

matrixpop: a matrix population model for plankton size classes

- model the size distribution of a plankton population

setup:

m number of size classes
 v_{min} minimum size
 Δ_v determines size class spacing
 Δ_v^{-1} must be an integer

model:

$w(t) \in \mathbb{R}^m$ population
 $A(t, \Delta_t) \in \mathbb{R}^{m \times m}$ time-propagation matrix
 $w_i(t)$ proportion of cells at time t with size between v_i and v_{i+1} ($\sum_{i=1}^m w_i(t) = 1$)

size classes:

$$v_i = v_{min} 2^{(i-1)\Delta_v} \text{ for } i = 1, \dots, m+1$$

integer $j = 1 + \Delta_v^{-1}$ with

$$\begin{aligned} v_{i+j-1} &= v_{min} 2^{(i+j-1-1)\Delta_v} \\ &= v_{min} 2^{(i+\Delta_v^{-1}-1)\Delta_v} \\ &= v_{min} 2^{(i-1)\Delta_v} \cdot 2 \\ &= 2 \cdot v_i \end{aligned}$$

$$w(t + \Delta_t) = \frac{A(t, \Delta_t)w(t)}{\|A(t, \Delta_t)w(t)\|_1}$$

structure of A

growth: with size-independent growth rate $\gamma(t, \Delta_t) = \Delta_t \gamma_{\max} \left(1 - \exp\left(-\frac{E(t)}{E^*}\right)\right)$ (2 parameters)

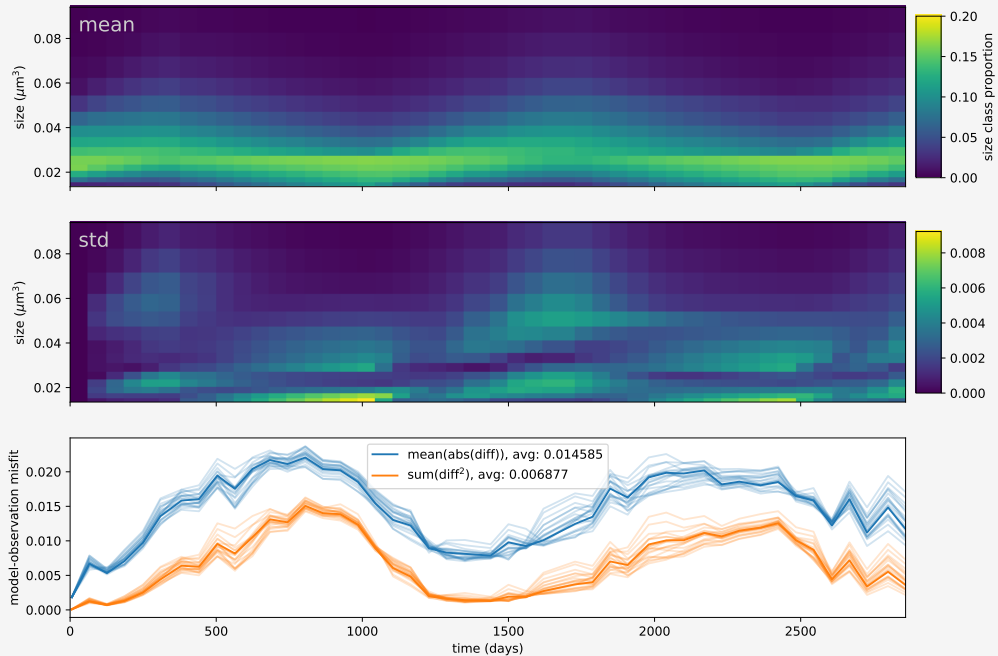
division: with size-dependent division rate $\delta_i(\Delta_t) = \Delta_t \delta_{\max} \frac{v_i^b}{1+v_i^b}$ (2 parameters)

