IMEX Runge-Kutta Schemes and Hyperbolic Systems of Conservation Laws with Stiff Diffusive Relaxation

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Abstract. Hyperbolic system of conservation laws often have relaxation terms that, under a suitable scaling, lead to a reduced system of parabolic or hyperbolic type. The development of numerical methods to solve systems of this form his an active area of research. These systems in addition to the stiff relaxation term have the convection term stiff too. In this paper we will mainly concentrate on the study of the stiff regime. In fact in this stiff regime most of the popular methods for the solution of these system fail to capture the correct behavior of the relaxation limit unless the small relaxation rate is numericaly resolved. We will show how to overcome this difficulties and how to construct numerical schemes with the correct asymnptotic limit, i.e., the correct zero-relaxation limit should be preserved at a discrete level.

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INTRODUCTION

In this article we will focus our attention on two simple prototype of hyperbolic systems with diffusive relaxation given by

$$\begin{aligned} u_t + v_x &= 0 \\ \varepsilon^2 v_t + u_x &= -v \end{aligned} \tag{1}$$

and

$$\begin{aligned} u_t + v_x &= 0 \\ \varepsilon^2 v_t + \mu u_x &= -(v-u). \end{aligned} \tag{2}$$

where the parameter $\varepsilon > 0$ is ussually called relaxation parameter. When $\varepsilon \to 0$ the two systems (1), (2) relax, respectively, towards the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ v &= -u_x \end{aligned} \tag{3}$$

and convection-diffusion equation

$$\begin{aligned} u_t + u_x &= \mu u_{xx} \\ v &= u - \mu u_x. \end{aligned} \tag{4}$$

The development of numerical methods to solve systems of this form is an active area of research (see, e.s.,[6, 7]). In general, numerical approaches that works with good properties for only hyperbolic system with stiff relaxation term do not apply to these problems since this system in addition to the stiff relaxation term has the convection term stiff too. Then special care must be taken to assure that the schemes possess the correct zero-relaxation limit, in the sense that the asymptotic limit from systems (1) and (2) to (3), (4) should be preserved at a discrete level.
Now we will manly concentrate on the study of the stiff regime for systems (1) and (2) that is to say when $\varepsilon << 1$.

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In fact in this stiff regime most of the popular methods for the solution of these systems fail to capture the right behavior of the relaxation limit unless the small relaxation rate is numericaly resolved. We will show how to overcome this difficulties and how to construct numerical schemes with the correct asymptotic limit.
In this paper, our idea is to reformulate the problem (1) and (2) such that it allows us to design a class of IMplicit-EXplicit (IMEX) Runge-Kutta schemes that work with high order accuracy in time in the zero-diffusion limit, i.e. when ε is very small.
We note again that solving (1) and (2) numerically is challeging due to the stiffness of the problem both in the convection and in the relaxation terms. Moreover, since the characteristic speed of the hyperbolic part is of order $1/\varepsilon$, the usual approach used in [1] for IMEX R-K schemes leads to a CFL condition like $\Delta t \approx \varepsilon \Delta x$. Instead in the diffusive regime where $\varepsilon < \Delta x$, this is unnecessary since a parabolic condition $\Delta t \approx \Delta x^2$ should suffice.
In [6], [7], the authors developed a class of numerical schemes, called *diffusive relaxation scheme*, for system (1) and (2), and other related problems, that have parabolic CFL condition ($\Delta t \approx \Delta x^2$), with coarse grid Δt, $\Delta x >> \varepsilon$, where Δt and Δx are respectively the time step and the mash size.
In this paper, our new approach gives a class of IMEX R-K scheme that have as CFL condition $\Delta t \approx \Delta x$ when applied to (1) and (2) in the diffusive regime.
System (1) and (2) are *toy models* where one can easily see the major difficulty we encounter while dealing with hyperbolic problems with stiff diffusive relaxation.
The aim in this paper is to analyze the accuracy of different types of IMEX Runge-Kutta schemes when applied to systems (1), (2) in the stiff regime ($\varepsilon \to 0$).

