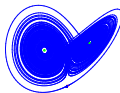# THE LORENZ SYSTEM   Math118, O. Knill

ABSTRACT. In this lecture, we have a closer look at the Lorenz system.

THE LORENZ SYSTEM. The differential equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz .$$

are called the Lorenz system. There are three parameters. For $\sigma = 10, r = 28, b = 8/3$, Lorenz discovered in 1963 an interesting long time behavior and an aperiodic "attractor". The picture to the right shows a numerical integration of an orbit for $t \in [0, 40]$.

DERIVATION. Lorenz original derivation of these equations are from a model for fluid flow of the atmosphere: a two-dimensional fluid cell is warmed from below and cooled from above and the resulting convective motion is modeled by a partial differential equation. The variables are expanded into an infinite number of modes and all except three of them are put to zero. One calls this a Galerkin approximation. The variable x is proportional to the intensity of convective motion, y is proportional to the temperature difference between ascending and descending currents and z is proportional to the distortion from linearity of the vertical temperature profile. The parameters $\sigma > 1, r > 0, b > 0$ have a physical interpretation. σ is the Prandl number, the quotient of viscosity and thermal conductivity, r is essentially the temperature difference of the heated layer and b depends on the geometry of the fluid cell.

SYMMETRIES. The equations are invariant under the transformation $S(x, y, z) = (-x, -y, z)$. That means that if $(x(t), y(t), z(t))$ is a solution, then $(-x(t), -y(t), z(t))$ is a solution too.
If $(x_0, y_0, z_0) = (0, 0, z_0)$, then the equations are $\dot{z} = -bz$. Therefore, we stay on the z axes and to the equilibrium point $(0, 0, 0)$.

VOLUME. The Lorenz flow is dissipative: indeed, the divergence of F is negative. The flow contracts volume.

$$\text{div}(F) = -1 - \sigma - b$$

A TRAPPING REGION.

A region Y in space which has the property that if $x(t) \in Y$ then for all $s > t$ also $x(s) \in Y$ is called a **trapping region**. A function, which is nondecreasing along the flow is also called a **Lyapunov function**. Don't confuse this with the **Lyapunov exponent**.
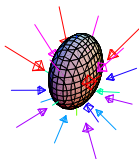
LEMMA. There exists a bounded ellipsoid E which is a trapping region for the Lorenz flow. The time-one map T of the Lorenz flow maps E into the interior of E.

PROOF. We show that the function $V = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$ is a Lyapunov function outside some ellipsoid. Indeed, the time derivative satisfies

$$\dot{V} = -2\sigma(rx^2 + y^2 + bz^2 - 2brz) .$$

Define $D = \{\dot{V} \geq 0\}$. This is a bounded region. If c the maximum of V in D and $E = \{V \leq c + \epsilon\}$ for some $\epsilon > 0$. then E is a region containing D. Outside this ellipsoid E, we have $\dot{V} \leq -\delta$ for some positive δ. With an initial condition $\vec{x}_0$ outside E, the vlaue of $V(x(t))$ decreases and within finite time, the trajectory will enter the ellipsoid E. All trajectories pass inwards through the boundary of E so that a trajectory which is once within E, remains there forever.

GLOBAL EXISTENCE. Remember that nonlinear differential equations do not necessarily have global solutions like $d/dt x(t) = x^2(t)$. If solutions do not exist for all times, there is a finite τ such that $|x(t)| \to \infty$ for $t \to \tau$.

LEMMA. The Lorenz system has a solution $x(t)$ for all times.

Since we have a trapping region, the Lorenz differential equation exist for all times $t > 0$. If we run time backwards, we have $\dot{V} = 2\sigma(rx^2 + y^2 + bz^2 - 2brz) \leq cV$ for some constant c. Therefore $V(t) \leq V(0)e^{ct}$.

THE ATTRACTING SET. The set $K = \bigcap_{t>0} T_t(E)$ is invariant under the differential equation. It has zero volume and is called the **attracting set** of the Lorenz equations. It contains the unstable manifold of O.

EQUILIBRIUM POINTS. Besides the origin $O = (0, 0, 0$, we have two other equilibrium points. $C^\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. For $r < 1$, all solutions are attracted to the origin. At $r = 1$, the two equilibrium points appear with a **period doubling bifurcation**. They are stable until some parameter r^*. The picture to the right shows the unstable manifold of the origin for $\sigma = 10, b = 8/3, r = 10$ which end up as part of the stable manifold of the two equilibrium points.

HYPERBOLICITY IN THREE DIMENSIONS. An equilibrium point is called **hyperbolic** if there are no eigenvalues on the imaginary axes. This is quite a wide notion and includes attractive or repelling equilibrium points as well as the possibility to have a one dimensional stable and two dimensional unstable direction or a two dimensional stable and a one dimensional unstable direction.
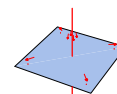
THE JACOBEAN. The Lorenz differential equations $\dot{x} = F(x)$ has the Jacobean $DF(x, y, z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$.

THE ORIGIN. At the equilibrium point $(0, 0, 0)$, the Jacobean $DF(0, 0, 0)$ is block diagonal. The eigenvalues are $-b, \frac{-1-s\pm\sqrt{(1-s)^2+4rs}}{2}$. For $r < 1$, where $\sqrt{(1-s)^2 + 4rs} < (1+s)$, all three eigenvalues are negative. For $r > 1$, we have one positive eigenvalues and two negative eigenvalue. To the positive eigenvalue belongs an unstable manifold which is part of the Lorenz attractor.
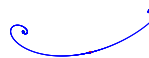
THE TWO OTHER POINTS. At the two other equilibrium points, the eigenvalues are the roots of a polynomial of degree 3. For $\sigma > b + 1$ and $1 < r < r^* = (\sigma(\sigma + b + 3))/(\sigma - b - 1)$, all eigenvalues have negative a real part and the two points $C^\pm$ are stable. At $r = r^*$, a **Hopf bifurcation** happens. The two stable points $C^\pm$ collide each with an unstable cycle and become unstable. For $\sigma = 10, b = 8/3$ we have $r^* = 470/19 = 24.7$.
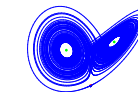
PERIODIC ORBITS. For large r parameters, the attractor can be single periodic orbit. Known windows are $99.534 < r < 100.795, 145.96 < r < 166.07, 214.364 < r < \infty$. Some periodic solutions are knots.

LYAPUNOV EXPONENTS OF DIFFERENTIAL EQUATIONS. If $T_t(x_0) = x_t$ is the time t map defined by the differential equation $\frac{d}{dt}x = F(x)$, then

$$\lambda(F, x) = \lim_{t\to\infty} \frac{1}{t}\log||DT_t(x)||$$

is called the **Lyapunov exponent** of the orbit. It is always ≥ 0. The Lyapunov exponent is for non-periodic orbits only accessible numerically.