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# Tractable Probabilistic Description Logic

## *Algorithms and Implementation*

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# Resumo

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Lógicas de descrição (LDs) são uma família de linguagens de representação de conhecimento. Uma delas, o $\mathcal{EL}^{++}$, está dentre uma das LDs mais expressivas cuja complexidade de raciocínio inferencial é tratável. Entretanto, simplesmente adicionar restrições probabilísticas deixa complexidade de decisão intratável. Esse trabalho apresenta algoritmos para um raciocínio probabilístico em um fragmento do $\mathcal{EL}^{++}$, chamado Graphic $\mathcal{EL}$. Os algoritmos modelam seu problema de satisfatibilidade probabilística como um programa linear, que pode ser resolvido por uma adaptação do método simplex com geração de colunas. Então, é possível reduzir o problema de geração de colunas para o de satisfatibilidade máxima partial ponderada para $\mathcal{GEL}$ ($\mathcal{GEL}$-MaxSAT). Esse fragmento permite modelar axiomas como arestas de um grafo com pesos, o que motiva o uso de técnicas baseadas em grafos, calculando cortes mínimos, para desenvolver um algoritmo tratável para $\mathcal{GEL}$-MaxSAT e, consequentemente, uma lógica de descrição probabilística tratável. Esses algoritmos foram implementados e, para essa solução, um limite super teórico foi estimado e análises experimentais confirmam a complexidade polinomial do tempo de execução.

**Palavras-chave:** Lógicas de descrição. Polinomial. Tratável. PSAT. MaxSAT. Corte Mínimo. Programação Linear. Geração de colunas.

# Abstract

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Description logics (DLs) are a family of knowledge representation languages. One of them, the $\mathcal{EL}^{++}$, is among the most expressive DLs in which the complexity of inferential reasoning is tractable. However, simply adding probabilistic constraints leaves its decision complexity unfeasible. This work presents tractable algorithms for a probabilistic reasoning in a fragment of $\mathcal{EL}^{++}$, called Graphic $\mathcal{EL}$ ($\mathcal{GEL}$). The algorithms model its probabilistic satisfiability problem as a linear program, which can be solved by an adaptation of the simplex method with column generation. Thus, it is possible to reduce the column generation problem to the weighted partial maximum satisfiability for $\mathcal{GEL}$ ($\mathcal{GEL}$-MaxSAT). This fragment allows modeling axioms as edges in a weighted graph, which motivates the use of graph-based machinery, calculating minimal cuts, to develop a tractable algorithm for the $\mathcal{GEL}$-MaxSAT and, as a result, a tractable probabilistic description logic. These algorithms were implemented and, for this solution, a theoretical upper bound was estimated and experimental analysis confirm the polynomial complexity of the run time.

**Keywords:** Description logics. Polynomial. Tractable. PSAT. MaxSAT. Minimal Cut. Linear Programming. Column generation.

# Abbreviations

| | |
|---|---|
| DL | Description logic |
| SAT | Satisfiability problem |
| PSAT | Probailistic satisfatibility problem |
| $\mathcal{GEL}$ | Graphic $\mathcal{EL}$ |
| $\mathcal{GEL}$-MaxSAT | Weighted partial maximum satisfiability problem for $\mathcal{GEL}$ |
| PGEL-SAT | Probablistic satisfiability problem for $\mathcal{GEL}$ |
| KB | Knowledge base |
| GCI | General concept inclusion |
| RI | Role inclusion |

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# Chapter 1

# Introduction

Description logics are a family of formal knowledge representation languages, being of particular importance in providing a logical formalism for ontologies and the Semantic Web. Also, they are notable in biomedical informatics for assisting the codification of biomedical knowledge. Due to these uses, there is a great demand to find tractable (i.e., polynomial-time decidable) description logics.

One of them, the logic $\mathcal{EL}^{++}$, is one of the most expressive description logics in which the complexity of inferential reasoning is tractable (Baader *et al.*, 2005). Even though it is expressive enough to deal with several practical applications, there was also a need to model situations in which an axiom is not always true, which has already been proposed in the literature (Boole, 1854).

**Example 1.1**

> Consider a medical situation in which a patient may have non-specific symptoms, such as high fever, cough, and headache. Also, COVID-19, a severe acute respiratory syndrome caused by the SARS-CoV-2 virus, is a disease that can account for those symptoms, but not all patients present all symptoms. Such an uncertain situation is suitable for probabilistic modeling.
>
> In a certain hospital, a patient with a high fever has some probability of having COVID-19, but that probability is 20% larger if the patient has a cough too. On the other hand, COVID-19 is not very prevalent and is not observed in the hospital 90% of the time. If those probabilistic constraints are satisfiable, one can also ask the minimum and maximum probability that a hospital patient Mary, with fever and cough, is a suspect of suffering from COVID-19.

For classical propositional formulas, this problem, called *probabilistic satisfiability* (PSAT), has already been presented with tractable fragments (Andersen and Pretolani, 2001). On the other hand, in description logics, most studies result in intractable reasoning; moreover, by adding probabilistic reasoning capabilities to $\mathcal{EL}^{++}$, to model such situation, the complexity reaches NP-completeness (Finger, 2019).

To solve this problem, ongoing studies of Finger and Lopes (n.d.) propose a fragment of $\mathcal{EL}^{++}$ called *Graphic* $\mathcal{EL}$ ($\mathcal{GEL}$), which does not allow axioms with conjunctions or existential restrictions in the left-hand side. With this logic, using probabilistic constraints applied to axioms, its probabilistic satisfaction can be seen in a linear algebraic view. Furthermore, it can be reduced to an optimization problem, which can be solved by an adaptation of the simplex method with column generation. Thus, it is possible to reduce the column generation problem to the *weighted partial maximum satisfiability for $\mathcal{GEL}$* ($\mathcal{GEL}$-MaxSAT). Another property of this fragment is that axioms can be modeled as edges in a graph, as opposed to hyperedges in a hypergraph, which is the case of $\mathcal{EL}^{++}$. This allows the use of graph-based machinery to develop a tractable algorithm for the $\mathcal{GEL}$-MaxSAT and, as a result, a tractable probabilistic description logic.

## 1.1 Objectives

The main objective of this work is contribute to the studies of Finger and Lopes (n.d.) in order to find and implement algorithms for a tractable probabilistic description logic. This goal can be summarized in the following points:

1. Investigate a potentially tractable fragment of $\mathcal{EL}^{++}$;
2. Study and implement tractable algorithms for the problem of $\mathcal{GEL}$-MaxSAT;
3. Study and implement algorithms for the problem of *probabilistic satisfiability for $\mathcal{GEL}$* (PGEL-SAT), using the $\mathcal{GEL}$-MaxSAT solver as a subroutine. Thus, it is excepted to achieve a tractable algorithm for a probabilistic description logic.

## 1.2 Structure of this work

This work is organized as follows. Chapter 2 presents the theoretical background about description logics and the logical fragment of interest. Chapter 3 describe the development of a tractable algorithm for solving probabilistic satisfiability in description logics. Chapter 4 presents experiments to analyse the tractability of this algorithms and discusses its results. After that, Chapter 5 revises the current studies about this topic. Then, Chapter 6 summarizes the achievements and discusses further research.

# Chapter 2

# Background

In this chapter, we present the theoretical background of description logics and the fragment of logic focused in this work. The main satisfatibility problems are defined and detailed.

## 2.1 Description logics

Description logics (DLs) are used to represent knowledge, such as the semantic of words, people and their relations, and medical terms. Such set of terms are called an *ontology*. These scenarios require precise specification and meaning so that different systems behave the same way. The first DL modeling languages appeared in the mid-1980s and have an important role in the context of the Semantic Web, an initiative to represent web content in a form that is more machine friendly (Krötzsch *et al.*, 2012).

As their name suggests, DLs are logics; indeed, most of them are fragments of first-order logic. This relation with logics is what provides their precise specification, called *formal semantics*. Also, it equips their languages with a formal deduction to *infer* additional information, and the computation of these inferences is called *reasoning*. The performance of algorithms for reasoning strongly relies on the expressiveness of the DL: fast algorithms usually exist for lightweight logics. Then, there is not just a single DL because the balance between expressiveness and performance depends on the application. (Krötzsch *et al.*, 2012)

### 2.1.1 Building blocks of description logic ontologies

A DL is composed of concepts, roles, and individual names. Concepts are sets of individuals, roles are binary relations between individuals, and individual names are single individuals in the domain.

For example, an ontology modeling the situation in Example 1.1 can use the concepts Patient, to represent the set of all patients in the hospital, and Symptom, to represent the set of all symptoms; roles such hasSymptom, to represent the binary relation between patients and symptoms; and individual names such as mary and s1, to represent the individuals Mary and Mary's symptoms.

Additionally, DLs allows us to describe more complex situations, creating new concepts and roles from the previously defined ones.

Some concept constructors provide boolean operations similar to that found in set theory and logic expressions. For example, if we want to describe the set of individuals that are both fever and cough, we could use the *conjunction* operator, as follows

$$\mathsf{Fever} \sqcap \mathsf{Cough}.$$

We can link concepts and roles using role restrictions. For example, to describe all individuals that are suspect of some disease that is COVID-19, we use the *existential restriction*

$$\exists \mathsf{suspectOf}.\mathsf{COVID\text{-}19}.$$

Also, to define concepts with only one individual we use *nominals* like $\{\mathsf{mary}\}$.

More expressive logics can have other operations such as *disjunction* ($C \sqcup D$), *negation* ($\neg C$), *universal restriction* ($\forall r.C$) and *number restrictions* ($\leq n\ r.C$).

To capture knowledge about the world, DL ontologies also allow us to describe relations between concepts, roles, and individual names. For example, the fact that all fevers are symptoms is represented by the *concept inclusion*

$$\mathsf{Fever} \sqsubseteq \mathsf{Symptom};$$

the knowledge that someone that has a symptom which is caused by some disease is suspect of that disease can be expressed by the *role inclusion* with a *role composition*

$$\mathsf{hasSymptom} \circ \mathsf{hasCause} \sqsubseteq \mathsf{suspectOf};$$

and the fact that Mary is a patient of the hospital and has symptoms is represented by the *assertions* $\mathsf{Patient}(\mathsf{mary})$ and $\mathsf{hasSymptom}(\mathsf{mary}, \mathsf{s1})$.

