MATH 20D: INTRO TO DIFFERENTIAL EQUATIONS
FINAL EXAM PRACTICE EXERCISES – SOLUTIONS

Problem 0. The final exam is cumulative, so please make sure you review all the exercises from the practice packages for the two midterms.

Problem 1. Use the integral definition to compute the Laplace transform for t^n for $t = 1, 2$, and for a general value n.

Solution.

a. Let $f(t) = t$ and $s > 0$ then the Laplace transform of $f(t)$ is given by

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} t \, dt = \lim_{A\to\infty} \int_0^A e^{-st} t \, dt \\
&= \lim_{A\to\infty} \left(-\frac{te^{-st}}{s}\Big|_0^A - \frac{e^{-st}}{s^2}\Big|_0^A \right) \\
&= \lim_{A\to\infty} \left(\frac{1}{s^2} - \frac{e^{-As}}{s^2} - \frac{Ae^{-As}}{s} \right) = \frac{1}{s^2}.
\end{aligned}$$

b. Let $f(t) = t^2$ and $s > 0$ then the Laplace transform of $f(t)$ is given by

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} t^2 \, dt = \lim_{A\to\infty} \int_0^A e^{-st} t^2 \, dt \\
&= \lim_{A\to\infty} \left(-\frac{t^2 e^{-st}}{s}\Big|_0^A + \frac{2}{s}\int_0^A te^{-st} dt \right) \\
&= \lim_{A\to\infty} \left(-\frac{A^2 e^{-As}}{s} \right) + \frac{2}{s}\int_0^\infty te^{-st} dt \\
&= \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}.
\end{aligned}$$

c. Let $f(t) = t^n$ where n is a positive integer, and $s > 0$. We will show by induction that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

The base cases for $n = 1$ and $n = 2$ are confirmed in the previous parts. Suppose that the formula is true for $n = k$, that is, $\mathcal{L}\{t^k\} = \dfrac{k!}{s^{k+1}}$, we will show that it also holds for $n = k + 1$.

using integration by parts, we have

$$
\begin{aligned}
\mathcal{L}\{t^{k+1}\} &= \int_0^\infty e^{-st}t^{k+1}\,dt = \lim_{A\to\infty}\int_0^A e^{-st}t^{k+1}\,dt \\
&= \lim_{A\to\infty}\left(-\frac{t^{k+1}e^{-st}}{s}\Big|_0^A + \frac{k+1}{s}\int_0^A t^k e^{-st}dt\right) \\
&= \lim_{A\to\infty}\left(-\frac{A^{k+1}e^{-As}}{s}\right) + \frac{k+1}{s}\int_0^\infty t^k e^{-st}dt \\
&= \frac{k+1}{s}\mathcal{L}\{t^k\} = \frac{k+1}{s}\cdot\frac{k!}{s^k} = \frac{(k+1)!}{s^{k+1}},
\end{aligned}
$$

as required. Hence, $\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$ for all positive integer n.

Problem 2.

(a) Let $b \in \mathbb{R}$ be a constant. Recall that $\cosh(b) = \dfrac{e^{bt}+e^{-bt}}{2}$ and $\sinh(b) = \dfrac{e^{bt}-e^{-bt}}{2}$. Find the Laplace transform of $\cosh(bt)$ and $\sinh(bt)$.

(b) Let $b \in \mathbb{R}$ be a constant. We have the following formulas for $\sin(bt)$ and $\cos(bt)$:

$$
\cos(bt) = \frac{e^{ibt}+e^{-ibt}}{2}, \quad \text{and} \quad \sin(bt) = \frac{e^{ibt}-e^{-ibt}}{2}.
$$

Find the Laplace transform of $\cos(bt)$ and $\sin(bt)$.

(c) Find the Laplace transform of the stepwise function $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}$

Solution.

