

## matrixpop: a matrix population model for plankton size classes

- model the size distribution of a plankton population

setup:

$m$  number of size classes  
 $v_{min}$  minimum size  
 $\Delta_v$  determines size class spacing  
 $\Delta_v^{-1}$  must be an integer

model:

$w(t) \in \mathbb{R}^m$  population  
 $A(t, \Delta_t) \in \mathbb{R}^{m \times m}$  time-propagation matrix  
 $w_i(t)$  proportion of cells at time  $t$  with size between  $v_i$  and  $v_{i+1}$  ( $\sum_{i=1}^m w_i(t) = 1$ )

size classes:

$$v_i = v_{min} 2^{(i-1)\Delta_v} \text{ for } i = 1, \dots, m+1$$

integer  $j = 1 + \Delta_v^{-1}$  with

$$\begin{aligned} v_{i+j-1} &= v_{min} 2^{(i+j-1-1)\Delta_v} \\ &= v_{min} 2^{(i+\Delta_v^{-1}-1)\Delta_v} \\ &= v_{min} 2^{(i-1)\Delta_v} \cdot 2 \\ &= 2 \cdot v_i \end{aligned}$$

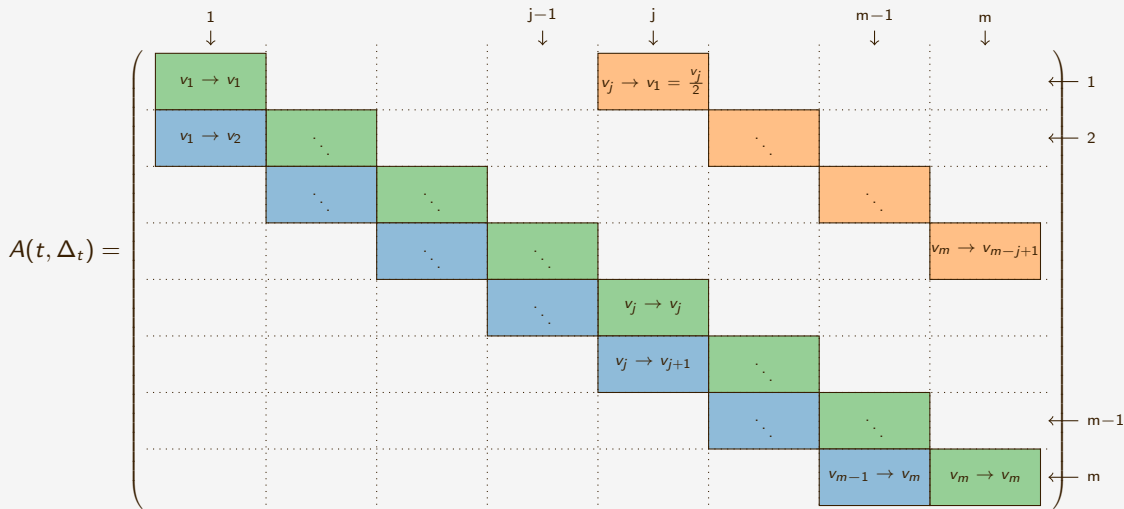
$$w(t + \Delta_t) = \frac{A(t, \Delta_t)w(t)}{\|A(t, \Delta_t)w(t)\|_1}$$

# structure of $A$

**growth:** with size-independent growth rate  $\gamma(t, \Delta_t) = \Delta_t \gamma_{\max} \left(1 - \exp\left(-\frac{E(t)}{E^*}\right)\right)$  (2 parameters)

**division:** with size-dependent division rate  $\delta_i(\Delta_t) = \Delta_t \delta_{\max} \frac{v_i^b}{1+v_i^b}$  (2 parameters)

