Announcements

- Homework 1 online due today
- Homework 2 online due next Friday
- Remember FinAid survey Due Today on canvas!!!
- Minor office hour schedule changes *this week*
  – Akhila: Thursday 4-6pm -> Friday 7-9pm

Last Time

- Strongly connected components and metagraphs

Strongly Connected Components

Definition: In a directed graph G, two vertices v and w are in the same Strongly Connected Component (SCC) if v is reachable from w *and* w is reachable from v.

Metagraph

Definition: The metagraph of a directed graph G is a graph whose vertices are the SCCs of G, where there is an edge between C_1 and C_2 if and only if G has an edge between some vertex of C_1 and some vertex of C_2.

Result

Theorem: The metagraph is any directed graph is a DAG.

Computing SCCs

Problem: Given a directed graph G compute the SCCs of G and its metagraph.

Observation

Suppose that SCC(v) is a sink in the metagraph.

- G has no edges from SCC(v) to another SCC.
- Run `explore(v)` to find all vertices reachable from v.
  - Contains all vertices in SCC(v).
  - Contains no other vertices.
- If v in sink SCC, `explore(v)` finds *exactly* v's component.

Today

- Computing SCCs
- Shortest Paths in Graphs (Ch 4)
  - BFS

Strategy

- Find v in a sink SCC of G.
- Run `explore(v)` to find component C_1.
- Repeat process on $G-C_1$.

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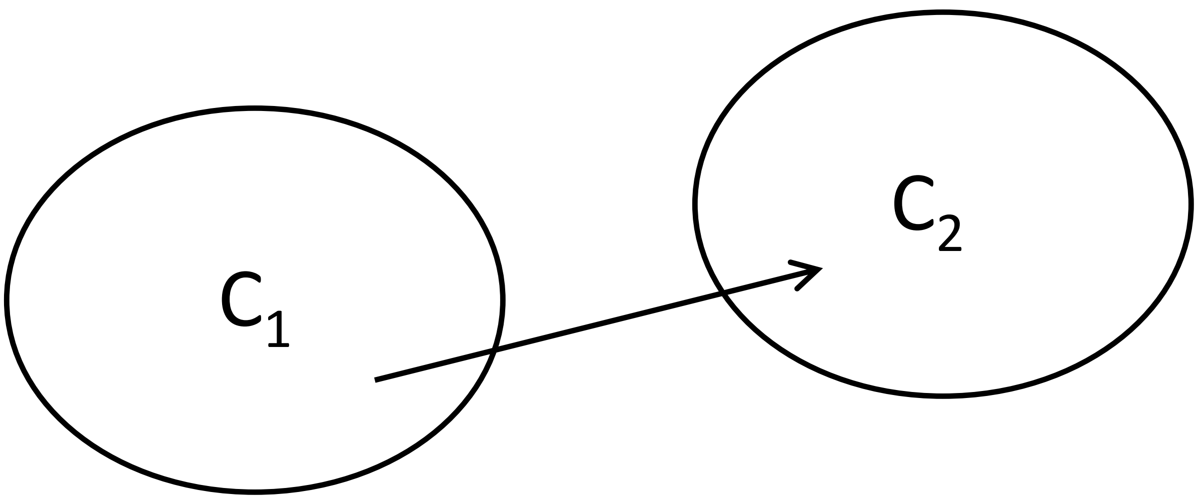
Problem: How do we find v?

Result

<u>Proposition:</u> Let C_1 and C_2 be SCCs of G with an edge from C_1 to C_2. If we run `DFS` on G, the largest postorder number of any vertex in C_1 will be larger than the largest postorder number in C_2.



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  - There is no edge to SCC(v) from any other SCC.
  - SCC is a <u>source</u> SCC.
- But we want a *sink* SCC.
- A sink is like a source, only with the edges going in the opposite direction.

Reverse Graph

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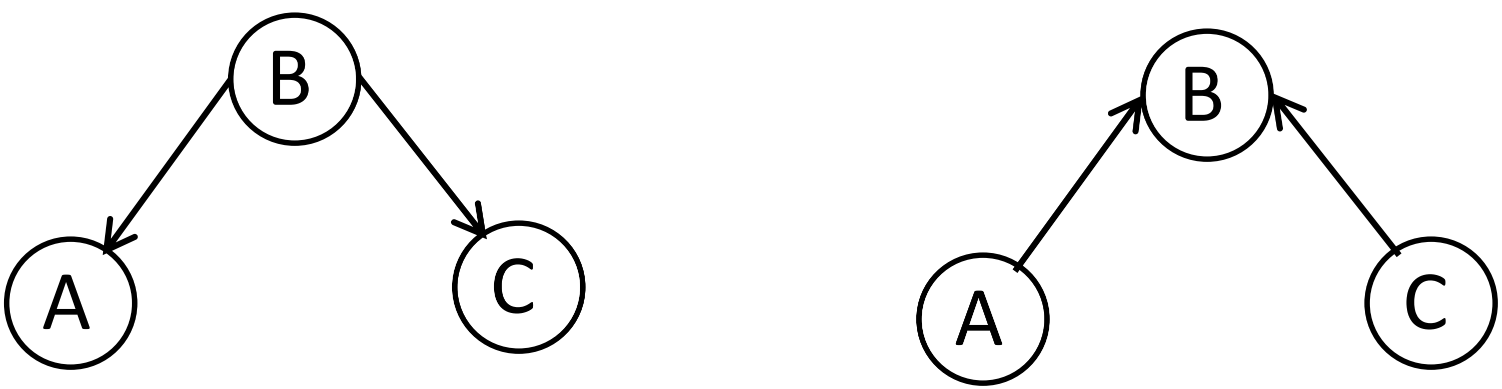

Question: Reverse Graph Properties

Which of the following are NOT true about reverse graphs?

A) G and G^R have the same number of vertices.

B) G and G^R have the same number of edges.

C) $G = (G^R)^R$.

D) A vertex has as many ingoing/outgoing edges in G as it does in G^R.

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Other Properties of Reverse Graphs

Given a directed graph G and its reverse graph G^R:

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- The source SCCs of G are the sink SCCs of G^R.

Other Properties of Reverse Graphs

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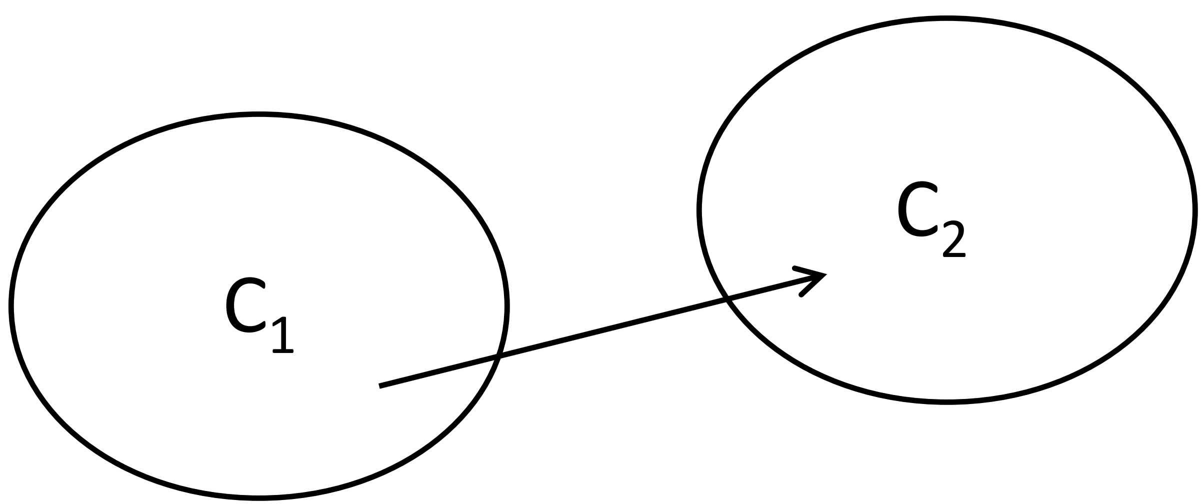
- G and G^R have the *same* SCCs.
- The sink SCCs of G are the source SCCs of G^R.
- The source SCCs of G are the sink SCCs of G^R.

So we can find a sink SCC of G, by finding a source SCC of G^R!

Proposition Reminder

Proposition: Let C_1 and C_2 be SCCs of G with an edge from C_1 to C_2. If we run `DFS` on G, the largest postorder number of any vertex in C_1 will be larger than the largest postorder number in C_2.



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- v has largest postorder in C_1 or C_2.

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If `DFS` discovers a vertex in C_2 before C_1:

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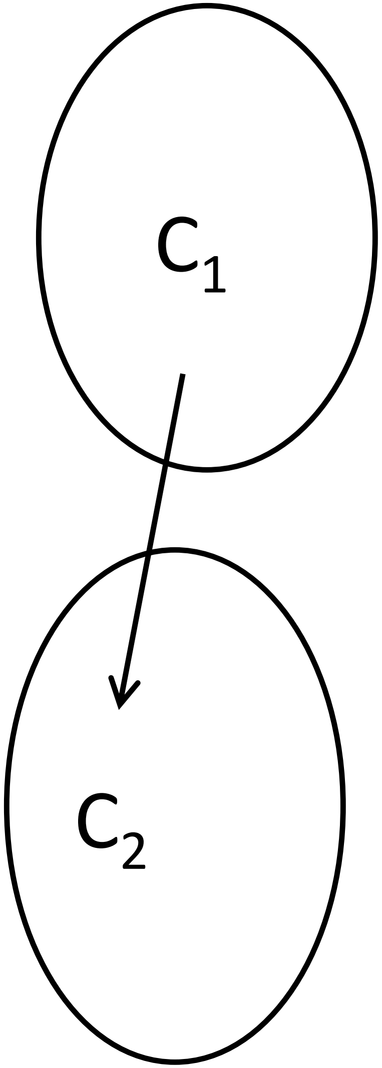

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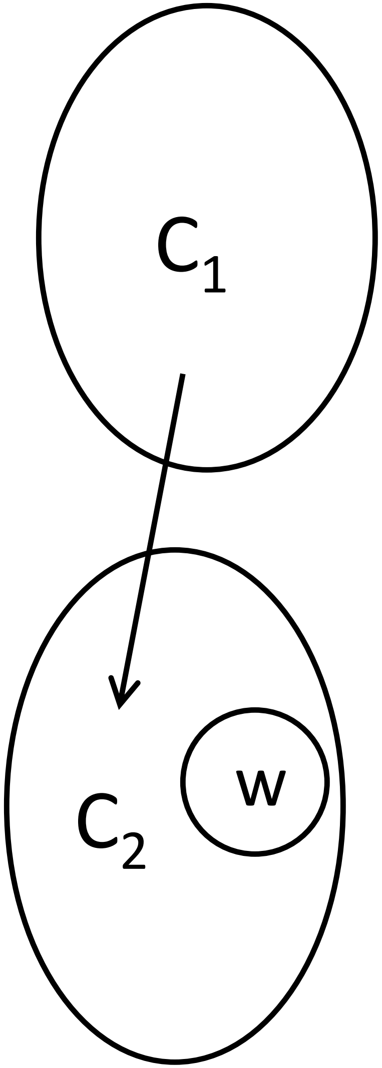

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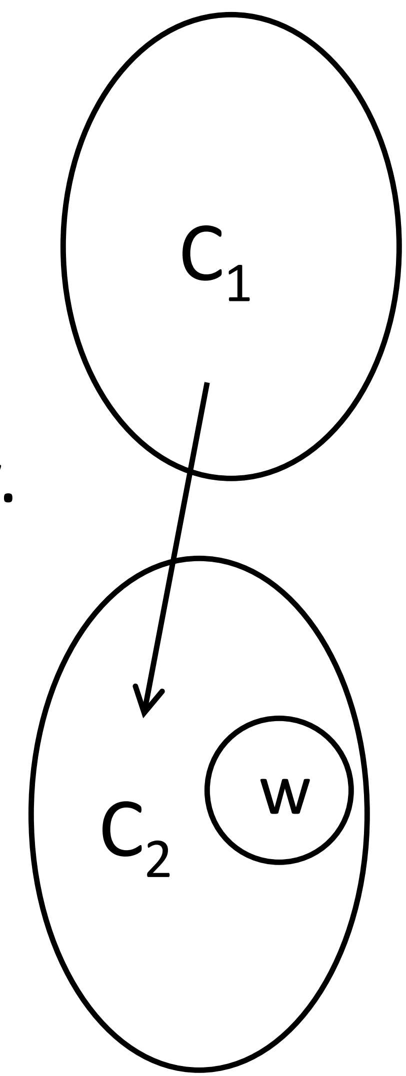

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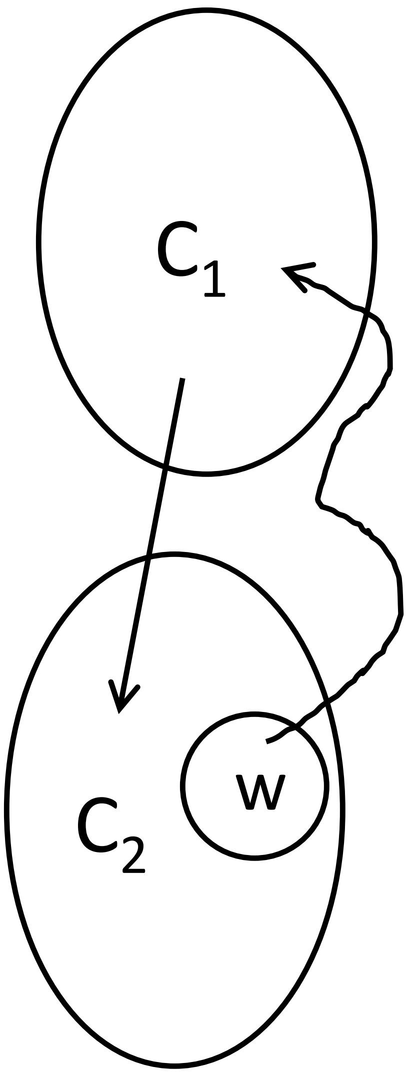
- C_1 cannot be reached from w.



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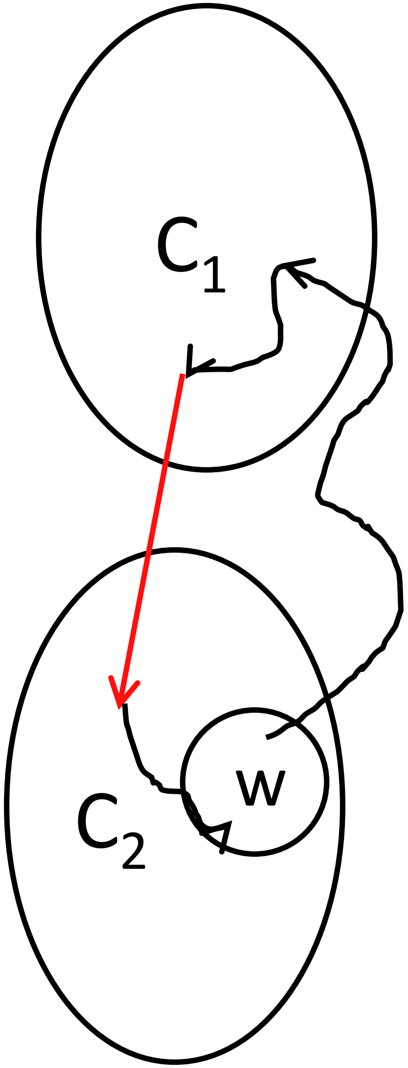

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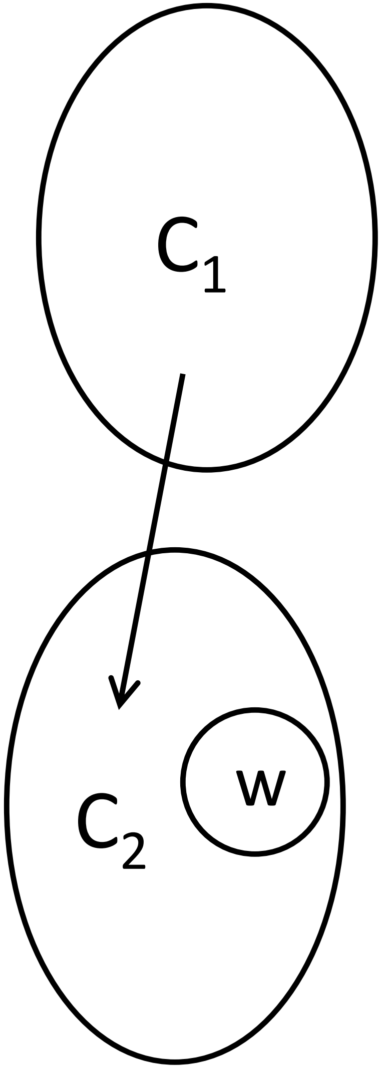

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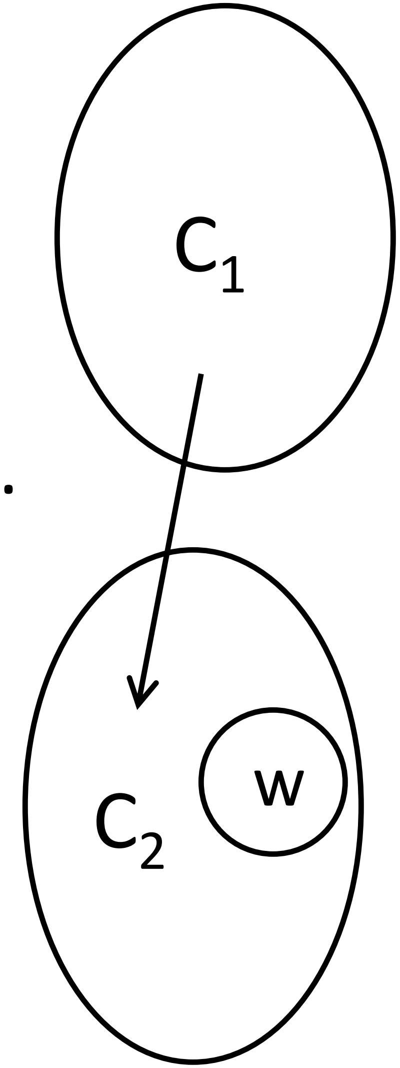
- Every vertex in C_1 discovered after w finished.
- C_1 has larger posts than C_2.



