ME 406
The Lorenz Equations

sysid

Mathematica 4.1.5, DynPac 10.67, 4/8/2002

intreset;

plotreset;

■ 1. Introduction

This notebook contains all of the material given in class on the Lorenz equations, and it constitutes section 2.5 of the class notes. The Lorenz equations are given by

$$\dot{x} = \sigma(y - x) , \quad \dot{y} = rx - y - xz , \quad \dot{z} = xy - bz . \tag{1}$$

These equations contain three parameters: σ, r and b. In what follows, we will always assume that these parameters are positive. In all of the numerical calculations below, we take $\sigma = 10.0$, and $b = 8/3$. These are the most used values in the study of the Lorenz equations. We will vary the parameter r over a wide range, and study how the solutions depend on r.

These equations, which are simple in appearance, have solutions with extraordinary properties. They were first studied in the 1960's by the M.I.T. meteorologist Edward Lorenz. He developed the equations as a model for the modal amplitudes in a nonlinear thermal convection problem. Lorenz recognized that the solutions of the equations can exhibit an unusual form of behavior which we now call chaos. It took time for others to realize exactly what Lorenz had discovered. Lorenz has told the story of the discovery in his book **The Essence of Chaos,** University of Washington Press, 1993. For a very readable and basic treatment of the equations, see Chapter 9 of **Nonlinear Dynamics and Chaos**, S.H. Strogatz, Addison-Wesley, 1994. For a general work containing a more advanced treatment, see **Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields**, J. Guckenheimer and P. Holmes, Springer-Verlag, 1983. For a book-length treatment containing many detailed results, see **The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors**, C. Sparrow, Springer-Verlag, 1982. In our study here, we will continue our primarily experimental approach and use the computer to learn about the system. We begin by defining the equations for DynPac.

setstate[{x, y, z}]; setparm[{σ, r, b}]; sysname = "Lorenz";

slopevec = {σ * (y – x), r * x – y – x * z, x * y – b * z};

■ 2. Basic Properties

■ 2.1 Symmetry

The equations are invariant to the transformation

$$(x, y, z) \longrightarrow (-x, -y, z) \ . \tag{2}$$

Thus if $x(t)$, $y(t)$, $z(t)$ is a solution, so is $-x(t)$, $-y(t)$, $z(t)$. As we shall see, this symmetry shows up in a number of ways, including the location of the equilibrium points of the system.

■ 2.2 The z-Axis is Invariant

We see from equations (1) that if $x(0) = 0$ and $y(0) = 0$, then x and y remain zero for all t. Thus the z-axis is an orbit, on which

$$\dot{z} = -bz \ , \text{ hence } z(t) = z(0)e^{-bt} \ , \text{ for } x, y = 0. \tag{3}$$

Thus the z-axis is always a part of the stable manifold for the equilibrium at the origin.

■ 2.3 The System is Dissipative

The divergence of the slopevector is

```
div[slopevec]
```

$$-1 - b - \sigma$$

and this is always negative. As we saw earlier in class, for any given volume V of phase points moving with the flow, we have

$$\frac{dV}{dt} = V \, \text{div}[\text{slopevec}] = -(1 + b + \sigma)V, \tag{4}$$

hence $V(t) = V(0)e^{-(1+b+\sigma)t}$.

With our canonical values of 10 for σ and 8/3 for b, this is $V(t) = V(0)e^{-13.67\,t}$, so that volumes of initial points are reduced by a factor of e in a time

```
1 / 13.67
```

```
0.0731529
```

■ 2.4 The Solutions are Bounded

It is not hard to prove that the solutions of the Lorenz equations are bounded. Consider first the case when $r < 1$. We examine a potential Liapunov function

```
V = x² + σ * (y² + z²);
```

The orbital derivative is

```
Simplify[orbdt[V]]
```

$$-2 \, (x^2 - (1 + r) \ x \, y + y^2 + b \, z^2) \ \sigma$$

The z-term is negative. We get for the x and y terms

$$x^2 - (1 + r) \, x \, y + y^2 \ = \ \left[x - \frac{1}{2} \ (1 + r) \, y\right]^2 + \frac{1}{4} \, (1 - r)\,(3 + r)\,y^2,$$

and this quantity is postive except for $x = y = 0$. Thus $\dot{V} < 0$, and this shows that the origin is a global attractor for the system when $r < 1$.

When $r > 1$, we have to work a little harder to show the solutions are bounded. We start with a new Liapunov function

$$V = r*x^2 + \sigma*y^2 + \sigma*(z - 2*r)^2;$$

Level surfaces $V = V_0$ are ellipsoids with center at $(0, 0, 2r)$, and semi-axes ($\sqrt{V_0/r}$, $\sqrt{V_0/\sigma}$, $\sqrt{V_0/\sigma}$). The orbital derivative is

$$Vdot = Simplify[orbdt[V]]$$

$$-2 \ (y^2 + b \ z^2 + r \ (x^2 - 2 \ b \ z)) \ \sigma$$

We work on this a bit.

$$Vdotmod = Vdot \ / \ (2 \ \sigma \ b \ r^2)$$

$$-\frac{y^2 + b \ z^2 + r \ (x^2 - 2 \ b \ z)}{b \ r^2}$$

We can write this as

$$-\frac{x^2}{br} - \frac{y^2}{b \ r^2} - \frac{(z-r)^2}{r^2} + 1 \tag{5}$$

The expression in equation (5) is zero on the ellipsoid given by

$$\frac{x^2}{br} + \frac{y^2}{b \ r^2} + \frac{(z-r)^2}{r^2} = 1 \ , \tag{6}$$

and is positive inside that ellipsoid and negative outside that ellipsoid. By choosing V_0 large enough, we can make the level surface $V = V_0$ lie entirely outside the ellipsoid (6). Any orbit starting outside that level surface will cross to the inside, and any orbit starting inside that level surface will remain inside. Thus all of the solutions are bounded, subject only to the assumption we made at the beginning that σ, r, and b are positive. Unlike the case $r < 1$, we cannot in this case draw any conclusions about a global attractor or even a stable equilibrium.

▪ 2.5 Equilibria

We use findpolyeq to look for equilibria.

$$eqpoints = findpolyeq$$

$$\left\{ \{0, 0, 0\}, \ \left\{ -\sqrt{-b+b \ r}, \ -\sqrt{-b+b \ r}, \ -1+r \right\}, \right.$$
$$\left. \left\{ \sqrt{-b+b \ r}, \ \sqrt{-b+b \ r}, \ -1+r \right\} \right\}$$

We see that the origin is an equilibrium for any values of the parameters. The other two equilibria are real if and only if $r \geq 1$. In fact for $r = 1$, the three equilibria coincide at $\{0,0,0\}$, and we therefore have a pitchfork bifurcation at $r = 1$. We will look at this in more detail later. We name the three equilibria for later reference.

```
C0 = eqpoints[[1]]
```

{0, 0, 0}

```
C1 = eqpoints[[2]]
```

$\left\{-\sqrt{-b + b\,r}\,,\ -\sqrt{-b + b\,r}\,,\ -1 + r\right\}$

```
C2 = eqpoints[[3]]
```

$\left\{\sqrt{-b + b\,r}\,,\ \sqrt{-b + b\,r}\,,\ -1 + r\right\}$

Note that the *x* and *y* components of C1 and C2 are the negatives of one another, consistent with the symmetry discussed earlier. In addition, for each equilibrium we have $x = y$, which is a direct consequence of $\dot{x} = \sigma(y - x)$ at equilibrium.

■ 3. Bifurcations with Changing r

We now begin the major task in this notebook, namely the study of the solutions as the parameter *r* is changed.

■ 3.1 The Range 0 < r < 1.

We look at the eigenvalues at the origin, which is the only equilibrium in this range of *r*-values.

```
eigval[C0]
```

$\left\{-b,\ \frac{1}{2}\left(-1 - \sigma - \sqrt{1 - 2\,\sigma + 4\,r\,\sigma + \sigma^2}\,\right),\ \frac{1}{2}\left(-1 - \sigma + \sqrt{1 - 2\,\sigma + 4\,r\,\sigma + \sigma^2}\,\right)\right\}$

The quantity under the square root sign can be written as $(1 - \sigma)^2 + 4r\sigma$, which shows that the square root is real. The first and second eigenvalues are obviously negative. The condition for the third eigenvalue to be negative is

$(1 - 2\sigma + 4r\sigma + \sigma^2) < (1 + \sigma)^2$,

and this reduces to $r < 1$. Thus the origin is a strictly stable node in this range. We look at some typical values, choosing first $r = 0.1$.

```
parmval = {10.0,0.1,8./3.};
```

```
eigsys[C0]
```

{{-10.1098, -2.66667, -0.890228}, {{-0.99994, 0.0109766, 0.},
   {0., 0., 1.}, {-0.739246, -0.673436, 0.}}}

We see the numerical confirmation of our prediction of a strictly stable node. There is a considerable variation in the eigenvalues -- a factor of over 10 between largest and smallest. As we predicted earlier, one of the eigenvectors is along the *z*-axis. We look at two other values of *r* -- 0.5 and 0.9.

```
parmval = {10.0,0.5,8./3.};

eigsys[C0]

{{-10.5249, -2.66667, -0.475062}, {{-0.998625, 0.0524216, 0.},
   {0., 0., 1.}, {-0.724097, -0.689698, 0.}}}


parmval = {10.0,0.9,8./3.};

eigsys[C0]

{{-10.9083, -2.66667, -0.0916731}, {{-0.9959, 0.0904603, 0.},
   {0., 0., 1.}, {-0.710355, -0.703843, 0.}}}
```

One of the eigenvalues is approaching zero as we approach the bifurcation at $r = 1$.

We use portrait3D to construct a simple phase portrait for the representative case $r = 0.5$.

```
parmval = {10.0,0.5,8./3.};

intlist = {{5,5,10},{5,-5,10},{-5,-5,10},{-5,5,10},
             {0,5,10},{5,0,10},{-5,0,10},{0,-5,10}};

t0 = 0.0;

h = 0.015;

nsteps = 200;

boxrat = {1,1,1};

plrange3D = {{-5,5},{-5,5},{0,10}};

setcolor[
   {Red, Blue, Green, Cyan, Magenta, Yellow, BurntSienna, Purple}];
```

```
portrait3D[intlist,t0,h,nsteps,1,2,3];
```


Lorenz {σ, r, b}={ 10.00,   0.50,   2.67}
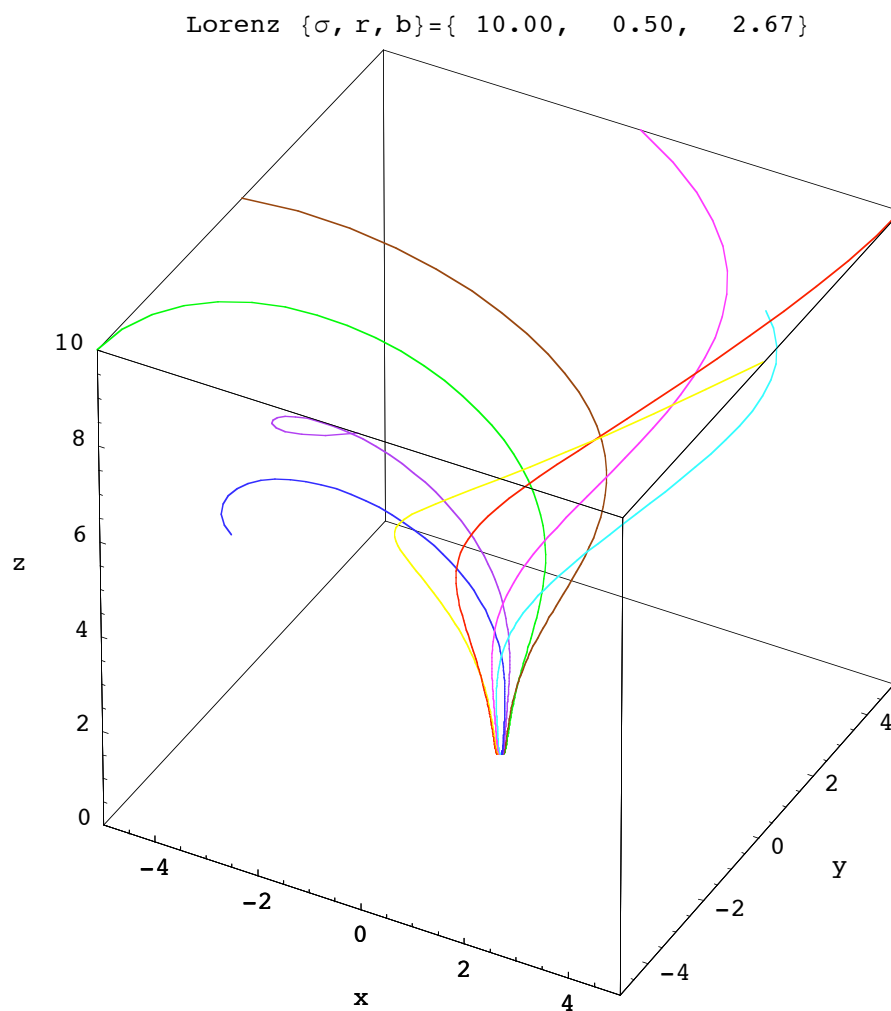
■ 3.2 The Pitchfork Bifurcation at r = 1.

For r < 1, there is one real equilibrium. For r > 1, there are three. At r = 1, the three equilibria are all at the origin, so we have a pitchfork bifurcation. We examine the stability near r = 1.

```
parmval = {10.0,1.01,8./3.};
```

```
classify[C0]
```

unstable

```
classify[C1]
```

strictly stable

```
classify[C2]
```

strictly stable

```
eigval[C0]
```

{-11.0091, -2.66667, 0.00908341}

```
eigval[C1]
```

{-11.0026, -2.64573, -0.0183213}

```
eigval[C2]
```

{-11.0026, -2.64573, -0.0183213}

Thus the origin (C0) goes from stable to unstable as r increases through 1 -- it is now a saddle with a one-dimensional unstable manifold and a two-dimensional stable manifold. The two new equilibria are stable nodes, so we have a supercritical pitchfork bifurcation at $r = 1$. For r near 1, the new equilibria are close to the origin:

```
N[eqstateval[C1]]
```

{-0.163299, -0.163299, 0.01}

```
N[eqstateval[C2]]
```

{0.163299, 0.163299, 0.01}

■ 3.3 The Node-Spiral Transition at r = 1.346.

