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EXACT AND SUPERLATIVE INDEX NUMBERS*

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The paper rationalizes certain functional forms for index numbers with functional forms for the underlying aggregator function. An aggregator functional form is said to be 'flexible' if it can provide a second order approximation to an arbitrary twice differentiable linearly homogeneous function. An index number functional form is said to be 'superlative' if it is exact (i.e., consistent with) for a 'flexible' aggregator functional form. The paper shows that a certain family of index number formulae is exact for the 'flexible' quadratic mean of order r aggregator function, $(\Sigma_i\Sigma_j a_{ij}x_i^{r/2}x_j^{r/2})^{1/r}$, defined by Denny and others. For r equals 2, the resulting quantity index is Irving Fisher's ideal index. The paper also utilizes the Malmquist quantity index in order to rationalize the Törnqvist–Theil quantity index in the nonhomothetic case. Finally, the paper attempts to justify the Jorgenson–Griliches productivity measurement technique for the case of discrete (as opposed to continuous) data.

1. Introduction

One of the most troublesome problems facing national income accountants and econometricians who are forced to construct some data series, is the question of which functional form for an index number should be used. In the present paper, we consider this question and relate functional forms for the underlying production or utility function (or aggregator function, to use a neutral terminology).

First, define a *quantity index* between periods 0 and 1, $Q(p^0, p^1; x^0, x^1)$, as a function of the prices in periods 0 and 1, $p^0 > 0_N$ and $p^1 > 0_N$ (where 0_N is an N-dimensional vector of zeroes), and the corresponding quantity vectors, $x^0 > 0_N$ and $x^1 > 0_N$, while a *price index* between periods 0 and 1, $P(p^0, p^1; x^0, x^1)$, is a function of the same price and quantity vectors. Given either a price index or a quantity index, the other function can be defined implicitly by the following equation [Fisher's (1922) weak factor reversal test]:

$$(1.1) \qquad P(p^0, p^1; x^0, x^1)\, Q(p^0, p^1; \quad x^0, x^1) = p^1 \cdot x^1 / p^0 \cdot x^0,$$

i.e., the product of the price index times the quantity index should yield the

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expenditure ratio between the two periods. (We indicate the inner product of two vectors as $p \cdot x$ or $p^T x$.)

Examples of price indices are

$$P_{\mathrm{La}}(p^0, p^1; x^0, x^1) \equiv p^1 \cdot x^0 / p^0 \cdot x^0 \qquad \text{[Laspeyres price index]},$$

$$P_{\mathrm{Pa}}(p^0, p^1; x^0, x^1) \equiv p^1 \cdot x^1 / p^0 \cdot x^1 \qquad \text{[Paasche price index]}.$$

The geometric mean of the Paasche and Laspeyres indices has been suggested as a price index by Bowley and Pigou (1920, p. 84), but it is Irving Fisher (1922) who termed the resulting index *ideal*:

$$(1.2) \qquad P_{\mathrm{Id}}(p^0, p^1; x^0, x^1) \equiv [p^1 \cdot x^0 p^1 \cdot x^1 / p^0 \cdot x^0 p^0 \cdot x^1]^{\frac{1}{2}}.$$

The Laspeyres, Paasche and ideal quantity indices are defined in a similar manner – quantities and prices are interchanged in the above formulae. In particular, the ideal quantity index is defined as

$$(1.3) \qquad Q_{\mathrm{Id}}(p^0, p^1; x^0, x^1) \equiv [p^1 \cdot x^1 p^0 \cdot x^1 / p^1 \cdot x^0 p^0 \cdot x^0]^{\frac{1}{2}}.$$

Notice that $P_{\mathrm{Id}} Q_{\mathrm{Id}} = p^1 \cdot x^1 / p^0 \cdot x^0$; i.e., the ideal price and quantity indexes satisfy the 'adding up' property (1.1). The following theorem shows that the ideal quantity index may be used to compute the quantity aggregates $f(x^r)$.

(1.4) *Theorem* [Byushgens (1925), Konyus and Byushgens (1926), Frisch (1936, p. 30), Wald (1939, p. 331), Afriat (1972, p. 45) and Pollak (1971)]. *Let* $p^r \gg 0_N$ *for periods* $r = 0, 1, 2, \ldots, R$, *and suppose that* $x^r > 0_N$ *is a solution to* $\max_x \{f(x) : p^r \cdot x \leqq p^r \cdot x^r, x \geqq 0_N\}$, *where* $f(x) \equiv (x^T A x)^{\frac{1}{2}} \equiv (\sum_{j=1}^N \sum_{k=1}^N x_j a_{jk} x_k)^{\frac{1}{2}}$, $a_{jk} = a_{kj}$, *and the maximization takes place over a region where* $f(x)$ *is concave and positive* (*which means* A *must have* $N-1$ *zero or negative eigenvalues and one positive eigenvalue*). *Then*

$$(1.5) \qquad f(x^r)/f(x^0) = Q_{\mathrm{Id}}(p^0, p^r; x^0, x^r), \qquad r = 1, 2, \ldots, R.$$

Thus given the *base period normalization* $f(x^0) = 1$, the ideal quantity index may be used to calculate the aggregate $f(x^r) = (x^{rT} A x^r)^{\frac{1}{2}}$ for $r = 1, 2, \ldots, R$, and *we do not have to estimate the unknown coefficients in the* A *matrix*. This is the major advantage of this method for determining the aggregates $f(x^r)$ [as opposed to the econometric methods suggested by Arrow (1972)], and it is particularly important when N (the number of goods to be aggregated) is large compared to R (the number of observations in addition to the base period observation p^0, x^0).

If a quantity index $Q(p^0, p^r; x^0, x^r)$ and a functional form for the aggregator function f satisfy eq. (1.5) then we say that Q is *exact* for f. Konyus and Byushgens (1926) show that the geometric quantity index $\prod_{i=1}^N (x_i^1/x_i^0)^{s_i}$ (where $s_i \equiv p_i^0 \cdot x_i^0 / p^0 \cdot x^0$) is exact for a Cobb–Douglas aggregator function, while Afriat (1972), Pollak (1971) and Samuelson–Swamy (1974) present other examples of exact index numbers. However, it appears that out of all the exact index numbers thus far exhibited, *only* the ideal index corresponds to a functional form for f which is capable of providing a second-order approximation to an arbitrary twice-differentiable linear homogeneous function. For a proof that the functional form $(x^T A x)^{\frac{1}{2}}$ can provide such a second-order approximation, see Diewert (1974a).

Let us call a quantity index Q '*superlative*' [see Fisher (1922, p. 247) for an undefined notion of a superlative index number] if it is *exact* for an f which can provide a second-order approximation to a linear homogeneous function.

In the following section, we show that the Törnqvist (1936), Theil (1965, 1967) and Kloek (1966, 1967) quantity index [which has been used by Christensen and Jorgenson (1970), Star (1974), Jorgenson and Griliches (1972, p. 83), Star and Hall (1973) as a discrete approximation to the Divisia (1926) index] is also a superlative index number. In section 3, we use the results of section 2 to provide a rigorous interpretation of the Jorgenson–Griliches method of measuring technical progress for discrete data.

In section 4, we introduce an entire family of superlative index numbers. Section 5 presents some conclusions which tend to support the use of Fisher's ideal quantity index in empirical applications and the final section is an appendix which sketches the proofs of various theorems developed in the following sections.

2. The Törnqvist—Theil 'Divisia' index and the translog function

Before stating our main results, it will be necessary to state a preliminary result which is extremely useful in its own right. Let z be an N-dimensional vector and define the *quadratic function* $f(z)$ as

$$(2.1) \qquad f(z) \equiv a_0 + a^T z + \tfrac{1}{2} z^T A z$$

$$= a_0 + \sum_{j=1}^{N} a_i z_i + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} z_i z_j,$$

where the a_i, a_{ij} are constants and $a_{ij} = a_{ji}$ for all i, j.

The following lemma is a global version of the Theil (1967, pp. 222–223) and Kloek (1966) local result.

(2.2) *Quadratic approximation lemma. If and only if the quadratic function f is defined by* (2.1), *then,*

(2.3) $f(z^1)-f(z^0) = \frac{1}{2}[\nabla f(z^1)+\nabla f(z^0)]^T(z^1-z^0)$,
where $\nabla f(z^r)$ is the gradient vector of f evaluated at z^r.

The above result should be contrasted with the usual Taylor series expansion for a quadratic function,

$$f(z^1)-f(z^0) = [\nabla f(z^0)]^T(z^1-z^0)+\tfrac{1}{2}(z^1-z^0)^T\nabla^2 f(z^0)(z^1-z^0),$$

where $\nabla^2 f(z^0)$ is the matrix of second-order partial derivatives of f evaluated at an initial point z^0. In the expansion (2.3), a knowledge of $\nabla^2 f(z^0)$ is not required, but a knowledge of $\nabla f(z^1)$ is required. It must be emphasized that (2.3) holds as an equality for all z^1, z^0 belonging to an open set if and only if f is a quadratic function.

Actually, the quadratic approximation lemma (2.2) is closely related to the following result which we will prove as a corollary to (2.2):

(2.4) *Lemma* [Bowley (1928, pp. 224–225)]. *If a consumer's preferences can be represented by a quadratic function f, defined by* (2.1); $x^1 \gg 0_N$ *is a solution to the utility maximization problem*

(2.5) $\max_z\{f(z): p^1\cdot z = Y^1, z \geqq 0_N\}$, where $p^1 \gg 0_N$, $Y^1 \equiv p^1\cdot x^1$, the inner product of p^1 and x^1; $x^0 \gg 0_N$ (i.e., each component of x^0 is positive) is a solution to the utility maximization problem

(2.6) $\max_z\{f(z): p^0\cdot z = Y^0, z \geqq 0_N\}$, where $p^0 \gg 0_N$ and $Y^0 \equiv p^0\cdot x^0$; then the change in utility between periods 0 and 1 is

(2.7) $f(x^1)-f(x^0) = \frac{1}{2}[\lambda_1^* p^1+\lambda_0^* p^0]\cdot(x^1-x^0)$, where λ_i^* is the marginal utility of income in period i for $i = 0, 1$; i.e., λ_i^* is the optimal value of the Lagrange multiplier for the maximization problem (2.5), and λ_0^* is the Lagrange multiplier for (2.6).

Bowley's lemma is frequently used in applied welfare economics and cost–benefit analysis, while the quadratic approximation lemma is frequently used in index number theory, which indicates the close connection between the two fields.

Suppose that we are given a homogeneous translog aggregator function [Christensen, Jorgenson and Lau (1971)] defined by

$$\ln f(x) \equiv \alpha_0+\sum_{n=1}^{N}\alpha_n \ln x_n+\tfrac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\gamma_{jk}\ln x_j \ln x_k,$$

where $\sum_{n=1}^N\alpha_n = 1$, $\gamma_{jk} = \gamma_{kj}$ and $\sum_{k=1}^N\gamma_{jk} = 0$ for $j = 1, 2, \ldots, N$.

