DERIVATION OF THE LORENZ EQUATIONS

I. GOVERNING EQUATIONS

Consider the incompressible Navier-Stokes equations within the Boussinesq approximation, modeling the thermal convection between two parallel horizontal flat plates separated by a distance d and kept at temperatures T_0 (bottom plate) and T_1 (top plate, with $T_1 < T_0$):

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho_0} \boldsymbol{\nabla} p + \nu \nabla^2 \mathbf{u} + \alpha g (T - T_0) \hat{\mathbf{z}}, \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T = \kappa \nabla^2 T. \tag{3}$$

Here, ρ_0 is the density of the fluid at temperature T_0, ν is the kinematic viscosity of the liquid, α is the coefficient of thermal expansion, κ is the thermal diffusivity, and $\hat{\mathbf{z}}$ is a unit vector in the vertical direction. We introduce the following dimensionless variables:

$$\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad \mathbf{u}^* = \frac{d}{\kappa} \mathbf{u}, \quad t^* = \frac{\kappa}{d^2} t, \quad p^* = \frac{1}{\rho_0} \left(\frac{d}{\kappa} \right)^2 p. \tag{4}$$

We also introduce a dimensionless temperature disturbance θ^*, defined by

$$\theta^* = \frac{T - T_0}{T_1 - T_0} - z^*, \tag{5}$$

where z^* is the (dimensionless) vertical coordinate.

Upon nondimensionalization, the governing equations become

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{6}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla} p + \mathrm{Pr} \nabla^2 \mathbf{u} + \mathrm{Pr}\, \mathrm{Ra}\, \theta\, \hat{\mathbf{z}}, \tag{7}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \theta = w + \nabla^2 \theta. \tag{8}$$

We have omitted the asterisks (*) for dimensionless variables. We denote by (u, v, w) the velocity components in the x, y, and z directions. In equation (7), we have introduced the Prandtl number $\mathrm{Pr} = \nu/\kappa$ (which is a material property), and the Rayleigh number Ra defined as

$$\mathrm{Ra} = \frac{\alpha g (T_0 - T_1) d^3}{\nu \kappa}. \tag{9}$$

We assume that the boundaries at $z = 0$ and $z = 1$ are stress-free, rigid, and isothermal, leading the boundary conditions

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad \text{and} \quad \theta = 0 \quad \text{at } \ z = 0 \ \text{ and } \ z = 1. \tag{10}$$

II. TWO-DIMENSIONAL CONVECTION ROLLS

We now consider the convection in the form of two-dimensional convection rolls, say aligned with the x axis. In this case, $u = 0$, and v, w, and θ only depend on y, z, and t. We introduce the two-dimensional streamfunction $\psi(y, z, t)$ such that

$$v = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial y}, \tag{11}$$

in which case the continuity equation $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ is automatically satisfied.

In two-dimensions, the vorticity only has one non-zero component ξ in the x direction:

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial x} = -\nabla_2^2 \psi \quad \text{where} \quad \nabla_2^2 \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{12}$$

By taking the curl of the momentum equation (7) and projecting in the x direction, we can eliminate the pressure p from the equations and obtain the following equation for ξ:

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \xi = \Pr \nabla_2^2 \xi + \Pr \mathrm{Ra} \frac{\partial \theta}{\partial y}. \tag{13}$$

Also observe that for any scalar quantity χ we have

$$\mathbf{u} \cdot \boldsymbol{\nabla} \chi = v \frac{\partial \chi}{\partial y} + w \frac{\partial \chi}{\partial z} = \frac{\partial \psi}{\partial z}\frac{\partial \chi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \chi}{\partial z} = J(\chi, \psi), \tag{14}$$

where $J(\chi, \psi)$ is the Jacobian of (χ, ψ) with respect to (y, z).

Using these observations, the governing set of equations for the two-dimensional convections rolls becomes:

$$\frac{\partial \xi}{\partial t} + J(\xi, \psi) = \Pr \nabla_2^2 \xi + \Pr \mathrm{Ra} \frac{\partial \theta}{\partial y}, \tag{15}$$

$$\frac{\partial \theta}{\partial t} + J(\theta, \psi) = \nabla_2^2 \theta - \frac{\partial \psi}{\partial y}, \tag{16}$$

$$\xi = -\nabla_2^2 \psi. \tag{17}$$

III. TRUNCATED GALERKIN EXPANSION

We now proceed to reduce this set of PDEs to a set of ODEs, which will lead to the Lorenz equations. The simplest approach is to consider a severely truncated Galerkin expansion:

$$\psi(y, z, t) = a(t) \sin \pi z \sin k\pi y + \ldots \tag{18}$$

$$\theta(y, z, t) = b(t) \sin \pi z \cos k\pi y + c(t) \sin 2\pi z + \ldots \tag{19}$$