The next event along the r-trail is a change in the character of the equilibria C1 and C2 from nodes to spirals. As shown below, this occurs at about r = 1.346. This is the first appearance of oscillatory behavior in the system as r increases from 0.

```
parmval = {10.0,1.345,8./3.};
eigval[C1]
```

{-11.0878, -1.34549, -1.23337}

```
parmval = {10.0,1.347,8/3};
eigval[C1]
```

{-11.0883, -1.28918 + 0.0839114 i, -1.28918 - 0.0839114 i}

The eigenvalues for C2 are the same. We see that the oscillations are very weak (i.e. low frequency) just above the transition. Now we look at the parameter value $r = 10$ which is well into the oscillatory range.

```
parmval = {10.0, 10.0,8./3.};
```

```
N[eqstateval[C1]]
```

```
{- 4.89898, - 4.89898, 9.}
```

```
eigval[C1]
```

```
{- 12.4757, - 0.595497 + 6.17416 ⅈ, - 0.595497 - 6.17416 ⅈ}
```

We now see a damped oscillation with an oscillation period of

```
2 π / 6.17416
```

```
1.01766
```

Let's do an integration so that we can look at the solution for $r = 10$ in detail.

```
initvec = {1,1,1};
```

```
t0 = 0.0;
```

```
h = 0.02;
```

```
nsteps = 400;
```
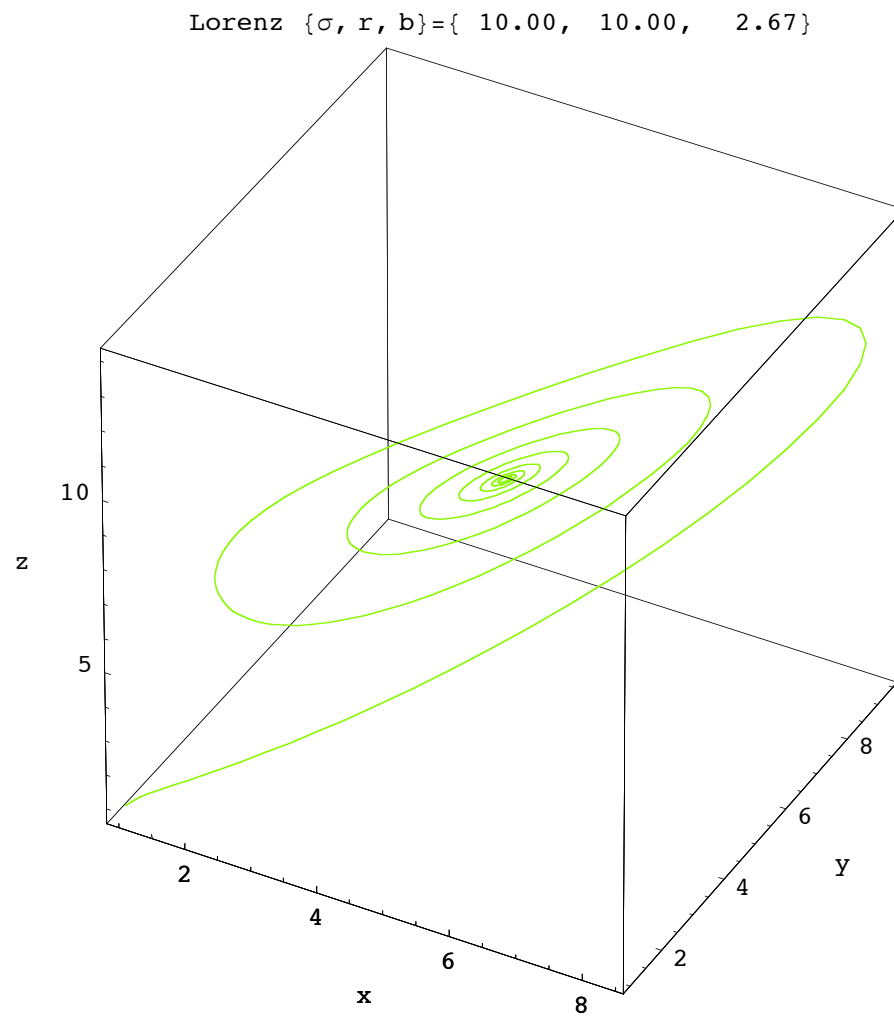
```
sol1 = integrate[initvec,t0,h,nsteps];
```

```
plotreset;
```

```
boxrat = {1,1,1};
```

```
setcolor[{LawnGreen}];
```

```
graph1 = phaser3D[sol1];
```


Lorenz $\{\sigma, r, b\} = \{ 10.00, 10.00, 2.67\}$

A nice spiral into the equilibrium C2. Let's look at the x-y phase plane plot, followed by a plot of x versus time.

```
plrange = {{0, 10}, {0, 10}};
```

```
graph1xy = phaseplot[sol1,1,2];
```


Lorenz $\{\sigma, r, b\} = \{ 10.00, \ 10.00, \ 2.67\}$
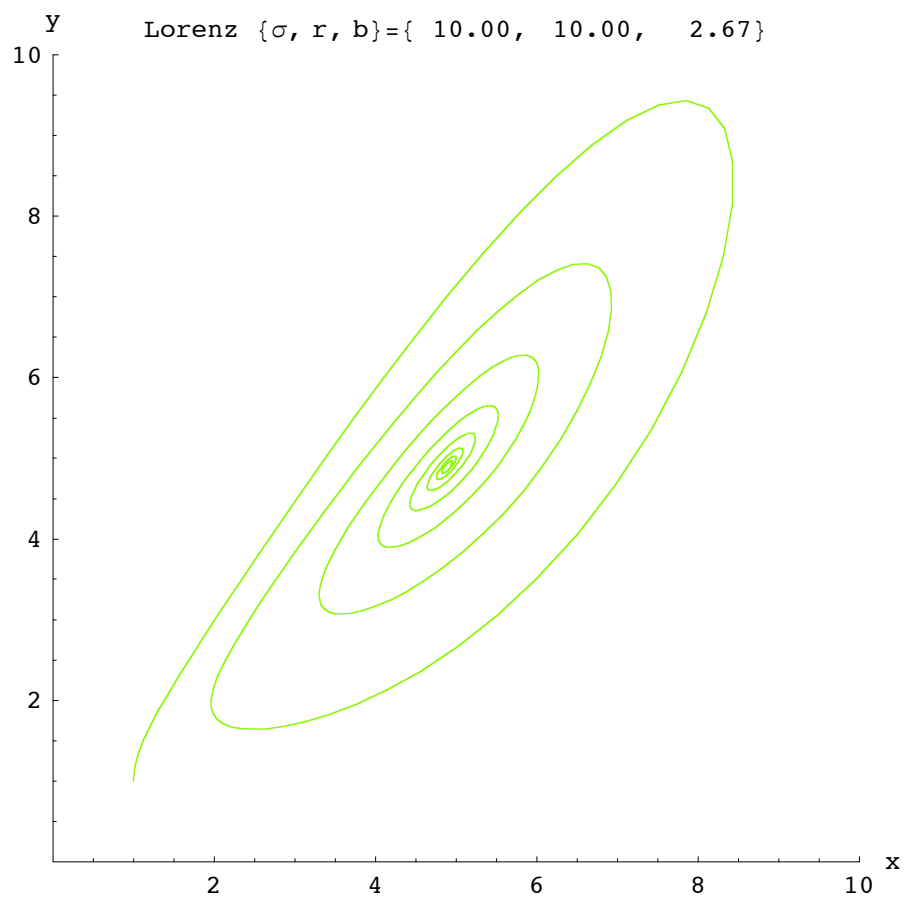
```
asprat = 0.5;

setcolor[{Red}];

plrange = {{0, 9}, {0, 10}};

graph1time = timeplot[sol1,1];
```


Lorenz $\{\sigma, r, b\} = \{ 10.00, \ 10.00, \ 2.67\}$
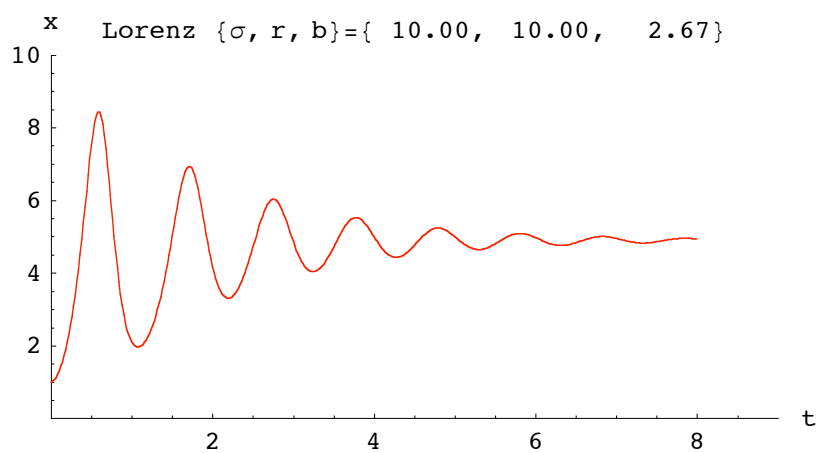
■ **3.4 The Homoclinic Orbit at r = 13.926.**

For values of r < 13.926, the solution leaving the origin along the local unstable manifold spirals into the nearer of C1 and C2. For values of r > 13.926, the solution leaving the origin loops around the nearer spiral equilibrium and then settles into the further one. Let's verify that. We begin by finding the local direction of the unstable manifold leaving the origin.

```
parmval = {10.0, 12.0, 8./3.};

eiger = eigsys[C0]

{{-17.3427, 6.34272, -2.66667}, {{-0.806044, 0.591856, 0.},
   {-0.521936, -0.852985, 0.}, {0., 0., 1.}}}
```

Now we take an initial condition slightly displaced from the origin, along the eigenvector of the unstable manifold -- that is, the eigenvector associated with the eigenvalue 6.34272, namely

```
unstabvec = eiger[[2, 2]]

{-0.521936, -0.852985, 0.}


t0 = 0.0;

h = 0.02;

nsteps = 600;

startvec = h*unstabvec

{-0.0104387, -0.0170597, 0.}


sol2 = integrate[startvec,t0,h,nsteps];

setcolor[{Magenta}];
```

```
graph2 = phaser3D[sol2];
```

Lorenz {σ, r, b}={ 10.00, 12.00,   2.67}
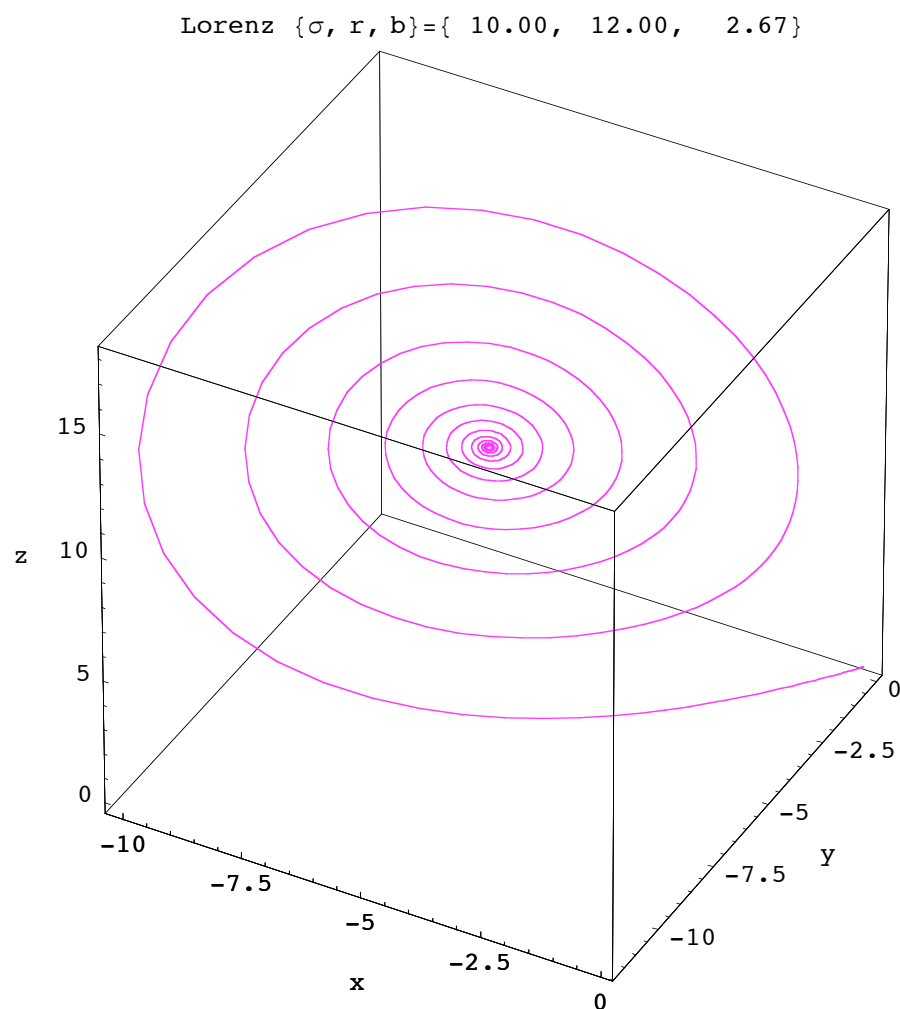
   We see that the solution leaving the origin spirals into the nearest equilibrium (C1 in this case). Let's look at a 2D view -- in the xy phase plane.

```
asprat = 1.0;
```

```
plrange = {{-12, 0}, {-12, 0}};
```

```
graph2xy = phaseplot[sol2,1,2];
```

Lorenz $\{\sigma, r, b\}=\{$ 10.00, 12.00, 2.67$\}$
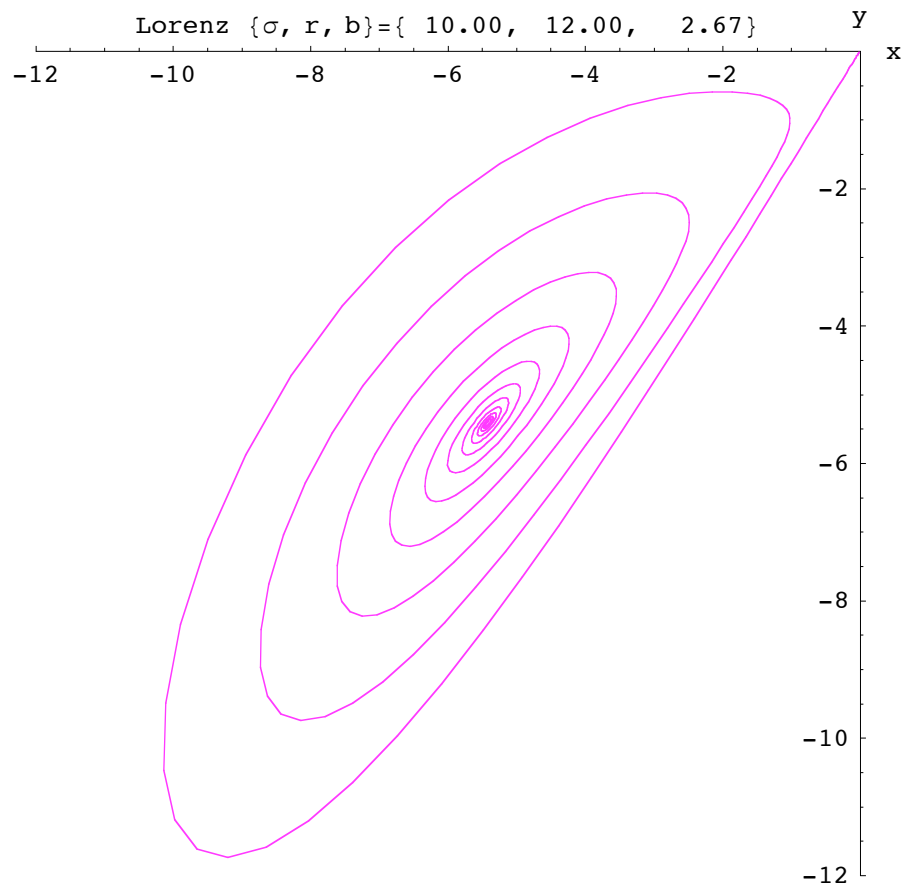
If we leave the origin on the unstable manifold in the other direction, we go to the equilibrium C2. We carry that out.

```
startvec = -h*unstabvec
```

$\{0.0104387, 0.0170597, 0.\}$

```
sol2a = integrate[startvec,t0,h,nsteps];
```

```
plrange = {{0, 12}, {0, 12}};
```

```
graph2xya = phaseplot[sol2a,1,2];
```
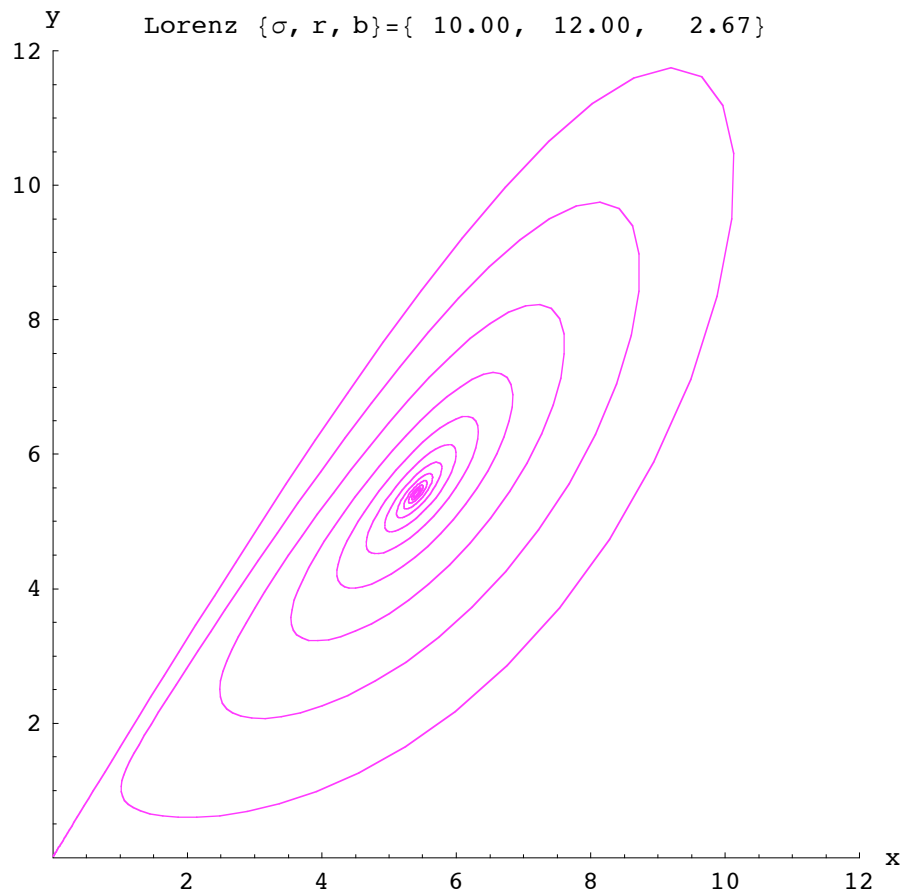

