Probabilistic Description Logic: *Algorithms and Implementation***. Capstone Project Report (Bachelor). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2020.

Elemento obrigatório, elaborado com as mesmas características do resumo em língua portuguesa. De acordo com o Regimento da Pós-Graduação da USP (Artigo 99), deve ser redigido em inglês para fins de divulgação. É uma boa ideia usar o sítio www.grammarly.com na preparação de textos em inglês.

Keywords: Description logics. Keyword2. Keyword3.

Lista de Abreviaturas

DL Description Logic

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Todo list

Chapter 1

Introduction

Need to rewrite, but the example is OK

Description logics are a family of formal knowledge representation languages, being of particular importance in providing a logical formalism for ontologies and the Semantic Web. Also, they are notable in biomedical informatics for assisting the codification of biomedical knowledge. Due to these uses, there is a great demand to find tractable (i.e., polynomial-time decidable) description logics.

One of them, the logic $\mathcal{EL}^{++}$, is one of the most expressive description logics in which the complexity of inferential reasoning is tractable (Baader *et al.*, 2005). Even though it is expressive enough to deal with several practical applications, there was also a need to model uncertain knowledge.

Example 1.1

> Consider a medical situation, in which a patient may have non-specific symptoms, such as high fever, cough, and headache. Also, COVID-19, a severe acute respiratory syndrome caused by the SARS-CoV-2 virus, is a disease that can account for those symptoms, but not all patients present all symptoms. Such an uncertain situation is suitable for probabilistic modeling.
>
> In a certain hospital, a patient with a high fever has some probability of having COVID-19, but that probability is 20% larger if the patient has a cough too. On the other hand, COVID-19 is not very prevalent and is not observed in the hospital 90% of the time. If those probabilistic constraints are satisfiable, one can also ask the minimum and maximum probability that a hospital patient Mary, with fever and cough, is a suspect of suffering from COVID-19.

For classical propositional formulas, this problem, called *probabilistic satisfiability* (PSAT), has already been presented with tractable fragments (Andersen and Pretolani, 2001). On the other hand, in description logics, most studies result in intractable reasoning; moreover, by adding probabilistic reasoning capabilities to $\mathcal{EL}^{++}$, to model such situation, the complexity reaches NP-completeness (Finger, 2019).

To solve this problem, probabilistic constraints can be applied to axioms and its probabilistic satisfaction can be seen in a linear algebraic view. Furthermore, it can be reduced to an optimization problem, which can be solved by an adaptation of the simplex method with column generation (Finger, 2019). Thus, it is possible to reduce the column generation problem to the *weighted partial maximum satisfiability*.

Moreover, recent studies show that it is necessary to focus on a fragment of $\mathcal{EL}^{++}$ for obtain tractable probabilistic reasoning, which will be called Graphic $\mathcal{EL}^{++}$ ($\mathcal{GEL}^{++}$) (Finger, n.d.). Therefore, this fragment allows axioms to be seen as edges in a graph, as opposed to hyperedges in a hypergraph, which is the case of $\mathcal{EL}^{++}$. This allows the use of graph-based machinery to develop a tractable algorithm for the *weighted partial Maximum SATisfiability* for $\mathcal{GEL}^{++}$ (Max $\mathcal{GEL}^{++}$-SAT) and, as a result, a tractable probabilistic description logic.

1.1 Objective

Then, the objective of this project is to propose and implement tractable algorithms for weighted partial Max-SAT and Probabilistic SAT for a fragment of $\mathcal{EL}^{++}$ description logic.

1.2 Structure

In this paper, we describe the implementation of these algorithms [1] and is organized as follows: Section **??** highlights related results in the literature. The basic definition of $\mathcal{GEL}^{++}$ with its algorithms for MaxSAT and PSAT are described in Section **??** and followed by Section **??**, which presents details about the implementation and its experimental evaluation.

[1] Available at https://github.com/AndrewIjano/pgel-sat

Chapter 2

Background

In this chapter, we present the theoretical background of DLs and some definitions.