After that, if we have a set of these relations, one could ask if there is a set of individuals, or instances, that satisfies these relations, which is called an *interpretation*. Interpretations can be understood as the assignment of meaning to logical terms in an ontology. Because a DL usually considers all the possibles situations, property that is sometimes referred to as *open world assumption*, an ontology can have multiple satisfiable interpretations. The fewer restrictions it has, the more interpretations satisfy this ontology. The computational complexity to find the existence of these interpretations is one of the key aspects to choose different DL fragments.

These terms will be formally defined in the Section 2.2, in the case of the DL $\mathcal{EL}^{++}$.

### 2.1.2 Description logic fragments and OWL

There are many DL fragments. Each subset of features, like those described previously, can lead to different fragments of first-order logic. For example, the logic $\mathcal{ALC}$ does not allow role inclusions and admits only $\sqcap, \sqcup, \neg, \exists$ and $\forall$ as concept constructors; their best reasoning algorithms, however, are worst-case exponential time. On the other hand, the $\mathcal{EL}$ logic allows only $\sqcap$ and $\exists$ as concept constructors, and its reasoning algorithms are polynomial time.

To express DL ontologies, the World Wide Web Consortium (W3C) designed the OWL 2 Web Ontology Language (OWL 2) (Hitzler *et al.*, 2009). This declarative language is part of the W3C's Semantic Web technology stack and comes with various syntaxes, such as RDF/XML. Because of this use on the web, names in OWL are *international resource identifiers* (IRIs).

## 2.2 The description logic $\mathcal{EL}^{++}$

The logic $\mathcal{EL}^{++}$ is an extension of the DL $\mathcal{EL}$ (Baader *et al.*, 2005). It was created with large bio-health ontologies in mind, such as SNOMED-CT, the NCI thesaurus, and Galen, and became an official OWL 2 profile (Hitzler *et al.*, 2009). We concentrate on presenting $\mathcal{EL}^{++}$ without concrete domains.

### 2.2.1 Syntax

In $\mathcal{EL}^{++}$, *concept descriptions* are defined inductively from a set $\mathsf{N_C}$ of *concept names*, a set $\mathsf{N_R}$ of *role names* and set $\mathsf{N_I}$ of *individual names* as follows:

- $\top$, $\bot$ and concept names in $\mathsf{N_C}$ are concept descriptions;
- if $C$ and $D$ are concept descriptions, $C \sqcap D$ is a concept description;
- if $C$ is a concept description and $r \in \mathsf{N_R}$, $\exists r.C$ is a concept description;
- if $a \in \mathsf{N_I}$, $\{a\}$ is a concept description.

To represent knowledge using concept descriptions, we need to define facts (axioms and role inclusions) and assertions.

An *axiom*, or a *general concept inclusion* (GCI), is an expression of the form $C \sqsubseteq D$, where $C$ and $D$ are concept inclusions. Also, we write $C \equiv D$ to represent the axioms $C \sqsubseteq D$ and $D \sqsubseteq C$. A *role inclusion* (RI) is an expression of the form $r_1 \circ \cdots \circ r_k \sqsubseteq r$, where $r_1, \ldots r_k, r \in \mathsf{N_R}$. The symbol "$\circ$" denotes composition of binary relations. A *constraint box* (CBox) is a finite set of GCIs and a finite set of RIs.

Similarly, a *concept assertion* is an expression of the form $C(a)$ and a *role assertion*, $r(a, b)$, where $C$ is a concept description, $a, b \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$. A finite set of concept assertions and role assertions is an *assertional box* (ABox).

Then, an $\mathcal{EL}^{++}$ *knowledge base* $\mathcal{K}$ (KB) is a pair $(\mathcal{C}, \mathcal{A})$, where $\mathcal{C}$ is a CBox and $\mathcal{A}$ is an ABox.

### 2.2.2 Semantics

The semantics of $\mathcal{EL}^{++}$ are given by *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. The *domain* $\Delta^{\mathcal{I}}$ is a non-empty set of individuals, and the *interpretation function* $\cdot^{\mathcal{I}}$ maps each concept name $A \in \mathsf{N_C}$ to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, each role name $r$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and each individual name $a$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The extension of $\cdot^{\mathcal{I}}$ for an arbitrary concept description is inductively defined by the third column of Table 2.1.

| Name | Syntax | Semantics |
|---|---|---|
| top | $\top$ | $\Delta^{\mathcal{I}}$ |
| bottom | $\bot$ | $\emptyset$ |
| nominal | $\{a\}$ | $\{a^{\mathcal{I}}\}$ |
| conjunction | $C \sqcap D$ | $C^{\mathcal{I}} \cap D^{\mathcal{I}}$ |
| existential restriction | $\exists r.C$ | $\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} \;:\; (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$ |
| GCI | $C \sqsubseteq D$ | $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ |
| RI | $r_1 \circ \cdots \circ r_k \sqsubseteq r$ | $r_1^{\mathcal{I}} \circ \cdots \circ r_k^{\mathcal{I}} \subseteq r^{\mathcal{I}}$ |
| concept assertion | $C(a)$ | $a^{\mathcal{I}} \in C^{\mathcal{I}}$ |
| role assertion | $r(a, b)$ | $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ |

**Table 2.1:** *Syntax and semantics of $\mathcal{EL}^{++}$ without concrete domains*

The interpretation $\mathcal{I}$ *satisfies*:

- an axiom $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ (represented as $I \models C \sqsubseteq D$);
- a RI $r_1 \circ \cdots \circ r_k \sqsubseteq r$ if $r_1^{\mathcal{I}} \circ \cdots \circ r_k^{\mathcal{I}} \subseteq r^{\mathcal{I}}$ (represented as $I \models r_1 \circ \cdots \circ r_k \sqsubseteq r$);
- an assertion $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (represented as $I \models C(a)$);
- an assertion $r(a, b)$ if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ (represented as $I \models r(a, b)$).

Also, we say that $\mathcal{I}$ is a *model* of:

- a CBox $\mathcal{C}$ if it satisfies every axiom and RI in $\mathcal{C}$ (represented as $\mathcal{I} \models \mathcal{C}$);
- an ABox $\mathcal{A}$ if it satisfies every assertion in $\mathcal{A}$ (represented as $\mathcal{I} \models \mathcal{A}$);

Then, an important problem in $\mathcal{EL}^{++}$ is to determine its *consistency*, that is if $\mathcal{A}$ and $\mathcal{C}$ have a common model, which is in PTime (Baader *et al.*, 2005).

### 2.2.3 Normal form

We can convert an $\mathcal{EL}^{++}$ knowledge base into a normal form, in polynomial time, by introducing new concept and role names (Baader *et al.*, 2005).

First, there is no need of explicit ABox, because $\mathcal{I} \models C(a) \iff \mathcal{I} \models \{a\} \sqsubseteq C$ and $\mathcal{I} \models r(a, b) \iff \{a\} \sqsubseteq \exists r.\{b\}$. In other words, a knowledge base can be represented by just a CBox, by transforming assertions in axioms.

In addition, given a CBox $\mathcal{C}$, consider the set $\mathsf{BC}_{\mathcal{C}}$ of *basic concept descriptions*, which is the smallest set of concept descriptions that contains the top concept $\top$, all concept names used in $\mathcal{C}$ and all concepts of the form $\{a\}$ used in $\mathcal{C}$.

Then, every axiom can be represented in the following normal form, where $C_1, C_2 \in \mathsf{BC}_{\mathcal{C}}$, $D \in \mathsf{BC}_{\mathcal{C}} \cup \{\bot\}$:

$$C_1 \sqsubseteq D \quad \text{(simple)}$$
$$C_1 \sqsubseteq \exists r.C_2 \quad \text{(existential-head)}$$
$$\exists r.C_1 \sqsubseteq D \quad \text{(existential-body)}$$
$$C_1 \sqcap C_2 \sqsubseteq D \quad \text{(conjunctive-body)}$$

And every RI are of the form $r \sqsubseteq s$ or $r_1 \circ r_2 \sqsubseteq s$.

**Example 2.1**

Consider the following CBox $\mathcal{C}_{exa}$ representing the situation in Example 1.1. On the left, we have basic knowledge of diseases and, on the right, the specific knowledge about Mary. Note that, for simplicity, it is not in normal form.

$\mathsf{Fever} \sqsubseteq \mathsf{Symptom}$
$\mathsf{Cough} \sqsubseteq \mathsf{Symptom}$
$\mathsf{COVID\text{-}19} \sqsubseteq \mathsf{Disease}$
$\mathsf{Symptom} \sqsubseteq \exists \mathsf{hasCause}.\mathsf{Disease}$
$\mathsf{Patient} \sqsubseteq \exists \mathsf{hasSymptom}.\mathsf{Symptom}$
$\mathsf{hasSymptom} \circ \mathsf{hasCause} \sqsubseteq \mathsf{suspectOf}$

$\{\mathsf{mary}\} \sqsubseteq \mathsf{Patient}$
$\{\mathsf{s1}\} \sqsubseteq \mathsf{Fever} \sqcap \mathsf{Cough}$
$\{\mathsf{mary}\} \sqsubseteq \exists \mathsf{hasSymptom}.\{\mathsf{s1}\}$

Because CBoxes can only represent facts, there is no way to describe uncertain knowledge. Even though, in cases when which of them are true, we could define three axioms

$Ax_1$ := $\mathsf{Fever} \sqsubseteq \exists \mathsf{hasCause}.\mathsf{COVID\text{-}19}$, when fever is actually caused by COVID-19;

$Ax_2$ := $\mathsf{Fever} \sqcap \mathsf{Cough} \sqsubseteq \exists \mathsf{hasCause}.\mathsf{COVID\text{-}19}$ , when both fever and cough are caused by COVID-19;

$Ax_3$ := $\mathsf{COVID\text{-}19} \sqsubseteq \bot$, when there are no presence of COVID-19 in the hospital.

In the following sections, it will be presented how to add these axioms in a KB with probabilistic properties.

We want to model uncertain information using DLs. However, it has been proved that, by adding probabilistic reasoning capabilities to $\mathcal{EL}^{++}$, the complexity reaches NP-completeness (Finger, 2019). Then, it is necessary to reduce the expressiveness of this language.

## 2.3 Graphic $\mathcal{EL}$ ($\mathcal{GEL}$)

*Graphic* $\mathcal{EL}$ ($\mathcal{GEL}$) is a fragment of $\mathcal{EL}^{++}$ in which every axiom and RI are in normal form and only simple and existential-head axioms are allowed (Finger and Lopes, n.d.).

The semantics are the same as that of $\mathcal{EL}^{++}$.