Now we look at the same sequence of calculations for r = 16.

```
parmval = {10.0, 16.0, 8./3.};
```

```
eiger2 = eigsys[C0]
```

```
{{-18.9257, 7.92572, -2.66667}, {{-0.746043, 0.665897, 0.},
   {-0.487178, -0.873302, 0.}, {0., 0., 1.}}}
```

```
unstabvec = eiger2[[2, 2]]
```

```
{-0.487178, -0.873302, 0.}
```

```
startvec = h*unstabvec
```

```
{-0.00974357, -0.017466, 0.}
```

```
sol3 = integrate[startvec,t0,h,nsteps];
```

```
graph3 = phaser3D[sol3];
```

Lorenz {σ, r, b}={ 10.00, 16.00,   2.67}
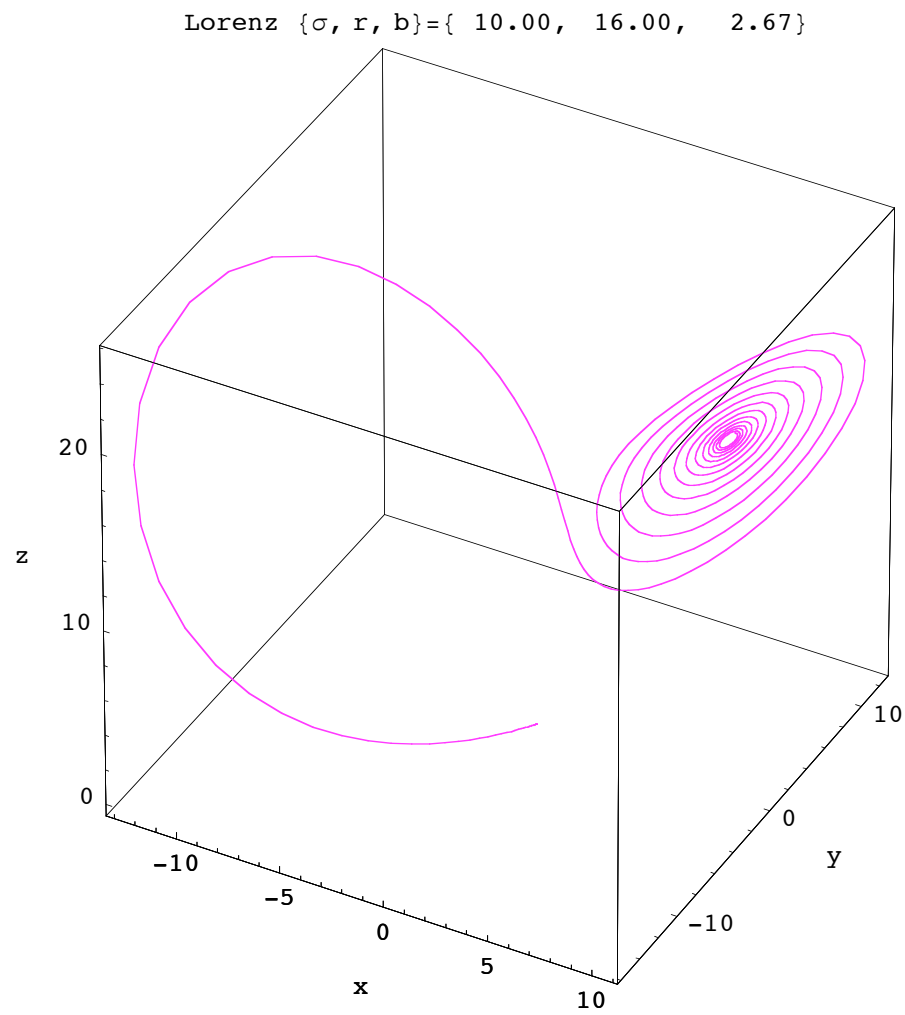
Now the solution leaving the origin makes one loop around the nearest equilibrium (C1), and then travels to C2 where it spirals in. The 2D view is given below.

```
plrange = {{-15, 15}, {-15, 15}};
```

```
graph3xy = phaseplot[sol3,1,2];
```


Lorenz {σ, r, b}={ 10.00, 16.00,   2.67}
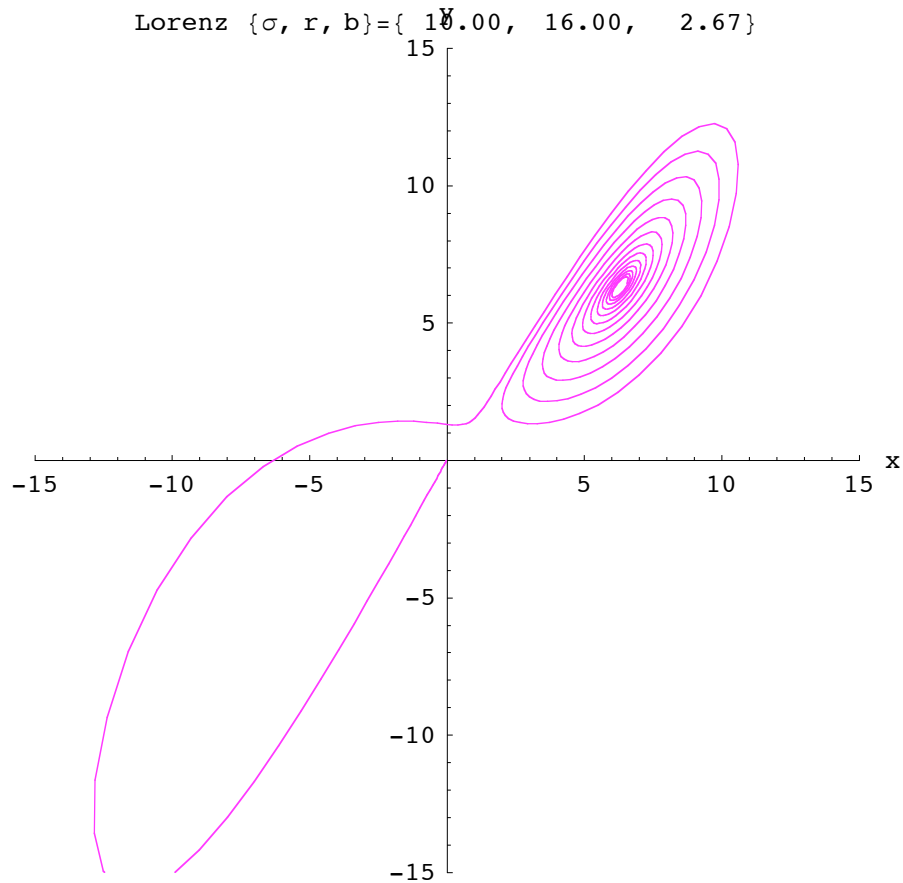
By symmetry, we will reach C1 instead of C2 if we leave the origin in the opposite direction along the unstable manifold.

What is the transition case between the two cases we have just looked at? As numerical exploration has shown, the transition occurs for r = 13.926..., and the solution leaving the origin along the local unstable manifold returns to the origin along the local stable manifold (tangent to the z-axis). It is hard to construct this homoclinic orbit numerically, because it is unstable. We take an indirect approach, and look at a sequence of plots for a range of r values bracketing 13.926. We construct the orbits using in each case an initial point displaced h from the origin along the local unstable manifold.

```
initvec := h*Part[Part[eigsys[C0],2],2];

parmlist = {{10.0, 13.84, 8./3.},{10.0, 13.88, 8./3.},
            {10.0, 13.92, 8./3.},{10.0, 13.96, 8./3.},
            {10.0, 14.00, 8./3.}};

t0 = 0.0;

h = 0.01;

nsteps = 300;

plrange = {{-15,15},{0,25}};
```
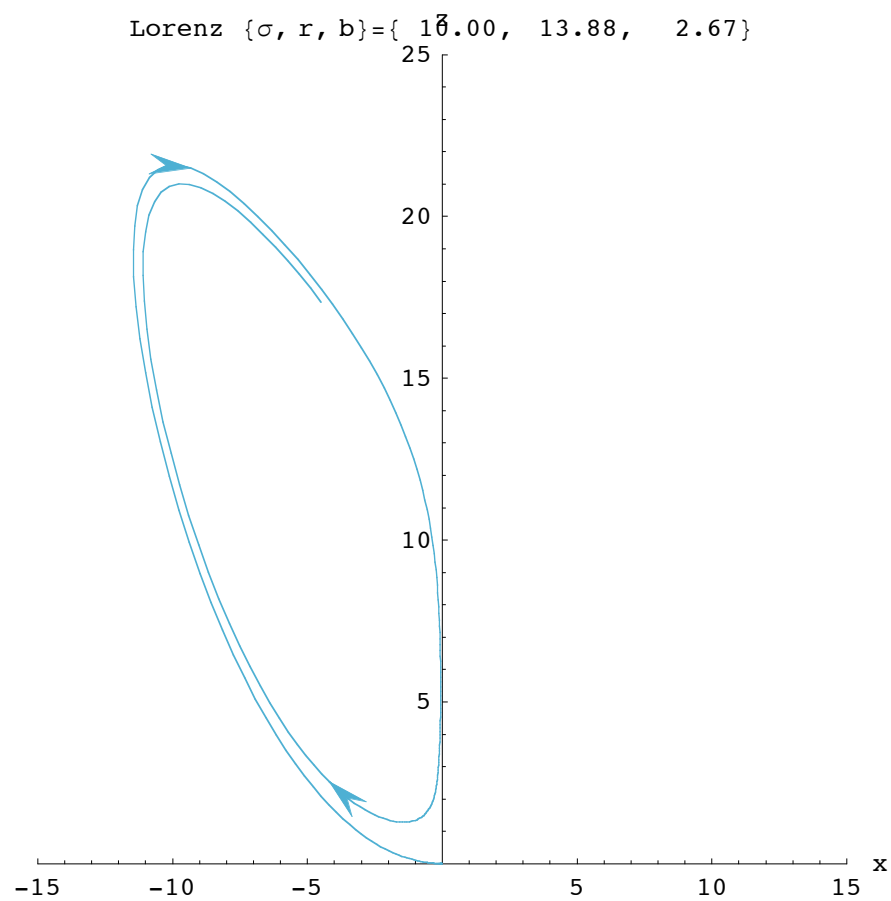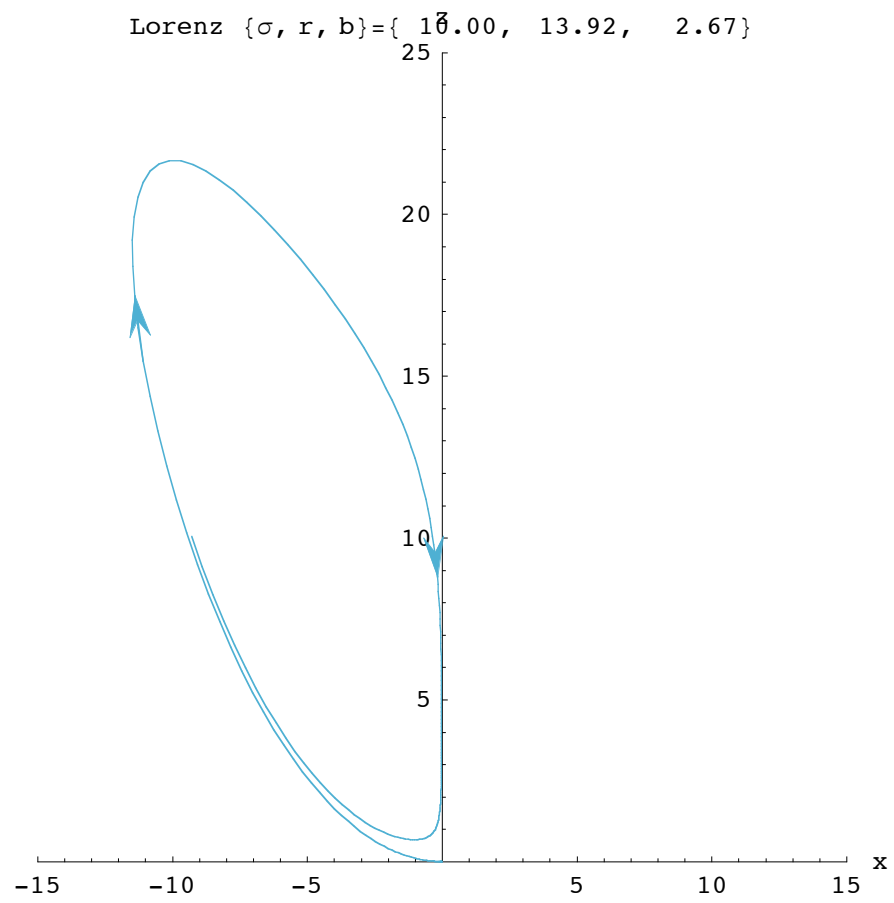
```
setcolor[{Peacock}];
```

