Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2015	COMP1021WE01

Title: Mathematics for Computer Science

Time Allowed:	2 hours		
Additional Material provided:			
Materials Permitted:			
Calculators Permitted:	No	Models Permitted:	
Visiting Students may use dictionaries: Yes			

Instructions to Candidates:	Answer FOUR questions. (ONE from each Section) ANSWER EACH SECTION IN A SEPARATE ANSWER BOOK

Revision:	

Section A Logic and Discrete Structures
(Dr. M. Gadouleau and Dr. G. Mertzios)

Question 1

(a) This question is about $\oplus$ (Exclusive-Or).

i. Give the truth table of $X \oplus Y$. Express this formula in disjunctive normal form and in conjunctive normal form. **[5 Marks]**

ii. Prove that $X \wedge (Y \oplus Z) \equiv (X \wedge Y) \oplus (X \wedge Z)$. **[3 Marks]**

iii. Prove that $X \oplus (Y \Leftrightarrow Z) \equiv (X \oplus Y) \Leftrightarrow Z$. **[3 Marks]**

(b) What is the Law of the Excluded Middle? Prove it by natural deduction, without using any derived rules. **[5 Marks]**

(c) For each of the following rules of inference, give its definition and prove it by Resolution.

i. Modus Ponens. **[3 Marks]**

ii. Modus Tollens. **[3 Marks]**

iii. Hypothetical Syllogism. **[3 Marks]**

continued

Question 2

(a) Let A, B, and C be arbitrary sets. Prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

[7 Marks]

(b) Let $\mathcal{P}(X)$ denote the power set of a set X. Let A and B be two sets.

i. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. **[3 Marks]**

ii. If $A \nsubseteq B$ and $B \nsubseteq A$, prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ and that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$. **[4 Marks]**

iii. If $A \nsubseteq B$ and $A \cap B \neq \varnothing$, prove that $\mathcal{P}(A \setminus B) \setminus \{\varnothing\} \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$ and that $\mathcal{P}(A \setminus B) \setminus \{\varnothing\} \neq \mathcal{P}(A) \setminus \mathcal{P}(B)$. **[4 Marks]**

(c) i. Give the definition of an injective function, of a surjective function, and of a bijective function. **[3 Marks]**

ii. Let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = 2x + 1$. Is g injective? Is g surjective? Justify your answers. **[2 Marks]**

iii. Let $h : \mathbb{Z} \to \mathbb{Z}$ be such that $h(x) = 5x + 2$. Is h injective? Is h surjective? Justify your answers. **[2 Marks]**

Section B Logic and Discrete Structures
(Dr. K. Dabrowski and Dr. A. Erickson)

Question 3

(a) Define a binary relation from a set A to a set B. **[2 Marks]**

(b) Which of the following binary relations are graphs of functions from $\mathbb{R}$ to $\mathbb{R}$? Justify your answers.

i. $\{(x, y) \ : \ x, y \in \mathbb{R},\ 2y = 4x + 100\}$;

ii. $\{(x, y) \ : \ x, y \in \mathbb{R},\ y = x^{1000} + x^2 + x\}$;

iii. $\{(x, y) \ : \ x, y \in \mathbb{R},\ y^2 = 4x^2\}$. **[3 Marks]**

(c) When do we call a binary relation anti-symmetric? Let R and S be anti-symmetric binary relations. Is $R \cap S$ always an anti-symmetric relation? Is $R \cup S$ always an anti-symmetric relation? Justify your answers. **[5 Marks]**

(d) Let R be an irreflexive binary relation. Is $R \circ R$ (i.e., R composed with itself) always an irreflexive relation? Justify your answer. **[3 Marks]**

(e) Consider the binary relation "Has the same birthday as" defined on the set of all people. Is this relation an equivalence relation? Justify your answer. **[4 Marks]**

(f) Recall that a poset (A, R) consists of a set A together with a partial order R on A. Let $A = \{4, 6, 8, 10, 12, 14\}$. Let $R_1 = \{(x, y) \in A \times A \mid x \leq y+2\}$, and let $R_2 = \{(x, y) \in A \times A \mid x = y+2\}$. Is R_1 a reflexive relation? Is R_1 an anti-symmetric relation? Is R_1 a transitive relation? Answer the same three questions for R_2 as well. Determine whether (A, R_1) and (A, R_2) are posets. Justify your answers. **[8 Marks]**

continued

Question 4

(a) When do we say that a first-order formula is in *prenex normal form*? **[2 Marks]**