Algorithm

```
SCCs(G)
  Run DFS(G^R) record postorder
  Find v with largest v.post
  Set all vertices unvisited
  Run explore(v)
  Let C be the visited vertices
  Return SCCs(G-C) U {C}
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Final Runtime: $O((|V|+|E|)(\#SCCs))$

Still Too Slow

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Solution: We don't have to do this. After removing some SCCs to get G', the largest postorder number of vertices in G' is *still* in a sink component of G'.

Algorithm II

```
SCCs(G)
  Run DFS(G^R) record postorders
  Mark all vertices unvisited
  For v ∈ V in reverse postorder

    If v not in a component yet
      explore(v) on
      G-components found,
      marking new component
```

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SCCs(G)
  Run DFS(GR) record postorders
  Mark all vertices unvisited
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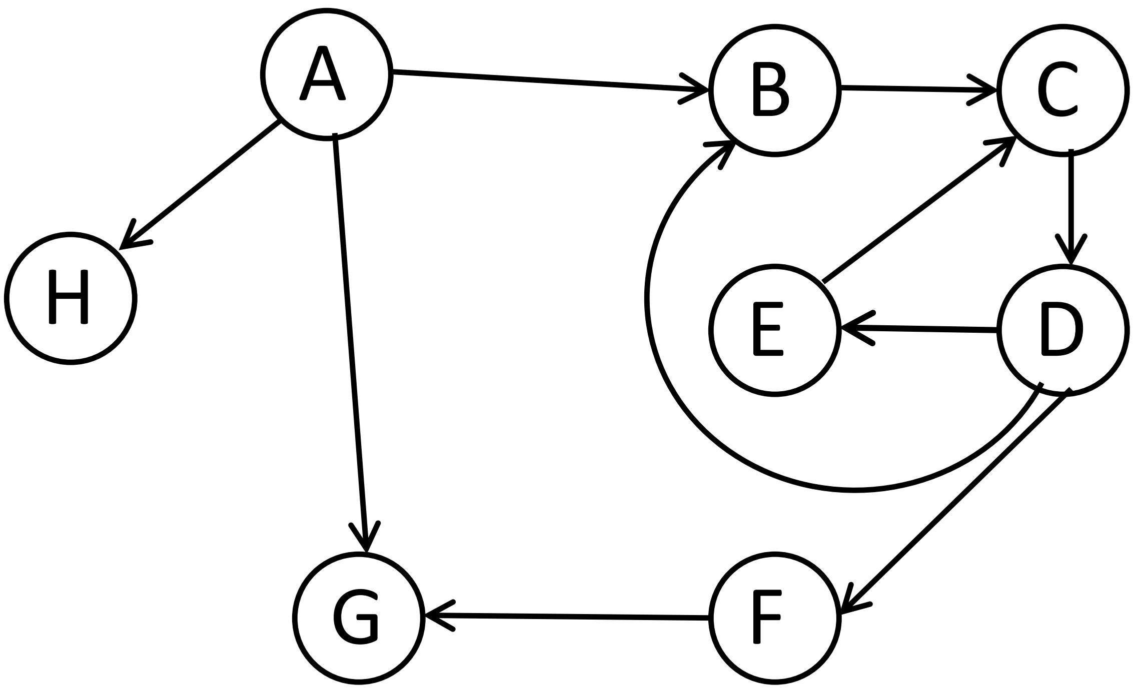
Just 2 DFSs! Runtime $O(|V|+|E|)$.

Example



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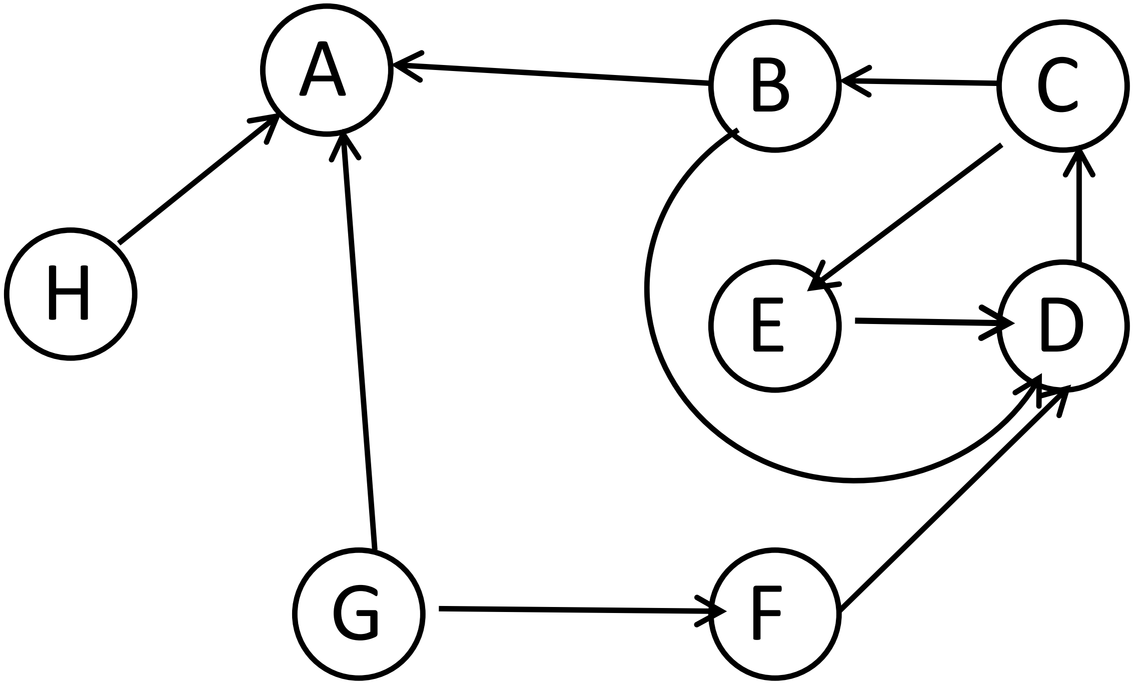

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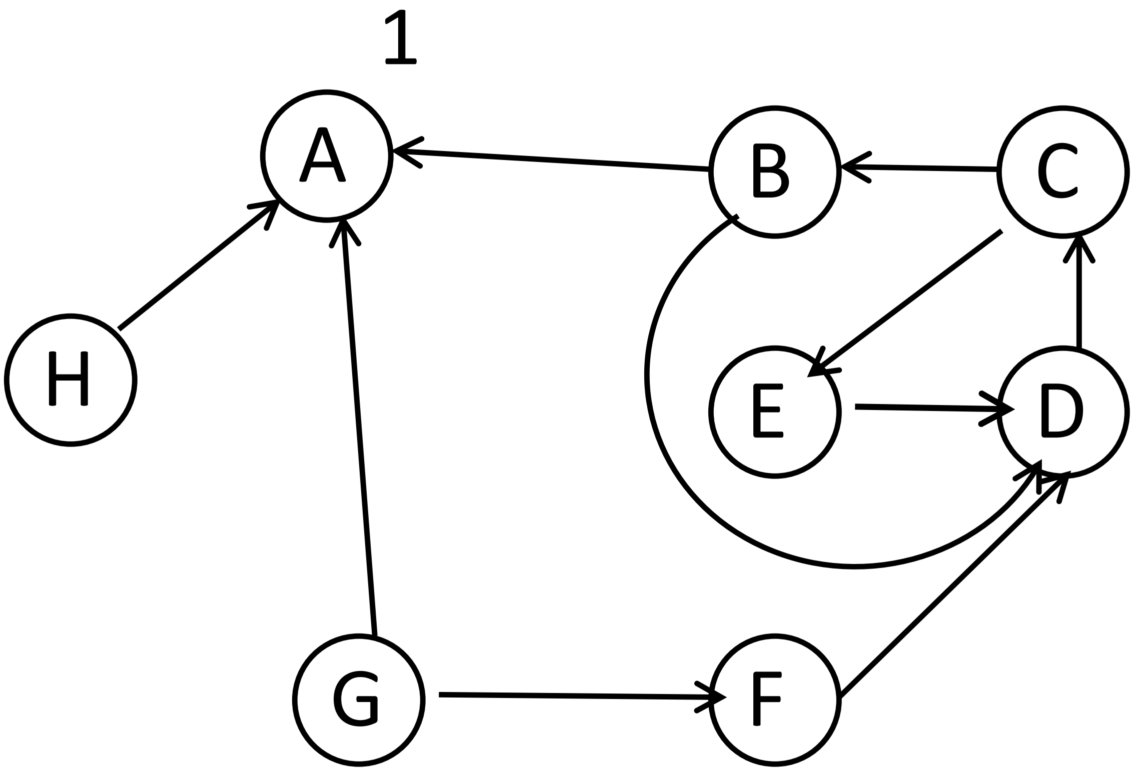

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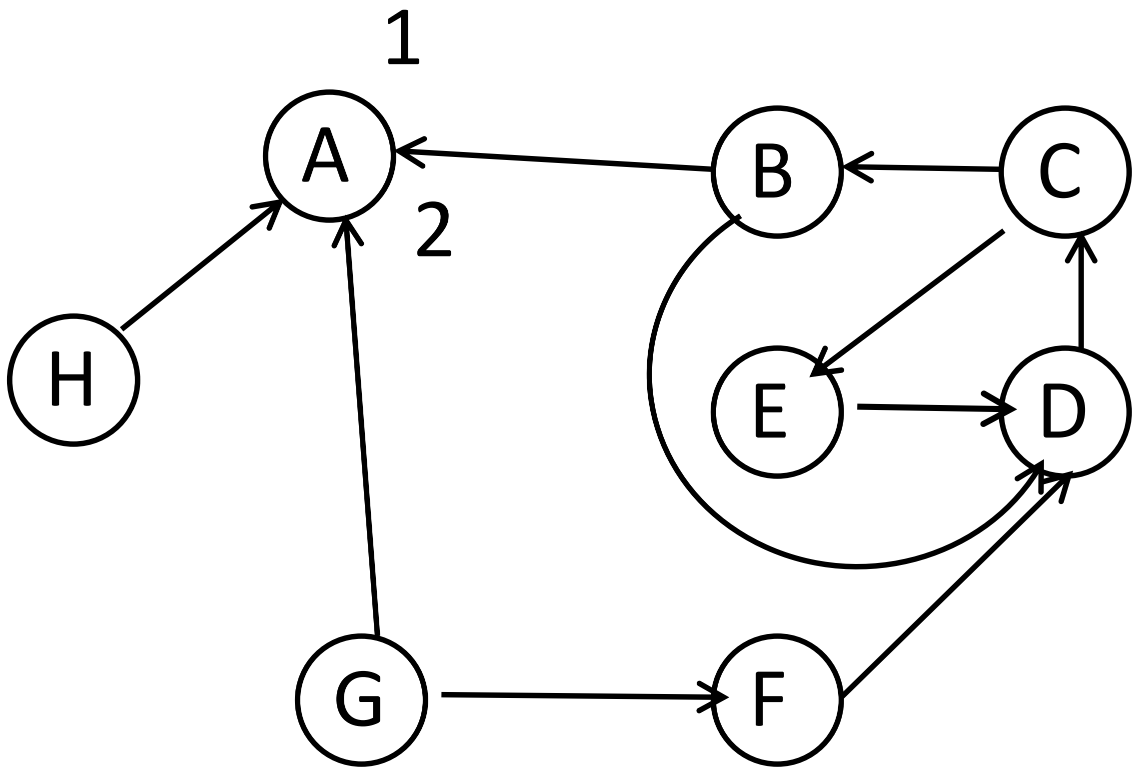

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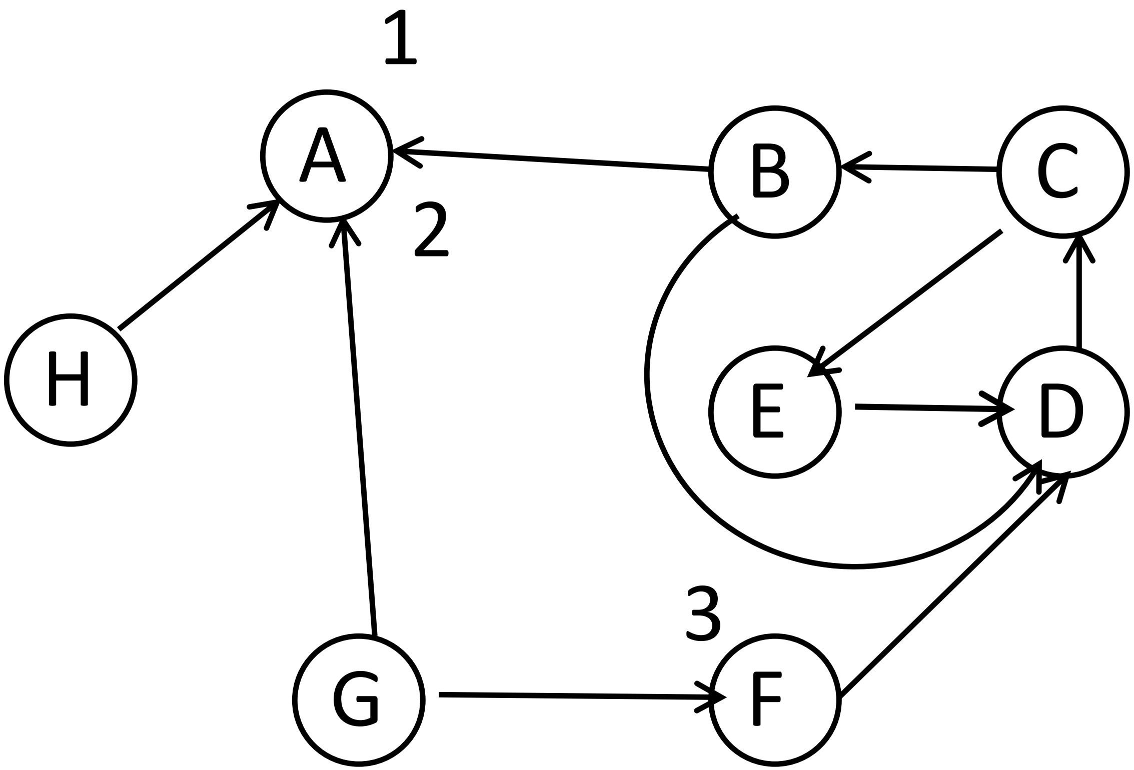

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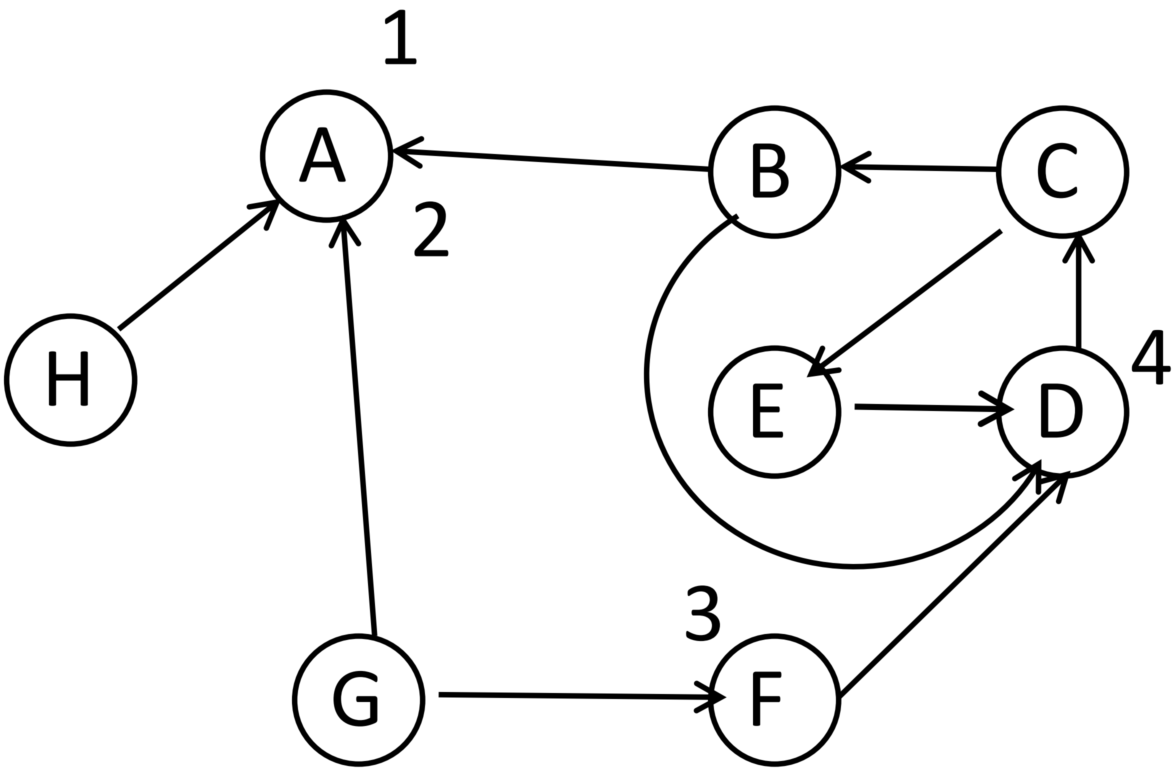

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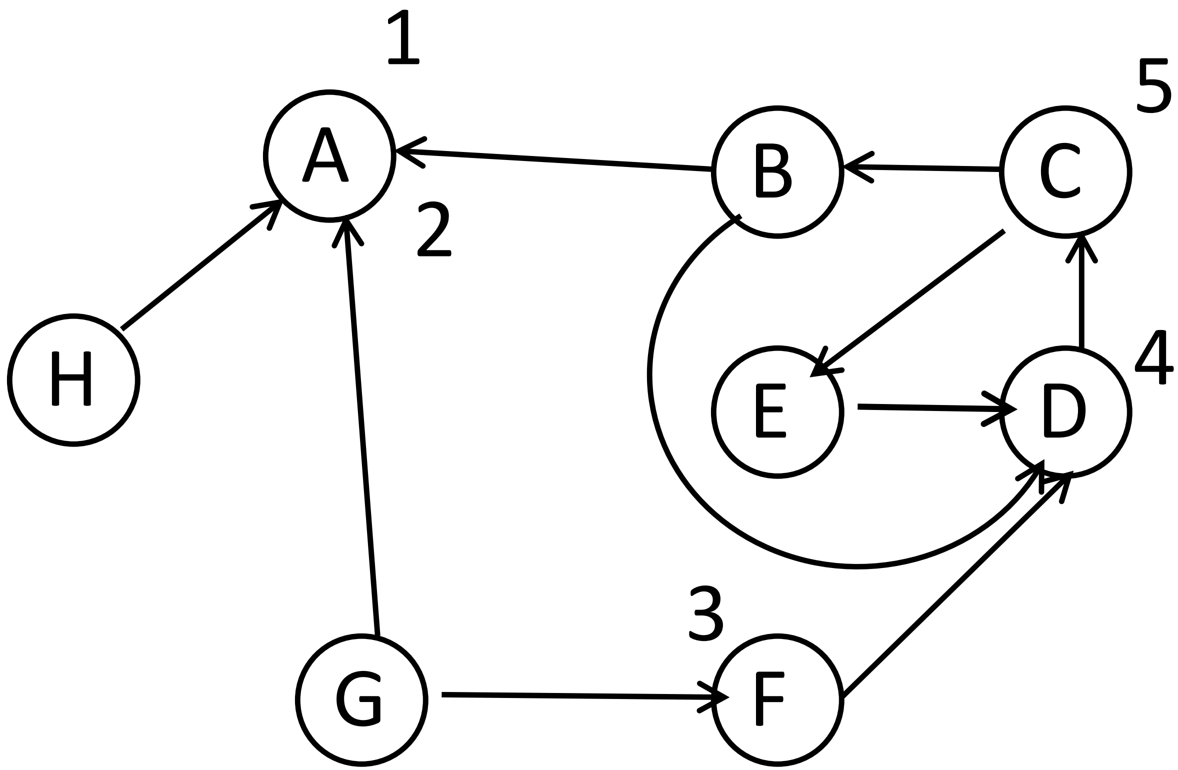