We use a Do loop to construct the orbits for all the above parameter values.
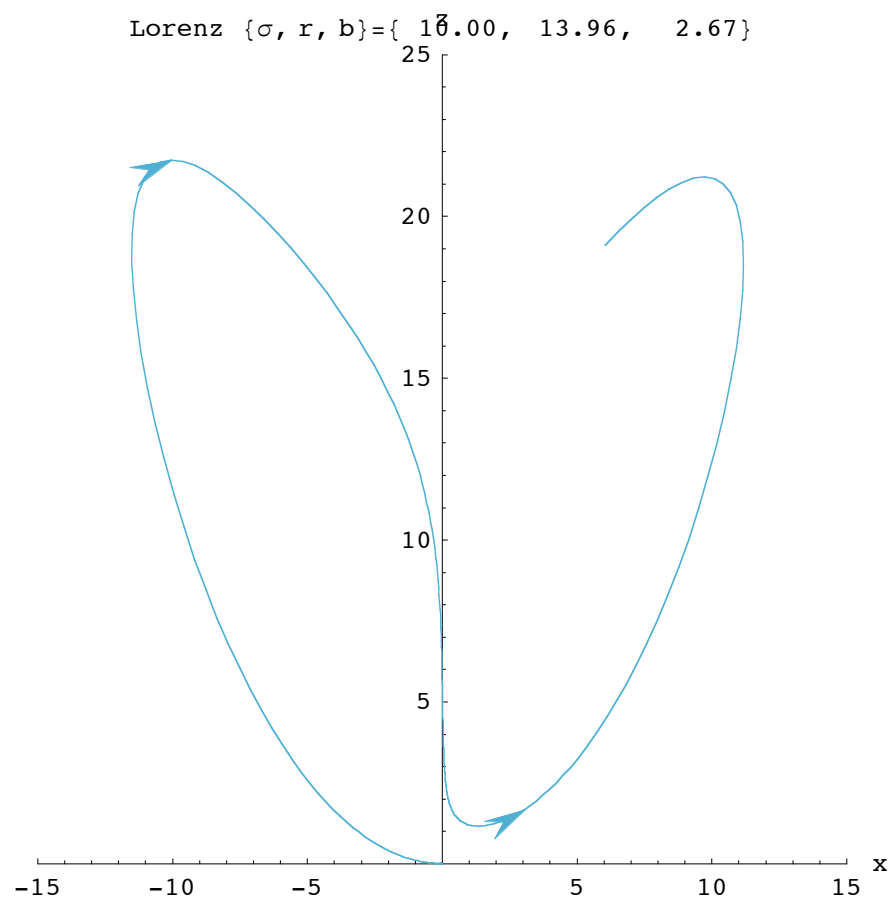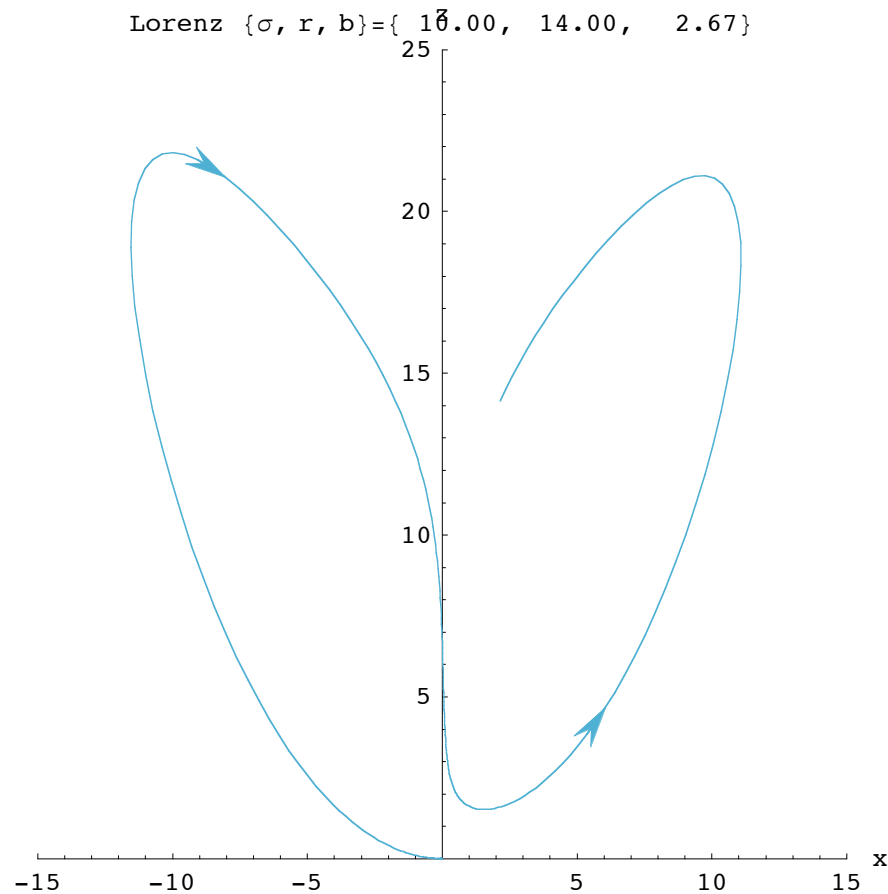
```
arrowflag = True; arrowvec = {1 / 3, 2 / 3};

Do[(parmval = parmlist[[q]];startvec = initvec;
    phaseplot[integrate[startvec,t0,h,nsteps],1,3]),
    {q,1,Length[parmlist]}]
```

Lorenz $\{\sigma, r, b\}=\{$ 10.00, 13.84, 2.67$\}$

Lorenz $\{\sigma, r, b\}=\{ 10.00, 13.88, 2.67\}$

Lorenz {σ, r, b}={ 10.00, 13.92, 2.67}

Lorenz $\{\sigma, r, b\} = \{\ 10.00,\ \ 13.96,\ \ \ 2.67\}$

Lorenz {σ, r, b}={ 10.00, 14.00, 2.67}
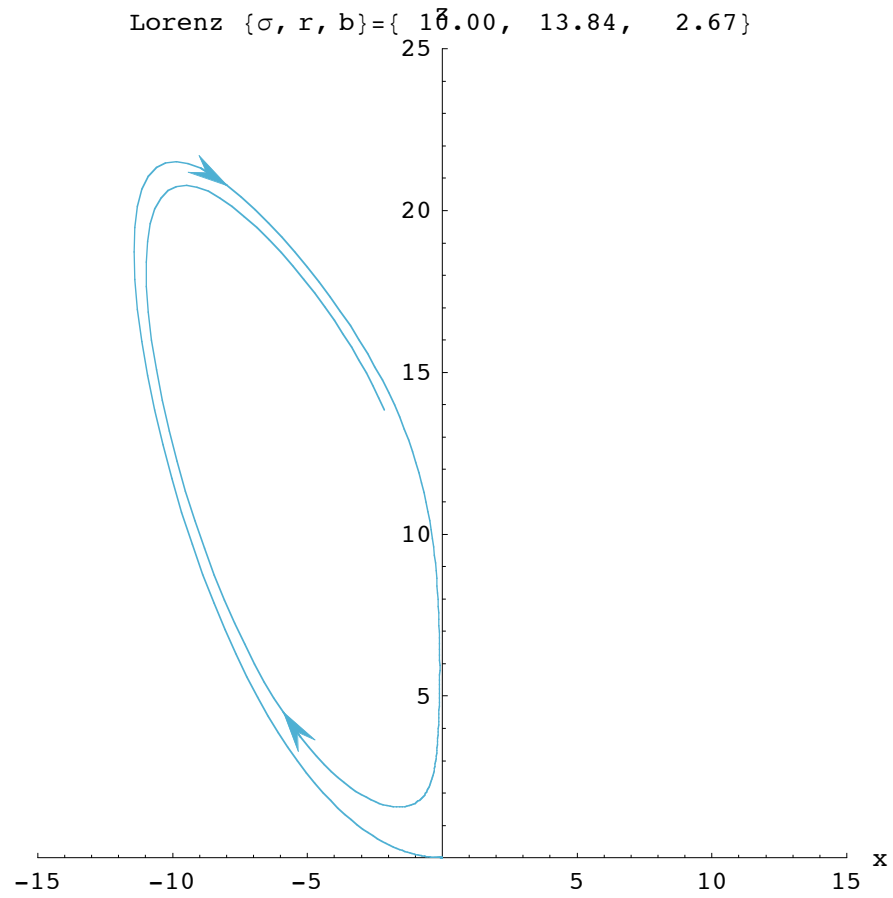
From these graphs, it is clear that somewhere between r = 13.92 and r = 13.96 there is a homoclinic orbit. When r exceeds the value corresponding to the homoclinic orbit, there also exists an unstable limit cycle, which is difficult to find numerically. This is an example of what is called a homoclinic bifurcation.

By symmetry, we get the mirror image of the above picture if we start on the other side of the origin.

■ 3.5 The Range 13.926 < r < 24.06 and Pre-Chaotic Transients.

The main event in this parameter range is the increasingly oscillatory nature of the solution as *r* increases. For the larger values of *r* in this range, some solutions go back and forth between C1 and C2 many times before finally spiraling into one of them. These kinds of solutions are called pre-chaotic transients. We will look at one of them shortly.

The other event in this range (beyond our capabilities here to look at numerically) is the shrinking of the unstable limit cycles around C1 and C2. The cycles are heading for a subcritical Hopf bifurcation which occurs at *r* = 24.74... .

We look at a pre-chaotic transient now.

```
parmval = {10.0,20.0,8./3.};

h = 0.02;

t0 = 0.0;
```

```
nsteps = 3000;

initvec = {14.0,12.0,10.0};

chaotrans = integrate[initvec,t0,h,nsteps];

plotreset;

plrange = {{0,60},{-15,15}};

asprat = 0.5;

setcolor[{Blue}];

imsize = 380;

chaotranstimex = timeplot[chaotrans,1];
```
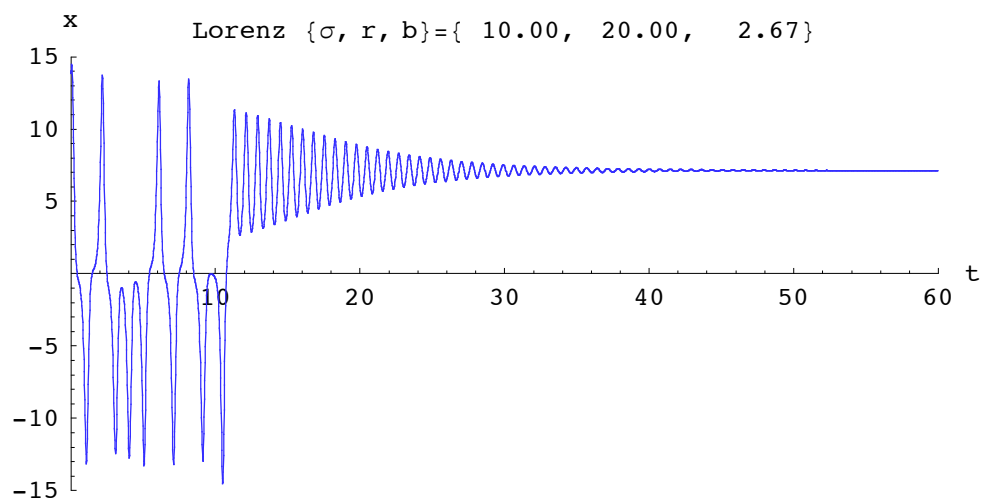

We see eight zero crossings, meaning eight transitions between C1 and C2, with the system eventually settling into C2. Now let's do this integration again with a smaller time step and the same final time.
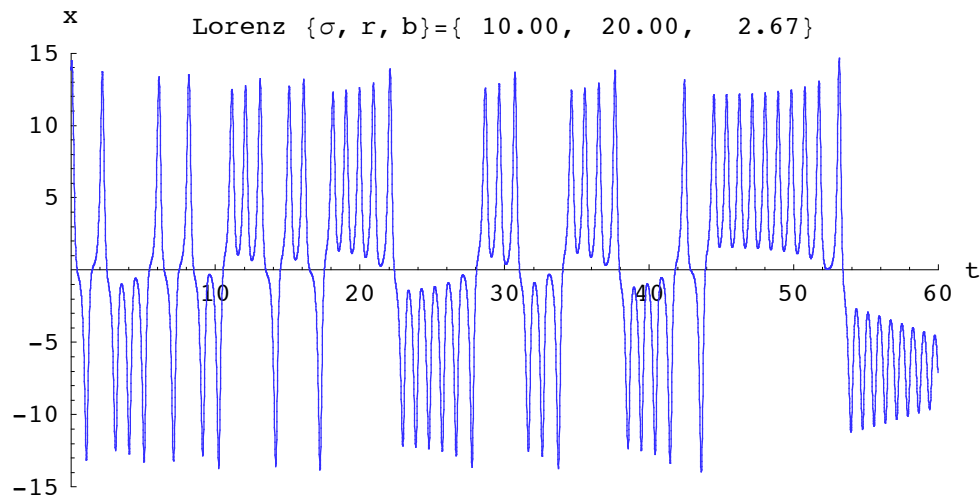
```
h = 0.01;

nsteps = 6000;

chaotrans2 = integrate[initvec,t0,h,nsteps];
```

```
trans2timex = timeplot[chaotrans2,1];
```


Lorenz {σ, r, b}={ 10.00, 20.00, 2.67}

Now we have 21 transitions, and the system ends up in C1 rather than C2, a different final state than before. What a sobering example! Halving the time step produced a completely different solution. We don't even know the true final fate of the system -- does it go to C1 or C2? The main lesson in these calculations is that these pre-chaotic transients share some properties with their even more difficult chaotic cousins. Perhaps you would want to explore the question of whether we can overcome the uncertainty about the final state with a sufficiently small time step.

Let's finish this off by looking at the 3D phase plot of this solution.

```
plotreset;

boxrat = {1,1,1};

plrange3D = Automatic;

setcolor[{Maroon}];
```

```
transgraph3D = phaser3D[chaotrans2];
```

Lorenz $\{\sigma, r, b\} = \{\ 10.00,\ \ 20.00,\ \ \ 2.67\}$
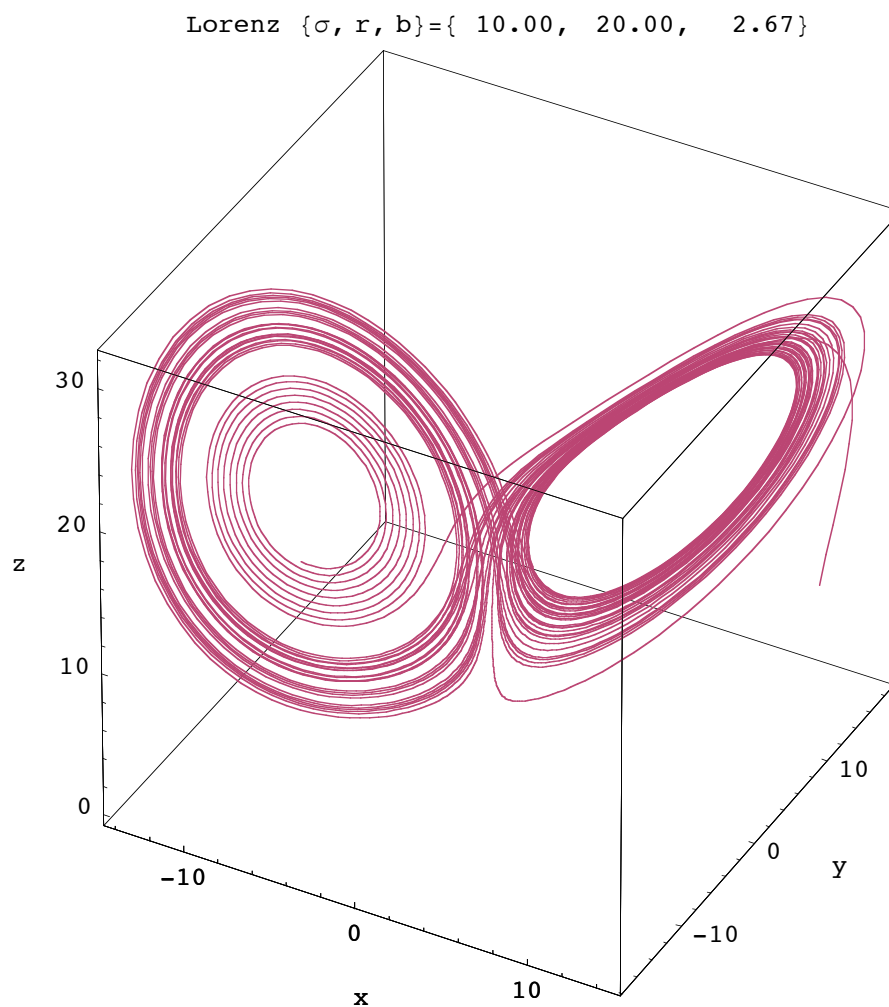
This is starting to look a lot like the famous chaotic Lorenz attractor.

▪ 3.6 Point and Chaotic Attractors for 24.06 < r < 24.74.

As *r* takes on values beyond 24.06, some of the pre-chaotic transients slip over the brink into true chaos. Interestingly enough, the equilibria C1 and C2 remain stable for 24.06 < *r* < 24.74. Thus in this small range, the system has three attractors: a chaotic attractor, and two stable spiral equilibria. We will verify the nature of the equilibria, and then look at a few interesting orbits.

```
parmval = {10.0,24.4,8./3.};

eigval[C0]

{-21.7558, 10.7558, -2.66667}
```

```
eigval[C1]
```

```
{-13.6462, -0.0102336 + 9.56315 i, -0.0102336 - 9.56315 i}
```

```
parmval = {10.0,24.73,8./3.};
```

```
eigval[C0]
```

```
{-21.857, 10.857, -2.66667}
```

```
eigval[C1]
```

```
{-13.6663, -0.000206807 + 9.62329 i, -0.000206807 - 9.62329 i}
```

```
parmval = {10.0,24.75,8./3.};
```

```
eigval[C0]
```

```
{-21.8631, 10.8631, -2.66667}
```

```
eigval[C1]
```

```
{-13.6675, 0.000397583 + 9.62692 i, 0.000397583 - 9.62692 i}
```

The above calculations verify that the equilibria C1 and C2 are stable spirals until we reach r = 24.74, at which point they become unstable spirals. Although the calculations of unstable limit cycles are beyond our capabilities here, it can be shown that in fact r = 24.74 is a subcritical Hopf bifurcation in which, as *r* is increased, the unstable limit cycles are absorbed by the equilibria which then become unstable themselves. The character of the equilibrium at the origin is unchanged. It remains a saddle with a one-dimensional unstable manifold.

