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SELF-SIMILAR SOLUTIONS FOR A MODIFIED BROADWELL MODEL AND ITS FLUID-DYNAMIC LIMITS*

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Abstract. The existence of self-similar solutions of the Riemann problem for a modified Broadwell model is established. Regularity estimates at the singular points of the problem are obtained. The passing of the fluid-dynamic limit is justified, which yields the Riemann problem for a system of conservation laws.

Key words. Broadwell model, Riemann problem, fluid-dynamic limits, kinetic theory

AMS subject classifications. Primary 82C40, 82C; Secondary 76P05

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1. Introduction.

In this paper, we study the Riemann problem of the Broadwell model and its fluid-dynamic limits. The Broadwell model, proposed by Broadwell [B], is a system of equations

$$\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} = \frac{1}{\epsilon}(f_3^2 - f_1 f_2),$$

$$\frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} = \frac{1}{\epsilon}(f_3^2 - f_1 f_2), \tag{1.1}$$

$$\frac{\partial f_3}{\partial t} = \frac{1}{2\epsilon}(f_1 f_2 - f_3^2)$$

that provides a simple statistical description of a gas of interacting particles. Here the functions f_1 and f_2 are the densities of particles moving in positive and negative x-directions, respectively, and f_3 is the density of particles moving in each of the positive or negative of y- or z-directions. The mean free path ϵ is the measure of average distance between successive collisions. An important feature of this system of equations is its asymptotic equivalence in small mean free path ϵ to the Euler of compressible fluid dynamics

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho g(u))_x &= 0, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} \rho &= (f_1 + f_2 + 4(f_1 f_2)^{1/2}), \\ m &= \rho u = f_1 - f_2, \\ g(u) &= \frac{1}{3}[2(1+3u^2)^{1/2} - 1]. \end{aligned} \tag{1.3}$$

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The justification of passing the fluid-dynamic limit in Boltzmann equation or some models of Boltzmann equation has been studied by several authors. The reader is referred to Cercignani [Ce] for a survey of the literature of the Boltzmann equation and to Platkowski and Illner [PI] for results on discrete models of kinetic theory. Recently, Bardos, Golse, and Levermore [BGL] proved the validity of the fluid-dynamic limit of the Boltzmann equation to the incompressible Navier–Stokes equations under some hypothesis. For the Broadwell model, Caflisch and Papanicolaou [CP] and Caflisch [Ca] showed that a given smooth solution of the limit equation can be approximated by a solution of the Broadwell model when ϵ is small. For solutions with shocks of the Broadwell model, there are studies on the stability in time for traveling waves [CL], [KM] and rarefaction waves [Ma]. Xin [X] proved that a given piecewise smooth solution with noninteracting shocks of the limit fluid equations can be approximated by solutions of the Broadwell model as $\epsilon \to 0+$. Recently, Liu and Xin [LX] studied the boundary-layer problems for the Broadwell model and revealed some interesting phenomena. They found that there exist boundary layers in the Broadwell model due to purely kinetic effects that cannot be detected by Chapman–Enskog expansion on the viscous level. They classified the boundary layers as compressive and expansive and showed that expansive boundary layers are stable while compressive boundary layers are stable before they leave the boundary. They also obtained the optimal rate of convergence in the L^∞-norm of kinetic solutions to fluid-dynamic solution in terms of ϵ if the interior fluid flow is smooth. Many of above-mentioned works belong to the approximation program, meaning that an admissible solution of the limit equation, which is a system of conservation laws, is used to construct solutions of the Broadwell model and is intended as a method to solve the Broadwell model. Another approach is to construct solutions of the limit conservation laws as the limit of solutions of the Broadwell model. Recently, Slemrod and Tzavaras [ST], [T] studied the self-similar fluid-dynamic limits of a modified Riemann problem of (1.1) with Maxwellian Riemann data. Work done by Chen and Liu [ChL] and Chen et al. [CLL] on relaxation also sheds light on this subject.

In an attempt to gain insight in the latter direction, Slemrod and Tzavaras [ST] studied the self-similar dynamic-limit approach. For Maxwellian Riemann data

$$f(x,0) = \begin{cases} f_+, & x > 0, \\ f_-, & x < 0, \end{cases} \tag{1.4a}$$

$$Q(f_+) = 0, \qquad Q(f_-) = 0, \quad f_{1\pm}, f_{2\pm}, f_{3\pm} > 0, \tag{1.4b}$$

where

$$Q(f) = f_3^2 - f_1 f_2, \tag{1.4c}$$

the solutions of the limit equation (1.2) are expected to be self-similar functions of $\xi = x/t$. Motivated by this reasoning, they considered the modified Broadwell system

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\epsilon t}(f_3^2 - f_1 f_2), \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\epsilon t}(f_3^2 - f_1 f_2), \\ \frac{\partial f_3}{\partial t} &= \frac{1}{2\epsilon t}(f_1 f_2 - f_3^2). \end{aligned} \tag{1.5}$$

By making the ansatz $f(x,t) = f(x/t)$ in (1.4) and (1.5), the Riemann problem (1.4)–(1.5) becomes a singular boundary-value problem:

$$
\begin{aligned}
(1-\xi)f_1' &= \frac{Q(f)}{\epsilon},\\
-(1+\xi)f_2' &= \frac{Q(f)}{\epsilon},\\
\xi f_3' &= \frac{Q(f)}{2\epsilon},\\
f(-1) = f_-, &\qquad f(+1) = f_+,\\
Q(f_+) = 0, &\qquad Q(f_-) = 0, \quad f_{1\pm}, f_{2\pm}, f_{3\pm} > 0,
\end{aligned}
\tag{1.6}
$$

for $\xi \in [-1,1]$. They proved that the total variations of solutions of (1.6) are bounded uniformly in ϵ, and hence there is a sequence $\{f^{\epsilon_n}\}$, $\epsilon_n \to 0+$, such that $f^{\epsilon_n} \to f$ almost everywhere, where f is a weak solution of the Riemann problem (1.2), (1.4). However, the existence of solutions of (1.6) was proved only in the case where $f_{1-} < f_{1+}$, $f_{2-} > f_{2+}$, and $f_{3-} = f_{3+}$. Under these assumptions on the initial data, the limit solution will be continuous and hence precludes the case where shocks are present.

In this paper, we shall prove the existence of a positive continuous solution of (1.6) and hence (1.4)–(1.5) with no restrictions attached. We also prove some regularity estimates for solutions of (1.6). The precise statement of these results is as follows.

THEOREM 1.1. *There exists a positive continuous solution of* (1.6). *Moreover, any positive solution $f(\xi)$ of* (1.6) *satisfies the following estimates:*

$$f_1(\xi) - f_{1+} = O(1)(1-\xi)^{\frac{f_{2+}}{\epsilon}} \quad \text{for } \xi \text{ near } \xi = 1, \tag{1.7a}$$

$$f_2(\xi) - f_{2-} = O(1)(\xi+1)^{\frac{f_{1-}}{\epsilon}} \quad \text{for } \xi \text{ near } \xi = -1, \tag{1.7b}$$

and

$$f_3(\xi) - f_3(0) = O(1)|\xi|^{\frac{\min(f_{3-}, f_3(0), f_{3+})}{\epsilon}} \quad \text{for } \xi \text{ near } \xi = 0, \tag{1.7c}$$

$$Q(f(\xi)) = O(1)(\xi+1)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2+}}{\epsilon}}|\xi|^{\frac{\min(f_{3-}, f_3(0), f_{3+})}{\epsilon}}. \tag{1.7d}$$

Combining Theorem 1.1 with the results from [ST], we prove the following.

COROLLARY 1.2. *For any Maxwellian Riemann data* (1.4), *there is a solution of* (1.4)–(1.5) $f^\epsilon(x/t)$. *Further, there is a sequence of solutions of* (1.4)–(1.5), $\{f^{\epsilon_n}\}$, $\epsilon_n \to 0+$, *such that f^{ϵ_n} converges almost everywhere to a weak solution of the limit equations* (1.2) *and* (1.4).

Remark. It is interesting to see that for system (1.1), the discontinuity at $x = 0$ at initial time will propagate along $x/t = \pm 1$ and $x = 0$, which are characteristics of (1.1), while for the modified system (1.5), there is no discontinuity at these locations. In fact, the discontinuities of (1.1) at $x/t = \pm 1$ and $x = 0$ are not intrinsic in the sense that in the limit $\epsilon \to 0+$, there is no shock at $x/t = \pm 1$ due to the stability condition, and there may not be a shock at $x = 0$ in the limit system (1.2).

We organize this paper as follows. In section 2, we recall some results from [ST] and state the main results of this paper. In section 3, we prove that (1.6) has a continuous positive solution. The method we use is a kind of shooting argument. Finally, we prove that for some sequence $\{\epsilon_n\}$, $\epsilon_n \to 0+$ as $n \to \infty$, the solutions of (1.6), f^{ϵ_n}, converge almost everywhere to a weak solution of (1.2), (1.4).

2. Preliminaries. We see that there are three singular points, $\xi = \pm 1, 0$, in the boundary-value problem (1.6). At $\xi = \pm 1$, two components of f are continuous. Thus the other component must also be continuous due to the boundary condition. Therefore, in what follows, we define the solutions of (1.6) to be weak solutions of (1.6) which are continuous on $[-1, 0) \cup (0, 1]$.

