Optimisation

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1 Question 1

Define L to be the following linear program

$$\begin{aligned}
&\text{min} && x_1 - 3x_2 && \text{(1a)}\\
&\text{subject to} && x_1 - x_2 \leq 1 && \text{(1b)}\\
& && x_1 - x_2 \geq -1 && \text{(1c)}\\
& && 2x_1 - x_2 \leq 3 && \text{(1d)}\\
& && x_1, x_2 \geq 0 && \text{(1e)}
\end{aligned}$$

We convert L to the linear program L' in Standard Equational Form:

$$\begin{aligned}
&\text{max} && 3x_2 - x_1 && \text{(1f)}\\
&\text{subject to} && x_1 - x_2 + x_3 = 1 && \text{(1g)}\\
& && x_2 + x_4 - x_1 = 1 && \text{(1h)}\\
& && 2x_1 - x_2 + x_5 = 3 && \text{(1i)}\\
& && x_1, x_2, x_3, x_4, x_5 \geq 0 && \text{(1j)}
\end{aligned}$$

Note we multiplied $(1c)$ from the original problem by -1 to make the computations easier later. We can view L' as:

$$z(\underline{x}) = \begin{pmatrix} -1 & 3 & 0 & 0 & 0 \end{pmatrix} \underline{x}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{pmatrix} \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Note that $B = \{2, 3, 4\}$ is a feasible basis with basic feasible solution $\begin{pmatrix} 0 & 0 & 1 & 1 & 3 \end{pmatrix}^T$. We therefore know L is feasible and can apply the simplex algorithm to it.

$t = min\{-, \frac{1}{1}, -\}$ so pick row 3. By Blands rule pick column 3 since it is only negative coefficient.

$$\left[\begin{array}{c|cccccc|c} 1 & 1 & -3 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 & 3 \end{array}\right] \tag{2}$$

$t = min\{\frac{2}{0}, -, \frac{4}{1}\}$ so pick row 3. By Blands rule pick column 2 since it is only negative coefficient.

$$\left[\begin{array}{c|cccccc|c} 1 & -2 & 0 & 0 & 3 & 0 & 3 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \end{array}\right] \tag{3}$$

Top row in (4) is now entirely positive so we terminate. We now see from the tableau we have an optimum value to L' of 11 with values $x_1 = 4, x_2 = 5$

$$\left[\begin{array}{c|ccccc|c} 1 & 0 & 0 & 0 & 5 & 2 & 11 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \end{array}\right] \tag{4}$$

Therefore the optimum value for L was -11 with $x_1 = 4, x_2 = 5$.

2 Question 2

It is given in the question that the problem has a basis feasible solution so we immediately begin applying the simplex algorithm.

$t = min\{-, \frac{2}{1}, \frac{6}{2}\}$ so pick row 3. By Blands rule we pick column 2.

$$\left[\begin{array}{c|cccccc|c} 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & -3 & -1 & 0 & 0 & 1 & 6 \end{array}\right] \tag{5}$$

$t = min\{-,-,-\}$ in column 2 (picked because of Blands rule) therefore the original problem is unbounded.

$$\left[\begin{array}{c|cccccc|c} 1 & 0 & -3 & 1 & 0 & 2 & 0 & 4 \\ \hline 0 & 0 & -1 & 1 & 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 & 0 & -2 & 1 & 2 \end{array}\right] \tag{6}$$

We now recover the certificate of unboundedness. From the slides $x'_B = \underline{b} - tA_k$, where $k = 2$ meaning $x_2 = t, x_3 = 0, x_5 = 0$.

$$\begin{pmatrix} x_4 \\ x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} - t \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$
$$\implies \begin{pmatrix} x_4 \\ x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2+t \\ 5+t \\ 2+t \end{pmatrix}$$
$$\implies x(t) = \begin{pmatrix} 2+t & t & 0 & 5+t & 0 & 2+t \end{pmatrix}^T$$
$$\implies x(t) = \begin{pmatrix} 2 & 0 & 0 & 5 & 0 & 2 \end{pmatrix}^T + t \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}^T$$

It is trivial to see as $t \to \infty$ that $z(x(t)) \to \infty$ and $\forall t : x(t) \geq \underline{0}$. It is also easy to verify that

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & -3 & -1 & 0 & 0 & 1 \end{pmatrix} x(t) = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$

Therefore the certificates of unboundedness are $\bar{x} = \begin{pmatrix} 2 & 0 & 0 & 5 & 0 & 2 \end{pmatrix}, d = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$.