**Example 2.2**

> Since there are conjunctive-body axioms in the CBox in Example 2.1, we need to modify this knowledge in order to represent it in $\mathcal{GEL}$. First, we substitute every concept description Fever ⊓ Cough by a new basic concept FeverAndCough. After that, we add axioms FeverAndCough ⊑ Fever and FeverAndCough ⊑ Cough to CBox.

The name of this fragment comes from the fact that each GCI can be represented as arrows in a graph where nodes are basic concepts. This representation is useful for the development of algorithms and it is used to define its SAT decision.

### 2.3.1 Graphical representation

Consider a $\mathcal{GEL}$-CBox $\mathcal{C}$ with $n_R$ roles, its graphical representation is a edge-labeled graph $G(\mathcal{C}) := (N, E, \ell)$, where $N$ is a set of nodes, $E \subseteq N^2$ is a set of directed edges and $\ell : E \longrightarrow \{0, 1, \dots, n_R\}$ is a labeling function.

In addition, Finger and Lopes (n.d.) defines the following notation. The set $E_i \subseteq E$ is the set of all edges $e$ such that $\ell(e) = i$. We write $X_1 \longrightarrow_i X_2$ if there is an edge $(X_1, X_2) \in E_i$, and $X_1 \not\longrightarrow_i X_2$ if $(X_1, X_2) \notin E_i$. The expression $X \longrightarrow_i^* Y$ represents the reflexive transitive closure of $\longrightarrow_i$, which is the existence of a path in the graph of size $\geq 0$, starting in $X$, ending in $Y$, and only going through edges in $E_i$, for $0 \leq i \leq n_R$. Finally, $X \rightsquigarrow Y$ represents a path from $X$ to $Y$ using any type of edge. Furthermore, we write $(u \longrightarrow_i v)$ to refer to an edge $e := (u, v)$ such that $\ell(e) = i$; and we write $(x \cdots u \longrightarrow_i v \longrightarrow_j w \cdots y)$ to refer to a path $(x, \dots, u, (u \longrightarrow_i v), v, (v \longrightarrow_j w), w, \dots, y)$.

Then, the graph $G(\mathcal{C}) = (N, E, \ell)$ can be constructed from a CBox $\mathcal{C}$ with the following steps:

1. the set $N$ of nodes is obtained from the basic concepts of $\mathcal{C}$, an initial-node symbol $Init$ and the bottom concept $\perp$ as follows:

$$N := \{Init, \perp\} \cup \mathrm{BC}_{\mathcal{C}};$$

2. if $C \sqsubseteq D \in \mathcal{C}$ then $C \longrightarrow_0 D$;
3. if $C \sqsubseteq \exists r_i.D$ then $C \longrightarrow_i D$;
4. If $\top$ occurs in $\mathcal{C}$, $Init \longrightarrow_0 \top$;
5. for every node of the form $\{a_i\} \in N$, $Init \longrightarrow_0 \{a_i\}$.

**Example 2.3**

> Consider the CBox in Example 2.2 and the uncertain information in Example 2.1. Its graphical representation is displayed in the Figure 2.1. The $\longrightarrow_0$-edges are represented by continuous black arrows, the $\longrightarrow_i$-edges, $i \geq 1$, are represented by blue labeled arrows and red dotted arrows indicate that source is uncertain information.

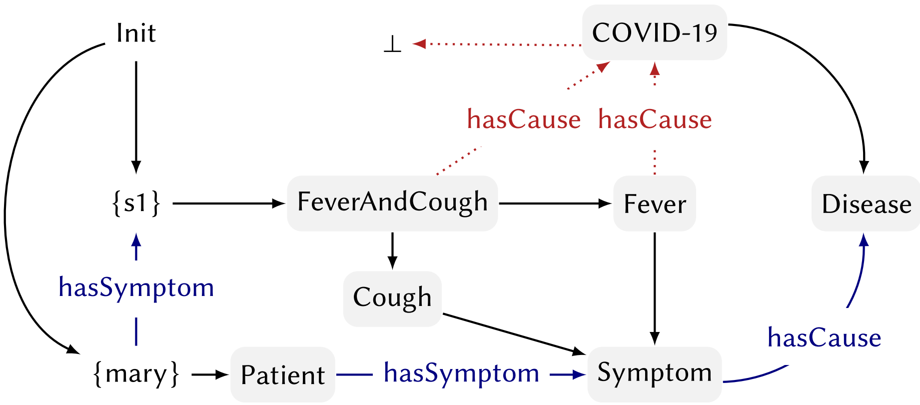


**Figure 2.1:** *Graphical representation of the ontology in Example 2.2.*

### 2.3.2 SAT decision

To calculate the satisfiability of a CBox $\mathcal{C}$, Finger and Lopes (n.d.) defined a completed graph $G^{\bullet}(\mathcal{C})$, obtained by applying the *graph completion rules* in Figure 2.2 until no more rules apply. Also, edges from the original graph $G(\mathcal{C})$ are called *basic edges* and the edges inserted from the rules in $G^{\bullet}(\mathcal{C})$ are called *derived edges*.

**GC1** If $C \longrightarrow_0 C'$, $C' \longrightarrow_i D$ but $C \not\longrightarrow_i D$; insert $(C \longrightarrow_i D)$ into $E_i$;

**GC2** If $C \longrightarrow_i C'$, $C' \longrightarrow_0 D$ but $C \not\longrightarrow_i D$; insert $(C \longrightarrow_i D)$ into $E_i$;

**GC3** If $D \longrightarrow_0^* \bot$, $C \longrightarrow_i D$ but $C \not\longrightarrow^* D$; insert $(C \longrightarrow_0 D)$ into $E_0$;

**GC4** If $C \longrightarrow_0^* \{a\}, D \longrightarrow_0^* \{a\}, Init \rightsquigarrow D$, $Init \rightsquigarrow C$ but $C \not\longrightarrow_0^* D$; insert $(C \longrightarrow_0 D)$ into $E_0$;

**GC5** If $r_i \sqsubseteq r_j \in \mathcal{C}$, $C \longrightarrow_i$ but $C \not\longrightarrow_j D$; insert $(C \longrightarrow_j D)$ into $E_j$;

**GC6** If $r_i \circ r_j \sqsubseteq r_k \in \mathcal{C}, C \longrightarrow_i D'$ and $D' \longrightarrow_j D$ but $C \not\longrightarrow_k D$; insert $(C \longrightarrow_k D)$ into $E_k$.

**Figure 2.2:** *Graph completion rules*

From Finger and Lopes (n.d.), we have the following lemma.

**Lemma 1.** *Given a $\mathcal{GEL}$-CBox $\mathcal{C}$, it is unsatisfiable iff $Init \longrightarrow_0^* \bot$ in the completed graph $G^{\bullet}(\mathcal{C})$. Furthermore, this decision can be made in polynomial time in $|\mathcal{C}|$, the number of symbols in $\mathcal{C}$.*

**Lemma 2.** *Given a $\mathcal{GEL}$-CBox $\mathcal{C}$, the following two statements are equivalent*

(a) *$Init \rightsquigarrow \bot$ in $G(\mathcal{C})$;*

(b) *$Init \longrightarrow_0^* \bot$ in $G^{\bullet}(\mathcal{C})$, which is $G(\mathcal{C})$ after completion.*

*Proof.* We break the proof in two parts.

$(a) \implies (b)$. Suppose that we have $Init \rightsquigarrow \bot$ in $G(\mathcal{C})$. In particular, consider one path $P := (Init, e_1, \ldots, e_m, \bot)$ in $G(\mathcal{C})$. Every edge $e_j$, $1 \le j \le m$, in this path or has a zero label ($\ell(e_j) = 0$) or a non-zero label ($\ell(e_j) \neq 0$).

Because the graph completion only add edges to the completed graph, there is also a path $P^{\bullet} := P$ in $G^{\bullet}(\mathcal{C})$.

By induction in the number $k$ of non-zero labeled edges in $P^{\bullet}$, we are going to prove that for every $k \geq 0$ there is a path $P' := (Init, e'_1, \dots, e'_{m'}, \bot)$ in $G^{\bullet}(\mathcal{C})$ such that $\ell(e_j) = 0\ \forall j,\ 1 \leq j \leq m'$.

*Base case*: If $k = 0$, every edge in $P^{\bullet}$ is labeled with 0. Thus, there is nothing to prove.

*Inductive case*: Assume the induction hypothesis that the proposition is true for $k = n$.

Consider a path $P$ with $n + 1$ non-zero labeled edges. So, there are vertices $u, v$ such that $(Init, e_1, \dots, u, e, v, \dots, \bot)$, $\ell(e) \neq 0$ and $v \longrightarrow_0^* \bot$. That is, $(u, v)$ is the last non-zero labeled edge in $P$.

By the *graph completion rule 3*, the completed graph $G^{\bullet}(\mathcal{C})$ will also have $u \longrightarrow_0 v$ by inserting an artificial edge $e'$. Then, we have a path $P' := (Init, \dots, u, e', v, \dots, \bot)$ in $G^{\bullet}(\mathcal{C})$, which has $n$ non-zero labeled edges.

By the induction hypothesis and $P'$, there is also a path $P''$ in $G^{\bullet}(\mathcal{C})$ such that every edge is labeled with 0, which completes the inductive step.

Then, we have that there is a path $P' := (Init, e'_1, \dots, e'_{m'}, \bot)$ in $G^{\bullet}(\mathcal{C})$ with every edge labeled with 0. That is, we have $Init \longrightarrow_0^* \bot$ in $G^{\bullet}(\mathcal{C})$.

$(a) \Longleftarrow (b)$. Suppose that we have $Init \longrightarrow_0^* \bot$ in $G^{\bullet}(\mathcal{C})$. In particular, consider one path $P^{\bullet} := (Init, e_1, \dots, e_m, \bot)$ such that $\ell(e_j) = 0\ \forall j,\ 1 \leq j \leq m$. This path may have derived edges, which are not in $G(\mathcal{C})$.

We can create a path $P'$ in $G^{\bullet}(\mathcal{C})$ without derived edges using the following algorithm: If there is an derived edge $e := (u \longrightarrow_i v)$ in $P^{\bullet}$, it was derived by some graph completion rule $N$. Then, modify $P^{\bullet}$ following the *path recovering rule $N$* (PR$N$). At the end, we obtain a path $P'$ without derived edges. Because there are a finite number of edges, this algorithm terminates.

**PR1** If $e$ was inserted by rule GC1, there is a vertex $u'$ such that $u \longrightarrow_0 u'$ and $u' \longrightarrow_i v$, for some $i \neq 0$.

