Irwin, Fraser, and Kapral Reply: Stark's description[1] of the precise mathematical properties that hold at resonance values appears to preclude the use of a limiting invariant density for the description of stochastically induced coherence[2] because the measure is totally (purely[3]) singular at resonance. We stress, however, that in any experiment the dynamical state of the system cannot be precisely known and thus the stationary state can only be determined to a finite precision.[4] A well defined coarse-grained-invariant density *does* exist in these circumstances, and this was the quantity computed for the figures in our paper (the figures referred to in the present text are those of Ref. 2). The density displayed in Fig. 2 is a histogram of 10^7 points in 2000 bins in the unit interval. Because the stochastic process can be realized in experiments, we chose to give a global description of the resonance phenomenon at this level of resolution for both x and λ. It is evident from Fig. 1 that the qualitative changes in the character of the coarse-grained density are organized by the systems of lines referred to in Ref. 2. Our justification for this presentation is that it is precisely these qualitative changes that would be observed in high-resolution experiments.

In light of Stark's comments it is useful to summarize these qualitative changes. We can couch this description in terms of the mass function which increases from zero at $\lambda=0$ to one at $\lambda=1$. In the Cantor set regime this function increases on a nowhere dense perfect set (the "thin fractal" Cantor set).[3] Since the mass function has zero derivative almost everywhere, the corresponding coarse-grained density has large gaps which are readily observable. In contrast, in the resonance regime the "fat fractal"[5] mass function is nowhere constant on the unit interval.[3] As a consequence, the coarse-grained density has no gaps, but may have zeros and in the neighborhood of these zeros there are deep cusp-shaped troughs. Abrupt changes in the appearance of the coarse-grained density take place at λ values corresponding to centrally periodic orbits (cf. Fig. 1). These λ values are reciprocal Pisot-Vijayarghavan (PV) numbers.[6]

An important qualitative change occurs at $\lambda=1/\sqrt{2}$ where the exponent $z_0=1$ and the absolutely continuous density has a trapezoidal form. For $\lambda>1/\sqrt{2}$, $z_0>1$ implying a smooth rather than a cusp-shaped coarse-grained density at the edges of the interval. In the resonance regime the cusps at the edges of the interval are replicated on the centrally periodic orbit under iteration of the functional equation. In contrast, under the functional equation, the images of the left- and right-hand parts of a smooth coarse-grained density overlap at these points. This was in fact the reason for our terminology *apparently smooth* which was stressed several times in our discussion of this regime. There are indeed λ values corresponding to PV numbers in this regime. One value is given by Erdös[3] and is the real root with magnitude less than 1 of the polynomial $\lambda^3+\lambda^2-1=0$. Dynamically this corresponds to a transient of two steps leading to the period-2 central periodic orbit. Another λ value corresponding to a PV number arises from a three-step transient that backtracks and leads to a period-2 central periodic orbit. The corresponding polynomial is $\lambda^4+\lambda-1=0$.[5] However, the coarse-grained density at these "singular" λ values has very weakly structured appearance lacking deep fissures. In contrast to the resonance regime, no dramatic changes take place in the coarse-grained density as λ changes. Hence in the context of even high-resolution physical experiments the distinction between these two regimes is useful.

It is evident that the coarse-grained density at resonance values has a self-similar structure (cf. Fig. 2), and should be treated as a fractal measure in which the singular (multifractal) character of the density would show itself under suitable rescaling.[7] Space limitations prevented our describing this aspect of the "density" in our Letter, but experiments could easily be constructed to imitate this invariance under rescaling.

After all, as Stark points out, little of a precise mathematical nature is known about the invariant measure corresponding to this stochastic process, although at certain λ values it has a totally singular character which is already described by Erdös.[3] In spite of this it is still possible to discuss the bounded objects that would be observed in experiments.

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