**stasis:** the rest of the cells remain at their size class



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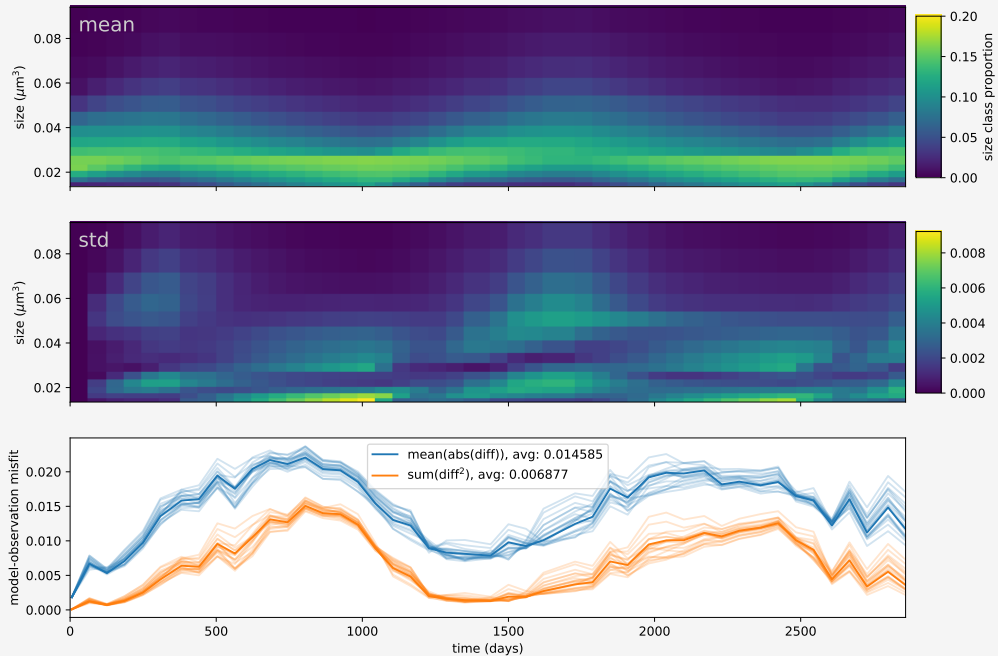
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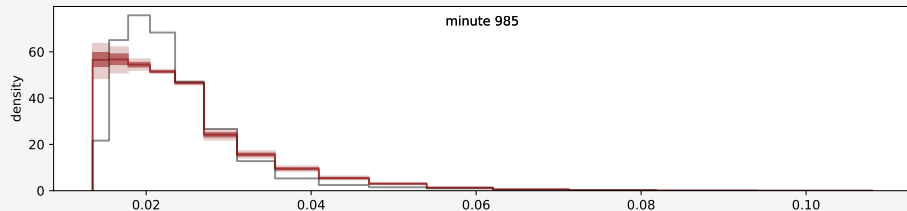
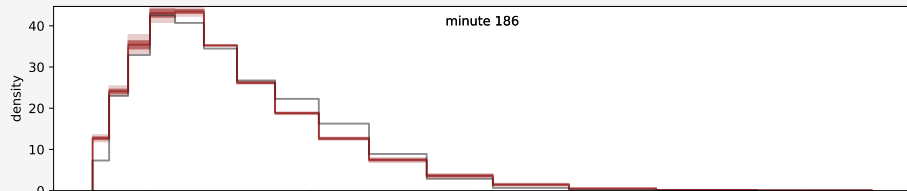
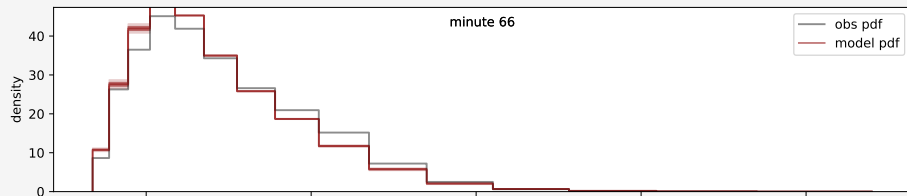
**stasis:** the rest of the cells remain at their size class

$$A(t, \Delta_t) = \begin{pmatrix} \begin{array}{cccccc} \downarrow 1 & & & & & \\ (1-\gamma) & & & & & \\ \gamma & \ddots & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & (1-\gamma) & & \\ & & & \gamma & \begin{array}{l} (1-\delta_j) \\ (1-\gamma) \end{array} & \\ & & & \begin{array}{l} (1-\delta_j) \\ \gamma \end{array} & \ddots & \\ & & & & \ddots & \begin{array}{l} (1-\delta_{m-1}) \\ (1-\gamma) \end{array} \\ & & & & & \begin{array}{l} (1-\delta_{m-1}) \\ \gamma \end{array} & 1-\delta_m \end{array} \\ \left. \begin{array}{l} \leftarrow 1 \\ \leftarrow 2 \\ \\ \\ \\ \leftarrow m-1 \\ \leftarrow m \end{array} \right\} \end{array}$$

- So far the plots only show the posterior mean, add some plots that emphasize that it is a posterior distribution.

