THE VOLTERRA GYROSTAT: A JOURNEY FROM THE CLASSICAL EULER GYROSCOPE TO CONTEMPORARY GEOPHYSICAL LOW-ORDER MODELS

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In this paper, the author reviews results on the dynamics and integrability of Volterra gyrostats from the dynamical systems viewpoint. Recall that the Volterra gyrostat is a gyrostat for which the distribution of relative velocities does not vary. In particular, the Euler gyroscope is a Volterra gyrostat with no gyrostatic motion. In the 70's Obukhov suggested that systems of coupled Euler gyroscopes could model homogeneous fluid motions. In recent years, Gluhovsky noticed that, in order to preserve the physics of the original equations, the corresponding geophysical low-order models should be coupled Volterra gyrostats.

Keywords: Volterra gyrostat, geophysical low-order model, dynamical system, first integral, Hamiltonian.

1. INTRODUCTION

The Volterra gyrostat [35] is a rigid body with an axi-symmetric rotor that rotates with a constant angular velocity about an axis fixed in the carrier. If we denote the principal moments of inertia of the gyrostat by I_i, i = 1, 2, 3, the components of the angular velocity vector by ω_i, i = 1, 2, 3, and the components of the fixed angular momentum caused by the relative motion of the rotor by h_i, i = 1, 2, 3, then the equations of the Volterra gyrostat are the following:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + h_2\omega_3 - h_3\omega_2, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + h_3\omega_1 - h_1\omega_3, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + h_1\omega_2 - h_2\omega_1. \end{aligned} \tag{1}$$

The Volterra gyrostat is a generalization of the free rigid body, introduced by Euler in 1765 (see e.g. [8,2]). Indeed, in the absence of the linear gyrostatic terms ($h_i = 0$, i = 1, 2, 3), system (1) becomes:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2. \end{aligned} \tag{2}$$

System (2), describing the dynamics of the Euler (free) rigid body (the Euler gyroscope), can be also be written [8] equivalently in terms of the angular momenta , i = 1, 2, 3:

$$\begin{aligned} \dot{M}_1 &= a_1 M_2 M_3, \\ \dot{M}_2 &= a_2 M_3 M_1, \\ \dot{M}_3 &= a_3 M_1 M_2, \end{aligned} \tag{3}$$

where $a_1 = (I_2 - I_3)/I_2 I_3$, $a_1 = (I_2 - I_3)/I_2 I_3$, $a_1 = (I_2 - I_3)/I_2 I_3$.

In the last ten years the Volterra gyrostat has become an important tool in modeling energy conserving geophysical low order models [10-13]. In this paper we go back to its origins and give a short review of its journey from the classical Euler gyroscope to contemporary geophysical models.

2. RESULTS ON THE EULER GYROSCOPE

There is a rich and growing literature on the Euler gyroscope (see e.g. [16,2,3,10-13,21-25,27] and in Romanian [14,30,31,33,34]). The fact that the three-dimensional system (2) has the two independent first integrals

$$K = \sum I_i \omega_i^2 / 2 , \tag{4}$$

and

$$C = \sum \left(I_i \omega_i \right)^2 , \tag{5}$$

allows one to completely describe the qualitative behavior its solutions [1]. The solutions of system (2) can be described both geometrically and analytically [16].

The Lagrangian form of system (2)

$$\frac{d}{dt}\frac{\partial L}{\partial \bar{\omega}} = \frac{\partial L}{\partial \bar{\omega}} \times \bar{\omega} ,$$

where $\bar{\omega} = (\omega_1, \omega_2, \omega_3)$ is the body angular velocity vector, $L = \sum I_i \omega_i^2 / 2 = K$ of (4), and $\frac{\partial L}{\partial \bar{\omega}} = \left(\frac{\partial L}{\partial \omega_1}, \frac{\partial L}{\partial \omega_2}, \frac{\partial L}{\partial \omega_3} \right)$, appears explicitly in [15] and [29].

The Hamiltonian form of system (2) is:

$$\dot{M}_i = \left[M_i , H \right] , \tag{6}$$

where $M_i = I_i \omega_i$, (i = 1, 2, 3), $H = K$ given by (4) and

$$\left[f , g \right] = -\varepsilon_{ijk} \left(\partial f / \partial M_i \right) \left(\partial g / \partial M_j \right) M_k = -\bar{M} \cdot \left(\nabla f \times \nabla g \right) ,$$

is the Lie-Poisson bracket. The function C of (5) is a Casimir function, because it satisfies the equation $\left[C, f \right] = 0$ for any function $f\left(M_1, M_2, M_3 \right)$ (see [26] for definitions of Hamiltonian system, Lie-Poisson bracket and Casimir functions).

