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The Relaxation Limit for Systems of Broadwell Type

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This paper considers the Cauchy problem for the following systems of Broadwell's type

$$\left.\begin{aligned} f_{1t} + f_{1x} &= \frac{F(f_1, f_2, f_3)}{\tau} \\ f_{2t} - f_{2x} &= \frac{F(f_1, f_2, f_3)}{\tau} \\ f_{3t} &= -\frac{F(f_1, f_2, f_3)}{2\tau} \end{aligned}\right\} \tag{1}$$

When the nonlinear function F takes the special form $f_1 f_2 - f_3^2$, (1) is a simple mathematical model of gas kinetics, the so called Broadwell model [1] (see also [2], [6] [13], [16], [17], [18] and the references therein). It describes an idealization of a discrete velocity gas of particles in one dimension subject to a simple binary collision mechanism.

Let $\rho = f_1 + f_2 + 4f_3$, $m = f_1 - f_2$, $s = f_3$. (1) may be written as follows:

$$\left.\begin{aligned} \rho_t + m_x &= 0 \\ m_t + (\rho - 4s)_x &= 0 \\ s_t + \frac{\bar{F}(\rho, m, s)}{\tau} &= 0 \end{aligned}\right\} \tag{2}$$

The conditions for (ρ, m, s) to be a local Maxwellian are

$$s = \frac{1}{6}\left(2 - \sqrt{1 + 3\frac{m^2}{\rho^2}}\right)\rho \tag{3}$$

for the Broadwell case $F = f_1 f_2 - f_3^2$. If the nonlinear collision function F can be written as $F = h_1(f_3) - h_2(f_1 + f_2 + 4f_3)$ for some nonlinear functions h, h_1 and h_2 the conditions on (ρ, m, s) to be a local Maxwellian are

$$s = h(\rho) \; . \tag{4}$$

The equilibrium systems corresponding to (3) and (4) are the following Euler equations (5) and p-system (6):

$$\left.\begin{aligned} \rho_t + m_x &= 0 \\ m_t + (\rho G(u))_x &= 0 \end{aligned}\right\} \tag{5}$$

where

$$G(u) = \frac{1}{3}\left(2(1+3u^2)^{\frac{1}{2}} - 1\right) ;$$

$$\left.\begin{aligned} \rho_t + m_x &= 0 \\ m_t + (\rho - 4h(\rho))_x &= 0 \end{aligned}\right\} \tag{6}$$

The asymptotic relationship between the solutions of the Broadwell model (1) and the solutions of the Euler equations (5) as τ goes to zero has been investigated by many authors [2], [13], [16], [18] (also see references therein). All authors considered the limit assuming some special structure of the solution, such as continuity [2], Riemann solution [16], finite number of shock waves [18].

In this paper, we study the Cauchy problem (2) with bounded L^2 measurable initial data

$$(\rho, m, s)_{|t=0} = \left(\rho_0(x), m_0(x), s_0(x)\right) . \tag{7}$$

When the local Maxwellian is given by (4), we show that the solution of the equilibrium system (6) is given by the limit of the solutions of the viscous approximation

$$\left.\begin{aligned} \rho_t + m_x &= \epsilon\rho_{xx} \\ m_t + (\rho - 4s)_x &= \epsilon m_{xx} \\ s_t + \frac{\bar{F}(\rho, m, s)}{\tau} &= \epsilon s_{xx} \end{aligned}\right\} \tag{8}$$

as ϵ and τ go to zero. Our method is the compensated compactness. This method has shown itself powerful in solving some relaxation limit problems [4], [3], [5], [8], [9], [10], [12]. When dealing with systems of more than two equations it is well known that the one basic difficulty is the a priori estimate independent of the approximate parameter ϵ in a suitable L^p space ($p > 1$). Since system (2) in general can not be diagonalized by using Riemann invariants, it is not to be expected that viscosity solutions $(\rho^\epsilon, m^\epsilon, s^\epsilon)$ of the Cauchy problem (8) will be bounded in L^∞, uniformly in ϵ, by using the invariant region principle. We have to search for solutions of the system (2) in L^p space. Similar results about zero relaxation systems of three equations are discussed in [12]. In paper [12], we studied the following system:

$$\left.\begin{aligned} v_t - u_x &= 0 \\ u_t - \sigma(v, s)_x &= 0 \\ s_t + \frac{s - f(v)}{\tau} &= 0 \end{aligned}\right\} \tag{9}$$

where $\sigma(v, s)$ is a nonlinear function of v and s, but $f(v)$ must be a linear function cv in order to make the technique used in [12] work. System (2) is of a different form. The flux functions are linear, but the zero-th order term is *non*linear. In [7], the authors suggested to solve the following system:

$$\left.\begin{aligned} u_t^\tau + div(v^\tau) &= 0 \\ v_t^\tau + \mu\, div(u^\tau) &= \frac{1}{\tau}\big(f(u^\tau) - v^\tau\big) \end{aligned}\right\} \tag{10}$$

as an approximation to the general nonlinear hyperbolic system

$$u_t + div\big(f(u)\big) = 0\,. \tag{11}$$

So in some sense, the study of the system (2) is more significant than that of (9) in comparing the relationship between (10) and (11).

The difficulty in applying the compensated compactness to the system (2) is the compactness analysis of the viscosity solutions of the Cauchy problem (8) in L^p. To overcome this difficulty, we adopt the method used in [12] to reduce the equations to two equations and then use the entropy-entropy flux pairs of system (6) as constructed by Jim Shearer [15] and the framework given by Serre and Shearer [14] to realize our aim.

In this paper we make the following assumptions on $\bar{F}$ in equation (2) and on the initial data:

($\mathbf{A_1}$): $\bar{F}(\rho, m, s) = H(s) - \rho$; $H(s) \in C^3(R)$, $H'(s) \geq 4 + c$ for some constants $c > 0$.

Since $H'(s) > 0$, $H(s) = \rho$ has an inverse function $H^{-1}(\rho) = s$.

Let $\sigma(\rho) = \rho - 4h(\rho)$, $h(\rho) = H^{-1}(\rho)$ and $\sigma(\rho)$ satisfy all the conditions in [14], namely

($\mathbf{A_2}$): Strict hyperbolicity: $\sigma'(\rho) \geq \sigma_0 > 0$ with $\sigma_0 =$ constant.

($\mathbf{A_3}$): Genuine nonlinearity except at a point: $\sigma''(\lambda_0) = 0$ and $\sigma''(\lambda) \neq 0$ for $\lambda \neq \lambda_0$.

($\mathbf{A_4}$): Growth constraints: $\dfrac{\sigma''}{(\sigma')^{\frac{5}{4}}}, \dfrac{\sigma'''}{(\sigma')^{\frac{7}{4}}} \in L^2$; $\dfrac{\sigma''}{(\sigma')^{\frac{3}{2}}}$, $\dfrac{\sigma'''}{(\sigma')^2} \in L^\infty$ $\dfrac{\sigma(\rho)}{\Sigma(\rho)} \to 0$ as $|\rho| \to \infty$ and there are constants c_1, c_2 with $c_1 > \frac{1}{2}$ such that $\big(\sigma'(\rho)\big)^{c_1} \leq c_2\big(1 + \Sigma(\rho_0)\big)$, where $\Sigma(\rho_0) = \int_0^\rho \sigma(s)ds$.

We have the following assumption about the initial data (7):

($\mathbf{A_5}$) : $\rho_0(x)$, $m_0(x)$, $s_0(x)$ are all bounded in $L^2(R)$ and tend to zero as $|x| \to \infty$ sufficiently fast such that the smooth functions given in (17) satisfy

$$\lim_{|x|\to\infty}\Big(\frac{d^i\rho_0^\epsilon(x)}{dx^i}, \frac{d^i m_0^\epsilon(x)}{dx^i}, \frac{d^i s_0^\epsilon(x)}{dx^i}\Big) = (0,0,0) \quad i = 0,1 \tag{12}$$

$$|\rho_0^\epsilon(x)|_{H^1(R)} \leq M(\epsilon), |m_0^\epsilon(x)|_{H^1(R)} \leq M(\epsilon), |s_0^\epsilon(x)|_{H^1(R)} \leq M(\epsilon)\,. \tag{13}$$