Add initial description of the chapter

2.1 Description logics

Add section about DLs in general

2.2 The description logic $\mathcal{EL}^{++}$

$\mathcal{EL}^{++}$ is an extension of the DL $\mathcal{EL}$ (Baader *et al.*, 2005). It was created with large bio-health ontologies in mind, such as SNOMED-CT, the NCI thesaurus, and Galen, and became an official OWL 2 profile (Hitzler *et al.*, 2009). We concentrate on presenting $\mathcal{EL}^{++}$ without concrete domains.

2.2.1 Syntax

In $\mathcal{EL}^{++}$, *concept descriptions* are defined inductively from a set $\mathsf{N_C}$ of *concept names*, a set $\mathsf{N_R}$ of *role names* and set $\mathsf{N_I}$ of *individual names* as follows:

- $\top$, $\bot$ and concept names in $\mathsf{N_C}$ are concept descriptions;
- if C and D are concept descriptions, $C \sqcap D$ is a concept description;
- if C is a concept description and $r \in \mathsf{N_R}$, $\exists r.C$ is a concept description;
- if $a \in \mathsf{N_I}$, $\{a\}$ is a concept description.

To represent knowledge using concept descriptions, we need to define facts (axioms and role inclusions) and assertions.

An *axiom*, or a *general concept inclusion* (GCI), is an expression of the form $C \sqsubseteq D$, where C and D are concept inclusions. A *role inclusion* (RI) is an expression of the form $r_1 \circ \cdots \circ r_k \sqsubseteq r$, where $r_1, \cdots r_k, r \in \mathsf{N_R}$. The symbol "$\circ$" denotes composition of binary relations. A *constraint box* (CBox) is a finite set of GCIs and a finite set of RIs.

Similarly, a *concept assertion* is an expression of the form $C(a)$ and a *role assertion*, $r(a, b)$, where C is a concept description, $a, b \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$. A finite set of concept assertions and role assertions is an *assertional box* (ABox).

Then, an $\mathcal{EL}^{++}$ *knowledge base* $\mathcal{K}$ (KB) is a pair $(\mathcal{C}, \mathcal{A})$, where $\mathcal{C}$ is a CBox and $\mathcal{A}$ is an ABox.

2.2.2 Semantics

The semantics of $\mathcal{EL}^{++}$ are given by *interpretations* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$. The *domain* $\Delta^\mathcal{I}$ is a non-empty set of individuals, and the *interpretation function* $\cdot^\mathcal{I}$ maps each concept name $A \in \mathsf{N_C}$ to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, each role name r to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and each individual name a to an element $a^\mathcal{I} \in \Delta^\mathcal{I}$. The extension of $\cdot^\mathcal{I}$ for an arbitrary concept description is inductively defined by the third column of Table 2.1.

Name	Syntax	Semantics
top	$\top$	$\Delta^\mathcal{I}$
bottom	$\bot$	$\emptyset$
nominal	$\{a\}$	$\{a^\mathcal{I}\}$
conjunction	$C \sqcap D$	$C^\mathcal{I} \cap D^\mathcal{I}$
existential restriction	$\exists r.C$	$\{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I} \ : \ (x, y) \in r^\mathcal{I} \wedge y \in C^\mathcal{I}\}$
GCI	$C \sqsubseteq D$	$C^\mathcal{I} \subseteq D^\mathcal{I}$
RI	$r_1 \circ \cdots \circ r_k \sqsubseteq r$	$r_1^\mathcal{I} \circ \cdots \circ r_k^\mathcal{I} \subseteq r^\mathcal{I}$
concept assertion	$C(a)$	$a^\mathcal{I} \in C^\mathcal{I}$
role assertion	$r(a, b)$	$(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$

Table 2.1: *Syntax and semantics of $\mathcal{EL}^{++}$ without concrete domains*

The interpretation $\mathcal{I}$ *satisfies*:

- an axiom $C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ (represented as $I \models C \sqsubseteq D$);
- a RI $r_1 \circ \cdots \circ r_k \sqsubseteq r$ if $r_1^\mathcal{I} \circ \cdots \circ r_k^\mathcal{I} \subseteq r^\mathcal{I}$ (represented as $I \models r_1 \circ \cdots \circ r_k \sqsubseteq r$);
- an assertion $C(a)$ if $a^\mathcal{I} \in C^\mathcal{I}$ (represented as $I \models C(a)$);
- an assertion $r(a, b)$ if $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$ (represented as $I \models r(a, b)$).