IMEX RUNGE KUTTA SCHEMES ANALYSIS

Concerning systems (1) and (2), the key ingridient in our approach requires only to add and remove the second spacial derivative of the variable u in the first equation.
In fact, we consider the two equivalent systems to (1) and (2)

$$\begin{aligned} u_t + v_x &= u_{xx} - u_{xx} \\ \varepsilon^2 v_t + u_x &= -v \end{aligned} \tag{5}$$

and

$$\begin{aligned} u_t + v_x &= \mu u_{xx} - \mu u_{xx} \\ \varepsilon^2 v_t + \mu u_x &= -(v-u). \end{aligned} \tag{6}$$

Now, we look for a Fourier solution of the form $u = \hat{u}(t)\exp(i\xi x)$, $v = \hat{v}(t)\exp(i\xi x)$. Inserting the *ansatz* into systems (1), (2) one obtains

$$\begin{aligned} \hat{u}_t &= -i\xi\hat{v} + \xi^2\hat{u} - \xi^2\hat{u} \\ \varepsilon^2\hat{v}_t &= -i\xi\hat{u} - \hat{v} \end{aligned} \tag{7}$$

and

$$\begin{aligned} \hat{u}_t &= -i\xi\hat{v} + \mu\xi^2\hat{u} - \mu\xi^2\hat{u} \\ \varepsilon^2\hat{v}_t &= -\mu i\xi\hat{u} - (\hat{v} - \hat{u}). \end{aligned} \tag{8}$$

Now, we consider an IMEX Runge-Kutta (R-K) scheme applied to system (7), where we treat the quantities $(-i\xi\hat{v} + \xi^2\hat{u})$ explicitly, and $-\xi^2\hat{u}$ and $(-i\xi\hat{u} - \hat{v})$ implicitly. Then we get

$$\begin{aligned} \hat{u}_{n+1} &= \hat{u}_n - \Delta t \sum_{k=1}^{s} \tilde{b}_k (i\xi\hat{V}_k - \xi^2\hat{U}_k) - \Delta t \sum_{k=1}^{s} b_k \xi^2 \hat{U}_k \\ \varepsilon^2\hat{v}_{n+1} &= \varepsilon^2\hat{v}_n - \Delta t \sum_{k=1}^{s} b_k (i\xi\hat{U}_k + \hat{V}_k), \end{aligned} \tag{9}$$

where

$$\begin{aligned}\hat{U}_k &= \hat{u}_n - \Delta t \sum_{i=1}^{k-1} \tilde{a}_{kj}(i\xi\hat{V}_j - \xi^2\hat{U}_j) - \Delta t \sum_{j=1}^{k} a_{kj}\xi^2\hat{U}_j \\ \varepsilon^2\hat{V}_k &= \varepsilon^2\hat{v}_n - \Delta t \sum_{j=1}^{k} a_{kj}(i\xi\hat{U}_j + \hat{V}_j),\end{aligned} \tag{10}$$

are the internal stages of the IMEX R-K scheme. Analogously, we can apply an IMEX R-K scheme to system (8) where we treat $(-i\xi\hat{v}+\mu\xi^2\hat{u})$, explicitly, $-\xi^2\hat{u}$ and $(-i\xi\hat{u}-\hat{v}-\hat{u})$ implicitly. In this section we apply the different types of IMEX schemes to system (7), analogous analysis and results we obtain with system (8).
In this section we study the behavior of different types of IMEX R-K schemes when $\varepsilon \to 0$ in particular for $\varepsilon = 0$. IMEX R-K schemes can be classified in three different types (A [3], CK [5], and ARS [2]) characterized by the structure of the matrix A of the implicit scheme (see [1]).
Let us start to consider the type A IMEX R-K scheme. In the limiting case $\varepsilon = 0$ such scheme gives us the corresponding system

$$\begin{aligned}\hat{u}_{n+1} &= \hat{u}_n - \Delta t \sum_{k=1}^{s} b_k\xi^2\hat{U}_i \\ \hat{v}_{n+1} &= R(\infty)\hat{v}_n + \Delta t \sum_{k=1}^{s} b_k\omega_{kj}\hat{V}_k,\end{aligned} \tag{11}$$

with

$$\begin{aligned}\hat{U}_k &= \hat{u}_n - \Delta t \sum_{j=1}^{k} a_{kj}\xi^2\hat{U}_j \\ \hat{V}_k + i\xi\hat{U}_k &= 0, \;\; k = 1,\ldots,s\end{aligned} \tag{12}$$

