Measuring Treasury Debt and Market Depth

January 3, 2024

Abstract

This paper introduces index number theory to our analysis of US Treasuries, the depth of their market, and fiscal capacity; more generally. Using the user cost of Treasuries derived herein, I construct an index that tracks the true quantity of US Treasury debt and the monetary service flows provided by Treasuries. The importance of this view can be seen in the differing Treasury market reactions to the European debt crisis and COVID-19's "dash for cash" scenario. Despite large amounts of deficit spending in both situations, this measure reveals that market depth rose dramatically in light of the former crisis and fell during the latter.

Keywords: US Treasuries, Market Depth, Aggregation, Fiscal Capacity, Index Number Theory

1 Introduction

COVID-19, industrial policy, and inflation have thrust the US government's debt burden back into the spotlight in recent years. The predominant view of fiscal debt, however, focuses on the size of the market and ignores its depth. A "deep market" is typically synonymous with high levels of liquidity, but it can incorporate any attribute that generates market participation and a reduction in price sensitivity, overall. Krishnamurthy and Vissing-Jorgensen (2012) Vissing-Jorgensen and Krishnamurthy (2013), and Nagel (2016)—to name just a few—empirically show that liquidity and safety are key qualities of the Treasury market.[1] Brunnermeier, Merkel and Sannikov (2022) expand on this, demonstrating that these additional properties (which they call "transaction services") have implications for fiscal capacity.[2][3] This paper introduces index number theory to the fiscal debt discussion, continuing the move from simple accounting to a measure of the full economic value of the underlying securities.

The contribution of this paper is three-fold. First, expanding on the existing literature, I provide empirical evidence suggesting that the various types of Treasuries that constitute the debt should not be treated as perfect substitutes and therefore should not be aggregated in the standard, simple-sum manner. Next, I motivate the use of an index number to account for this imperfect subsitutability; deriving the user cost for long-term Treasuries and creating a Fisher ideal quantity index that tracks the true aggregate. Lastly, I show that the value of these monetary services directly increases Treasury market depth and fiscal capacity, more generally. Together, these results reveal

[1] Caballero, Farhi and Gourinchas (2017) build on this idea to explain historically low borrowing costs in the face of historically high debt levels.

[2] I will refer to these transaction services more generally as "monetary services" These monetary services generally include the aforementioned transactions services as well as other attributes such as liquidity, safety, use as collateral, etc.

[3] Further, they show that the inclusion of these monetary services disrupt the traditional FTPL channel (see Leeper, 1991, as a seminal example) and, along with the empirical results of Jiang *et al.* (2019), suggest that this channel is important in the valuation of Treasury debt.

that aspects of our standard view of fiscal debt can be misleading, which is problematic for the policy/discussion surrounding it.

The first step in establishing a new measurement of fiscal debt is to extend the work of Amihud and Mendelson (1991), showing that the differing types of nominal Treasury securities (bills, notes, and bonds) are not perfect subsitutes as the simple-sum framework implicitly assumes. Regressing Treasuries' relative yields to maturity (YTM) against their relative liquidities, coupons, etc.; I find that bills benefit from a liquidity premium even relative to defacto zero-coupon notes with the same time to maturity.[4] Bonds, on the other hand, primarily act as a savings vehicle rather than a liquidity hedge. These results provide evidence for the use of a more theoretically rigorous aggregation approach.

I then propose—via the findings of Diewert (1976, 1978) and Dumagan (2002)—the use of a Fisher ideal index number to track the true quantity of fiscal debt and derive the one-period user cost of the underlying securities. The Fisher ideal index is not only superlative, but it also handles zero quantity values better than the Törnqvist-Theil Divisia quantity index ubiquitous to the monetary aggregation literature (e.g. Barnett and Serletis, 2000). This quantity index tracks outstanding fiscal debt in the United States since 1977 and is used to derive the growth rate of the monetary services provided by this stock of fiscally-provided assets. Having this measure of fiscal monetary services provides a new data point in evaluating the impact of fiscal deficits and debt on the economy at large.

Lastly, I follow the Brunnermeier *et al.* (2022) template to show that the full value of these fiscally-supplied monetary services—and the resulting market depth—increases fiscal capacity. This is important because the economic value of US fiscal debt has

[4] That is, matched notes and bills with less than six months to maturity.

historically outpaced the simple accounting view of outstanding principal. At first glance, this could lead many to think that this new metric means there is *less* fiscal capacity, but the monetary services add demand and depth to the market, increasing the government's ability to borrow.

The importance of this alternative view of Treasury debt can be seen in the US government's responses to the Great Financial Crisis and COVID-19. Both saw substantial spikes in deficit spending, but were set against different global economic environments. The former coincided with the European debt crisis that sent investors flooding into the safer Treasuries market. The latter came with what we now call the "dash for cash," in which even US Treasuries were not liquid enough. The view of market depth derived here shows a large, sustained growth in monetary services during the former period, and a sharp decrease in monetary services during the latter period. When combined with the fiscal-capacity connection explored in this paper, it is straightforward to see why a market-based measure like this is important to ongoing fiscal policy discussions.

The remainder of this paper is organized as follows. Section 2 motivates—both empirically and theoretically—the use of index numbers in the fiscal debt realm. Section 3 derives the needed user costs of longer-term Treasury securities. Section 4 considers the data and methodology for constructing the index number. Section 5 presents the Fisher ideal index and compares it to the ubiquitous simple sum aggregate. Section 6 explores the relationship between fiscally-provided monetary services and fiscal capacity. Section 7 concludes.

2 Motivation

Measuring the monetary services of fiscal debt specifically is a novel concept, but the idea of considering the monetary services of financial assets in general is not new. In this section I briefly touch on the monetary aggregation literature, motivate the need for a new measure of fiscal debt, and consider the hurdles in applying the theory to this data specificially.

2.1 Bond Market Segmentation

Fiscal debt is ubiquitously measured as the sum of outstanding principal values, but simple-sum aggregates come with an underlying assumption that the underlying assets are perfect substitutes. As Serletis and Robb (1986, pg 431) explain: "Simple-sum aggregation (the usual procedure) is justified [...] only if the compoent assets are perfect substitutes (implying linear indifference surfaces) one for one. If this is condition of perfect substituability is violated, it is inappropriate to form a quantity index by giving an equal weight to each asset component."

Preliminary signs of *imperfect* substitutability would include a yield-to-maturity (YTM) spread between assets that are virtually identical in everything but name. On the secondary market, simple equilibrium conditions would suggest that, even in the face of differing coupon payments, the prices of two securirties with the same issuing entity and remaining time to maturity would be bid up or down to equalize their yields. So generally speaking, the only difference between these securities on the secondary market is the label attached to their initial maturity. However, using monthly CRSP data at the CUSIP level, Figure 1 shows a persistent gap in the median yields to maturity of Treasury notes and bonds with roughly the same time to maturity for the 1970–2020 period. That is, the time series compare the YTM of a Treasury note that matures within t periods with a Treasury bond that also matures with t periods. As can be seen, these YTMs are not equal, and the spread between them is not constant.

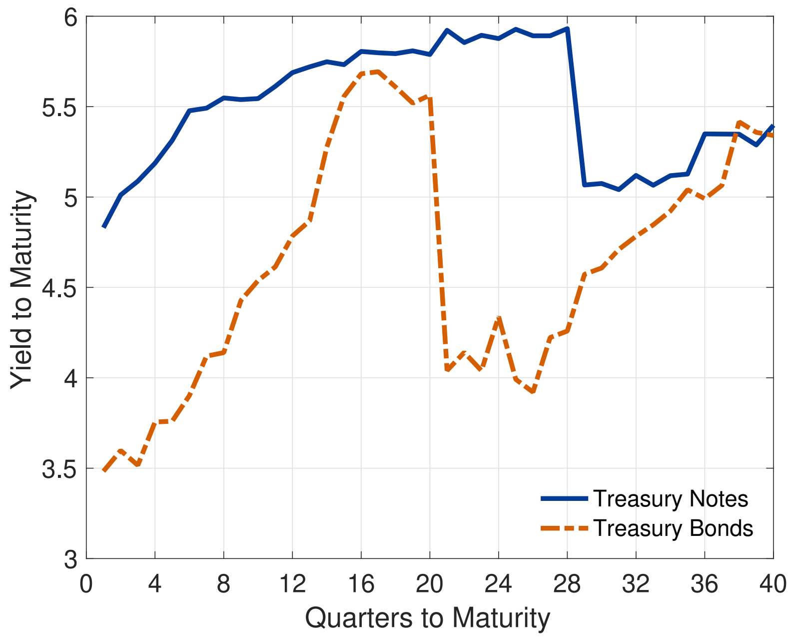

Figure 1: Median Treasury Bond and Note Yields with Same Time to Maturity (1970-2020)