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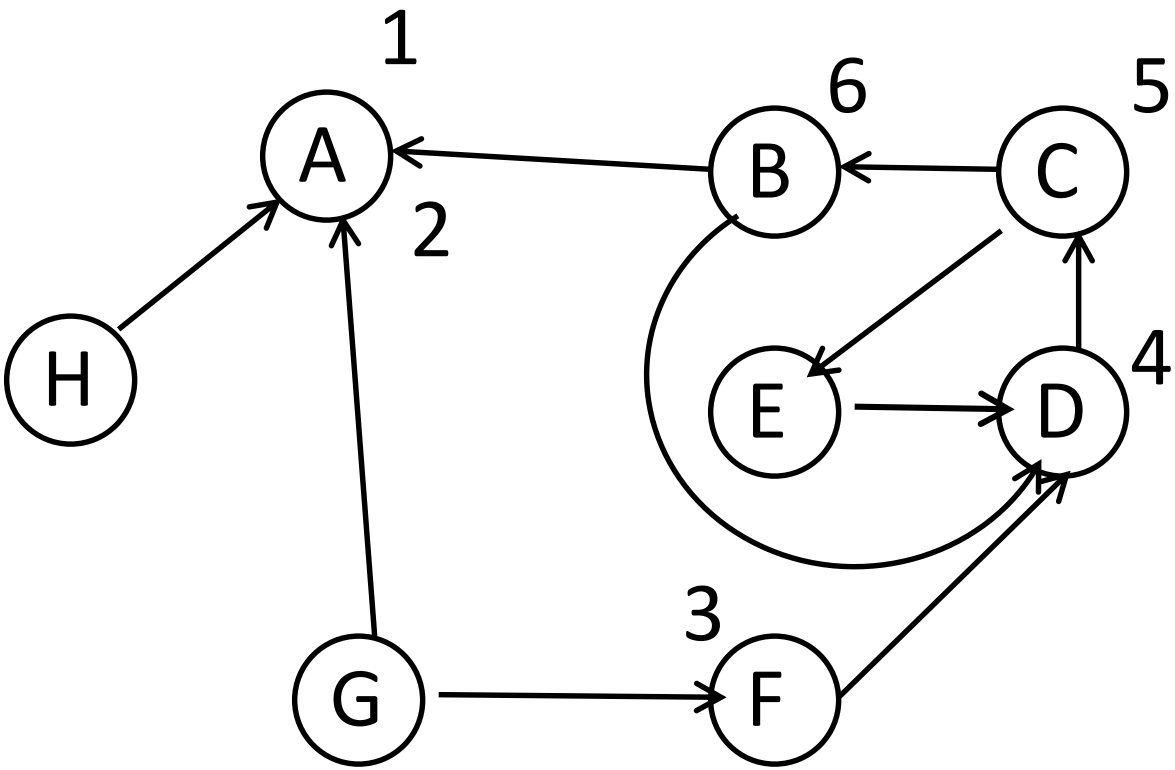

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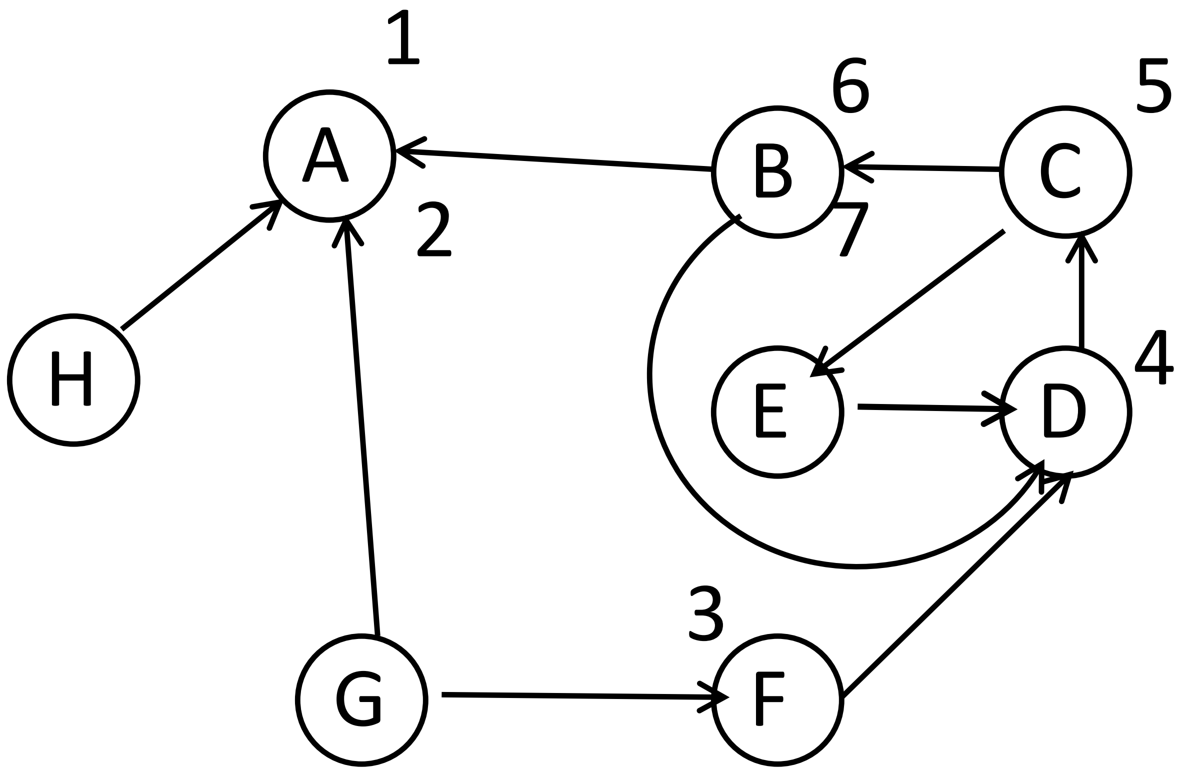

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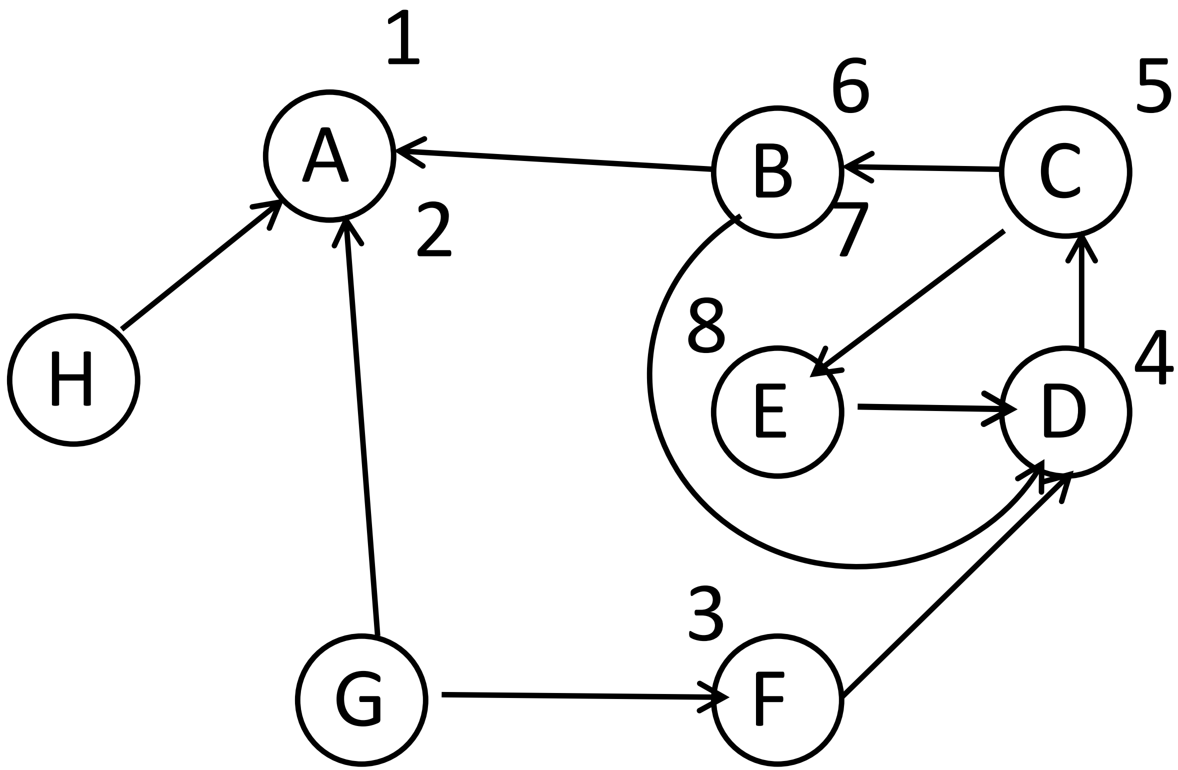

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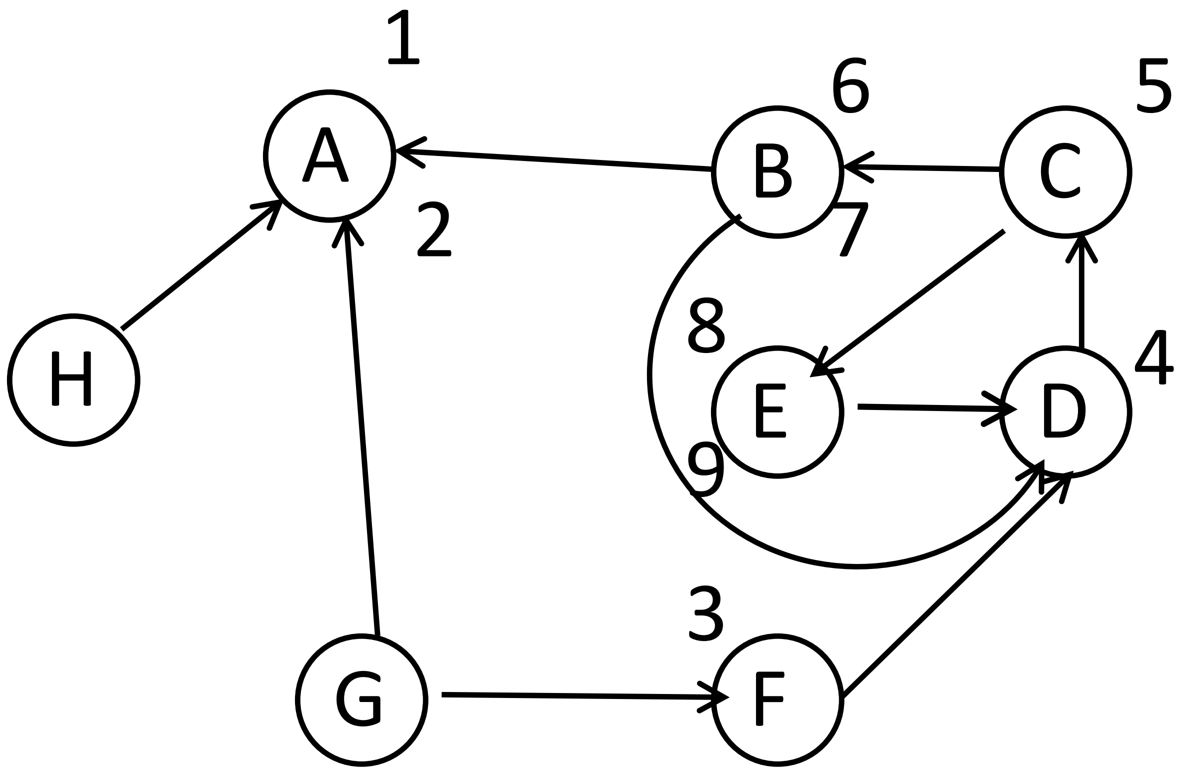

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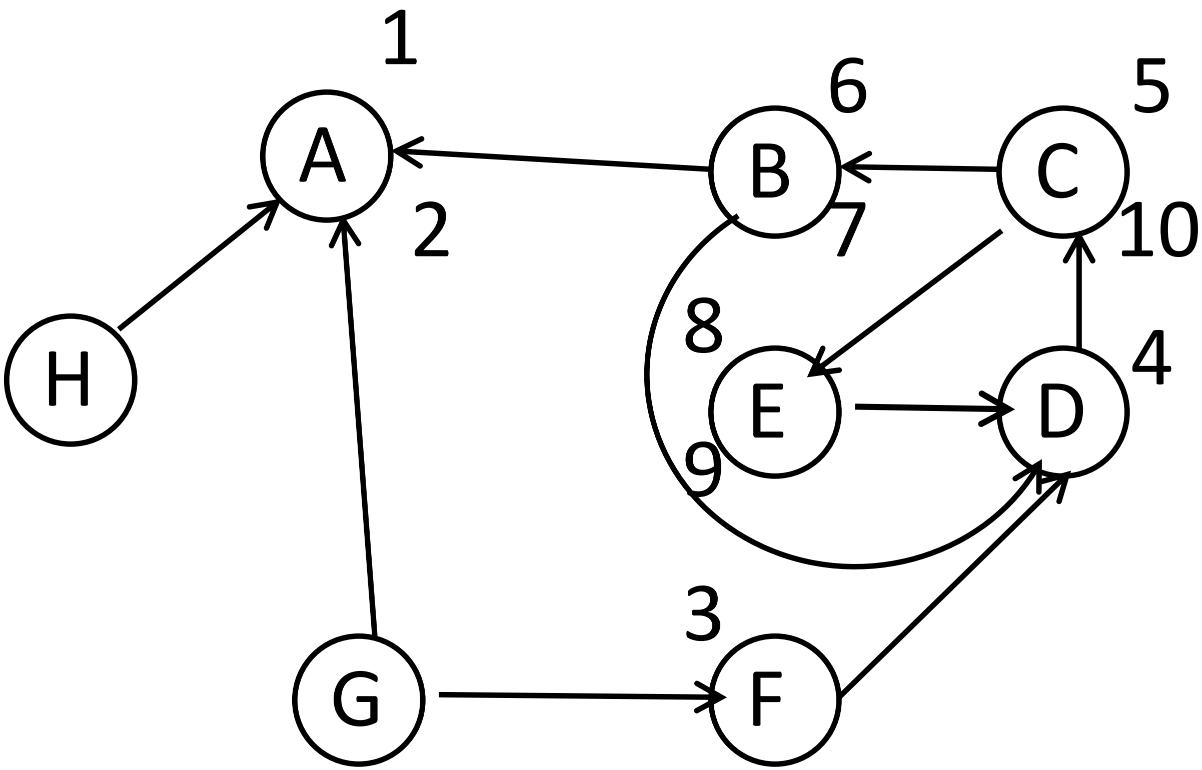

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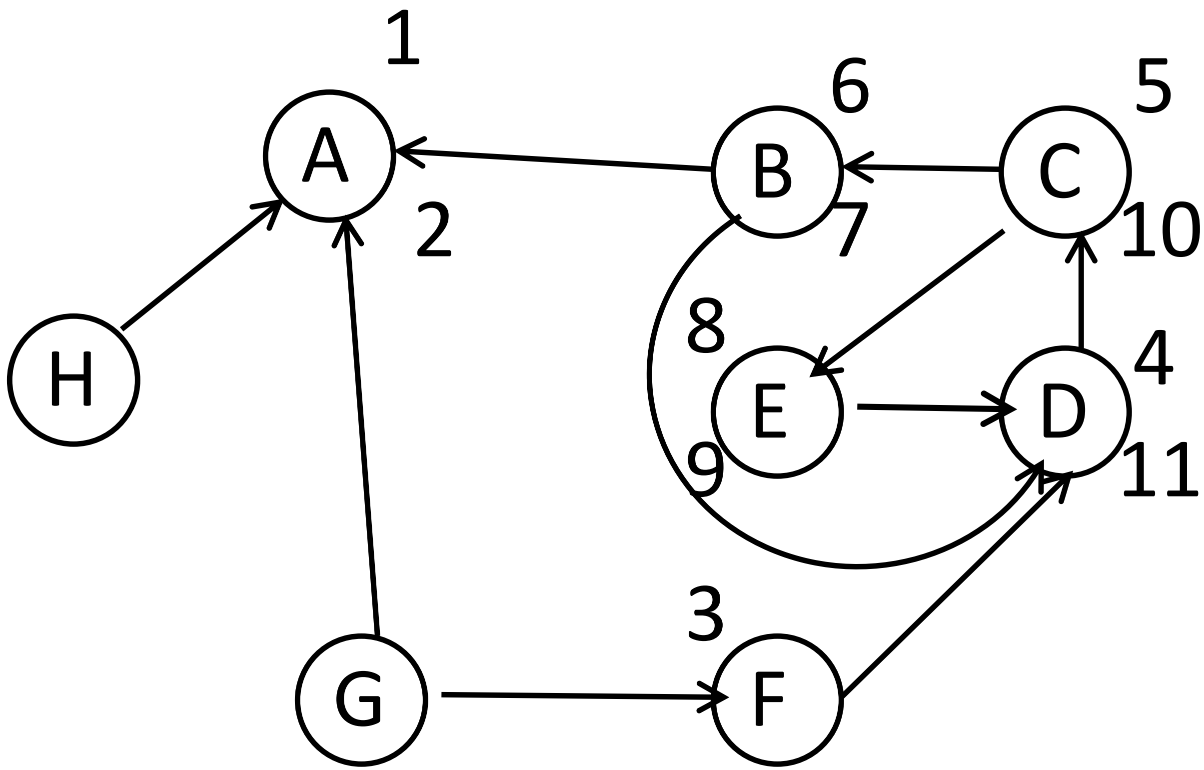

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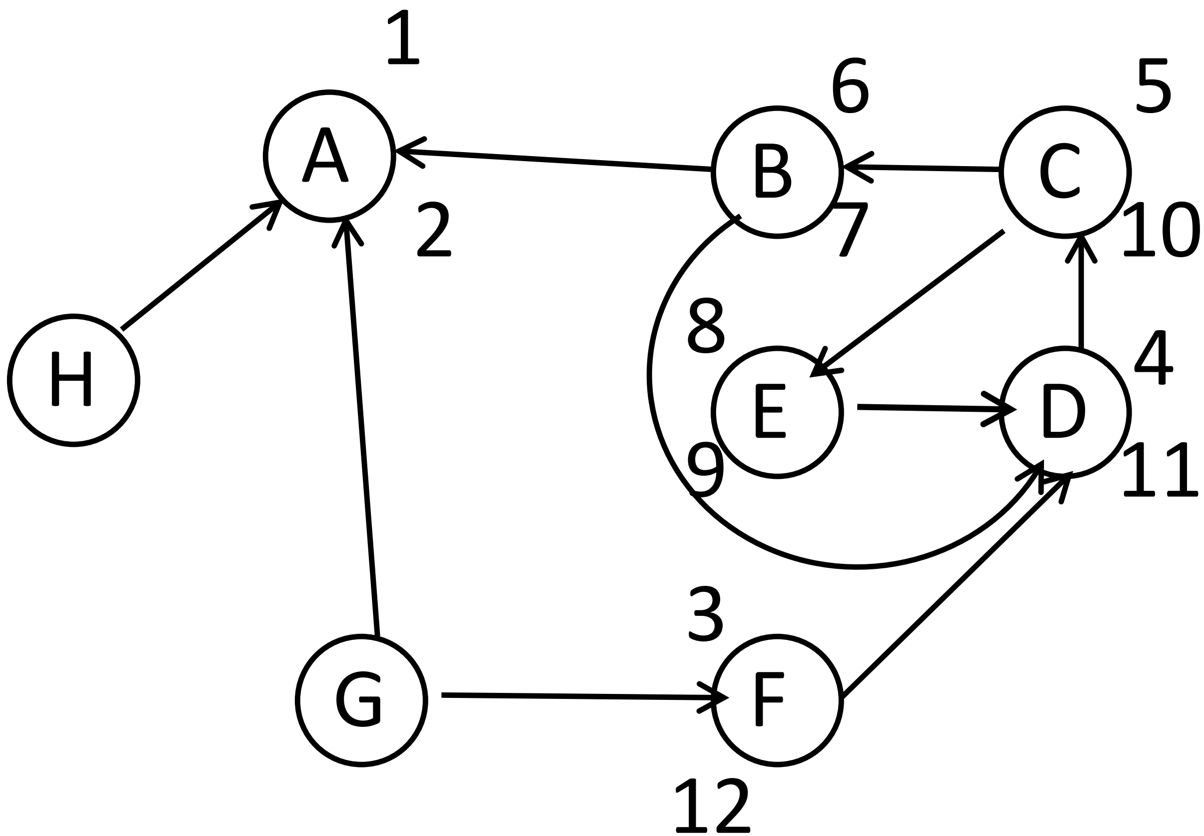

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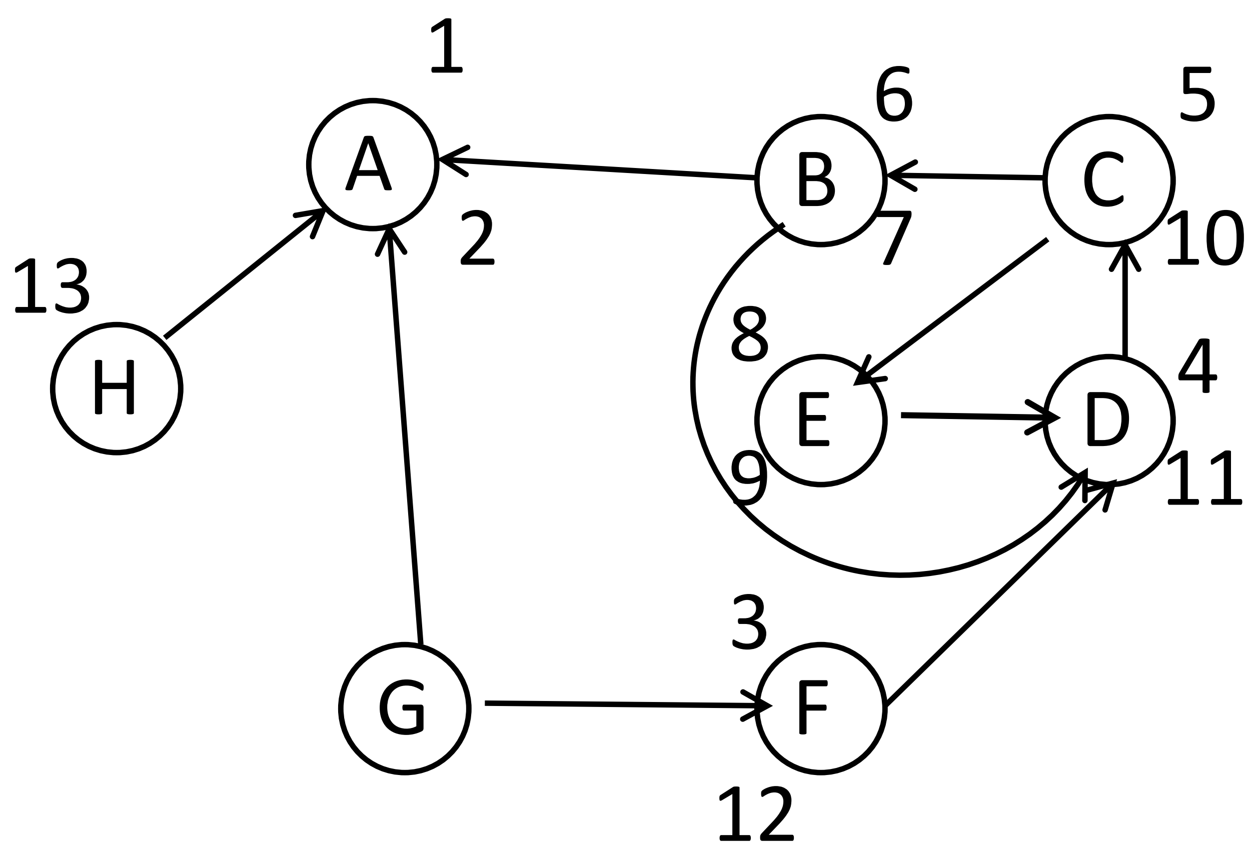