We set an *r* value midway in this range, and see if we can find orbits leading to each of three attractors (by symmetry, we can just do one of C1 or C2). For the spiral point attractor, we will cheat a little by starting very near the attractor, and then integrating backwards in time, to construct in that way an orbit which connects some distant point to the attractor.

```
parmval = {10.0,24.4,8./3.};
```

```
N[eqstateval[C1]]
```

```
{-7.89937, -7.89937, 23.4}
```

```
initvec = % + 0.001*{1,1,1}
```

```
{-7.89837, -7.89837, 23.401}
```

```
h = -0.001;
```

```
t0 = 0.0;
```

```
nsteps = 1000;
```
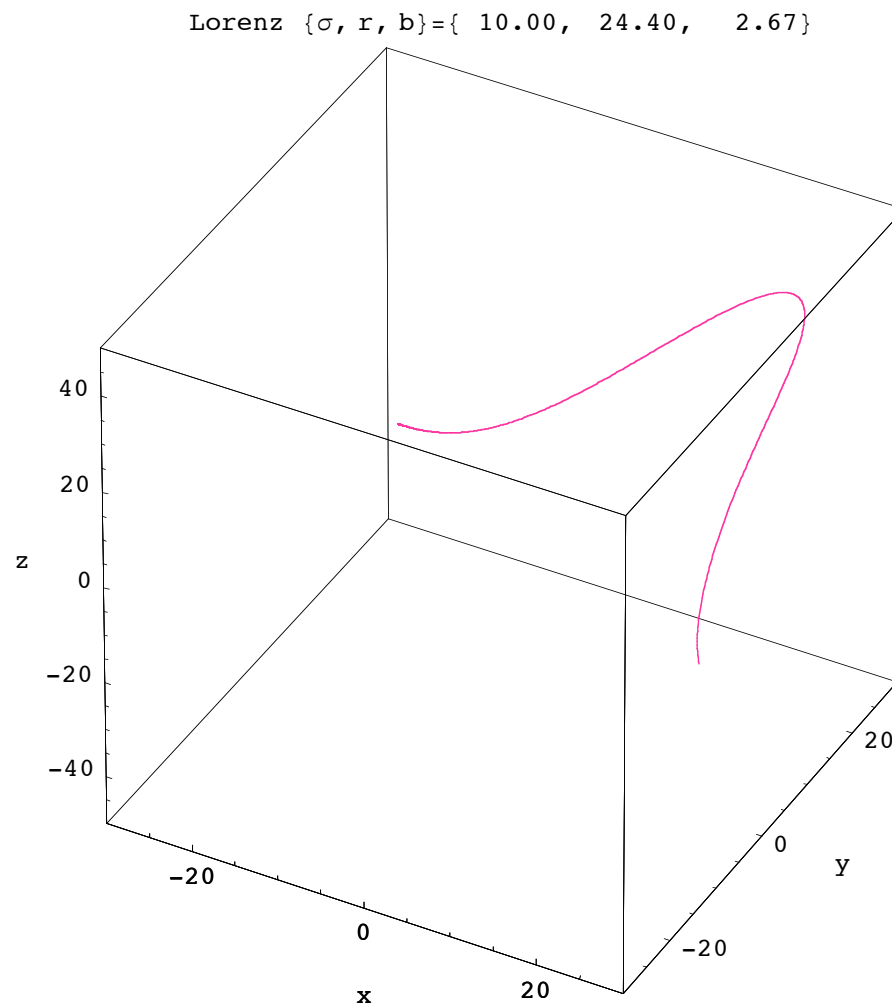
```
rangeflag = True;

ranger = {{-30,30},{-30,30},{-50,50}};

plrange3D = ranger;

boxrat = {1,1,1};

ans1 = integrate[initvec,t0,h,nsteps];

 setcolor[{DeepPink}];

graphans1 = phaser3D[ans1];
```


Lorenz {σ, r, b}={ 10.00, 24.40, 2.67}

Let's see where this solution went out of bounds.

```
firstx
```

{30.4011, -14.574, 2.15898}

The x-coordinate went out of bounds in the backward integration.

      The stable spirals have eigenvalues with very small real parts. The one large negative real eigenvalue tells us that the solutions rapidly approach the plane of the spiral, after which they slowly spiral in. Let's try to observe at least part of the spiral by integrating forward in time, and magnifying our view. For this direction in time, we can turn off range checking to speed up the integration.

```
rangeflag = False;

h = 0.02;

ans2 = integrate[initvec,t0,h,nsteps];

staterange[ans2]

 {{x, {-7.90032, 0.44}, {-7.89827, 0.1}},
  {y, {-7.90068, 0.36}, {-7.89805, 0.7}},
  {z, {23.3983, 0.86}, {23.4017, 0.54}}}


plrange3D = {{-7.901,-7.898},{-7.901,-7.898},{23.398,23.402}};

boxrat = {1,1,1};

 setcolor[{PeachPuff}];
```

```
graphans2 = phaser3D[ans2];
```

Lorenz {σ, r, b}={ 10.00, 24.40,   2.67}
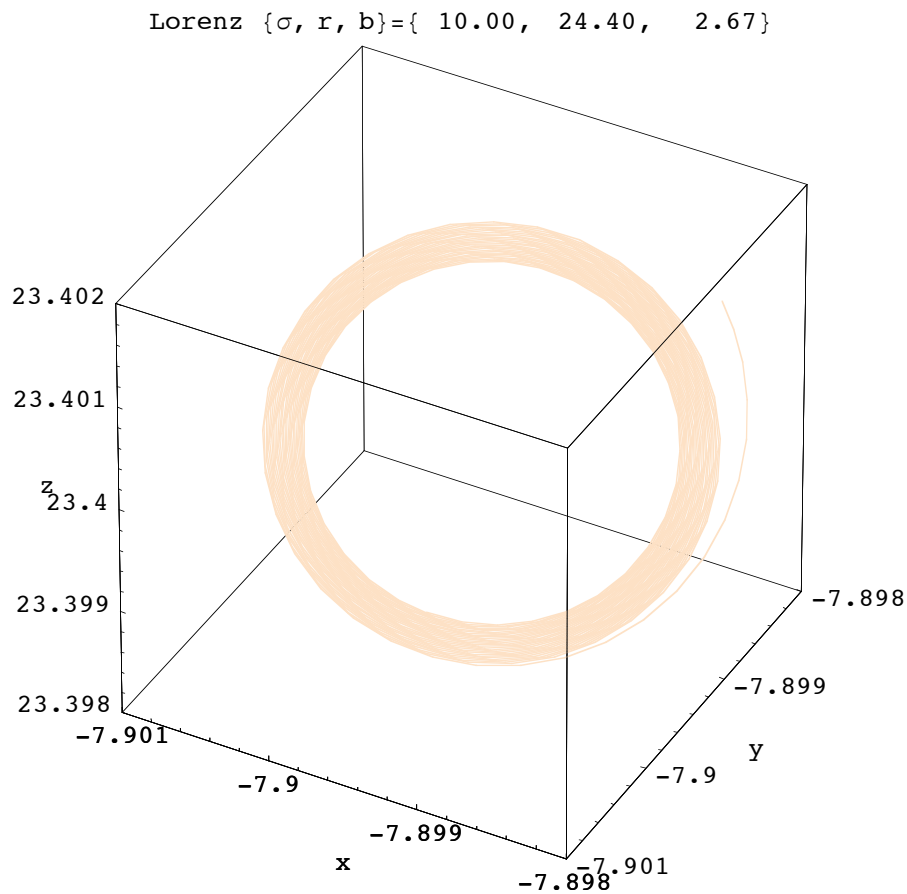
A VERY slow spiral, which we could have predicted from the eigenvalues.

Now we look for a chaotic solution. We will use plots of x versus time to track our progress.
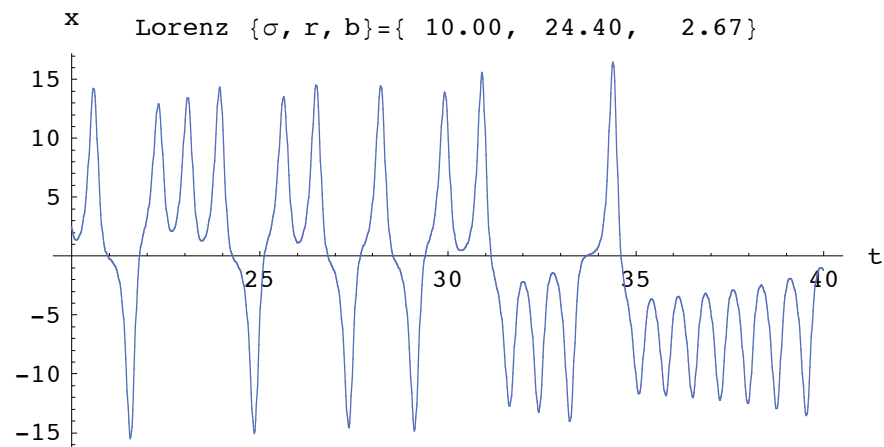
```
h = 0.02;

nsteps = 1000;

t0 = 0.0;

initvec = {10,10,10};

intreset;

plotreset;

asprat = 0.5;

cha1 = integrate[initvec,t0,h,nsteps];
```

```
setcolor[{Cobalt}];

chagraph1 = timeplot[cha1,1];
```
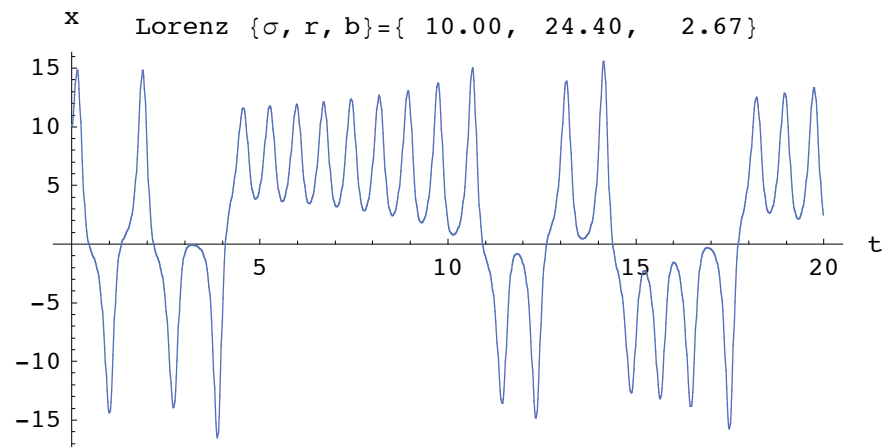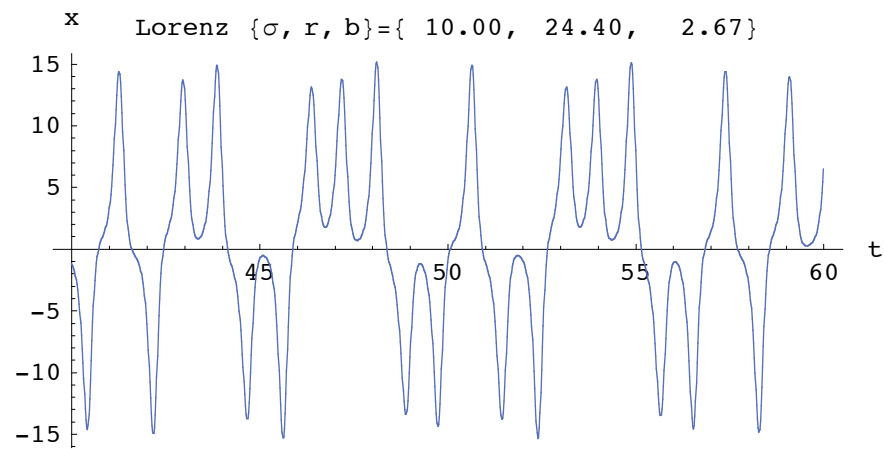

We continue the integration.

```
cha2 = integrate[lastx,lastt,h,nsteps];

chagraph2 = timeplot[cha2,1];
```


One more time.

```
cha3 = integrate[lastx,lastt,h,nsteps];
```

```
chagraph3 = timeplot[cha3,1];
```


At this point, we can be reasonably sure that this is a chaotic solution. Let's combine our three pieces and make a phase plot.
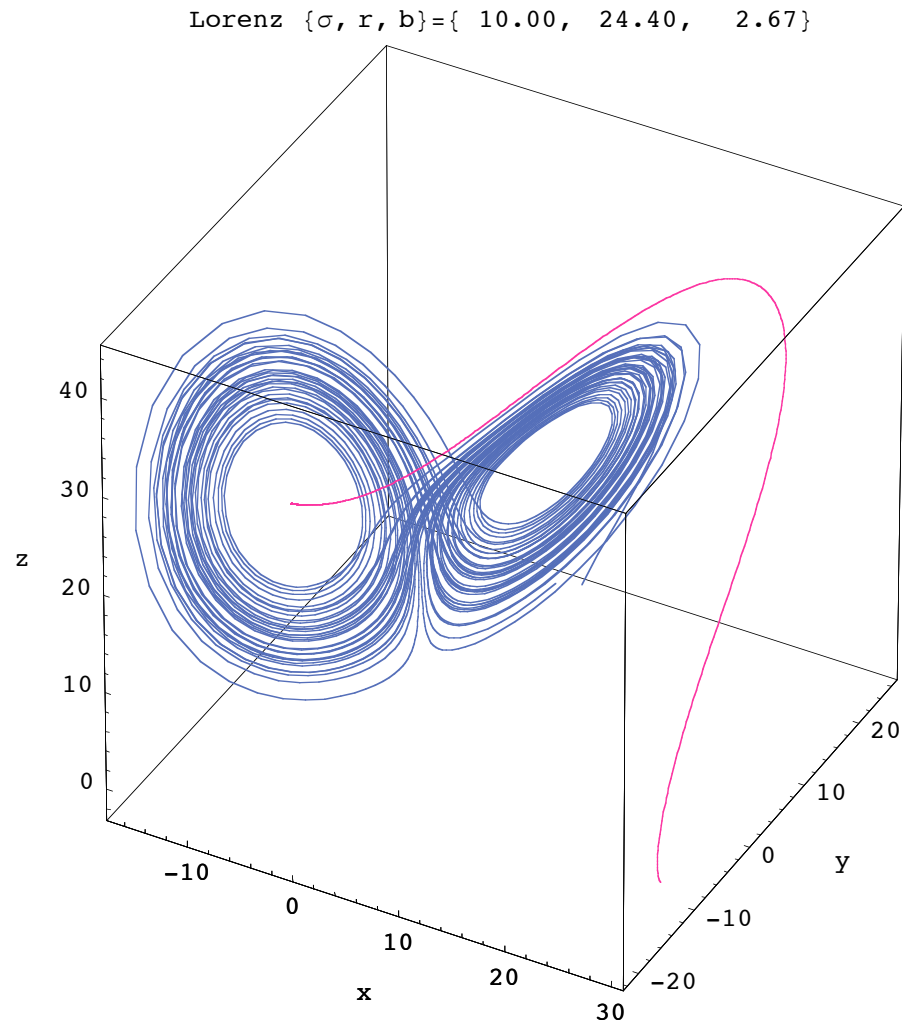
```
bigcha = joinsol[cha1,joinsol[cha2,cha3]];

boxrat = {1,1,1};
```

```
bigchagraph = phaser3D[bigcha];
```