We recall some results on (1.6) obtained in [ST].

LEMMA 2.1. *Let* $f = (f_1, f_2, f_3)$ *be a continuous solution of* (1.6). *Then*

(i) $Q(f(\xi))$ *does not change sign on the intervals* $(-1, 0)$ *or* $(0, 1)$,

(ii) $Q(f(-1)) = Q(f(0)) = Q(f(+1)) = 0$, *and*

(iii) f_1, f_2, *and* f_3 *are uniformly bounded from above and below by positive constants and of uniformly bounded total variation on* $[-1, 1]$. *These bounds are independent of* $\epsilon > 0$.

In fact, assertion (i) of Lemma 2.1 does not require the continuity of f. For our later use, we revise it as follows.

LEMMA 2.2. *Let* f *be a solution of* (1.6). *Then* $Q(f(\xi))$ *does not change sign on the interval* $(-1, 0)$ *and* $(0, 1)$. *Furthermore, each component of* f *is monotone on each interval* $(-1, 0)$ *and* $(0, 1)$.

Proof. A straightforward calculation based on (3.1) shows that

$$\frac{dQ}{d\xi} = \frac{1}{\epsilon}\left(\frac{f_1(\xi)}{\xi+1} + \frac{f_2(\xi)}{\xi-1} + \frac{f_3(\xi)}{\xi}\right) Q(\xi), \tag{2.1}$$

from which the assertion follows. ☐

3. Existence of solutions of (1.6). In this section, when we refer to solutions of a system of ordinary differential equations, we always mean continuous solutions unless otherwise indicated. We intend to prove the existence of (1.6) by a kind of shooting argument. For this purpose, we consider the trajectories of

$$\begin{aligned} (1-\xi) f_1' &= \frac{Q(f)}{\epsilon}, \\ -(\xi+1) f_2' &= \frac{Q(f)}{\epsilon}, \\ \xi f_3' &= \frac{Q(f)}{2\epsilon}, \quad \xi \in (-1, 1), \end{aligned} \tag{3.1}$$

issued from

$$(f_{1-}, f_{2-}, f_{3-}) \quad \text{at } \xi = -1 \tag{3.2}$$

and

$$(f_{1+}, f_{2+}, f_{3+}) \quad \text{at } \xi = +1, \tag{3.3}$$

respectively, where $f_{3\pm} = (f_{1\pm} f_{2\pm})^{1/2}$. It is clear that (3.1) has three singular points, $\xi = -1, 0, +1$. We first need some regularity results of trajectories of (3.1) near these points.

LEMMA 3.1.

(i) *Let* f *be a bounded solution of* (3.1) *and* (3.2) *on* $[-1, 0)$. *Then*

(3.4a)

$$\begin{aligned} Q = {} & C(1+\xi)^{\frac{f_{1-}}{\epsilon}} (1-\xi)^{\frac{f_{2+}}{\epsilon}} |\xi|^{\frac{\min(f_{3-}, f_3(0-))}{\epsilon}} \\ & \times \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{f_1(\zeta) - f_{1-}}{\zeta+1} + \frac{f_2(\zeta) - f_{2+}}{\zeta-1} + \frac{f_3(\zeta) - \min(f_{3-}, f_3(0-))}{\zeta}\right) d\zeta\right]. \end{aligned}$$

In fact, the numbers f_1*,* f_{2+}*, and* $\min(f_{3-}, f_3(0-))$ *in* (3.4a) *can be replaced by any other numbers.*

(ii) *Let* f *be a bounded solution of* (3.1) *and* (3.2) *on* $(0,1]$*. Then*

(3.4b)
$$Q = C(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2+}}{\epsilon}}|\xi|^{\frac{\max(f_{3+}, f_3(0+))}{\epsilon}} \times \exp\left[\frac{1}{\epsilon}\int_1^{\xi}\left(\frac{f_1(\zeta)-f_{1-}}{\zeta+1} + \frac{f_2(\zeta)-f_{2+}}{\zeta-1} + \frac{f_3(\zeta)-\max(f_{3+}, f_3(0+))}{\zeta}\right)d\zeta\right].$$

In fact, the numbers f_{1-}*,* f_{2+}*, and* $\max(f_{3+}, f_3(0+))$ *in* (3.4b) *can be replaced by any other numbers.*

(iii) *Let* f *be a bounded solution of* (3.1) *and* (3.2) *on* $[-1,0)$*. Then*

$$Q \le O(1)|\xi|^{\frac{\min(f_{3-}, f_3(0-))}{\epsilon}}. \tag{3.5a}$$

(iv) *Let* f *be a bounded solution of* (3.1) *and* (3.3) *on* $(0,1]$*. Then*

$$Q \le O(1)|\xi|^{\frac{\max(f_{3+}, f_3(0+))}{\epsilon}}. \tag{3.5b}$$

Proof. (i) Since f_3 is bounded and monotone on each of the intervals $[-1,0)$ and $(0,1]$, the one-sided limits $f_3(0\pm)$ are well defined and finite. Equation (3.4a) can be obtained from (2.1) as follows:

(3.6)
$$\begin{aligned}
Q(f(\epsilon)) &= Q\left(f\left(-\frac{1}{2}\right)\right) \times \exp\left[\frac{1}{\epsilon}\int_{-\frac{1}{2}}^{\xi}\left(\frac{f_1(\zeta)}{\zeta+1} + \frac{f_2(\zeta)}{\zeta-1} + \frac{f_3(\zeta)}{\zeta}\right)d\zeta\right] \\
&= Q\left(f\left(-\frac{1}{2}\right)\right)(\xi+1)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2+}}{\epsilon}}|\xi|^{\frac{\min(f_{3-}, f_3(0-)}{\epsilon}} \\
&\quad \times \exp\left[\frac{1}{\epsilon}\int_{\frac{-1}{2}}^{\xi}\left(\frac{f_1(\zeta)-f_{1-}}{\zeta+1} + \frac{f_2(\zeta)-f_{2+}}{\zeta-1} + \frac{f_3(\zeta)-\min(f_{3-}, f_3(0-))}{\zeta}\right)d\zeta\right] \\
&= C(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2+}}{\epsilon}}|\xi|^{\frac{\min(f_{3-}, f_3(0-))}{\epsilon}} \\
&\quad \times \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{f_1(\zeta)-f_{1-}}{\zeta+1} + \frac{f_2(\zeta)-f_{2+}}{\zeta-1} + \frac{f_3(\zeta)-\min(f_{3-}, f_3(0-))}{\zeta}\right)d\zeta\right].
\end{aligned}$$

The proof of (ii) is similar.

(iii) If $Q \ge 0$ on $[-1,0)$, then f_2 and f_3 are decreasing and

$$f_1' = \frac{Q}{\epsilon(1-\xi)} \le \frac{f_3^2}{\epsilon(1-\xi)} \le \frac{f_{3-}^2}{\epsilon(1-\xi)},$$

and hence the integral in (3.4a) is either negative or bounded. Thus we have the estimate

$$Q \le O(1)|\xi|^{\frac{\min(f_{3-}, f_3(0-))}{\epsilon}}.$$

If $Q < 0$ on $[-1,0)$, then f_1 is decreasing and f_2 is increasing; hence $f_1(\xi) < f_{1-}$ and $f_2(\xi > f_{2-}$ for $\xi \in [-1,0)$. A variation of (3.4a) states that

$$Q = C(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2-}}{\epsilon}}|\xi|^{\frac{\min(f_{3-}, f_3(0-))}{\epsilon}} \times \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{f_1(\zeta)-f_{1-}}{\zeta+1} + \frac{f_2(\zeta)-f_{2-}}{\zeta-1} + \frac{f_3(\zeta)-\min(f_{3-}, f_3(0-)}{\zeta}\right)d\zeta\right].$$

Since every term of the integrand is negative, we also have

$$Q = O(1)|\xi|^{\frac{\min(f_{3-}, f_3(0-))}{\epsilon}}.$$

We can prove (iv) similarly. ∎

Now we need some existence results for the trajectories of (3.1) and (3.2) and those of (3.1) and (3.3). For definiteness, we consider (3.1) and (3.2). The other part can be handled similarly. In view of Lemma 3.1, we let

$$P(\xi) := \frac{Q(f(\xi))}{(1+\xi)^{\frac{f_{1-}}{\epsilon}}}. \tag{3.7}$$

Under this transformation, the initial-value problem (3.1)–(3.2) becomes a regular problem:

$$\begin{aligned} f_1' &= \frac{(1+\xi)^{\frac{f_{1-}}{\epsilon}} P}{\epsilon(1-\xi)}, \\ f_2' &= \frac{-(1+\xi)^{\frac{f_{1-}}{\epsilon}-1} P}{\epsilon}, \\ P' &= \frac{P(\xi)}{\epsilon}\left(\frac{1}{1+\xi}\int_{-1}^{\xi} \frac{(1+s)^{\frac{f_{1-}}{\epsilon}}}{(1-s)\epsilon} P(s)ds \right. \\ &\qquad \left. + \frac{f_2}{\xi-1} + \frac{f_{3-} + \int_{-1}^{\xi} \frac{1}{2\epsilon s}(1+s)^{\frac{f_{1-}}{\epsilon}} P(s)ds}{\xi}\right) \end{aligned} \tag{3.8}$$

with initial conditions

$$f_1(-1) = f_{1-}, f_2(-1) = f_{2-}, P(-1) = P_-. \tag{3.9}$$

LEMMA 3.2. *The system of equations* (3.8)–(3.9) *is equivalent to* (3.1)–(3.2).

