Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2014	1 0 1 0 2 1 / 0 1

Title: Mathematics for Computer Science

Time Allowed:	2 hours
Examination Material Provided:	None
Additional Materials Permitted:	None
Instructions:	Answer FOUR questions. (ONE from EACH Section) ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK Calculators are NOT allowed Erasmus/Visiting students can use a dictionary

Revision:	

ED01/2014

Section A Logic and Discrete Structures (Dr. M. Gadouleau)

Question 1

(a) Give the resolution rule of inference. Justify it using truth tables. **[4 Marks]**

(b) Explain the resolution proof system. **[7 Marks]**

(c) Prove or disprove by resolution:
$((p \vee q) \wedge (p \vee r) \wedge (\neg p \vee \neg r)) \Rightarrow q.$ **[6 Marks]**

(d) Prove or disprove by resolution:
$((t \Rightarrow \neg y) \wedge (x \Rightarrow ((\neg y \vee z) \wedge s)) \wedge (x \vee t) \wedge (\neg y \Rightarrow (u \wedge \neg w))) \Rightarrow (s \vee \neg w).$
[8 Marks]

continued

Question 2

(a) State the two De Morgan's laws and the two distribution laws. **[4 Marks]**

(b) Reduce $(X \Rightarrow Y) \Rightarrow Z$ to disjunctive normal form and to conjunctive normal form. **[6 Marks]**

(c) Use De Morgan's and the distribution laws to simplify the formula: $p \wedge (q \vee \neg(p \vee r))$. **[3 Marks]**

(d) Determine whether the following sets of connectives are functionally complete: $\{\oplus, \wedge\}$ and $\{\Leftrightarrow, \vee, F\}$. **[6 Marks]**

(e) Is $(X \Leftrightarrow Y) \Leftrightarrow Z$ logically equivalent to $X \Leftrightarrow (Y \Leftrightarrow Z)$?
Is $(X \Rightarrow Y) \Rightarrow Z$ logically equivalent to $X \Rightarrow (Y \Rightarrow Z)$? **[6 Marks]**

End of Section A

Section B Logic and Discrete Structures (Dr.K.Dabrowski and Dr.G.Mertzios)

Question 3

(a) Let A and B be two arbitrary sets. What is a binary relation from A to B? Let $f : A \to B$ be a function. What is the graph of f? Give an example of a binary relation from A to B that is **not** the graph of some function. **[6 Marks]**

(b) Let A be a set, and let R be a relation on A. When do we call R symmetric? When do we call R anti-symmetric? Give an example of a set A and a relation R on A that is neither symmetric nor anti-symmetric (you should justify your answer). **[6 Marks]**

(c) Which of the following binary relations R are equivalence relations? You should justify your answers.

i. $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x < y+5\}$.

ii. $R = \{(x,y) \in \mathbb{N} \times \mathbb{N} :$ there exist $m,n \in \mathbb{N}$ such that $mx = ny\}$.

[7 Marks]

(d) A poset is a set A together with a partial order R on A. Are the following posets? You should justify your answers.

i. $A = \{1,2,3,4,5,6\}$ and $R = \{(1,1),(2,2),(3,3),(3,5),(4,4),(5,5)\}$.

ii. $A = \{1,2,3,4,5,6\}$ and
$R = \{(1,1),(1,6),(2,2),(3,3),(4,4),(5,5),(6,6)\}$. **[6 Marks]**

Question 4

(a) Explain how formulas in First-Order logic are recursively constructed from the atoms. **[5 Marks]**

(b) Let $\langle S, A, B, C, D\rangle$ be a signature where S is a constant, A, B, C are unary relations, and D is a binary relation. Consider the interpretation of this signature over the set $\mathbb{N}$ of natural numbers, which is defined as follows:

$$A = \{x : \exists k \in \mathbb{N}(x = 2k-1)\}$$

$$B = \{x : \exists k \in \mathbb{N}(x = 2k)\}$$

$$C = \{x : \exists k \in \mathbb{N}(x = 3k)\}$$

$$D = \{(x,y) : \exists k \in \mathbb{N}(y = kx)\}$$

$$S = 6$$

Express in English the meaning of the following First-Order sentences:

i. $\forall x \forall y((D(x,y) \wedge B(x)) \Rightarrow B(y))$ **[3 Marks]**

ii. $\forall x \neg \exists y(A(x) \wedge B(y) \wedge D(y,x))$ **[4 Marks]**

iii. $\forall x \forall y \forall z((D(x,y) \wedge D(y,z) \wedge B(x) \wedge C(y)) \Rightarrow D(S,z))$ **[5 Marks]**

