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Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2017	COMP1021-WE01

Title: Mathematics for Computer Science

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX-83 GTPLUS or Casio FX-85GTPLUS
Visiting Students may use dictionaries:		

Instructions to Candidates:	Answer FOUR questions. (ONE from each Section) PLEASE ANSWER EACH SECTION IN A SEPARATE ANSWER BOOKLET

Revision:	

Section A Logic and Discrete Structures (Prof. I.A. Stewart)

Question 1

(a) State De Morgan's Laws and the Distribution Laws. **[4 Marks]**

(b) What do we mean when we say that a formula of Propositional Logic is: in conjunctive normal form; or in disjunctive normal form? **[2 Marks]**

(c) Use the laws you stated in (a) to obtain formulae in conjunctive normal form that are equivalent to the following formulae:

i. $(p \wedge \neg(s \Rightarrow r)) \vee q$ **[3 Marks]**

ii. $(a \Leftrightarrow \neg b) \vee \neg a$ **[3 Marks]**

iii. $((p \wedge q) \vee \neg(r \vee s)) \wedge \neg q$ **[4 Marks]**

(d) Associated with every boolean formula is a corresponding truth-table. Prove that for every truth-table, there is a corresponding boolean formula. **[5 Marks]**

(e) By using your construction in (d) or otherwise, construct a formula whose truth-table is that below.

X	Y	Z	
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

[4 Marks]

continued

Question 2

(a) Explain the terms syntax, semantics, and proof system in relation to any logic. What properties would we like from any proof system? **[5 Marks]**

(b) Describe the proof system for Propositional Logic called Resolution. Explain how given some formula, you would use the proof system Resolution to decide whether the formula is a tautology or not. **[12 Marks]**

(c) Use Resolution to decide whether the formulae defined below are tautologies.

i. $(\neg p \wedge \neg r) \vee (\neg q \wedge r) \vee q \vee \neg(p \Rightarrow t) \vee s \vee \neg(t \Rightarrow s)$ **[4 Marks]**

ii. $(\neg p \wedge \neg r) \vee (\neg q \wedge r) \vee q \vee \neg(p \Rightarrow t) \vee \neg(t \Rightarrow s)$ **[4 Marks]**

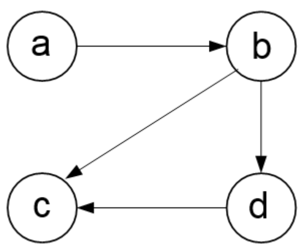
Section B Logic and Discrete Structures (Dr. R. Powell)

Question 3

(a) Let A and B be two arbitrary sets, and let $f : A \to B$ be a function. What is the graph of f? **[2 Marks]**

(b) Let $A = \{-2, 0, 1, 3\}$ and $B = \{-3, -1, 2, 4\}$. Let R be a binary relation from A to B where for all $(x, y) \in A \times B$, $(x, y) \in R$ if and only if $|x| \leq y$. Determine which pairs of $A \times B$ are in R. **[3 Marks]**

(c) Given below is a directed graph G, with vertex set $V = \{a, b, c, d\}$ and edge set $E = \{(a, b), (b, c), (b, d), (d, c)\} \subseteq V \times V$. What is $E \circ E$? (You should explicitly identify the elements of $E \circ E$ and describe what these elements represent in the graph.)



[5 Marks]

(d) Define the terms partial order and equivalence relation. **[2 Marks]**

(e) Which of the following binary relations R are equivalence relations? (You should justify your answers.)

i. $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x + y| \geq 1\}$ **[3 Marks]**

ii. $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x^2 = y^2\}$ **[4 Marks]**

this question is continued on the next page

(f) Let $A = \{1, 2, 3, 4, 5\}$ and let

$$R = \{(1, 1), (1, 2), (2, 3), (2, 4), (4, 4)\} \subseteq A \times A.$$

i. Define the reflexive closure of a relation, and give the reflexive closure of R. **[3 Marks]**

ii. Define the symmetric closure of a relation, and give the symmetric closure of R. **[3 Marks]**

Question 4

(a) What does it mean if we say a first-order formula is in prenex normal form? **[3 Marks]**

(b) Consider the first-order formula ϕ defined as: $\forall x(\forall y P(y,x) \Rightarrow \exists y E(x,y))$. Give an equivalent formula in prenex normal form. **[5 Marks]**

(c) Show that the following first-order formula is well-formed by drawing the parse-tree: $\forall x(\forall y(\neg P(x,x,y) \Leftrightarrow Q(y,x))) \vee \exists x(P(y,x,x) \wedge \neg Q(x,y))$ **[4 Marks]**

(d) Consider the following structure X over the signature $\langle P, Q, R, S\rangle$ where P is a ternary relation, Q and R are binary relations and S is a unary relation.

