**Learning Outcomes:**

For the first midterm, students are expected to be able to ...

1. Identify whether certain functions are solutions to a given ODE, as well as distinguish the difference between implicit vs explicit, general vs particular solutions.
2. Analyze the behavior of solutions to autonomous equations and classify equilibria in terms of asymptotically stable (sink), unstable (source), or semi-stable (node).
3. Identify and solve the following classes of first-order ODEs: separable (and homogeneous), linear, and exact.
4. Solve homogeneous second-order linear equations with constant coefficients in the case where the corresponding characteristic equation has distinct real-valued roots, repeated roots, and complex roots.
5. Apply the Method of Undetermined Coefficients and the Principle of Superposition to obtain an appropriate "guess" for the particular solution of a non-homogeneous 2nd-order linear equation.

The following integration formulas will be provided on the exam and you may apply any of these results directly without showing work/proof:

$$\int (ax+b)^n \; dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + C$$

$$\int e^{ax+b} \; dx = \frac{1}{a} \cdot e^{ax+b} + C$$

$$\int u^{ax+b} \; dx = \frac{1}{a \ln u} \cdot u^{ax+b} + C \text{ for } u > 0, u \neq 1$$

$$\int \frac{1}{ax+b} \; dx = \frac{1}{a} \cdot \ln|ax+b| + C$$

$$\int \sin(ax+b) \; dx = -\frac{1}{a} \cdot \cos(ax+b) + C$$

$$\int \cos(ax+b) \; dx = \frac{1}{a} \cdot \sin(ax+b) + C$$

$$\int \sec^2(ax+b) \; dx = \frac{1}{a} \cdot \tan(ax+b) + C$$

$$\int \csc^2(ax+b) \; dx = \frac{1}{a} \cdot \cot(ax+b) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1}(x) + C$$

$$\int \frac{1}{1+x^2} \; dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \; dx = \sinh^{-1}(x) + C$$

# 1 Autonomous Differential Equation & Equilibrium Solutions

**Section 1.3:** Autonomous equations mostly can be solved using separable techniques. They are differential equations of the form

$$\frac{dy}{dt} = f(y)$$

where the right-hand side depends only on $y$.

If $y_0$ is a solution of $f(y) = 0$ then the constant solution $y \equiv y_0$ is the *equilibrium solution*.

We can easily classify the behavior of equilibrium solutions by simply looking at the direction field. However, if the direction field is unavailable, then we need to consider the sign of $f(y)$ for $y$ "near" $y_0$. Specifically, we can classify the equilibrium into three types as follows:

- If $f(y) < 0$ for $y < y_0$ and $f(y) > 0$ for $y > y_0$ then $y_0$ is an **unstable** equilibrium or a source.
- If $f(y) > 0$ for $y < y_0$ and $f(y) < 0$ for $y > y_0$ then $y_0$ is an **asymptotically stable** equilibrium or a sink.
- If the sign of $f(y)$ is **unchanged as we pass through** $y_0$ then $y_0$ is a **semi-stable** equilibrium or a node.

## 2 First Order Linear Differential Equation

**Section 2.3:** Differential equations of the form

$$\frac{dy}{dt} + p(t)\ y = g(t) \tag{1}$$

where $t$ is the independent variable and $y(t)$ is the unknown function.

Solution steps:

0. Warning: If you are given an equation in the form $a(t)y' + b(t)y = c(t)$, where $a(t) \neq 0$, remember to transform it into the form given in (1) before doing the steps below.

1. Find the integrating factor $\mu(t)$ given by

$$\mu(t) = \exp\left(\int p(t)dt\right).$$

   Under the right integrating factor, we should have $\mu'(t) = \mu(t)p(t)$.

2. Multiplying $\mu(t)$ to both sides of (1) to obtain a product rule on the left-hand side, as follows.

$$\begin{aligned} \mu(t)\frac{dy}{dt} + \mu(t)p(t)\ y = \mu(t)g(t) &\Rightarrow \mu(t)\frac{dy}{dt} + \mu'(t)y = \mu(t)g(t) \\ &\Rightarrow (y \cdot \mu(t))' = \mu(t)g(t). \end{aligned}$$

   Again, one should be able to observe the Product Rule for derivative during this step is using the correct integrating factor.

3. Integrate both side of the result above with respect to (w.r.t) $t$ to obtain the solution.

$$y \cdot \mu(t) = \int \mu(t)g(t)dt \Rightarrow y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$$

   where $C$ is a constant.

4. Apply the initial condition (if any) to solve for the constant $C$.

## 3 Separable Differential Equation

**Section 2.2:** Differential equations of the form

$$\frac{dy}{dt} = g(t)h(y). \tag{2}$$

Solution steps:

1. Solve $h(y) = 0$ for any constant solution.

2. Assuming that $h(y) \neq 0$, transform (2) into the following

$$\frac{dy}{h(y)} = g(t)dt.$$

3. Integrate both side (w.r.t $y$ on the left and to $t$ on the right) and solve for $y$ in terms of $t$. Remember any constant that may resulted from the integration!

4. If given an initial condition, use this to solve for the constants from step 3.

### 3.1 Homogeneous Differential Equation

This is not entirely a new kind of differential equation and can be transformed into a separable one. Let $x$ be the independent variable and $y(x)$ be the unknown function. Homogeneous differential equations are of the form

$$\frac{dy}{dx} = f(x, y) \tag{3}$$

where $f(x, y)$ is homogeneous. That is, $f(tx, ty) = f(x, y)$ for any number $t$. An alternative definition of homogeneous is that the function $f(x, y)$, after some transformations, should depend only on the ratio $y/x$.