3 Question 3

3.1 Formulation

Since it is possible to make fractional units we do not restrict variable to integers, however all variables are nonnegative since you can't have negative amounts of paint. There are four recipe paint colours you can make Red (R), Green (G), Blue (B) and Black (K) and 3 Base colours Cyan, Yellow and Magenta.

First we enforce not using more paint than we have. For each recipe r we create a variable to denote the amount of the base colour used in r. For example the variable Y_R denotes the amount of yellow used to make red. We can't use more paint than we have, so for each base colour c we sum it's respective variables used in every r and bound it from above by the amount of c we own. E.g. Yellow is used in mixing Red, Green and Black so we have $Y_R + Y_G + Y_K \leq 11$, and so any valid solution must use at most 11 unit of yellow.

To ensure the colours are made correctly observe in any r the amount of any base colour used must be equal. For example the amount of yellow used to make red must be equal to the amount of magenta used. Therefore we add constraints for each r to ensure the constituent base colour amounts used are equal, for example $Y_R = M_R$.

For each recipe colour c we create a variable to denote the amount of c we make. E.g. let R be the amount of red we make $R := Y_R + M_R$ as defined in the recipe, the conversion of "making 2 units" is satisfied by this recipe since wlog we can see if $Y_R = 1, M_R$ then $R = 2$. Our objective function is $max(10R+15G+25B+25K)$ where the coefficient of each variables is the corresponding price per unit for that colour. The image of the objective function will therefore be the total amount of income got by mixing the corresponding amounts of R, G, B and K where our earlier constraints ensure the recipes are followed correctly and no additional paint is bought. Therefore our description captures the original problem.

3.2 Solution

The optimal solution is making 10 units of green and 15 units of black leaving you with a total income of 525. To make the black you would use 5 units of yellow, magenta and cyan. To make the red you would use 5 units of yellow and magenta. This leaves 1 unit of yellow left over but all other paint has been exhausted.

4 Question 4

4.1 Part a

4.1.1 Formulation

Since it is an integer problem we restrict every variable to the integer domain. For each item i let it's weight be w_i and value be v_i. To represent the choice of whether to take an item or not we use a variable $x_i \in \{0, 1\}$ where 0 will indicate we do not take the item i and 1 we do. The objective function then is $\sum_i x_i v_i$. E.g. The solution on 3 items of $x_1 = 1, x_2 = 0, x_3 = 1$ would represent taking items 1 and 3 with our value being $1v_1 + 0v_2 + 1v_3 = v_1 + v_3$. Since x_i is restricted to only 0 or 1 we can never take an item more than once.

To not exceed the weight requirement we add the constraint $\sum_i x_i w_i \leq 20$. This constraint represents the sum of the weights of the selected items being less than or equal to 20 which is the limit.

4.1.2 Solution

The solution is $x_A = 0, x_B = 1, x_C = 1, x_D = 0, x_E = 0, x_F = 1$. This means picking items B, C and F which would give you a total value of 210 and a total weight of 19.

4.2 Part b

4.2.1 Formulation

To ensure C is only taken with D we add the constraint $x_D \geq x_C$. In the illegal scenario C is taken without D $x_C = 1, x_D = 0 \implies 0 \geq 1$ which is not allowed. It is trivial to see all other scenarios, which are allowed, satisfy this inequality.

4.2.2 Solution

The solution is $x_A = 0, x_B = 0, x_C = 0, x_D = 1, x_E = 1, x_F = 1$. This means picking items D, E and F which would give you a total value of 186 and a total weight of 20.

4.3 Part c

4.3.1 Formulation

We add a new variable o to our problem to represent the overflow of the weight over 20. We amend our objective function to be $\sum_i x_i v_i - 15o$. Each kilogram over 20 will penalise 15 to the cost of the solution. We add the constraint $o \geq 0$, intuitively it doesn't make sense to define a negative overflow but also if o was negative $-15o$ would be positive and the problem would be unbounded.

We amend the constraint $\sum_i x_i w_i \leq 20$ to $\sum_i x_i w_i \leq 20 + o$. If the optimal solution requires packing less than 20 in total then o will be set to 0 by the solver but if the optimal solution is over 20 then to satisfy the constraint the solver will set o to be how far over 20 it is, which will then be penalised in the objective function accordingly.

4.3.2 Solution

The solution was $x_A = 1, x_B = 1, x_C = 1, x_D = 0, x_E = 0, x_F = 0$. This means picking items A, B, C for a total value of 215 and total weight 21.0