Lorenz $\{\sigma, r, b\} = \{10.00, 24.40, 2.67\}$
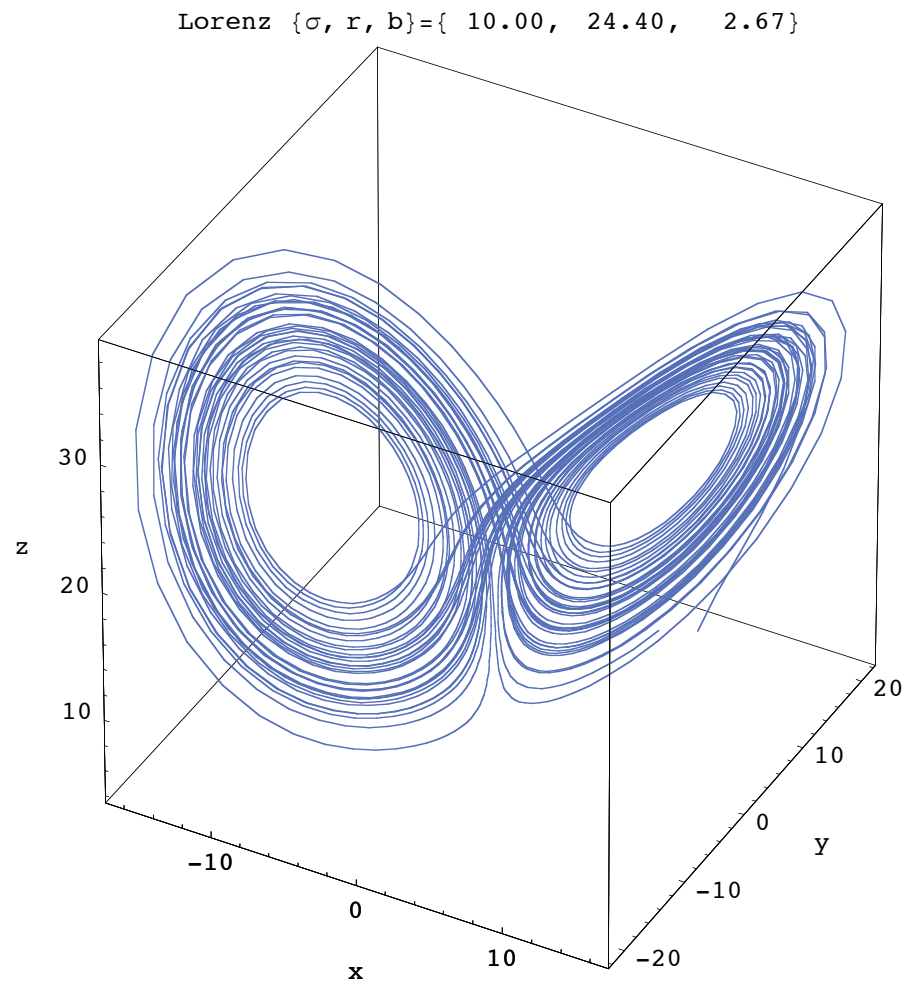
Let's add to this picture our earlier orbit which reached the attractor at C1.

show3D[graphans1,bigchagraph];

Lorenz {σ, r, b}={ 10.00, 24.40, 2.67}

■ **3.7 Chaotic Solutions for r = 28.**

We begin by resetting our plotting and integration defaults. Then we set our parameter values and examine the three equilibrium points.

intreset;

plotreset;

parmval = {10.0,28.0,8./3.};

eigval[C0]

{-22.8277, 11.8277, -2.66667}

```
eigval[C1]
```

$\{-13.8546,\ 0.0939556 + 10.1945\ \mathbb{i},\ 0.0939556 - 10.1945\ \mathbb{i}\}$

```
eigval[C2]
```

$\{-13.8546,\ 0.0939556 + 10.1945\ \mathbb{i},\ 0.0939556 - 10.1945\ \mathbb{i}\}$

Now all three equilibrium points are unstable, so the attractor is something other than a stable equilibrium. We construct and plot a solution in order to get some idea of what the attractor might be. We carry out an integration of 6000 steps.

```
h = 0.01;

t0 = 0.0;

nsteps = 6000;

initvec = {10,10,10};

lor1 = integrate[initvec,t0,h,nsteps];
```

We plot the solution, both in a 3D phase plot and in three 2D projections.
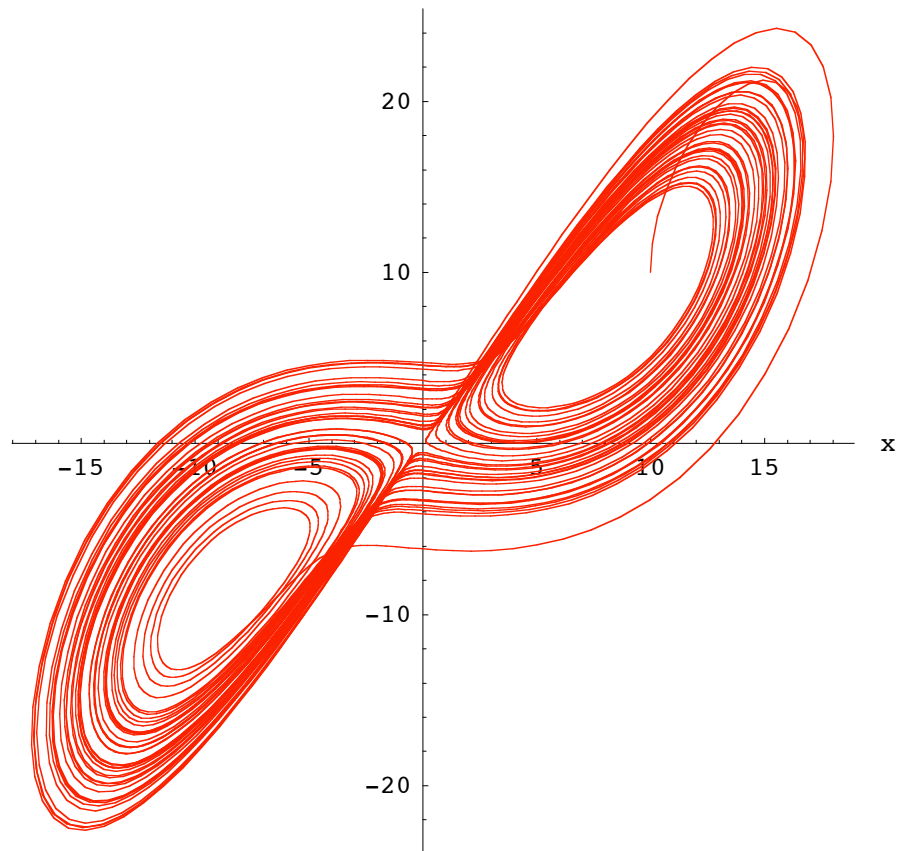
```
boxrat = {1,1,1};

setcolor[{Red}];
```

```
lorgraph1 = phaser3D[lor1];
```

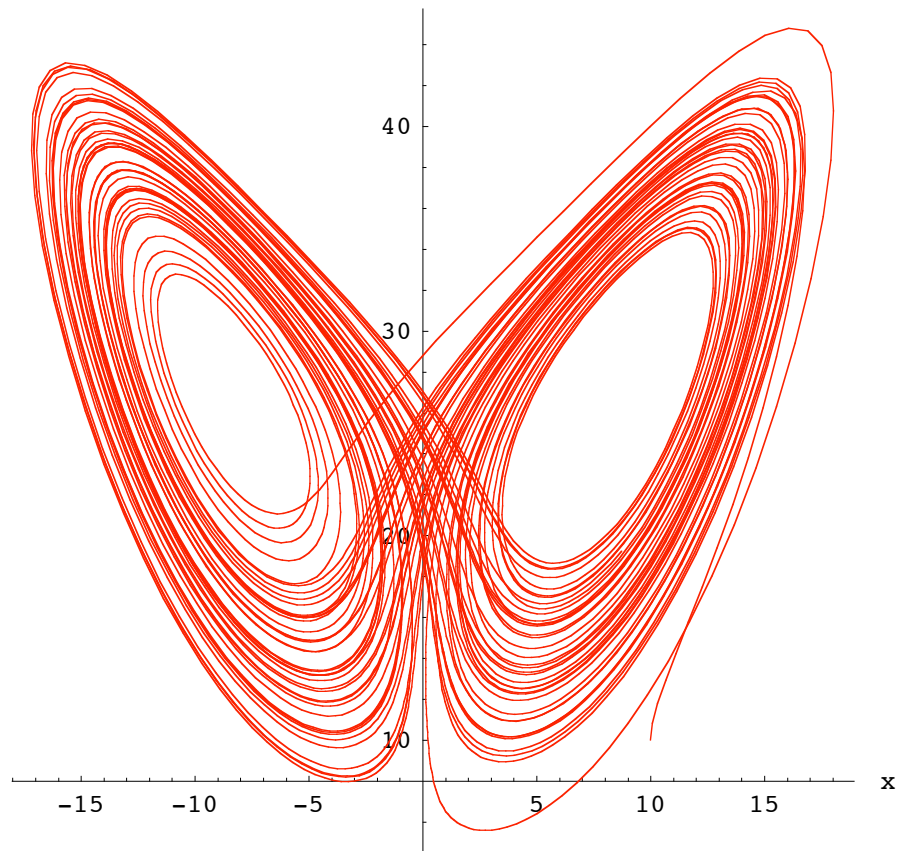Lorenz $\{\sigma, r, b\} = \{ 10.00, 28.00, 2.67\}$
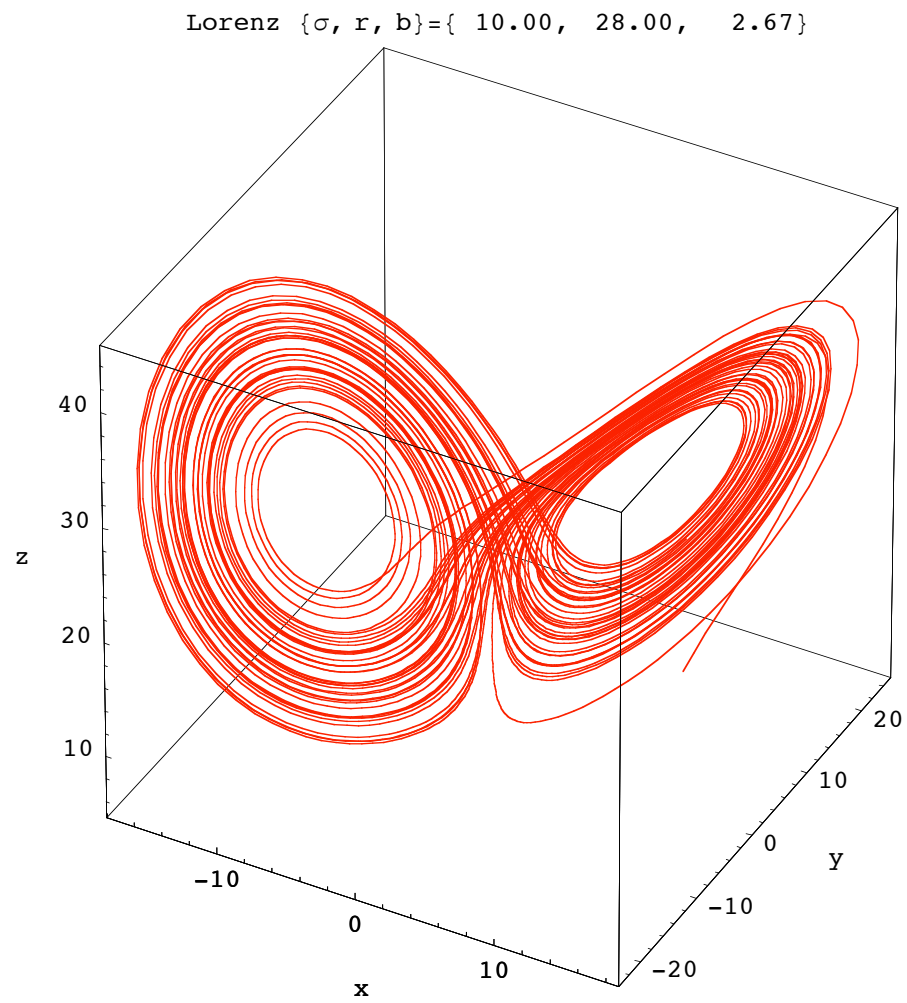

```
asprat = 1.0;
```
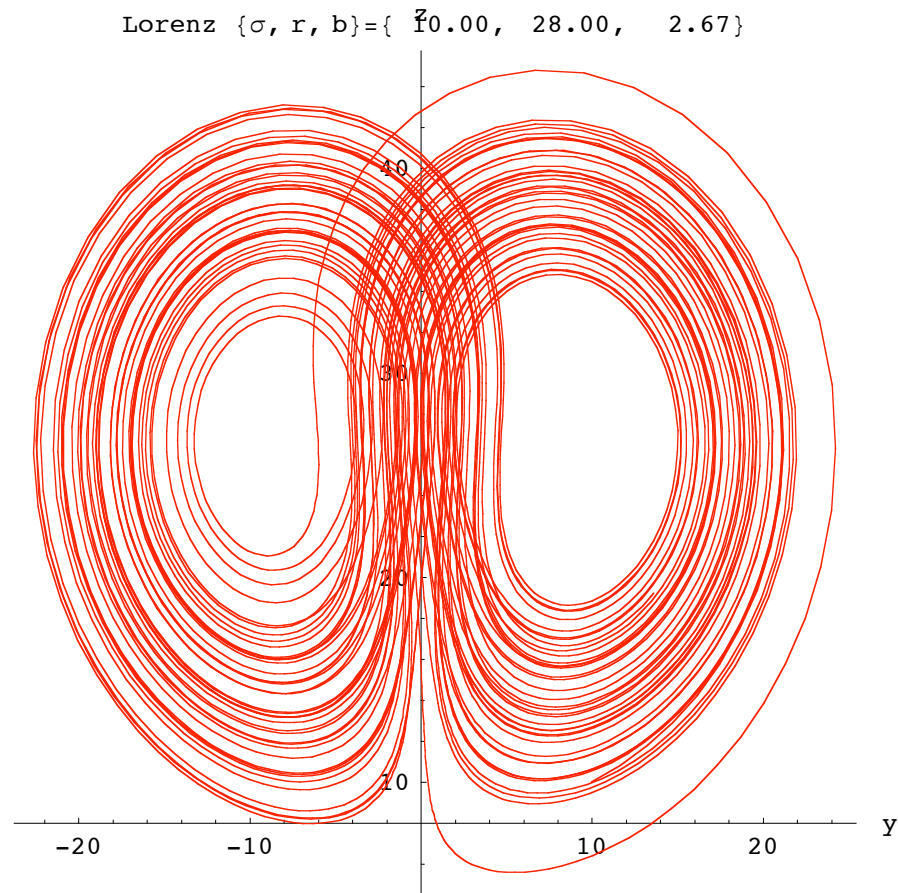
`lorgraph2 = phaseplot[lor1,1,2];`

Lorenz $\{\sigma, r, b\}=\{\ 10.00,\ 28.00,\ 2.67\}$

```
lorgraph3 = phaseplot[lor1,1,3];
```

Lorenz $\{\sigma, r, b\}=\{\ 10.00,\quad 28.00,\quad 2.67\}$

lorgraph4 = phaseplot[lor1,2,3];

Lorenz {σ, r, b}={ 10.00, 28.00, 2.67}

Clearly the attractor for this chaotic system is not a simple set. We discuss the nature of the attractor briefly in section 4.3 below.

■ 3.8 A Large-r Limit Cycle.