where $R(\infty) = 1 - \sum_{k,j=1} b_k\omega_{kj}$ with ω_{kj} elements of the inverse of (a_{kj}). Moreover if the implicit scheme is *stiffly accurate* with A invertible one has $R(\infty) = 0$, and, by (12), we get $\Delta t\xi^2\hat{U}_k = \sum_{k,j=1}\omega_{kj}(\hat{u}_n - \hat{U}_j)$ that gives

$$\begin{aligned}\hat{u}_{n+1} &= \hat{U}_s, \\ \hat{v}_{n+1} &= \hat{V}_s,\end{aligned} \tag{13}$$

with $\hat{V}_s = -i\xi\hat{U}_s$. We note that concerning the system (7) when $\varepsilon = 0$ we obatin a corresponding *reduced* system

$$\begin{aligned}\hat{u}_t &= -\xi^2\hat{u} \\ \hat{v} &= -i\xi\hat{u},\end{aligned} \tag{14}$$

where $\hat{v}$ depends on $\hat{u}$.
Now if we apply to system (14) an implicit time discretization we can have a scheme that is unconditionally stable. This guarantees that no diffusive restrictions of the form $\Delta t \approx \Delta x^2/2$ on the time step (due to the stability of the explicit part of the IMEX R-K scheme) becomes necessary. Obviously this statement follows from the above analysis that reduces the system (12), (11) to system (13) where only the implicit part of the IMEX scheme is applied, under the assumption that the implicit part of the scheme is stiffly accurate.
Now it would be interesting to know how the numerical solution of the IMEX schemes of type A is related to the exact solution about the rate of convergence. In fact, by Theorem 3.1 in [1], it is proved that for higher IMEX R-K scheme of type A the numerical solutions of the system (12), (11) exhibits order reduction. Notice that these schemes suffer from the phenomenon of order reduction in the stiff regime when the classical order is greater than two. In the following we will show this throught numerical results.
We now analize the IMEX R-K scheme of type CK where by the defination 2.4 in [1] we assume that the submatrix $\hat{A}$ is invertible and $a_{11} = 0$. Furthermore, we consider that the implicit part of the scheme is *stiffly accurate*. Then applying

a scheme of type CK to system (7) setting $\varepsilon = 0$ and considering the assumption *stiffly accurate* for the implicit part of the scheme, and making use of condition $b_1 + \sum_{k=2}^{s} b_k \alpha_k = 0$, (see Lemma 5.1 in [1]) we obtain

$$\begin{aligned} \hat{u}_{n+1} &= \hat{u}_n - \Delta t \tilde{b}_1 (i\xi \hat{v}_n - \xi^2 \hat{u}_n) - \Delta t \sum_{k=2}^{s} \tilde{b}_k (i\xi \hat{V}_k - \xi^2 \hat{U}_k) - \Delta t b_1 \xi^2 \hat{u}_n - \Delta t \sum_{k=1}^{s} b_k \xi^2 \hat{U}_i \\ \hat{v}_{n+1} &= R(\infty)\hat{v}_n + \Delta t \sum_{k=2}^{s} b_k \hat{\omega}_{kj} \hat{V}_j, \end{aligned} \tag{15}$$

where $\alpha_k = -\sum_{j=2}^{s} \hat{\omega}_{kj} a_{j1}$ for $k = 2, ..., s$ with $\hat{\omega}_{kj}$ elements of the inverse matrix of $\hat{A}$, with

$$\begin{aligned} \hat{U}_k &= \hat{u}_n - \Delta t \tilde{a}_{k1} (i\xi \hat{v}_n - \xi^2 \hat{u}_n) - \Delta t \sum_{j=2}^{k-1} \tilde{a}_{kj} (i\xi \hat{V}_j - \xi^2 \hat{U}_j) - \Delta t a_{k1} \xi^2 \hat{u}_n - \Delta t \sum_{j=2}^{k} \hat{a}_{kj} \xi^2 \hat{U}_j \\ (i\xi \hat{U}_k + \hat{V}_k) &= \alpha_k (i\xi \hat{u}_n + \hat{v}_n), \text{for } k = 2, ..., s. \end{aligned}$$