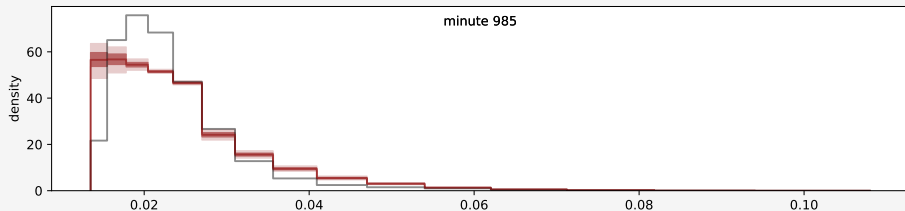
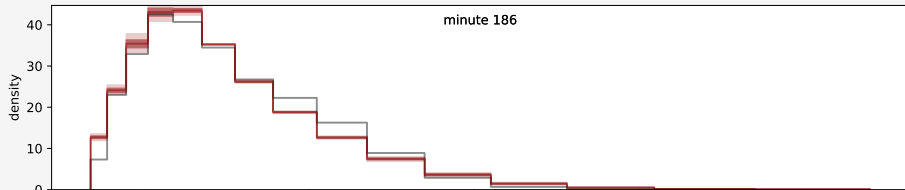
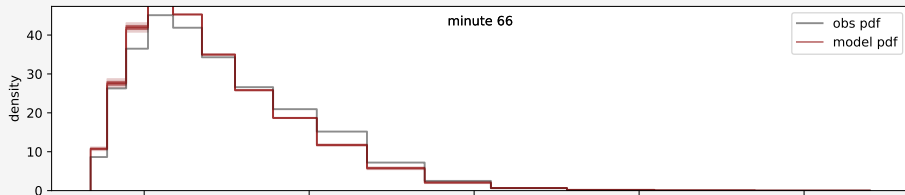
stasis: the rest of the cells remain at their size class

$$A(t, \Delta_t) = \begin{pmatrix} \begin{array}{cccccc} \downarrow 1 & & & & & \\ (1-\gamma) & & & & & \\ \gamma & \ddots & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & (1-\gamma) & & \\ & & & \gamma & \begin{array}{l} (1-\delta_j) \\ (1-\gamma) \end{array} & \\ & & & \begin{array}{l} (1-\delta_j) \\ \gamma \end{array} & \ddots & \\ & & & & \ddots & \begin{array}{l} (1-\delta_{m-1}) \\ (1-\gamma) \end{array} \\ & & & & & \begin{array}{l} (1-\delta_{m-1}) \\ \gamma \end{array} & 1-\delta_m \end{array} \\ \left. \begin{array}{l} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow j-1 \\ \leftarrow j \\ \leftarrow m-1 \\ \leftarrow m \end{array} \right) \end{array}$$

- So far the plots only show the posterior mean, add some plots that emphasize that it is a posterior distribution.

exercise





some ideas

- ① adding respiration
- ② making growth size-dependent
- ③ size-dependent respiration
- ④ a better way to compare model an observation
- ⑤ multiple populations

idea: adding respiration

- let cells “shrink” (move down one size class) at a specific rate

$$A = \begin{pmatrix} \begin{array}{c} \downarrow 1 \\ (1 - \gamma_1) \end{array} & \begin{array}{c} \downarrow 2 \\ \rho \end{array} & & \begin{array}{c} \downarrow j-1 \\ \end{array} & \begin{array}{c} \downarrow j \\ 2\delta_j \end{array} & & \begin{array}{c} \downarrow m-1 \\ \end{array} & \begin{array}{c} \downarrow m \\ \end{array} \\ \hline \begin{array}{c} \gamma_1 \\ (1-\rho) \end{array} & \begin{array}{c} (1-\gamma_2) \\ (1-\rho) \end{array} & \ddots & & & & & \\ \hline & \ddots & \begin{array}{c} \rho \end{array} & & & & & \\ \hline & & \ddots & \begin{array}{c} (1-\gamma_{j-1}) \\ (1-\rho) \end{array} & \begin{array}{c} (1-\delta_j) \\ \rho \end{array} & & & 2\delta_m \\ \hline & & & \begin{array}{c} \gamma_{j-1} \\ (1-\rho) \end{array} & \begin{array}{c} (1-\delta_j) \\ (1-\gamma_j) \\ (1-\rho) \end{array} & \ddots & & \\ \hline & & & & \begin{array}{c} (1-\delta_j) \\ \gamma_j \\ (1-\rho) \end{array} & \ddots & \begin{array}{c} (1-\delta_{m-1}) \\ (1-\gamma_{m-1}) \\ (1-\rho) \end{array} & \begin{array}{c} (1-\delta_m) \\ \rho \end{array} \\ \hline & & & & & \begin{array}{c} (1-\delta_{m-1}) \\ \gamma_{m-1} \\ (1-\rho) \end{array} & \begin{array}{c} 1-\delta_m \\ (1-\rho) \end{array} \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \\ \leftarrow j-1 \\ \leftarrow j \\ \\ \leftarrow m-1 \\ \leftarrow m \end{matrix}$$