We have seen that as we increased r, chaos first appeared at $r = 24.06$, and by the time $r = 24.74$, it was the only possibility. One might think that the larger the value of r, the more chaotic the solution, but nonlinear sytems are full of non-monotonic surprises. It turns out that there are narrow windows of large r values in which the solutions are not chaotic, and the attractor is a stable limit cycle! We explore one such case here, namely $r = 100$.

parmval = {10.0,100.0,8./3.};

Now we check the equilibrium states.

eigval[C0]

{-37.4414, 26.4414, -2.66667}

eigval[C1]

{1.15811 + 18.1387 i, 1.15811 - 18.1387 i, -15.9829}

All three equilibria are unstable. We begin an integration with a more or less arbitrary initial condition.
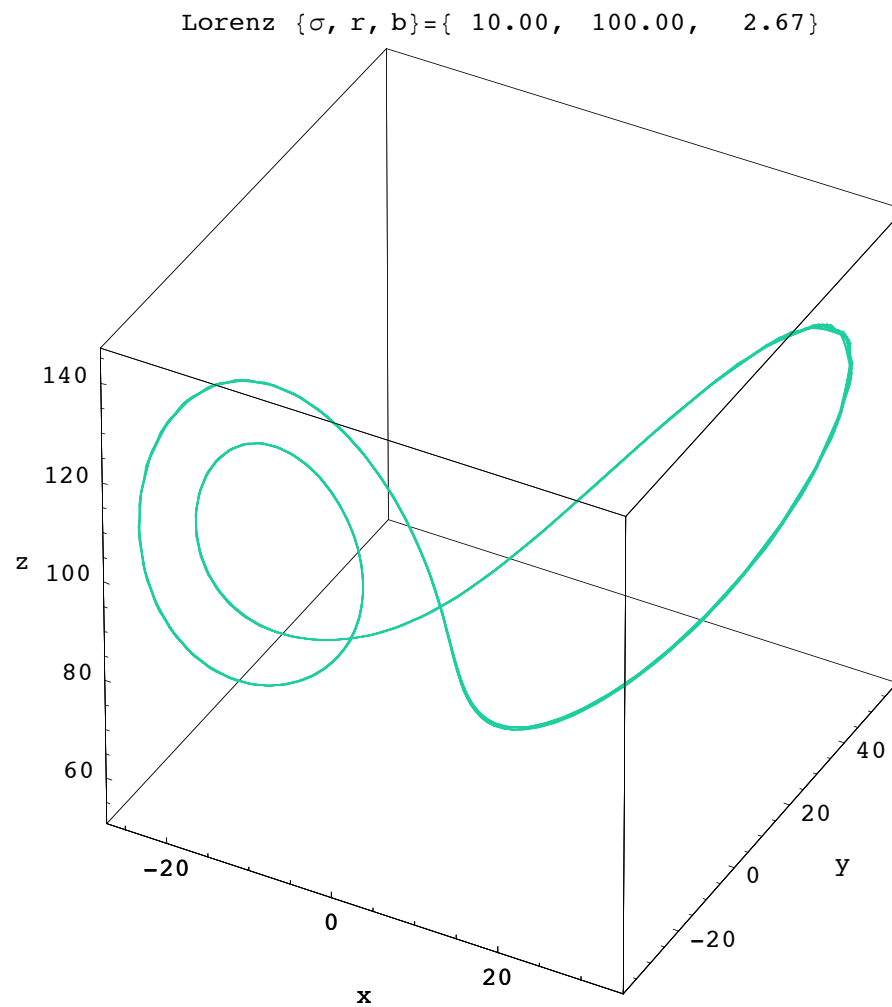
```
initvec = {10.0,10.0,10.0};

h = 0.01;

t0 = 0.0;

nsteps = 1000;

lorlim = integrate[initvec,t0,h,nsteps];

boxrat = {1,1,1};

 setcolor[{TurquoiseBlue}];

limgraph1 = phaser3D[lorlim];
```


Lorenz {σ, r, b}={ 10.00, 100.00, 2.67}
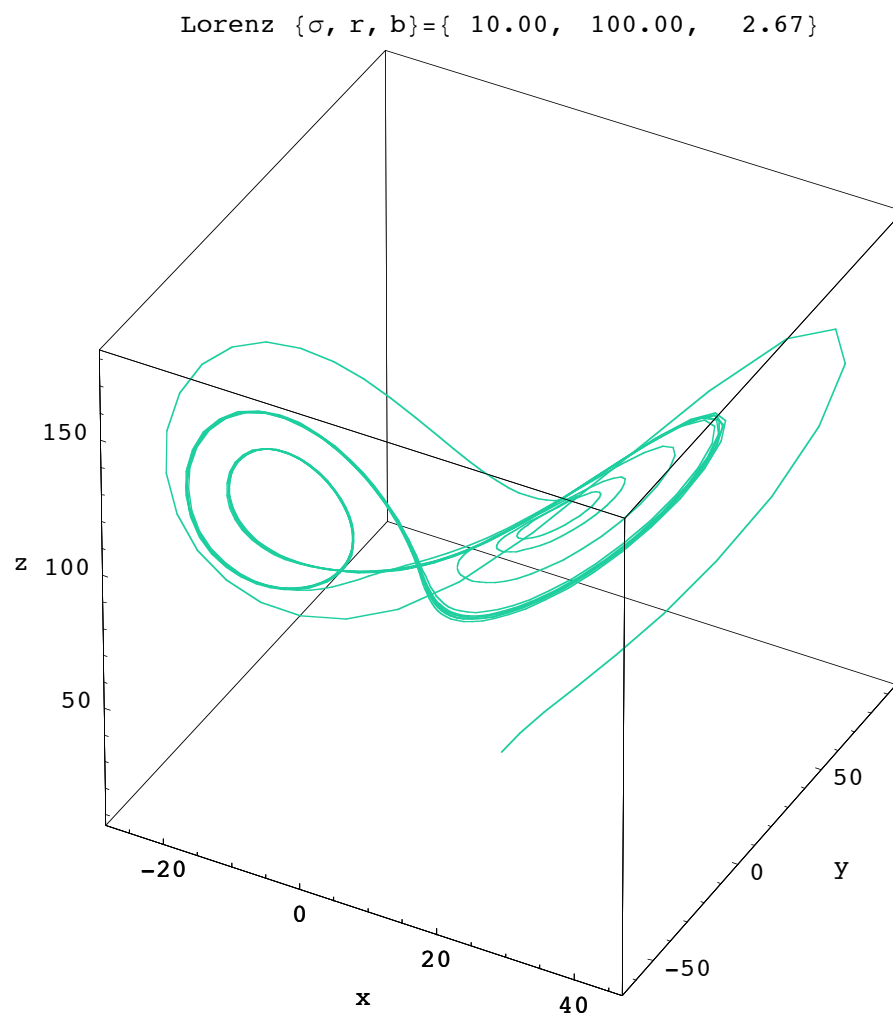
Hard to tell if there is a limit cycle because of the transients. We continue the integration to get a better idea.

```
lorlim2 = integrate[lastx,lastt,h,nsteps];
```

```
limgraph2 = phaser3D[lorlim2];
```
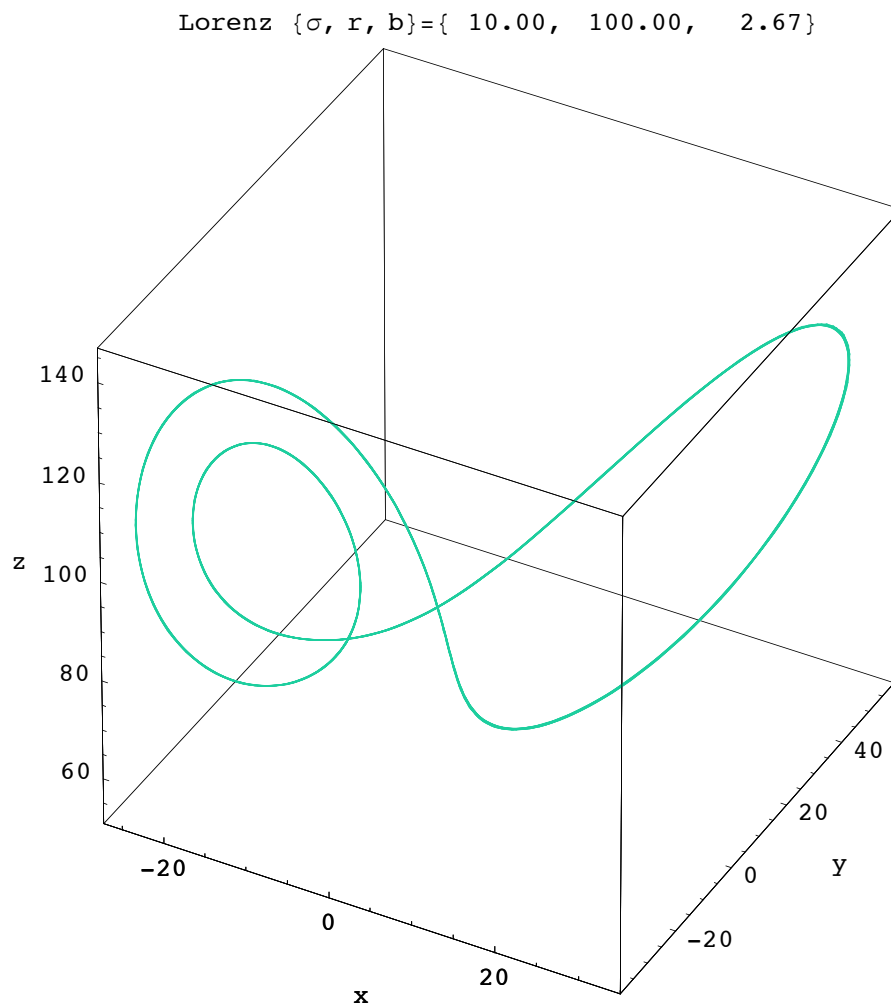

Now we clearly have a rather exotic-looking limit cycle. We clean it up a little by using a smaller time step and integrating once more.

```
h = 0.005; t0 = 0.0; nsteps = 2000;

lorlim3 = limcyc[lastx, t0, h, nsteps];
```

```
limgraph3 = phaser3D[lorlim3];
```


Lorenz {σ, r, b}={ 10.00, 100.00,   2.67}

■ 4. The Nature of the Chaos and the Chaotic Attractor

■ 4.1 Sensitive Dependence on Initial Conditions.

Perhaps the most characteristic feature of a chaotic system is its long-term unpredictability. This comes about because of sensistive dependence of solutions on initial conditions. Two different initial conditions, no matter how close, will eventually become widely separated. Thus for any finite number of digits of accuracy in an initial condition, there will be a future time at which no accurate predictions can be made about the state of the system. We illustrate this with plots of x versus t for solutions of the Lorenz equations with slightly differing initial conditions. We use the canonical values for σ and b, and the value $r = 28$.
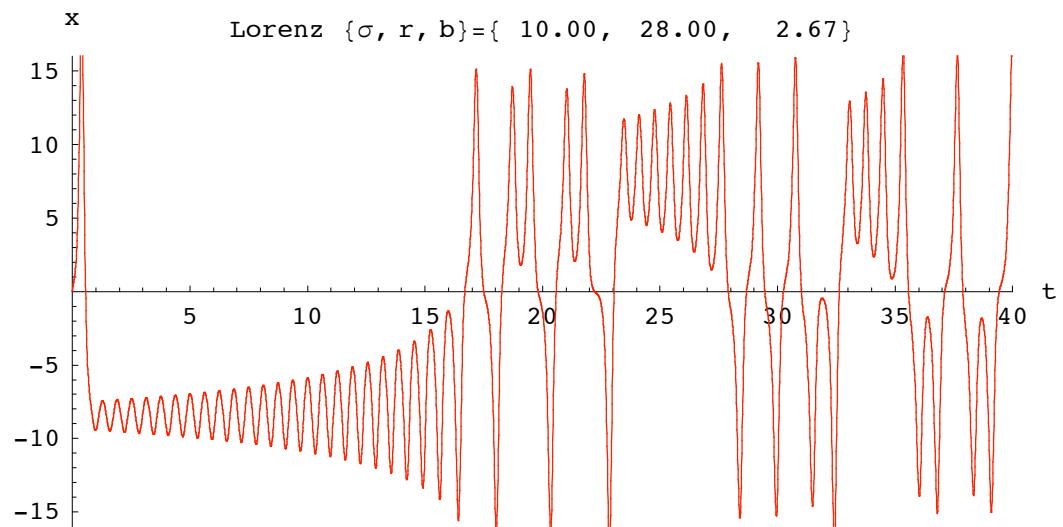
```
parmval = {10.0,28.0,8./3.};
```

We use two initial vectors which differ only by one part in a thousand in the *y* coordinate and are indentical in *x* and *z*.

```
initialvec1 = {0.0, 1.0, 0.0};

initialvec2 = {0.0, 1.001, 0.0};

h = 0.01;

nsteps = 4000;

t0 = 0.0;

sens1 = integrate[initialvec1, t0, h, nsteps];

asprat = 0.5; plrange = { {0, 40}, {-16, 16}};

imsize = 400;

setcolor[{Red}];

graphsens1 = timeplot[sens1, 1];
```
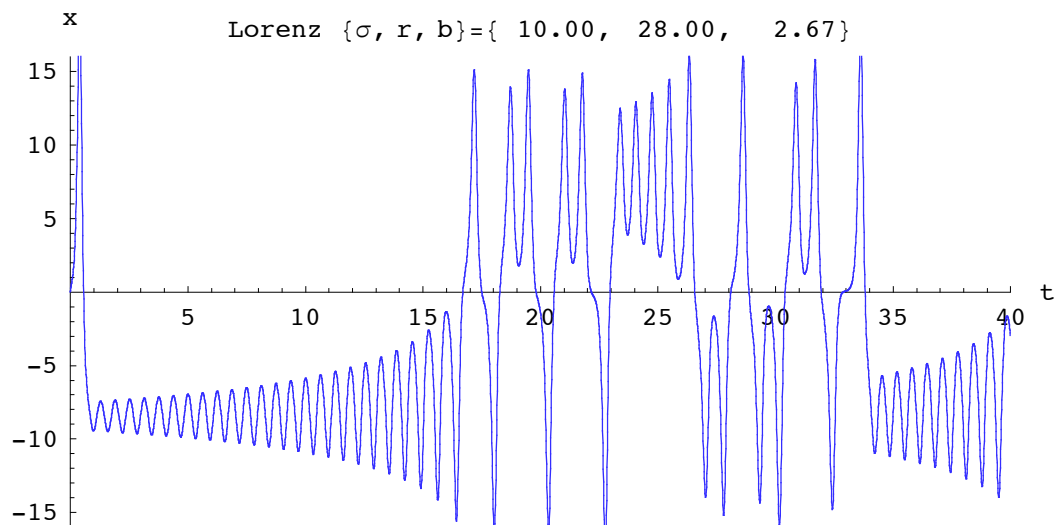


```
sens2 = integrate[initialvec2, t0, h, nsteps];

setcolor[{Blue}];
```
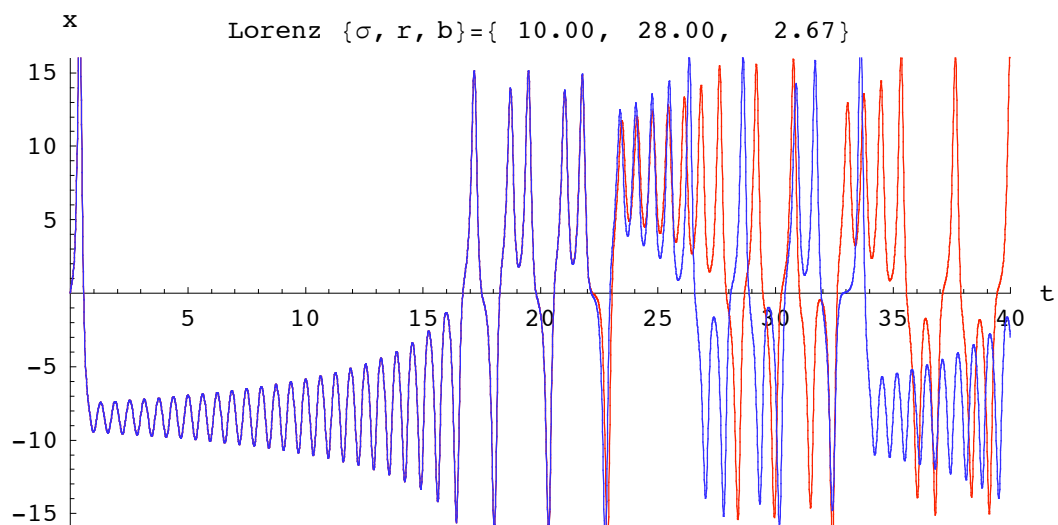
graphsens2 = timeplot[sens2, 1];

x   Lorenz {σ, r, b}={ 10.00, 28.00,   2.67}

Now we show these graphs together.