Jorgenson and Lau have shown that the homogeneous translog function can provide a second-order approximation to an arbitrary twice-continuously-differentiable linear homogeneous function. Let us use the parameters which occur in the translog functional form in order to define the following function, f^*:

$$(2.8)\qquad f^*(z) \equiv \alpha_0 + \sum_{j=1}^{N} \alpha_i z_i + \tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_{ij} z_i z_j .$$

Since the function f^* is quadratic, we can apply the quadratic approximation lemma (2.2), and we obtain

$$(2.9)\qquad f^*(z^1) - f^*(z^0) = \tfrac{1}{2}[\nabla f^*(z^1) + \nabla f^*(z^0)] \cdot (z^1 - z^0).$$

Now we relate f^* to the translog function f. We have

$$(2.10)\qquad \begin{aligned} &\partial f^*(z^r)/\partial z_j = \partial \ln f(x^r)/\partial \ln x_j = [\partial f(x^r)/\partial x_j][x_j^r/f(x^r)]. \\ &f^*(z^r) = \ln f(x^r), \\ &z_j^r = \ln x_j^r, \qquad \text{for} \quad r = 0, 1 \quad \text{and} \quad j = 1, 2, \ldots, N. \end{aligned}$$

If we substitute relation (2.10) into (2.9), we obtain

$$(2.11)\qquad \ln f(x^1) - \ln f(x^0) = \tfrac{1}{2}\left[\hat{x}^1 \frac{\nabla f(x^1)}{f(x^1)} + \hat{x}^0 \frac{\nabla f(x^0)}{f(x^0)}\right] \cdot [\ln x^1 - \ln x^0],$$

where $\ln x^1 \equiv [\ln x_1^1, \ln x_2^1, \ldots, \ln x_N^1]$, $\ln x^0 \equiv [\ln x_1^0, \ln x_2^0, \ldots, \ln x_N^0]$, $\hat{x}^1 \equiv$ the vector x^1 diagonalized into a matrix, and $\hat{x}^0 \equiv$ the vector x^0 diagonalized into a matrix.

Assume that $x^r \gg 0_N$ is a solution to the aggregator maximization problem $\max_x\{f(x)\colon p^r \cdot x = p^r \cdot x^r, x \geqq 0_N\}$, where $p^r \gg 0_N$ for $r = 0, 1$, and f is the homogeneous translog function. The first-order conditions for the two maximization problems, after elimination of the Lagrange multipliers [Konyus and Byushgens (1926, p. 155), Hotelling (1935, pp. 71–74), Wold (1944, pp. 69–71) and Pearce (1964, p. 59) lemma], yield the relations $p^r/p^r \cdot x^r = \nabla f(x^r)/x^r \cdot \nabla f(x^r)$ for $r = 0, 1$. Since f is linear homogeneous, $x^r \cdot \nabla f(x^r)$ may be replaced by $f(x^r)$ in the above, and substitution of these last two relations into (2.11) yields

$$\begin{aligned} \ln [f(x^1)/f(x^0)] &= \tfrac{1}{2}\left[\frac{\hat{x}^1 p^1}{p^{1T} x^1} + \frac{\hat{x}^0 p^0}{p^{0T} x^0}\right] \cdot [\ln x^1 - \ln x^0] \\ &= \sum_{n=1}^{N} \tfrac{1}{2}[s_n^1 + s_n^0] \ln [x_n^1/x_n^0], \end{aligned}$$

or

$$(2.12)\qquad f(x^1)/f(x^0) = \prod_{n=1}^{N} [x_n^1/x_n^0]^{\frac{1}{2}[s_n{}^1+s_n{}^0]} \equiv Q_0(p^0, p^1; x^0, x^1),$$

where $s_n^r \equiv p_n^r x_n^r / p^r \cdot x^r$, the nth share of cost in period r.

The right-hand side of (2.12) is the quantity index which corresponds to Irving Fisher's (1922) price index number 124, using (1.1). It has also been advocated as a quantity index by Törnqvist (1936) and Theil (1965, 1967, 1968). It has been utilized empirically by Christensen and Jorgenson (1969, 1970) as a discrete approximation to the Divisia (1926) index and by Star (1974) and Star and Hall (1973) in the context of productivity measurement. The above argument shows that this quantity index is *exact* for a homogeneous translog aggregator function, and in view of the second-order approximation property of the homogeneous translog function, we see that the right-hand side of (2.12) is a *superlative* quantity index.

It can also be seen [using the if and only if nature of the quadratic approximation lemma (2.2)] that the homogeneous translog function is the only differentiable linear homogeneous function which is exact for the Törnqvist–Theil quantity index.

The above argument can be repeated (with some changes in notation) if the unit cost function for the aggregator function is the translog unit cost function defined by

$$\ln c(p) \equiv \alpha_0^* + \sum_{j=1}^{N} \alpha_n^* \ln p_n + \tfrac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk}^* \ln p_j \ln p_k,$$

where $\sum_{n=1}^{N} \alpha_n^* = 1$, $\gamma_{jk}^* = \gamma_{kj}^*$ and $\sum_{k=1}^{N} \gamma_{jk}^* = 0$ for $j = 1, 2, \ldots, N$. We also need the following results.

(2.13) *Lemma* [Shephard (1953, p. 11), Samuelson (1947)]. *If f is positive, linearly homogeneous and concave; if*

$$p^r \cdot x^r = \min_x \{p^r \cdot x : f(x) \geqq f(x^r)\} \equiv c(p^r) f(x^r), \qquad \textit{for} \quad r = 0, 1,$$

and if the unit cost function c is differentiable at p^r, then

$$x^r = \nabla c(p^r) f(x^r), \qquad \textit{for} \quad r = 0, 1.$$

(2.14) *Corollary.*[1] $x^r / p^r \cdot x^r = \nabla c(p^r)/c(p^r)$, *for* $r = 0, 1$.

Now under the assumption of cost-minimizing behavior in periods 0 and 1 [which implies (2.14)], we have upon applying the quadratic approximation lemma (2.2) to the translog unit function,

[1] *Proof:* divide (2.13) by $p^r \cdot x^r = c(p^r) f(x^r)$.

$$\ln c(p^1) - \ln c(p^0) = \tfrac{1}{2}\left[\hat{p}^1 \frac{\nabla c(p^1)}{c(p^1)} + \hat{p}^0 \frac{\nabla c(p^0)}{c(p^0)}\right] \cdot [\ln p^1 - \ln p^0]$$

$$= \tfrac{1}{2}\left[\hat{p}^1 \frac{x^1}{p^1 \cdot x^1} + \hat{p}^0 \frac{x^0}{p^0 \cdot x^0}\right] \cdot [\ln p^1 - \ln p^0] \qquad \text{(using 2.14)}$$

$$= \sum_{n=1}^{N} [s_n^1 + s_n^0] \ln [p_n^1 / p_n^0],$$

or

$$\text{(2.15)} \qquad c(p^1)/c(p^0) = \prod_{n=1}^{N} [p_n^1/p_n^0]^{\frac{1}{2}[s_n{}^1 + s_n{}^0]},$$

where $s_n^r = p_n^r x_n^r / p^r \cdot x^r$ (the nth share of cost in period r), $p^r \gg 0_N$ (period r prices, $r = 0, 1$), $x^r \geqq 0_N$ (period i quantities, $i = 0, 1$), and $c(p) =$ the translog unit cost function.[2]

The right-hand side of (2.15) corresponds to Irving Fisher's (1922) price index 123. The above argument shows that this price index is *exact* for a translog unit cost function, and that this is the only differentiable unit cost function which is exact for this price index.

Let us denote the right-hand side of (2.15) as the price index function $P_0(p^0, p^1; x^0, x^1)$, and denote the right-hand side of (2.12) as the quantity index $Q_0(p^0, p^1; x^0, x^1)$. It can be verified that $P_0(p^0, p^1; x^0, x^1) Q_0(p^0, p^1; x^0, x^1) \neq p^1 \cdot x^1 / p^0 \cdot x^0$ in general; i.e., the price index P_0 and the quantity index Q_0 do not satisfy the weak factor reversal test (1.1). This is perfectly reasonable, since the quantity index Q_0 is consistent with a homogeneous translog (direct) aggregator function, while the price index P_0 is consistent with an aggregator function which is dual to the translog unit cost function, and *the two aggregator functions do not in general coincide*; i.e., they correspond to different (aggregation) technologies. Thus, given Q_0, the corresponding price index, which satisfies (1.1), is defined by $\tilde{P}_0(p^0, p^1; x^0, x^1) \equiv p^1 \cdot x^1 / [p^0 \cdot x^0 Q^0(p^0, p^1; x^0, x^1)]$. The quantity index Q_0 and the corresponding (implicit) price index $\tilde{P}_0$ were used by Christensen and Jorgenson (1969, 1970) in order to measure U.S. real input and output. On the other hand, given P_0, the corresponding (implicit) quantity index, which satisfies (1.1), is defined by $\tilde{Q}_0(p^0, p^1; x^0, x^1) \equiv p^1 \cdot x^1 / [p^0 \cdot x^0 P_0(p^0, p^1; x^0, x^1)]$. The price–quantity index pair $(P_0, \tilde{Q}_0)$ was advocated by Kloek (1967, p. 2) over the pair $(\tilde{P}_0, Q_0)$ on the following grounds: as we disaggregate more and more, we can expect the individual consumer or producer to utilize positive amounts of fewer and fewer goods (i.e., as N grows, components of the vectors x^0 and x^1 will tend to become zero), but the prices which the producer or consumer faces are generally positive irrespective of

[2]Note that the validity of (2.15) depends crucially on the validity of (2.14), which will be valid if p^0 and p^1 belong to an open convex set of prices P, such that the translog $c(p)$ satisfies the regularity conditions of positivity, linear homogeneity and concavity over P.

the degree of disaggregation. Since the logarithm of zero is not finite, Q_0 will tend to be indeterminate as the degree of disaggregation increases, but P_0 will still be well defined (provided that all prices are positive).

Theil (1968) and Kloek (1967) provided a somewhat different interpretation of the indices P_0 and Q_0, an interpretation which does not require the aggregator function to be linear homogeneous. Let the aggregate u be defined by $u = f(x)$, where f is a not necessarily homogeneous aggregator function which satisfies for example the Shephard (1970) or Diewert (1971) regularity conditions for a production function. For $p \gg 0_N$, $Y > 0$, define the total cost function by $C(u;p) \equiv \min_x\{p\cdot x : f(x) \geqq u; x \geqq 0_N\}$ and the indirect utility function by $g(p/Y) \equiv \max_x\{f(x): p\cdot x \leqq Y, x \leqq 0_N\}$. The true cost of living price index evaluated at 'utility' level u is defined as $P(p^0, p^1; u) \equiv C(u;p^1)/C(u;p^0)$ and the Theil index of quantity (or 'real income') evaluated at prices p is defined as $Q_T(p; u^0, u^1) \equiv C(u^1;p)/C(u^0;p)$. The Theil–Kloek results are that: (i) $P_0(p^0, p^1; x^0, x^1)$ is a second-order approximation to $P(p^0, p^1; g(v^*))$, where the nth component of v^* is $v_n^* \equiv (p_n^0 p_n^1/p^0\cdot x^0 p^1\cdot x^1)^{\frac{1}{2}}$, for $n = 1, 2, \ldots, N$, and (ii) $Q_0(p^0, p^1; x^0, x^1)$ is a second-order approximation to $Q_T(p^*; g(p^0/p^0\cdot x^0), g(p^1/p^1\cdot x^1))$, where the nth component of p^* is $p_n^* \equiv (p_n^0 p_n^1)^{\frac{1}{2}}$.