Then, we substitute $(u \longrightarrow_i v)$ by $(u \longrightarrow_0 u' \longrightarrow_i v)$ in $P^{\bullet}$.

**PR2** If $e$ was inserted by rule GC2, there is a vertex $u'$ such that $u \longrightarrow_i u'$ and $u' \longrightarrow_0 v$, for some $i \neq 0$.

Then, we substitute $(u \longrightarrow_i v)$ by $(u \longrightarrow_i u' \longrightarrow_0 v)$ in $P^{\bullet}$.

**PR3** If $e$ was inserted by rule GC4, we also have $u \longrightarrow_i v$, for some $i \neq 0$.

Then, we substitute $(u \longrightarrow_0 v)$ by $(u \longrightarrow_i v)$ in $P^{\bullet}$.

**PR4** If $e$ was inserted by rule GC5, we have a path $P_v$ from $Init$ to $v$, because $Init \rightsquigarrow v$.

Then, we substitute $(Init, \cdots, u)$ by $P_v$ in $P^{\bullet}$.

**PR5** If $e$ was inserted by rule GC6, $e$ is of the form $(u \longrightarrow_j v)$, we have $r_i \sqsubseteq r_j \in \mathcal{C}$ and $u \longrightarrow_i v$, for some $i, j \neq 0$.

Then, we substitute $(u \longrightarrow_j v)$ by $(u \longrightarrow_i v)$ in $P^{\bullet}$.

**PR6** If $e$ was inserted by rule GC7, $e$ is of the form $(u \longrightarrow_k v)$, we have $r_i \circ r_j \sqsubseteq r_k \in \mathcal{C}$, and a vertex $u'$ such that $u \longrightarrow_i u'$ and $u' \longrightarrow_j v$, for some $i, j, k \neq 0$.

Then, we substitute $(u \longrightarrow_k v)$ by $(u \longrightarrow_i u' \longrightarrow_j v)$ in $P^{\bullet}$.

Because $P'$ does not have any derived edge, there is a path $P := P'$ in $G(\mathcal{C})$. That is, we have $Init \rightsquigarrow \bot$ in $G(\mathcal{C})$.

□

**Theorem 1.** *Given a $\mathcal{GEL}$-CBox $\mathcal{C}$, it is unsatisfiable iff $Init \rightsquigarrow \bot$ in $G(\mathcal{C})$. Furthermore, this decision can be made in polynomial time in $|\mathcal{C}|$, the number of symbols in $\mathcal{C}$.*

*Proof.* By Lemmas 1 and 2, we have that a $\mathcal{GEL}$-CBox $\mathcal{C}$ is unsatisfiable iff $Init \rightsquigarrow \bot$ in $G(\mathcal{C})$. Also, by Lemma 1, the graph construction can be made in polynomial time, and finding a path in a graph is also computed in polynomial time, and so is the satisfiability decision. □

Then, we have that the unsatisfiability decision in $\mathcal{GEL}$ can be reduced to finding a path $Init - \bot$ in the graph $G(\mathcal{C})$.

## 2.4 MaxSAT for $\mathcal{GEL}$

Before focusing on the probabilistic extension for $\mathcal{GEL}$, we need to define its MaxSAT problem, which will compose further the probabilistic reasoner.

The *weighted partial maximum satisfiability problem for $\mathcal{GEL}$* ($\mathcal{GEL}$-MaxSAT) can be defined as follows: given a potentially inconsistent weighted CBox $\mathcal{C}$, we want to find the maximal satisfiable subset of axioms; since it is partial, some axioms must be present in this subset. Usually, partiality can be modeled assigning infinite weights to the axioms that must not be excluded.

A *weighted* CBox is a pair $\langle \mathcal{C}, w \rangle$ where $\mathcal{C}$ is a CBox and $w : \mathcal{C} \longrightarrow \mathbb{Q} \cup \{\infty\}$ is a weight function, which maps axioms in $\mathcal{C}$ to weights. The infinite weight is used to represent axioms that must not be excluded in the maximal satisfiable subset. Also, it is defined that RIs of the form $r \sqsubseteq s$ and $r_1 \circ r_2 \sqsubseteq s$ always have infinite weight.

Then, given a weighted CBox $\langle \mathcal{C}, w \rangle$, a solution for the weighted partial $\mathcal{GEL}$-MaxSAT problem is a set $\mathcal{C}_{max} \subseteq \mathcal{C}$ such that:

- $\mathcal{C}_{max}$ is satisfiable; and
- $\mathcal{C}_{max} \models C \sqsubseteq D$ if $w(C \sqsubseteq D) = \infty$; and
- the sum of finite weights in $\mathcal{C}_{max}$ is maximal.

To handle evenly finite and infinite weights, and given that $\mathcal{C}$ is finite, we could use an alternative definition. The solution for the weighted partial $\mathcal{GEL}$-MaxSAT problem is a satisfiable set $C_{max}$ such that the sum of weights in the deleted set $C_{del} := C \setminus C_{max}$ is minimal and finite.

## 2.5 Probabilistic $\mathcal{GEL}$

Probability in $\mathcal{GEL}$ is constructed from a *probability function* $P$ (Finger and Lopes, n.d.). Consider a finite number of interpretation, $\mathcal{I}_1, \dots, \mathcal{I}_m$, we define the probability function $P : \{\mathcal{I}_1, \dots, \mathcal{I}_m\} \longrightarrow \mathbb{Q}$, such that $P(\mathcal{I}_i) \geq 0$ and $\sum_{i=1}^{m} P(\mathcal{I}_i) = 1$. We can also define the probability of an axiom $C \sqsubseteq D$ as follows

$$P(C \sqsubseteq D) = \sum_{\mathcal{I}_i \models C \sqsubseteq D} P(\mathcal{I}_i). \tag{2.1}$$

A *probabilistic knowledge base* is a pair $\langle \mathcal{C}, \mathcal{P} \rangle$, where $\mathcal{C}$ is a CBox and $\mathcal{P}$ is a PBox. A PBox is a set of $k$ linear constraints over $n$ axioms, of the form

$$\sum_{j=1}^{n} a_{ij} \cdot P(C_j \sqsubseteq D_j) \leq b_i; \quad 1 \leq i \leq k. \tag{2.2}$$

We can define the *satisfiability problem* for this probabilistic KB (PGEL-SAT) as deciding if it is consistent or not. If it is consistent, the solution is a set of interpretations $\{\mathcal{I}_1, \dots, \mathcal{I}_m\}$ and a probability function $P : \{\mathcal{I}_1, \dots, \mathcal{I}_m\} \longrightarrow \mathbb{Q}^+$ such that $\sum_{i=1}^{m} P(\mathcal{I}_i) = 1$, $P(C \sqsubseteq D) = 1$ for $C \sqsubseteq D \in \mathcal{C}$ (axioms in CBox are certain) and P verifies all linear constraints in $\mathcal{P}$.

**Example 2.4**

Now we can model the uncertain situation stated in Example 1.1 using the probability knowledge base $\langle \mathcal{C}_{exa}, \mathcal{P}_{exa} \rangle$, where $\mathcal{C}_{exa}$ is the CBox from Example 2.2 and $\mathcal{P}_{exa}$ is given by

$$\begin{aligned} \mathcal{P}_{exa} := \{ \quad & P(Ax_2) - P(Ax_1) = 0.2, \\ & P(Ax_3) = 0.9 \quad \}. \end{aligned}$$

Then, we need a polynomial algorithm to find if this probabilistic KB is consistent.

### 2.5.1 Linear algebraic view

The PGEL-SAT problem was also defined by Finger and Lopes (n.d.) in a linear algebraic view, which is useful to develop its polynomial reasoning algorithm. It was shown that a probabilistic KB $\langle \mathcal{C}, \mathcal{P} \rangle$ is satisfiable iff the linear equation $C \cdot x = d$ has a solution $x \geq 0$, where

$$C := \begin{bmatrix} -I_n & M_{n\times m} \\ A_{k\times n} & 0_{k\times m} \\ 0'_n & 1'_m \end{bmatrix} \quad x := \begin{bmatrix} p_n \\ \pi_m \end{bmatrix} \quad d := \begin{bmatrix} 0_n \\ b_k \\ 1 \end{bmatrix} \tag{2.3}$$

and

- $A_{k\times n}$ is a $k \times n$ matrix whose elements $a_{ij}$ are given by Equation (2.2);
- $b_k$ is a $k$ vector whose elements $b_i$ are also given by Equation (2.2);
- $M_{n\times m}$ is a $n \times m$ matrix given by the following steps:

  Consider an interpretation $\mathcal{I}$ model of $C$, also called $C$-*satisfiable interpretation*, its corresponding vector in $\mathcal{P}$ is a $\{0, 1\}$-vector $y$ such that $y_i = 1$ iff $\mathcal{I} \models C_i \sqsubseteq D_i$, for $1 \leq i \leq n$.

  Then, given a set of interpretations $\mathcal{I}_1, \ldots, \mathcal{I}_m$, we define $M_{n\times m}$ a matrix whose column $M^j$ is $\mathcal{I}_j$'s corresponding vector in $\mathcal{P}$;
- $I_n$ is the $n$-dimensional identity matrix;
- $0_n$ is a column 0-vector of size $n$ (similarly for $1_n$);
- $0'_n$ is the previous vector's transpose;
- $0_{k\times m}$ is a 0-matrix of shape $k \times m$;
- $p_n$ is a vector of size $n$ which corresponds to the probability of axioms occurring in Equation (2.2);
- $\pi_m$ is a vector of size $m$ which corresponds to the probability distribution over interpretations $\mathcal{I}_1, \ldots, \mathcal{I}_m$.

To add some intuition about this linear algebraic view, each row of matrices in $C$ represents some linear restriction of the problem. The first one describes Equation (2.1), the second row codifies Equation (2.2) and last represents the restriction $\sum_{i=1}^{m} P(\mathcal{I}_i) = 1$.

Thus, we can use techniques for solving linear equations to find a tractable algorithm for PGEL-SAT.

# Chapter 3

# Development

In this section, we describe the development of a tractable algorithm for PGEL-SAT[1], described by Finger and Lopes (n.d.). It was implemented using Python programming language (Van Rossum and Drake, 2009).

## 3.1 Input and output format

The algorithm accepts as input a P$\mathcal{GEL}$ KB encoded in an OWL 2 ontology. Both certain and uncertain knowledge must be a $\mathcal{GEL}$-CBox in the normal form, with the additional support of equivalence axioms. Due to limitations in the OWL parser used, which will be detailed further, RDF/XML, OWL/XML and NTriples are the only file formats supported.

