Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2016	COMP1021-WE01

Title:

MATHEMATICS FOR COMPUTER SCIENCE

Time Allowed:	2 hours		
Additional Material provided:	None		
Materials Permitted:	None		
Calculators Permitted:	No	Models Permitted:	
Visiting Students may use dictionaries: Yes			

Instructions to Candidates:	Answer FOUR questions. (ONE from each Section) PLEASE ANSWER EACH SECTION IN A SEPARATE ANSWER BOOKLET

Revision:	

Section A Logic and Discrete Structures
(Prof. I.A. Stewart)

Question 1

(a) What is a theorem in the proof system Natural Deduction? What do we mean when we say that Natural Deduction is both sound and complete? **[3 Marks]**

(b) Describe the rule of Natural Deduction for introducing elimination and explain how it is used in proofs. **[5 Marks]**

(c) Give a proof in Natural Deduction of the following propositional formulae:

i. $X \Rightarrow (Y \wedge W) \vdash (X \Rightarrow W) \wedge (X \Rightarrow Y)$ **[3 Marks]**

ii. $A \Rightarrow B \vdash B \vee \neg A$ **[7 Marks]**

(You should only use the rules of Natural Deduction in your derivations. If you wish to use Natural Deduction proofs of certain formulae, e.g., the Law of the Excluded Middle, then you should derive these proofs before using the formulae.)

(d) What is conjunctive normal form? Explain how the proof system Resolution is used to prove a propositional formula (be sure to describe the actual resolution rule of inference). **[7 Marks]**

Question 2

(a) i. With respect to Propositional Logic, what do we mean when we say that a set of logical connectives is functionally complete? **[1 Mark]**

ii. Explain whether the following sets of logical connectives are functionally complete:

1. $\{\wedge, \Rightarrow\}$ **[4 Marks]**
2. $\{\uparrow\}$ where $X \uparrow Y \equiv \neg(X \wedge Y)$ **[4 Marks]**

(b) i. Explain how we can give semantics to a propositional formula involving the Boolean variables $X_1, X_2, \ldots, X_n$ by using sets and a Venn diagram. **[4 Marks]**

ii. Give the interpretation of the formula $X_1 \wedge (X_2 \vee \neg X_3)$ using a Venn diagram. **[2 Marks]**

(c) i. Let $A_1, A_2, \ldots, A_n$ be sets, for some $n \geq 1$. Define the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$. **[1 Marks]**

ii. Let A_1, A_2 and A_3 be non-empty sets. Explain why the Cartesian product $(A_1 \times A_2) \times A_3$ is different from the Cartesian product $A_1 \times (A_2 \times A_3)$. **[3 Marks]**

(d) Let $X = \{1, 2, ..., n\}$, for some $n \geq 1$. Describe a bijection from the power set $\mathcal{P}(X)$ to the set of all n-bit strings $\{0, 1\}^n$ (you should explain why your function is a bijection). **[6 Marks]**

Section B Logic and Discrete Structures
(Dr. M. Johnson)

Question 3

(a) Define a binary relation from a set A to a set B. **[2 Marks]**

(b) Let $A = \{-3, 0, 3, 6\}$ and $B = \{-4, 0, 4, 8\}$. Let R be a binary relation from A to B where for all $(x, y) \in A \times B$, $(x, y) \in R$ if and only if $|x| < |y|$. Determine which pairs of $A \times B$ are in R. **[3 Marks]**

(c) Let R be a binary relation on the set of natural numbers $\mathbb{N}$ such that $(a, b) \in R$ if and only if $a + b$ is even. Is R reflexive? Is R symmetric? Is R transitive? Justify your answers. **[5 Marks]**

(d) Let R be a symmetric and transitive binary relation on a set A such that there exists $a \in A$ such that $(a, x) \in R$ for all $x \in A$. Prove that R is reflexive. **[5 Marks]**

(e) Let $A = \{a, b, c, d, e\}$. Let R be an equivalence relation on A such that R has three equivalence classes. Suppose also that $(a, d) \in R$ and $(b, c) \in R$. Write out R as a set. **[5 Marks]**

(f) Recall that a poset (A, R) consists of a set A together with a partial order R on A. Let A be the set of positive integers and let (x, y) be in R if and only if x is a multiple of y (that is, if and only if x is equal to y multiplied by a positive integer). Is (A, R) a poset? Justify your answer. **[5 Marks]**

Question 4

(a) How can formulae of first-order logic be constructed from atomic formulae? **[5 Marks]**

(b) Let ϕ be this formula of first-order logic: $\forall x \forall y (A(x) \wedge B(x, y) \Rightarrow A(y))$. For each of the following interpretations: express in words the meaning of ϕ; state whether ϕ it is true or false; and, if it is false, give witnesses for x and y.