(a) Recall from an example in class that $\mathcal{L}\{e^{at}\}(s) = \dfrac{1}{s-a}$ for all $s > a$. Thus,

$$
\mathcal{L}\{\cosh(bt)\} = \mathcal{L}\left\{\frac{e^{bt}}{2} + \frac{e^{-bt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{bt}\} + \frac{1}{2}\mathcal{L}\{e^{-bt}\} = \frac{1}{2(s-b)} + \frac{1}{2(s+b)} = \frac{s}{s^2-b^2}, \text{ and}
$$

$$
\mathcal{L}\{\sinh(bt)\} = \mathcal{L}\left\{\frac{e^{bt}}{2} - \frac{e^{-bt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{bt}\} - \frac{1}{2}\mathcal{L}\{e^{-bt}\} = \frac{1}{2(s-b)} - \frac{1}{2(s+b)} = \frac{b}{s^2-b^2}
$$

for all $s > |b|$.

(b) We can check that

$$
\int_0^\infty e^{(u+iv)t}e^{-st}dt = \frac{1}{s-u-iv}
$$

for all $s > u$. Thus, if a is a real number, then the Laplace's transform of e^{iat} is given by $\mathcal{L}\{e^{iat}\} = \dfrac{1}{s-ia}$, for all $s > 0$. Hence,

$$\begin{aligned}\mathcal{L}\{\cos(bt)\} &= \mathcal{L}\Big\{\frac{e^{ibt}}{2i} + \frac{e^{-ibt}}{2i}\Big\} = \frac{1}{2i}\mathcal{L}\{e^{ibt}\} + \frac{1}{2i}\mathcal{L}\{e^{-ibt}\}\\ &= \frac{1}{2i(s-ib)} + \frac{1}{2i(s+ib)} = \frac{s}{s^2+b^2}, \text{ and}\end{aligned}$$

$$\begin{aligned}\mathcal{L}\{\sin(bt)\} &= \mathcal{L}\Big\{\frac{e^{ibt}}{2i} - \frac{e^{-ibt}}{2i}\Big\} = \frac{1}{2i}\mathcal{L}\{e^{ibt}\} - \frac{1}{2i}\mathcal{L}\{e^{-ibt}\}\\ &= \frac{1}{2i(s-ib)} - \frac{1}{2i(s+ib)} = \frac{b}{s^2+b^2},\end{aligned}$$

for all $s > 0$.

(c) Let $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}$. The Laplace transform of $f(t)$ is given by

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} t\, dt = \int_0^1 e^{-st} t\, dt\\ &= -\frac{te^{-st}}{s}\Big|_0^1 - \frac{e^{-st}}{s^2}\Big|_0^1\\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} = \frac{1-e^{-s}(1+s)}{s^2}.\end{aligned}$$

Problem 3. Find the inverse Laplace transform of the following functions:

$$(a)\ \frac{3}{s^2+4} \qquad (b)\ \frac{2s+2}{s^2+2s+5} \qquad (c)\ \frac{2s-3}{s^2-4}$$

Solution.

(a) $\mathcal{L}^{-1}\Big\{\dfrac{3}{s^2+4}\Big\} = \dfrac{3}{2}\mathcal{L}^{-1}\Big\{\dfrac{2}{s^2+2^2}\Big\} = \dfrac{3}{2}\sin(2t)$.

(b) $\mathcal{L}^{-1}\Big\{\dfrac{2s+2}{s^2+2s+5}\Big\} = 2\mathcal{L}^{-1}\Big\{\dfrac{(s+1)}{(s+1)^2+2^2}\Big\} = 2e^{-t}\cos(2t)$.

(c) We use partial fraction and the linearity of Laplace's transform, as follows.

$$\mathcal{L}^{-1}\Big\{\frac{2s-3}{s^2-4}\Big\} = \mathcal{L}^{-1}\Big\{\frac{1/4}{s-2} + \frac{7/4}{s+2}\Big\} = \frac{1}{4}\mathcal{L}^{-1}\Big\{\frac{1}{s-2}\Big\} + \frac{7}{4}\mathcal{L}^{-1}\Big\{\frac{1}{s+2}\Big\} = \frac{e^{2t}}{4} + \frac{7e^{-2t}}{4}.$$

Remark:

Alternatively, if we use the result we obtained from the previous problem for the Laplace transform for hyperbolic trigonometric functions, we can also find the inverse Laplace transform of the given function as follows.

$$\mathcal{L}^{-1}\Big\{\frac{2s-3}{s^2-4}\Big\} = \mathcal{L}^{-1}\Big\{\frac{2s}{s^2-4} - \frac{(3/2)(2)}{s^2-4}\Big\} = 2\mathcal{L}^{-1}\Big\{\frac{s}{s^2-4}\Big\} - \frac{3}{2}\mathcal{L}^{-1}\Big\{\frac{2}{s^2-4}\Big\} = 2\cosh(2t) - \frac{3}{2}\sinh(2t)$$

Through some algebraic manipulations, one can show that the two answers are equal.