division: with size-dependent division rate δ_i

growth: with growth rate γ

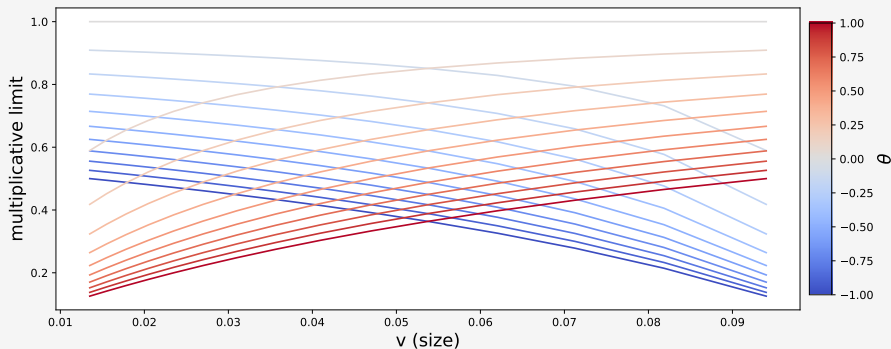
respiration: with size-independent respiration rate ρ

stasis: the rest of the cells remain at their size class

idea: making growth size-dependent

- turn the γ parameter into γ_i
- an approach for data-driven size-dependence:

$$\text{"size limit"} = \begin{cases} \frac{v_i/v_m}{\theta + v_i/v_m} & \text{if } \theta > 0 \\ \frac{1 + \frac{v_{\min} - v_i}{v_m}}{1 + \frac{v_{\min} - v_i}{v_m} - \theta} & \text{otherwise} \end{cases} \quad \text{for } \theta \in [-1, 1]$$



- combine the previous 2 ideas, to add size-dependence to the respiration rate

idea: a better way to compare model and observations

- find a better way to compare model with observations

```
// fitting observations
for (it in 1:nt_obs){
  diff = 0.0;
  for (iv in 1:m){
    diff += fabs(mod_obspos[iv,it] - obs[iv,it]);
  }
  diff = diff/sigma;
  diff ~ normal(0.0, 1.0) T[0,];
}
```

- the standard, even more naive approach leads to significantly worse results

```
// fitting observations
for (it in 1:nt_obs){
  for (iv in 1:m){
    obs[iv,it] ~ normal(mod_obspos[iv,it], sigma);
  }
}
```

- in a follow-up paper, Hunter-Cevera et al. (2014) “Diel size distributions reveal seasonal growth dynamics of a coastal phytoplankter”, the authors use 2 subpopulations to model the data
- it is easy to add an `npop` parameter and use an iteration turning

```
for (it in 1:nt){ // time-stepping loop
  // compute gamma
  gamma = gamma_max * dt_days * (1.0 - exp(-E[it]/E_star));
  ...
  for (i in 1:m){ // size-class loop
    ...
```

into

```
for (ipop in 1:npop){
  ...
  for (it in 1:nt){ // time-stepping loop
    // compute gamma
    gamma = gamma_max[ipop] * dt_days * (1.0 - exp(-E[it]/E_star[ipop]));
    ...
    for (i in 1:m){ // size-class loop
      ...
```