In addition, uncertain axioms must have an annotation (`rdfs:comment`) with its unique numerical index. That is, given an uncertain axiom $Ax_i$, its annotation must be of the following form in Figure 3.1.

`#!pbox-id` $i$

**Figure 3.1:** *Annotation format for an uncertain axiom*

PBox restrictions are represented with annotations in the $\top$ concept (`owl:Thing`). That is, given a restriction of the form $a_0P(Ax_0) + a_1P(Ax_1) + \cdots + a_{n-1}P(Ax_{n-1}) = b$, its annotation must be of the form in Figure 3.2. Also, inequalities $\leq$ and $\geq$ can be represented, respectively, with the symbols <= and >=.

The output of the algorithm is `True` if the given KB has a solution, and `False` if not.

[1] Available at https://github.com/AndrewIjano/pgel-sat.

```
#!pbox-restriction
0 a_0
1 a_1
...
n − 1 a_{n−1}
==
b
```

**Figure 3.2:** *Annotation format for a PBox restriction*

## 3.2 Knowledge base representation

To read the ontology in OWL 2, it was used the Python module *Owlready2* (Lamy, 2017).

The probabilistic KB is represented as the edge-labeled graph in Section 2.3.1 with three matrices for the inequalities in Equation (2.2). The graph implemented is slightly different from the definition but the property in Theorem 1 is maintained. The vertex $\top$ is always present but the implementation will not insert arrows ($\top \longrightarrow C$) for every vertex $C$, because it is unnecessary for the algorithm. Also, the matrices are represented and manipulated using the numerical computing package Numpy (Walt *et al.*, 2011).

The probabilistic KB representation allows us to develop a PGEL-SAT solver, using this data structure as input.

## 3.3 PGEL-SAT solver

In order to understand the PGEL-SAT solver implemented, we need to find a solution for the linear system stated in Section 2.5.1.

First, consider the matrix $C$ in Equation (2.2), it has $p$-columns and $\pi$-columns, referring to which part of $x$ they multiply. We say that a $p$-column $pc_i$ is *well-formed* if it is of the form in Equation (2.2), starting with the $i$-th column of $-I_n$, followed by the $i$-th column of $A$ and with one 0 at the end. Also, we say that a $\pi$-column is *well-formed* if it starts with a corresponding vector of an interpretation model of $C$, followed by $k$ zeros and with one 1 at the end. A column that is not well-formed is *ill-formed.*

Now, consider that the matrix $C$ may have ill-formed columns. We define a binary *cost vector* $c$ such that each element $c_i = 1$ iff column $C^i$ is ill-formed; otherwise $c_i = 0$. Then, the linear system stated in Section 2.5.1 has a solution iff the following minimization problem has minimum 0.

$$\begin{aligned} \text{minimize} \quad & c' \cdot x \\ \text{subject to} \quad & C \cdot x = d \\ & x \geq 0 \end{aligned} \tag{3.1}$$

This problem can be solved by a linear algebraic solver but we need to find beforehand

$C$ and $d$ that reaches the minimum 0. Since matrix $A$ and vector $b$ are generated from PBox, we need to choose a set of interpretations that provides the solution. However, we potentially have an exponential collection of possible interpretations, which is encoded in $M$, and calculating all of them would make the algorithm untractable.

Even though, from Carathéodory's Theorem (Eckhoff, 1993), Finger and Lopes (n.d.) states that if constraints in Equation (2.2) are solvable then there exists a solution where $x$ has at most $n+k+1$ values such that $x_j > 0$. This allows us to avoid generating an exponential number of columns of $C$ by using a linear algebraic technique called *column generation* (Gilmore and Gomory, 1961; Gilmore and Gomory, 1963). With this technique, we can create an algorithm PGEL-SAT-Solver which generates well-formed $\pi$-columns of $C$ on the fly by solving an auxiliary problem and uses a linear solver in each iteration to verify if the solution for Equation (3.1) has reached minimum 0. This solution leads to the algorithm implemented, which is shown in Algorithm 1.

**Algorithm 1** The PGEL-SAT solver algorithm

**function** PGEL-SAT-Solver($\langle \mathcal{C}, \mathcal{P} \rangle$)
    $c \leftarrow$ Initialize-c($\mathcal{P}$)
    $C \leftarrow$ Initialize-C($\mathcal{P}$)
    $d \leftarrow$ Initialize-d($\mathcal{P}$)

    $lp \leftarrow$ LinProg($c, C, d$)

    **while** $lp.minCost \neq 0$ **do**
        $result \leftarrow$ GenerateColumn($\langle \mathcal{C}, \mathcal{P} \rangle, lp$)
        **if** $result.isSuccess$ = False **then**
            **return** False     ▷ $\langle \mathcal{C}, \mathcal{P} \rangle$ is unsatisfiable
        **end if**

        $y \leftarrow result.column$     ▷ well-formed $\pi$-column
        $C \leftarrow (C \,|\, y)$
        $c \leftarrow (c \,|\, 0)$
        $lp \leftarrow$ LinProg($c, C, d$)
    **end while**

    **return** (True, $lp$)     ▷ $\langle \mathcal{C}, \mathcal{P} \rangle$ is satisfiable
**end function**

The algorithm PGEL-SAT-Solver in Algorithm 1 receives a potentially unsatisfiable probabilistic knowledge base and returns *True* if it is satisfiable, and *False*, otherwise.

It starts with matrices $c$, $C$ and $d$. Because the algorithm needs to start with a cost vector with non-zero elements, the $C$ matrix is initialized with $n+k+1$ ill-formed columns, represented by the identity matrix $I_{n+k+1}$, and well-formed $p$-columns as shown in Equation (2.3); therefore, the cost vector $c$ is initialized with $n+k+1$ 1-elements plus $n$ 0-elements. The $d$ vector is unaltered during the execution and is given by Equation (2.3). These initializations are shown in Algorithm 2.

**Algorithm 2** Matrices initializations

**function** INITIALIZE-c($\mathcal{P}$)
  **return** $(\, 1'_{n+k+1} \,|\, 0'_n \,)'$
**end function**

**function** INITIALIZE-C($\mathcal{P}$)
  Get $A$ from PBox $\mathcal{P}$ restrictions
  $PC \leftarrow \begin{pmatrix} -I_n \\ A \\ 1'_n \end{pmatrix}$ ▷ $p$-columns
  **return** $(\, I_{n+k+1} \,|\, PC \,)$
**end function**

**function** INITIALIZE-d($\mathcal{P}$)
  Get $b$ from PBox $\mathcal{P}$ restrictions
  **return** $(\, 0'_n \,|\, b' \,|\, 1 \,)'$
**end function**

In addition, the algorithm uses a program LINPROG. It receives matrices $c$, $C$ and $d$ and returns the solution for the optimization problem in Equation (3.1). Its output $lp$ needs to contain the minimal cost ($lp.minCost$), the primal solution ($lp.primal$) and dual solution ($lp.dual$).

Then, the algorithm executes the column generation process while the minimal cost is not zero. In each iteration, a well-formed $\pi$-column $y$ is generated by an auxiliary program GENERATECOLUMN. After that, the column is appended to $C$, with its respective 0 cost in $c$, and a new minimal cost is calculated. If no more columns can be generated with a minimal cost higher than 0, represented by $result.isSuccess = False$, the KB is unsatisfiable.

## 3.4 Linear solver

The program LINPROG was implemented using the open-source GNU linear program kit (MAKHORIN, 2001) and the Python wrapper for this library, the *swiglpk* package (SONNENSCHEIN, n.d.). The library was chosen because it is open-source, has tractable algorithms for large-scale problems, and provides a solution that contains the required elements described in Section 3.3.

There are several methods for solving a linear optimization problem. The *simplex method* is known to be very efficient in practice (BORGWARDT, 2012) although its worst time complexity is untractable (KLEE and MINTY, 1972). On the other hand, the modern *interior point method* has a proven polynomial-time complexity and is especially ideal for very sparse problems (BOYD *et al.*, 2004) but may not be as efficient as the simplex method in all cases.

Because of the theoretical tractability, the linear solver LINPROG implemented uses the interior point method.

# 3.5 Column generation

The program GenerateColumn is responsible for generating a well-formed $\pi$-column for $C$ that reduces $c'x$ until it is minimal; in particular, when the KB is satisfiable, this value is equal to 0. To find such a column, we need to analyze the optimization problem we are solving from the theory of linear programming (Bertsimas and Tsitsiklis, 1997).

## 3.5.1 Cost reduction

Suppose that the matrix $C$ is composed of $I_{n+k+1}$ and all well-formed $p$-columns $PC$, which are result of Initialize-C, concatenated with all possible (ill and well formed) $\pi$-columns $y^{(j)}$, as illustrated below.

$$C := \left( I_{n+k+1} \,|\, PC \,|\, y^{(1)} \,|\, y^{(2)} \cdots \right)$$

This matrix can be rearranged in the form $C = (B|N)$, where $B$ is a linear independent square matrix called *basis matrix*. This allows obtaining a *basic feasible solution* $x = (\, x'_B \,|\, x'_N \,)'$ where $x_N = 0$ and $x_B = B^{-1}d$. In particular, if $B = I_{n+k+1}$ we have $x_B = d$.

However, this solution may, and probably will, not be optimal. Then, we need to find some column $C^j$ in $N$ and increase $x_j$ by some positive value $\theta$, moving $x$ to $x + \theta s$, for some $s$. We create this vector $s$ such that $s_j = 1$, $s_i = 0$ for every non basic index $i \neq j$ and its basic part $s_B = (s_{B(1)}, s_{B(2)}, \ldots, s_{B(m)})$.

Since we want feasible solution, we need $C(x + \theta s) = d$; given that $x$ is feasible, we also have $Cx = d$. Thus, we need $C \cdot s = C \cdot (x + \theta s - x) = d - d = 0$. Also, because $s_j = 1$ and $s_i = 0$ for every non basic index $i \neq j$, we have that

$$0 = Cs = \sum_{i=1}^{n+k+1} C^i s_i = \sum_{i=i}^{m} C^{B(i)} s_{B(i)} + C^j = Bs_B + C^j.$$

This follows that $s_B = -B^{-1}C^j$.

Then, we could calculate the variation of the objective function by:

$$c'(x + \theta s) - c'x = \theta c's = \theta(c_j + c'_B s_B) = \theta(c_j - c'_B B^{-1} C^j).$$

This value $r_j := c_j - c'_B B^{-1} C^j$ is called *reduced cost* of the variable $x_j$. The quantity is useful for finding columns in $C$ that lead to an optimal solution. In case of nondegeneracy, if $r_j < 0$ for some $j$, we could minimize the goal increasing $x_j$. Also, from duality properties, given an optimal dual variable $w$, we have $w = c'_B B^{-1}$.