- Let the domain of discourse D be the set of natural numbers $\mathbb{N}$.
- The ternary relation $P = \{(x,y,z) : x,y,z \in D, x+y > z\}$
- The binary relation $Q = \{(x,y) : x,y \in D, x = y^2\}$
- The binary relation $R = \{(x,y) : x,y \in D, x < y\}$
- The unary relation $S = \{(x) : x \in D, x \text{ is even}\}$

Is the structure X a model of the following sentences? (You should justify your answers.)

i. $\forall a \forall b \exists c(P(a,b,c) \wedge R(a,c) \wedge R(c,b))$ **[3 Marks]**

ii. $\forall a \forall b \exists c(R(a,b) \wedge S(a) \wedge S(b) \Rightarrow (R(a,c) \wedge R(c,b)))$ **[4 Marks]**

iii. $\exists a \exists b \exists c((P(a,c,b) \wedge Q(b,c) \wedge R(a,c)) \vee (Q(a,b) \wedge R(a,b)))$ **[6 Marks]**

End of Section B

continued

Section C Discrete Mathematics and Linear Algebra (Dr. I. Ivrissimtzis)

Question 5

(a) How many permutations of the letters in the string ABCDEFG contain:

i. the string CFG? **[2 Marks]**

ii. the strings DB and GF? **[2 Marks]**

iii. the strings AC and CFB? **[2 Marks]**

(b) How many solutions are there to the equation $x_1 + x_2 + x_3 = 5$ where x_1, x_2 and x_3 are:

i. non-negative integers? **[4 Marks]**

ii. positive integers? **[2 Marks]**

(c) State Bayes' theorem. **[3 Marks]**

(d) In a bit string generated by a random source, the probability of a 1 is 0.6 and the probability of a 0 is 0.4. The bits are transmitted over a noisy communications channel and they are received as sent with probability 0.8, but errors occur with probability 0.2.

i. What is the probability that a 1 was sent given that we received a 1? **[5 Marks]**

ii. To improve the reliability of the channel, we generate a digit and send this digit three times. What is the probability that 111 was sent given that we received 010? **[5 Marks]**

continued

Question 6

(a) Define the following graphs:

i. the path on n vertices P_n **[1 Mark]**

ii. the cycle on n vertices C_n **[1 Mark]**

iii. the complete graph on n vertices K_n **[1 Mark]**

iv. the n-cube Q_n **[2 Marks]**

(b) i. How many spanning trees does the graph P_n have? **[3 Marks]**

ii. How many spanning trees does the graph C_n have? **[3 Marks]**

(Justify your answers.)

(c) What is the maximum number of edges in a graph that has six vertices and two connected components? **[6 Marks]**

(d) Prove that a graph G is bipartite if and only if it does not have a circuit of odd length. **[8 Marks]**

End of Section C

Section D Discrete Mathematics and Linear Algebra (Dr. I. Ivrissimtzis)

Question 7

(a) i. Define Euler's ϕ-function. **[2 Marks]**

ii. Explain how $\phi(n)$ can be computed from the prime decomposition of n, i.e. from the representation $n = p_1^{k_1} \cdot \cdots \cdot p_l^{k_l}$ where $p_1 < \cdots < p_l$ are primes. **[3 Marks]**

iii. Determine $\phi(600)$. **[3 Marks]**

(b) i. State Fermat's Little Theorem. **[2 Marks]**

ii. Find a positive integer x such that $x^{102} \equiv 4 \bmod 11$. **[3 Marks]**

(c) Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

[6 Marks]

(d) Determine which of the following sets of matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are subspaces of $\mathbb{M}_{22}$:

i. the set of all matrices with $a + d = 0$ **[3 Marks]**

ii. the set of all matrices with $a + d = 1$ **[3 Marks]**

Question 8

(a) Define a basis of a vector space. Define the dimension of a finite-dimensional vector space. **[4 Marks]**

(b) Find the rank and nullity of the matrix

$$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

(Justify your answer.) **[4 Marks]**

(c) Find the determinant of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 & 4 \\ 3 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

[5 Marks]

(d) What is the maximum number of zeros that a 5×5 matrix can have without having a zero determinant? (Justify your answer.) **[4 Marks]**

(e) Define eigenvectors and eigenvalues of a square matrix. Find the eigenvalues and describe all eigenvectors of the following matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

[8 Marks]

END OF PAPER