3. FROM THE EULER GYROSCOPE TO GEOPHYSICAL LOW ORDER MODELS

In the 1950s a general method of studying Partial Differential Equations (PDEs) became popular in fluid dynamics [9], by which the qualitative study of the initial PDEs is reduced to the study of a finite-dimensional system of ordinary differential equations (ODEs), called low-order model (LOM). The LOM is obtained from the initial PDEs by the Galerkin projection technique and by retaining only a finite number of projections [19].

In 1960 Lorenz [17] applied this method to a PDE called the barotropic vorticity equation, which approximately governs the vertically –averaged horizontal flow in the atmosphere, in order to show that the even the simplest resulting LOM still captures some nonlinear properties of the original PDE.

More precisely, Lorenz derived the LOM known as the minimum hydrodynamic equations:

$$\begin{aligned}\dot{x} &= -\left(\frac{1}{k^2}-\frac{1}{k^2+m^2}\right)kmyz,\\ \dot{y} &= \left(\frac{1}{m^2}-\frac{1}{k^2+m^2}\right)kmzx,\\ \dot{z} &= -\frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{k^2}\right)kmxy,\end{aligned} \tag{7}$$

describing the iteraction between a zonal flow, of amplitude *y(t)*, and a large-scale wave disturbance, of amplitude *x(t)* and phase *z(t)*; the constants k and m define the wavelength of the disturbance $2\pi/k$ and the distance $2\pi/m$ between successive zonal wind maxima.

Note that system (7) reminds of the Euler gyroscope (3), but is different from it because of the ½ factor. Lorenz [17] chose to study system (7) by using the analytical form of its solutions (elliptic functions of time t) and by integrating (7) numerically. However, in [7] we determined the phase portrait of system (7) by first rescaling it, and then, by using two of its first integrals (the Manley-Rowe quantities of (7)). If $k > m$ in (7), then the rescaled form of system (7) is the following:

$$\begin{aligned}\dot{x} &= -yz,\\ \dot{y} &= zx,\\ \dot{z} &= -xy,\end{aligned} \tag{8}$$

and its Manley-Rowe quantities are

$$H_1 = x^2 + y^2, \text{ and } H_2 = y^2 + z^2.$$

Lorenz and Krishnamurthy [20] found the one-parameter ε family of exact solution

$$x = \varepsilon\,\mathrm{sech}(\varepsilon t),\ y = \varepsilon\tanh(\varepsilon t),\ z = -\varepsilon\,\mathrm{sech}(\varepsilon t)$$

of system (8), while Boyd [4] gave the three-parameter (ε, L, ϕ) family of solutions of system (8):

$$\begin{aligned}x &= \varepsilon\sum_{m=-\infty}^{\infty}\mathrm{sech}\left(\varepsilon[t+\phi-mL]\right),\\ y &= -\varepsilon\sum_{m=-\infty}^{\infty}(-1)^m\tanh\left(\varepsilon[t+\phi-mL]\right),\\ y &= -\varepsilon\sum_{m=-\infty}^{\infty}(-1)^m\,\mathrm{sech}\left(\varepsilon[t+\phi-mL]\right).\end{aligned}$$

Another geophysical context in which the Euler gyroscope (7) occurs is that of resonant Rossby waves. Recall that Rossby (or planetary) waves are large-scale meanders of the jet stream and play an important role in determining the weather at midlatitudes [21]. At resonance, three Rossby waves can interact such that any two of them produce the third one. Their amplitudes $a_1(t)$, $a_2(t)$, and $a_3(t)$ satisfy [28, 21] the differential system:

$$\dot{a}_1 + \frac{B_1}{K_1^2 + F} a_2 a_3 = 0,$$
$$\dot{a}_2 + \frac{B_2}{K_2^2 + F} a_3 a_1 = 0, \tag{9}$$
$$\dot{a}_3 + \frac{B_3}{K_3^2 + F} a_3 a_1 = 0,$$

where the interaction coefficients B_i, i = 1,2,3 are $B_1 = 1/2\left(k_2 l_3 - k_3 l_2\right)\left(K_2^2 - K_3^2\right)$ with B_2, and B_3 defined by cyclic permutation of the indices. For each i = 1,2,3, K_i^2, the square of the total wavenumber of the wave amplitude $a_i\left(t\right)$, is given by $K_i^2 = k_i^2 + l_i^2$; the parameter F is the inverse of the Rossby radius of deformation. Assuming that the waves are ordered such that the third one has an intermediate horizontal scale i.e. $k_1 < k_3 < k_2$ and $l_1 < l_3 < l_2$, system (8) can be written [21], after a rescaling, under the form:

$$\begin{aligned} \dot{x} &= -yz, \\ \dot{y} &= -zx, \\ \dot{z} &= \quad xy. \end{aligned} \tag{10}$$

Clearly, the phase portraits of systems (10) and (8) are the same.