Solution steps:

1. Let $v = \dfrac{y}{x}$ so that $y = x \cdot v$. Differentiate (implicitly) both sides w.r.t $x$ to transform the left-hand side of (3) into

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Notice that both $y$ and $v$ here are functions of $x$.

2. For the right-hand side of (3), apply the substitution $v = y/x$ to obtain $f(1, y/x) = f(1, v) = g(v)$.

3. Solve the equation

$$v + x\frac{dv}{dx} = g(v)$$

using separable techniques for $v$. Then go back to $y$ by replacing $v$ with $y/x$.

4. Use initial conditions to solve for the constants.

# 4 Exact Differential Equation

**Section 2.4:** Differential equations of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \text{ or equivalently } M(x,y)dx + N(x,y)dy = 0$$

where

$$M_y(x,y) := \frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial y}(x,x) =: N_x(x,y).$$

Solution steps:

0. Check that $M_y(x,y) = N_x(x,y)$ to confirm that we indeed have an exact equation. If not, multiply the integrating factor to both sides to obtain an exact differential equation. (Note: obtaining the integrating factor for exact equation will not be asked on this exam.)

1. Compute the function $F(x,y)$ using the following two formulas

$$F(x,y) = \int M(x,y)dx = F_M(x,y) + c_1(y),$$
$$F(x,y) = \int N(x,y)dy = F_N(x,y) + c_2(x).$$

Remark: the constant term coming from the above integrations is not entirely a constant, but a function depending on the remaining variable.

2. Set $F_M(x,y) + c_1(y) = F_N(x,y) + c_2(x)$ and solve for $F(x,y)$. The general solution will be given by

$$F(x,y) = C$$

for some constant $C$.

3. Use the initial conditions to solve for $C$, if needed.

# 5 Second Order Linear Differential Equation

**Sections 4.2, 4.3:** We consider the homogeneous second order linear differential equation with constant coefficients given by

$$ay'' + by' + cy = 0 \tag{4}$$

where $a, b$, and $c$ are constants. The characteristic equation of (4) is given by

$$ar^2 + br + c = 0. \tag{5}$$

There are three possibilities:

- If (5) has two distinct real solutions $r_1$ and $r_2$ then the general solution of (4) is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  for any constants $c_1$ and $c_2$.

- If (5) has repeated root $r_1 = r_2 = r$ then the general solution of (4) is given by

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

  for any constants $c_1$ and $c_2$.

- If (5) has complex roots $r_{1,2} = u \pm iv$ then the general solution of (4) is given by

$$y(t) = c_1 e^{ut} \cos(vt) + c_2 e^{ut} \sin(vt)$$

  for any constants $c_1$ and $c_2$.

## 5.1 Principle of Superposition

**Sections 4.2 & 4.5:** In general, given the second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0. \tag{6}$$

If $y_1(t)$ and $y_2(t)$ are two solution of (6) such that the Wronskian determinant

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0$$

for some value of $t \in I$ then we say that $y_1(t)$ and $y_2(t)$ are linearly independent on $I$ and they form a *fundamental set of solutions*. In this case, any linear combination of $y_1(t)$ and $y_2(t)$ will be a solution to (6). That is, the general solution of (6) is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for any $c_1$ and $c_2$.

# 6 Method of Undetermined Coefficients

**Section 4.4, 4.5:** Consider the hon-homogeneous second order linear equation

$$y'' + p(t)y' + q(t)y = g(t), \text{ where } g(t) \neq 0. \tag{7}$$

Then the solution to (7) can be expressed in the form

$$y = y_H + y_P$$

where

$$y_H = C_1 y_1 + C_2 y_2$$

is the general solution to the corresponding homogeneous part $y'' + p(t)y' + q(t)y = 0$, and $y_P$ is **any** specific function that satisfies (7).

In general, it is not easy to find the particular solution $y_P$. However, depending on the choice of the right-hand side $g(t)$, we may be able to "guess" the general form/type of $y_P$. Below are several cases for $g(t)$ that we can apply the method to obtain a "first guess" for $y_P$.

| $g(t)$ | First guess for $y_P$ |
|---|---|
| $ke^{rt}$ | $Ae^{rt}$ |
| $k\cos(\omega t)$ or $k\sin(\omega t)$ | $A\cos(\omega t) + B\sin(\omega t)$ |
| $P_n(t)$ | $A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$ |
| $P_n(t)e^{rt}$ | $\left(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0\right) e^{rt}$ |
| $P_n(t)e^{rt}\cos(\omega t)$ or $P_n(t)e^{rt}\sin(\omega t)$ | $\left(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0\right) e^{rt}\cos(\omega t) +$ $\left(B_n t^n + B_{n-1}t^{n-1} + \cdots + B_1 t + B_0\right) e^{rt}\sin(\omega t)$ |

Here, $P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ represents an $n$-th degree polynomial.

If any term in the first guess has been included in the solution to the corresponding homogeneous equation, we need to **multiply the entire guess** by $t$. If there is still repeated term in the second guess, multiply by $t$ again and keep doing this until there is no repeated term.

Finally, substitute the general form of $y_P$ into (7) and find solve for the coefficients.