In particular by previous formula we get

$$-\Delta t \xi^2 \hat{U}_k = \sum_{j=2}^{k} \hat{\omega}_{kj} (\hat{U}_k - \hat{u}_n) + \Delta t \sum_{j=2}^{k} \hat{\omega}_{kj} \tilde{a}_{j1} (i\xi \hat{v}_n - \xi^2 \hat{u}_n) - \Delta t \alpha_k \xi^2 \hat{u}_n + + \Delta t \sum_{j=2}^{k} \hat{\omega}_{kj} \tilde{a}_{jl} \alpha_l (i\xi \hat{u}_n + v_n),$$

and substituting in (15) we obtain

$$\begin{aligned} \hat{u}_{n+1} &= R(\infty)\hat{u}_n + \Delta t \sum_{k=2}^{s} b_k \hat{\omega}_{kj} \xi^2 \hat{U}_k - \Delta t (b_1 + \sum_{k=2}^{s} b_k \alpha_k) \xi^2 \hat{u}_n \\ &- \Delta t (\tilde{b}_1 + \sum_{k=2}^{s} b_k \tilde{\alpha}_k)(i\xi \hat{v}_n - \xi^2 \hat{u}_n) + \Delta t \sum_{k=2}^{s} b_k \hat{\omega}_{kj} \tilde{a}_{jl} \alpha_l (i\xi \hat{u}_n + \hat{v}_n) - \Delta t \sum_{k=2}^{s} \tilde{b}_k \alpha_k (i\xi \hat{u}_n + \hat{v}_n), \end{aligned} \tag{16}$$

where $\tilde{\alpha}_k = -\sum_{j=2}^{k} \hat{\omega}_{kj} \tilde{a}_{j1}$. Now, if we consider the assumption that the scheme is *stiffly accurate* and put $\tilde{a}_{s1} = \tilde{b}_1$ we get $b_1 + \sum_{k=2}^{s} b_k \alpha_k = 0$ and $\tilde{b}_1 + \sum_{k=2}^{s} b_k \tilde{\alpha}_k = 0$. This allows us to obtain

$$\hat{u}_{n+1} \quad = \hat{U}_s - \Delta t (\textstyle\sum_{k=2}^{s} \tilde{b}_k \alpha_k - \sum_{k=2}^{s} \tilde{a}_{sk} \alpha_k)(i\xi \hat{u}_n + \hat{v}_n)$$

and if we choose $\tilde{b}_k = \tilde{a}_{sk}$ for $k = 2, ..., s-1$ we have $\hat{u}_{n+1} = \hat{U}_s$. Then, the type CK IMEX R-K scheme converge and we are able to give estimates about the numerical solution of u and v-variables otherwise convergence is in general not guaranteeed. In particular, assuming $\tilde{b}_k = \tilde{a}_{sk}$, we obtain $\hat{u}_{n+1} = \hat{U}_s$ with $\hat{V}^s = -i\xi \hat{U}^s$. Then, we can produce the same considerations as done for the type A.

NUMERICAL RESULTS

In this section we show the time accuracy of several IMEX R-K scheme of different type when $\varepsilon \to 0$ and we demostrate that the assumptions introduced for each type of scheme agree with the numerical results. We choose second order type A and type ARS IMEX R-K schemes.
We first test these schemes on (5) with the initial data $u(x,0) = \cos(x)$, $v(x,0) = \sin(x)$ on the spatial interval $[0, 2\pi]$, at the final time $t = 1$ with periodic boundary condition. We choose $\varepsilon = 10^{-8}$ and $\Delta t \approx \Delta x$. The results are given in Table 1.
We remark that any SSP2 IMEX R-K scheme presented in [3] produces the same results obtained from the SSP2-322 scheme. Below, MARS(2,2,2) stands for a Modified type ARS IMEX R-K scheme where the condition $\tilde{a}_{s1} = \tilde{b}_1$ is satisfied and the double *tableau* Butcher is

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & \gamma & 0 & 0 \\ 1 & \tilde{b}_1 & \tilde{a}_{32} & 0 \\ \hline & \tilde{b}_1 & \tilde{b}_2 & 0 \end{array} \qquad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ 1 & 0 & 1-\gamma & \gamma \\ \hline & 0 & 1-\gamma & \gamma \end{array} \quad \gamma = 1 - \frac{\sqrt{2}}{2}, \;\; \tilde{b}_2 = \frac{1}{2\gamma}, \;\; \tilde{b}_1 = 1 - \tilde{b}_2, \;\; \tilde{a}_{32} = 1 - \tilde{b}_1.$$

TABLE 1. Convergence rate of u in L_2-norm, N.R. stands for not requested.