Amihud and Mendelson (1991) find that the YTM spread between notes and bills is driven by relative liquidity. Again using monthly data on Treasury securities identified at the CUSIP level, I extend their work to empirically test this claim across the three common types of nominal securities. The data are first separated by their type label: bills, notes, and bonds. Then, for every month in the Oct 1996–Dec 2020 subsample period, each "senior" security (bond/note) is matched to a "junior" security (note/bill) that matures within one day of said "senior" security, allowing for exact matches if they exist.[5] Securities without a match are removed from the analysis. The securities' YTM

[5] The October 1996 cutoff for this analysis comes from the source data in the CRSP database. Prior to October 16, 1996, CRSP data on bid and ask prices were sourced from the FRBNY; while sourced from GovPX and its aquiring institutions since then. According to the CRSP documentation: "The FRBNY described its listed bid price as '...the most widely quoted price from the range of quotations

spreads are then regressed against the spreads in their relative bid-ask ratios, which are calculated as

$$\text{relative bid-ask ratio} = \frac{\text{ask price} \; - \; \text{bid price}}{\text{ask price} \; + \; \text{accrued interest}} \times 100.$$

An increase in the bid–ask spread implies a relative decrease in the liquidity of the "senior" security. I also control for the spread in their respective coupon rates, a general time to maturity (months/years) for any term effects, and the expected state of the economy at the time through the 10yr–2yr yield curve spread.[6] Table 1 presents the results for the notes–bills, bonds–notes, and bonds–bills analyses.

Matching the findings of Amihud and Mendelson (1991), the differences in note and bill YTM can be explained by their relative liquidities. That is, bills benefit from a liquidity premium and cannot be considered as perfect substitutes for notes, and vice versa.

The results of the bonds–notes and bonds–bills analyses paint bonds primarily as a savings vehicle in financial markets. Neither the bond's coupon rate nor the expected state of the economy impact its YTM spread versus bills, suggesting that bonds are largely purchased to hold. Additionally, a decrease in the liquidity of bonds relative to bills causes the YTM spread to decrease, opposite of the notes result. A plausible explanation for this is that the demand for bonds reduces the YTM relative to bills, but that these bonds are purchased for savings purposes and are more likely held to maturity, reducing the liquidity of the market overall. The positive, yet not statistically

received'. The ask price was determined by the FRBNY based on what they expect a typical bid-ask spread to be. The rule used to make this derivation was not public domain. GovPX described its listed bid and ask prices as the 'best price'. To determine their 'best price' they observe the prices from the five inter-dealer brokers and report the bid and ask prices that produce the smallest bid-ask spread." Use of latter period data ensures that these irregularities in the data do not interfere with the analysis.

[6] Note that the coupon rate for bills is zero. The 10yr–2yr spread is used because it is a spread between two notes, attempting to get at the general expectations of the future economy without crossing the asset types in the process. Standard errors are heteroskedasitcally robust (HC1).

Table 1: Relative Liquidity and Yield to Maturity[a]

	Notes–Bills[b]	Bonds–Notes	Bonds–Bills[b]
Relative Bid-Ask Spread	0.4163**	0.0981	−0.1424**
	(0.173)	(0.074)	(0.074)
Coupon Rate Spread	0.0210***	0.0170***	0.0076
	(0.001)	(0.005)	(0.013)
Months to Maturity	−0.0152***		−0.1028***
	(0.002)		(0.018)
Years to Maturity		−0.0049***	
		(0.001)	
10y-2y Spread	0.0234***	−0.0124***	−0.0484
	(0.003)	(0.002)	(0.033)
Constant	0.0076	−0.0554**	−0.3244**
	(0.010)	(0.030)	(0.154)
Observations	2250	7430	78
R-Squared	0.207	0.505	0.399
F-statistic	99.48	207.2	20.19

[a] The dependent variable is the yield-to-maturity spread between securities matched by thier respective days to maturity. Standard errors are heteroskedasticity robust (HC1). Designations ***, **, and * represent results that are statistically significant at the one, five, and ten percent levels, respecitivly.
[b] Analyses considering Treasury bills are limited to those securities with six months to maturity or less. This ensures that both securities have no other payments remaining until the maturity date.

significant bonds–notes liquidity result also suggests that bonds occupy a part of the market that is less concerned with the liquidity of the investment and more about the safety and return.

Overall, these results confirm the generally accepted theory that bills are primarily used as a liquidity hedge while bonds are primarily a savings vehicle. Notes seem to fall somewhere in the middle, being more sensitive to market liquidity relative to bills, and more sensitive to differences in return relative to bonds. These results also confirm that, when aggregating, none of these assets should be treated as perfect substitutes for the others.

2.2 Finding the Proper Quantity Index

Given that these Treasury securities are not perfect substitutes, the next step is selecting the best quantity index for proper aggregation. Christensen *et al.* (1971) show that the homogenous tanslog function

$$\ln f(x_t) = \alpha_0 + \sum_{n=1}^{N} \alpha_n \ln x_t^n + \frac{1}{2} \sum_{j,k=1}^{N} \gamma_{j,k} \ln x_t^j \ln x_t^k,$$

can provide a second-order approximation to an unknown, arbitrary twice-continuously-differentiable linear homogenous aggregator. Here, $\sum_{n=1}^{N} \alpha_n = 1$, $\gamma_{j,k} = \gamma_{k,j}$, and $\sum_{k=1}^{N} \gamma_{j,k} = 0$ for $j \in [1, N]$. Boisvert (1982) explains that the translog function is is more flexible than the Cobb-Douglas or constant elasticity of substitution (CES) functional forms by allowing the partial elastiticies of substituion between the assets to vary. He also goes on to explain that this functional form can be viewed as a) an exact aggregator function, b) a second-order Taylor approximation to a general, but unknown aggregator function, or c) a second-order approximation to a CES function. These properties make the translog function the perfect starting point in deriving the best quantity index.

Diewert (1976, Section 2) shows that the Törnqvist-Theil Divisia quantity index

$$\ln M_t^d = \ln M_{t-1}^d + \sum_{n=1}^{N} \frac{1}{2}(s_{n,t} + s_{n,t-1}) \left(\ln x_t^n - \ln x_{t-1}^n\right),$$

can be derived directly from a translog aggregator function. Here, M_t^d is the level of the Divisia quantity index, x_t^n is the nominal value of asset n, and

$$s_{n,t} = \frac{\eta_t^n x_t^n}{\boldsymbol{\eta}_t \mathbf{x'}_t}$$

corresponds to the weight on the marginal change in financial asset n. Here, $\mathbf{x}_t$ and $\boldsymbol{\eta}_t$ are $N \times 1$ vectors of nominal quantities and user costs, respectively. This type of quantity index is ubiquitous in the monetary aggregation literature.

One drawback of the Törnqvist-Theil Divisia index is that it does not handle assets coming into and out of the market ($x_t^n = 0$ for some n, t) very well. Given the frequent changes in Treasury policy and the potential categorizations I will use, zero values are likely to be commonplace in this analysis. When calculating general monetary aggregates, this problem does not arise often and is managed by imputing a reservation price and switching to a Fisher ideal index

$$M_t^f = M_{t-1}^f \left[\frac{(\boldsymbol{\eta}_t \mathbf{x'}_t)(\boldsymbol{\eta}_{t-1} \mathbf{x'}_t)}{(\boldsymbol{\eta}_t \mathbf{x'}_{t-1})(\boldsymbol{\eta}_{t-1} \mathbf{x'}_{t-1})}\right]^{\frac{1}{2}} \tag{1}$$

for those time periods. Diewert (1976) shows that this quantity index is also superlative, with Diewert (1978) and Dumagan (2002) showing that the two indexes will appriximate each other numerically and mathematically, respectively. Additionally, Diewert (1976) shows that the ideal index is derived from a quadratic mean of order 2 quantity index, which corresponds to the assumed translog aggregator function and is even recommended as the preferred index to use (Diewert, 1976, Section 5). Therefore, a Fisher ideal index is used in the remaining analysis.

3 The User Cost of Treasury Securities

As Barnett (1980) outlines, aggregation theory relies on known, exact functional forms with estimable parameters. The functions of interest are typically utility and production functions, which are often impossible to know. This is why I rely on statistical index numbers, whose theory only relies on the existence of maximizing behavior. From this optimizing behavior we can derive the user costs of the underlying components and bypass the unknown parameters because the resulting index numbers are not dependent on any specialized properties of the aggregator function. That is, while I'm not deriving the true aggregate itself, the resulting quantity index will track the true aggregate. A complete guide to index number theory and its application to monetary aggregation can be found in Barnett and Serletis (2000).