Problem 4. Use Laplace transform to solve the following IVPs:

(a) $y'' - 2y' + 2y = 0,\ y(0) = 0,\ y'(0) = 1$

(b) $y^{(4)} - y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 0$

(c) $y'' + 2y' + y = 4e^{-t},\ y(0) = 2,\ y'(0) = -1$

Solution.

(a) Consider the IVP $y'' - 2y' + 2y = 0,\ y(0) = 0,\ y'(0) = 1$. Applying the Laplace's transform to both sides yields

$$\begin{aligned} 0 = \mathcal{L}\{0\} &= \mathcal{L}\{y'' - 2y' + 2y\} \\ &= (s^2\mathcal{L}\{y\} - sy(0) - y'(0)) - 2(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} \\ &= (s^2 - 2s + 2)\mathcal{L}\{y\} - 1. \end{aligned}$$

Thus,

$$\mathcal{L}\{y\} = \frac{1}{s^2 - 2s + 2}.$$

The solution to the IVP is given by

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 2s + 2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2 + 1^2}\right\} = e^t \sin(t).$$

(b) Consider the IVP $y^{(4)} - y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 0$. Applying the Laplace's transform to both sides yields

$$\begin{aligned} 0 = \mathcal{L}\{0\} &= \mathcal{L}\{y^{(4)} - y\} \\ &= (s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) - \mathcal{L}\{y\} \\ &= (s^4 - 1)\mathcal{L}\{y\} - s^3 - s. \end{aligned}$$

Thus,

$$\mathcal{L}\{y\} = \frac{s^3 + s}{s^4 - 1} = \frac{s}{s^2 - 1} = \frac{1/2}{s-1} + \frac{1/2}{s+1}.$$

The solution to the IVP is given by

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2 - 1}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s-1} + \frac{1/2}{s+1}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = \frac{e^t}{2} + \frac{e^{-t}}{2}.$$

(c) Consider the IVP $y'' + 2y' + y = 4e^{-t},\ y(0) = 2,\ y'(0) = -1$. Applying the Laplace's transform to both sides yields

$$\begin{aligned} \frac{4}{s+1} = \mathcal{L}\{4e^{-t}\} &= \mathcal{L}\{y'' + 2y' + y\} \\ &= (s^2\mathcal{L}\{y\} - sy(0) - y'(0)) + 2(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} \\ &= (s^2 + 2s + 1)\mathcal{L}\{y\} - 2s - 3 \\ &= (s+1)^2\mathcal{L}\{y\} - 2s - 3 \end{aligned}$$

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Thus,

$$\mathcal{L}\{y\} = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2} = \frac{4}{(s+1)^3} + \frac{1}{(s+1)^2} + \frac{2}{s+1}.$$

The solution to the IVP is given by

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\Big\{\frac{4}{(s+1)^3} + \frac{1}{(s+1)^2} + \frac{2}{s+1}\Big\} \\ &= 2\mathcal{L}^{-1}\Big\{\frac{2!}{(s+1)^3}\Big\} + \mathcal{L}^{-1}\Big\{\frac{1!}{(s+1)^2}\Big\} + 2\mathcal{L}^{-1}\Big\{\frac{1}{s+1}\Big\} \\ &= 2t^2e^{-t} + te^{-t} + 2e^{-t}. \end{aligned}$$

MORE EXERCISES ON LAPLACE TRANSFORM

Problem I. Find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ for the following functions. Simplify your answer, if able.