In view of the Theil–Kloek approximation results, we might be led to ask whether the index number P_0 is exact for any general (non-homothetic) functional forms for the cost function $C(u;p)$. The following theorem answers this question in the affirmative:

(2.16) *Theorem. Let the functional form for the cost function be a general translog of the form*

$$\ln C(u;p) \equiv \alpha_0^* + \sum_{i=1}^{N} \alpha_i^* \ln p_i + \tfrac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} \gamma_{jk}^* \ln p_j \ln p_k + \beta^* \ln u + \delta^*(\ln u)^2 + \sum_{k=1}^{N} \varepsilon_i^* \ln u \ln p_i,$$

where $\sum_{i=1}^N \alpha_i^* = 1$, $\gamma_{jk}^* = \gamma_{kj}^*$, $\sum_{k=1}^N \gamma_{jk}^* = 0$, *for* $j = 1, 2, \ldots, N$, *and* $\sum_{i=1}^N \varepsilon_j^* = 0$.[3] *Let* $(u^0; p^0)$ *and* $(u^1; p^1)$ *belong to a* $(u;p)$ *region where* $C(u;p)$ *satisfies the appropriate regularity conditions for a cost function [e.g., see Shephard* (1970), *Hanoch* (1970) *or Diewert* (1971)] *and define the quantity vectors* $x^0 \equiv \nabla_p C(u^0;p^0)$ *and* $x^1 \equiv \nabla_p C(u^1;p^1)$. *Then*

$$P_0(p^0, p^1; x^0, x^1) = C(u^*;p^1)/C(u^*;p^0),$$

where $u^* \equiv (u^0 u^1)^{\frac{1}{2}}$ *and* P_0 *is defined by the right-hand side of* (2.15).

In contrast to the case of a linear homogeneous aggregator function where the cost function takes the simple form $C(u;p) = c(p)u$, theorem (2.16) is

[3] These restrictions ensure the linear homogeneity of $C(u;p)$ in p.

not an if and only if result; that is, the index number $P_0(p^0, p^1; x^0, x^1)$ is exact for functional forms for $C(u; p)$ other than the translog. In fact, theorem (2.16) remains true if: (i) we define C as $\ln C(u; p) \equiv \alpha_0(u) + \sum_{i=1}^N [\alpha_i + \varepsilon_i h(u)] \ln p_i + \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \gamma_{jk} \ln p_j \ln p_k$, where $\sum_{i=1}^N \alpha_i = 1$, $\sum_{i=1}^N \varepsilon_i = 0$, $\gamma_{jk} = \gamma_{kj}$, $\sum_{k=1}^N \gamma_{jk} = 0$, for $j = 1, 2, \ldots, N$, and h is a montonically increasing function of one variable, and (ii) define the reference utility level u^* as the solution to the equation $2h(u^*) = h(u^1) + h(u^0)$. [In the translog case, $h(u) \equiv \ln u$.]

Thus the same price index P_0 is exact for more than one functional form (and reference utility level) for the true cost of living.

We can also provide a justification for the quantity index Q_0 in the context of an aggregator function f which is not necessarily linearly homogeneous. In order to provide this justification, it is necessary to define the quantity index which has been proposed by Malmquist (1953) and Pollak (1971) in the context of consumer theory, and by Bergson (1961) and Moorsteen (1961) in the context of producer theory.

Given an aggregator function f and an aggregate $u \equiv f(x)$, define f's *distance function* as $D[u; x] \equiv \max_k\{k : f(x/k) \geqq u\}$. To use the language of utility theory, the distance function tells us by what proportion one has to deflate (or inflate) the given consumption vector x in order to obtain a point on the utility surface indexed by u. It can be shown that if f satisfies certain regularity conditions, then f is completely characterized by D [see Shephard (1970), Hanoch (1970) and McFadden (1970)]. In particular, $D[u; x]$ is linear homogeneous non-decreasing and concave in the vector of variables x and non-increasing in u in Hanoch's formulation.

Now define the Malmquist quantity index as $Q_M(x^0, x^1; u) \equiv D[u; x^1]/D[u; x^0]$. Note that the index depends on x^0 (the base period quantities), x^1 (the current period quantities) and on the base indifference surface (which is indexed by u) onto which the points x^0 and x^1 are deflated. The following theorem relates the translog functional form to the Malmquist quantity index:

(2.17) *Theorem. Let an aggregator function f satisfying the Hanoch* (1970) *and Diewert* (1971) *regularity conditions be given such that f's distance function D is a general translog of the form*

$$\ln D[u; x] = \alpha_0 + \sum_{i=1}^N \alpha_i \ln x_i + \tfrac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \gamma_{jk} \ln x_j \ln x_k + \beta \ln u + \delta(\ln u)^2$$
$$+ \sum_{i=1}^N \varepsilon_i \ln u \ln x_i,$$

where $\sum_{i=1}^N \alpha_i = 1$, $\gamma_{jk} = \gamma_{kj}$, $\sum_{k=1}^N \gamma_{jk} = 0$, *for* $j = 1, 2, \ldots, N$, *and* $\sum_{i=1}^N \varepsilon_i = 0$. *Suppose that the quantity vector* x^0 *is a solution to the aggregator maximization problem* $\max_x\{f(x) : p^0 \cdot x = p^0 \cdot x^0\}$, *while*

x^1 is a solution to $\max_x\{f(x): p^1\cdot x = p^1\cdot x^1\}$ *and* $u^0 \equiv f(x^0)$, $u^1 \equiv f(x^1)$. *Then*

$$Q_0(p^0, p^1; x^0, x^1) = D[u^*; x^1]/D[u^*; x^0] \equiv Q_M(x^0, x^1; u^*),$$

where $u^* \equiv (u^0u^1)^{\frac{1}{2}}$ *and* Q_0 *is defined in* (2.12).

As was the case with the price index P_0, the quantity index Q_0 is equal to Malquist quantity indexes which are defined by non-translog distance functions; i.e., theorem (2.17) is not an if and only if result.

However, theorems (2.16) and (2.17) do provide a rather strong justification for the use of P_0 or Q_0 since the translog functional form provides a second-order approximation to a general cost or distance function (which in turn are dual to a general non-homothetic aggregator function).

Finally, note that theorems (2.16) and (2.17) have a 'global' character to them in contrast to the Theil–Kloek 'local' results.

3. Productivity measurement and 'Divisia' indexes

Jorgenson and Griliches (1972, pp. 83–84) have advocated the use of the indexes $\tilde{P}_0$, Q_0 in the context of *productivity measurement*. It is perhaps appropriate to review their procedure in the light of the results of the previous section.

First, we note (by a straightforward computation) that it is not in general true that 'a discrete Divisia index of discrete Divisia indexes is a discrete Divisia index of the components' [Jorgenson and Griliches (1972, p. 83)], where the 'Divisia' quantity index is defined to be Q_0. In view of the one-to-one nature of the index number Q_0 with the translog functional form for the aggregator function f in the linear homogeneous case, it can be seen that the Jorgenson–Griliches assertion will be true if the producer or consumer is maximizing an aggregator function f subject to an expenditure constraint, where f is *both* a homogeneous translog function *and* a translog of micro-translog aggregator functions. The set of such translog functions is not empty since it contains the set of Cobb–Douglas functions. Thus if cost share's are approximately constant (which corresponds to the Cobb–Douglas case), then the Jorgenson–Griliches assertion will be approximately true.

It can be similarly shown that in general, it is not true that a discrete 'Divisia' price index of discrete 'Divisia' indexes is a discrete 'Divisia' price index of the components, where the 'Divisia' price index is defined to be P_0: the first method of constructing a price index is justified if the aggregator function has a unit cost function dual of the form $\hat{c}[c^1(p^1), c^2(p^2), \ldots, c^J(p^J)]$, where $(p^1, p^2, \ldots, p^J)$ represents a partition of the price vector p and the functions $\hat{c}$, c^1, $c^2, \ldots, c^J$ are all translog unit cost functions, while the second method of constructing a price index is justified if the aggregator function has a unit cost function dual, $c(p)$, which is translog.

Jorgenson and Griliches (1972) use the index number formula $Q_0(p^0, p^1; x^0, x^1)$ defined by the right-hand side of (2.12) not only to form an index of real input, but also to form an index of real output. Just as the aggregation of inputs into a composite input rests on the duality between unit cost and homogeneous production functions, the aggregation of outputs into a composite output can be based on the duality between unit revenue and homogeneous factor requirements functions.[4] We briefly outline this latter duality.

Suppose that a producer is producing M outputs, $(y_1, y_2, \ldots, y_M) \equiv y$, and the technology of the producer can be described by a *factor requirements function*, g, where $g(y)$ = the minimum amount of aggregate input required to produce the vector of outputs y.[5] The producer's *unit* (aggregate input) *revenue function*[6] is defined for each price vector $w \geqq 0_M$ by

$$(3.1) \qquad r(w) \equiv \max_y\{w \cdot y : g(y) \leqq 1, y \geqq 0_M\}.$$

Thus given a factor requirements function g, (3.1) may be used to define a unit revenue function. On the other hand, given a unit revenue function $r(w)$ which is a positive, linear homogeneous, convex function for $w \gg 0_M$, a factor requirements function g^* consistent with r may be defined for $y \gg 0_M$ by[7]

$$\begin{aligned}(3.2) \qquad g^*(y) &\equiv \min_\lambda\{\lambda : w \cdot y \leqq r(w)\lambda \quad \text{for every} \quad w \geqq 0_M\} \\ &= \min_\lambda\{\lambda : 1 \leqq r(w)\lambda \quad \text{for every} \quad w \geqq 0_M \quad \text{such that} \quad w \cdot y = 1\} \\ &= 1/\max_w\{r(w) : w \cdot y = 1, w \geqq 0_M\}.\end{aligned}$$

The translog functional form may be used to provide a second-order approximation to an arbitrary twice-differentiable factor requirements function. Thus assume that g is defined (at least over the relevant range of y's) by

$$(3.3) \qquad \ln g(y^r) \equiv a_0 + \sum_{m=1}^{M} a_m \ln y_m^r + \tfrac{1}{2} \sum_{j=1}^{M} \sum_{k=1}^{M} c_{jk} \ln y_j^r \ln y_k^r, \qquad \text{for} \quad r = 0, 1,$$

where

$$\sum_{m=1}^{M} a_m = 1, \quad c_{jk} = c_{kj}, \quad \sum_{k=1}^{M} c_{jk} = 0, \quad \text{for} \quad j = 1, 2, \ldots, M.$$

[4]See Diewert (1969, 1974a), Fisher and Shell (1972) and Samuelson and Swamy (1974) on this topic.

[5]Assume g is defined for $y \geqq 0_M$, and has the following properties: (i) $g(y) > 0$ for $y \gg 0_M$ (*positivity*), (ii) $g(\lambda y) = \lambda g(y)$ for $\lambda \geqq 0$, $y \geqq 0$ (*linear homogeneity*), and (iii) $g(\lambda y^1 + (1-\lambda)y^2) \leqq \lambda g(y^1) + (1-\lambda)g(y^2)$ for $0 \leqq \lambda \leqq 1, y^1 \geqq 0_M, y^2 \geqq 0_M$ (*convexity*).

[6]If g satisfies the 3 properties listed in footnote 5, then r also has those 3 properties.