A proper quantity index requires a price that is derived from an optimizing agent. Thus, motivating the measurement of marketable government debt requires a partial equilibrium model that derives both the period-by-period user cost of holding government debt as well as the budgetary constraints the fiscal authority encounters. This model incorporates both short- and long-term debt issued by the government, as well as an alternative long-term asset that acts as a pure savings vehicle and can be used as the benchmark asset. In this model, I refer to the alternative asset as "capital," though it could also be motivated in some other manner.

3.1 Long-Term Asset Dynamics

The long-term government bonds and capital both evolve in a similar fashion. As described by Krause and Moyen (2016), each period new nominal long-term government bonds $B_t^{L,n}$ are issued, which are added to the stock of outstanding long-term debt B_t^L. A portion $\alpha \in (0,1)$ of the previous period's stock of long-term bonds mature, while the remaining $(1-\alpha)$ remain in the stock of outstanding long-term debt. The maturity of these bonds is therefore $1/\alpha$. Together, the stock of outstanding long-term debt evolves such that

$$B_t^L = (1-\alpha)B_{t-1}^L + B_t^{L,n}. \tag{2}$$

The average nominal interest rate paid on the stock of outstanding long-term debt r_t^L evolves such that

$$r_t^L B_t^L = (1-\alpha)r_{t-1}^L B_{t-1}^L + r_t^{L,n} B_t^{L,n}, \tag{3}$$

where $r_t^{L,n}$ is the interest rate on newly-issued long-term bonds.

Capital is structured in a similar way, with the total stock of outstanding capital evolving according to

$$K_t = (1-\delta)K_{t-1} + I_t, \tag{4}$$

where I_t is new investment in capital and $\delta \in (0,1)$ designates the portion of the outstanding stock of capital that matures each period. This implies that the maturity of capital is $1/\delta$. The average interest rate paid on the capital stock outstanding R_t is derived by

$$R_t K_t = (1-\delta)R_{t-1}K_{t-1} + R_t^n I_t, \tag{5}$$

where R_t^n is the interest rate on newly-issued capital.

3.2 Representative Household

The representative household in this model works l_t hours each period for nominal wage W_t. It also earns income through it's previous investments in capital $(\delta + R_{t-1})K_{t-1}$, short-term (one-period) bonds $(1+r_t)B_{t-1}$, long-term bonds $(\alpha + r^L_{t-1})B^L_{t-1}$, and profits from a continuum of intermediate goods-producing firms $\int_0^1 \Pi_t(s)\mathrm{d}s$.[7] This income is spread between a real consumption good c_t at price p_t as well as new investments in nominal short-term bonds B_t, long-term bonds $B^{L,n}_t$, capital I_t, and a nominal lump sum tax $P_t\tau_t$. Combined, the household's period-by-period budget constraint can be expressed as

$$B_t + B^{L,n}_t + p_t c_t + I_t = (\delta + R_{t-1})K_{t-1} + (1 + r_{t-1})B_{t-1} + \\ (\alpha + r^L_{t-1})B^L_{t-1} + W_t l_t + \int_0^1 \Pi_t(s)\mathrm{d}s - P_t\tau_t. \quad (6)$$

The household objective is to maximize utility over the real consumption good, the real monetary services provided by its portfolio of government bonds, and leisure

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + v\left(m_t\right) + x(1 - l_t) \right\}, \quad (7)$$

where $u(\cdot)$, $v(\cdot)$, and $x(\cdot)$ are increasing, concave functions.[8] Real monetary services are approximated using a second-order translog function

$$\ln m_t = \theta_0 + \theta_s \ln b_t + \theta_L \ln b^L_t + \frac{1}{2}\omega_{s,s}\left(\ln b_t\right)^2 + \frac{1}{2}\omega_{L,L}\left(\ln b^L_t\right)^2 + \omega_{s,L}\ln b_t \ln b^L_t. \quad (8)$$

[7] The δ considered here can deviate from that in (4) if depreciation is considered. In this present study, however, we assume zero deprecation and treat "capital" as an alternative long-term security similar to the long-term government bond. Additionally, the inclusion/exclusion of the corporate profit term does not alter the results of this partial equilibrium setup, but would be needed in a full, general equilibrium presentation.

[8] The assumption of additive separability in c_t and m_t is stronger than the needed weakly separable assumption for this analysis. This is for better tractability and does not change the result in the end.

Inclusion of the asset of focus into the utility function is not new, and has been used in the literature to motivate the original user cost of money (Barnett, 1978). Modeling demand in this way is important for two reasons.

First, in Section 2.1, I delineate the distinctions among various Treasury securities. However, this interpretation applies solely to that context. For instance, bonds might be seen more as a savings tool than bills, but they are still typically viewed as more liquid than other debt securities available. Bills, though largely influenced by their inherent liquidity, are also widely recognized as a safe asset. To accurately portray Treasury securities without making sweeping generalizations about their financial market position, this modeling method is essential.

Second, if liquidity/safety were explicitly modeled in a comprehensible/tractable manner, the outcome woud remain consistent. Generally, it is evident that when transaction costs or cash-in-advance components are added to the budget constraint, there exists a representation with money/bonds in the utility function can yield the same first-order conditions. The main aim of this study is to construct a nonparametric quantity index monitoring US Treasury debt. As illustrated in Appendix B, the user cost's derivation stems from the equivalency between marginal benefit and cost. The nonparametric user cost emerges from the marginal cost side, whereas the parametrized modeling choices appear on the marginal benefit side. This underscores the necessity for a well-defined index number, which should originate from some form of agent maximization, enabling its derivation even without an explicit understanding of the aggregation function.

Solving the household's problem results in the following dynamic conditions:

$$1 + \frac{\gamma_{2,t}}{b_t} \left(\theta_s + \omega_{s,s} \ln b_t + \omega_{s,L} \ln b_t^L \right) = \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{1 + r_t}{\pi_{t+1}} \right], \tag{9}$$

$$1 + \frac{\gamma_{2,t}}{b_t^L}\left(\theta_L + \omega_{L,L}\ln b_t^L + \omega_{s,L}\ln b_t\right) + \gamma_{4,t}\left(r_t^L - r_t^{L,n}\right)$$
$$= \beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{1}{\pi_{t+1}}\left\{1 + r_t^L + (1-\alpha)\gamma_{4,t+1}\left(r_t^L - r_{t+1}^{L,n}\right)\right\}\right], \quad (10)$$

and

$$1 + \gamma_{3,t}\left(R_t - R_t^n\right) = \beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{1}{\pi_{t+1}}\left\{1 + R_t + (1-\delta)\gamma_{3,t+1}\left(R_t - R_{t+1}^n\right)\right\}\right], \quad (11)$$

The solution technique to the household's problem can be found in Appendix A. Here, $\mu_{1,t} = u'(c_t)$, $\gamma_{2,t} = {}^{v'(\cdot)}/_{u'(\cdot)}$, and $\gamma_{3,t}$ and $\gamma_{4,t}$ are the relative prices of the long-term assets B_t^L and K_t, respectively. These prices evolve according to

$$\gamma_{3,t} = \beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{1}{\pi_{t+1}}\left\{1 + (1-\delta)\gamma_{3,t+1}\right\}\right], \quad (12)$$

and

$$\gamma_{4,t} = \beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{1}{\pi_{t+1}}\left\{1 + (1-\alpha)\gamma_{4,t+1}\right\}\right]. \quad (13)$$

The user costs of the short- and long-term bonds are

$$\eta_t = \frac{\mathbb{E}_t\left[\frac{\mu_{t+1}}{\mu_t}\frac{1}{\pi_{t+1}}\left\{R_t^n - (1-\delta)\gamma_{3,t+1}\Delta R_{t+1}^n - r_t\right\}\right]}{\mathbb{E}_t\left[\frac{\mu_{t+1}}{\mu_t}\frac{1}{\pi_{t+1}}\left\{1 + R_t^n - (1-\delta)\gamma_{3,t+1}\Delta R_{t+1}^n\right\}\right]}, \quad (14)$$

and

$$\eta_t^L = \frac{\mathbb{E}_t\left[\frac{\mu_{t+1}}{\mu_t}\frac{1}{\pi_{t+1}}\left\{R_t^n - (1-\delta)\gamma_{3,t+1}\Delta R_{t+1}^n - r_t^{L,n} + (1-\alpha)\gamma_{4,t+1}\Delta r_{t+1}^{L,n}\right\}\right]}{\mathbb{E}_t\left[\frac{\mu_{t+1}}{\mu_t}\frac{1}{\pi_{t+1}}\left\{1 + R_t^n - (1-\delta)\gamma_{3,t+1}\Delta R_{t+1}^n\right\}\right]}, \quad (15)$$

respectively. The derivations of these user costs can be found in Appendix B.