Proof. It is clear that (3.1) implies (3.8). To see that (3.8) implies (3.1), we let

$$\begin{aligned} Q &= (1+\xi)^{\frac{f_{1-}}{\epsilon}} P(\xi), \\ f_3 &:= f_{3-} + \int_{-1}^{\xi} \frac{Q}{\epsilon s} ds. \end{aligned} \tag{3.10}$$

A straightforward calculation based on (3.8) yields (3.1) and

$$\begin{aligned} \frac{dQ}{d\xi} &= \frac{1}{\epsilon}\left(\frac{f_1}{\xi+1} + \frac{f_2}{\xi-1} + \frac{f_3}{\xi}\right)Q \\ &= -f_1 f_2' - f_2 f_1' + (f_3^2)' \\ &= \frac{d}{d\xi}(f_3^2 - f_1 f_2), \end{aligned}$$

which implies

$$\begin{aligned} Q &= f_2^2 - f_1 f_2 + f_3^2(-1) - f_1(-1) f_2(-1) \\ &= f_3^2 - f_1 f_2. \end{aligned} \tag{3.11}$$ ∎

The following lemma indicates how $f(\xi, P_-)$ can be extended to $[-1, 0]$.

LEMMA 3.3. *For any* $f_{1-} > 0$, $f_{2-} > 0$, *and* $P_- = (Q(f(\xi))/(1+\xi)^{\frac{f_{1-}}{\epsilon}})\big|_{\xi=-1}$, *problem* (3.8)–(3.9) *has a unique solution on* $[-1, \xi_0)$ *for some* $\xi_0 \in (-1, 0]$.

Proof. System (3.8)–(3.9) is an initial-value problem for systems of regular differential integral equations to which the standard contraction-mapping argument applies, and hence we have the assertion. ∎

For convenience, we shall denote the trajectory of (3.1)–(3.2) that satisfies (3.18) by

$$f(\xi; f_{1-}, f_{2-}, P_-)$$

or by $f(\xi; P_-)$ when no confusion will arise.

LEMMA 3.4. *Let* $f_{1-} > 0$ *and* $f_{2-} > 0$.

(i) *If* $P_- \leq 0$, *then problem* (3.8)–(3.9) *has a unique solution on* $[-1, 0]$.

(ii) *For* $P_- > 0$, *the solution of* (3.8)–(3.9) *exists on* $[-1, 0]$ *if and only if* $f_2(\xi) > 0$ *on* $[-1, 0) \cap$ (*domain of* f).

(iii) *If the solution of* (3.1)–(3.2) *exists on* $[-1, 0]$, *then* f *is positive on* $[-1, 0)$, *i.e., all of the components of* f *are positive on* $[-1, 0)$.

Proof. From the structure of (3.1), we can see that if $Q(f(\xi))$ is bounded on $[-1, 0]$ in supremum norm, then $f(\xi)$ is bounded in $[-1, \xi_1]$ for any $\xi_1 \in (-1, 0)$. Then by the standard argument of continuation of contraction mapping, the existence and uniqueness result of Lemma 3.3 can be extended to $[-1, 0)$.

(i) In the case where $P_- \leq 0$, we have similarly to (3.4a) that

$$Q(f(\xi)) = P_-(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2-}}{\epsilon}}|\xi|^{\frac{f_{3-}}{\epsilon}} \times \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{f_1(\zeta) - f_{1-}}{\zeta+1} + \frac{f_2(\zeta) - f_{2-}}{\zeta - 1} + \frac{f_3(\zeta) - f_{3-}}{\zeta}\right)d\zeta\right] < 0. \tag{3.12}$$

Then equation (3.1) implies that

$$\begin{aligned} f_1(\zeta) &< f_{1-} = f_1(-1), \\ f_2(\zeta) &> f_{2-} = f_2(-1), \\ f_3(\zeta) &> f_{3-} \end{aligned} \tag{3.13}$$

for $\zeta \in (-1, 0]$ and therefore

$$|Q(f(\xi))| \leq |P_-|(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2-}}{\epsilon}}|\xi|^{\frac{f_{3-}}{\epsilon}}.$$

Hence f is bounded in $[-1, 0]$, and thus the solution of (3.8)–(3.9) exists on $(-1, 0)$. Further, because of (3.13), the existence can be extended to $[-1, 0]$.

(ii) In the case where $P_- > 0$, we can derive from (3.12) that $Q(f(\xi)) > 0$ on $[-1, 0]$. Then (3.1) shows that

$$\begin{aligned} f_1(\xi) &> f_{1-} > 0, \\ f_2(\xi) &< f_{2-}, \\ f_3(\xi) &< f_{3-} \end{aligned}$$

for $\xi \in (-1, 0) \cap$ (domain of existence of f). If $f_2(\xi) > 0$ for $\xi \in (-1, 0)$, then

$$f_{3-} > f_3(\xi) > 0, \quad \xi \in (-1, 0),$$

because otherwise equation $(3.1)_3$ shows that at ξ_0, the infimum of points at which $f_3(\xi) = 0$,

$$f_3'(\xi_0) = \frac{-f_1(\xi_0) f_2(\xi_0)}{\epsilon \xi_0} > 0,$$

which cannot be true at ξ_0. Thus we have

$$0 < Q(f(\xi)) < f_3^2(\xi) < f_{3-}^2. \tag{3.14}$$

Conversely, if $f_2(\xi_0) < 0$ for some $\xi_0 \in (-1, 0)$, then

$$f_2'(\xi) = \frac{Q(f)}{\xi} < \frac{-f_1(\xi_0) f_2(\xi_0)}{\xi} < 0,$$

where we have used the monotonicity of f_1 and f_2 on $(-1, 0)$. It can be seen that $f_2(\xi) \to -\infty$ and hence $Q(f(\xi)) \to \infty$ as ξ approaches $0-$. Then the assertion in (ii) follows.

(iii) We first claim that $f_1(\xi) > 0$ on $[-1, 0)$. Indeed, otherwise, there would be a point $\xi_0 \in [-1, 0)$ such that $f_1(\xi_0) = 0$ and $f_1(\xi) > 0$ for $\xi \in [-1, \xi_0)$. Then we have $f_1'(\xi_0) \le 0$. On the other hand, however, equation $(3.1)_1$ implies that $0 \le (1 - \xi_0) f_1'(\xi_0) = Q(f(\xi_0)) = f_3^2(\xi_0) \ge 0$. From (3.4a), we see that $Q(f(\xi)$ remains positive or negative or 0 on $(-1, 0)$, which in our case says that $Q \equiv 0$ on $(-1, 0)$. This implies that $f_1 \equiv f_{1-}$ on $(-1, 0)$, which is a contradiction.

When $P_- \le 0$, our assertion holds in view of (3.13).

When $P_- > 0$, assertion (ii) says that $f_2(\xi) > 0$ on $[-1, 0)$. Further, the proof of (ii) indicates that if $f_2(\xi) > 0$ on $[-1, 0)$, then so does f_3. This completes the proof of (iii). $\square$

THEOREM 3.5. *For any $f_{1-} > 0$ and $f_{2-} > 0$, there is P_0, $0 < P_0 < \infty$, such that problem* (3.8)–(3.9) *has a unique solution on $[-1, 0]$ for $P_- \in (-\infty, P_0]$ and*

$$f_2(0; f_{1-}, f_{2-}, P_0) = 0\,. \tag{3.15}$$

Furthermore, the solution $f(\xi, P_-)$ is positive on $[-1, 0)$ for all $P_- \in (-\infty, P_0]$.

Proof. From Lemma 3.4(i), we see that problem (3.8)–(3.9) has a unique solution on $[-1, 0]$ for all $P_- \le 0$.