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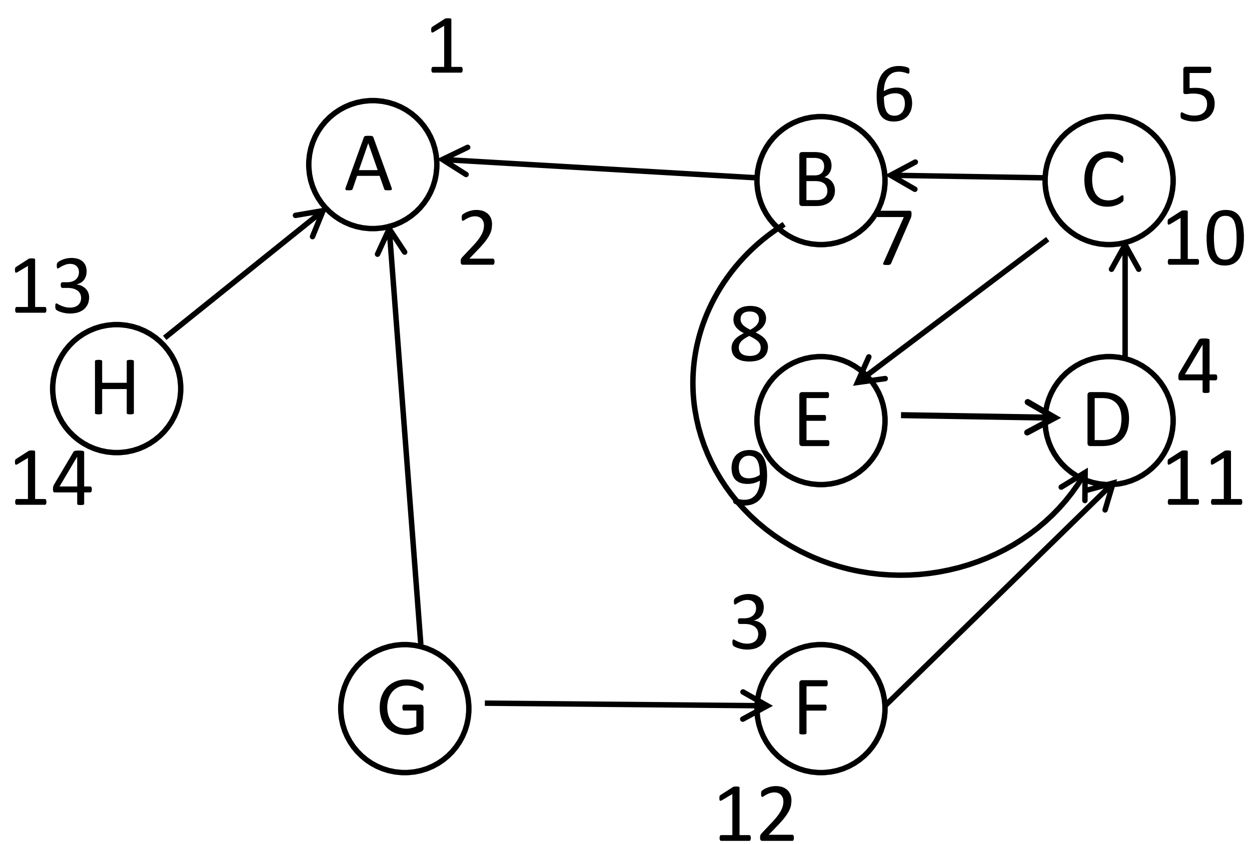

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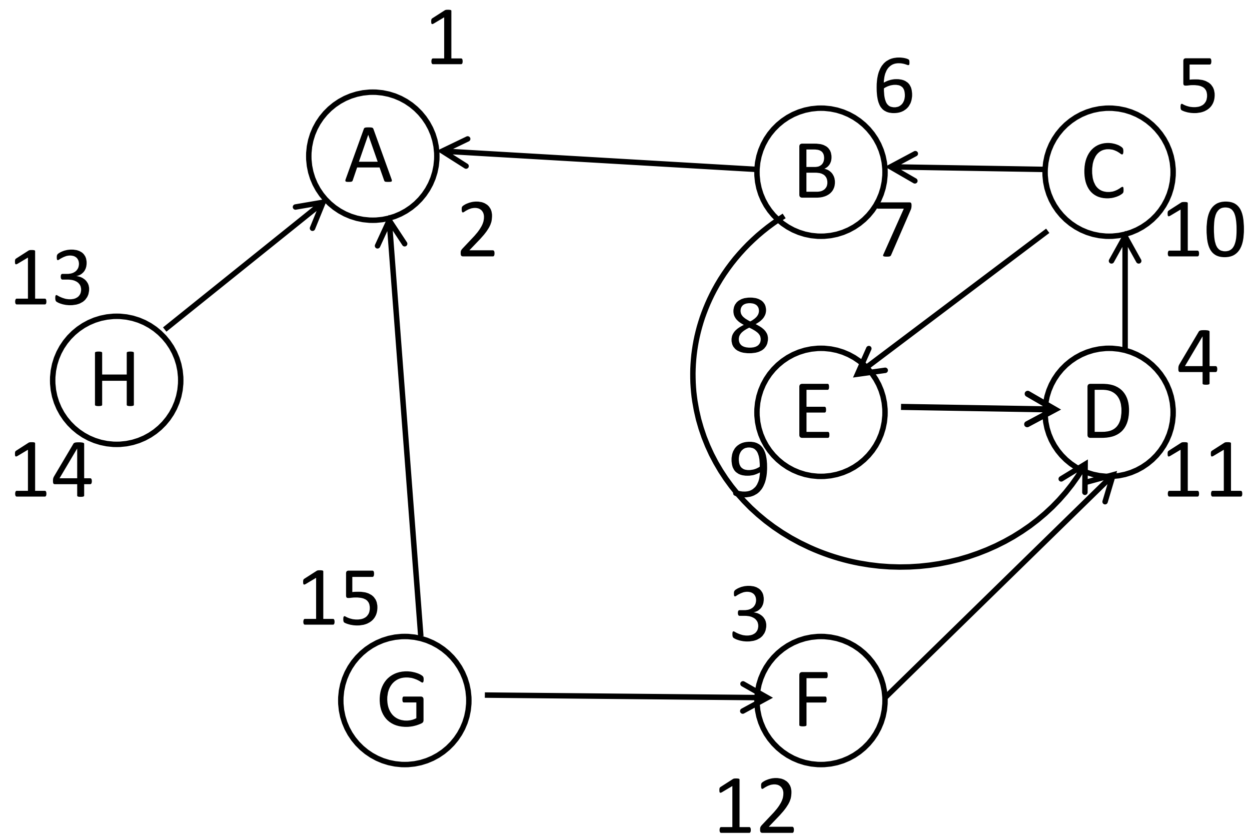

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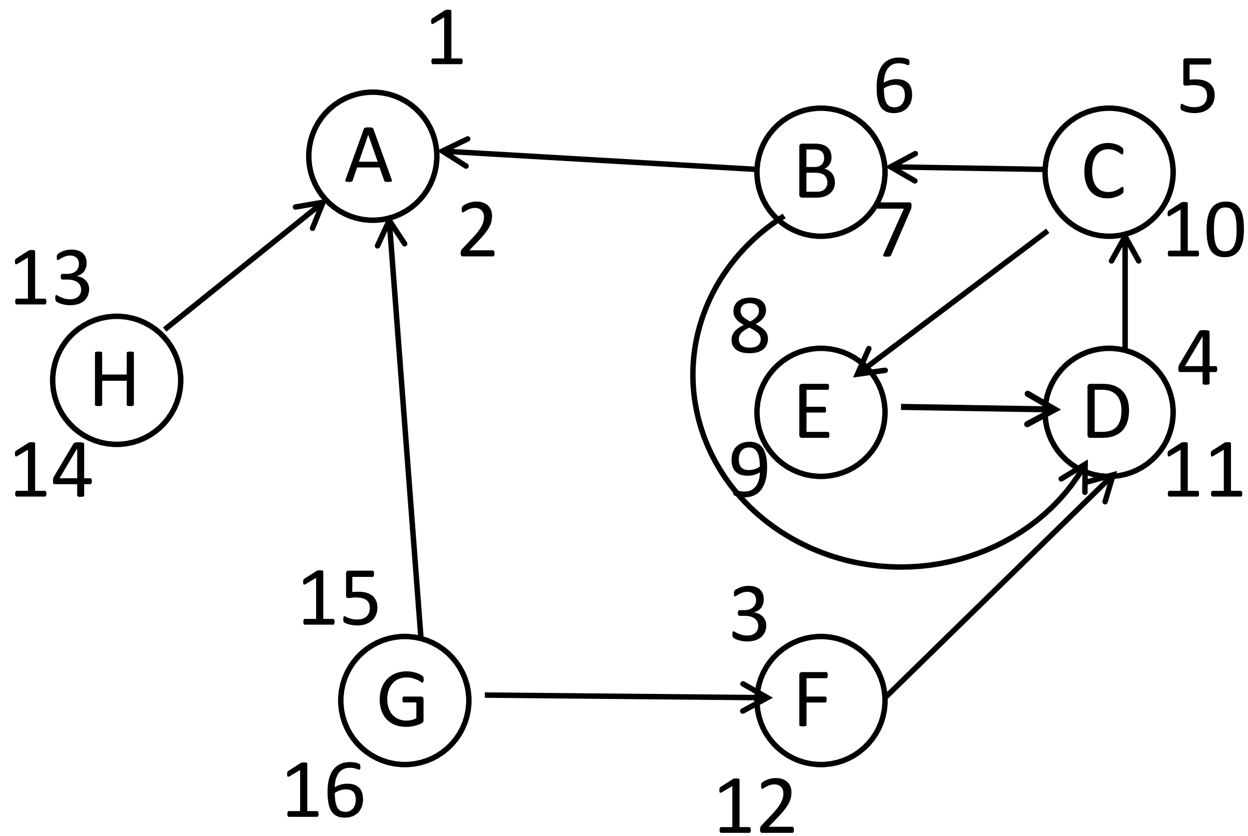

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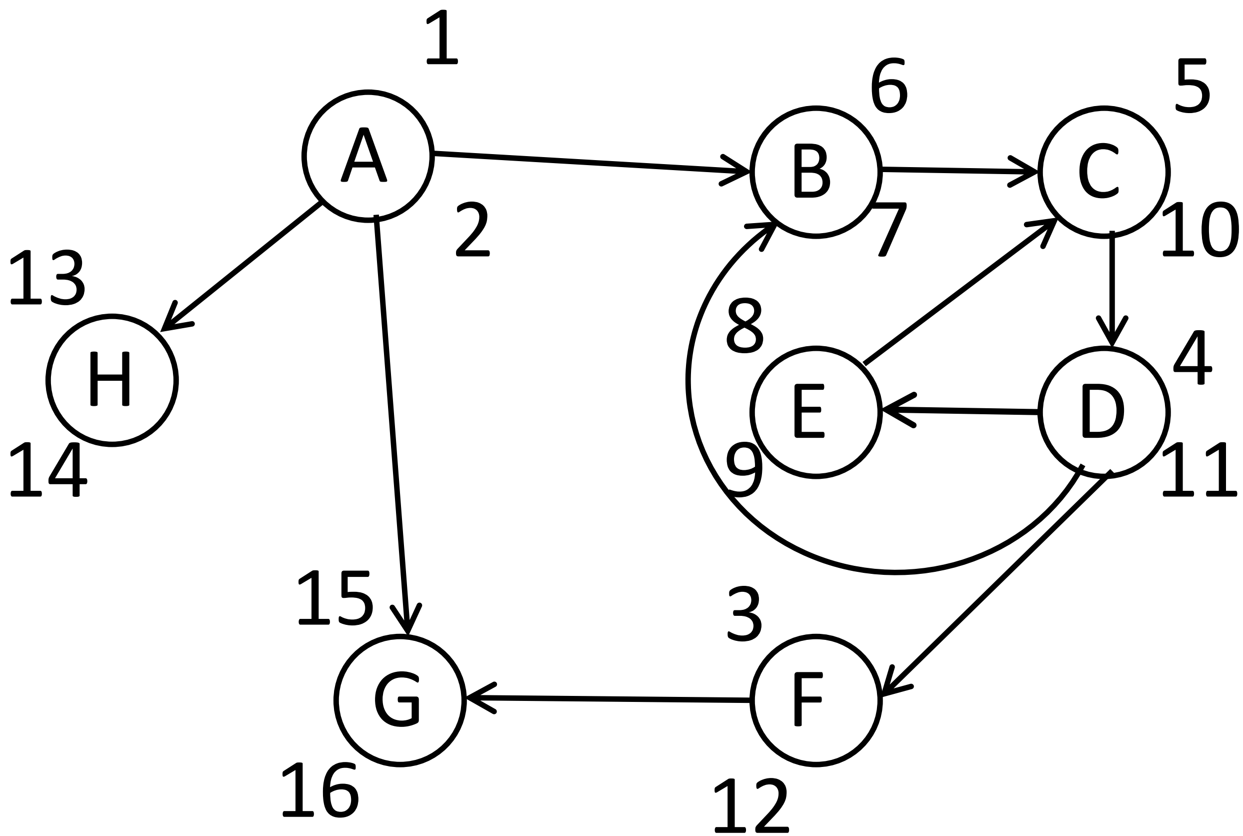

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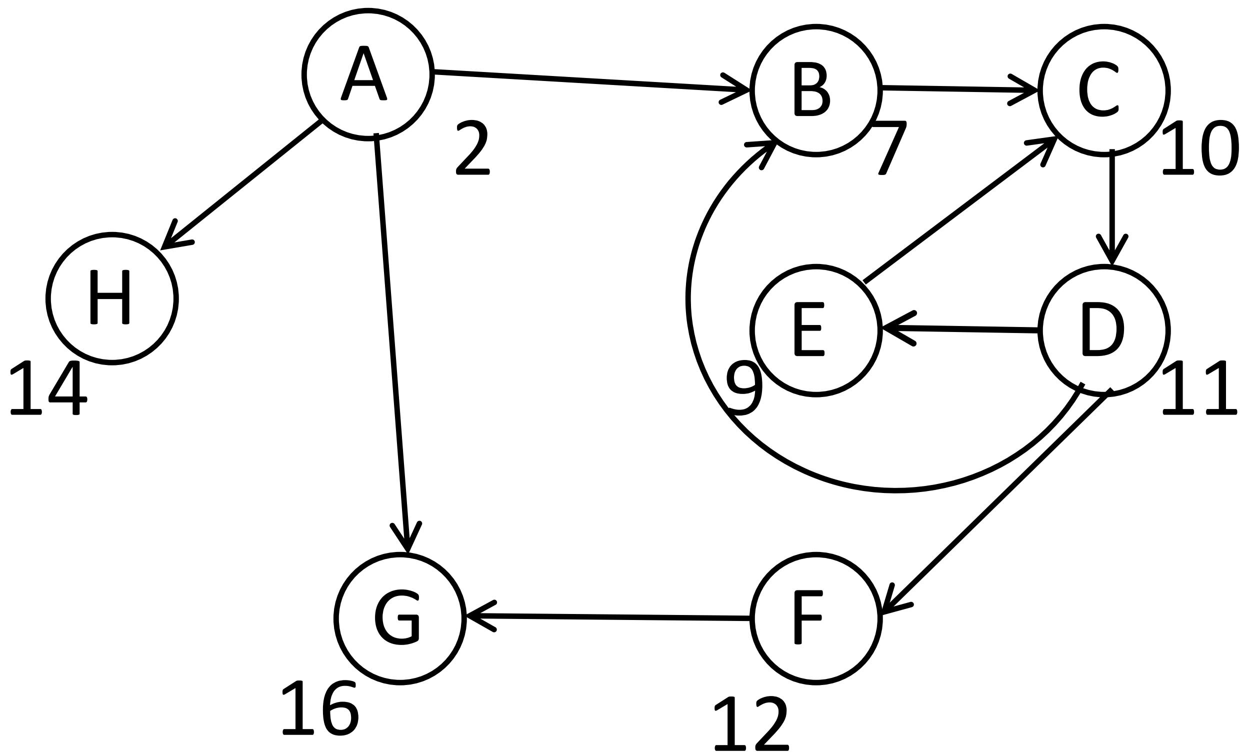

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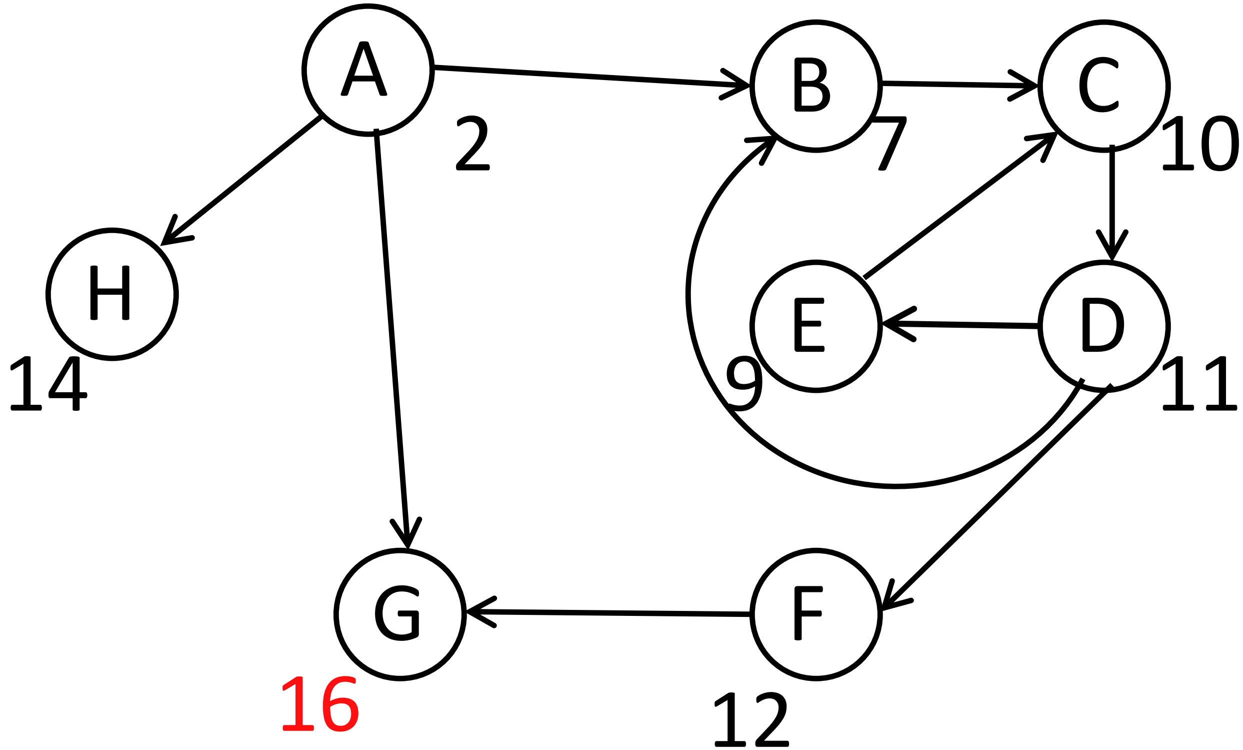

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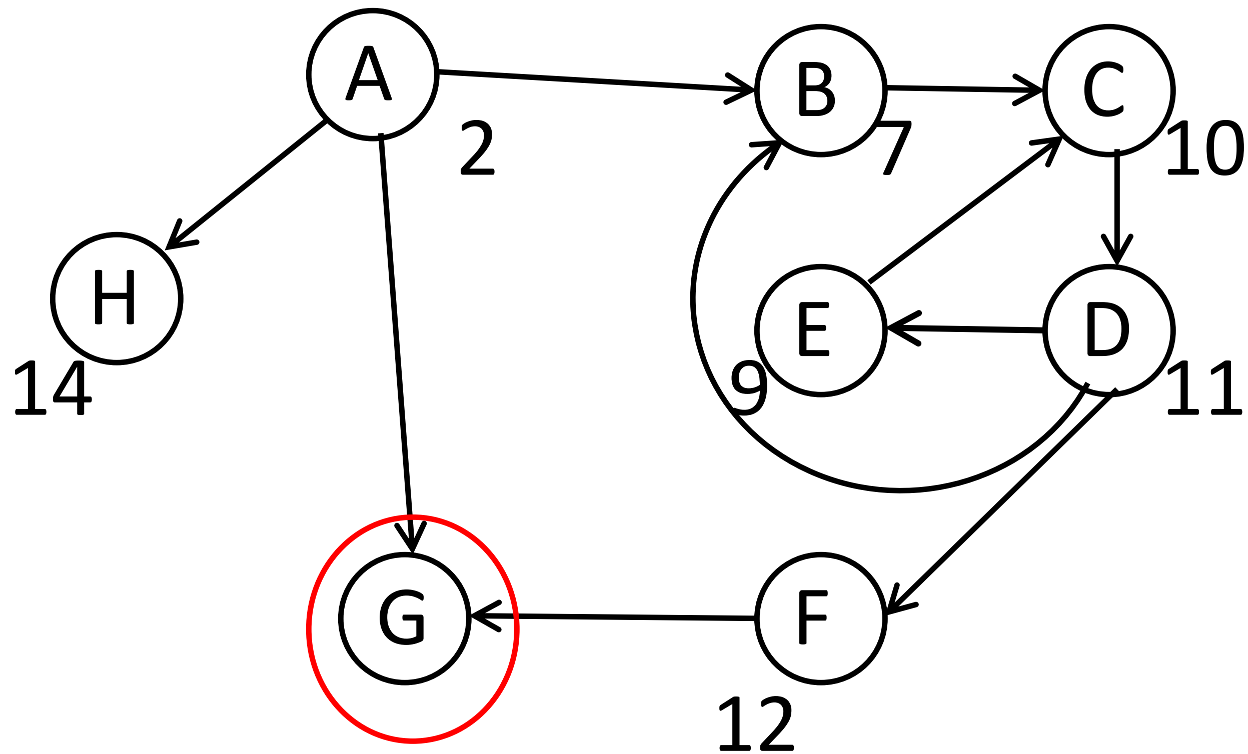

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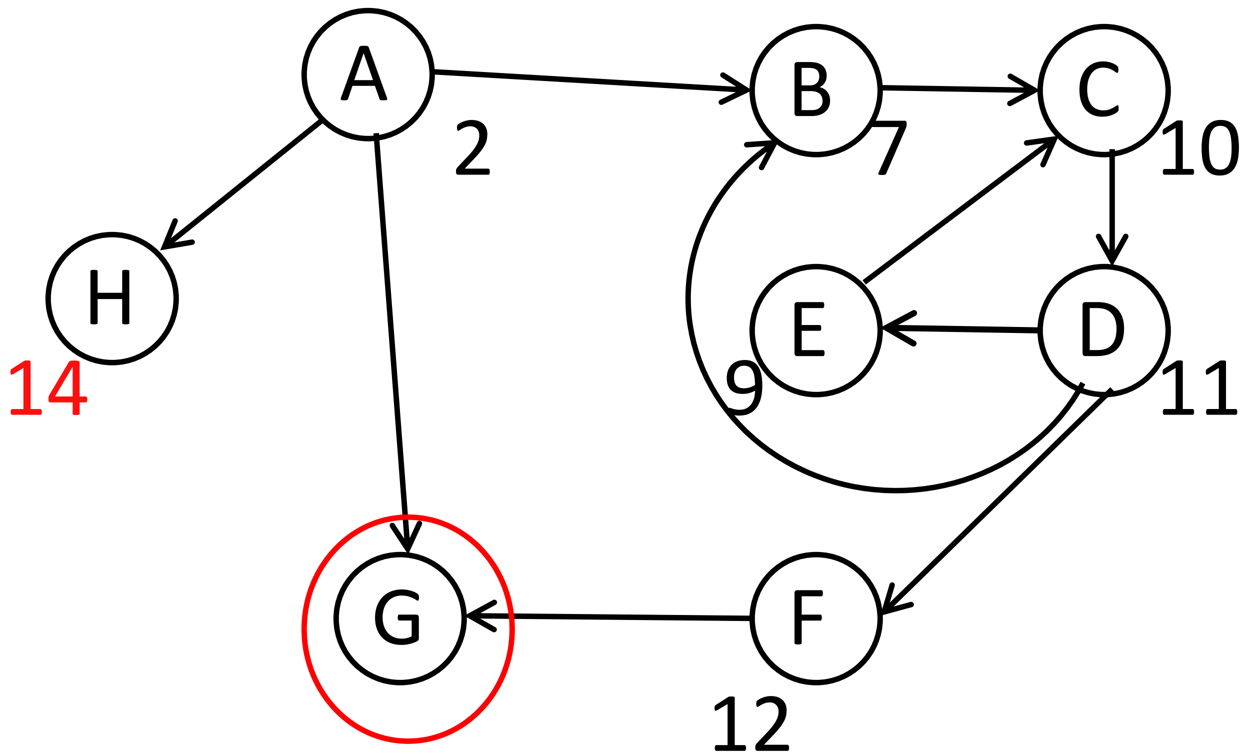