Note that system (3) can be generalized to the n-component Euler gyroscope

$$\begin{aligned} \dot{M}_1 \quad &= (1/I_3 - 1/I_2) M_2 M_3, \\ \dot{M}_2 \quad &= (1/I_4 - 1/I_3) M_3 M_4, \\ &\dots \\ \dot{M}_{n-1} &= (1/I_1 - 1/I_n) M_n M_1, \\ \dot{M}_n \quad &= (1/I_2 - 1/I_1) M_1 M_2. \end{aligned}$$

Obukhov [23, 24] showed that the simplest hydrodynamic-type system is equivalent to the Euler gyroscope (2). Obukhov [25] also suggested systems of coupled Euler gyroscopes for modeling homogeneous flows.

Gluhovsky and his collaborators showed that low order models arising in fluid dynamics and atmospheric problems are equivalent to coupled Volterra gyrostats. For example, Gluhovsky and Agee [11] showed that the famous Lorenz-63 model [18]

$$\begin{aligned} \dot{x} &= s\left(y - x\right), \\ \dot{y} &= -y + rx - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{11}$$

may be considered as a Volterra gyrostat

$$\begin{aligned} I_1 \dot{\omega}_1 &= \left(I_1 - I_3\right) \omega_2 \omega_3 - \alpha_1 \omega_1 + N_1, \\ I_1 \dot{\omega}_2 &= \left(I_3 - I_1\right) \omega_3 \omega_1 - h_1 \omega_3 - \alpha_2 \omega_2, \\ I_3 \dot{\omega}_3 &= h_1 \omega_2 - \alpha_3 \omega_3, \end{aligned} \tag{12}$$

with one pair of nonlinear terms (the terms $\left(I_1 - I_3\right)\omega_2\omega_3$ and $\left(I_3 - I_1\right)\omega_3\omega_1$) and one pair of linear terms (the terms $-h_1\omega_3$ and $h_1\omega_2$), in a forced (the N_1 term) dissipative regime (the terms $-\alpha_i\omega_i$, i = 1,2,3).

The three-dimensional analog of the Lorenz model [10]

$$\begin{array}{l|l|l|}
\dot{x}_1 = f - x_1 & -x_2 x_3 & -x_4 x_5 \\
\dot{x}_2 = -\alpha_2 x_2 & +x_3 x_1 - x_3 & \\
\dot{x}_3 = -\alpha_3 x_3 & +x_2 & \\
\dot{x}_4 = -\alpha_4 x_4 & & +x_5 x_1 - x_5 \\
\dot{x}_5 = -\alpha_5 x_5 & & +x_4
\end{array} \tag{13}$$

is a coupling between two Volterra gyrostats of type (12).

The LOM for quasigeostrophic atmospheric circulation at midlatitudes developed by Charney and DeVore in 1979 [5] also has a modular structure of coupled Volterra gyrostats [13]. Dobrovolschi and Llibre [6] studied the three-dimensional Charney DeVore model from the dynamical systems viewpoint. In the conservative case, the system is reduced to systems without parameters. Among these, two of them are interesting: one is homeomorphic to the Euler gyroscope and the other is a perturbation of it. In the dissipative case, all trajectories enter into an ellipsoid for any values of the parameters.

The Volterra gyrostat (1) is an integrable system, in the sense that it has two independent first integrals: the kinetic energy invariant (4), and the square of the angular momentum invariant,

$$C = \sum \left(I_i \omega_i + h_i \right)^2 . \tag{14}$$

System (1) is Hamiltonian (Gluhovsky et al. 2002, Gluhovsky 2006), because it can be written in the form (6) with $M_i = I_i \omega_i$, (i = 1, 2, 3), $H = K$ given by (4), and

$$[f, g] = -\varepsilon_{ijk} \left(\partial f / \partial M_i \right) \left(\partial g / \partial M_j \right) \left(M_k + h_k \right)$$

is the Lie-Poisson bracket. The function C given by (14) is a Casimir function of (1).

4. METHODS OF STUDY

The local stability type of the equilibrium points of the Euler gyroscope was studied in [34] by using *linear stability analysis* in the case of hyperbolic equilibrium points, and by constructing suitable *Lyapunov functions* in the non-hyperbolic case. The nature of the equilibrium points of system (3) can also be studied by determining the intersection of the level sets of the *first integrals* of the system [6,7]. This geometric approach allows to completely determine the phase portrait of the system. *Rescalings* and *changes of variables* simplify the problem of determining the phase portrait of a differential system by replacing the initial system with a system with the same dynamical behaviour but with fewer parameters.

5. CONCLUSIONS

The rescaling technique is simple, but powerful. In our case, it reveals various physical contexts in which the Euler gyroscope occurs. Consequently, the properties are transferable from one system to its equivalent (e.g. the precession of the swinging spring holds also for the Rossby triad [21]).

The geometric method used for determining the phase portrait of various three-dimensional geophysical low order models depends strongly on the simplicity of the first integrals and of the dimension of the phase space. An open problem is to find methods suitable for determining the phase portrait of differential systems with the number of dimensions higher than three.

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