[7]The proof is analogous to the proof of the Samuelson–Shephard duality theorem presented in Diewert (1974b); alternatively, see Samuelson and Swamy (1974).

Now assume that $y^r \gg 0_M$ is a solution to the aggregate input minimization problem $\min_y\{g(y): w^r \cdot y^r = w^r \cdot y^r, y \geqq 0_M\}$, where $w^r \gg 0_M$ for $r = 0, 1$, and g is the translog function defined by (3.3). Then the first-order necessary conditions for the minimization problems along with the linear homogeneity of g yield the relations $w^r/w^r \cdot y^r = \nabla g(y^r)/g(y^r)$, for $r = 0, 1$, and using these two relations in lemma (2.2) applied to (3.3),

$$\text{(3.4)} \qquad g(y^1)/g(y^0) = \prod_{m=1}^{M} [y_m^1/y_m^0]^{\frac{1}{2}[w_m{}^1 y_m{}^1/w^1 \cdot y^1 + w_m{}^0 y_m{}^0/w^0 \cdot y^0]}$$

$$\equiv Q_0^*(w^0, w^1; y^0, y^1).$$

Thus again the Törnqvist formula can be used to aggregate quantities consistently, provided that the underlying aggregator function is homogeneous translog.

Similarly if the revenue function $r(w)$ is translog over the relevant range of data and if the producer is in fact maximizing revenue, then we can show that $r(w^1)/r(w^0) = P^*(w^0, w^1; y^0, y^1) \equiv Q_0^*(y^0, y^1; w^0, w^1)$, the Törnqvist price index.

Using the above material, we may now justify the Jorgenson–Griliches (1972) method of measuring technical progress. Assume that the production possibilities efficient set can be represented as the set of outputs y and inputs x such that

$$\text{(3.5)} \qquad g(y) = f(x),$$

where g is the homogeneous translog factor requirements function defined by (3.3), and f is the homogeneous translog production function defined in section 2. Let $w^r \gg 0_M$, $p^r \gg 0_N$, $r = 0, 1$ be vectors of output and input prices during periods 0 and 1, and assume that $y^0 \gg 0_M$ and $x^0 \gg 0_N$ is a solution to the period 0 profit maximization problem,

$$\text{(3.6)} \qquad \max_{y,x}\{w^0 \cdot y - p^0 \cdot x : g(y) = f(x)\}.$$

Suppose that 'technical progress' occurs between periods 0 and 1 which we assume to be a parallel outward shift of the 'isoquants' of the aggregator function f; i.e., we assume that the equation which defines the efficient set of outputs and inputs in period 1 is $g(y) = (1+\tau)f(x)$ where τ represents the amount of 'technical progress' if $\tau > 0$ or 'technical regress' if $\tau < 0$. Finally, assume that $y^1 \gg 0_M$ and $x^1 \gg 0_N$ is a solution to the period 1 profit maximization problem,

$$\text{(3.7)} \qquad \max_{y,x}\{w^1 \cdot y - p^1 \cdot x : g(y) = (1+\tau)f(x)\}.$$

Thus we have $g(y^0) = f(x^0)$ and $g(y^1) = (1+\tau)f(x^1)$. It is easy to see that $y^r \gg 0_M$ is a solution to the aggregate input minimization problem $\min_y\{g(y)\colon w^r \cdot y = w^r \cdot y^r, y \geqq 0_M\}$, for $r = 0, 1$, and thus (3.4) holds. Similarly, $x^r \gg 0_N$ is a solution to the aggregator maximization problem $\max_x\{f(x)\colon p^r \cdot x = p^r \cdot x^r, x \geqq 0_N\}$, for $r = 0, 1$, and thus (2.12) holds. Substitution of (2.12) and (3.4) into the identity $g(y^1)/g(y^0) = (1+\tau)f(x^1)/f(x^0)$ yields the following expression for $(1+\tau)$ in terms of observable prices and quantities:

$$(3.8) \qquad (1+\tau) = \prod_{m=1}^{M} [y_m^1/y_m^0]^{\frac{1}{2}[w_m{}^1 y_m{}^1/w^1 \cdot y^1 + w_m{}^0 y_m{}^0/w^0 \cdot y^0]} \Big/ \prod_{n=1}^{N} [x_n^1/x_n^0]^{\frac{1}{2}[p_n{}^1 x_n{}^1/p^1 \cdot x^1 + p_n{}^0 x_n{}^0/p^0 \cdot x^0]}$$

Thus the Jorgenson–Griliches method of measuring technical progress can be justified if: (i) the economy's production possibilities set can be represented by a *separable* transformation surface defined by $g(y) = f(x)$, where the input aggregator function f and the output aggregator function g are both homogeneous translog functions, (ii) producers are maximizing profits and (iii) technical progress takes place in the '*neutral*' manner postulated above.

Since the separability assumption $g(y) = f(x)$ is somewhat restrictive from an a priori theoretical point of view, it would be of some interest to devise a measure of technical progress which did not depend on this separability assumption. This can be done, but only at a cost as we shall see below.

Suppose that technology can be represented by a transformation function,[8] where $y_1 = t(y_2, y_3, \ldots, y_M; x_1, x_2, \ldots, x_N) \equiv t(\tilde{y}; x) \equiv t(z)$ is the maximum amount of output one that can be produced, given that the vector of other outputs $\tilde{y} = (y_2, y_3, \ldots, y_M)$ is to be produced by the vector of inputs $x = (x_1, x_2, \ldots, x_N)$. Assume that t is a positive, linear homogeneous, concave function over a convex set of the non-negative orthant S in $K \equiv M-1+N$ dimensional space. Assume also that $t(\tilde{y}; x)$ is non-increasing in the components of the other outputs vector $\tilde{y}$ and non-decreasing in the components of the input vector x. Suppose that the transformation function t is defined for z belonging to S by

$$(3.9) \qquad \ln t(z) \equiv \alpha_0 + \sum_{k=1}^{K} \ln z_k + \tfrac{1}{2} \sum_{j=1}^{K} \gamma_{jk} \ln z_j \ln z_k,$$

where $\sum_{k=1}^{K} \alpha_k = 1$, $\gamma_{jk} = \gamma_{kj}$ and $\sum_{k=1}^{K} \gamma_{jk} = 0$, for $j = 1, 2, \ldots, K$; i.e., t is a translog transformation function over the set S.

Suppose that $y^r \equiv (y_1^r, y_2^r, \ldots, y_M^r) \gg 0_M$ (output vectors), $x^r \equiv (x_1^r, x_2^r, \ldots, x_N^r) \gg 0_N$ (input vectors), $w^r \gg 0_M$ (output price vectors), $p^r \gg 0_N$ (input

[8]For a more detailed discussion of transformation functions and their properties, see Diewert (1973a).

price vectors), and $w^r \cdot y^r = p^r \cdot x^r$ (value of outputs equals value of inputs) for periods $r = 0, 1$. Assume that $z^0 \equiv (y_2^0, \ldots, y_M^0, x_1^0, \ldots, x_N^0) \equiv (\tilde{y}^0, x^0)$ is a solution to the following output maximization subject to an expenditure constraint problem in period 0:

$$\max_z\{t(z): q^0 \cdot z = q^0 \cdot z^0, \quad z \text{ belongs to } S\} \tag{3.10}$$

where t is the translog transformation function defined by (3.9), $q^0 \equiv (-w_2^0, -w_3^0, \ldots, -w_M^0; p_1^0, p_2^0, \ldots, p_N^0) \equiv (-\tilde{w}^0; p^0)$ and

$$y_1^0 = t(z^0).\text{[9]} \tag{3.11}$$

The first-order conditions for the maximization problem (3.10) imply that [Konyus–Byushgens (1926) lemma]

$$q^0/q^0 \cdot z^0 = \nabla t(z^0)/t(z^0). \tag{3.12}$$

As before, we assume that 'neutral' input augmenting technical progress takes place between periods 0 and 1; i.e., if $(y; x)$ was an efficient vector of outputs and inputs in period 0, then $(y; (1+\tau)^{-1}x)$ is on the efficiency surface in period 1. Thus the efficiency surface in period 1 can be defined as the set of $(y_1, y_2, \ldots, y_M; x_1, x_2, \ldots, x_N)$ which satisfy the following equation:

$$y_1 = t(y_2, y_3, \ldots, y_M; (1+\tau)x_1, (1+\tau)x_2, \ldots, (1+\tau)x_N). \tag{3.13}$$

Assume that $(y_2^1, y_3^1, \ldots, y_M^1; x_1^1, x_2^1, \ldots, x_N^1) \equiv (\tilde{y}^1; x^1)$ is a solution to the period 1 output maximization subject to an expenditure constraint problem $\max_{y,x}\{t(\tilde{y}; (1+\tau)x): -\tilde{w}^1 \cdot y + p^1 \cdot x = -\tilde{w}^1 \cdot y^1 + p^1 \cdot x^1; (\tilde{y}: (1+\tau)x) \text{ belongs to } S\}$. Then $z^1 \equiv (\tilde{y}^1; (1+\tau)x^1)$ is a solution to the following output maximization problem:

$$\max_z\{t(z): q^1 \cdot z = q^1 \cdot z^1, \quad z \text{ belongs to } S\}, \tag{3.14}$$

where t is the translog function defined by (3.9), $q^1 \equiv (-\tilde{w}^1; p^1)$, and

$$y_1^1 = t(z^1) = t(\tilde{y}^1; (1+\tau)x^1). \tag{3.15}$$

Again, the Konyus–Byushgens–Hotelling lemma applied to the maximization problem (3.14), using the linear homogeneity of t, implies that[10]

$$q^1/q^1 \cdot z^1 = \nabla t(z^1)/t(z^1). \tag{3.16}$$

[9]Note that $q^0 \cdot z^0 = w_1{}^0 \cdot y_1{}^0 > 0$, since $w^0 \cdot y^0 = p^0 \cdot x^0$.
[10]We assume that τ is small so that $q^1 \cdot z^1 \equiv -\tilde{w}^1 \cdot \tilde{y}^1 + p^1 \cdot (1+\tau)x^1 > 0$.

Now substitute (3.12) and (3.16) into the identity (2.11), except that t replaces f and z replaces x, and we obtain

$$
\begin{aligned}
(3.17)\qquad t(z^1)/t(z^0) &= \sum_{k=1}^{K} [z_k^1/z_k^0]^{\frac{1}{2}[q_k{}^1 z_k{}^1/q^1\cdot z^1 + q_k{}^0 z_k{}^0/q^0\cdot z^0]} \\
&= t[\tilde{y}^1; (1+\tau)x^1]/t(y^0; x^0).
\end{aligned}
$$

Combining (3.13), (3.15) and (3.17), we obtain the following equation in τ:

$$
(3.18)\qquad y_1^1/y_1^0 = \prod_{n=1}^{N} [(1+\tau)x_n^1/x_n^0]^{\frac{1}{2}[p_n{}^1(1+\tau)x_n{}^1/V^1(\tau) + p_n{}^0 x_n{}^0/V^0]} \Big/ \prod_{m=2}^{M} [y_m^1/y_m^0]^{\frac{1}{2}[w_m{}^1 y_m{}^1/V^1(\tau) + w_m{}^0 y_m{}^0/V^0]},
$$

where $V^0 \equiv -\sum_{m=2}^{M} w_m^0 y_m^0 + \sum_{n=1}^{N} p_n^0 x_n^0$ = net cost of producing output y_1 in period 0, and $V^1(\tau) \equiv -\sum_{m=2}^{M} w_m^1 y_m^1 + \sum_{n=1}^{N} p_n^1(1+\tau)x_n^1$.