Also, we say that $\mathcal{I}$ is a *model* of:

- a CBox $\mathcal{C}$ if it satisfies every axiom and RI in $\mathcal{C}$ (represented as $\mathcal{I} \models \mathcal{C}$);
- an ABox $\mathcal{A}$ if it satisfies every assertion in $\mathcal{A}$ (represented as $\mathcal{I} \models \mathcal{A}$);

Then, an important problem in $\mathcal{EL}^{++}$ is to determine its *consistency*, that is if $\mathcal{A}$ and $\mathcal{C}$ have a common model, which is in PTime (Baader *et al.*, 2005).

2.2.3 Normal form

We can convert an $\mathcal{EL}^{++}$ knowledge base into a normal form, in polynomial time, by introducing new concept and role names (Baader *et al.*, 2005).

First, there is no need of explicit ABox, because $\mathcal{I} \models C(a) \iff \mathcal{I} \models \{a\} \sqsubseteq C$ and $\mathcal{I} \models r(a, b) \iff \{a\} \sqsubseteq \exists r.\{b\}$. In other words, a knowledge base can be represented by just a CBox, by transforming assertions in axioms.

In addition, given a CBox $\mathcal{C}$, consider the set $\mathsf{BC}_{\mathcal{C}}$ of *basic concept descriptions*, which is the smallest set of concept descriptions that contains the top concept $\top$, all concept names used in $\mathcal{C}$ and all concepts of the form $\{a\}$ used in $\mathcal{C}$.

Then, every axiom can be represented in the following normal form, where $C_1, C_2 \in \mathsf{BC}_{\mathcal{C}}$, $D \in \mathsf{BC}_{\mathcal{C}} \cup \{\bot\}$:

$$
\begin{aligned}
C_1 &\sqsubseteq D && \text{(simple)}\\
C_1 &\sqsubseteq \exists r.C_2 && \text{(existential-head)}\\
C_2 &\sqsubseteq D && \text{(existential-body)}\\
C_1 \sqcap C_2 &\sqsubseteq D && \text{(conjunctive-body)}
\end{aligned}
$$

And every RI are of the form $r \sqsubseteq s$ or $r_1 \circ r_2 \sqsubseteq s$.

Example 2.1

Consider the following CBox $\mathcal{C}_{exa}$ representing the situation in Example 1.1. On the left, we have basic knowledge of diseases and, on the right, the specific knowledge about Mary. Note that, for simplicity, it is not in normal form.

$$
\begin{array}{ll}
\mathsf{Fever} \sqsubseteq \mathsf{Symptom} & \\
\mathsf{Cough} \sqsubseteq \mathsf{Symptom} & \\
\mathsf{COVID\text{-}19} \sqsubseteq \mathsf{Disease} & \{\mathsf{mary}\} \sqsubseteq \mathsf{Patient}\\
\mathsf{Symptom} \sqsubseteq \exists \mathsf{hasCause}.\mathsf{Disease} & \{\mathsf{s1}\} \sqsubseteq \mathsf{Fever} \sqcap \mathsf{Cough}\\
\mathsf{Patient} \sqsubseteq \exists \mathsf{hasSymptom}.\mathsf{Symptom} & \{\mathsf{mary}\} \sqsubseteq \exists \mathsf{hasSymptom}.\{\mathsf{s1}\}\\
\mathsf{CovidPatient} \equiv \exists \mathsf{suspectOf}.\mathsf{COVID\text{-}19} & \\
\mathsf{hasSymptom} \circ \mathsf{hasCause} \sqsubseteq \mathsf{suspectOf} &
\end{array}
$$

Because CBoxes can only represent facts, there is no way to describe uncertain knowledge. Even though, in cases when which of them are true, we could define three axioms

Ax_1 := $\mathsf{Fever} \sqsubseteq \exists\mathsf{hasCause}.\mathsf{COVID\text{-}19}$, when fever is actually caused by COVID-19;

Ax_2 := $\mathsf{Fever} \sqcap \mathsf{Cough} \sqsubseteq \exists\mathsf{hasCause}.\mathsf{COVID\text{-}19}$, when both fever and cough are caused by COVID-19;

Ax_3 := $\mathsf{COVID\text{-}19} \sqsubseteq \bot$, when there are no presence of COVID-19 in the hospital.