Schemes	$\tilde{a}_{sk} = \tilde{b}_k$	N	$L_2(u)$	Order
ARS(2,2,2) [2]	YES	20	2.173e-03	–
		40	7.634e-04	1.508
		80	1.984e-04	1.943
		160	5.003e-05	1.988
		320	1.2562e-05	1.994
MARS(2,2,2)	YES	20	2.172e-03	–
		40	7.634e-04	1.508
		80	1.984e-04	1.943
		160	5.003e-05	1.988
		320	1.2562e-05	1.994
SSP2-332 [3]	N.R.	20	2.363e-03	–
		40	7.910e-04	1.578
		80	2.046e-04	1.950
		160	5.1546e-05	1.988
		320	1.2942e-05	1.994

TABLE 2. Convergence rate of u in L_2-norm

Schemes	$\tilde{a}_{sk} = \tilde{b}_k$	N	$L_2(u)$	Order
ARS(2,2,2) [2]	YES	40	3.867e-03	–
		80	9.457e-04	2.031
		160	2.330e04	2.020
		320	5.798e-005	2.001
MARS(2,2,2)	YES	40	3.867e-03	–
		80	9.457e-04	2.031
		160	2.330e-04	2.020
		320	5.798e-05	2.001
SSP2-332 [3]	N.R.	40	2.615e-03	–
		80	6.243e-04	2.067
		160	1.543e-04	2.015
		320	3.850e-05	2.003

Next we consider system (6) and the limiting equation for this model is the advection-diffusion equation (4). We test the schemes proposed before with the initial data $u(x,0) = \exp(-(1-\cos(x-\pi))/\sigma)$, $v(x,0) = u(x,0)(1+\mu\sin(x-\pi)/\sigma)$ with $\sigma = 0.05$ and $\mu = 1$ (we choose this value of μ because we want that the dominant term in the advection-diffusion equation is the diffusive term) on the spatial interval $[0,2\pi]$, at the final time $t = 0.3$ with periodic boundary condition. We choose $\varepsilon = 10^{-8}$ and $\Delta t \approx \Delta x$. The results are given in the Table 2.

Now, we consider tree different types of third order IMEX R-K schemes, and we test the time accuracy. We consider again system (6). Concerning the type CK, the third order IMEX scheme ARK3(2)4L[2]SA doesn't work because the following assumption $\tilde{b}_k = \tilde{a}_{sk}$ for $k = 2,\ldots,s-1$ is not satisfied. Then we construct a particular type CK IMEX R-K scheme that satisfies the previous assumption in order to have convergence and accuracy in time. The result are given in Table 3.

MCK(5,5,3) stands for a Modified type CK IMEX R-K scheme where the condition $\tilde{a}_{sk} = \tilde{b}_k$ for $k = 1,\ldots,s-1$ is satisfied and the double *tableau* Butcher is

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2/3 & 4/9 & 2/9 & 0 & 0 & 0 \\
1 & 1/4 & 0 & 3/4 & 0 & 0 \\
1 & 1/4 & 0 & 3/4 & 0 & 0 \\
\hline
 & 1/4 & 0 & 3/4 & 0 & 0
\end{array}
\qquad
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1/2 & 1/2 & 0 & 0 & \\
2/3 & 5/18 & -1/9 & 1/2 & 0 & 0 \\
1 & 1/2 & 0 & 0 & 1/2 & 0 \\
1 & 1/4 & 0 & 3/4 & -1/2 & 1/2 \\
\hline
 & 1/4 & 0 & 3/4 & -1/2 & 1/2
\end{array}
$$

TABLE 3. Convergence rate of u in L_2-norm

Schemes	$\tilde{a}_{sk} = \tilde{b}_k$	N	$L_2(u)$	**Order**
ARS(4,4,3) [2]	YES	40	4.297e-04	–
		80	5.770e-05	2.89
		160	7.922e-06	2.86
		320	1.256e-06	2.65
SSP(4,3,3) [3]	N.R	40	3.172e-04	–
		80	2.753e-05	3.52
		160	3.039e-06	3.17
		320	8.738e-07	1.80
MCK(5,5,3) [5]	YES	40	8.300e-04	
		80	1.167e-04	2.83
		160	1.603e-05	2.86
		320	2.230e-06	2.85

It is worth mentioning that SSP(4,3,3) scheme exibits order reduction, this behavior appears beause SPP(4,3,3) doesn't satify the conditions requested in the Theorem 3.1 in [1] and the stiffly accurate condition for the implicit scheme (see also [4]).

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