1. $f(t) = t^3 - t^2 + 5t + 2$
2. $f(t) = -2\cos(6t) + 5\sin(6t)$
3. $f(t) = 3e^t - 4e^{2t} + 2e^{-4t}$
4. $f(t) = 7t + 6e^t - 2e^{-t} - 10$
5. $f(t) = \sin(2t)\sin(3t)$
6. $f(t) = \cos^2(5t)$
7. $f(t) = t\cos^2(5t)$
8. $f(t) = te^{at}\left(\cos(bt) + \sin(bt)\right)$
9. $f(t) = t^3\cos(2t)$
10. $f(t) = \cos(\alpha t + \beta)$
11. $f(t) = \sin(\alpha t + \beta)$

Solution.

1. $f(t) = t^3 - t^2 + 5t + 2 \Rightarrow F(s) = \dfrac{2s^3 + 5s^2 - 2s + 6}{s^4}$
2. $f(t) = -2\cos(6t) + 5\sin(6t) \Rightarrow F(s) = \dfrac{-2s + 30}{s^2 + 36}$
3. $f(t) = 3e^t - 4e^{2t} + 2e^{-4t} \Rightarrow F(s) = \dfrac{s^2 - 12s - 4}{(s-1)(s-2)(s+4)}$
4. $f(t) = 7t + 6e^t - 2e^{-t} - 10 \Rightarrow F(s) = \dfrac{-6s^3 + 15s^2 + 10s - 7}{s^2(s-1)(s+1)}$
5. $f(t) = \sin(2t)\sin(3t) \Rightarrow F(s) = \dfrac{1}{2}\left(\dfrac{s}{s^2+1} - \dfrac{s}{s^2+25}\right)$
6. $f(t) = \cos^2(5t) \Rightarrow F(s) = \dfrac{1}{2}\left(\dfrac{1}{s} + \dfrac{s}{s^2+100}\right)$
7. $f(t) = t\cos^2(5t) \Rightarrow F(s) = \dfrac{1}{2}\left(\dfrac{1}{s^2} + \dfrac{s^2-100}{(s^2+100)^2}\right)$
8. $f(t) = te^{at}\left(\cos(bt) + \sin(bt)\right) \Rightarrow F(s) = \dfrac{(s-a)^2 - b^2 + 2b(s-a)}{((s-a)^2 + b^2)^2}$

9. $f(t) = t^3 \cos(2t) \Rightarrow F(s) = \dfrac{6s^4 - 144s^2 + 96}{(s^2+4)^4}$

10. $f(t) = \cos(\alpha t + \beta) \Rightarrow F(s) = \dfrac{s\cos(\beta) - \alpha \sin(\beta)}{s^2 + \alpha^2}$

11. $f(t) = \sin(\alpha t + \beta) \Rightarrow F(s) = \dfrac{s\sin(\beta) + \alpha \cos(\beta)}{s^2 + \alpha^2}$

Problem II. Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for the following functions. Simplify your answer, if able.

1. $F(s) = \dfrac{4s+2}{s^2+6s+34}$

2. $F(s) = \dfrac{1}{(s-2)(s-4)(s-8)}$

3. $F(s) = \dfrac{3s-1}{4s^2+1}$

4. $F(s) = \dfrac{s^2+s-6}{s^3+2s^2+s}$

5. $F(s) = \dfrac{1}{s^4-81}$

6. $F(s) = \dfrac{s^3}{s^4-16}$

7. $F(s) = \dfrac{12}{s^3-8}$

Solution.

1. $F(s) = \dfrac{4s+2}{s^2+6s+34} \Rightarrow f(t) = 4e^{-3t}\cos(5t) - 2e^{-3t}\sin(5t)$

2. $F(s) = \dfrac{1}{(s-2)(s-4)(s-8)} \Rightarrow f(t) = \dfrac{e^{2t}}{12} - \dfrac{e^{4t}}{8} + \dfrac{e^{8t}}{24}$

3. $F(s) = \dfrac{3s-1}{4s^2+1} \Rightarrow f(t) = \dfrac{3}{4}\cos(t/2) - \dfrac{7}{2}\sin(t/2)$

4. $F(s) = \dfrac{s^2+s-6}{s^3+2s^2+s} \Rightarrow f(t) = 7e^{-t} + 6te^{-t} - 6$