> In the following sections, it will be presented how to add these axioms in a KB with probabilistic properties.

We want to model uncertain information using DLs. However, it has been proved that, by adding probabilistic reasoning capabilities to $\mathcal{EL}^{++}$, the complexity reaches NP-completeness (Finger, 2019). Then, it is necessary to reduce the expressiveness of this language.

2.3 Graphic $\mathcal{EL}$ ($\mathcal{GEL}$)

Graphic $\mathcal{EL}$ ($\mathcal{GEL}$) is a fragment of $\mathcal{EL}^{++}$ in which every axiom and RI are in normal form and no conjunctive-body axiom is allowed (Finger, n.d.). The semantics are the same as that of $\mathcal{EL}^{++}$.

Example 2.2

> Since there are conjunctive-body axioms in the CBox in Example 2.1, we need to modify this knowledge in order to represent it in $\mathcal{GEL}$. First, we substitute every concept description $\mathsf{Fever} \sqcap \mathsf{Cough}$ by a new basic concept FeverAndCough. After that, we add axioms $\mathsf{FeverAndCough} \sqsubseteq \mathsf{Fever}$ and $\mathsf{FeverAndCough} \sqsubseteq \mathsf{Cough}$ to CBox.

The name of this fragment comes from the fact that each GCI can be represented as arrows in a graph where nodes are basic concepts. This representation is useful for the development of algorithms and it is used to define its SAT decision.

2.3.1 Graphical representation

Consider a $\mathcal{GEL}$-CBox $\mathcal{C}$ with n_R roles, its graphical representation is a edge-labeled graph $G(\mathcal{C}) := (N, E, \ell)$, where N is a set of nodes, $E \subseteq N^2$ is a set of directed edges and $\ell : E \longrightarrow \{0, 1, \dots, n_R\}$ is a labeling function.

Do I define the graph construction here? It is the same process that I use in the implementation and I could describe it there

Example about graph completion

2.3.2 SAT decision

If I explain the graph construction in the Development section, where is the best place to explain the SAT decision?

Graph completion rules; SAT is path init-bot in completed graph

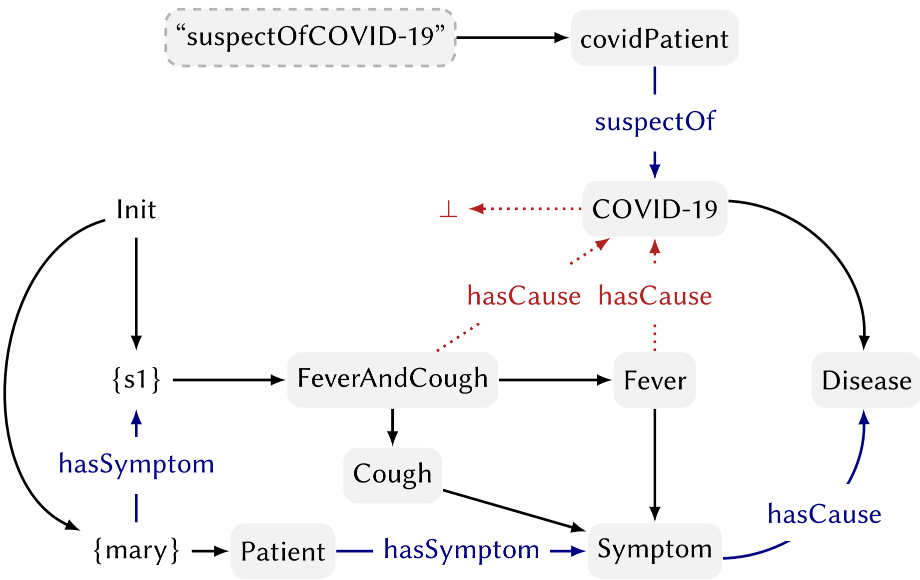

Figure 2.1: *Graphical representation of the ontology in Example 2.2.*

2.4 MaxSAT for $\mathcal{GEL}$

Before focusing on the probabilistic extension for $\mathcal{GEL}$, we need to define its MaxSAT problem, which will compose further the probabilistic reasoner.