From that, we define that the *reduced cost* $r_y$ of a column $y$ to be added to the solution is

$$r_y = c_y - w \cdot y.$$

Because $y$ is well-formed, we have $c_y = 0$. Then, to reduce the cost of the objective function, we need $r_y \leq 0$ and, therefore, the new column must satisfy

$$w \cdot y \geq 0. \tag{3.2}$$

### 3.5.2 Intermediate problem

In this algorithm, only the $n$ first elements, which correspond to a column of $M$, must be generated, since the last elements are composed of $k$ 0-elements and one 1-element. This new column $M^j$ is the corresponding vector in $\mathcal{P}$ of a $\mathcal{C}$-satisfiable interpretation. Then, to find such column, we can solve an intermediate problem.

Define $\mathcal{C} \cup \mathcal{P}$ as a potentially unsatisfiable CBox with all axioms of $\mathcal{C}$ and axioms from restrictions of $\mathcal{P}$. We want to find a new CBox $\mathcal{C}^{\bullet} \subseteq \mathcal{C} \cup \mathcal{P}$ such that $\mathcal{C}^{\bullet}$ is satisfiable and $\mathcal{C}^{\bullet} \models C \sqsubseteq D$ if $C \sqsubseteq D \in \mathcal{C}$. An interpretation model of $\mathcal{C}^{\bullet}$ will be $\mathcal{C}$-satisfiable and can generate a corresponding vector $M^j$ in $\mathcal{P}$. Also, the new column $y$ created from $M^j$ must obey the restriction in Equation (3.2).

Instead of generating a column $y$ that just satisfies the conditions above, we could generate one that maximizes the cost reduction in Equation (3.2). This can be done maximizing $\sum_{i=1}^{n} w_i \cdot y_i$ which is choosing axioms $Ax_i$ for $\mathcal{C}^{\bullet}$ based on weights $w_i$, $1 \leq i \leq n$.

Then, this problem can be modeled as a $\mathcal{GEL}$-MaxSAT for $\mathcal{C} \cup \mathcal{P}$ where axioms of $\mathcal{C}$ have infinite weight and every axiom $Ax_i$ from $\mathcal{P}$ has weight $w_i$.

### 3.5.3 Generating column

This leads to the solution in Algorithm 3. We generate a new column solving the intermediate problem with $\mathcal{GEL}$-Max-SAT-Solver and an auxiliary function ExtractColumn. If the $\mathcal{GEL}$-MaxSAT problem does not have a solution or Equation (3.2) cannot be satisfied, it returns False; otherwise, it returns True and the generated column $y$.

**Algorithm 3** The algorithm of column generation

**function** GENERATECOLUMN($\langle \mathcal{C}, \mathcal{P} \rangle$, $lp$)
  $w \longleftarrow lp.dual$
  $result \longleftarrow \mathcal{GEL}$-MAX-SAT-SOLVER($\mathcal{C} \cup \mathcal{P}, (w_1, \dots, w_n)$)
  **if** $result.isSuccess$ = False **then**
    **return** False
  **end if**

  $y \longleftarrow$ EXTRACTCOLUMN($result$)
  **if** $w \cdot y < 0$ **then**
    **return** False
  **end if**

  **return** (True, $y$)
**end function**

**function** EXTRACTCOLUMN($result$)
  $\mathcal{C}_{max} \longleftarrow result.solution$
  Define $n$-vector $m_n$ such that $m_i = 1$ iff we have $Ax_i$ in $\mathcal{C}_{max}$
  $y \longleftarrow (\, m'_n \,|\, 0'_k \,|\, 1 \,)'$
  **return** $y$
**end function**

## 3.6 $\mathcal{GEL}$-MaxSAT solver

The function $\mathcal{GEL}$-MAX-SAT-SOLVER is responsible for generating a solution to the $\mathcal{GEL}$-MaxSAT problem. To find such solution, we need to look at the graph modeling used for its SAT decision.

A weighted $\mathcal{GEL}$-CBox $\langle \mathcal{C} \cup \mathcal{P}, w \rangle$ can be represented as a weighted directed graph $G_w(\mathcal{C} \cup \mathcal{P}) = (N, E, \ell, w)$, following the representation in Section 2.3, where $(N, E, \ell)$ are the same as before and $w : E \longrightarrow \mathbb{Q} \cup \{\infty\}$ is an edge weight function with $w((C \longrightarrow_0 D)) = \infty$ iff $C \sqsubseteq D \in \mathcal{C}$. We also write $w(u, v)$ to represent $w((u, v))$, with $u, v \in N$.

In this problem, we want to find a satisfiable subset of $\mathcal{C} \cup \mathcal{P}$. From Theorem 1, we know that a $\mathcal{GEL}$-CBox $\mathcal{C}$ is satisfiable iff there is a $Init$-$\bot$ path in $G(\mathcal{C})$. Thus, to find a satisfiable $\mathcal{C}^\bullet \subseteq \mathcal{C} \cup \mathcal{P}$, we need to remove every $Init$-$\bot$ path in $G_w(\mathcal{C} \cup \mathcal{P})$ which can be done by removing some edges in these paths. For maximality, we need to remove edges that minimize the sum of weights of the deleted set.

Also, given a weighted directed graph $G_w(\mathcal{C})$ and nodes $s, t \in N$, a $s$-$t$ *cut* in $G_w(\mathcal{C})$ is a partition $(S, T)$ such that $s \in S$ and $t \in T$. A *cut set* $Cut(S, T)$ is the set of edges of $G_w(\mathcal{C})$ with one end in $S$ and other end in $T$; we use $Cut(s, t)$ to refer to a cut set from a $s$-$t$ cut. To remove all paths from $s$ to $t$, we need to remove all edges of a cut set $Cut(s, t)$. The weight of a cut set is given by the sum of weights of its edges:

$$w(Cut(s,t)) = \sum_{(u,v) \in Cut(s,t)} w(u,v).$$

A graph usually has many $s$-$t$ cuts; a $s$-$t$ cut is *minimal* (*min cut*) if it has minimal weight; it is finite if its cut set has only edges of finite weight.

Then, the $\mathcal{GEL}$-MaxSAT problem can be modeled as finding a min cut in a weighted directed graph, which leads to the solution in Algorithm 4. It receives a CBox $\mathcal{C} \cup \mathcal{P}$ and an $n$-vector $w$. First, we compute $G_w(\mathcal{C} \cup \mathcal{P})$ where the weights are defined as $w(C_i, D_i) = w_i$ if $C_i \sqsubseteq D_i$ appears in $\mathcal{P}$; otherwise, $w(C, D) = \infty$. After that, we use the program MinCut to calculate the cut set of a $Init$-$\bot$ min cut. If the weight of the minimal cut set is infinite, meaning that we need to remove a certain axiom to keep satisfiability, then there is no solution; otherwise, return $E$ without the edges in the cut set.

**Algorithm 4** The $\mathcal{GEL}$-MaxSAT solver algorithm

**function** $\mathcal{GEL}$-Max-SAT-Solver($\mathcal{C} \cup \mathcal{P}, w$)
  Compute $G_w(\mathcal{C} \cup \mathcal{P}) \longleftarrow (N, E, \ell, w)$
  $Cut(Init, \bot) \longleftarrow$ MinCut($G_w(\mathcal{C} \cup \mathcal{P}), Init, \bot$)
  **if** $w(Cut(Init, \bot)) = \infty$ **then**
    **return** False
  **end if**

  **return** (True, $E \setminus Cut(Init, \bot)$)
**end function**

The solution in Algorithm 5 calculates the $Init$-$\bot$ min cut in $G_w(\mathcal{C} \cup \mathcal{P})$, based on Edmonds-Karp algorithm for maximum-flow (Cormen *et al.*, 2009). We create a *residual graph* $G_R$ where its weights $w_R$ represent its *residual capacity*. In each iteration we search for the shortest $Init$-$\bot$ path $P$ with positive weights (augment path), calculated with a *breadth first search*. We get a augment flow $\alpha := \{ w_R(u,v) \ : \ (u,v) \in P \}$ and propagate the flow through the path, which is decrementing by $\alpha$ the residual capacity forward, and incrementing backwards.

After executing the iterations in lines 5-10, we have a partition $(S, T)$, where $Init \in S$, $\bot \in T$ and every edge $(u, v)$ such that $u \in S$ and $v \in T$ have residual capacity 0. Also, the sum of residual capacities from edges from $T$ to $S$ is equal to the flow $f$ passing from $S$ to $T$. Then, because every edge from $S$ to $T$ have residual capacity 0, $f$ is the sum of capacities (or weights) of these edges in $G_w(\mathcal{C} \cup \mathcal{P})$ which means that $Cut(S, T)$ is minimal (Ford and Fulkerson, 1962).

Also, note that edges with negative weight are removed beforehand from $G_w(\mathcal{C} \cup \mathcal{P})$ for maximality.

**Algorithm 5** The minimal cut algorithm

**function** MinCut($G_w(\mathcal{C} \cup \mathcal{P})$, $Init$, $\perp$)
  remove edges with negative weight in $G_w(\mathcal{C} \cup \mathcal{P})$
  define the residual graph $G_R := (N_R, E_R, \ell_R, w_R)$ as a copy of $G_w(\mathcal{C} \cup \mathcal{P})$
  **while** there is an augment shortest path $P$ in $G_R$ **do**
    $\alpha \longleftarrow \min \{ w_R(u, v) \ : \ (u, v) \in P \}$ ▷ $\alpha$ is the augment flow in $P$
    **for** each $(u, v) \in P$ **do**
      decrement $\alpha$ of $w_R(u, v)$
      increment $\alpha$ of $w_R(v, u)$
    **end for**
  **end while**
  get the set $S$ of nodes reachable from $Init$ by edges with positive weight
  **return** edges $(u, v) \in E$ such that $u \in S$ and $v \notin S$
**end function**

## 3.7 Time complexity analysis

In this section, we concentrate on presenting a polynomial time complexity bound of PGEL-SAT in Algorithm 1. Consider a probabilistic $\mathcal{GEL}$-knowledge base with $n$ concepts, $m$ certain axioms, $p$ uncertain axioms and $k$ probabilistic restrictions. Also, define $N = p + k + 1$.