show[graphsens1, graphsens2];

x   Lorenz {σ, r, b}={ 10.00, 28.00,   2.67}

The graphs are essentially identical until about $t = 21$ or so. By the time t has reached 27, the curves are widely separated. Clearly there is no predictability for t greater than about 25. Can we improve the situation by knowing more about the initial conditions? We try the experiment again, this time separating the initial conditions by only one part in 10,000.
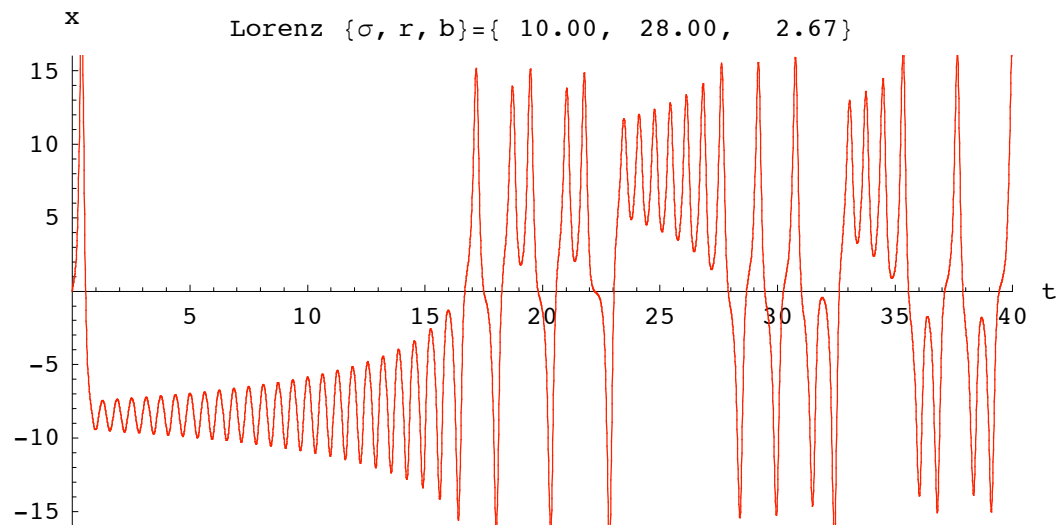
initialvec1 = {0.0, 1.0, 0.0};

initialvec2 = {0.0, 1.0001, 0.0};

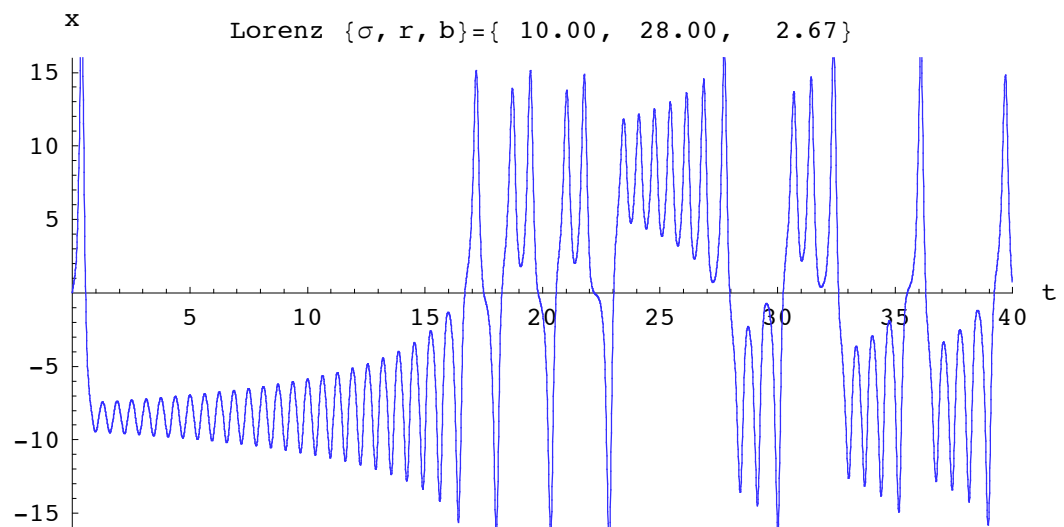**sens1 = integrate[initialvec1, t0, h, nsteps];**

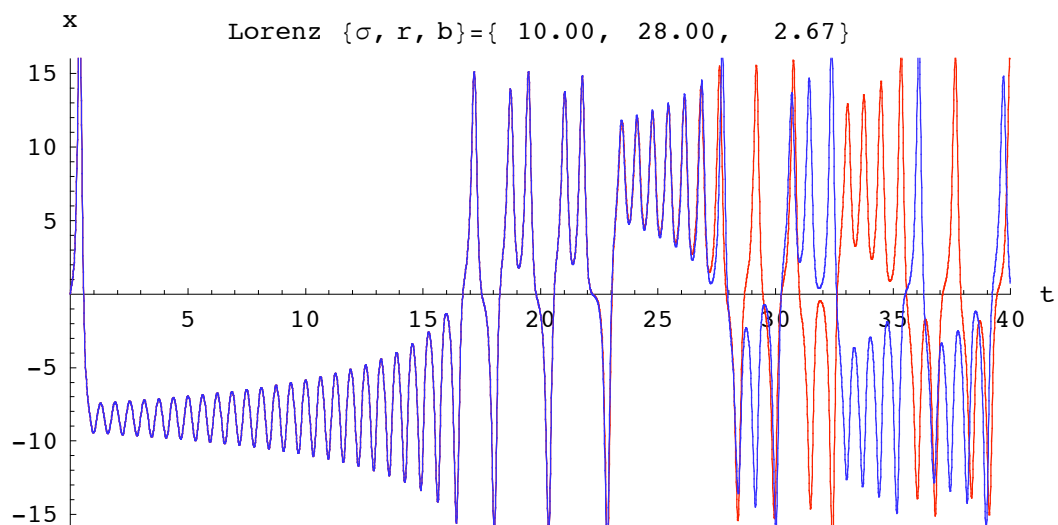**sens2 = integrate[initialvec2, t0, h, nsteps];**

```
setcolor[{Red}];

sensgraph1 = timeplot[sens1, 1];
```


Lorenz {σ, r, b}={ 10.00, 28.00, 2.67}

```
setcolor[{Blue}];

sensgraph2 = timeplot[sens2, 1];
```


Lorenz {σ, r, b}={ 10.00, 28.00, 2.67}

```
show[sensgraph1, sensgraph2];
```


The improvement is slight. We still have a wide separation, now beginning at about $t = 28$. Thus an increase in the accuracy of the intial condition by a factor of 10 has given us an increase in predictibility time of only around 10%. It turns out that the relation is logarithmic, and we have to pay a large price in accuracy of initial conditions to get a modest gain in predictibility. The main point is that unless the initial conditions are known with perfect accuracy, the system will become unpredictable at a finite time.

■ 4.2 Animating the Attractor.

In section 3.7, we constructed a three-dimensional phase portrait of a chaotic solution of the Lorenz equations. This gave us a numerical view of the attractor in this case. We use a longer integration here to make a more complete view, and then use the function animate to produce a sequence of graphs which can then be used to visualize a spatial rotation of the attractor. In that way, we can get an idea of the three-dimensional structure of the attractor.

```
initialvec = {10, 10, 10}; t0 = 0.0; h = 0.01; nsteps = 5000;

lorsol = integrate[initialvec, t0, h, nsteps];
```

phaser3D[lorsol];

Lorenz {σ, r, b}={ 10.00, 28.00, 2.67}
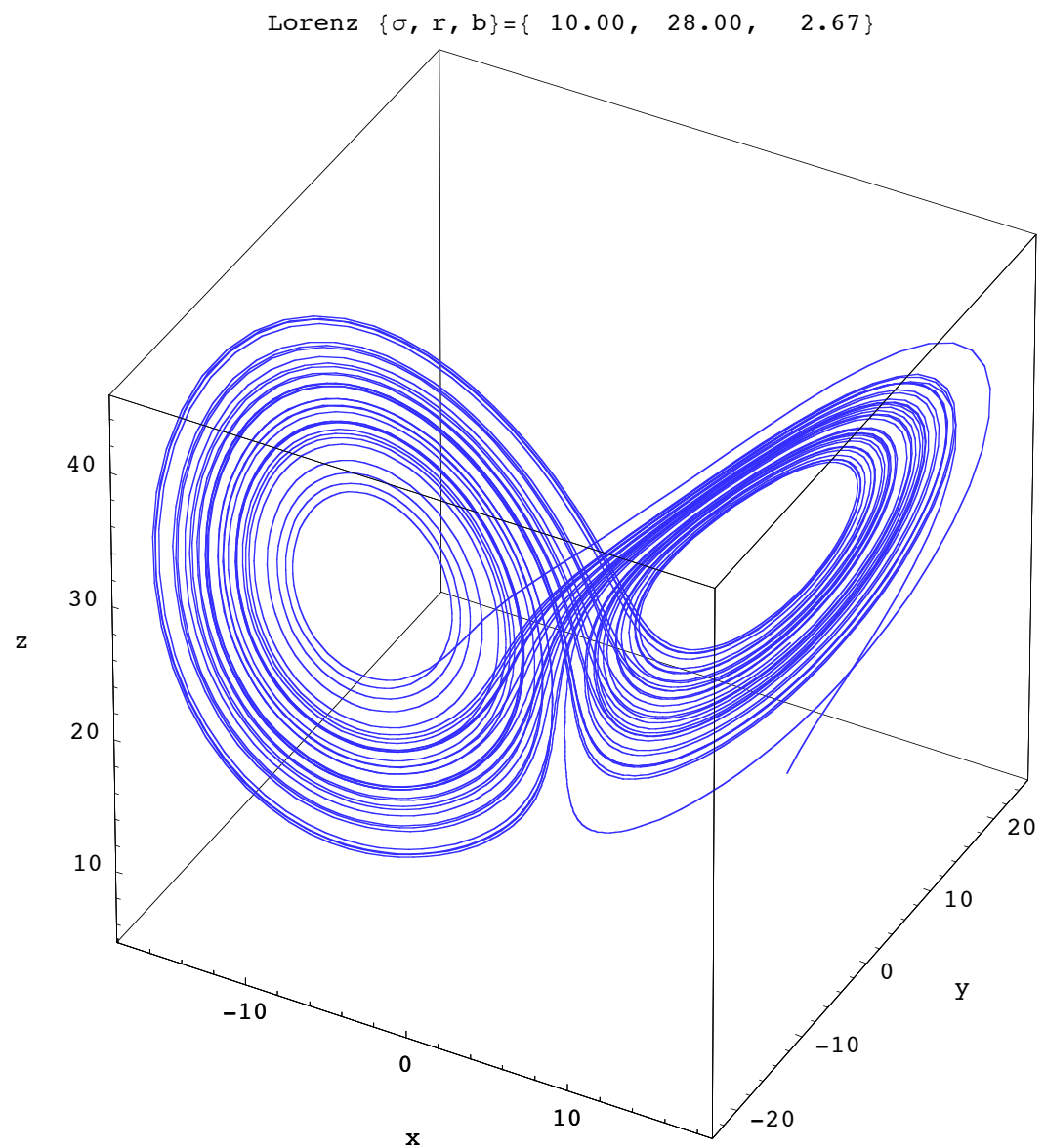
Now we animate this by producing a sequence of 30 graphs. Each graph in the sequence is rotated 12 degrees about the *z*-axis relative to the preceeding graph. You can animate the sequence by double-clicking on any graph in the sequence. In the printed version of this notebook, only the first graph of the sequence is shown.

animate[lorsol, 1, 2, 3, 30];
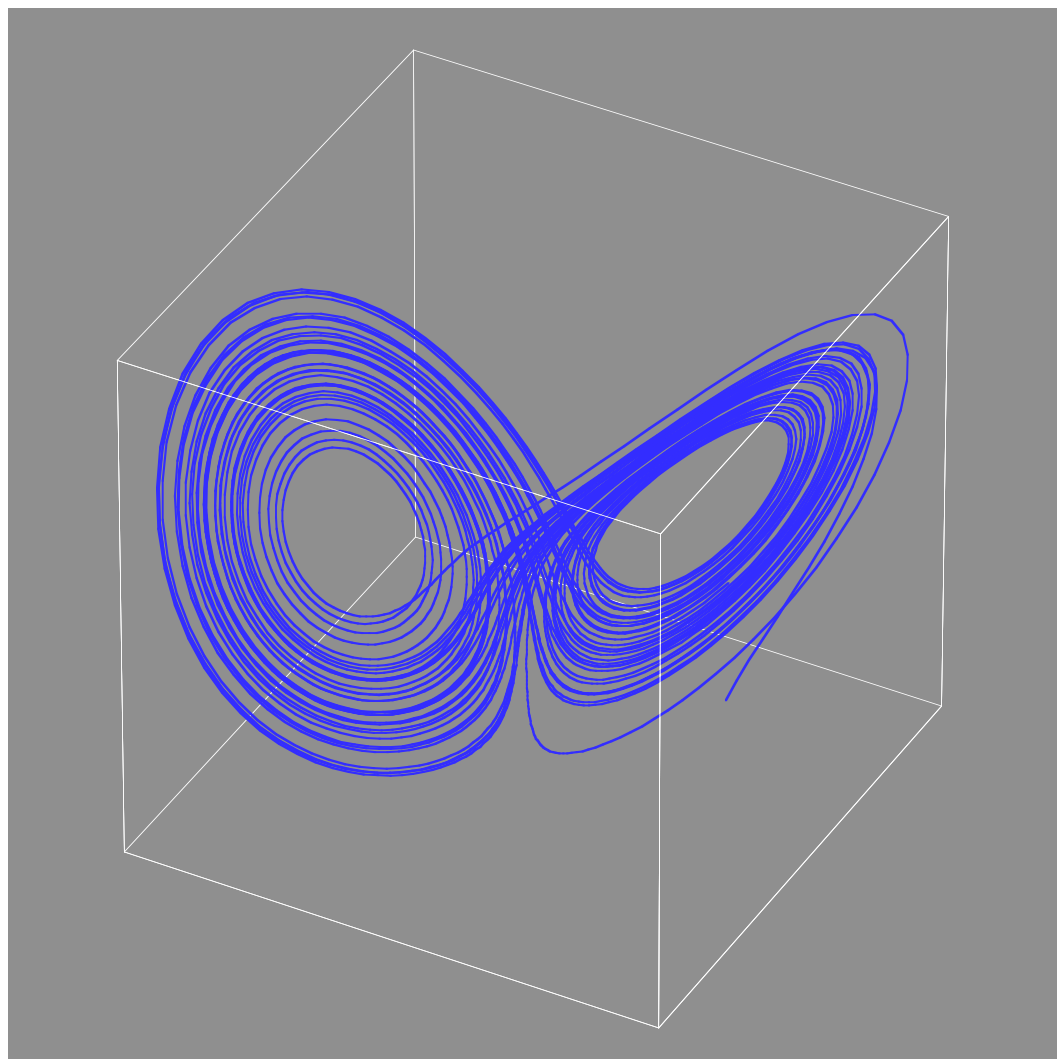
▪ 4.3 What is the Lorenz Attractor?

We just give here a very brief summary, based mostly on the discussion in section 9.2 of **Nonlinear Dynamics and Chaos**, by Steven Strogatz, Addison-Wesley, 1994.

Consider the situation when $\sigma = 10$, $r = 28$, and $b = 8/3$. Some solutions were calculated for this case in sections 3.7 and 4.2 above. For that set of parameter values, the three equilibria are all unstable, so none of them is a possible attractor. The solutions are bounded, so the system cannot run off to infinity. Whatever the attractor is, it must have zero volume, because under the flow, volumes contract exponentially as $\exp[-(1 + \sigma + b)t]$. What kind of attractors have zero volume? In two dimensions, we can have a point attractor or a limit cycle, and both of those have zero volume. In the present case, we have already ruled out point attractors. Could there be a stable limit cycle that just hasn't yet been found? Although there is no rigorous proof that such a cycle does not exist, there are strong arguments against it. With all the work that has been done on the Lorenz equations, it is unlikely that such an interesting object would have escaped discovery. A stronger argument, given in Strogatz' discussion, is based on the Lorenz map which we have not yet discussed. Even though there is not a rigorous proof, there is a strong belief that no such limit cycle exists. In two dimensions this would be paradoxical -- in fact the Poincare-Bendixson theorem essentially says that periodic solutions and point attractors are the only possibilities. The truth in this situation emerged only gradually, beginning with the work of Lorenz. The surface you see above is a

numerical approximation to this attractor. The orbits move on it forever, never intersecting themselves and never intersecting each other. The essence of the motion is an oscillation around one of the equilibria C1 or C2, followed by a transition to an oscillation around the other, with the cycle continuing in a somewhat irregular way forever. None of this has answered the question: What kind of object is the attractor? It is in fact an example of something called a fractal -- a geometric object with fractional dimension. In this case of the Lorenz attractor, the dimension is believed to be about 2.05. Thus it is something more than a surface (dimension 2) and less than a volume (dimension 3). Unfortunately we won't get to fractals in our course, but there is a very nice introduction in Chapter 11 of Strogatz.