5. $F(s) = \dfrac{1}{s^4-81} \Rightarrow f(t) = \dfrac{e^{3t}}{108} - \dfrac{e^{-3t}}{108} - \dfrac{\sin(3t)}{54}$

6. $F(s) = \dfrac{s^3}{s^4-16} \Rightarrow f(t) = \dfrac{e^{2t}}{4} + \dfrac{e^{-2t}}{4} - \dfrac{\cos(2t)}{2}$

7. $F(s) = \dfrac{12}{s^3 - 8} \Rightarrow f(t) = e^{2t} - e^{-t}\cos(\sqrt{3}t) - \sqrt{3}e^{-t}\sin(\sqrt{3}t)$

Problem III. Find the following Laplace transforms:

1. $\mathcal{L}\{u_\pi(t)t^2\}$
2. $\mathcal{L}\{u_4(t)t^2e^{5t}\}$
3. $\mathcal{L}\{u_{\pi/2}(t)e^{-t}\cos(2t)\}$
4. $\mathcal{L}\{u_3(t)(t^2 - t + 2)e^{-5t}\}$
5. $\mathcal{L}\{\delta_{\pi/3}(t)\cos^2(t)\sin(2t)\}$
6. $\mathcal{L}\{\delta_2(t)t^3e^{-t}\cos^3(\pi t)\}$
7. $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} t^2 + t, & 0 \le t < 2 \\ 1 - e^{-4t}, & 2 \le t < 5 \\ 0, & 5 \le t \end{cases}$

Solution.

1. $\mathcal{L}\{u_\pi(t)t^2\} = e^{-\pi s}\left(\dfrac{2}{s^3} + \dfrac{2\pi}{s^2} + \dfrac{\pi^2}{s}\right)$
2. $\mathcal{L}\{u_4(t)t^2e^{5t}\} = e^{-4s}e^{20}\left(\dfrac{2}{(s-5)^3} + \dfrac{8}{(s-5)^2} + \dfrac{16}{s-5}\right)$
3. $\mathcal{L}\{u_{\pi/2}(t)e^{-t}\cos(2t)\} = -e^{-(s+1)\pi/2}\dfrac{s+1}{(s+1)^2 + 4}$
4. $\mathcal{L}\{u_3(t)(t^2 - t + 2)e^{-5t}\} = e^{-3s}e^{-15}\left(\dfrac{2}{(s+5)^3} + \dfrac{5}{(s+5)^2} + \dfrac{8}{s+5}\right)$
5. $\mathcal{L}\{\delta_{\pi/3}(t)\cos^2(t)\sin(2t)\} = \dfrac{\sqrt{3}}{8}e^{-s\pi/3}$
6. $\mathcal{L}\{\delta_2(t)t^3e^{-t}\cos^3(\pi t)\} = 8e^{-2s}e^{-2}$
7. $f(t) = \begin{cases} t^2 + t, & 0 \le t < 2 \\ 1 - e^{-4t}, & 2 \le t < 5 \\ 0, & 5 \le t \end{cases}$

   $$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{1}{s^2} - e^{-2s}\left(\frac{e^{-8}}{s+4} + \frac{2}{s^3} + \frac{5}{s^2} + \frac{5}{s}\right) + e^{-5s}\left(\frac{e^{-20}}{s+4} - \frac{1}{s}\right)$$

Problem IV. Find the following inverse Laplace transforms $f(t) = \mathcal{L}\{F(s)\}$:

1. $F(s) = e^{-4s}\dfrac{3s+22}{s^2+3s-10}$ and $F(s) = e^{-4s}\dfrac{3s}{s^2+3s-10}$
2. $F(s) = e^{-6s}\dfrac{4s+11}{s^2+6s+9}$
3. $F(s) = e^{-2s}\dfrac{2s-14}{s^2+2s+17}$
4. $F(s) = e^{-8s}\dfrac{3s^2-10s+8}{s^3+4s}$
5. $F(s) = \dfrac{2s^2+s-7}{s^2+49}$
6. $F(s) = \dfrac{3s^3+3s^2+6s}{(s+1)^3}$

Solution.