4 Data and Methodology

In this section, I construct the Fisher ideal quantity index described in Section 2 from data compiled by the Center for Research in Securities Prices on marketable Treasury securities. Non-marketable debt is typically held by intergovernmental agencies, and is therefore not typically considered to be a significant burden.[9] The Treasury began its "regular and predictable" debt issuance campaign in the 1970s and data regarding the spot and forward interest rates goes back to January 1977. Therefore the variables constructed here will cover the February 1977 to December 2020 period.

The first issue that needs to be addressed is essentially analogous to new-product bias. As (1) shows, the calculation of the growth rates requires both current and lagged quantities and user costs. If I were to treat each issuance as a separate $m_{i,t}$, there would be distortions at issuance and maturity months. For example, in the issuance month of $m_{i,t}$, the preceding month's yield $r_{i,t-1}$ of that particular issuance doesn't exist, which means its user cost $\eta_{i,t-1}$ also doesn't exist. Therefore, an assumption would need to be made regarding this lagged user cost. Feenstra (1994) and others have outlined a theoretically justified solution to this problem in which the price is set at its reservation level, where the quantity demanded would equal zero in the preceding month.[10] It's also well documented that on-the-run bonds sell at a premium over their off-the-run counterparts, suggesting $\eta_{i,t-1} > \eta_{i,t}$, but by how much? Estimating the reservation price of each issuance in each month over the sample period would seem to be too technically burdensome. With how distortionary this bias can be, some adjustments

[9] The inclusion of only publicly-held, marketable US Treasury debt would be the optimal choice here, but data restrictions keep me from assessing this. For instance, the CRSP data doesn't include the amount of each T-bill issuance held publicly. 0 Another option could be to exclude the issuances held by the Federal Reserve, since that is the primary non-public holder of US Treasuries, but the SOMA data only goes back to 2003. Future research could apply this methodology to the shortened timeframe.

[10] The Center for Financial Stability, which publishes data on Divisia monetary aggregates, delays the inclusion of new monetary assets for a few months. In this situation, however, the regular issuance and maturity of short-term debt make this strategy problematic.

and/or assumptions are needed.

To reduce the magnitude of this issue, I cluster the assets into groups that are most likely to be perfect substitutes in any given month. While there may be discrepancies in on-the-run/off-the-run securities, they should average out within the clusters. Based on the work of Amihud and Mendelson (1991) as well as the results in Table 1, the securities need to be separated across their designation of *bills*, *notes*, and *bonds*. Since 1997, the Treasury has issued Treasury Inflation-Protected Securities (TIPS), which yield a real rate of interest instead of the traditional nominal yield. These have been shown to be less liquid than their nominal counterparts, so I create two additional categories of *TIPS notes* and *TIPS bonds*. Lastly, as these securities mature, their yields tend to fall with the trend of the constant maturity yield curve, reflecting the changing monetary services they provide. Therefore, each of these five categories needs to be further segmented by their time to maturity. Since bonds and longer notes are issued on a quarterly basis, it's common to have individual months where there are no securities of a certain type.[11] A fine-grained segmentation such as this would again be subject to a large amount of new-product bias, so I define each sub-category as the quarters-to-maturity. That is, for securities that will mature over the next one-to-three months, they are categorized has maturing within one quarter. A thirty-year threshold on bonds suggests 120 quarters-worth of categories, but there were a series of longer-term bonds that were issued between 1953 and 1965, which could impact the first years of the analysis. Therefore, the number of quarters-to-maturity categories is set at 160, or forty years. Overall, when accounting for both the types of securities and the quarters to maturity, there are 800 categories.[12]

[11] These notes and bonds are typically subject to reopenings in the months following the initial issuance, but the maturity dates are unchanged. For example, if a 30-year bond is issued in February, a reopening of that issuance may be offered in March, but the time-to-maturity on that reopening is 29 years, 11 months.

[12] Note that, based on (1), baskets that are consistently empty do not impact the measurement, allowing me to keep it consistent across the types of securities. So while there are 800 total baskets, a large number of them will not contribute to the measurement, but it does provide the flexibility needed for the aggregate over the almost fifty-year analysis.

The next step in this process is to identify the proper interest rates from the theory above. Since the user costs represent the holding period return and incorporate the interest rates paid on newly-issued securities (r_t and $r_t^{L,n}$) and not on the average interest rate paid out on outstanding debt (r_t^L), the most accurate interest rates to consider are the coupon rates. Additionally, the term $(1-\alpha)\gamma_{4,t+1}\Delta r_{t+1}^{L,n}$ represents the expected capital gains of the long-term bond. To account for these expectations, I use the current price of the bonds adjusted for its maturity, multiplied by the difference between the one-month-ahead forward rate and spot rate for each bond's particular month and maturity. For TIPS, I consider the real spot and forward yield curves in the calculation of expected capital gains. While forward rates incorporate more than just future expectations of current spot rates, the impact of any term premium looking only one month ahead should be negligible.

The rate considered for each basket of securities is the quantity-weighted average over the issuances therein, though a further assumption is needed to address missing values/new-product bias. As discussed above, even the broader categories used here do not ensure that there are no issues with new-product bias. Since most of the empty categories in this situation are simply a cluster of maturing debt through time, and not new issuances of debt, I use the linear interpolation of the rates across the maturities instead of calculating a reservation rate. This ensures that I'm capturing the true shape of the yield curves used.

The benchmark rate used here is the maximum holding-period return of the baskets of securities considered each month, plus twenty-five basis points.[13] This is similar to the

[13] Previous iterations of this measurement attempted to use the Baa Corporate Bond yield, which would have accounted for both the liquidity and safety attributes of Treasury securities. However, even the addition of 200 basis points did not ensure that all user costs were positive across the sample period, especially in the volatile years of 1979–1981 and long periods of time between 2014 and 2016. Adjustments to create that kind of spread for the early years also had the drawback of washing out the differences in user costs in the later parts of the sample period.

strategy used by the Federal Reserve in their calculation of Divisia monetary aggregates and is common in other derivations of user costs.

5 Results

The Fisher ideal quantity index of US fiscal debt—and its corresponding simple-sum aggregate—is presented in Figure 3. This value is constructed from the month-over-month growth rates calculated via the logarithm of (1).[14] Multiple theories have claimed that Treasury securities provide services above and beyond their principal value, but this is the first attempt to measure such services. The base month considered here is August 1989 due to the relative flatness of the yield curve at that time. When the yield curve is flat, the differences in the user costs are minimal, suggesting that the underlying assets are perceived as near-perfect substitutes and collapsing the Fisher ideal index into a simple sum aggregate. Using the respective growth rates of both measures, I estimate the growth rate of the fiscally-supplied monetary services. Since the true aggregate incorporates both the quantities of the underlying assets and the monetary services they provide, the difference in their growth rates reveals the growth of the monetary services alone. That is, I am essentially factoring-out the growth that comes purely from the issued principal values. This series is then used to constuct a quantity index for the monetary services (Figure 3), separated from the quantity of the underlying principal value. As can be seen, these monetary services have generally grown as principal levels have risen, though it is not uncommon for the monetary services of fiscal debt to decrease even in the face of deficit spending.

The year-over-year growth rate of these monetary services is easier to analyze and

[14] This indirect approach to the aggregates' levels is necessary as there is no initial value from which to begin the calculation of (1). These month-over-month growth rates can be found in Appendix C.