Suppose (3.8)–(3.9) has a solution for some $\bar{P}_- > 0$. We claim that (3.8)–(3.9) has a solution for all $P_- \in (-\infty, \bar{P}_-]$. To this end, without loss of generality, we consider only $\bar{P}_- > P_- > 0$. Let

$$\bar{f}(\xi) := f(\xi; f_{1-}, f_{2-}, \bar{P}_-),$$
$$f(\xi) := f(\xi; f_{1-}, f_{2-}, P_-)\,.$$

We claim that

$$Q(\bar{f}(\xi)) > Q(f(\xi)) \tag{3.16}$$

for all $\xi \in (-1, 0)$ if $f(\xi)$ exists. To see this, we recall that

$$\bar{P}(-1) = \frac{Q(f(\xi))}{(1+\xi)^{\frac{f_{1-}}{\epsilon}}}\bigg|_{\xi=-1} = \bar{P}_- > P_- = P(-1) = \frac{Q(f(\xi))}{(1+\xi)^{\frac{f_{1-}}{\epsilon}}}\bigg|_{\xi=-1}.$$

Then there is a $\xi_0 \in (-1, 0]$ such that for $\xi \in (-1, \xi_0)$, the solutions $\bar{P}(\xi)$ and $P(\xi)$ to (3.8)–(3.9), with $\bar{P}(-1) = \bar{P}_-$ and $P(-1) = P_-$, respectively, satisfy

$$\bar{P}(\xi) > P(\xi)$$

and hence

$$Q(\bar{f}(\xi)) > Q(f(\xi)).$$

Without loss of generality, we can assume ξ_0 to be the maximum of ξ_0 in the above statement. Then

$$Q(\bar{f}(\xi_0)) = Q(f(\xi_0)) \tag{3.17}$$

and

$$Q(\bar{f}(\xi)) > Q(f(\xi)) \tag{3.18}$$

for $\xi \in (-1, \xi_0)$. From (3.1), we see that

$$\begin{aligned} \bar{f}_1(\xi) &> f(\xi), \\ \bar{f}_2(\xi) &< f_2(\xi), \\ \bar{f}_3(\xi) &< f_3(\xi) \end{aligned} \tag{3.19}$$

for $\xi \in (-1, \xi_0)$. Using the same technique used to derive (3.4), we obtain

$$\begin{aligned} 0 &= Q(\bar{f}(\xi_0)) - Q(f(\xi_0)) \\ &= \bar{P}_-(1+\xi)^{\frac{f_{1-}}{\epsilon}} \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi_0}\left(\frac{\bar{f}_1(\zeta) - f_{1-}}{\zeta+1} + \frac{\bar{f}_2(\zeta)}{\zeta-1} + \frac{\bar{f}_3(\zeta)}{\zeta}\right)d\zeta\right] \\ &\quad - P_-(1+\xi)^{\frac{f_{1-}}{\epsilon}} \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{f_1(\zeta) - f_{1-}}{\zeta+1} + \frac{f_2(\zeta)}{\zeta-1} + \frac{f_3(\zeta)}{\zeta}\right)d\zeta\right] \\ &= Q(f(\xi_0))\left\{\frac{\bar{P}_-}{P_-}\exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{\bar{f}_1(\zeta) - f_1(\zeta)}{\zeta+1} + \frac{\bar{f}_2(\zeta) - f_2(\zeta)}{\zeta-1}\right.\right.\right. \\ &\qquad\left.\left.\left. + \frac{\bar{f}_3(\zeta) - f_3(\zeta)}{\zeta}\right)d\zeta\right] - 1\right\} > 0, \end{aligned}$$

where in the last inequality we used (3.19), which is a contradiction. This proves that if $\bar{P}_- > P_- > 0$,

$$0 < Q(f(\xi)) < Q(\bar{f}(\xi))\,. \tag{3.20}$$

Then by (3.1) and the assumption that $\bar{f}(\xi)$ is a solution of (3.1)–(3.2) on $[-1, 0]$, we have

$$0 \le \bar{f}_2(\xi) < f_2(\xi),$$

and hence $f(\xi, P_-)$ exists in view of Lemma 3.4(ii).

Now we denote

$$\begin{aligned} P_0 := \sup\{\bar{P}_- \ge 0;\ &(3.8)\text{–}(3.9) \text{ has a solution on} \\ &[-1, 0] \text{ with } P_- = \bar{P}_- \text{ on } [-1, 0]\} \ge 0. \end{aligned} \tag{3.21}$$

Then we have proved that (3.8)–(3.9) has a solution on $[-1,0]$ with $P_- \in (-\infty, P_0)$. It remains to prove that $P_0 < +\infty$ and that $f(\xi, f_{1-}, f_{2-}, P_0)$ exists on $[-1,0]$ and $f_2(0, P_0) = 0$. We claim that

$$\lim_{P_- \to P_0} f(\xi; P) \tag{3.22}$$

exists for all $\xi \in [-1,0]$ and is the solution $f(\xi, f_{1-}, f_{2-}, P_0)$. To this end, we observe that $P_0 \geq 0$. If $-1 \leq P \leq 0$, then the proof of Lemma 3.4(i) states that $f(\xi)$ is bounded on $[-1,0]$. If $P_0 > P > 0$, then by (3.19) we have

$$0 \leq Q(f(\xi; P_-)) < f_3^2(\xi; P_-) < f_{3-}^2, \tag{3.23}$$

and hence $f_1(\xi; f_{1-}, f_{2-}, P_-)$ is bounded uniformly in P_-, and so are f_2 and f_3 since

$$\begin{aligned} &0 < f_2(\xi; P_-) < f_{2-}, \\ &0 < f_3(\xi; P_-) < f_{3-}. \end{aligned} \tag{3.24}$$

Also, f_1, f_2, and f_3 are monotone in ξ on $[-1,0]$ and hence have total variation bounded independently of $P \in [-1, P_0)$, and hence (3.22) exists. The conclusion that the limit is the solution $f(\xi; f_{1-}, f_{2-}, P_0)$ is obvious from the integral form of (3.8). We claim that $P_0 < +\infty$. Indeed, otherwise, we would have similarly to (3.4) that

$$\begin{aligned} Q(f(\xi; P_-)) &= P_-(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2-}}{\epsilon}}|\xi|^{\frac{f_{3-}}{\epsilon}} \\ &\qquad \times \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{f_1(\zeta)-f_{1-}}{\zeta+1} + \frac{f_2(\zeta)-f_{2-}}{\zeta-1} + \frac{f_3(\zeta)-f_{3-}}{\zeta}\right)d\zeta\right] \\ &> P_-(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2-}}{\epsilon}}|\xi|^{\frac{f_{3-}}{\epsilon}} \to \infty \quad \text{ as } P_- \to +\infty \end{aligned}$$

for $\xi \in (-1,0)$, where we used (3.24) when $P_- > 0$. Then from $(3.1)_2$, we have

$$f_2(\xi; P_-) = f_{2-} - \int_{-1}^{\xi} \frac{Q(f(\zeta; P_-))}{\zeta+1} d\zeta \to -\infty \qquad \text{as } P_- \to +\infty \tag{3.25a}$$

for $\xi \in (-1,0)$. On the other hand, we have

$$f_2(\xi, P_0) \geq f_2(0; P_0) = \lim_{P_- \to P_0} f_2(0; P_-) \geq 0 \tag{3.25b}$$

for $\xi \in [-1,0]$. The contradiction between (3.25a) and (3.25b) proves the claim that $P_0 < +\infty$. Finally, we verify that

$$f_2(0, P_0) = 0\,.$$

For contradiction, we assume the contrary, which is, in light of (3.25b), that

$$f_2(0; P_0) > 0. \tag{3.26}$$

Consider $f(\xi; f_{1-}, f_{2-}, P_-)$ for any P_- close to P_0 and $P_- > P_0 \geq 0$. The solution $f(\xi; P_-)$ of (3.8)–(3.9) cannot be defined on $[-1,0]$, for otherwise the definition of P_0, (3.21), would be violated. Then by Lemma 3.4(ii),

$$f_2(\xi_0; f_{1-}, f_{2-}, P_-) = 0 \tag{3.27}$$

for some $\xi_0 = \xi_0(P_-) \in (-1, 0)$. Take a sequence of $\{P_{-,k}\}_{k=1}^{\infty}$ such that $P_{-,k} \to P_0+$ and $\xi_k = \xi_0(P_{-,k}) \to \eta$ as $k \to +\infty$. For any small $\tau > 0$, by the definition of η, $f(\xi; f_{1-}, f_{2-}, P_{-,k})$ exists on $[-1, -\tau + \eta]$. Since the right-hand side of (3.8) is continuous and hence its solution is continuous in initial value (3.9), we have

$$|f(-\tau + \eta, P_0) - f(-\tau + \eta, P_{-,k})| \to 0 \quad \text{as } k \to \infty. \tag{3.28}$$

Since $f_2(\xi)$ is decreasing when $P_- > 0$, we infer from (3.26) that

(3.29), $$f_2(-\tau + \eta, f_{1-}, f_{2-}, P_0) =: f_0 > 0,$$

and hence

$$f_2(-\tau + \eta, P_{-,k}) > \frac{1}{2} f_0 \tag{3.30}$$

for large k. Then the mean-value theorem implies that

$$\begin{aligned} |f_2'(\theta; P_{-,k})| &= \left| \frac{f_2(\xi_0(P_{-,k}), P_{-,k}) - f_2(-\tau + \eta, P_{-,k})}{\xi_0(P_{-,k}) + \tau - \eta} \right| \\ &\geq \frac{|0 - \frac{1}{2} f_0|}{2\tau} = \frac{f_0}{4\tau} \end{aligned} \tag{3.31}$$

for some $\theta \in (-\tau + \eta, \xi_0(P_{-,k})) \subset (-1, 0)$. On the other hand, we also have

$$\begin{aligned} |f_2'(\theta; P_{-,k})| &= \frac{Q(f(\theta; P_{-,k}))}{(\theta + 1)\epsilon} \\ &< \frac{f_3^2(\theta, P_{-,k})}{(\theta + 1)\epsilon} < \frac{f_{3-}^2}{(\theta + 1)\epsilon} < \frac{f_{3-}^2}{(1 - \tau + \eta)\epsilon}, \end{aligned} \tag{3.32}$$

where we have used the fact that $f_2(\xi, P_{-,k}) > 0$ and $0 < f_3(\xi, P_{-,k}) < f_{3-}$ for $\xi \in (-\tau + \eta, \xi_0(P_{-,k}))$. Combining (3.31) and (3.32), we obtain a contradiction:

$$\frac{f_{3-}^2}{(1 - \tau + \eta)\epsilon} > \frac{f_0}{4\tau}, \tag{3.33}$$

where $\tau > 0$ can be arbitrarily small. This contradiction proves (3.15). ☐

We shall establish the Lipschitz-continuous dependence on P_- of $f(\xi, P_-)$. Since the right-hand side of (3.8) is not Lipschitz continuous at $\xi = -1$ and 0, we have to prove it directly.