i. The domain is the natural numbers $\mathbb{N}$, A is the set of even numbers and $B = \{(u, v) \in \mathbb{N} \times \mathbb{N} : \ v \text{ is a multiple of } u\}$.

ii. The domain is the natural numbers $\mathbb{N}$, A is the set of even numbers and $B = \{(u, v) \in \mathbb{N} \times \mathbb{N} : \ u \text{ is a multiple of } v\}$.

iii. The domain is the natural numbers $\mathbb{N}$, A is the set of odd prime numbers and $B = \{(u, v) \in \mathbb{N} \times \mathbb{N} : v = u + 38\}$. **[10 Marks]**

(c) Let $\langle P, E, R, S \rangle$ be a signature, where P and E are unary relations, R is a binary relation and S is a ternary relation. Consider the following interpretation. Let the domain of discourse be the natural numbers $\mathbb{N}$. Let P be the set of prime numbers. Let E be the set of even numbers. Let R be the set $\{(u, v) \in \mathbb{N} \times \mathbb{N} : u \geq v\}$. Let S be the set $\{(u, v; w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : u + v = w\}$. Express each of the following sentences by a sentence in first-order logic:

i. There is a least natural number.

ii. There is no largest prime number.

iii. Every even number is the sum of two prime numbers. **[10 Marks]**

Section C Discrete Mathematics and Linear Algebra (Prof. A. Krokhin)

Question 5

(a) A sequence $s = (a_1, a_2, a_3, a_4, a_5)$ of 5 integers such that $0 \leq a_i \leq 9$ for $i = 1, \ldots, 5$ is chosen at random.

i. Find the probability that some number appears in s two times or more. **[3 Marks]**

ii. Find the probability that some number appears in s three times or more. **[4 Marks]**

iii. What is the probability that some number appears in s three times or more if we know that $a_1 = a_2 = 0$? **[5 Marks]**

iv. What is the conditional probability that some number appears in s three times or more given that $a_1 = a_2$? **[7 Marks]**

v. If X is the random variable equal to the sum of all *odd* numbers in s (i.e. of all $a_i \in \{1, 3, 5, 7, 9\}$), find the expected value of X. (Hint: Represent X as a sum of several random variables.) **[6 Marks]**

Show your working. You do not need to simplify numerical expressions in the answers, that is, an answer of the form $1 - \frac{123! \cdot 456!}{789!}$ would be fine.

Question 6

(a) Describe the structure of the graph G in the following cases:

i. G is not connected, but, for any pair of non-neighbours u, v in G, G becomes connected after the edge uv is added to G. **[5 Marks]**

ii. The number of connected components in G increases after removing any edge from G. **[4 Marks]**

iii. After removing any edge from G, one obtains a tree. **[5 Marks]**

Justify your answers.

(b) Define the height of a rooted tree. Prove by induction on h that an m-ary tree of height h has at most m^h leaves. **[4 Marks]**

(c) Define the n-cube graph Q_n. Find the number of edges in Q_n. Define the diameter of a graph and determine the diameter of Q_n. Justify your answers. **[7 Marks]**

Section D Discrete Mathematics and Linear Algebra (Prof. A. Krokhin)

Question 7

(a) Let $0 \leq a < m$ be integers. What is a multiplicative inverse of a number a modulo m? State a necessary and sufficient condition for a multiplicative inverse to exist. Show that a has at most one multiplicative inverse $\bar{a}$ modulo m in the range $[0, m-1]$ (i.e. such that $0 \leq \bar{a} \leq m-1$).
[7 Marks]

(b) Find the multiplicative inverse of 11 modulo 15 in the range $[0, 14]$.
[5 Marks]

(c) Use the inversion algorithm to find the inverse (if it exists) of the matrix

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 2 & 3 \\ 9 & 5 & 7 \end{pmatrix}.$$

[6 Marks]

(d) Explain why the inversion algorithm for matrices always gives the correct solution. You may refer to the theorem about invertible matrices in your explanation. **[7 Marks]**

Question 8

(a) Define a basis of a vector space. Prove that the vectors $v_1 = (9, 0, 1), v_2 = (4, -2, 3), v_3 = (3, 2, -1)$ form a basis in $\mathbb{R}^3$ and find the coordinates of $u = (9, 4, 2)$ in this basis. Show your working. **[5 Marks]**

(b) Show that if $S = \{v_1, \ldots, v_n\}$ is a basis for a vector space V then each vector $v \in V$ can be expressed as $v = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n$ (where $k_i \in \mathbb{R}$ for $i = 1, \ldots, n$) in exactly one way. **[7 Marks]**

(c) Explain how matrices of size $m \times n$ are in one-to-one correspondence with linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. **[5 Marks]**

(d) Define eigenvectors and eigenvalues of a square matrix. Find the eigenvalues and describe all eigenvectors of the following matrix

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}.$$

[8 Marks]

END OF PAPER