1. $F(s) = e^{-4s}\dfrac{3s+22}{s^2+3s-10} \Rightarrow f(t) = u_4(t)\left(4e^{2(t-4)} - e^{-5(t-4)}\right)$

   $F(s) = e^{-4s}\dfrac{3s}{s^2+3s-10} \Rightarrow f(t) = u_4(t)\left(\dfrac{6}{7}e^{2(t-4)} + \dfrac{15}{7}e^{-5(t-4)}\right)$
2. $F(s) = e^{-6s}\dfrac{4s+11}{s^2+6s+9} \Rightarrow f(t) = u_6(t)\left(4e^{-3(t-6)} - (t-6)e^{-3(t-6)}\right)$
3. $F(s) = e^{-2s}\dfrac{2s-14}{s^2+2s+17} \Rightarrow f(t) = u_2(t)\left(2e^{-(t-2)}\cos(4(t-2)) - 4e^{-(t-2)}\sin(4(t-2))\right)$
4. $F(s) = e^{-8s}\dfrac{3s^2-10s+8}{s^3+4s} \Rightarrow f(t) = u_8(t)\left(2+\cos(2(t-8))\right) - 5\sin(2(t-8))\right)$
5. $F(s) = \dfrac{2s^2+s-7}{s^2+49} \Rightarrow f(t) = 2\delta(t) + \cos(7t) - 15\sin(7t)$
6. $F(s) = \dfrac{3s^3+3s^2+6s}{(s+1)^3} \Rightarrow f(t) = \delta(t) + 3te^{-2} - 2t^2e^{-t}$

Problem V. Use Laplace transform to solve the following IVP:

1. $y' + 10y = t^2, \quad y(0) = 0$
2. $y' + 2y = te^{-t}, \quad y(0) = 2$
3. $y' - 6y = 2\sin(3t), \quad y(0) = -1$
4. $y'' + 4y = 8\cos(2t) - 8e^{-2t}, \quad y(0) = -2, y'(0) = 0$
5. $y''' + 4y'' - 5y' = 0, \quad y(0) = 4, y'(0) = -7, y''(0) = 23$
6. $y''' - y'' + 4y' - 4y = 26e^{3t}, \quad y(0) = -2, y'(0) = 3, y''(0) = 1$

7. $y'' + 6y' + 9y = u_5(t)e^{-t}, \quad y(0) = 10, y'(0) = 0$

8. $y' + y = u_4(t)t - \delta(t-10), \quad y(0) = -2$

9. $y'' - 6y' = \delta(t-1) - u_4(t), \quad y(0) = 2, y'(0) = 5$

10. $y'' - 6y' + 9y = 4 - \delta(t-7), \quad y(0) = 2, y'(0) = 0$

Solution.

1. $y = \dfrac{t^2}{10} - \dfrac{t}{50} + \dfrac{1}{500} - \dfrac{e^{-10t}}{500}$

2. $y = te^{-t} - e^{-t} + 3e^{-2t}$

3. $y = 6 - te^{-t}$

4. $y = -\cos(2t) - \sin(2t) + 2t\sin(2t) - e^{-2t}$

5. $y = 5 - 2e^{t} + e^{-5t}$

6. $y = -4e^{t} + e^{3t} + \cos(2t) + 2\sin(2t)$

7. $y = 10e^{-3t} + 30te^{-3t} + \dfrac{e^{-5}u_5(t)}{4}\left(e^{-t+5} + (9-2t)e^{-3t+15}\right)$

8. $y = -2e^{-2} + u_4(t)(t - 1 - 3e^{-t+3}) - u_{10}(t)e^{-t+10}$

9. $y = \dfrac{7}{6} + \dfrac{5e^{6t}}{6} + \dfrac{u_1(t)}{6}\left(-1 + e^{6t-6}\right) + \dfrac{u_4(t)}{36}\left(6t - 23 - e^{6t-24}\right)$

10. $y = \dfrac{4}{9} + \dfrac{14e^{3t}}{9} - \dfrac{14te^{3t}}{3} - u_7(t)(t-7)e^{3t-21}$