LEMMA 3.6. *Suppose* $f(\xi) = f(\xi; f_{1-}, f_{2-}, P_-)$ *and* $\bar{f}(\xi) = f(\xi; f_{1-}, f_{2-}, \bar{P}_-)$ *exist on* $[-1, 0]$ *and* $f_2(0-) > 0$ $\bar{f}_2(0-) > 0$. *Then for all* $\xi \in [-1, 0]$,

$$|f(\xi; f_{1-}, f_{2-}, P_-) - f(\xi; f_{1-}, f_{2-}, \bar{P}_-)| \leq C|P_- - \bar{P}_-| \tag{3.34}$$

for some constant $C > 0$.

Proof. By the definition in (3.7), we have

$$P(\xi) = P_- \exp\left[\frac{1}{\epsilon} \int_{-1}^{\xi} \left(\frac{f_1(\zeta) - f_{1-}}{\zeta + 1} + \frac{f_2(\zeta)}{\zeta - 1} + \frac{f_3(\zeta)}{\zeta} \right) d\zeta \right].$$

Consequently,

$$\bar{P}(\xi)-P(\xi)=\frac{P(\xi)}{P_-}\Big\{\bar{P}_-\exp\Big[\frac{1}{\epsilon}\int_{-1}^{\xi}\Big(\frac{\bar{f}_1(\zeta)-f_1(\zeta)}{\zeta+1}+\frac{\bar{f}_2(\zeta)-f_2(\zeta)}{\zeta-1}+\frac{\bar{f}_3(\zeta)-f_3(\zeta)}{\zeta}\Big)d\zeta\Big]-P_-\Big\},$$

where we assumed without loss of generality that $P_-\neq 0$. This leads to

$$\begin{aligned}\bar{P}(\xi)-P(\xi)&=\frac{P(\xi)}{P_-}(\bar{P}_--P_-)\\&\quad+\frac{P(\xi)\bar{P}_-}{P_-}\Big\{\exp\Big[\frac{1}{\epsilon}\int_{-1}^{\xi}\Big(\frac{\bar{f}_1(\zeta)-f_1(\zeta)}{\zeta+1}+\frac{\bar{f}_2(\zeta)-f_2(\zeta)}{\zeta-1}+\frac{\bar{f}_3(\zeta)-f_3(\zeta)}{\zeta}\Big)d\zeta\Big]-1\Big\}\\&=\frac{P(\xi)}{P_-}(\bar{P}_--P_-)\\&\quad+\frac{P(\xi)\bar{P}_-}{P_-}\Big\{\exp\Big[\frac{1}{\epsilon^2}\int_{-1}^{\xi}\int_{-1}^{\zeta}\Big(\frac{(1+s)^{\frac{f_{1-}}{\epsilon}}}{(\zeta+1)(1-s)}+\frac{(1+s)^{\frac{f_{1-}}{\epsilon}-1}}{(\zeta-1)}\\&\quad+\frac{(1+s)^{\frac{f_{1-}}{2}}}{s\zeta}\Big)(\bar{P}(s)-P(s))ds\,d\zeta\Big]-1\Big\}.\end{aligned}\tag{3.35}$$

We denote

$$\alpha=\frac{\min(f_{3-},f_3(0-),\bar{f}_3(0-))}{\epsilon}.$$

By the structure of (3.1), the assumption that $f_2(0-)>0$ and $\bar{f}_2(0-)>0$ implies that $f_3(0-)>0$ and $\bar{f}_3(0-)>0$ and hence that $\alpha>0$. Furthermore, from (3.5), we see that $P(\xi)/\xi^\alpha$ is bounded. We divide (3.35) by ξ^α to obtain

$$\begin{aligned}\Big|\frac{\bar{P}(\xi)-P(\xi)}{\xi^\alpha}\Big|&\le|\bar{P}_--P_-|\Big|\frac{P(\xi)}{P_-\xi^\alpha}\Big|\\&\quad+\Big|\frac{P(\xi)\bar{P}_-}{\xi^\alpha P_-}\Big|\Big\{\exp\Big[\frac{1}{\epsilon^2}\int_{-1}^{\xi}d\zeta\int_{-1}^{\zeta}\Big(\frac{\frac{(1+s)^{f_{1-}}}{2}}{(\zeta+1)(1-s)}\\&\quad+\frac{(1+s)^{\frac{f_{1-}}{\epsilon}}-1}{1-\zeta}-\frac{(1+s)^{\frac{f_{1-}}{2}}}{\zeta}\Big)s^{\alpha-1}\Big|\frac{\bar{P}(s)-P(s)}{s^\alpha}\Big|ds\Big]-1\Big\}\\&\le|\bar{P}_--P_-|\Big|\frac{P(\xi)}{\xi^\alpha P_-}\Big|\\&\quad+\Big|\frac{P(\xi)\bar{P}_-}{\xi^\alpha P_-}\Big|\Big\{\exp\Big[M(\alpha)\int_{-1}^{\xi}\max_{s\in[-1,\zeta]}\Big|\frac{\bar{P}(s)-P(s)}{s^\alpha}\Big|d\zeta\Big]-1\Big\},\end{aligned}\tag{3.36}$$

where $M(\alpha)$ is a constant bounded for $\alpha > \delta > 0$. This yields that for $\xi \in [-1, 0]$,

$$
\begin{aligned}
\max_{s\in[-1,\xi]} \left| \frac{\bar{P}(s) - P(s)}{s^\alpha} \right| &\le |P_- - \bar{P}_-| \max_{s\in[-1,\xi]} \left| \frac{P(s)}{s^\alpha P_-} \right| \\
&\quad + \max_{s\in[-1,\xi]} \left| \frac{P(s)\bar{P}_-}{s^\alpha P_-} \right| \exp\left(M \int_{-1}^{\xi} \max_{s\in[-1,\zeta]} \left| \frac{\bar{P}(s) - P(s)}{s^\alpha} \right| d\zeta \right) \\
&\quad \times \int_{-1}^{\xi} \max_{s\in[-1,\zeta]} \left| \frac{\bar{P}(s) - P(s)}{s^\alpha} \right| d\zeta \\
&\le A + B \int_{-1}^{\xi} \max_{s\in[-1,\zeta]} \left| \frac{\bar{P}(s) - P(s)}{s^\alpha} \right| d\zeta.
\end{aligned} \tag{3.37}
$$

Applying Gronwall's inequality to the above expression, we obtain

$$\max_{s\in[-1,\xi]} \left| \frac{\bar{P}(s) - P(s)}{s^\alpha} \right| \le |\bar{P}_- - P_-| A \exp[B(\xi + 1)]$$

for $\xi \in (-1, 0]$. Then the assertion follows from (3.8), (3.7), and (3.1). □

COROLLARY 3.7. *Let P_0 be as in Theorem 3.5. Then $f(0, f_{1-}, f_{2-}, P_-)$ is Lipschitz continuous in P_- for $P_- \in (-\infty, P_0 - \delta]$ for any $\delta > 0$.*

We see from the last theorem that

$$\big\{(f_1(0; f_{1-}, f_{2-}, P_-), f_2(0; f_{1-}, f_{2-}, P_-)) : P_- \in (-\infty, P_0)\big\} =: C_-(f_{1-}, f_{2-}) \tag{3.38}$$

is a continuous curve in the first quadrant of the (f_1, f_2)-plane for each fixed (f_{1-}, f_{2-}). Taking $P_- = 0$, we see that

$$(f_{1-}, f_{2-}) \in C_-(f_{1-}, f_{2-}). \tag{3.39}$$

We study the range of this curve in the following theorem.

THEOREM 3.8. *Let $C_-(f_{1-}, f_{2-})$ be defined as in (3.36). Then*

$$C_-(f_{1-}, f_{2-}) \subset \big\{(f_1, f_2) \in \mathbb{R}^2 : 0 < f_1 \le f_1(0; P_0), 0 = f_2(0; P_0) \le f_2\big\}. \tag{3.40}$$

Furthermore,

$$f_2(0, P_-) \to +\infty \text{ as } P_- \to -\infty. \tag{3.41}$$

Proof. From Lemma 3.4(ii), we see that $f_2(0; P_-) \ge 0$ for $P_- \in (-\infty, P_0]$. The structure of (3.1) guarantees

$$f_1(0, P_-) > 0.$$

To prove (3.40), it suffices to prove that

$$f_1(0; P_-) \le f_1(0; P_0). \tag{3.42}$$

If $P_- \le 0$, then $Q(f(\xi, P_-)) \le 0$ on $(-1, 0)$. Equation (3.1) then implies that

$$f_1(0; P_-) \le f_{1-} = f_1(-1; 0). \tag{3.43}$$

For $P_- \in [0; P_0]$, we recall that $Q(f(\xi, P_-))$ is monotone increasing with respect to $P_- > 0$; see (3.16). Equation (3.1) then states that $f_1(0, P_-)$ is increasing with respect to $P_- > 0$. Thus

$$f_{1-} \le f_1(0; P_-) \le f_1(0; P_0) \tag{3.44}$$

for $P_- \in [0, P_0]$. Inequality (3.42) follows from (3.43) and (3.44).