## exercise





## some ideas

- ① adding respiration
- ② making growth size-dependent
- ③ size-dependent respiration
- ④ a better way to compare model an observation
- ⑤ multiple populations

# idea: adding respiration

- let cells “shrink” (move down one size class) at a specific rate

$$A = \begin{pmatrix} \begin{matrix} \downarrow 1 & \downarrow 2 & & \downarrow j-1 & \downarrow j & & \downarrow m-1 & \downarrow m \end{matrix} \\ \begin{matrix} (1-\gamma_1) & \rho & & & 2\delta_j & & & \\ \gamma_1(1-\rho) & (1-\gamma_2)(1-\rho) & \ddots & & & & & \\ & \ddots & \ddots & \rho & & & & \\ & & \ddots & (1-\gamma_{j-1})(1-\rho) & (1-\delta_j)\rho & & & 2\delta_m \\ & & & \gamma_{j-1}(1-\rho) & (1-\delta_j)(1-\gamma_j)(1-\rho) & \ddots & & \\ & & & & (1-\delta_j)\gamma_j(1-\rho) & \ddots & & \\ & & & & & \ddots & (1-\delta_{m-1})(1-\gamma_{m-1})(1-\rho) & (1-\delta_m)\rho \\ & & & & & & (1-\delta_{m-1})\gamma_{m-1}(1-\rho) & (1-\delta_m)(1-\rho) \end{matrix} \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \\ \\ \\ \\ \leftarrow m-1 \\ \leftarrow m \end{matrix}$$

**division:** with size-dependent division rate  $\delta_i$

**growth:** with growth rate  $\gamma$

**respiration:** with size-independent respiration rate  $\rho$

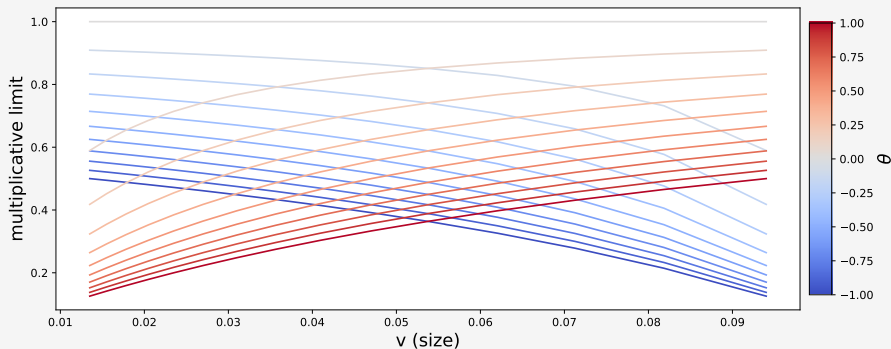
**stasis:** the rest of the cells remain at their size class



## idea: making growth size-dependent

- turn the  $\gamma$  parameter into  $\gamma_i$
- an approach for data-driven size-dependence:

$$\text{"size limit"} = \begin{cases} \frac{v_i/v_m}{\theta + v_i/v_m} & \text{if } \theta > 0 \\ \frac{1 + \frac{v_{\min} - v_i}{v_m}}{1 + \frac{v_{\min} - v_i}{v_m} - \theta} & \text{otherwise} \end{cases} \quad \text{for } \theta \in [-1, 1]$$



- combine the previous 2 ideas, to add size-dependence to the respiration rate

## idea: a better way to compare model and observations

- find a better way to compare model with observations

```
// fitting observations
for (it in 1:nt_obs){
  diff = 0.0;
  for (iv in 1:m){
    diff += fabs(mod_obspos[iv,it] - obs[iv,it]);
  }
  diff = diff/sigma;
  diff ~ normal(0.0, 1.0) T[0,];
}
```

- the standard, even more naive approach leads to significantly worse results

```
// fitting observations
for (it in 1:nt_obs){
  for (iv in 1:m){
    obs[iv,it] ~ normal(mod_obspos[iv,it], sigma);
  }
}
```

- in a follow-up paper, Hunter-Cevera et al. (2014) “Diel size distributions reveal seasonal growth dynamics of a coastal phytoplankter”, the authors use 2 subpopulations to model the data
- it is easy to add an `npop` parameter and use an iteration turning

```
for (it in 1:nt){ // time-stepping loop
  // compute gamma
  gamma = gamma_max * dt_days * (1.0 - exp(-E[it]/E_star));
  ...
  for (i in 1:m){ // size-class loop
    ...
```

into

```
for (ipop in 1:npop){
  ...
  for (it in 1:nt){ // time-stepping loop
    // compute gamma
    gamma = gamma_max[ipop] * dt_days * (1.0 - exp(-E[it]/E_star[ipop]));
    ...
    for (i in 1:m){ // size-class loop
      ...
```