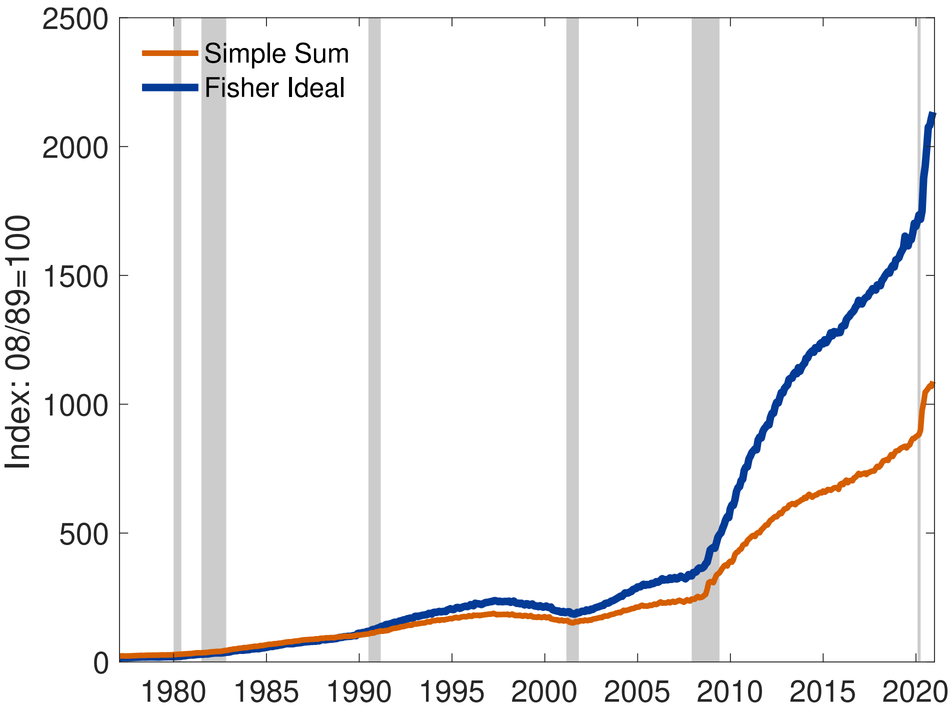

Figure 2: Monetary Services Level

shown in Figure 4. Eras with large financial market turbulance led to increased demand for the safety and liquidity of Treasury debt. For instance, there are surges in the growth of these fiscally-supplied monetary services in the mid-late 1980s around the peak of the Savings and Loan Crisis and the 1987 stock market crash. One of the largest and most sustained periods of growth align with the passage of the American Recovery and Reinvestment Act of 2009 and the Euro-zone debt crisis in the early 2010s. Sustained decreases can be seen during the bugdet surplus years of the late 1990s and in the late 2010s as the Treasury attempted to lengthen the maturity structure of its outstanding debt. The latter results from a move from highly-liquid bills to less-liquid notes and bonds. Most notably, and despite substantial deficit spending, the "dash for cash" in 2020 is seen as a sharp decrease in these monetary services as businesses and households moved to pure liquidity to smooth spending during the COVID-19 shutdowns. The alignment of these economic experiences with the more dramatic swings in the Fisher

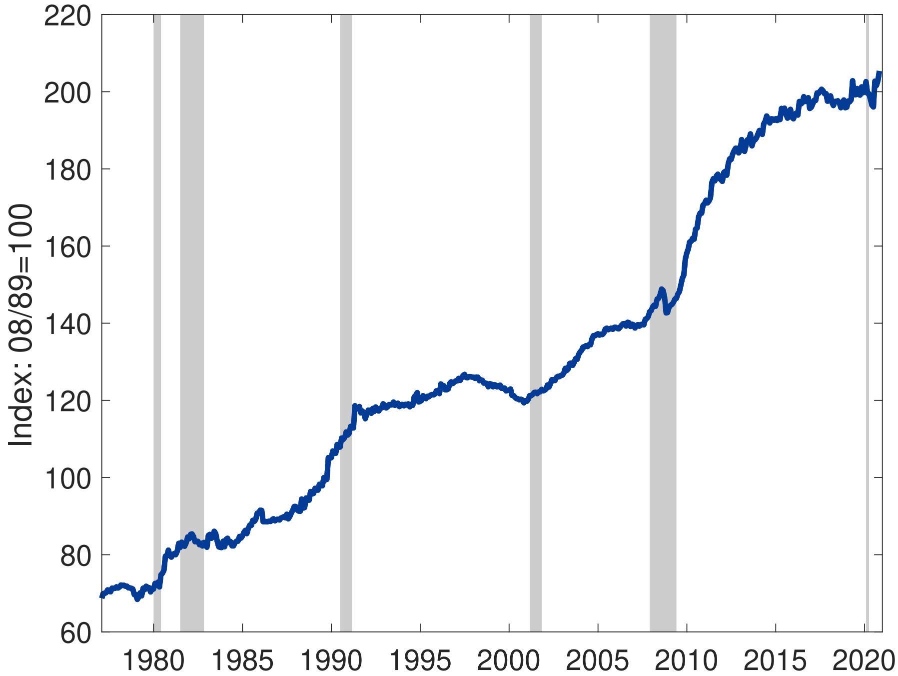

Figure 3: Derived Monetary Services Quantity Index

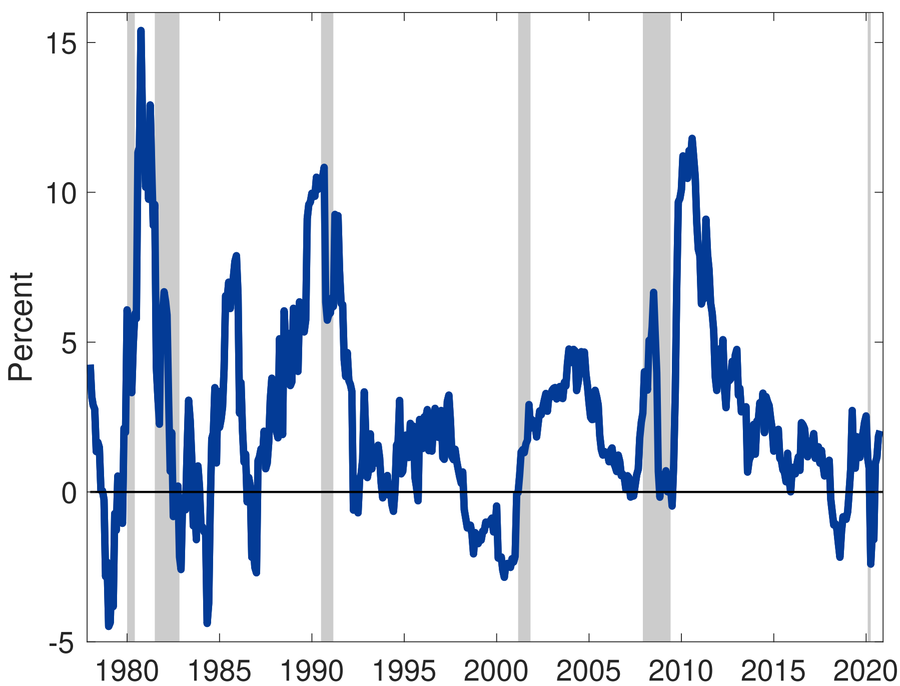

Figure 4: Year-over-year Growth of Fiscally-Supplied Monetary Services

ideal quantity index suggest that this measure is capturing what is intended.

6 The Treasury Market: Accounting vs. Economic Value

Discussions of fiscal debt often turn into discussions of fiscal capacity. Focusing on the simple-sum value of the underlying principal, however, considers only the purely accounting aspects of Treasury debt. In this section I consider the differences between the accounting (simple-sum) approach and an approach that considers the economic value of the debt (index-number).

6.1 Debt Dynamics, Explored

Brunnermeier, Merkel and Sannikov (2022) have shown that government debt constraints need to be augmented for the service flows (transaction/monetary services) that the securities provide. Here I expand upon this idea with the specifics of the model above, providing a deeper look at how we can ascertain those services flows in particular.[15]

Consider the simple budget constraint of a government that corresponds to the household side of the model above

$$(1 + r_{t-1})\frac{B_{t-1}}{p_t} + (\alpha + r^L_{t-1})\frac{B^L_{t-1}}{p_t} = \frac{B_t}{p_t} + \frac{B^{L,n}_t}{p_t} + s_t,$$

where s_t is denotes the real primary surplus. Now incorporate (2) into the above equation to get

$$(1 + r_{t-1})\frac{B_{t-1}}{p_t} + (1 + r^L_{t-1})\frac{B^L_{t-1}}{p_t} = \frac{B_t}{p_t} + \frac{B^L_t}{p_t} + s_t. \tag{16}$$

[15] This is identical to the exercise conducted by Brunnermeier *et al.* (2020), though I abstract from the bubble term to focus solely on the monetary services component.

For simplicity and without loss of generality, I assume that the true monetary services aggregate is a constant elastisticy of subsitution function

$$M_t = \left[\lambda^{\frac{1}{\sigma}} B_t^{\frac{\sigma-1}{\sigma}} + (1-\lambda)^{\frac{1}{\sigma}} B_t^{L\,\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{17}$$

where $\lambda \in [0,1]$ dictates the weight of each government debt security and σ dictates the elasticity of substitution between the two assets.