To prove (3.41), we let

$$P_- < 0$$

and consider the following application of (3.4).

$$\begin{aligned}
f_2(\xi; P_-) - f_{2-} &= \frac{-1}{\epsilon}\int_{-1}^{\xi} \frac{Q(f(\zeta))}{\zeta+1} d\zeta \\
&= \frac{-P_-}{\epsilon}\int_{-1}^{\xi} (1+\zeta_1)^{\frac{f_{1-}}{\epsilon}-1}(1-\zeta_1)^{\frac{f_{2-}}{\epsilon}}|\zeta_1|^{\frac{f_{3-}}{\epsilon}} \\
\times \exp\Bigg[\frac{1}{\epsilon}\int_{-1}^{\zeta_1}&\left(\frac{f_1(\zeta_2)-f_{1-}}{\zeta_2+1} + \frac{f_2(\zeta_2)-f_{2-}}{\zeta_2-1} + \frac{f_3(\zeta_2)-f_{3-}}{\zeta_2}\right) d\zeta_2\Bigg] d\zeta_1 \\
&= \frac{-P_-}{\epsilon}\int_{-1}^{\xi} (1+\zeta_1)^{\frac{f_{1-}}{\epsilon}-1}(1-\zeta_1)^{\frac{f_{2-}}{\epsilon}}|\zeta_1|^{\frac{f_{3-}}{\epsilon}} \\
\times \exp\Bigg[\frac{1}{\epsilon^2}\int_{-1}^{\zeta_1}\int_{-1}^{\zeta_2}&\left(\frac{1}{(1+\zeta_2)(1-\zeta_3)} + \frac{1}{(1-\zeta_2)(1+\zeta_3)} + \frac{1}{\zeta_2\zeta_3}\right) Q(f(\zeta_3)) d\zeta_3 d\zeta_2\Bigg].
\end{aligned}$$

Invoking (3.13),

$$Q(f(\zeta)) \ge P_-(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2-}}{\epsilon}}|\xi|^{\frac{f_{3-}}{\epsilon}}$$

for $P_- < 0$ in the above, we obtain

$$\begin{aligned}
f_2(\xi) - f_{2-} \ge{}& \frac{-P_-}{\epsilon}\int_{-1}^{\xi} (1+\zeta_1)^{\frac{f_{1-}}{\epsilon}-1}(1-\zeta_1)^{\frac{f_{2-}}{\epsilon}}|\zeta|^{\frac{f_{3-}}{\epsilon}} \\
&\times \exp\Bigg[\frac{1}{\epsilon^2}\int_{-1}^{\zeta_1}\int_{-1}^{\zeta_2}\left(\frac{1}{(1+\zeta_2)(1-\zeta_3)} + \frac{1}{(1-\zeta_2)(1+\zeta_3)} + \frac{1}{\zeta_2\zeta_3}\right) \\
&\times P_-(1+\zeta_3)^{\frac{f_{1-}}{\epsilon}}(1-\zeta_3)^{\frac{f_{2-}}{\epsilon}}|\zeta_3|^{\frac{f_{3-}}{\epsilon}} d\zeta_2 d\zeta_3\Bigg] d\zeta_1 \\
={}& \frac{-P_-}{\epsilon}O(1)\int_{-1}^{\xi}(1+\zeta_1)^{\frac{f_{1-}}{\epsilon}-1}\exp\Bigg\{\frac{P_-}{\epsilon^2}\int_{-1}^{\zeta_1}\int_{-1}^{\zeta_2}\Bigg[O(1)(1+\zeta_3)^{\frac{f_{1-}}{\epsilon}-1} \\
&+ \frac{O(1)(1+\zeta_3)^{\frac{f_{1-}}{\epsilon}}}{1+\zeta_2} + O(1)(1+\zeta_3)^{\frac{f_{1-}}{\epsilon}}\Bigg] d\zeta_2 d\zeta_3\Bigg\} d\zeta_1
\end{aligned}$$

for $\xi \in [-1, -1/2]$. A further calculation of the above yields

$$f_2(\xi) - f_{2-} \ge -P_- A(1+\xi)^{\frac{f_{1-}}{\epsilon}} \exp\left[P_- B(1+\xi)^{\frac{f_{1-}}{\epsilon}+1}\right] \tag{3.45}$$

for $\xi \in [-1, -1/2]$, where $A, B > 0$ are constants depending only on f_- and ϵ. For any $N > 0$, let $P_- < -N^{\frac{\epsilon+f_{1-}}{\epsilon}}$. Then $(N/-P_-)^{\frac{\epsilon}{f_{1-}}} < 1/N$. Choosing

$$\xi_0 = -1 + \left(\frac{N}{-P_-}\right)^{\frac{\epsilon}{f_{1-}}} < -1 + \frac{1}{N},$$

we have

$$-P_-(1+\xi_0)^{\frac{f_{1-}}{\epsilon}} = N,$$
$$P_-(1+\xi_0)^{\frac{f_{1-}}{\epsilon}+1} = -N(1+\xi_0) > -N\cdot\frac{1}{N} = -1.$$

Then (3.45) reads

$$f_2(\xi_0) - f_{2-} \geq NA\exp(-B).$$

Since $P_- < 0$, the function $f_2(\xi; P_-)$ is increasing on $[-1, 0]$, and hence

$$f_2(0; P_-) - f_{2-} \geq NA\exp(-B)$$

if $P_- < -M^{\frac{\epsilon+f_{1-}}{\epsilon}}$. Thus

$$\lim_{P_-\to-\infty} f_2(0; P_-) = \infty. \qquad \square$$

Now we consider trajectories of (3.1) on $[0, 1]$ with initial condition (3.3). Under the transformation

$$(\xi, f_1, f_2, f_3) \mapsto (-\xi, f_2, f_1, f_3), \tag{3.46}$$

problem (3.1), (3.3) becomes the initial-value problem of equation (3.1) with initial value $f(-1) = (f_{2+}, f_{1+}, f_{3+})$, which we have already studied. Our previous results then yield the following theorem.

THEOREM 3.9. *There is a $P_{0+} > 0$ such that the initial-value problem* (3.1), (3.3) *has a unique, positive solution $f^+(\xi; P_+)$ on $[0, 1]$ for any given*

$$P_+ = \left.\frac{Q(f(\xi))}{(1-\xi)^{\frac{f_{2-}}{\epsilon}}}\right|_{\xi=1}, \quad P_+ \in (-\infty, P_{0+}].$$

Furthermore, we have the following:

(i)

$$f_1^+(0; P_{0+}) = 0. \tag{3.47}$$

(ii) *$f^+(0; P_+)$ is continuous with respect to $P_+ \in (-\infty, P_{0+}]$.*

(iii)

(3.48a)
$$\begin{aligned} C_+(f_{1+}, f_{2+}) &:= \left\{(f_1^+(0; P_+), f_2^+(0, ; P_+)) \in \mathbb{R}^2 : P_+ \in (-\infty, P_0)\right\} \\ &\subset \left\{(f_1, f_2) \in \mathbb{R}^2 : f_1 \geq f_1^+(0; P_{0+}) = 0, f_2^+(0; P_{0+}) \geq f_2 > 0\right\}. \end{aligned}$$

(iv)

$$\lim_{P_-\to-\infty} f_1^+(0; P_-) = +\infty. \tag{3.48b}$$

THEOREM 3.10. *Problem* (1.6) *always has a continuous positive solution.*

Proof. For any given $f_\pm > 0$, the curve $C_-(f_{1-}, f_{2-})$, parametrized as $(f_1(0; P_-), f_2(0; P_-))$, $P_- \in (-\infty, P_0]$, runs from $f_2(0, P_0) = 0$ to $f_2(0; -\infty) = +\infty$ and $0 <$

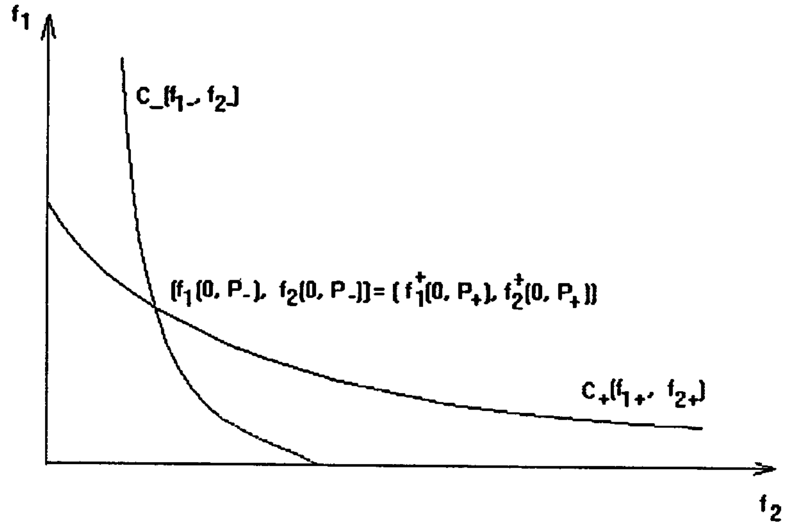

Fig. 1.