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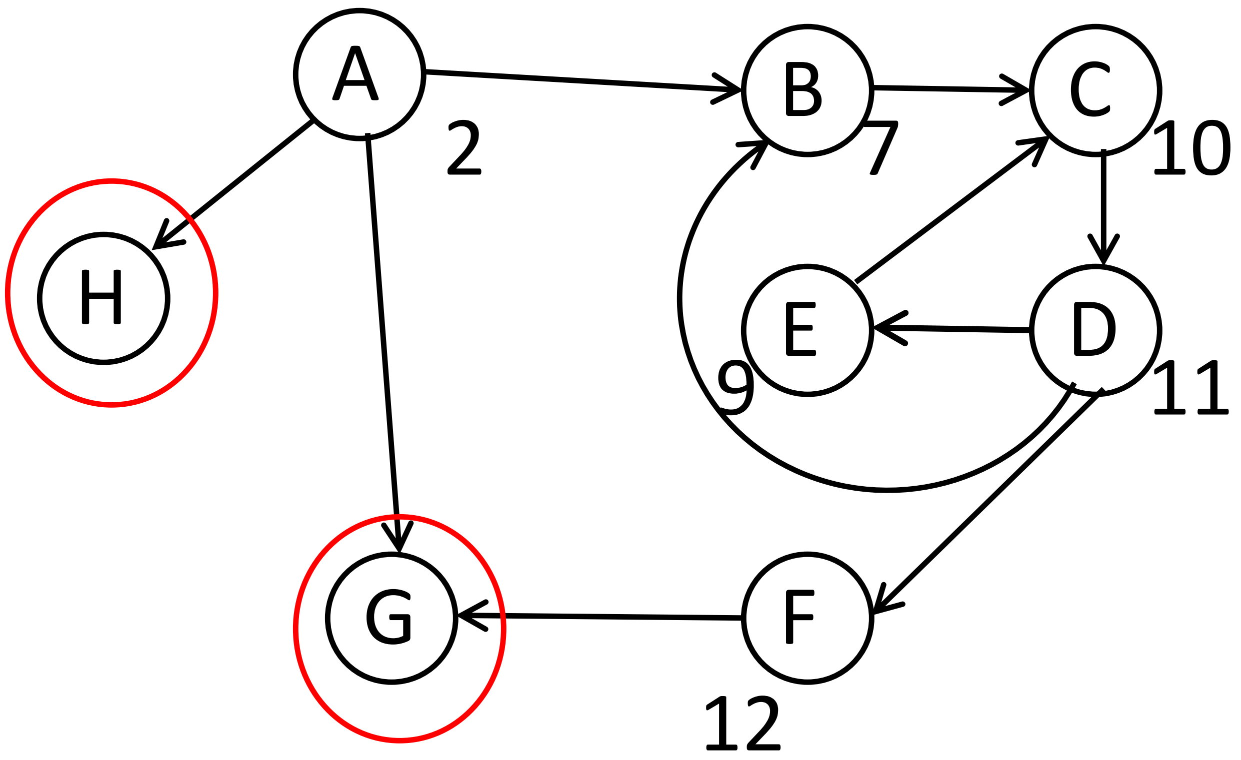

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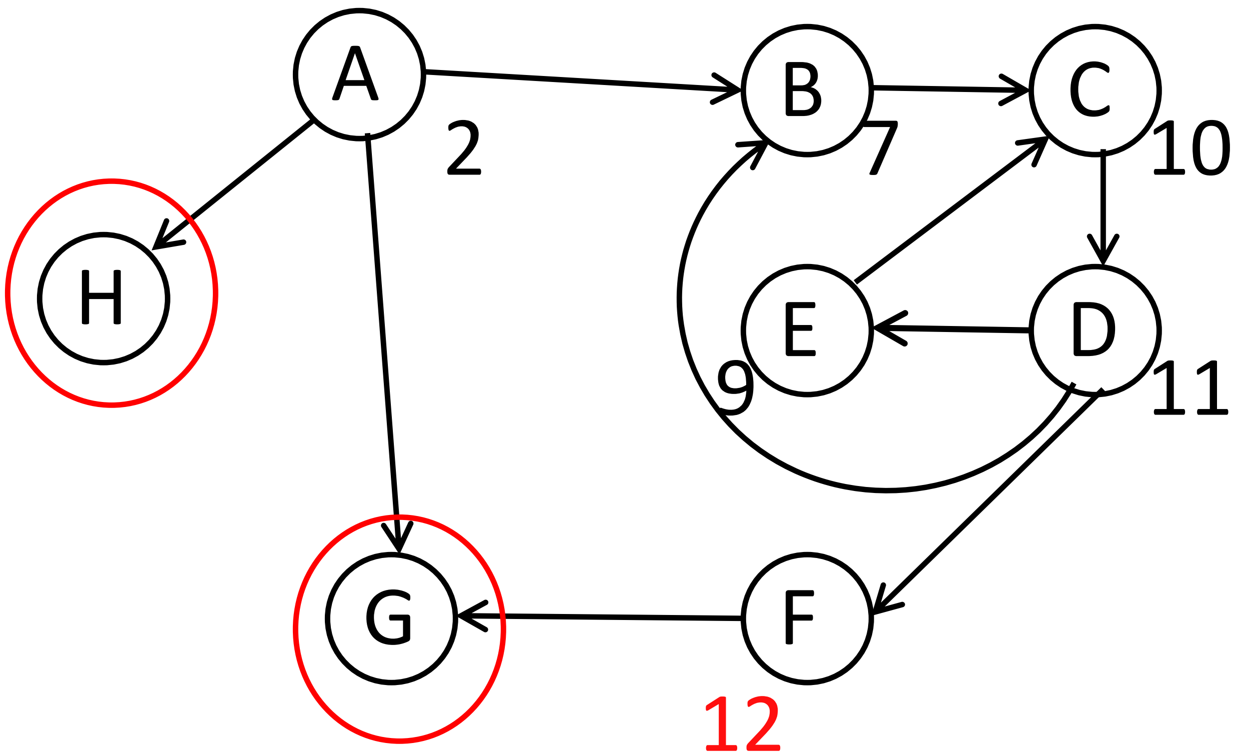

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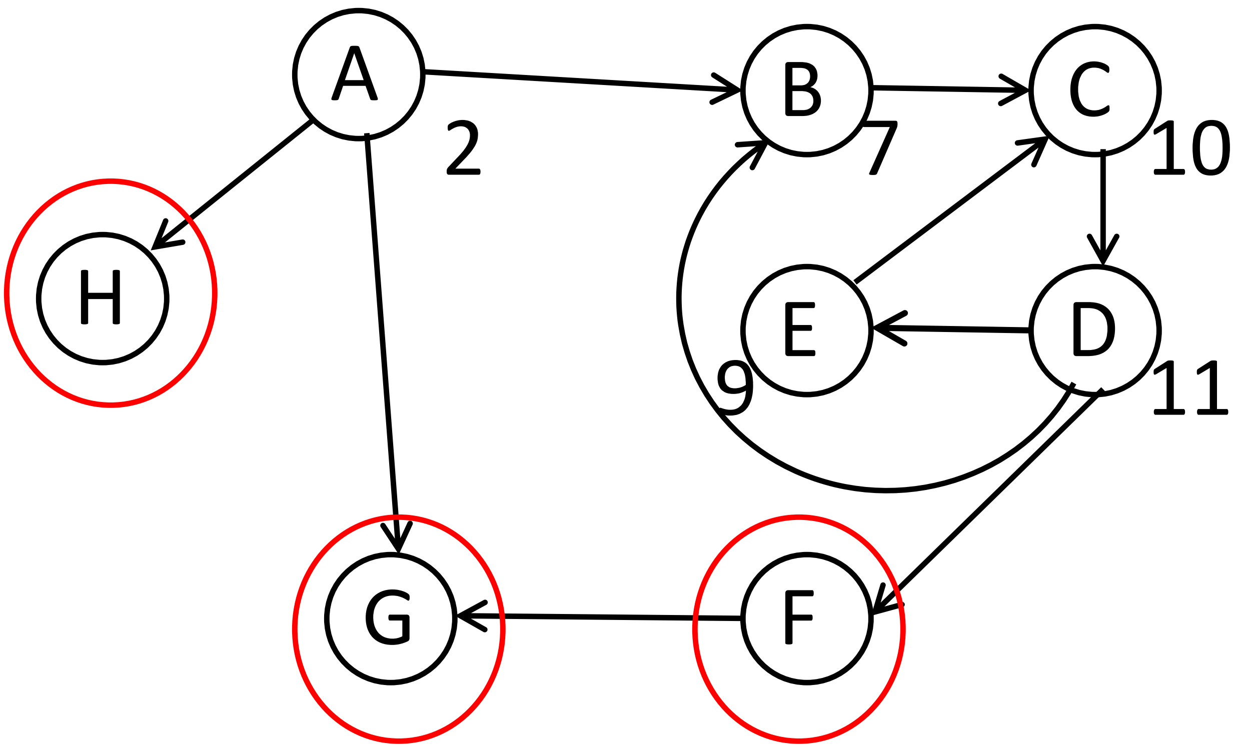

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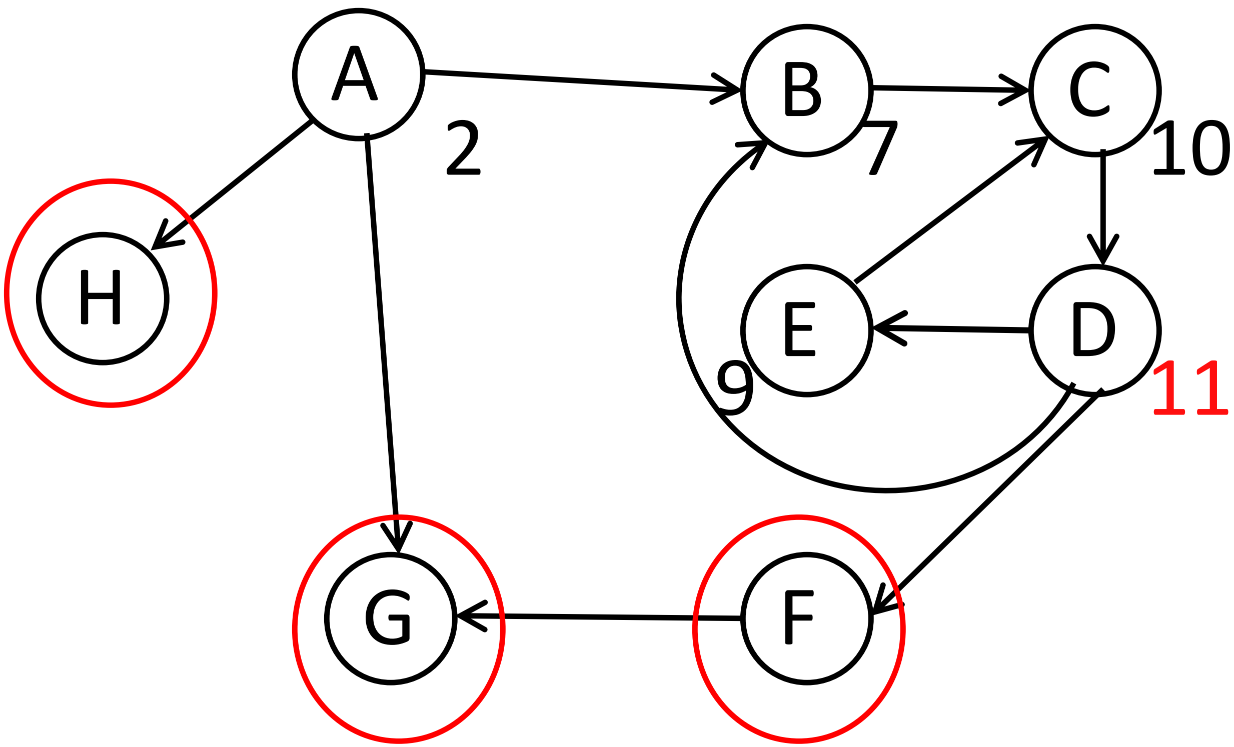

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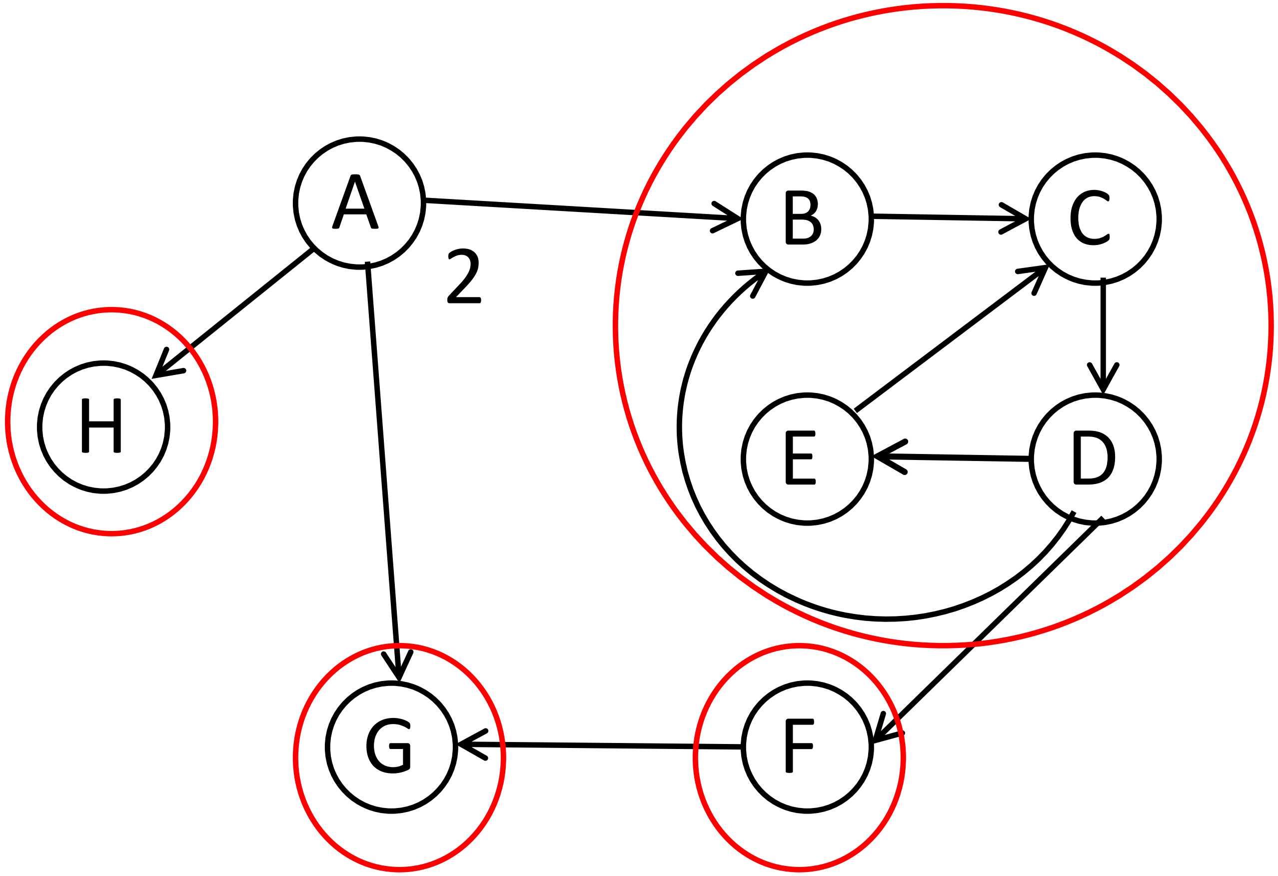

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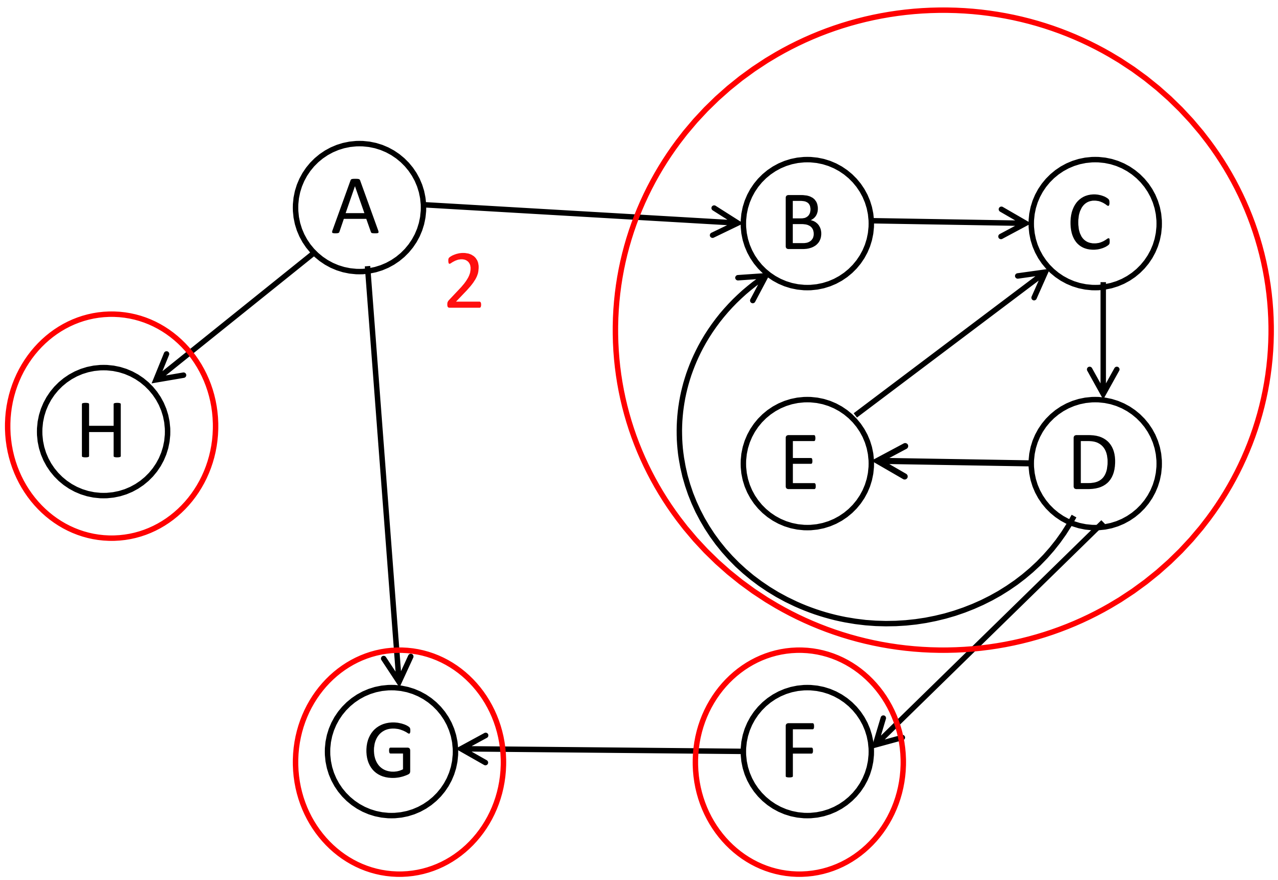

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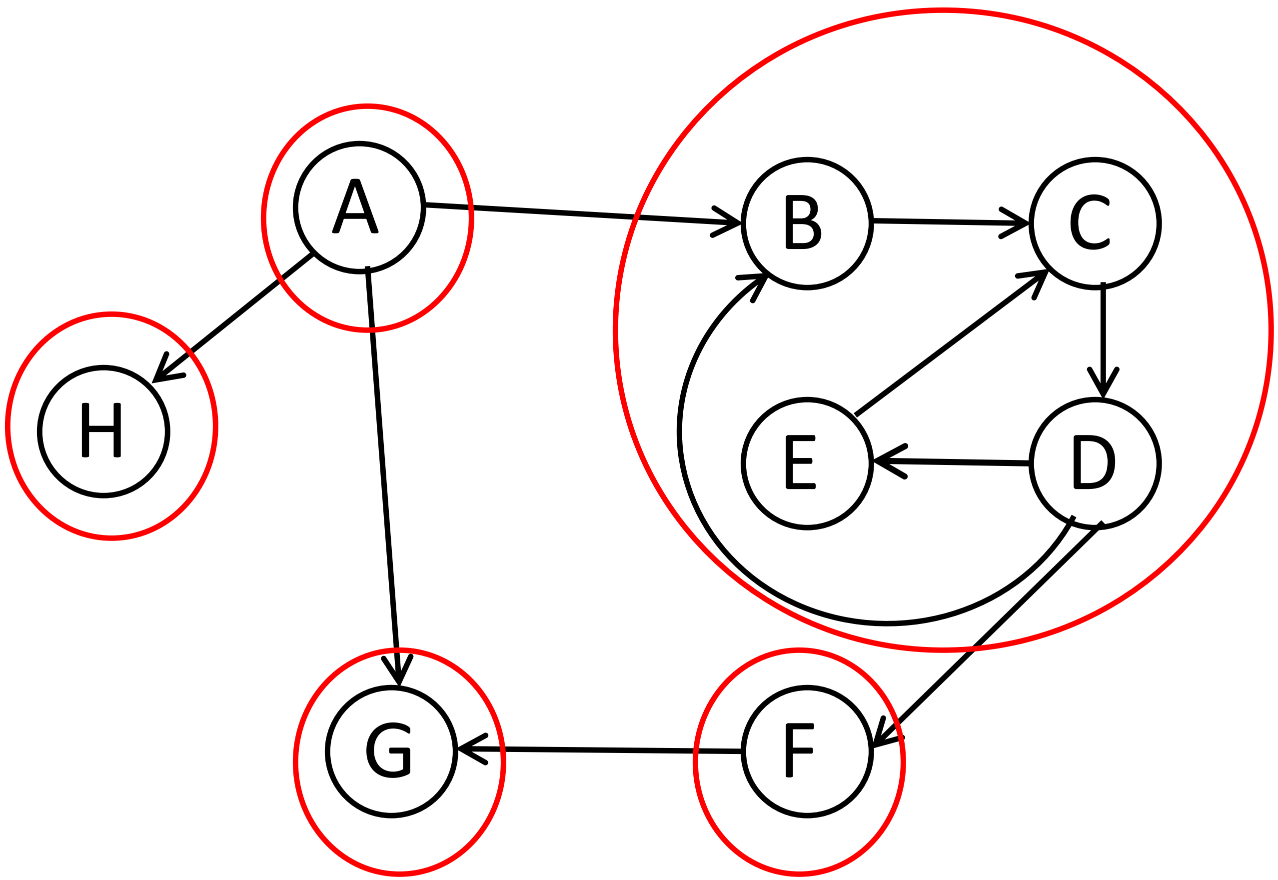

Paths in Graphs (Ch 4)

- Breadth First Search
- Dijkstra
  - Priority Queues
- Bellman-Ford
- Shortest Paths in DAGs

Motivation