From the basic property of the mollifier, we have that

$$\big(\rho_0^\epsilon(x), m_0^\epsilon(x), s_0^\epsilon(x)\big) \to \big(\rho_0(x), m_0(x), s_0(x)\big) \tag{14}$$

uniformly on any compact set in R as $\epsilon \to 0$, and also the following properties:

$$\left.\begin{aligned} |\rho_0^\epsilon(x)|_{L^2} \le |\rho_0(x)|_{L^2} \le M \\ |m_0^\epsilon(x)|_{L^2} \le |m_0(x)|_{L^2} \le M \\ |s_0^\epsilon(x)|_{L^2} \le |s_0(x)|_{L^2} \le M \end{aligned}\right\} \tag{15}$$

$$|\frac{d^i\rho_0^\epsilon(x)}{dx^i}|\,, \quad |\frac{d^i m_0^\epsilon(x)}{dx^i}|\,, \quad |\frac{d^i s_0^\epsilon(x)}{dx^i}| \le M(\epsilon) \quad i = 0, 1, 2 \tag{16}$$

We now proceed as follows: At first we consider the existence of viscosity solutions of system (8) with initial data

$$\left(\rho^\epsilon, m^\epsilon, s^\epsilon\right)_{|t=0} = \left(\rho_0^\epsilon, m_0^\epsilon, s_0^\epsilon\right) \tag{17}$$

where $(\rho_0^\epsilon, m_0^\epsilon, s_0^\epsilon)$ are smooth functions obtained by smoothing the initial data (7) with a mollifier. The existence is based on the standard local existence theory by using the contraction mapping principle to an integral representation of (8) and an a-priori estimation of the local solution depending on ϵ and τ. The a-priori bound is obtained by the energy method.

$$|\rho(.,t)|_{L^2(R)} \le M\,, \quad |m(.,t)|_{L^2(R)} \le M\,, \quad |s(.,t)|_{L^2(R)} \le M \tag{18}$$

$$|\frac{(H(s)-\rho)^2}{\tau}|_{L^1(R\times[0,T])} \le M \tag{19}$$

$$|\epsilon\rho_x^2|_{L^1(R\times[0,T])} \le M\,, \quad |\epsilon m_x^2|_{L^1(R\times[0,T])} \le M\,, \quad |\epsilon s_x^2|_{L^1(R\times[0,T])} \le M \tag{20}$$

This gives the following theorem:

Theorem 1. *If the initial data (17) satisfies (12), (13), (15) and $\bar{F}(\rho, m, s)$ satisfies the condition (A_1). Then for any fixed ϵ, $\tau > 0$, there is a global solution of the Cauchy problem (8), (17) such that all the estimates in (18), (19), (20) hold.*

In the next step, the compensated compactness method is used to study convergence of the viscosity solutions $\left(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}, s^{\epsilon,\tau}\right)$. First the convergence of $(\rho^{\epsilon,\tau}, m^{\epsilon,\tau})$ is shown, and then, using the estimate (19) the convergence of $s^{\epsilon,\tau}$ is shown. When taking $\delta = O(\epsilon)$, the global weak solution of the equilibrium (6) is obtained as ϵ goes to zero.

We thus arrive at the main theorem:

Theorem 2. *The solutions $\left(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}, s^{\epsilon,\tau}\right)$ of the Cauchy problem (8), (17) with the assumptions (A_1)-(A_5) converge almost everywhere in a compact set $\Omega \in R \times R^+$ to a L^2 bounded function triple (τ, m, s) as ϵ, τ go to zero related by $\tau = O(\epsilon)$. Moreover (ρ, m) is a weak solution of the Cauchy problem (6) with initial data $\left(\rho_0(x), m_0(x)\right)$ in (7).*

For proofs please refer to [11].

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