The equilibrium conditions analogous to (9) and (10)

$$1 - \gamma_{2,t} \left(\frac{\lambda M_t}{B_t} \right)^{\frac{1}{\sigma}} = \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{1+r_t}{\pi_{t+1}} \right],$$

and

$$\begin{aligned} 1 - \gamma_{2,t} \left(\frac{(1-\lambda) M_t}{B_t^L} \right)^{\frac{1}{\sigma}} &+ \gamma_{4,t} \left(r_t^L - r_t^{L,n} \right) \\ &= \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{1}{\pi_{t+1}} \left\{ 1 + r_t^L + (1-\alpha)\gamma_{4,t+1} \left(r_t^L - r_{t+1}^{L,n} \right) \right\} \right], \end{aligned}$$

must therefore hold. Incorporating these into (16) yields

$$\begin{aligned} \frac{B_{t-1} + B_{t-1}^L}{p_t}(1 + r_{t-1}) = s_t - (r_{t-1}^L - r_{t-1}) \frac{B_{t-1}^L}{p_t} &+ \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{B_t + B_t^L}{p_{t+1}} (1+r_t) \right] \\ - \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{B_t^L}{p_{t+1}} (1+r_t) \right] &+ \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{B_t^L}{p_{t+1}} (1+r_t^L) \right] \\ + \gamma_{2,t} \left(\frac{\lambda M_t}{B_t} \right)^{\frac{1}{\sigma}} \frac{B_t}{p_t} &+ \gamma_{2,t} \left(\frac{(1-\lambda) M_t}{B_t^L} \right)^{\frac{1}{\sigma}} \frac{B_t^L}{p_t} \\ + \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{B_t^L}{p_{t+1}} \left\{ 1 + (1-\alpha)\gamma_{4,t+1} \left(r_t^L - r_{t+1}^{L,n} \right) \right\} \right] &- \gamma_{4,t} \left(r_t^L - r_t^{L,n} \right) \frac{B_t^L}{p_t} \end{aligned} \tag{18}$$

Subsituting (13) into this—and some rearranging—we can simplify the above equation to

$$\frac{B_{t-1}+B_{t-1}^L}{p_t}(1+r_{t-1}) = s_t - (r_{t-1}^L - r_{t-1})\frac{B_{t-1}^L}{p_t} + \beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{B_t+B_t^L}{p_{t+1}}(1+r_t)\right]$$
$$+\beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{B_t^L}{p_{t+1}}(1+r_t^L-(1-\alpha)\gamma_{4,t+1}\Delta r_{t+1}^{L,n}-r_t)\right]$$
$$+\gamma_{2,t}\left(\frac{\lambda M_t}{B_t}\right)^{\frac{1}{\sigma}}\frac{B_t}{p_t}+\gamma_{2,t}\left(\frac{(1-\lambda)M_t}{B_t^L}\right)^{\frac{1}{\sigma}}\frac{B_t^L}{p_t}. \quad (19)$$

The last two terms are of particular importance here. The assumed-true monetary services aggregate (17) reduces the above expression to

$$\frac{B_{t-1}+B_{t-1}^L}{p_t}(1+r_{t-1}) = s_t - (r_{t-1}^L - r_{t-1})\frac{B_{t-1}^L}{p_t} + \beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{B_t+B_t^L}{p_{t+1}}(1+r_t)\right]$$
$$+\beta\mathbb{E}_t\left[\frac{\mu_{1,t+1}}{\mu_{1,t}}\frac{B_t^L}{p_{t+1}}(1+r_t^L-(1-\alpha)\gamma_{4,t+1}\Delta r_{t+1}^{L,n}-r_t)\right]+\gamma_{2,t}\frac{M_t}{p_t}, \quad (20)$$

where it can be shown that

$$\gamma_{2,t} = \left[\lambda\eta_t^{\sigma-1}+(1-\lambda)\left(\eta_t^L\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} \quad (21)$$

is the price dual for the quantity aggregate M_t.

There are a couple of items to point out with regards to the above expression. First, fiscal capacity intuitively diminishes as long-term rates rise relative to short-term rates. This term is reflecting the convenience yields that the government enjoys on its short-term debt. Second, the fourth term on the right side suggests long-term debt can improve fiscal capacity via its longer maturity and a higher expected one-period return over the short-term debt. That is, while capacity shrinks with higher long-term interest rates, demand for the debt increases with higher expected holding-period returns. Lastly, the borrowing ability of the Treasury increases with the value of the monetary/transaction services the debt provides. Thus, the monetary services provided by government debt issuances is directly captured in the government's budget constraint in a similar fashion to that shown in Brunnermeier *et al.* (2022). Analyses of market depth—and the fiscal capacity that comes with it— that focus only on the accounting view of debt and omits the economic value of the underlying securities will inevitably lead to a biased view of the fiscal situation.

Fisher's factor reversal can be used to calculate the non-parametric version of the price dual (21). The full economic value of the Treasury debt is shown in Figure 5. The data does not go back far enough to know whether the higher values in the late 1970s are a one-off spike or a sustained trend. The sudden decrease in this index in the 1989–1990 period comes from a sudden decrease in the price of obtaining these monetary services. While the reasoning for this dramatic shift is beyond the scope of this paper, this does generally align with some later estimates of the Great Moderation as well as the surge in the information technology realm.[16] One could imagine that the decrease in general market volatility, coupled with an increase in the available information about other financial assets, decimated the Treasury's comparative advantage in safety in the financial world.

The value of the monetary services aggregate was relatively stable until the financial crisis of 2007-2009 and the Euro debt crisis that followed, when that value increased approximately 378 percent from January 2007 to January 2014. This increase in the market depth would help explain the lack of inflationary pressure during that period of expansion despite increases in federal spending and monetary stimulus over that time. The stimulus passed during the 2020 pandemic, however, did not provide the same

[16] While most of the work on the Great Moderation centers around 1984:Q4 as the break point, Stock and Watson (2003) find a relatively wide 95% confidence interval that expands all the way to 1989:Q4. The first web browser was also introduced in 1990, providing information in an easy-to-use format to the masses.

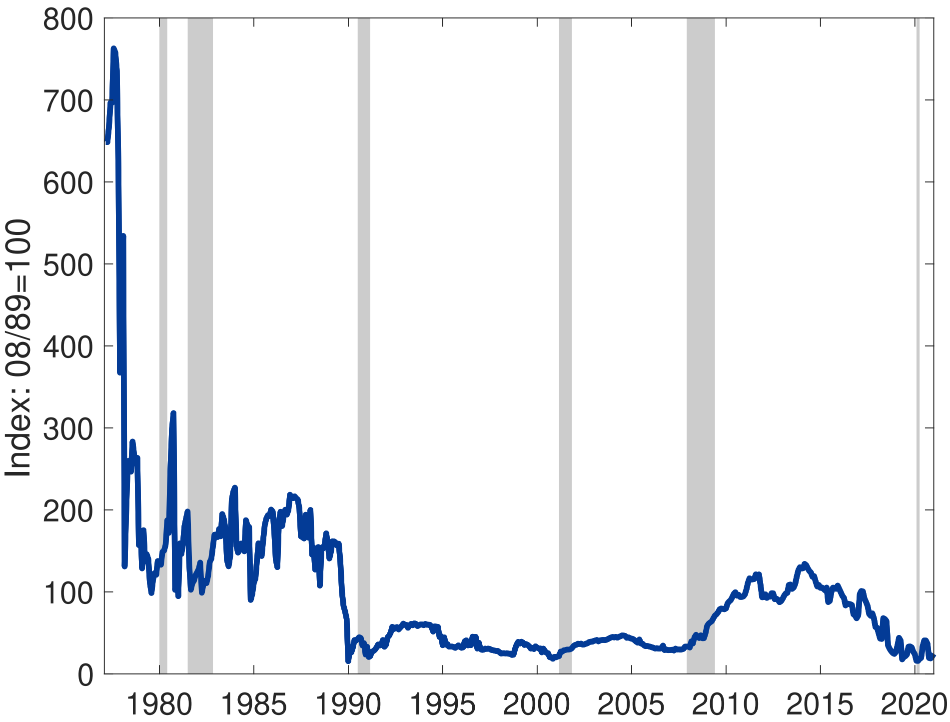

Figure 5: Value of Fiscal Monetary Services

increase in total value, as the private sector moved towards pure liquidity ("dash for cash") in the face of lockdowns. In that type of environment, even Treasury securities do not provided the needed liquidity and safety features.

7 Concluding Remarks

The three primary contributions of this paper center around how we view fiscal debt and Treasury market depth. First, I show empirically and theoretically that these securities should not be treated as perfect substitutes and therefore cannot be aggregated in a simple-sum manner. Next, I motivate the use of index number theory to account for this imperfect subsitutability; deriving the user cost for fiscal debt securities and creating a Fisher ideal quantity index. Lastly, I reveal that a failure to account for the monetary services provided by fiscal debt ultimately leads to a misunderstanding of a nation's fiscal capacity.