The *weighted partial maximum satisfiability problem for* $\mathcal{GEL}$ ($\mathcal{GEL}$-MaxSAT) can be defined as follows: given a potentially inconsistent weighted CBox $\mathcal{C}$, we want to find the maximal satisfiable subset of axioms; since it is partial, some axioms must be present in this subset. Usually, partiality can be modeled assigning infinite weights to the axioms that must not be excluded.

A *weighted* CBox is a pair $\langle \mathcal{C}, w \rangle$ where $\mathcal{C}$ is a CBox and $w : \mathcal{C} \longrightarrow \mathbb{Q} \cup \{\infty\}$ is a weight function, which maps axioms in $\mathcal{C}$ to weights. The infinite weight is used to represent axioms that must not be excluded in the maximal satisfiable subset. Also, it is defined that RIs of the form $r \sqsubseteq s$ and $r_1 \circ r_2 \sqsubseteq s$ always have infinite weight.

Then, given a weighted CBox $\langle \mathcal{C}, w \rangle$, a solution for the weighted partial $\mathcal{GEL}$-MaxSAT problem is a set $\mathcal{C}_{max} \subseteq \mathcal{C}$ such that:

- $\mathcal{C}_{max}$ is satisfiable; and
- $\mathcal{C}_{max} \models C \sqsubseteq D$ if $w(C \sqsubseteq D) = \infty$; and
- the sum of finite weights in $\mathcal{C}_{max}$ is maximal.

2.5 Probabilistic $\mathcal{GEL}$

Probability in $\mathcal{GEL}$ is constructed from a *probability function* P (Finger, n.d.). Consider a finite number of interpretation, $\mathcal{I}_1, \dots, \mathcal{I}_m$, we define the probability function $P : \{\mathcal{I}_1, \dots, \mathcal{I}_m\} \longrightarrow \mathbb{Q}$, such that $P(\mathcal{I}_i) \geq 0$ and $\sum_{i=1}^{m} P(\mathcal{I}_i) = 1$. We can also define the probability of an axiom $C \sqsubseteq D$ as follows

$$P(C \sqsubseteq D) = \sum_{\mathcal{I}_i \models C \sqsubseteq D} P(\mathcal{I}_i).$$

A *probabilistic knowledge base* is a pair $\langle \mathcal{C}, \mathcal{P} \rangle$, where $\mathcal{C}$ is a CBox and $\mathcal{P}$ is a PBox. A PBox is a set of k linear constraints over n axioms, of the form

$$\sum_{j=1}^{n} a_{ij} \cdot P(C_j \sqsubseteq D_j) \leq b_i; \quad 1 \leq i \leq k. \tag{2.1}$$

We can define the *satisfiability problem* for this probabilistic KB (PGEL-SAT) as deciding if it is consistent or not. If it is consistent, the solution is a set of interpretations $\{\mathcal{I}_1, \ldots, \mathcal{I}_m\}$ and a probability function $P : \{\mathcal{I}_1, \ldots, \mathcal{I}_m\} \longrightarrow \mathbb{Q}^+$ such that $\sum_{i=1}^{m} P(\mathcal{I}_i) = 1$, $P(C \sqsubseteq D) = 1$ for $C \sqsubseteq D \in \mathcal{C}$ (axioms in CBox are certain) and P verifies all linear constraints in $\mathcal{P}$.

Example 2.3

Now we can model the uncertain situation stated in Example 1.1 using the probability knowledge base $\langle \mathcal{C}_{exa}, \mathcal{P}_{exa} \rangle$, where $\mathcal{C}_{exa}$ is the CBox from Example 2.2 and $\mathcal{P}_{exa}$ is given by

$$\begin{aligned} \mathcal{P}_{exa} \;:= \{ \quad & P(Ax_2) - P(Ax_1) = 0.2, \\ & P(Ax_3) = 0.9 \quad \}. \end{aligned}$$

I don't think this is the right way to express a PBox

Then, we need a polynomial algorithm to find if this probabilistic KB is consistent.