Given data on outputs, inputs and prices, eq. (3.18) can be solved for the unknown rate of technical progress τ. Note that eq. (3.18) is quite different from the Jorgenson–Griliches equation for τ defined by (3.8) (except that the two equations are equivalent when $M = 1$, i.e., when there is only one output).

However, it should be pointed out that our more general measure of technical progress, which is obtained by solving (3.18) for τ, suffers from some disadvantages: (i) our procedure is computationally more difficult,[11] and (ii) our procedure is not symmetric in the outputs; that is, the first output y_1 is asymmetrically singled out in (3.18). Thus different orderings of the outputs could give rise to different measures of technical progress. This is because each ordering of the outputs corresponds to a *different* translog assumption about the underlying technology and thus different measures of τ can be obtained. However, all of these measures should be close in empirical applications since the different translog functions are all approximating the same technology to the second order.

4. Quadratic means of order r and exact index numbers

For $r \neq 0$, the (homogeneous) *quadratic mean of order r aggregator function* is defined by

$$
(4.1)\qquad f_r(x) \equiv \left[\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} x_i^{r/2} x_j^{r/2}\right]^{1/r},
$$

[11]Furthermore, we cannot a priori rule out the possibility that eq. (3.18) will have either multiple solutions for τ or no solutions at all. The Fisher measure of technical progress, to be introduced in sections 5, overcomes these difficulties.

where $a_{ij} = a_{ji}$, $1 \leqq i, j \leqq N$, are parameters, and the domain of definition of f_r is restricted to $x \equiv (x_1, x_2, \ldots, x_N) \gg 0_N$ such that $\sum\sum a_{ij} x_i^{r/2} x_j^{r/2} > 0$, and f_r is concave. The above functional form is due to McCarthy (1967), Kadiyala (1971–2), Denny (1972, 1974) and Hasenkamp (1973). Denny also defined *the quadratic mean of order r unit cost function*,

$$(4.2) \qquad c_r(p) \equiv \left[\sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} p_i^{r/2} p_j^{r/2} \right]^{1/r}, \qquad b_{ij} = b_{ji}, \quad r \neq 0.$$

Denny noted that if $r = 1$, then (4.1) reduces to the *generalized linear* functional form [Diewert (1969, 1971)], (4.2) reduces to the *generalized Leontief* functional form [Diewert (1969, 1971)], and if all $a_{ij} = 0$ for $i \neq j$, then (4.1) reduces to the C.E.S. functional form [Arrow, Chenery, Minhas and Solow (1961)], while if all $b_{ij} = 0$ for $i \neq j$, then (4.2) reduces to the C.E.S. unit cost function.

We may also note that when $r = 2$, (4.1) reduces to the Konyus–Byushgens (1926) homogeneous quadratic production or utility function, while (4.2) reduces to the Konyus–Byushgens unit cost function. This functional form has also been considered by Afriat (1972, p. 72) and Pollak (1971) in the context of utility functions and by Diewert (1969, 1974a) in the context of revenue and factor requirements functions.

Lau (1973) has shown that the limit as r tends to zero of the quadratic mean of order r aggregator function (4.1) is the homogeneous translog aggregator function and similarly that the limit as r tends to zero of (4.2) is the translog unit cost function.

This completes our discussion of special cases of the above family of functional forms. The following theorem shows that the functional form is 'flexible'.

(4.3) *Theorem. Let f be any linear homogeneous, twice-continuously-differentiable, positive function defined over an open subset of the positive orthant in N-dimensional space. Then for any $r \neq 0$, f_r defined by* (4.1) *can provide a second-order differential approximation to f.*

By a second-order differential approximation to f at a point $x^* \gg 0_N$,[12] we mean that there exists a set of a_{ij} parameters for f_r defined by (4.1), such that $f_r(x^*) = f(x^*)$, $\nabla f_r(x^*) = \nabla f(x^*)$, and $\nabla^2 f_r(x^*) = \nabla^2 f_r(x^*)$; i.e., the values of f_r and f and their first- and second-order partial derivatives at x^* all coincide.

Define the *quadratic mean of order r quantity index Q_r* for $x^0 \gg 0_N$, $x^1 \gg 0_N$, $p^0 > 0_N$, $p^1 > 0_N$, for $r \neq 0$, as

$$(4.4) \qquad Q_r(p^0, p^1; x^0, x^1) \equiv \left\{ \frac{\sum_{i=1}^{N} (x_i^1/x_i^0)^{r/2} (p_i^0 x_i^0 / p^0 \cdot x^0)}{\sum_{k=1}^{N} (x_k^0/x_k^1)^{r/2} (p_k^1 x_k^1 / p^1 \cdot x^1)} \right\}^{1/r}$$

[12] This terminology follows Lau (1974).

$$= \left[\sum_{i=1}^{N} (x_i^1/x_i^0)^{r/2} s_i^0 \right]^{1/r} \left[\sum_{k=1}^{N} (x_k^1/x_k^0)^{-r/2} s_k^1 \right]^{-1/r}.$$

Thus for any $r \neq 0$, Q_r may be calculated as a function of observable prices and quantities in two periods. Note that Q_r can be expressed as the product of a mean of order r [13] in the square roots of the quantity relatives $(x_i^1/x_i^0)^{\frac{1}{2}}$ (using base period cost share as weights) times a mean of order $-r$ in the square roots of the quantity relatives $(x_i^1/x_i^0)^{\frac{1}{2}}$ (using period one cost shares as weights).

It is perhaps of some interest to note which of Irving Fisher's (1911, 1922) *tests* are satisfied by the quantity index Q_r. It can be verified that Q_r satisfies: (i) *the commodity reversal test*, i.e., the value of the index number does not change if the ordering of the commodities is changed; (ii) *the identity test*, i.e., $Q_r(p^0, p^0; x^0, x^0) \equiv 1$ [in fact $Q_r(p^0, p^1; x^0, x^0) \equiv 1$, the quantity index equals one if all quantities remain unchanged]; (iii) *the commensurability test*, i.e., $Q_r(D^{-1}p^0, D^{-1}p^1; Dx^0, Dx^1) = Q_r(p^0, p^1; x^0, x^1)$ where D is a diagonal matrix with positive elements down the main diagonal; thus the quantity index remains invariant to changes in units of measurement; (iv) *the determinateness test*, i.e., $Q_r(p^0, p^1; x^0, x^1)$ does not become zero, infinite or indeterminate if an individual price becomes zero for any $r \neq 0$ and $Q_r(p^0, p^1; x^0, x^1)$ does not become zero, infinite or indeterminate if an individual *quantity* becomes zero if $0 < r \leqq 2$;[14] (v) *the proportionality test*, i.e., $Q_r(p^0, p^1; x^0, \lambda x^0) = \lambda$ for every $\lambda > 0$; and (vi) *the time or point reversal test*, i.e., $Q_r(p^0, p^1; x^0, x^1) Q_r(p^1, p^0; x^1, x^0) \equiv 1$.

Define the *quadratic mean of order r price index* P_r for $p^0 \gg 0_N, p^1 \gg 0_N, x^0 > 0_N, x^1 > 0_N$, for $r \neq 0$, as

$$(4.5) \qquad P_r(p^0, p^1; x^0, x^1) \equiv \left\{ \frac{\sum_{i=1}^{N} (p_i^1/p_i^0)^{r/2} (p_i^0 x_i^0 / p^0 \cdot x^0)}{\sum_{k=1}^{N} (p_k^0/p_k^1)^{r/2} (p_k^1 x_k^1 / p^1 \cdot x^1)} \right\}^{1/r}$$

$$= Q_r(x^0, x^1; p^0, p^1).$$

It is easy to see that P_r will also satisfy Fisher's tests (i) to (vi). The only Fisher tests *not* satisfied by the indexes P_r and Q_r are: (vii) *the circularity test*, i.e., $P_r(p^0, p^1; x^0, x^1) P_r(p^1, p^2; x^1, x^2) \not\equiv P(p^0, p^2; x^0, x^2)$; and (viii) *the factor reversal test*, i.e., $P_r(p^0, p^1; x^0, x^1) Q_r(p^0, p^1; x^0, x^1) \not\equiv p^1 \cdot x^1 / p^0 \cdot x^0$, except that P_2 and Q_2 (the 'ideal' price and quantity index) satisfy the factor reversal test.

[13] Ordinary, as opposed to quadratic means of order r, were defined by Hardy, Littlewood and Polya (1934).

[14] Thus the quantity indices Q_r, for $0 < r \leqq 2$, are somewhat more satisfactory than the Törnqvist–Theil index Q_0 defined by (2.12).

For $r \neq 0$ define the *implicit quadratic mean of order r price index* $\tilde{P}_r$ as

(4.6) $$\tilde{P}_r(p^0, p^1; x^0, x^1) \equiv p^1 \cdot x^1/[p^0 \cdot x^0 Q_r(p^0, p^1; x^0, x^1)],$$

and define the implicit quadratic mean of order r quantity index $\tilde{Q}_r$ as

(4.7) $$\tilde{Q}_r(p^0, p^1; x^0, x^1) \equiv p^1 \cdot x^1/[p^0 \cdot x^0 P_r(p^0, p^1; x^0, x^1)].$$

Thus the two pairs of indexes $(Q_r, \tilde{P}_r)$ and $(\tilde{Q}_r, P_r)$ will satisfy the weak factor reversal test (1.1).

The following theorem relates the aggregator function f_r to the quantity index Q_r:

(4.8) *Theorem. Suppose that* (i) $f_r(x)$ *is defined by* (4.1), *where* $r \neq 0$; (ii) $x^0 \gg 0_N$ *is a solution to the maximization problem* $\max_x\{f_r(x)$: $p^0 \cdot x \leqq p^0 \cdot x^0$, *x belongs to* $S\}$, *where S is a convex subset of the non-negative orthant in* $\mathbf{R}^N$, $f_r(x^0) > 0$ *and the price vector* p^0 *is such that* $p^0 \cdot x^0 > 0$; *and* (iii) $x^1 \gg 0_N$ *is a solution to the maximization problem* $\max_x\{f_r(x)$: $p^1 \cdot x \leqq p^1 \cdot x^1$, *x belongs to* $S\}$, $f_r(x^1) > 0$ *and the price vector* p^1 *is such that* $p^1 \cdot x^1 > 0$; *then*

(4.9) $$f_r(x^1)/f_r(x^0) = Q_r(p^0, p^1; x^0, x^1).$$

Thus the quadratic mean of order r quantity index Q_r is exact for the quadratic mean of order r aggregator function, which in view of theorem (4.3) implies that Q_r is a *superlative* index number.