`DFS/explore` allow us to determine *if* it is possible to get from one vertex to another, and using the DFS tree, you can also find *a* path. But this often is not an efficient path.

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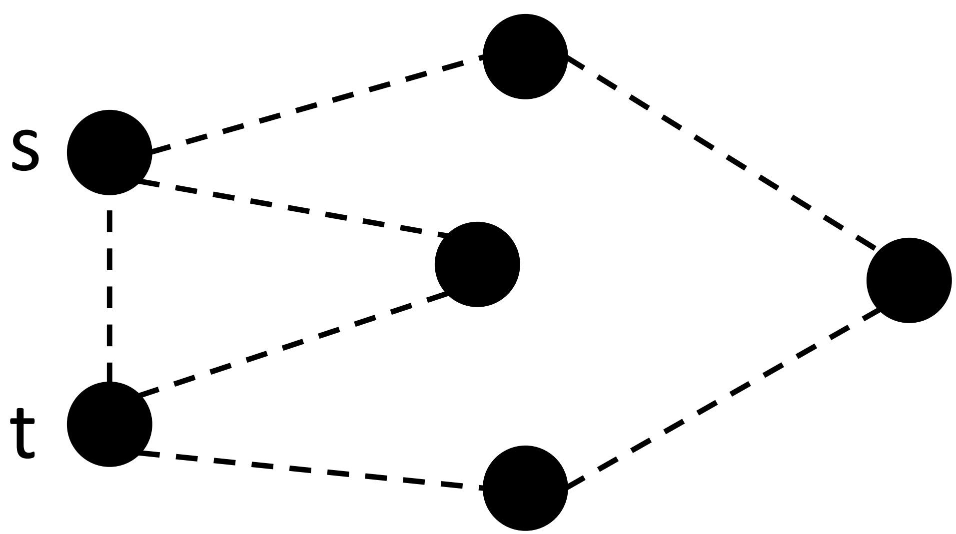

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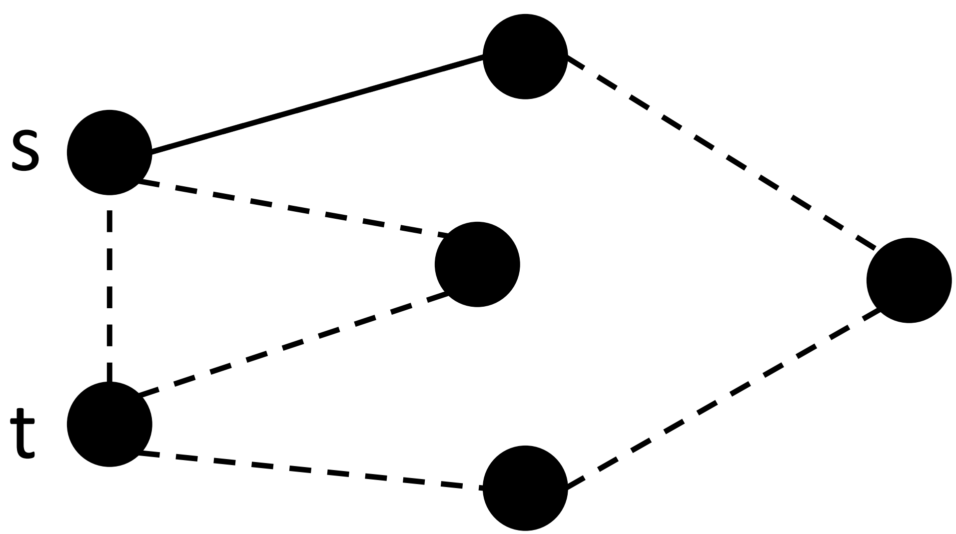

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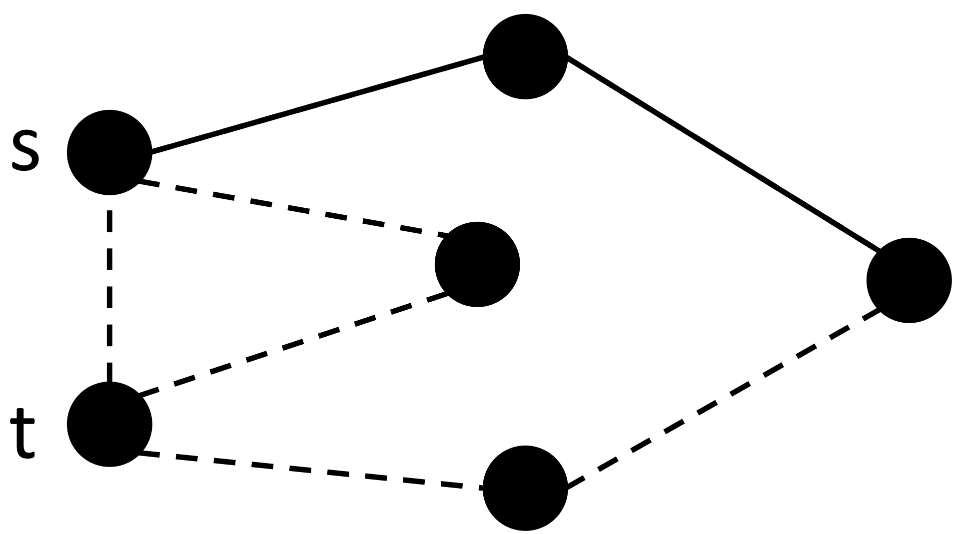

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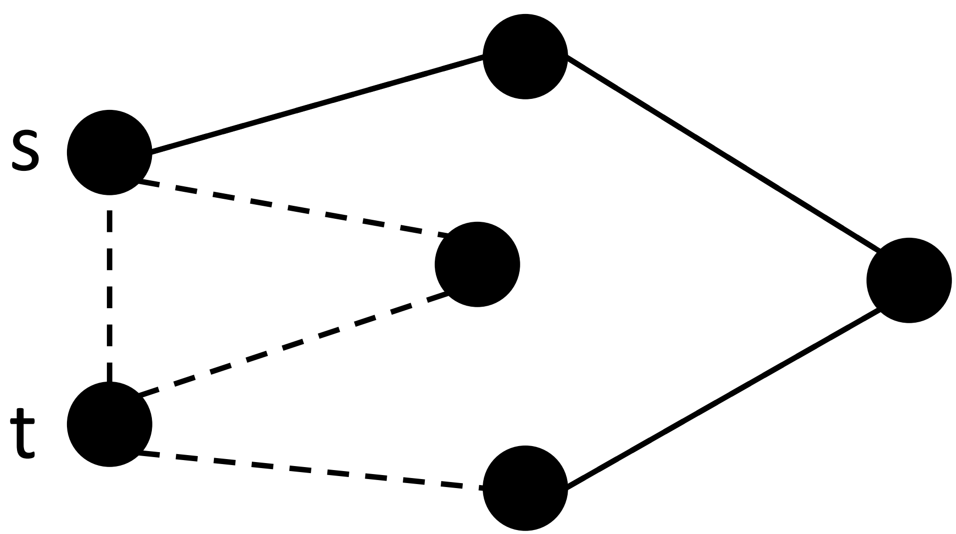

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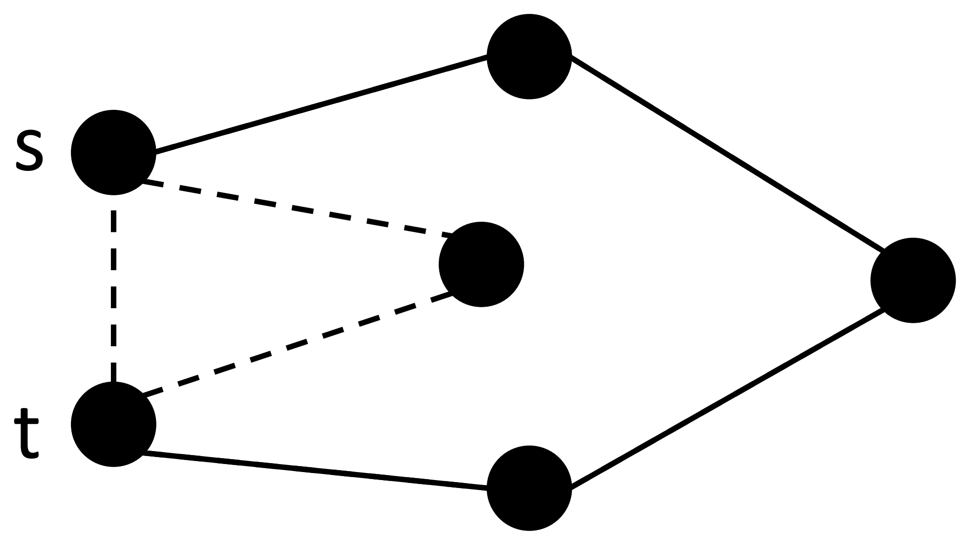

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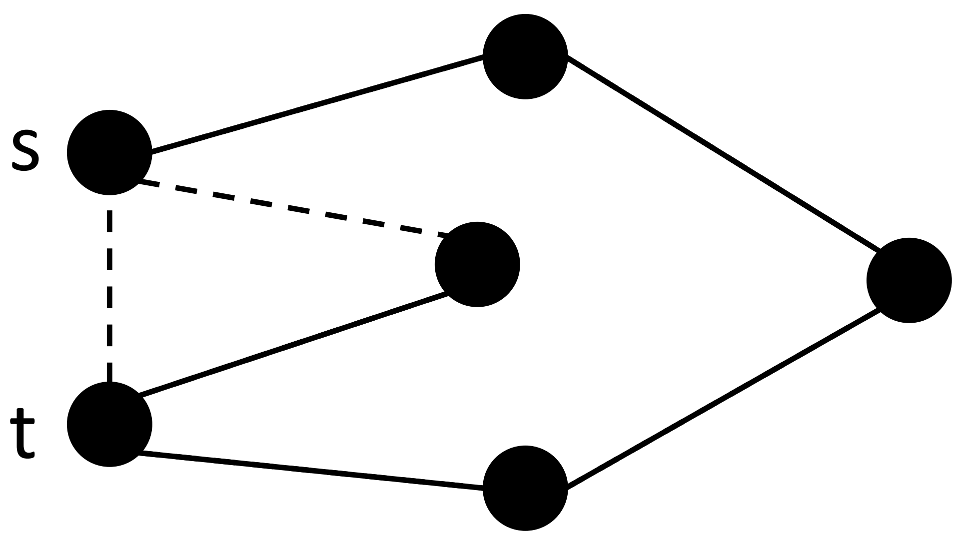

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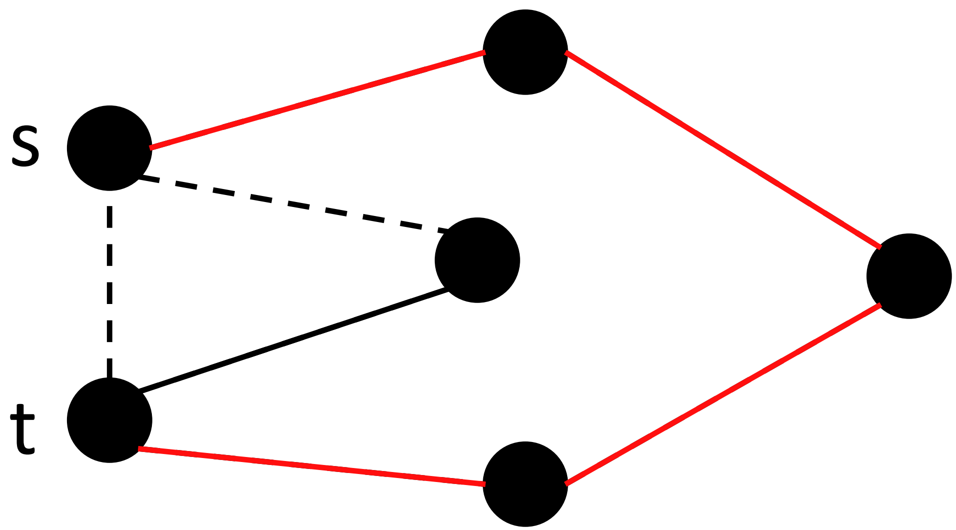