(b) When do we say that a first-order formula is valid? Give an example of a valid first-order formula. **[4 Marks]**

(c) Prove that the following two formulae are not valid:

i. $(\forall x A(x) \vee \forall x B(x)) \Rightarrow (\exists x A(x) \wedge \exists x B(x))$. **[4 Marks]**

ii. $(\forall x A(x) \Rightarrow \forall x B(x)) \Rightarrow \forall x (A(x) \Rightarrow \neg B(x))$. **[5 Marks]**

(d) Let $\langle C, D, E \rangle$ be a signature, where C, D are constants and E is a binary relation. Consider the following interpretation. Let the domain of discourse be the set of vertices of some directed graph. The constants C and D are two distinct vertices of the directed graph. The binary relation E is the set of edges of the directed graph, that is, $E(x, y)$ is true if and only if there is a directed edge from vertex x to vertex y. Give an equivalent sentence in first-order logic for each of the following sentences:

i. There exists a vertex equal to neither C nor D that has a directed edge to C and a directed edge to D. **[2 Marks]**

ii. There is a directed edge from vertex C to every other vertex except to vertex D. **[4 Marks]**

iii. There exists a vertex that has exactly one directed edge going out and no directed edges coming in. **[4 Marks]**

Section C Discrete Mathematics and Linear Algebra (Professor A. Krokhin)

Question 5

(a) A fair coin is tossed four times. Are the following events E and F independent?

- E: Both heads and tails come up at least once.
- F: Heads comes up at most once.

Justify your answer. **[8 Marks]**

(b) Define mutually independent Bernoulli trials. What is the probability of exactly k successes in n mutually independent Bernoulli trials? Justify your answer. **[6 Marks]**

(c) A box contains 3 white and 7 black balls. A ball is chosen at random from the box. What is the probability that the ball is black? **[2 Marks]**

(d) We have two boxes. The first box contains 3 white and 7 black balls, and the second box contains 6 white and 3 black balls. A ball is chosen at random from the first box, and, without looking at its colour, put into the second box. Then a ball is chosen at random from the second box, and it is white. Is it more likely than not that the ball moved from the first box to the second was black? Justify your answer. **[9 Marks]**

Question 6

(a) Define an Eulerian circuit in a graph. Prove that each connected graph, in which each vertex has even degree, has an Eulerian circuit. **[6 Marks]**

(b) Assume that a connected graph G on n vertices has $k \geq 0$ vertices of degree 1 and $n - k$ vertices of degree 2.

i. Determine the possible value(s) of k if G has no cycles. **[7 Marks]**

ii. Determine the possible value(s) of k if G has at least one cycle. **[7 Marks]**

Justify your answers.

(c) Define a full m-ary tree. How many nodes does a full m-ary tree with i internal nodes have? Justify your answer. **[5 Marks]**

Section D Discrete Mathematics and Linear Algebra (Professor A. Krokhin)

Question 7

(a) Define Euler's ϕ-function, explaining all notions used in the definition. Explain how $\phi(n)$ can be computed from the prime decomposition of n, i.e. from the representation $n = p_1^{k_1} \cdot \ldots \cdot p_l^{k_l}$ where $p_1 < \ldots < p_l$ are primes. Prove your formula for $n = p^k$ (p prime) and determine $\phi(1750)$. Show your working. **[8 Marks]**

(b) State the Chinese Remainder Theorem. **[5 Marks]**

(c) Consider the following matrix

$$A = \begin{pmatrix} 0 & 3 & -3 & 2 \\ 1 & 5 & 2 & 0 \\ 2 & -4 & 2 & 1 \\ 7 & -1 & 0 & 0 \end{pmatrix}$$

Find the determinant of A

i. by using cofactors **[6 Marks]**

ii. by using elementary row operations **[6 Marks]**

Question 8

(a) Define a subspace of a vector space. Do the invertible 2×2 matrices form a subspace of the space of all 2×2 matrices? Justify your answer. **[5 Marks]**

(b) Define linear independence of a set of vectors. Is the following set of vectors in $\mathbb{R}^4$ linearly independent? $\{v_1 = (0, 2, 7, 5), v_2 = (-1, 2, 4, -2), v_3 = (3, -4, -5, 11)\}$. Justify your answer. **[7 Marks]**

(c) Define a linear map between two vector spaces. Is the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (0, 1)$ linear? Justify your answer. **[5 Marks]**

(d) Define eigenvectors and eigenvalues of a square matrix. Find the eigenvalues and describe all eigenvectors of the following matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}.$$

[8 Marks]

END OF PAPER