Suppose that $x^s \gg 0_N$ is a solution to $\max_x\{f_r(x): p^s \cdot x \leqq p^s \cdot x^s$, x belongs to $S\}$, where $f_r(x) > 0$, $p^s \cdot x^s > 0$ for $s = 0, 1, 2$. Then using (4.9) three times, we find that

$$\begin{aligned} Q_r(p^0, p^1; x^0, x^1)\, Q_r(p^1, p^2; x^1, x^2) &= f_r(x^1)[f_r(x^0)]^{-1} f_r(x^2)[f_r(x^1)]^{-1} \\ &= f_r(x^2)/f_r(x^0) \\ &= Q_r(p^0, p^2; x^0, x^2). \end{aligned}$$

Thus under the assumption that the producer or consumer is maximizing $f_r(x)$ subject to an expenditure constraint each period, we find that Q_r will satisfy the *circularity test* in addition to the other Fisher tests which it satisfies. A similar proposition is true for any exact index number, a fact which was first noted by Samuelson and Swamy (1974). Since the circularity test is capable of empirical refutation, we see that we can empirically refute the hypothesis that an economic agent is maximizing $f_r(x)$ subject to an expenditure constraint. Thus violations of the circularity test could mean either that the economic agent was not engaging in maximizing behavior or that his aggregator function was not $f_r(x)$.

We note that theorem (4.8) did not require that all prices be non-negative, only that quantities be positive. f_r can also be a *transformation* function (recall section 2) which is non-decreasing in inputs and non-increasing in outputs. Theorem (4.8) will still hold except that prices of other outputs must be indexed negatively while prices of inputs are taken to be positive. The quantity index Q_r may be used in the context of productivity measurement just as we used the index Q_0 in section 3. We will return to this topic in section 5.

Theorem (4.8) tells us that f_r defined by (4.1) is exact for Q_r defined by (4.4). However, could there exist a linear homogeneous functional form f different from f_r which is also exact for Q_r? The answer is no, as the following theorem shows:

(4.10) *Theorem* [Generalization of Byushgens (1925), Konyus and Byushgens (1926)]. *Let S be an open subset of the positive orthant in* $\mathbf{R}^N$ *which is also a convex cone. Suppose f is defined over S and is* (i) *positive,* (ii) *once-differentiable,* (iii) *linear homogeneous, and* (iv) *concave. Suppose that f is exact for the quantity index Q_r defined by* (4.4) *for* $r \neq 0$ [*i.e., if x^s is a solution to* $\max_x\{f(x): p^s \cdot x \leqq p^s \cdot x^s$, x *belongs to* $S\}$ *for* $s = 0, 1$, *then* $Q_r(p^0, p^1; x^0, x^1) = f(x^1)/f(x^0)$]. *Then f is a quadratic mean of order r defined by* (4.1) *for some* a_{ij}, $1 \leqq i \leqq j \leqq N$.

We note that the functional form f_r defined by (4.1) may also be used as a factor requirements function, and that the quantity index Q_r defined by (4.4) will still be exact for f_r; i.e., theorems (4.8) and (4.10) will still hold except that the maximization problems $\max_x\{f_r(x): p^s \cdot x \leqq p^s \cdot x^s$, x belongs to $S\}$ are replaced by the minimization problems $\min_x\{f_r(x): p^s \cdot x \geqq p^s \cdot x^s$, x belongs to $S\}$ for $s = 0, 1$, and condition (iv) is changed from concavity to convexity. Thus the quadratic mean of order r quantity indexes Q_r can be used to aggregate either inputs or outputs provided that the functional form for the aggregator function is a quadratic mean of order r.

The above theorems have their counterparts in the dual space.

(4.11) *Theorem. Suppose that* (i) $c_r(p) \equiv [\Sigma_i \Sigma_j b_{ij} p_i^{r/2} p_j^{r/2}]^{1/r}$, *where* $b_{ij} = b_{ij}$ *for all* i, j, $r \neq 0$ *and* $(p_1, p_2, \ldots, p_N) \equiv p$ *belongs to S where S is an open, convex cone which is a subset of the positive orthant in* $\mathbf{R}^N$; (ii) $c_r(p)$ *is positive, linear homogeneous and concave over* S; (iii) $x^0/p^0 \cdot x^0 = \nabla c_r(p^0)/c_r(p)$, *where* $p^0 \gg 0_N$ *so that* [*using the corollary* (2.14) *to Shephard's lemma*] x^0 *is a solution to the aggregator maxi-*

mization problem $\max_x\{\tilde{f}_r(x): p^0\cdot x \leqq p^0\cdot x^0, x \leqq 0_N\}$, *where* $\tilde{f}_r$ [15] *is the direct aggregator function which is dual to* $c_r(p)$; *and* (iv) $x^1/p^1\cdot x^1 = \nabla c_r(p^1)/c_r(p^1)$, $p^1 \gg 0_N$ *so that* x^1 *is a solution to the aggregator maximization problem* $\max_x\{\tilde{f}_r(x): p^1\cdot x \leqq p^1\cdot x^1, x \geqq 0_N\}$. *Then*

$$c_r(p^1)/c_r(p^0) = P_r(p^0, p^1; x^0, x^1), \tag{4.12}$$

where P_r *is the quadratic mean of order r price index defined by* (4.5).

The proof of theorem (4.11) is analogous to the proof of theorem (4.8), except that p replaces x, c_r replaces f_r, and corollary (2.14) is used instead of the Konyus–Byushgens–Hotelling lemma.

Thus the quadratic mean of order r unit cost function c_r is *exact* for the price index P_r. Since by theorem (4.3), c_r can provide a second-order approximation to an arbitrary twice-differentiable unit cost function, we see that P_r is a *superlative* price index for each $r \neq 0$. We note also that there is an analogue to theorem (4.10) for P_r; i.e., c_r is essentially the only functional form which is exact for the price index function P_r.

However, if we relax the assumption that the underlying aggregator function be linear homogeneous, then the index numbers P_r and Q_r can be exact for a number of true cost of living price indexes and Malmquist quantity indexes, respectively; i.e., analogues to theorems (2.16) and (2.17) hold.

We have obtained two families of price and quantity indexes: P_r, $\tilde{Q}_r$ and $\tilde{P}_r$, Q_r defined by (4.6) and (4.4) for any $r \neq 0$. The first price–quantity family corresponds to an aggregator function $\tilde{f}_r$ which has the unit cost function c_r defined (4.2) as its dual, and the second price–quantity family corresponds to an aggregator function f_r defined by (4.1). Recall also that the price–quantity indexes P_0, $\tilde{Q}_0$ correspond to a translog unit cost function, while $\tilde{P}_0$, Q_0 correspond to a homogeneous translog aggregator function.

For various values of r, some of the indexes P_r or $\tilde{P}_r$ have been considered in the literature. For $r = 2$, $P_2 \equiv \tilde{P}_2$ becomes the Pigou (1920) and Fisher (1922) *ideal* price index which corresponds to the Konyus–Byushgens (1926) homogeneous quadratic aggregator function $f_2(x) \equiv [x^TAx]^{\frac{1}{2}}$, where $A = A^T$ is a symmetric matrix of coefficients and it also corresponds to the unit cost function $c_2(p) \equiv [p^TBp]^{\frac{1}{2}}$, where $B = B^T$ is a symmetric matrix of coefficients. If A^{-1} exists, then it is easy to show that the unit cost function which is dual to f_2 is $\tilde{c}_2(p) = (p^TA^{-1}p)^{\frac{1}{2}}$ (at least for a range of prices). However, if $f_2(x) \equiv [x^Taa^Tx]^{\frac{1}{2}} = a^Tx$, where $a \gg 0_N$ is a vector of coefficients (*linear aggregator function*), then $\tilde{c}_2(p) = \tilde{c}_2(p_1, p_2, \ldots, p_N) \equiv \min_i\{p_i/a_i: i = 1, 2, \ldots, N\}$ which is *not* a member of the family of unit cost functions defined by $c_2(p) \equiv [p^TBp]^{\frac{1}{2}}$. On the other hand, if $c_2(p) = (p^Tbb^Tp)^{\frac{1}{2}} = b^Tp$, where $b \gg 0_N$ is a vector of coefficients (*Leontief unit cost function*), then the dual aggregator function is $\tilde{f}_2(x_1, x_2, \ldots, x_N) = \min_i\{x_i/b_i: i = 1, 2, \ldots, N\}$, which is a

[15] Define $\tilde{f}_r(x) \equiv 1/\max_p\{c_r(p): p\cdot x = 1, p \text{ belongs to } \bar{S}\}$, where $\bar{S}$ is the closure of S.

Leontief aggregator function. Thus P_2 is exact for a Leontief aggregator function [a fact which was noted by Pollak (1971)] and since $P_2 \equiv \tilde{P}_2$, it is also exact for a linear aggregator function. This is an extremely useful property for an index number formula, since the two types of aggregator function correspond to *zero* substitutability between the commodities to be aggregated and *infinite* substitutability respectively.

For $r = 1$, the price index P_1 has been recommended by Walsh (1901, p. 105). P_1 is exact for the unit cost function c_1, whose dual $\tilde{f}_1$ is the *generalized Leontief aggregator function,* which has the Leontief aggregator function as a special case. Walsh also recommended the price index $\tilde{P}_1$, which is exact for the *generalized linear* aggregator function f_1, which of course has the linear aggregator function as a special case.

If, in fact, a producer or consumer was maximizing a linear homogeneous function subject to an expenditure constraint for a number of time periods, we would expect [in view of the approximation theorem (4.3)] that the price

Table 1
Comparison of some index numbers tabulated by Fisher.

Price index	Fisher number	1913	1914	1915	1916	1917	1918
P_{Pa}	54	100	100.3	100.1	114.4	161.1	177.4
P_{La}	53	100	99.9	99.7	114.1	162.1	177.9
P_0	123	100	100.1	99.9	113.8	162.1	177.8
$\tilde{P}_0$	124	100	100.16	99.85	114.25	161.74	178.16
$\tilde{P}_1$	1153	100	100.13	99.89	114.20	161.70	177.83
P_1	1154	100	100.12	99.90	114.24	161.73	177.76
P_2	353	100	100.12	99.89	114.21	161.56	177.65

indexes P_r and $\tilde{P}_r$ should more or less coincide, particularly if the variation in relative prices were small. However, since real world data is not necessarily consistent with this maximization hypothesis, let us consider some empirical evidence on this point.

Irving Fisher (1922, p. 489) tabled the wholesale prices and the quantities marketed for 36 primary commodities in the U.S. during the war years 1913–1918, a time of very rapid price and quantity changes. Fisher calculated and compared 134 different price indexes using this data. Table 1 reproduces Fisher's (1922, pp. 244–247) computations for the Paasche and Laspeyres price indexes, P_{Pa} and P_{La}, as well as for $P_0, \tilde{P}_0, P_1, \tilde{P}_1$ and $P_2 \equiv \tilde{P}_2 \equiv P_{Id}$. Fisher's identification number is given in column 2 of the table; e.g., P_2 or the 'ideal' price index was identified as number 353 by Fisher. All index numbers were calculated using 1913 as a base.