There are a multitude of extentions to this, including the calculation of indices for countries outside the United States. Of particular interest, if the data exist, would be countries that have undergone sovereign debt crises in the recent past. This would include countries such as Mexico, Argentina, and others, where an index like this may capture the erosion of the fiscal space better than the standard metrics. Additionally, one simplifying assumption taken here is that of risk neutrality or perfect certainty. While Treasury securities are considered to be the safest assets on the market, the inclusion of capital gains in the user costs means that there will always be some risk involved. An extension of this work, therfore, would be to derive the user costs under risk as Barnett *et al.* (1997) do for monetary aggregates.

In an economic and political environment in which fiscal debt is growing more and more salient, understanding how to properly track the metrics of choice and how they fit within the bigger picture becomes even more important. This paper shows how to view fiscal debt beyond simple accounting and why considering the true economic value of fiscal debt is important for future policy decisions.

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A Representative Household Solution

Combining (2) and (3)

$$\left(r_t^L - r_t^{L,n}\right) B_t^L = (1-\alpha)\left(r_{t-1}^L - r_t^{L,n}\right) B_{t-1}^L, \tag{A.1}$$

(4) and (5)

$$\left(R_t - R_t^n\right) K_t = (1-\delta)\left(R_{t-1} - R_t^n\right) K_{t-1}, \tag{A.2}$$

and (6), (2), and (4)

$$\begin{aligned} B_t + B_t^L - (1-\alpha)B_{t-1}^L + p_t c_t + K_t - (1-\delta)K_{t-1} &= (\delta + R_{t-1})K_{t-1} \\ &+ (1 + r_{t-1})B_{t-1} + (\alpha + r_{t-1}^L)B_{t-1}^L + W_t l_t + \int_0^1 \Pi_t(s)\mathrm{d}s - P_t\tau_t. \end{aligned} \tag{A.3}$$

A.1 Household's Problem

(7) subject to (8), (A.1), (A.2), and (A.3). Choosing $\left\{B_t, B_t^L, c_t, K_t, R_t, r_t^L, l_t, M_t\right\}$

A.2 Bellman Equation

$$
\begin{aligned}
&\mathbb{V}\big(B_{t-1}, B^L_{t-1}, K_{t-1}, R_{t-1}, r^L_{t-1}\big) \\
&\qquad = \max\left\{ u(c_t) + v\left(\frac{M_t}{P_t}\right) + x(l_t) + \beta\mathbb{E}_t\big[\mathbb{V}\big(B_t, B^L_t, K_t, R_t, r^L_t\big)\big]\right. \\
&\qquad + \frac{\mu_{1,t}}{p_t}\Big[(\delta + R_{t-1})K_{t-1} + (1 + r_{t-1})B_{t-1} + (\alpha + r^L_{t-1})B^L_{t-1} + W_t l_t \\
&\qquad + \int_0^1 \Pi_t(s)\mathrm{d}s - P_t\tau_t - B_t - B^L_t + (1-\alpha)B^L_{t-1} - p_t c_t - K_t + (1-\delta)K_{t-1}\Big] \\
&+ \frac{\mu_{2,t}}{p_t}\left[\theta_0 + \theta_s \ln b_t + \theta_L \ln b^L_t + \frac{1}{2}\omega_{s,s}\left(\ln b_t\right)^2 + \frac{1}{2}\omega_{L,L}\left(\ln b^L_t\right)^2 + \omega_{s,L}\ln b_t \ln b^L_t - \ln m_t\right] \\
&\qquad + \frac{\mu_{3,t}}{p_t}\Big[(1-\delta)\left(R_{t-1} - R^n_t\right)K_{t-1} - \left(R_t - R^n_t\right)K_t\Big] \\
&\qquad \left. + \frac{\mu_{4,t}}{p_t}\Big[(1-\alpha)\left(r^L_{t-1} - r^{L,n}_t\right)B^L_{t-1} - \left(r^L_t - r^{L,n}_t\right)B^L_t\Big]\right\}
\end{aligned}
\tag{A.4}
$$

A.3 First Order Conditions

$$u'(c_t) = \mu_{1,t} \tag{A.5}$$

$$v'\left(m_t\right) = \mu_{2,t} \tag{A.6}$$

$$x'(1 - l_t) = -\mu_{1,t} w_t \tag{A.7}$$

$$\beta\mathbb{E}_t\big[\mathbb{V}_1\left(B_t, B^L_t, K_t, R_t, r^L_t\right)\big] = \frac{\mu_{1,t}}{p_t} + \mu_{2,t}\left[\theta_s\frac{1}{B_t} + \omega_{s,s}\ln b_t\frac{1}{B_t} + \omega_{s,L}\ln b^L_t\frac{1}{B_t}\right] \tag{A.8}$$

$$\beta\mathbb{E}_t\big[\mathbb{V}_2\left(B_t, B^L_t, K_t, R_t, r^L_t\right)\big] = \frac{\mu_{1,t}}{p_t} + \mu_{2,t}\left[\theta_L\frac{1}{B^L_t} + \omega_{L,L}\ln b^L_t\frac{1}{B^L_t} + \omega_{s,L}\ln b_t\frac{1}{B^L_t}\right] \tag{A.9}$$

$$\beta\mathbb{E}_t\big[\mathbb{V}_3\left(B_t, B^L_t, K_t, R_t, r^L_t\right)\big] = \frac{\mu_{1,t}}{p_t} + \frac{\mu_{3,t}}{p_t}\left(R_t - R^n_t\right) \tag{A.10}$$

$$\beta\mathbb{E}_t\big[\mathbb{V}_4\left(B_t, B^L_t, K_t, R_t, r^L_t\right)\big] = \frac{\mu_{3,t}}{p_t}K_t \tag{A.11}$$

$$\beta \mathbb{E}_t \big[\mathbb{V}_5 \left(B_t, B_t^L, K_t, R_t, r_t^L \right) \big] = \frac{\mu_{4,t}}{p_t} B_t^L \tag{A.12}$$

A.4 Bienveniste-Scheinkman Conditions

$$\mathbb{V}_1 \left(B_{t-1}, B_{t-1}^L, K_{t-1}, R_{t-1}, r_{t-1}^L \right) = \frac{\mu_{1,t}}{p_t} (1 + r_{t-1}) \tag{A.13}$$

$$\mathbb{V}_2 \left(B_{t-1}, B_{t-1}^L, K_{t-1}, R_{t-1}, r_{t-1}^L \right) = \frac{\mu_{1,t}}{p_t} (1 + r_{t-1}^L) + \frac{\mu_{4,t}}{p_t} (1 - \alpha) \left(r_{t-1}^L - r_t^{L,n} \right) \tag{A.14}$$

$$\mathbb{V}_3 \left(B_{t-1}, B_{t-1}^L, K_{t-1}, R_{t-1}, r_{t-1}^L \right) = \frac{\mu_{1,t}}{p_t} (1 + R_{t-1}) + \frac{\mu_{3,t}}{p_t} (1 - \delta) \left(R_{t-1} - R_t^n \right) \tag{A.15}$$

$$\mathbb{V}_4 \left(B_{t-1}, B_{t-1}^L, K_{t-1}, R_{t-1}, r_{t-1}^L \right) = \left(\frac{\mu_{1,t}}{p_t} + (1 - \delta) \frac{\mu_{3,t}}{p_t} \right) K_{t-1} \tag{A.16}$$

$$\mathbb{V}_5 \left(B_{t-1}, B_{t-1}^L, K_{t-1}, R_{t-1}, r_{t-1}^L \right) = \left(\frac{\mu_{1,t}}{p_t} + (1 - \alpha) \frac{\mu_{4,t}}{p_t} \right) B_{t-1}^L \tag{A.17}$$