Algorithm 1 starts with $O(N^2)$ arithmetic operations plus a call to LinProg. After that, it executes several iterations with calls to GenerateColumn and LinProg. From Carathéodory's Theorem (Eckhoff, 1993), the algorithm stops after at most $N$ iterations.

The linear solver LinProg in this project implements an easy version of the primal-dual interior-point method based on Mehrotra's technique (Makhorin, 2001; Mehrotra, 1992). Although there are polynomial bounds for several variations of this method (Y. Zhang and D. Zhang, 1995; Salahi *et al.*, 2008; Teixeira and Almeida, 2012), we did not investigate the exact implementation to justify the use of a specific bound. Thus, for simplicity, we choose the known $O(N^3L)$ time complexity for interior-point methods (Goldfarb and Todd, 1989), where $L$ is the bit-length of input data.

The program GenerateColumn in Algorithm 3 executes $O(N)$ operations, to handle the column, plus a call to $\mathcal{GEL}$-Max-SAT-Solver. This function, in Algorithm 4, builds the weighted graph, whose complexity is $O(n + (m + p))$, and calculates the minimal cut with Algorithm 5. The program MinCut is based on Edmonds-Karp algorithm (Cormen *et al.*, 2009), which is $O(n(m + p)^2)$, with an additional search in the graph for the cut, which is $O(n+m+p)$. Then, we have that Algorithm 3 has an upper bound of $O(N+n(m+p)^2)$.

Finally, because one column is generated for each iteration, the size of the restriction matrix increases. Thus, the upper bound of the number of operations for solving the linear programs of every iteration can be calculated as

$$O(N^3L + N^3(L + N) + \cdots + N^3(L + N^2)) = O(N^4L + N^6).$$

Then, Algorithm 1 has an upper bound of $O(N^6 + N^4L + Nn(m + p)^2)$ operations. Therefore, the algorithm proposed to solve the PGEL-SAT problem is polynomial.

# Chapter 4

# Experiments and results

To understand the performance of the algorithm in practice, we proposed two experiments: one tries to identify phase transition behavior in the solution, and the other analyses its empirical run time complexity.

## 4.1 Phase transition analysis

The phenomenon of *phase transition* was conjectured as a property of all NP-complete problems (Cheeseman *et al.*, n.d.) and was further studied for PSAT problems (Finger and Bona, 2011). In the last study, experiments were made by the rate $m/n$, where $m$ is the number of 3-SAT clauses and $n$ is the number of variables; also, it was used $k$ fixed probability assignments to uniformly distributed random PSAT instances. Then, it was shown that, when $m/n$ is small, almost all instances are satisfied and, when $m/n$ is high, instances are unsatisfiable; this is called *first-order phase transition*. Also, for NP-Complete problems, harder instances concentrate between satisfiability and unsatisfiability; and this is called *second-order phase transition*.

Based on the work of Finger and Bona, we create a similar experiment for PGEL-SAT, with randomly generated P$\mathcal{GEL}$-KBs. Consider a probabilistic KB with $n$ concepts, $n_R$ roles, $m$ concept inclusions in the normal form, and $p$ uncertain axioms with $p$ probabilistic restrictions of the form $P(C \sqsubseteq D) \leq b$, where $b \in \mathbb{Q}$ and $C, D$ are concepts. For this experiment, fix $n$, $n_R$, $p$ and vary $m$. For each value of $m$, generate several instances of P$\mathcal{GEL}$-KBs as a random graph with linear restrictions.

The random KB generation is described as follows. Start with a graph $G$ with no edges and 3 vertices: $\bot, \top$ and $Init$; add $n$ vertices and $n_R$ roles to $G$. Then, add $m$ edges correspoding to certain axioms, and $p$ edges correspoding to uncertain axioms. Edges are added by choosing randomly two vertices $C$ and $D$ ($n + 3$ possibilities for each one), their role is chosen uniformly from $C \sqsubseteq D$ and the $n_R$ other roles ($C \sqsubseteq \exists r_i.D$). Also, a probabilistic restriction of the form $P(C \sqsubseteq D) \leq b$ is generated for each uncertain axiom $C \sqsubseteq D$, with a uniformly random $b \in [0, 2]$.

For this experiment, we fixed $n = 700$, $n_R = 3$, $p = 10$ and increased $m$ from 100 to 2000 in steps of 10. For each value of $m$ we generated 500 instances of PGEL-SAT, computed the ratio of satisfiable instances (%PGEL-SAT) and average run time. Also, it is applied a simple moving average with an window of 5 points to the curve. Then, we obtained the graphic in the Figure 4.1.

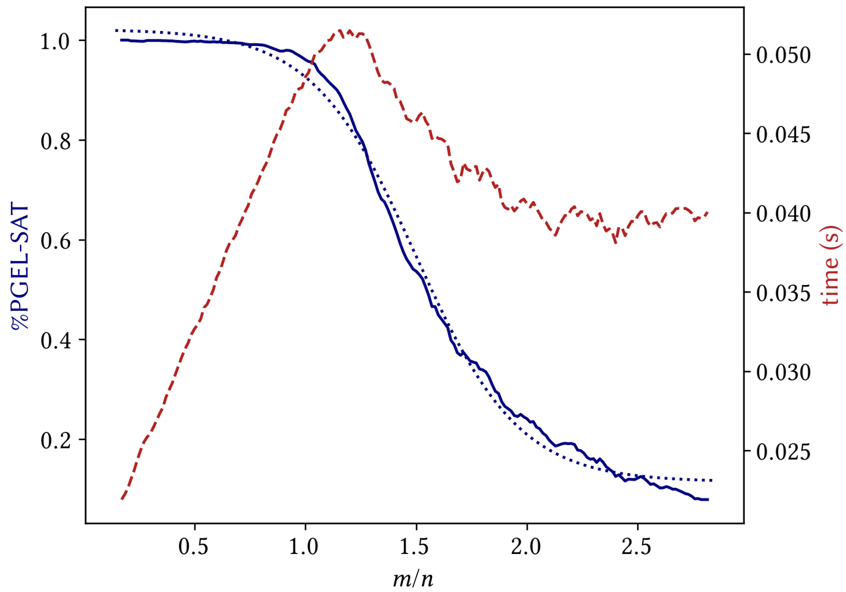


**Figure 4.1:** *%-satisfiable of PGEL-SAT and average run time for instances of formulas in several cases*

These experiments show that:

(a) There is an S-shaped first-order phase transition, from mostly satisfiable to mostly unsatisfiable, like its propositional problem;

(b) There is not a second-order phase transition, which is usually characterized by a high peak when satisfiability ratio is in 0.5. We can see that, when $m/n$ increases, the average time increases until the ratio of satisfiability starts to decrease. This can be explained because the algorithm tries to find an inconsistency. Consequently, when the axioms or restrictions are higher, it is easier to find such inconsistency.

## 4.2 Run time analysis

This experiment aims to estimate empirically the time complexity of the algorithm and verify its tractability.

Consider a probabilistic KB as defined in Section 4.1. Create three experiments for each parameter $n$, $m$ and $p$; varying one parameter and fixing others. For each value of the varying parameter, generate several instances of P$\mathcal{GEL}$-KBs as described in Section 4.1.

For this experiment, the fixed parameters had the values $n = 10$, $n_R = 3$, $m = 10$ and $p = 10$. Each parameter $n$, $m$ and $m$ varied from 10 to 400 in steps of 20. For each varying value, we generated 100 instances of PGEL-SAT, computed the average run time, average run time of a single iteration and number of iterations. Also, it is applied a simple moving average with an window of 5 points to the curves. Then, we obtained the graphics in Figures 4.2 to 4.4.

These experiments show that:

(a) The increase of axioms and concepts has low impact in the run time and the number of iterations;

(b) The increase of uncertain axioms has high impact in the run time;

(c) The number of iterations increases linearly with the number of uncertain axioms.

Then, we have a result similar to the estimated complexity in Section 3.7.

However, one could argue that the run time growth of varying uncertain axioms is exponential. Thus, we propose a new experiment to analyse this curve.

Using the same data from the previous experiment, we applied the Levenberg–Marquardt algorithm (Levenberg, 1944; Marquardt, 1963), which is a method for solving non-linear least squares problems. This method was applied with the SciPy's implemention called curve_fit (Virtanen *et al.*, 2020).

Define a polinomial $p(x)$ and an exponential function $e(x)$, such that

$$
\begin{aligned}
p(x) &:= A_7 \cdot x^7 + A_6 \cdot x^6 + A_5 \cdot x^5 + A_4 \cdot x^4 + A_3 \cdot x^3 + A_2 \cdot x^2 + A_1 \cdot x + A_0 \\
e(x) &:= B_2 \cdot 2^{B_2 \cdot x} + B_0.
\end{aligned}
$$

We want to find parameters $A_0, \ldots, A_7$ and $B_0, \ldots, B_2$ in such a way that $p(x)$ and $e(x)$ best approximate the experimental measures.

After applying the algorithm, we obtain $p(x)$ and $e(x)$ with the following approximated parameters, which are illustrated in the Figure 4.5,

$$
\begin{aligned}
p(x) = &- 1.5 \cdot 10^{-15} \cdot x^7 + 2 \cdot 10^{-12} \cdot x^6 - 1 \cdot 10^{-9} \cdot x^5 + 2.9 \cdot 10^{-7} \cdot x^4 \\
&- 4 \cdot 10^{-5} \cdot x^3 + 2.8 \cdot 10^{-3} \cdot x^2 - 0.08 \cdot x + 0.64 \\
e(x) = &\, 0.282 \cdot 2^{0.014 \cdot x} - 0.592.
\end{aligned}
$$

From this analysis, we have that:

(a) The terms that mostly contribute to the growth of $p(x)$ have a small degree ($x^4$ and $x^2$);

(b) The factor multipling $x$ in $e(x)$ is small ($\approx 0.014$); therefore, $e(x)$ corresponds to the function $f(x) = 0.282 \cdot (1.01)^x - 0.592$, which has a very small exponential growth rate.

Thus, there is no experimental results to refute that the run time growth of PGEL-SAT-Solver is not polynomial.

As expected, these algorithms do not have the phase transition behavior of NP-Complete problems and do not show any exponential run time growth rate. Then, these initial experiments confirm that PGEL-SAT-Solver is a promising algorithm for tractable probabilistic reasoning over description logics.