$f_1(0;P_-) \le f_1(0,P_0)$ while $C_+(f_{1+},f_{2+})$, parametrized by $(f_1^+(0;P_+),\ f_2^+(0;P_+))$, $P_+ \in (-\infty, P_{0+})$, runs from $f_1^+(0;P_{0+}) = 0$ to $f_1^+(0;-\infty) = +\infty$ with $0 < f_2^+(0;P_-) \le f_2^+(0;P_{0+})$. By using the Lipschitz continuity of $(f_{1\pm}(0;P), f_{2\pm}(0;P))$ obtained in Corollary 3.7, we can prove that

$$C_-(f_{1-},f_{2-}) \cap C_+(f_{1+},f_{2+}) \ne \emptyset, \tag{3.49}$$

as shown in Fig. 1.

The proof of (3.49) is given in Lemma 3.11 below. Let

$$\begin{aligned}\big(f_1(0;P_-), f_2(0;P_-)\big) &= \big(f_1^+(0;P_+), f_2^+(0;P_+)\big)\\ &\in C_-(f_{1-},f_{2-}) \cap C_+(f_{1+},f_{2+})\end{aligned} \tag{3.50}$$

for some $P_- \in (-\infty, P_0]$, $P_+ \in (-\infty, P_{0+}]$. Consider the function

$$f(\xi) = \begin{cases} f(\xi; f_{1-}, f_{2-}, P_-) & \text{if } \xi \in [-1,0],\\ f^+(\xi; f_{1+}, f_{2+}, P_+) & \text{if } \xi \in [0,1],\end{cases} \tag{3.51}$$

which is continuous and positive on $[-1,0)\cup(0,1]$. Furthermore, the matching condition (3.50) says that f_1 and f_2 are continuous on $[-1,1]$. Thus f_1 and f_2 are bounded and positive on $[-1,1]$, and hence so is f_3. Since f is positive on $[-1,0)\cup(0,1]$, Lemma 3.1 (iii) and (iv) imply that the function $Q(f(\xi)$ is continuous at $\xi = 0$ and $Q|_{\xi=0} = 0$. Then $f_3(\xi)$ must be continuous at $\xi = 0$ as well. Then f defined by (3.51) satisfies the integral form of equation (3.1) and hence (3.1) itself. Thus f is a positive, continuous solution of (1.6). □

Lemma 3.11.

$$C_-(f_{1-},f_{2-}) \cap C_+(f_{1+},f_{2+}) \ne \emptyset. \tag{3.52}$$

Proof. We recall from (3.36), (3.40), and (3.48a) that

$$\begin{aligned} C_-(f_{1-}, f_{2-}) &:= \big\{(f_1(0; f_{1-}, f_{2-}, P_-), f_2(0; f_{1-}, f_{2-}, P_-)) : P_- \in (-\infty, P_0)\big\} \\ &\subset \big\{(f_1, f_2) \in \mathbb{R}^2 : 0 < f_1 \le f_1(0; P_0), 0 = f_2(0; P_0) \le f_2\big\} \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} C_+(f_{1+}, f_{2+}) &:= \big\{(f_1^+(0; P_+), f_2^+(0, ; P_+)) \in \mathbb{R}^2 : P_+ \in (-\infty, P_0)\big\} \\ &\subset \big\{(f_1, f_2) \in \mathbb{R}^2 : f_1 \ge f_1^+(0; P_{0+}) = 0, f_2^+(0; P_{0+}) \ge f_2 > 0\big\}. \end{aligned} \tag{3.54}$$

By Corollary 3.7, the curves $C_-(f_{1-}, f_{2-})$ and $C_+(f_{1+}, f_{2+})$ are Lipschitz continuous in $P_\pm$, respectively. For the curve $C_-(f_{1-}, f_{2-})$, we define the following subsets of $(-\infty, P_0]$:

$$\mathcal{A} := \cup\big\{(p, q) \subset (-\infty, P_0] : (f_!(0; p), f_2(0; p)) = (f_!(0; q), f_2(0; q))\big\}, \tag{3.55}$$

$$\mathcal{B} := (-\infty, P_0]\backslash\mathcal{A}. \tag{3.56}$$

We further define

$$t(P) := P_0 + \int_{P_0}^{P} \chi_{\mathcal{B}}(p)dp, \tag{3.57}$$

$$P(t) := \inf\{P : t(P) = t\}, \tag{3.58}$$

$$\big(\bar{f}_1(t), \bar{f}_2(t)\big) := \big(f_1(0; P(t)), f_2(0; P(t))\big). \tag{3.59}$$

From the definitions in (3.55)–(3.59), we can see that the curve $\big(\bar{f}_1(t), \bar{f}_2(t)\big)$ cannot be self-intersecting. Indeed, otherwise, there would be t_1 and t_2, $t_1 < t_2$, such that

$$\big(\bar{f}_1(t_1), \bar{f}_2(t_1)\big) = \big(\bar{f}_1(t_2), \bar{f}_2(t_2)\big) \tag{3.60}$$

and hence

$$(P(t_1), P(t_2)) \subset \mathcal{A}. \tag{3.61}$$

Then we have from (3.57) and (3.61) that

$$t_2 - t_1 = \int_{P(t_1)}^{P(t_2)} \chi_{\mathcal{B}}(p)dp = 0,$$

which is a contradiction. We notice from (3.57) that $t(P)$ is an increasing function of P and hence $P(t)$ is also an increasing function. We further claim that $\big(\bar{f}_1(t), \bar{f}_2(t)\big)$ is continuous in t. To prove the claim, we let t_0 be any point in the domain of $\big(\bar{f}_1(t), \bar{f}_2(t)\big)$. Since $P(t)$ is increasing, $P(t) \to p_+ +$ (or $p_- -$) as $t \to t_0+$ (or t_0-). This implies that

$$\lim_{t \to t_0+} \bar{f}_1(t) = \lim_{t \to t_0+} f_1(0; P(t)) = f_1(0; p_+) \tag{3.62}$$

and

$$\lim_{t \to t_0-} \bar{f}_1(t) = \lim_{t \to t_0-} f_1(0; P(t)) = f_1(0; p_-). \tag{3.63}$$

From the definition in (3.57), we have

$$t_0 = P_0 + \int_{P_0}^{p_-} \chi_{\mathcal{B}}(y)dy$$

and

$$t_0 = P_0 + \int_{P_0}^{p_+} \chi_{\mathcal{B}}(y)dy,$$

which yield

$$\int_{p_-}^{p_+} \chi_{\mathcal{B}}(y)dy = 0. \tag{3.64}$$

Consider the open set $\mathcal{A} \cap (p_-, p_+)$. It is a union of countably many disjoint open intervals

$$\mathcal{A} \cap (p_-, p_+) = \cup_{n=1}^{\infty}(p_n, q_n).$$

Then (3.64) is equivalent to

$$\sum_{n=1}^{\infty}(q_n - p_n) = p_+ - p_-. \tag{3.65}$$

From the definition of $\mathcal{A}$, it is clear that $f_1(0;p_n) = f_1(0;q_n)$. We consider $\cup_{n=1}^{N}(p_n, q_n)$ for large integer N. We assume without loss of generality—rearranging the notation if necessary—that $p_1 < q_1 < p_2 < q_2 < \cdots < q_N$. The difference of $f_1(0;p_-)$ and $f_1(0;p_+)$ satisfies

$$\begin{aligned}
&|f_1(0;p_-) - f_1(0;p_+)| \\
&\quad = \left| f_1(0;p_-) - \sum_{n=1}^{N}\big(f_1(0;p_n) - f_1(q_n)\big) - f_1(0;p_+) \right| \\
&\quad = \left| f_1(0;p_-) - f_1(0;p_1) + \sum_{n=1}^{N-1}\big(f_1(0;q_n) - f_1(0;p_{n+1})\big) + f_1(0;q_N) - f_1(0;p_+) \right| \\
&\quad \le O(1)\left(|p_1 - p_-| + \sum_{n=1}^{N-1}|q_n - p_{n+1}| + |p_+ - q_N| \right) \\
&\quad = O(1)\left(p_+ - p_- - \sum_{n=1}^{N}(q_n - p_n) \right) \\
&\quad \to 0 \quad \text{as } N \to \infty,
\end{aligned} \tag{3.66}$$

where we used the Lipschitz continuity of f_1 on P and (3.65). This proves the continuity of $\bar{f}_1(t)$. We can prove the continuity of $\bar{f}_2(t)$ similarly.