Note that the Paasch and Laspeyres indexes coincide to about two significant figures, while the last 4 indexes mostly lie between the Paasche and Laspeyres indexes and coincide to 3 significant figures. Fisher (1922, p. 278) also calculated P_2 (and some of the other 'very good' index numbers) using different years as the base year and then he compared how the various series differed; that is, he tested for 'circularity'. Fisher found that average discrepancy was only about $\frac{1}{3}$ percent between any two bases. Thus as far as Fisher's *time series* data is concerned, it appears that any one of the price indexes P_r or $\tilde{P}_r$ gives the same answer to 3 significant figures, and that violations of circularity are only about $\frac{1}{3}$ percent so that the choice of base year is not too important.[16]

To determine how the price indexes P_r compare for different r's in the context of *cross section* data, one may look at Ruggle's (1967, pp. 189–190) paper which compares the consumer price indexes P_{Pa}, P_{La}, P_0 and P_2 for 19 Latin American countries for the year 1961. The indexes P_0 and P_2, using Argentina as a base, differed by about 1 percent per observation, while P_0 and P_2, using Venezuela as a base (the relative prices in the two countries differed markedly), differed by about 1.5 percent per observation. P_2 failed the circularity test (comparing values with Venezuela and Argentina as the base country) by an average of about 2 percent per observation, while P_0 failed the circularity test by about 3 percent per observation.[17] Thus it appears that the indexes P_r differ more and violate circularity more in the context of cross section analysis than in time series analysis. However, the agreement between P_0 and P_2 in the cross section context is still remarkable since the Paasch and Laspeyres indexes differed by about 50 percent per observation.

5. Concluding remarks

We have obtained two families of *superlative* price and quantity indexes, $(P_r, \tilde{Q}_r)$ and $(\tilde{P}_r, Q_r)$; that is, each of these index numbers is exact for a homogeneous aggregator function which is capable of providing a second-order approximation to an arbitrary twice-continuously-differentiable aggregator function (or its dual unit cost function). Moreover, $(P_r, \tilde{Q}_r)$ and $(\tilde{P}_r, Q_r)$ satisfy many of the Irving Fisher tests for index numbers in addition to their being consistent with a homogeneous aggregator function. Note also that if prices are varying proportionately, then the aggregates $\tilde{Q}_r$ and Q_r are consistent with Hicks' (1946) aggregation theorem.

Although any one of the index number pairs, $(P_r, \tilde{Q}_r)$ or $(\tilde{P}_r, Q_r)$, could be

[16]However, as a matter of general principle, it would seem that the chain method of calculating index numbers would be preferable, since over longer periods of time, the underlying functional form for the aggregator function may gradually change, so that for example (1.5) will only be approximately satisfied, the degree of approximation becoming better as r approaches 0.

[17]This failure of the circularity test should not be too surprising from the viewpoint of economic theory since we do not expect the aggregator function for the 270 consumer goods and services to be representable as a linear homogeneous function; that is, we do not expect all 'income' or expenditure elasticities to be unitary.

used in empirical applications, we would recommend the use of

$$(P_2, \tilde{Q}_2) \equiv (\tilde{P}_2, Q_2) \equiv ([p^1 \cdot x^1 p^1 \cdot x^0 / p^0 \cdot x^1 p^0 \cdot x^0]^{\frac{1}{2}}, \\ [x^1 \cdot p^1 x^1 \cdot p^0 / x^0 \cdot p^1 x^0 \cdot p^0]^{\frac{1}{2}}),$$

Irving Fisher's (1922) ideal index numbers, as the preferred pair of index numbers. There are at least three reasons for this selection.

(i) The functional form for the Fisher–Konyus–Byushgens ideal index number is *particularly simple* and this leads to certain simplifications in applications. For example, recall eq. (3.18) which we used in order to measure technical progress, $(1+\tau)$, in an economy whose transformation function could be represented by a (non-separable) homogeneous translog transformation function t. If we assume $t(z) = f_2(z)$, where f_2 is defined by (4.1) for $r = 2$, then the analogue to (3.18) is

$$\text{(5.1)} \qquad \frac{y_1^1}{y_1^0} = \frac{(-\tilde{w}^1 \cdot \tilde{y}^1 + p^1 \cdot (1+\tau)x^1)(-\tilde{w}^0 \cdot \tilde{y}^1 + p^0 \cdot (1+\tau)x^1)^{\frac{1}{2}}}{(-\tilde{w}^1 \cdot \tilde{y}^0 + p^1 \cdot x^0)(-\tilde{w}^0 \cdot y^0 + p^0 \cdot x^0)}.$$

If we square both sides of (5.1), the resulting quadratic equation in $(1+\tau)$ can easily be solved, given market data.

(ii) The indexes $P_2(p^0, p^1; x^0, x^1)$ and $\tilde{Q}_2(p^0, p^1; x^0, x^1)$ are functions of $p^0 \cdot x^1 / p^0 \cdot x^0$ and $p^1 \cdot x^0 / p^1 \cdot x^1$, which are 'sufficient statistics' for revealed preference theory,[18] and moreover $\tilde{Q}_2$ is *consistent with revealed preference theory* in the following sense: (a) if $p^0 \cdot x^1 < p^0 \cdot x^0$ and $p^1 \cdot x^0 \geqq p^1 \cdot x^1$ (i.e., x^0 revealed preferred to x^1), then $\tilde{Q}_2(p^0, p^1; x^0, x^1) < 1$; (b) if $p^0 \cdot x^1 \geqq p^0 \cdot x^0$ and $p^1 \cdot x^0 < p^1 \cdot x^1$ (i.e., x^1 revealed preferred to x^0), then $\tilde{Q}_2(p^0, p^1; x^0, x^1) > 1$ (i.e., the quantity index indicates an increase in the aggregate); and (c) if $p^0 \cdot x^1 = p^0 \cdot x^0$ and $p^1 \cdot x^0 = p^1 \cdot x^1$ (i.e., x^0 and x^1 revealed to be equivalent or indifferent), then $\tilde{Q}_2(p^0, p^1; x^0, x^1) = 1$ (i.e., the quantity index remains unchanged). Thus even if the true aggregator function f is non-homothetic, the quantity index $\tilde{Q}_2$ will correctly indicate the direction of change in the aggregate when revealed preference theory tells us that the aggregate is decreasing, increasing or remaining constant.

(iii) The index number pair (P_2, Q_2) is consistent with both a *linear aggregator*

[18] See Samuelson (1947), Houthakker (1950) and Afriat (1967). We should also mention the *non-parametric method* of price and quantity index number determination pioneered by Afriat (1967) which depends only on the R^2 inner products of the rth price vector p^r and the sth quantity vector x^s, $p^r \cdot x^s$, if there are R observations. See Diewert (1973b, p. 424, footnote 2) for an algorithm which would enable one to calculate a polyhedral approximation $\phi(x)$ to the 'true' linear homogeneous aggregator function $f(x)$.

function (infinite substitutability between the goods to be aggregated) and a *Leontief aggregator function* (zero substitutability between the commodities to be aggregated). No other $(P_r, \tilde{Q}_r)$ or $(\tilde{P}_r, Q_r)$ has this very useful property.

6. Proofs of theorems

Proof of (2.2)

$$\begin{aligned} f(z^1)-f(z^0) &= a^T z^1 + \tfrac{1}{2} z^{1T} A z^1 - a^T z^0 - \tfrac{1}{2} z^{0T} A z^0 \\ &= a^T(z^1 - z^0) + \tfrac{1}{2} z^{1T} A (z^1 - z^0) + \tfrac{1}{2} z^{0T} A (z^1 - z^0) \\ &= \tfrac{1}{2}[a + A z^1 + a + A z^0]^T (z^1 - z^0), \qquad \text{since} \quad A = A^T \\ &= \tfrac{1}{2}[\nabla f(z^1) + \nabla f(z^0)]^T (z^1 - z^0). \end{aligned}$$

Assume f is thrice-differentiable and satisfies the functional equation $f(x) - f(y) = \frac{1}{2}[\nabla f(x) + \nabla f(y)]^T (x-y)$, for all x and y, in an open neighbourhood. We wish to find the function that is characterized by the fact that its average slope between any two points equals the average of the endpoint slopes in the direction defined by the difference between the two points. If f is a function of one variable, the functional equation becomes $f(x) - f(y) = \frac{1}{2}[f'(x) + f'(y)]$ $(x-y)$. If we differentiate this last equation twice with respect to x, we obtain the differential equation $\frac{1}{2} f'''(x)(x-y) = 0$, which implies that $f(x)$ is a polynomial of degree two. The general case follows in an analogous manner using the directional derivative concept.

Proof of (2.4). λ_1^* and x^1 will satisfy the first-order necessary conditions for an interior maximum for the maximization problem (2.5),

$$(6.1) \qquad \nabla f(x^1) = \lambda_1^* p^1; \qquad p^1 \cdot x^1 = Y^1.$$

Similarly, λ_0^* and x^0 will satisfy the first-order conditions for the constrained maximization problem (2.6),

$$\nabla f(x^0) = \lambda_0^* p^0; \qquad p_0 \cdot x_0 = Y^0.$$

Now substitute the first parts of (6.1) into the right-hand side of the identity (2.3), and obtain (2.7).

Proof of (2.16). For a fixed u^*, $\ln C(u^*; p)$ is quadratic in the vector of variables $\ln p$ and we may apply the quadratic approximation lemma (2.2) to obtain

$$\ln C(u^*;p^1)-\ln C(u^*;p^0) = \tfrac{1}{2}\left[\hat{p}^1\frac{\nabla_p C(u^*;p^1)}{C(u^*;p^1)}+\hat{p}^0\frac{\nabla_p C(u^*;p^0)}{C(u^*;p^0)}\right]\cdot[\ln p^1-\ln p^0]$$

$$= \tfrac{1}{2}\left[\hat{p}^1\frac{\nabla_p C(u^1;p^1)}{C(u^1;p^1)}+\hat{p}^0\frac{\nabla_p C(u^0;p^0)}{C(u^0;p^0)}\right]\cdot[\ln p^1-\ln p^1]$$

(where the equality follows upon evaluating the derivatives of C and noting that $2\ln u^* = \ln u^1+\ln u^0$)

$$= \ln P_0(p^0,p^1;x^0,x^1)$$

(using the definitions of x^0, x^1 and P_0).

Proof of (2.17). It is first necessary to express the partial derivatives of D with respect to the components of x, $\nabla_x D[u^r;x^r]$, $r=0,1$, in terms of the partial derivatives of f. We have $D[u^r;x^r] \equiv \max_k\{k:f(x^r/k) \geqq u^r\} = 1$, for $r=0,1$, since each x^r is on the u^r 'utility' surface. To find out how the distance $D[u^0;x^0]$ changes as the components of x^0 change, apply the implicit function theorem to the equation $f(x^0/k) = u^0$ (where $k=1$ initially). We find that

$$\partial k/\partial x_j \equiv \partial D[u^0;x^0]/\partial x_j = f_j(x^0)\Big/\sum_{k=1}^{N} x_k^0 f_k(x^0), \qquad j=1,2,\ldots,N.$$

Similarly

$$\partial D[u^1;x^1]/\partial x_j = f_j(x^1)\Big/\sum_{k=1}^{N} x_k^1 f_k(x^1), \qquad j=1,2,\ldots,N.$$

Furthermore, the first-order conditions for the two aggregator maximization problems after elimination of the Lagrange multipliers yield the relations

$$p_j^0/p^0\cdot x^0 = f_j(x^0)\Big/\sum_{k=1}^{N} x_k^0 f_k(x^0), \qquad j=1,2,\ldots,N,$$

and

$$p_j^1/p^1\cdot x^1 = f_j(x^1)\Big/\sum_{k=1}^{N} x_k^1 f_k(x^1), \qquad j=1,2,\ldots,N.$$

Upon noting that the right-hand sides of the last set of relations are identical to the right-hand sides of the earlier relations, we obtain

$$(6.2) \qquad \nabla_x D[u^0;x^0] = p^0/p^0\cdot x^0 \quad \text{and} \quad \nabla_x D[u^1;x^1] = p^1/p^1\cdot x^1.$$

Now for a fixed u^*, $\ln D[u^*; x]$ is quadratic in the vector of variables $\ln x$ and we may again apply the quadratic approximation lemma (2.2) to obtain the following equality.

$$\ln D[u^*; x^1] - \ln D[u^*; x^0] = \tfrac{1}{2}\left[\hat{x}^1 \frac{\nabla_x D[u^*; x^1]}{D[u^*; x^1]} + \hat{x}^0 \frac{\nabla_x D[u^*; x^0]}{D[u^*; x^0]}\right] \cdot [\ln x^1 - \ln x^0]$$

$$= \ln Q_0(p^0, p^1; x^0, x^1),$$

where the equality follows upon evaluating the derivatives of D, noting that $2 \ln u^* = \ln u^1 + \ln u^0$, using (6.2), the equalities $D[u^1; x^1] = 1$, $D[u^0; x^0] = 1$ and the definition of Q_0.