A.5 Optimality Conditions

$$u'(c_t) = \mu_{1,t} \tag{A.18}$$

$$v' \left(\frac{M_t}{p_t} \right) = \mu_{2,t} \tag{A.19}$$

$$x'(1 - l_t) = -\mu_{1,t} w_t \tag{A.20}$$

$$1 + \frac{\gamma_{2,t}}{b_t} \left(\theta_s + \omega_{s,s} \ln b_t + \omega_{s,L} \ln b_t^L \right) = \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{1 + r_t}{\pi_{t+1}} \right], \tag{A.21}$$

$$\begin{aligned} 1 + \frac{\gamma_{2,t}}{b_t^L} \left(\theta_L + \omega_{L,L} \ln b_t^L + \omega_{s,L} \ln b_t \right) + \gamma_{4,t} \left(r_t^L - r_t^{L,n} \right) \\ = \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{p_{t+1}} (1 + r_t^L) + \frac{\mu_{4,t+1}}{p_{t+1}} (1 - \alpha) \left(r_t^L - r_{t+1}^{L,n} \right) \right] \end{aligned} \tag{A.22}$$

$$\frac{\mu_{1,t}}{p_t} + \frac{\mu_{3,t}}{p_t} \left(R_t - R_t^n \right) = \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{p_{t+1}} (1 + R_t) + \frac{\mu_{3,t+1}}{p_{t+1}} (1 - \delta) \left(R_t - R_{t+1}^n \right) \right] \tag{A.23}$$

$$\frac{\mu_{3,t}}{p_t} K_t = \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{p_{t+1}} + (1-\delta) \frac{\mu_{3,t+1}}{p_{t+1}} \right] K_t \tag{A.24}$$

$$\frac{\mu_{4,t}}{p_t} B_t^L = \beta \mathbb{E}_t \left[\frac{\mu_{1,t+1}}{p_{t+1}} + (1-\alpha) \frac{\mu_{4,t+1}}{p_{t+1}} \right] B_t^L \tag{A.25}$$

B Derivation of the User Costs

Combining (11) and (12) yields:

$$1 = \beta \mathbb{E}_t \left[\frac{\mu_{t+1}}{\mu_t} \frac{1}{\pi_{t+1}} \left\{ 1 + R_t^n - (1-\delta)\gamma_{3,t+1} \Delta R_{t+1}^n \right\} \right] \tag{B.1}$$

Substituting this for the 1 in (9) and rearranging yields the marginal benefit/marginal cost equilibrium:

$$\frac{\gamma_{2,t}}{b_t^L} \left(\theta_L + \omega_{L,L} \ln b_t^L + \omega_{s,L} \ln b_t \right) = \beta \mathbb{E}_t \left[\frac{\mu_{t+1}}{\mu_t} \frac{1}{\pi_{t+1}} \left\{ R_t^n - (1-\delta)\gamma_{3,t+1} \Delta R_{t+1}^n - r_t \right\} \right] \tag{B.2}$$

Now dividing both sides by (B.1) converts the right side to the standard user cost form

$$\frac{\gamma_{2,t}}{b_t^L} \left(\theta_L + \omega_{L,L} \ln b_t^L + \omega_{s,L} \ln b_t \right) = \frac{\mathbb{E}_t \left[\frac{\mu_{t+1}}{\mu_t} \frac{1}{\pi_{t+1}} \left\{ R_t^n - (1-\delta)\gamma_{3,t+1} \Delta R_{t+1}^n - r_t \right\} \right]}{\mathbb{E}_t \left[\frac{\mu_{t+1}}{\mu_t} \frac{1}{\pi_{t+1}} \left\{ 1 + R_t^n + \delta * - \delta - (1-\delta)\gamma_{3,t+1} \Delta R_{t+1}^n \right\} \right]} \tag{B.3}$$

Decoupling $\gamma_{2,t}$ shows that the right hand side is the marginal cost of holding the short-term asset, expressed in terms of utility.

$$\begin{aligned} v'(m_t) \frac{\left(\theta_L + \omega_{L,L} \ln b_t^L + \omega_{s,L} \ln b_t \right)}{b_t^L} \\ = u'(c_t) \frac{\mathbb{E}_t \left[\frac{\mu_{t+1}}{\mu_t} \frac{1}{\pi_{t+1}} \left\{ R_t^n - (1-\delta)\gamma_{3,t+1} \Delta R_{t+1}^n - r_t \right\} \right]}{\mathbb{E}_t \left[\frac{\mu_{t+1}}{\mu_t} \frac{1}{\pi_{t+1}} \left\{ 1 + R_t^n - (1-\delta)\gamma_{3,t+1} \Delta R_{t+1}^n \right\} \right]} \end{aligned} \tag{B.4}$$

An assumption of risk neutrality or perfect certainty effectively implies that that that $\frac{\mu_{1,t+1}}{\mu_{1,t}} \frac{1}{\pi_{t+1}}$ is independent of the one-period returns in brackets, simplifying this expression to

$$\eta_t = \frac{R_t^n - (1-\delta)\mathbb{E}_t[\gamma_{3,t+1} \Delta R_{t+1}^n] - r_t}{1 + R_t^n - (1-\delta)\mathbb{E}_t[\gamma_{3,t+1} \Delta R_{t+1}^n]}, \tag{B.5}$$

where η_t equals the left-hand side of (B.3) and represents the user cost of holding the short term-asset for one period. See Barnett, Liu and Jensen (1997) for an in-depth exploration of monetary index number theory under risk.

Beginning the same procedure from (10) will provide the analogous user cost for a long-term security

$$\eta_t^L = \frac{\mathbb{E}_t\left[\frac{\mu_{t+1}}{\mu_t}\frac{1}{\pi_{t+1}}\left\{R_t^n - (1-\delta)\gamma_{3,t+1}\Delta R_{t+1}^n - r_t^{L,n} + (1-\alpha)\gamma_{4,t+1}\Delta r_{t+1}^{L,n}\right\}\right]}{\mathbb{E}_t\left[\frac{\mu_{t+1}}{\mu_t}\frac{1}{\pi_{t+1}}\left\{1 + R_t^n - (1-\delta)\gamma_{3,t+1}\Delta R_{t+1}^n\right\}\right]}, \tag{B.6}$$

and again assuming independence as above yields

$$\eta_t^L = \frac{R_t^n - (1-\delta)\mathbb{E}_t[\gamma_{3,t+1}\Delta R_{t+1}^n] - r_t^{L,n} + (1-\alpha)\mathbb{E}_t[\gamma_{4,t+1}\Delta r_{t+1}^{L,n}]}{1 + R_t^n - (1-\delta)\mathbb{E}_t[\gamma_{3,t+1}\Delta R_{t+1}^n]}. \tag{B.7}$$

Since we're dealing with long-term assets here, the period-by-period user cost incorporates both the expected one-period payouts as well as the expected capital gains/loses. The capital gains are incorporated via the expected change in the one-period payouts, scaled by the expected future price of the asset.

C Initial Results

The figure below presents the differences in the index number and simple sum month-over-month growth rates, along with a 12-month moving average centered on the seventh month.

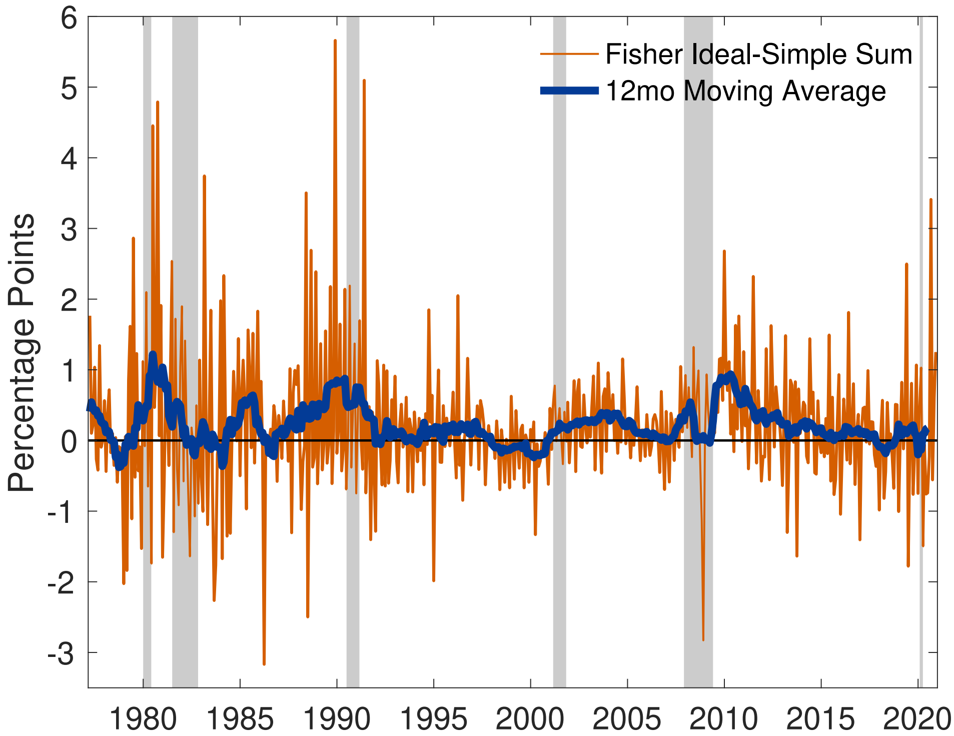

Figure 6: Month-over-month Growth Rate Spread