where the two terms involving $a(t)$ and $b(t)$ correspond to convection rolls with wavenumber k in the y direction, and the term involving $c(t)$ can be justified by considering the modification of the mean temperature profile due to convection. We then have:

$$\zeta(y, z, t) = \pi^2(1 + k^2) a(t) \sin \pi z \sin k\pi y + \ldots \tag{20}$$

$$\frac{\partial \zeta}{\partial y}(y, z, t) = k\pi^3(1 + k^2) a(t) \sin \pi z \cos k\pi y + \ldots \tag{21}$$

$$\frac{\partial \zeta}{\partial z}(y, z, t) = \pi^3(1 + k^2) a(t) \cos \pi z \sin k\pi y + \ldots \tag{22}$$

$$\frac{\partial \psi}{\partial y}(y, z, t) = k\pi a(t) \sin \pi z \cos k\pi y + \ldots \tag{23}$$

$$\frac{\partial \psi}{\partial z}(y, z, t) = \pi a(t) \cos \pi z \sin k\pi y + \ldots \tag{24}$$

$$\frac{\partial \theta}{\partial y}(y, z, t) = -k\pi b(t) \sin \pi z \sin k\pi y + \ldots \tag{25}$$

$$\frac{\partial \theta}{\partial z}(y, z, t) = \pi b(t) \cos \pi z \cos k\pi y + 2\pi c(t) \cos 2\pi z + \ldots \tag{26}$$

Recalling the definition of the Jacobian:

$$J(f,g) = \begin{vmatrix} \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \end{vmatrix} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}, \tag{27}$$

we also obtain:

$$J(\xi,\psi) = \frac{1}{4}k\pi^4(1+k^2)a^2(t)[\sin 2\pi z \sin 2k\pi y - \sin 2\pi z \cos 2k\pi y] + ... \tag{28}$$

$$J(\theta,\psi) = -\frac{1}{2}k\pi^2 a(t)b(t)[\sin 2\pi z \sin^2 k\pi y + \sin 2\pi z \cos^2 k\pi y] - 2k\pi^2 a(t)c(t)\sin \pi z \cos 2\pi z \cos k\pi y + ... \tag{29}$$

In a Galerkin method, the residual error in the equations is minimized by ensuring that the projection of the error on the basis functions retained is zero. The one-term Galerkin expansion of ξ is obtained by multiplying the ξ equation by $\sin \pi z \sin k\pi y$ and integrating in z over $[0,1]$ and integrating in y over $(0, 2/k)$, and gives after simplifications:

$$\frac{da}{dt} = -\mathrm{Pr}\,\pi^2(1+k^2)a(t) - \frac{k\pi}{\pi^2(1+k^2)}\mathrm{Pr}\,\mathrm{Ra}\,b(t). \tag{30}$$

Similarly, the two-term Galerkin expansion of θ gives:

$$\frac{db}{dt} + k\pi^2 a(t)c(t) = -\pi^2(1+k^2)b(t) - k\pi a(t), \tag{31}$$

$$\frac{dc}{dt} - \frac{1}{2}k\pi^2 a(t)b(t) = -4\pi^2 c(t). \tag{32}$$

This system of three coupled nonlinear ODEs is (in a rescaled form) the set introduced by Lorenz.

A. Rescaling

We rescale the equations as follows. Define:

$$\tau = \pi^2(1+k^2)t, \quad \hat{b}(\tau) = -\frac{k\mathrm{Ra}\,b(t)}{\pi^3(1+k^2)^2}, \quad \hat{c}(\tau) = -\frac{k^2\mathrm{Ra}\,c(t)}{\pi^3(1+k^2)^3}, \tag{33}$$

which gives:

$$\frac{da}{d\tau} = -\sigma a + \sigma\hat{b} \quad \text{where} \quad \sigma \equiv \mathrm{Pr}, \tag{34}$$

$$\frac{d\hat{b}}{d\tau} = -\hat{b} + ra - a\hat{c} \quad \text{where} \quad r \equiv k^2\mathrm{Ra}/\pi^4(1+k^2)^3, \tag{35}$$

$$\frac{d\hat{c}}{d\tau} = -s\hat{c} + \frac{k^2}{2(1+k^2)^2}a\hat{b} \quad \text{where} \quad s = 4/(1+k^2). \tag{36}$$

Rescaling further as

$$X = \frac{k}{\sqrt{2}(1+k^2)}a(\tau), \quad Y = \frac{k}{\sqrt{2}(1+k^2)}\hat{b}(\tau), \quad Z = \hat{c} \tag{37}$$

gives the standard form of the Lorenz equations:

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \tag{38}$$

$$\frac{dY}{d\tau} = -Y + rX - XZ, \tag{39}$$

$$\frac{dZ}{d\tau} = -sZ + XY. \tag{40}$$