7.2 Application to economics: Leontief Model

Wassily Leontief won the Nobel prize in economics in 1973.

The Leontief model is a model for the economics of a whole country or region. In the model there are n industries producing n different products such that the input equals the output or, in other words, consumption equals production. One distinguishes two models:

open model: some production consumed internally by industries, rest consumed by external bodies.

Problem: Find production level if external demand is given.

closed model: entire production consumed by industries.

Problem: Find relative price of each product.

The open Leontief Model

Let the n industries denoted by S_1, S_2, ..., S_n. The exchange of products can be described by an

input-output graph

Here, a_{ij} denotes the number of units produced by industry S_i necessary to produce one unit by industry S_j and b_i is the number of externally demanded units of industry S_i.

Example: *Primitive model of the economy of Kansas in the* 19^{th} *century.*

The following equations are satisfied:

Production of	*Total output*	=	*Internal consumption*	+	*External Demand*
farming industry (in tons):	x	=	$0.05\,x \;+\; 0.5\,y$	+	8000
horse industry:	y	=	$0.01\,x$	+	2000
(in 1000km horse rides)					

In general, let x_1, x_2, ..., x_n, be the total output of industry S_1, S_2, ..., S_n, respectively. Then

$$\begin{cases} x_1 &=& a_{11}\,x_1 + a_{12}\,x_1 + \cdots + a_{1n}\,x_1 + b_1 \\ x_2 &=& a_{21}\,x_1 + a_{22}\,x_1 + \cdots + a_{2n}\,x_2 + b_2 \\ & \cdots & \\ x_n &=& a_{n1}\,x_1 + a_{n2}\,x_1 + \cdots + a_{nn}\,x_n + b_n \end{cases},$$

since $a_{ij}\,x_j$ is the number of units produced by industry S_i and consumed by industry S_j. The total consumption equals the total production for the product of each industry S_i.

Let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

A is called the input-output matrix, B the external demand vector and X the production level vector. The above system of linear equations is equivalent to the matrix equation

$$X = AX + B.$$

In the **open Leontief model**, A and $B \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ are given and the problem is to determine X from this matrix equation.

We can transform this equation as follows:

$$\begin{aligned} I_n X - AX &= B \\ (I_n - A)X &= B \\ X &= (I_n - A)^{-1} B \end{aligned}$$

if the inverse of the matrix $I_n - A$ exists. ($(I_n - A)^{-1}$ is then called the Leontief inverse.) For a given realistic economy, a solution obviously must exist.

For our example we have:

$$A = \begin{pmatrix} 0.05 & 0.5 \\ 0.1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 8,000 \\ 2,000 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

We obtain therefore the solution

$$\begin{aligned} X &= (I_2 - A)^{-1}B \\ &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.05 & 0.5 \\ 0.1 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} 8,000 \\ 2,000 \end{pmatrix} \\ &= \begin{pmatrix} 0.95 & -0.5 \\ -0.1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 8,000 \\ 2,000 \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 10 & 5 \\ 1 & 9.5 \end{pmatrix} \begin{pmatrix} 8,000 \\ 2,000 \end{pmatrix} \\ &= \begin{pmatrix} 10,000 \\ 3,000 \end{pmatrix}, \end{aligned}$$

i.e., $x = 10,000$ *tons wheat and* $y = 3$ *Million km horse ride.*

If the external demand changes, ex. $B' = \begin{pmatrix} 7,300 \\ 2,500 \end{pmatrix}$, *we get*

$$\begin{pmatrix} x \\ y \end{pmatrix}' = (I_2 - A)^{-1}B' = \frac{1}{9}\begin{pmatrix} 10 & 5 \\ 1 & 9.5 \end{pmatrix}\begin{pmatrix} 7,300 \\ 2,500 \end{pmatrix} = \begin{pmatrix} 9,500 \\ 3,450 \end{pmatrix},$$

i.e., one doesn't need to recompute $(I_2 - A)^{-1}$.

One difficulty with the model: How to determine the matrix A from a given economy? Typically, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is known, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is known and $(a_{ij}x_j)_{i,j=1,\dots n}$ is known. One takes therefore the matrix $(a_{ij}x_j)_{i,j=1,\dots n}$ and divides the j-th column by x_j for $j = 1, \dots, n$ to get A.

Example: *An economy has the two industries* R *and* S*. The current consumption is given by the table*

	consumption R	S	external
Industry R *production*	50	50	20
Industry S *production*	60	40	100

Assume the new external demand is 100 *units of* R *and* 100 *units of* S*. Determine the new production levels.*

Solution: *The total production is* 120 *units for* R *and* 200 *units for* S*. We obtain* $X = \begin{pmatrix} 120 \\ 200 \end{pmatrix}$, $B = \begin{pmatrix} 20 \\ 100 \end{pmatrix}$, $A = \begin{pmatrix} \frac{50}{120} & \frac{50}{200} \\ \frac{60}{120} & \frac{40}{200} \end{pmatrix}$, *and* $B' = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$. *The solution is*

$$X' = (I_2 - A)^{-1}B' = \frac{1}{41}\begin{pmatrix} 96 & 30 \\ 60 & 70 \end{pmatrix}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 307.3 \\ 317.0 \end{pmatrix}.$$

The new production levels are 307.3 *and* 317.0 *for* R *and* S*, respectively.*

The closed Leontief Model

The closed Leontief model can be described by the matrix equation

$$X = AX,$$

i.e., there is no external demand. The matrix $I_n - A$ is usually **not** invertible. (Otherwise, the only solution would be $X = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.)

The input-output graph looks now as follows:

There is only internal consumption.