2.5.1 Linear algebraic view

The PGEL-SAT problem was also defined by Finger (n.d.) in a linear algebraic view, which is useful to develop its polynomial reasoning algorithm. It was shown that a probabilistic KB $\langle \mathcal{C}, \mathcal{P} \rangle$ is satisfiable iff the linear equation $C \cdot x = d$ has a solution $x \geq 0$, where

$$C := \begin{bmatrix} -I_n & M_{n\times m} \\ A_{k\times n} & 0_{k\times m} \\ 0'_n & 1'_m \end{bmatrix} \quad x := \begin{bmatrix} p_n \\ \pi_m \end{bmatrix} \quad d := \begin{bmatrix} 0_n \\ b_k \\ 1 \end{bmatrix} \tag{2.2}$$

and

- $A_{k\times n}$ is a $k \times n$ matrix whose elements a_{ij} are given by Equation 2.1;
- b_k is a k vector whose elements b_i are also given by Equation 2.1;
- $M_{n\times m}$ is a $n \times m$ matrix given by the following steps:

Consider an interpretation $\mathcal{I}$ model of $\mathcal{C}$, its corresponding vector in $\mathcal{P}$ is a $\{0, 1\}$-vector y such that $y_i = 1$ iff $\mathcal{I} \models C_i \sqsubseteq D_i$, for $1 \leq i \leq n$.

Then, given a set of interpretations $\mathcal{I}_1, \dots, \mathcal{I}_m$, we define $M_{n \times m}$ a matrix whose column M^j is $\mathcal{I}_j$'s corresponding vector in $\mathcal{P}$;

- I_n is the n-dimensional identity matrix;
- 0_n is a column 0-vector of size n (similarly for 1_n);
- $0'_n$ is the previous vector's transpose;
- $0_{k \times m}$ is a 0-matrix of shape $k \times m$;
- p_n is a vector of size n which corresponds to the probability of axioms occurring in Equation 2.1.
- π_m is a vector of size m which corresponds to the probability distribution over interpretations $\mathcal{I}_1, \dots, \mathcal{I}_m$.

Maybe define ill-formed and well-formed columns; Definition of cost vector; Definition of the linear system solution by the minimation problem

Chapter 3

Development

3.1 Input and output format

3.2 OWL parser

3.3 Knowledge Base

3.4 GEL-MaxSAT

3.5 Linear solver

3.6 PGEL-SAT reasoner

Chapter 4

Experiments

Chapter 5

Results

Chapter 6

Related work

The problem of probabilistic reasoning and extensions in logics to deal with uncertainty have been studied for several decades. The first known proposal of PSAT, for propositional formulas, is attributed to Boole (1854) and it has already been shown to be NP-Complete (Georgakopoulos *et al.*, 1988).

In the relational domain, the literature contain several logics with probabilistic reasoning capabilities although they have led to intractable decision problems. Some of them extend the already intractable $\mathcal{ALC}$, with probabilistic constrains over concepts (Heinsohn, 1994; Lukasiewicz, 2008; Gutiérrez-Basulto *et al.*, 2011). For the expressive and lightweight $\mathcal{EL}$-family, some extensions such as Gutiérrez-Basulto *et al.* (2017) and Ceylan and Peñaloza (2017) have led to ExpTime-hard or PP-complete probabilistic reasoning; futhermore, NP-completeness can be achieved with probability capabilities over axioms (Finger, 2019).

On the other hand, many results implies that the research on Max-SAT has a impact on the solutions of PSAT problems (Andersen and Pretolani, 2001). Also, there was already proposed a MaxSAT-solver for a propositional fragment of horn logic by a max-flow/min-cut formulation (Jaumard and Simeone, 1987). Thus, it is expected to ask if one could also take such results to a relational domain.

Chapter 7

Conclusion

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