Now we claim that $\bar{f}_2(t) \to \infty$ as $t \to \inf(\text{domain of definition of } (\bar{f}_1, \bar{f}_2))$. To this end, it suffices to prove that there is a sequence $\{P_n\} \subset \mathcal{B}$ such that $P_n \to -\infty$ as $n \to \infty$ since $f_2(0;P) \to \infty$ as $P \to -\infty$. Indeed, otherwise, by the definition of $\mathcal{B}$, there would be a sequence $\{\bar{P}_n\} \subset \mathcal{A}$ such that $\bar{P}_n \to -\infty$ as $n \to \infty$ and $f_2(0;\bar{P}_n)$

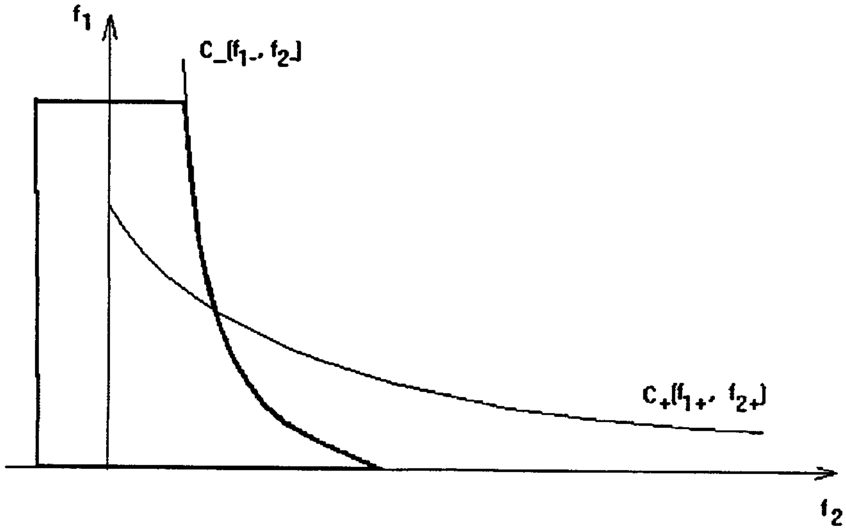

Fig. 2.

is bounded, which is a contradiction to the fact that $f_2(0;P) \to \infty$ as $P \to \infty$. The claim is proved.

The domain of definition of $(\bar{f}_1, \bar{f}_2)$ is the range of $t(P)$ defined in (3.57), which is an interval due to the continuity of $t(P)$. As P runs from P_0 to $-\infty$, the function $t(P)$ decreases from P_0 to inf(domain of definition of $(\bar{f}_1, \bar{f}_2)$) and $\bar{f}_2(t)$ runs from 0 to ∞. There is at least one point $T \in$(domain of definition of $(\bar{f}_1, \bar{f}_2)$) such that $\bar{f}_2(T) = 2f_{2+}(0; P_{0+})$. Let $\mathcal{T}$ be the set of all of these T's and $T_1 \in \mathcal{T}$ be such that $\min_{T\in\mathcal{T}} \bar{f}_1(T)$ is attained. We define a closed curve

$$\begin{aligned} \mathcal{J} := {} & \{(\bar{f}_1, \bar{f}_2)(t) : t \in [T_1, P_0]\} \\ & \cup \{(f_1, f_2) : f_2 = \bar{f}_2(T_1),\ -1 \le f_1 \le \bar{f}_1(T_1)\} \\ & \cup \{(-1, f_2) : 0 \le f_2 \le \bar{f}_2(T_1)\} \cup \{(f_1, 0) : -1 \le f_1 \le f_1(0; P_0)\}, \end{aligned} \tag{3.67}$$

which is depicted in Fig. 2.

From the definition in (3.67), the definition of T_1, and (3.53), we can see that the curve $\mathcal{J}$ is a simple closed curve. Jordan's curve theorem then states that the curve $\mathcal{J}$ divides the whole (f_1, f_2)-plane into two components, an interior component and an exterior component. The curve $C_+(f_{1+}, f_{2+})$ has a point $(0, f_2^+(0; P_{0+}))$ inside the interior component and a point in the exterior component since $(f_1^+(0;P) \to \infty$ as $P \to \infty$. Thus the curve $C_+(1+, f_{2+})$ must intersect the boundary of the interior component, which is the curve $\mathcal{J}$. From (3.54), we see that $C_+(f_{1+}, f_{2+})$ can intersect $\mathcal{J}$ only at the $\{(\bar{f}_1, \bar{f}_2)(t) : t \in [T_1, P_0]\}$ part of $\mathcal{J}$, which is a portion of the curve $C_-(f_{1-}, f_{2-})$. Thus we obtain the desired conclusion. □

The following corollary justifies the passing of the fluid dynamic limit in (1.5) to obtain the limit equation (1.2).

Corollary 3.12. *For any Maxwellian Riemann data* (1.4)*, there is a solution of the equation* (1.4)–(1.5)*, $f^\epsilon(x/t)$. Further, there is a sequence of solutions of* (1.4)–(1.5)*, $\{f^{\epsilon_n}\}$, $\epsilon_n \to 0+$, such that $f^{\epsilon_n} \to f(x/t)$ almost everywhere $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, where $f(x/t)$ is a solution of the limit equation* (1.2)*,* (1.4)*.*

Proof. The first assertion is a restatement of Theorem 3.10. In [ST], Slemrod and

Tzavaras proved that the total variation of solutions of (1.4)–(1.5) bounded uniformly in ϵ. Thus there is a sequence of solutions of (1.4)–(1.5), $\{f^{\epsilon_n}\}$, $\epsilon_n \to 0+$, such that $f^{\epsilon_n} \to f(x/t)$ almost everywhere $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. It is clear that the limit function is a weak solution of (1.2), (1.4). ∎

With help from Lemma 3.1, we can improve the result of Lemma 2.2 as follows.

THEOREM 3.13. *Let f be a positive solution of* (3.1) *with boundary condition* (3.2)–(3.3). *Then*

$$f_1(\xi) - f_{1+} = O(1)(1-\xi)^{\frac{f_{2+}}{\epsilon}} \quad \text{for } \xi \text{ near } \xi = 1, \tag{3.68}$$

$$f_2(\xi) - f_{2-} = O(1)(\xi+1)^{\frac{f_{1-}}{\epsilon}} \quad \text{for } \xi \text{ near } \xi = -1, \tag{3.69}$$

and

$$f_3(\xi) - f_3(0) = O(1)|\xi|^{\frac{\min(f_{3-}, f_3(0), f_{3+})}{\epsilon}} \quad \text{for } \xi \text{ near } \xi = 0, \tag{3.70}$$

$$Q(f\xi)) = O(1)(\xi+1)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2+}}{\epsilon}}|\xi|^{\frac{\min(f_{3-}, f_3(0), f_{3+})}{\epsilon}}. \tag{3.71}$$

Proof. From (3.5a, b), we know that

$$Q(f(\xi)) = O(1)|\xi|^{\frac{\min(f_{3-}, f_3(0), f_{3+})}{\epsilon}}. \tag{3.72}$$

Then by integrating $(3.1)_3$, we obtain

$$f_3(\xi) - f_3(0) = \int_0^\xi \frac{Q(f(\zeta)}{\epsilon\zeta} d\zeta = O(1)|\xi|^{\frac{\min(f_{3-}, f_3(0), f_{3+})}{\epsilon}},$$

which is (3.70).

To prove (3.68), we start with (3.4a),

$$\begin{aligned} Q &= C(1+\xi)^{\frac{f_{1-}}{\epsilon}}(1-\xi)^{\frac{f_{2+}}{\epsilon}}|\xi|^{\frac{\min(f_{3-}, f_3(0-))}{\epsilon}} \\ &\quad \times \exp\left[\frac{1}{\epsilon}\int_{-1}^{\xi}\left(\frac{f_1(\zeta) - f_{1-}}{\zeta+1} + \frac{f_2(\zeta) - f_{2+}}{\zeta - 1} + \frac{f_3(\zeta) - \min(f_{3-}, f_3(0-)}{\zeta}\right)d\zeta\right], \end{aligned} \tag{3.73}$$

which holds for $\xi \in [-1, 0)$. If $C \geq 0$ and hence $Q \geq 0$ on $[-1, 0)$, then f_2 and f_3 are decreasing on $[-1, 0)$, and hence

$$f_1' = \frac{Q}{\epsilon(1-\xi)} \leq \frac{f_3^2(\xi)}{\epsilon(1-\xi)} \leq \frac{f_{3-}^2}{\epsilon}.$$

Thus all of the terms in the integrand of (3.71) are either negative or finite, which implies that

$$Q(f(\xi)) = O(1)(1+\xi)^{\frac{f_{1-}}{\epsilon}} \quad \text{for } \xi \in [-1,). \tag{3.74}$$

This together with $(3.1)_2$ yields that

$$f_2(\xi) - f_{2-} = \int_{-1}^{\xi} \frac{-Q(f(\zeta))}{\epsilon(\zeta+1)} = O(1)(1+\xi)^{\frac{f_{1-}}{\epsilon}},$$

which proves (3.68). Similarly, we can prove that

$$Q(f(\xi)) = O(1)(1-\xi)^{\frac{f_{2+}}{\epsilon}} \quad \text{for } \xi \in (0, 1] \tag{3.75}$$

and thus (3.69). Combining (3.72), (3.74), and (3.75), we obtain (3.71). ∎

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