(c) Using the interpretation of the signature $\langle S, A, B, C, D\rangle$ from Question 4(b), give an equivalent sentence in First-Order logic for each of the following English sentences:

i. Every natural number divisible by 3 is odd or divisible by 6, or both. **[4 Marks]**

ii. There is an odd number x such that, for every y that is a multiple of x, there exists an even number z that is a multiple of y. **[4 Marks]**

Section C Discrete Mathematics and Linear Algebra (Professor A. Krokhin)

Question 5

(a) Four fair dice are thrown. What is more likely: that the four numbers are all (pairwise) different or that exactly two of the four numbers are odd? You should justify your answer. **[8 Marks]**

(b) There are 11 players in team A. It is known that four of them (call them high-scoring) score a penalty with probability 0.8 each and the other seven (low-scoring) with probability 0.5 each.

i. A player is chosen randomly from team A (without knowing his scoring ability), and the player scores a penalty. Use Bayes' Theorem to decide whether this player is more likely to be high-scoring than low-scoring. You should justify your answer. **[9 Marks]**

ii. There is a penalty shoot-out with another team, team B (of 11 players), in which every player from each team takes exactly one shot, so that 22 shots are to be taken. Every player in team B scores with probability 0.64. A team's score is the total number of goals scored by players from this team in the shoot-out. Which team is expected to have a higher score, A or B? You should justify your answer. **[8 Marks]**

continued

Question 6

(a) What is a spanning tree of a graph? Prove that every connected graph has a spanning tree. **[6 Marks]**

(b) In a shooting practice (where each shot uses up exactly one bullet), each time Jack hits the target, he gets three additional bullets. Jack started practice with exactly one bullet and took 43 shots when he ran out of bullets. How many times did Jack hit the target? You should justify your answer, modeling this situation as a tree. **[7 Marks]**

(c) There are five people at a dinner. Among each set of three of them, there is at least one pair of people who know each other and at least one pair of people who do not know each other. Model this situation as a graph and show that it is possible to arrange chairs for these people around a round table so that each person knows both people sitting next to her/him. **[6 Marks]**

(d) What is a bipartite graph? Describe the following graphs and decide which of them are bipartite:

i. the 6-vertex cycle C_6,

ii. the 6-cube Q_6,

iii. the 6-vertex complete graph K_6?

You should justify your answers. **[6 Marks]**

Section D Discrete Mathematics and Linear Algebra (Professor A. Krokhin)

Question 7

(a) Is it true that, for all **positive** integers a, b, c, m such that each of a, b, c is strictly smaller than m, it holds that

i. if $a \equiv b \mod m$ then $(a \cdot c) \equiv (b \cdot c) \mod m$?

ii. if $(a \cdot c) \equiv (b \cdot c) \mod m$ then $a \equiv b \mod m$?

You should justify your answers. **[4 Marks]**

(b) Find the multiplicative inverse of a modulo m (or show that it does not exist) if

i. $a = 7$ and $m = 25$; **[5 Marks]**

ii. $a = 6$ and $m = 21$. **[5 Marks]**

You should justify your answers.

(c) Let $A = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Compute, whenever possible, the following matrices:

i. AB, **[2 Marks]**

ii. $2A + B^T$, **[2 Marks]**

iii. $B^T A + AB^T$. **[2 Marks]**

You should justify your answers.

(d) Define elementary row operations and elementary matrices. Which square matrices can be represented as products of elementary matrices? **[5 Marks]**

continued

Question 8

(a) Find A^{-1} if

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}.$$

[5 Marks]

(b) Define a basis of a vector space, carefully explaining all notions that you use in the definition. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis of the vector space of all 2×2 matrices. **[6 Marks]**

(c) What is the null space of a matrix? Find a basis for the null space of the following matrix:

$$A = \begin{pmatrix} -1 & 2 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 3 & 1 & 3 \\ -5 & 8 & -2 & 1 \end{pmatrix}.$$

[9 Marks]

(d) Let $f : \mathbb{R}^6 \to \mathbb{R}^4$ be a linear map defined by

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = (x_2 - 2x_4 - x_5,\ 0,\ 2x_1 - x_6,\ x_1 + x_3 + x_4).$$

Find the standard matrix of this linear map. **[5 Marks]**

END OF PAPER