Comment on "Stochastically Induced Coherence in Bistable Systems"

In a recent Letter[1] Irwin, Fraser, and Kapral (IFK) describe a new model for stochastically forced bistable oscillators. This consists of two maps f_0, f_1 of the interval [0,1] given by

$$f_i(x) = \lambda x + i(1-\lambda) \quad \text{for } i=0,1\,, \tag{1}$$

where $0 < \lambda < 1$ is a parameter. One of the maps is applied randomly at each iteration, with probability p and $1-p$, respectively.

IFK then attempt to analyze the invariant density ρ^* of (1). This is given by the solution of the Chapman-Kolmogorov equation

$$\rho^*(x) = \lambda^{-1}[p\rho^*(\lambda^{-1}x) + (1-p)\rho^*(\lambda^{-1}(x+\lambda-1))]\,.$$

Unfortunately, it has long been known that there are many values of λ for which this has *no solution.* These values include precisely those λ at which the resonances studied by IFK are defined.

More precisely, Karlin[2] shows that (1) always has a unique invariant measure m^* which is the solution of

$$m^*(A) = pm^*(f_0^{-1}(A)) + (1-p)m^*(f_1^{-1}(A))$$

for every Borel measurable set $A \subset [0,1]$. Here $f_i^{-1}(A)$ is defined to be the set of points $x \in [0,1]$ such that $f_i(x) \in A$. This measure is either absolutely continuous[3] (AC) with respect to the Lebesque measure m on [0,1] or totally singular (TS). The former means that $m^*(A) = 0$ for every A such that $m(A) = 0$, in which case the Radon-Nikodym derivative[3] of m^* is precisely ρ^*. The definition of totally singular, on the other hand, is that there is a set A with $m^*(A) = 1$ but $m(A) = 0$. In such a case ρ^* cannot exist.

It is in general not known into which case m^* falls for particular values of λ and p. The only available results[4] are for $p = \frac{1}{2}$, which is the value analyzed by IFK. The only explicit values for which m^* is AC are $\lambda = 2^{-1/k}$ for $k = 1,2,\ldots$ and a set of algebraic numbers discovered by Garcia,[4] though Erdös[4] has proved that there exists $\delta > 0$ such that m^* is AC for almost every $\lambda \in (1-\delta, 1)$. Conversely, Salem[4] has shown that if λ is a Pisot-Vijayarghavan (PV) number[4] then m^* is TS. This includes the golden mean $\lambda = (\sqrt{5}-1)/2$ and all the other values of λ corresponding to the resonances defined by IFK. Their discussion of the behavior of ρ^* at these values is thus meaningless, though the scaling behavior of m^* is correct. Note that there are two PV numbers greater than $1/\sqrt{2}$, which invalidates IFK's assertion of smoothness for $\lambda > 1/\sqrt{2}$. Very little is in fact known about the continuity and smoothness of ρ^* except that for $\lambda = 2^{-1/k}$, ρ^* gets progressively smoother as $k \to \infty$. Also observe that m^* is TS for $\lambda < 1/2$.

It should also be mentioned that Eq. (1) is well known in a variety of other contexts. Apart from the work of Erdös already referred to in IFK, (1) was studied as an example of a stochastic learning automaton by Karlin,[2] while a deterministic realization is given by the fat-baker's transformation.[4] The Sinai-Bowen-Ruelle measure of this is thus given by $m^* \times m$. More recently, (1) occurs as an example of random maps in ergodic theory and more particularly of Barnsley's iterated function systems.[5,6]

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