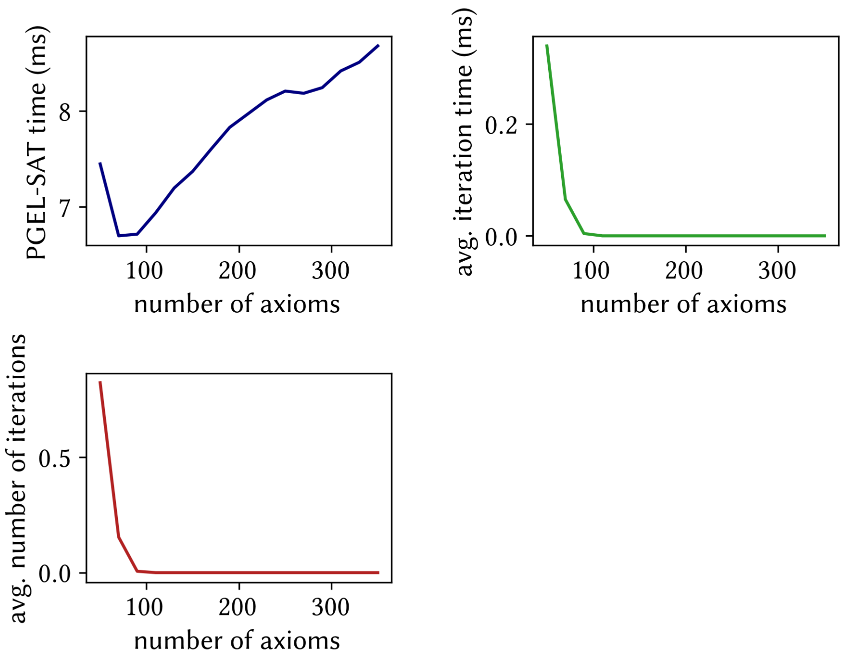


**Figure 4.2:** *Impact of the increase of certain axioms in the run time and number of iterations*

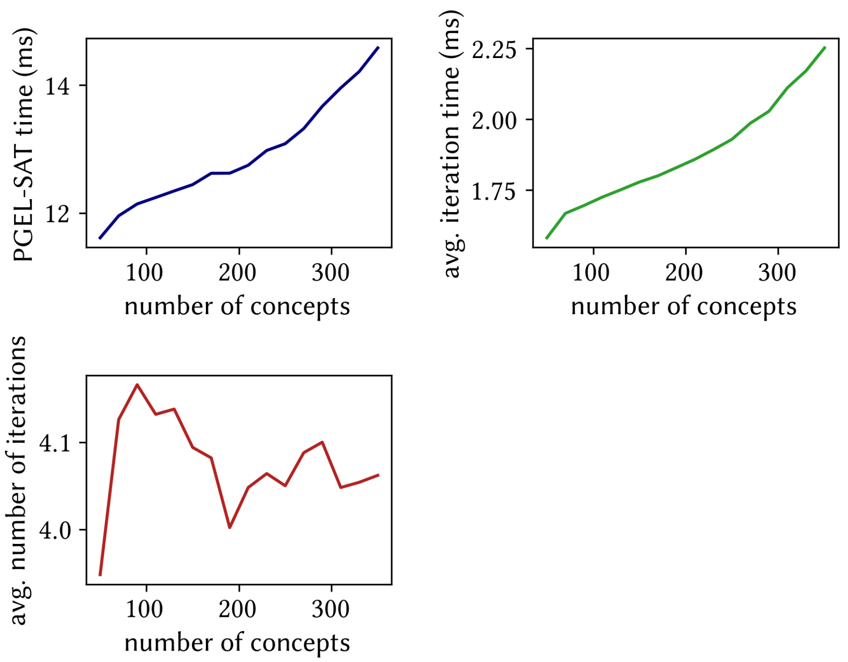


**Figure 4.3:** *Impact of the increase of concepts in the run time and number of iterations*

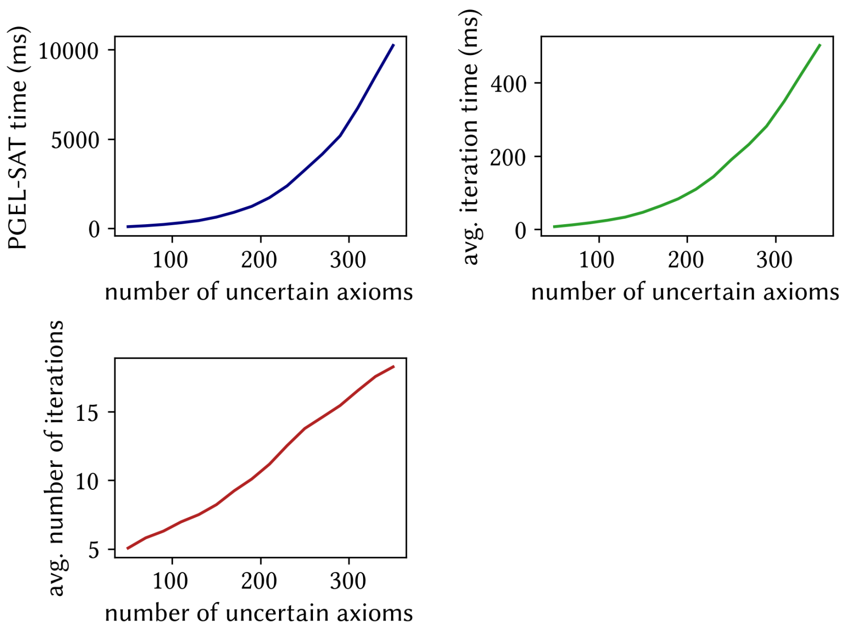


**Figure 4.4:** *Impact of the increase of uncertain axioms in the run time and number of iterations*

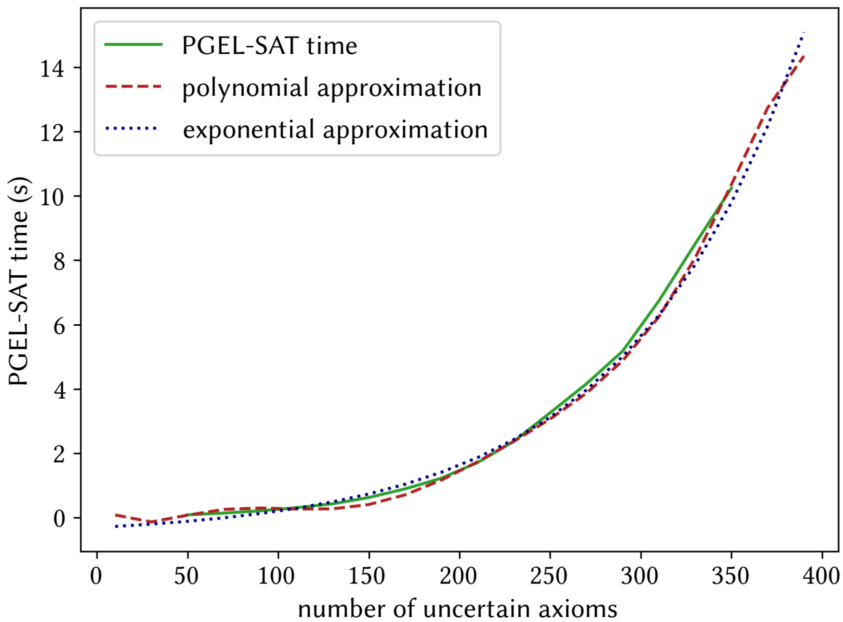


**Figure 4.5:** *Polynomial and exponential approximations of the run time growth by varying the number of uncertain axioms*

# Chapter 5

# Related work

The problem of probabilistic reasoning and extensions in logics to deal with uncertainty have been studied for several decades. The first known proposal of PSAT, for propositional formulas, is attributed to Boole (1854) and it has already been shown to be NP-Complete (Georgakopoulos *et al.*, 1988).

In the relational domain, the literature contain several logics with probabilistic reasoning capabilities although they have led to intractable decision problems. Some of them extend the already intractable $\mathcal{ALC}$, with probabilistic constrains over concepts (Heinsohn, 1994; Lukasiewicz, 2008; Gutiérrez-Basulto *et al.*, 2011). The probablistic Datalog$^{\pm}$ (Gottlob *et al.*, 2013), that combines Datalog$^{\pm}$ with Markov logic networks (Richardson and Domingos, 2006), has a query entailment in PP-hard (u. u. Ceylan *et al.*, 2016). For the expressive and lightweight $\mathcal{EL}$-family, some extensions such as Gutiérrez-Basulto *et al.* (2017) and I. I. Ceylan and Peñaloza (2017) have led to ExpTime-hard or PP-complete probabilistic reasoning; futhermore, NP-completeness can be achieved with probability capabilities over axioms (Finger, 2019).

On the other hand, tractable probabilistic reasoning was achieved by different ways. Domingos and W. Webb (2012) proposed the first tractable first-order probabilistic logic, called Tractable Markov Logic (TML), by imposing an hierarchical structure on its domain. A second aproach from TML led to a straightforward handling of existence uncertainty (W. A. Webb and Domingos, 2013). Besides that, a probabilistic extension of DL-Lite based on Bayesian networks allows satisfiability checking in LogSpace (d'Amato *et al.*, 2008).

Furthermore, some studies implies that the research on MaxSAT has a impact on the solutions of PSAT problems (Andersen and Pretolani, 2001). Also, there was already proposed a MaxSAT-solver for a propositional fragment of horn logic by a max-flow/min-cut formulation (Jaumard and Simeone, 1987), which inspired the solution described in this study.

# Chapter 6

# Conclusion and future work

Different studies focused on description logics with probabilistic reasoning. In particular, finding a logic that is both tractable and expressive enough became a problem of special interest in the field.

This work presented a fragment of $\mathcal{EL}^{++}$ with probabilistic capabilities called Probabilistic Graphic $\mathcal{EL}$. It was shown that its satisfiability problem can be modeled as a linear program, using a column generation technique. Each column generation is an instance of a MaxSAT problem, which can be solved by representing the knowledge base as a weighted graph and finding a minimal cut.

Also, all algorithms for solving PGEL-SAT were implemented and its tractability was presented. For this solution, a theoretical upper bound was estimated and experimental analysis confirm the polynomial complexity of the run time.

Furthermore, studies and implementations from this work contributed for improving the theory for this algorithm of Finger and Lopes (n.d.), which is under development.

Further research directions include finding tractable extensions of P$\mathcal{GEL}$. Existential body axioms and conjunctive axioms could be approximated using probabilistic restrictions; thus, it is expected to achieve tractable approximate reasoning for a probabilistic $\mathcal{EL}^{++}$. Also, conditional probabilities could be modeled using linear restrictions.

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