Proof of (4.3). Since both f and f_r are twice continuously differentiable, their Hessian matrices evaluated at x^*, $\nabla^2 f(x^*)$ and $\nabla^2 f_r(x^*)$, are both symmetric. Thus we need only show that $\partial^2 f(x^*)/\partial x_i \partial x_j = \partial^2 f_r(x^*)/\partial x_i \partial x_j$, for $1 \leqq i \leqq j \leqq N$. Furthermore, by Euler's theorem on linear homogeneous functions, $f(x^*) = x^{*T}\nabla f(x^*)$ and $f_r(x^*) = x^{*T}\nabla f_r(x^*)$. Since the partial derivative functions $\partial f(x)/\partial x_i$ are homogeneous of degree zero, application of Euler's theorem on homogeneous functions yields, for $i = 1, 2, \ldots, N$,

$$(6.3) \qquad \sum_{j=1}^{N} x_j^* \partial^2 f(x^*)/\partial x_i \partial x_j = 0 = \sum_{j=1}^{N} x_j^* \partial^2 f_r(x^*)/\partial x_i \partial x_j .$$

Thus the above material implies that $f_r(x^*) = f(x^*)$, $\nabla f_r(x^*) = \nabla f(x^*)$ and $\nabla^2 f_r(x^*) = \nabla^2 f(x^*)$ will be satisfied under our present hypothesis if and only if

$$(6.4) \qquad \partial f_r(x^*)/\partial x_i = f_i^* \equiv \partial f(x^*)/\partial x_i, \qquad \text{for} \quad i = 1, 2, \ldots, N,$$

$$(6.5) \qquad \partial^2 f_r(x^*)/\partial x_i \partial x_j = f_{ij}^* \equiv \partial^2 f(x^*)/\partial x_i \partial x_j, \qquad \text{for} \quad 1 \leqq i < j \leqq N.$$

Thus we need to choose the $N(N+1)/2$ independent parameters $a_{ij}(1 \leqq i \leqq j \leqq N)$, so that the $N + N(N-1)/2 = N(N+1)/2$ eqs. (6.4) and (6.5) are satisfied. Recall that $x^* \equiv (x_1^*, x_2^*, \ldots, x_N^*) \gg 0_N$ and that $y^* \equiv x^{*T}\nabla f(x^*) = f(x^*) > 0$, since f is assumed to be positive over its domain of definition. Thus since $y^* > 0$, $x_i^* > 0$ and $r \neq 0$, the numbers a_{ij}^*, for $1 \leqq i < j \leqq N$, can be defined by solving the following equations for a_{ij}^*:

$$(6.6)\qquad f_{ij}^* = \frac{1-r}{y^*} f_i^* f_j^* + \frac{r}{2} y^{*(1-r)} a_{ij}^* x_i^{*r/2-1} x_j^{*r/2-1}, \qquad 1 \leqq i < j \leqq N.$$

The system of eq. (6.6) is equivalent to (6.5) if we also make use of (6.4). Now define $a_{ji}^* = a_{ij}^*$, for $i \neq j$, and then a_{ii}^* is defined as the solution to the following equation:

$$(6.7)\qquad \sum_{j=1}^{N} a_{ij}^* x_i^{*r/2-1} x_j^{*r/2} y^{*(1-r)} = f_1^*, \qquad i = 1, 2, \ldots, N.$$

Now define $f_r(x) \equiv [\sum_{i=1}^N \sum_{j=1}^N a_{ij}^* x_i^{r/2} x_j^{r/2}]^{1/r}$, and it can be verified readily that eqs. (6.4) and (6.5) are satisfied by f_r as defined.

Proof of (4.8). Using assumptions (ii) and (iii) of (4.8) yields

$$(6.8)\qquad v^0 \equiv p^0/p^0 \cdot x^0 = \nabla f_r(x^0)/f_r(x^0),$$

$$(6.9)\qquad v^1 \equiv p^1/p^1 \cdot x^1 = \nabla f_r(x^1)/f_r(x^1).$$

Upon differentiating $f_r(x^0)$, the ith equation in (6.8) becomes

$$v_i^0 \equiv p_i^0/p^0 \cdot x^0 = (x_i^0)^{(r/2)-1} \sum_{j=1}^{N} a_{ij} x_j^{0r/2} \Big/ \sum_{k=1}^{N} \sum_{m=1}^{N} a_{km} x_k^{0r/2} x_m^{0r/2},$$

$$(6.10)\qquad \therefore \sum_{i=1}^{N} x_i^{1r/2} v_i^0 x_i^{01-r/2} = \sum_i \sum_j x_i^{1r/2} a_{ij} x_j^{0r/2} \Big/ \sum_k \sum_m a_{km} x_k^{0r/2} x_m^{0r/2}.$$

Similarly, using eq. (6.9), we obtain

$$(6.11)\qquad \sum_{i=1}^{N} x_i^{0r/2} v_i^{11-r/2} = \sum_i \sum_j x_i^{0r/2} a_{ij} x_j^{1r/2} \Big/ \sum_k \sum_m a_{km} x_k^{1r/2} x_m^{1r/2}.$$

Upon noting that $a_{ij} = a_{ji}$, take the ratio of (6.10) to (6.11),

$$(6.12)\qquad \frac{\sum_i (x_i^1/x_i^0)^{r/2} v_i^0 x_i^0}{\sum_j (x_j^0/x_j^1)^{r/2} v_j^1 x_j^1} = \frac{\sum_k \sum_m a_{km} x_k^{1r/2} x_m^{1r/2}}{\sum_k \sum_m a_{km} x_k^{0r/2} x_m^{0r/2}} = \left[\frac{f_r(x^1)}{f_r(x^0)}\right]^r.$$

Take the rth root of both sides of (6.12) and obtain (4.9).

Proof of (4.10). Let x, y be any two points belonging to S such that

$$(6.13) \qquad 1 = f(x) = f(y) = x \cdot \nabla f(x) = y \cdot \nabla f(y),$$

where the last two equalities follow from the linear homogeneity of f. Since f is a concave function over S, for every z belonging to S, $f(z) \leqq f(x) + \nabla f(x) \cdot (z-x) = f(x) + \nabla f(x) \cdot z - f(x) = \nabla f(x) \cdot z$, and similarly $f(z) \leqq \nabla f(y) \cdot z$. Thus x is a solution to $\max_z\{f(z): \nabla f(x) \cdot z \leqq \nabla f(x) \cdot x, z$ belongs to $S\}$, and y is a solution to $\max_z\{f(z): \nabla f(y) \cdot z \leqq \nabla f(y) \cdot y, z$ belongs to $S\}$. Since f is exact for Q_r for some $r \neq 0$ by assumption, we must have, using (6.13),

$$Q_r(\nabla f(x), \nabla f(y); x, y) = f(y)/f(x) = 1,$$

or

$$\sum_{i=1}^{N} (x_i/y_i)^{r/2} f_i(y) y_i / \nabla f(y) \cdot y = \sum_{n=1}^{N} (y_n/x_n)^{r/2} f_n(x) x_n / \nabla f(x) \cdot x,$$

or

$$(6.14) \qquad \sum_{n=1}^{N} y_n^{r/2}(f_n(x) x_n^{1-r/2}) = \sum_{n=1}^{N} y_n^{1-r/2} f_n(y) x_n^{r/2},$$

where $f_n(y) \equiv \partial f(y)/\partial y_n$, $f_n(x) \equiv \partial f(x)/\partial x_n$, and $x \cdot \nabla f(x) = 1 = y \cdot \nabla f(y)$. Replace the vector $y \equiv (y_1, y_2, \ldots, y_N)$, which occurs in (6.14) with the vector y^n belonging to S, where $f(y^n) = 1$, for $n = 1, 2, \ldots, N$. Regard the resulting system of N equations as N linear equations in the N unknowns,

$$f_1(x) x_1^{1-r/2}, f_2(x) x_2^{1-r/2}, \ldots, f_N(x) x_N^{1-r/2},$$

and since we can choose the vectors $y^1, y^2, \ldots, y^N$ to be such that the coefficient matrix on the left-hand side of the system of N equations is non-singular, we may invert the coefficient matrix and obtain the solution

$$(6.15) \qquad f_n(x) x_n^{1-r/2} = \sum_{j=1}^{N} A_{nj} x_j^{r/2}, \qquad n = 1, 2, \ldots, N,$$

for some constants, A_{ij}, $1 \leqq i, j \leqq N$. Eq. (6.15) is valid for any x belonging to S, such that $f(x) = 1$; in particular, (6.15) is true for $x = y$,

$$(6.16) \qquad f_n(y) y_n^{1-r/2} = \sum_{j=1}^{N} A_{nj} y_j^{r/2}, \qquad n = 1, 2, \ldots, N.$$

Now substituting (6.15) into the left-hand side of (6.14) and (6.16) into the right-hand side of (6.14), we obtain

$$\sum_{n=1}^{N} y_n^{r/2} \sum_{j=1}^{N} A_{nj} x_j^{r/2} = \sum_{n=1}^{N} x_n^{r/2} \sum_{j=1}^{N} A_{nj} y_n^{r/2},$$

or

$$\sum_{n}\sum_{j} y_n^{r/2} A_{nj} x_j^{r/2} = \sum_{n}\sum_{j} x_n^{s/2} A_{nj} y_j^{r/2}. \tag{6.17}$$

Since (6.17) is true for every x, y, such that $f(x) = 1 = f(y)$, we must have

$$A_{nj} = A_{nj}, \qquad \text{for} \quad 1 \leqq n, \quad j \leqq N. \tag{6.18}$$

Now take $x_n^{r/2}$ times (6.15) and sum over n,

$$\sum_{n=1}^{N} x_n^{r/2} f_n(x) x_n^{1-r/2} = \sum_{n=1}^{N} \sum_{n=1}^{N} A_{nj} x_n^{r/2} x_j^{r/2} = 1,$$

since $x \cdot \nabla f(x) = f(x) = 1$.

Thus if $f(x) = 1$, then x satisfies the equation $\sum_n \sum_j A_{nj} x_n^{r/2} x_j^{r/2} = 1$, where $A_{nj} = A_{jn}$. Since f is linear homogeneous by assumption, we must have for x belonging to S,

$$f(x) = \left[\sum_{n=1}^{N} \sum_{n=1}^{N} A_{nj} x_n^{r/2} x_j^{r/2} \right]^{1/r}. \tag{6.19}$$

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