Example: *Extended model of the economy of Kansas in the* 19^{th} *century including labor.*

The corresponding matrix equation is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.05 & 0.5 & 0.5 \\ 0.1 & 0 & 0.1 \\ 0.4 & 0.1 & \frac{1331}{1800} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If X is a solution, also $t \cdot X$ for every $t > 0$ is a solution. (Usually, one gets a one parameter family of solutions.) If $x \neq 0$, we can assume $x = 1{,}000$ by choosing the appropriate parameter t. One obtains then the solution

$$x = 1{,}000, \qquad y = \frac{2900}{11} \approx 263.63, \qquad z = \frac{18000}{11} \approx 1636.36.$$

For this computation, it is important to use rational numbers (i.e., fractions) as matrix entries since otherwise the approximation to the matrix $I_n - A$ usually will be invertible and only the trivial uninteresting solution $x = 0$, $y = 0$, and $z = 0$ will exist. This is also the reason, why the entry a_{33} has large numerator and denominator.

In a closed economy, the absolute units of output are less interesting. More important is the **relative consumption** of a product.

We can normalize therefore the matrix A such that the sum of every row is 1. This is a matrix $\widetilde{A}$, such that $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \widetilde{A} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. The recipe is: Divide the i-th row of A by the i-th component of $A \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ (that is the sum of the i-th row).

For our example, we have

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{21}{20} \\ \frac{1}{5} \\ \frac{2231}{1800} \end{pmatrix},$$

leading to the matrix

$$\widetilde{A} = \begin{pmatrix} \frac{1}{21} & \frac{10}{21} & \frac{10}{21} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{720}{2231} & \frac{180}{2231} & \frac{1331}{2231} \end{pmatrix}, \qquad \widetilde{A} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The entries of the matrix $\widetilde{A} = (\widetilde{a}_{i,j})_{i,j=1,\ldots,n}$ have the following meaning: $\widetilde{a}_{ij}$ is the relative consumption of the product of industry S_i by industry S_j.

Market prices

The consumption of products is regulated by prices. All income of an industry is used for buying other (or the own) products, i.e., income equals expenditure.

Let $P = (p_1, \ldots, p_n)$ the price vector; p_i is the relative price of the product of industry S_i. We can draw the flow of money into the input-output graph, the money flows in exchange for the products:

One has
$$\begin{cases} p_1 &= & a_{11}\,p_1 + a_{21}\,p_2 + \cdots + \widetilde{a}_{n1}\,p_n \\ p_2 &= & \widetilde{a}_{12}\,p_1 + \widetilde{a}_{22}\,p_2 + \cdots + \ a_{n2}\,p_2 \\ & \cdots & \\ p_n &= & \widetilde{a}_{1n}\,p_1 + \widetilde{a}_{n2}\,p_n + \cdots + \widetilde{a}_{nn}\,p_n \end{cases},$$
since $\widetilde{a}_{ij}\,p_i$ is the amount paid by industry S_j for products produced by industry S_i. The total income of industry S_j equals the total price S_j has to pay to all other industries.

Again, one can write this as a matrix equation:
$$PA = P.$$
This equation can be transformed in the following way
$$\begin{aligned} P \cdot I_n &= P \cdot \widetilde{A} \\ P \cdot (I_n - \widetilde{A}) &= (0, \ldots, 0). \end{aligned}$$
The matrix $I_n - \widetilde{A}$ is (similar as $I_n - A$) not invertible, since $(I_n - \widetilde{A}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

One can show that this implies that there is also a solution $P \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Since with P also $t \cdot P$ for $t > 0$ is a solution, only the relative price between the different products has a well-defined meaning.

Example (continued): *Assume* $p_1 = \$1,000$. *One gets* $p_2 = \$\,\frac{40000}{63} \approx \$\,634.92$ *and* $p_3 = \$\,\frac{11155500}{567} \approx \$\,1967.37$. *We can compare these relative prices with the production levels measured by the original units and obtain the following relative prices per unit:* $p_1/x = \frac{1000}{1000} = 1$ *for one ton of wheat,* $p_2/y \approx \frac{634.92}{263} \approx 2.4$ *for* 1000km *horse ride, and* $p_3/z \approx \frac{1967.37}{1636.36} \approx 1.2$ *for one man-year.*

Since the above matrix equation for P is not of the usual form which we have studied so far, we make a final modification. We define
$$\widetilde{\widetilde{A}} = (\widetilde{\widetilde{a}}_{i,j})_{i,\,j=1,\ldots,n}, \qquad \text{where } \widetilde{\widetilde{a}}_{i,j} = \widetilde{a}_{j,i}.$$

This gives us (just by switching the rôle of rows and columns) the price equation

$$\widetilde{P} = \widetilde{\widetilde{A}}\widetilde{P},$$

where $\widetilde{\widetilde{a}}_{i,j}$ is now the relative consumption of industry S_j by industry S_i, so that the sum of each column is 1, and $\widetilde{P} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ is the price column vector.

In the textbook, our matrix $\widetilde{\widetilde{A}}$ is again denoted by A and our $\widetilde{P}$ is denoted by X. The price equation is therefore $X = A \cdot X$. However, one has to keep in mind that this matrix A is different from the input-output matrix A we used in the open Leontief model!

Example: *Let*

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

Compute all wages, given that the wages for the 3*rd product is* \$ 30, 000.

Solution: *Let* $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *be the different wages with* $z = 30,000$. *We have to solve*

$$\begin{aligned} X &= AX \\ (I_3 - A)X &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{2} & -\frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{2}{3} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ 30,000 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This system of linear equations for x *and* y *has the solution* $x = 30,000$ *and* $y = 22,500$. *The wages for the first and second product are therefore* \$, 30, 000 *and* \$ 22, 500, *respectively.*