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What do we mean by best?

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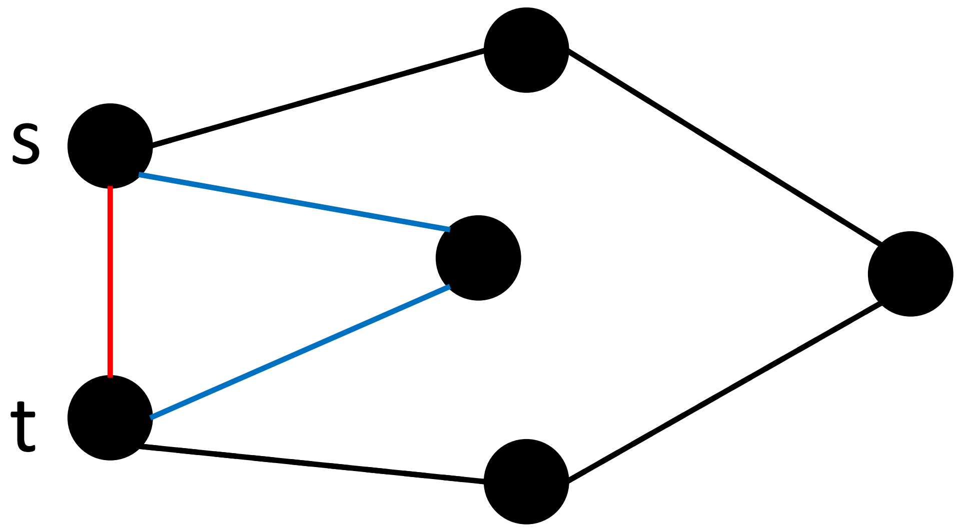
What do we mean by best?

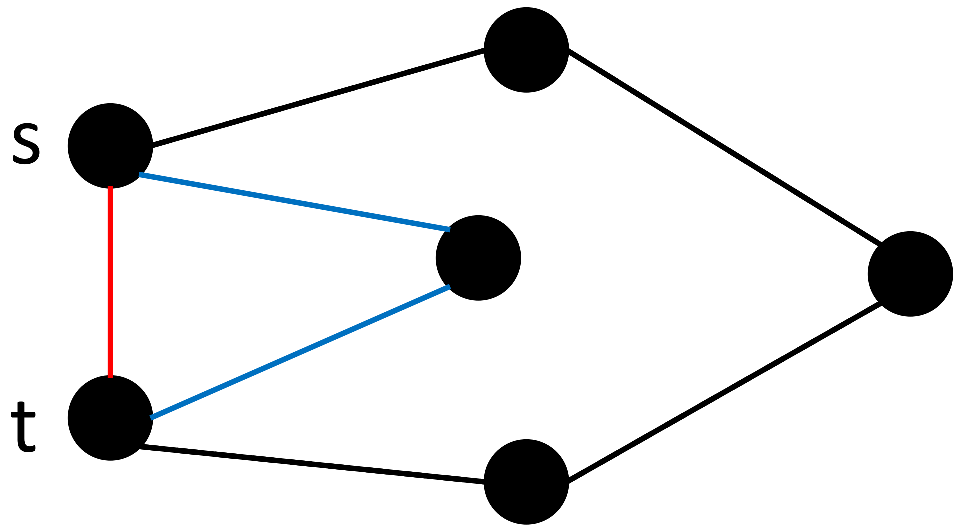
- Least expensive
- Best scenery
- Shortest
- For now: fewest edges

Example



4 edges
2 edges
1 edge

Example



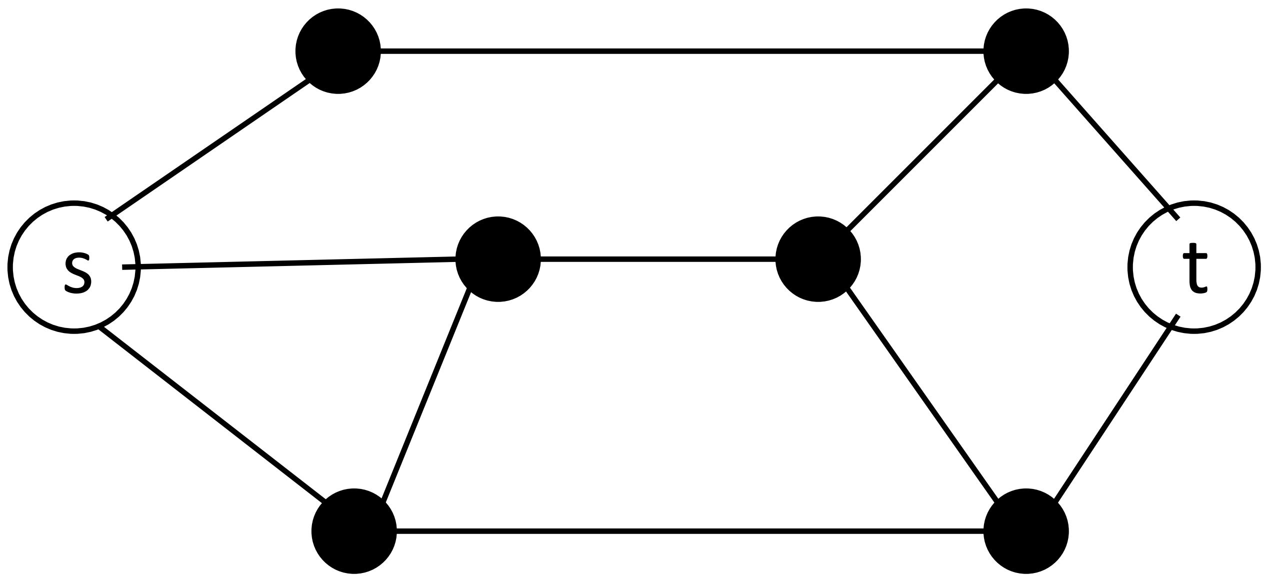
4 edges
2 edges
1 edge

Best is 1 edge.

Question: Shortest Path Length

What is the shortest path length from s to t in the graph below?

A) 2
B) 3
C) 4
D) 5
E) 6



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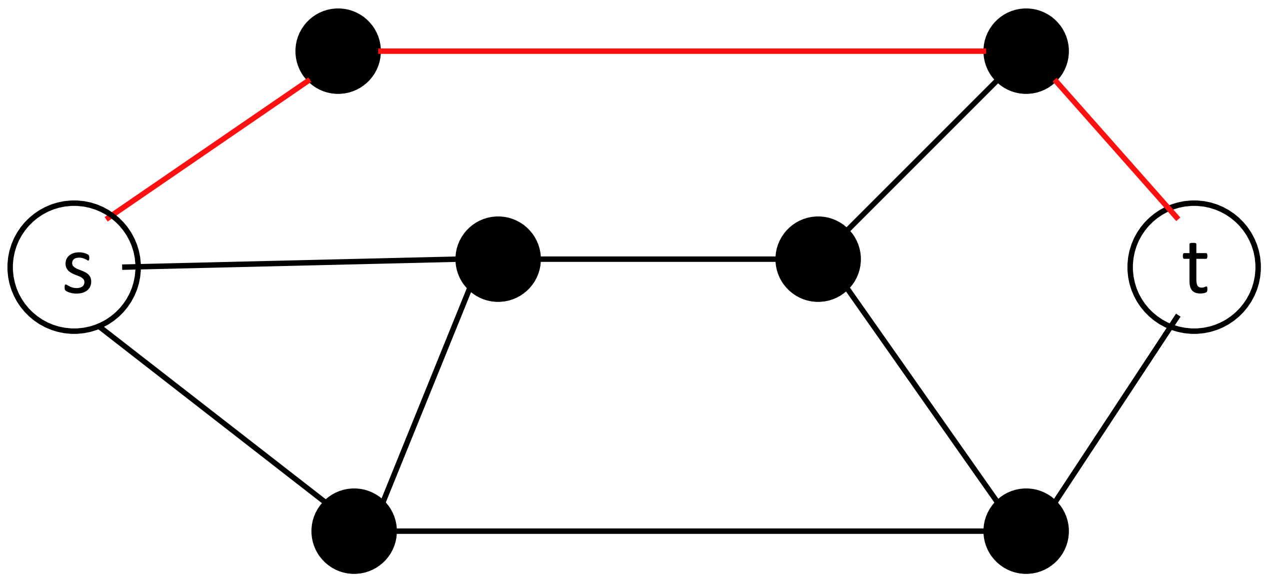

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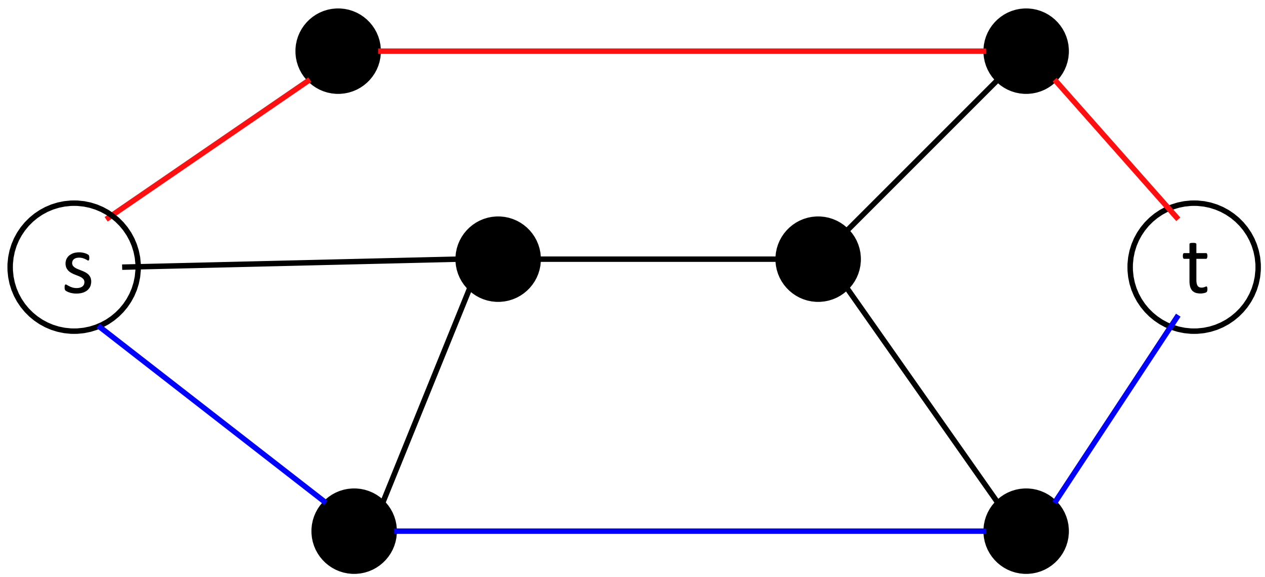
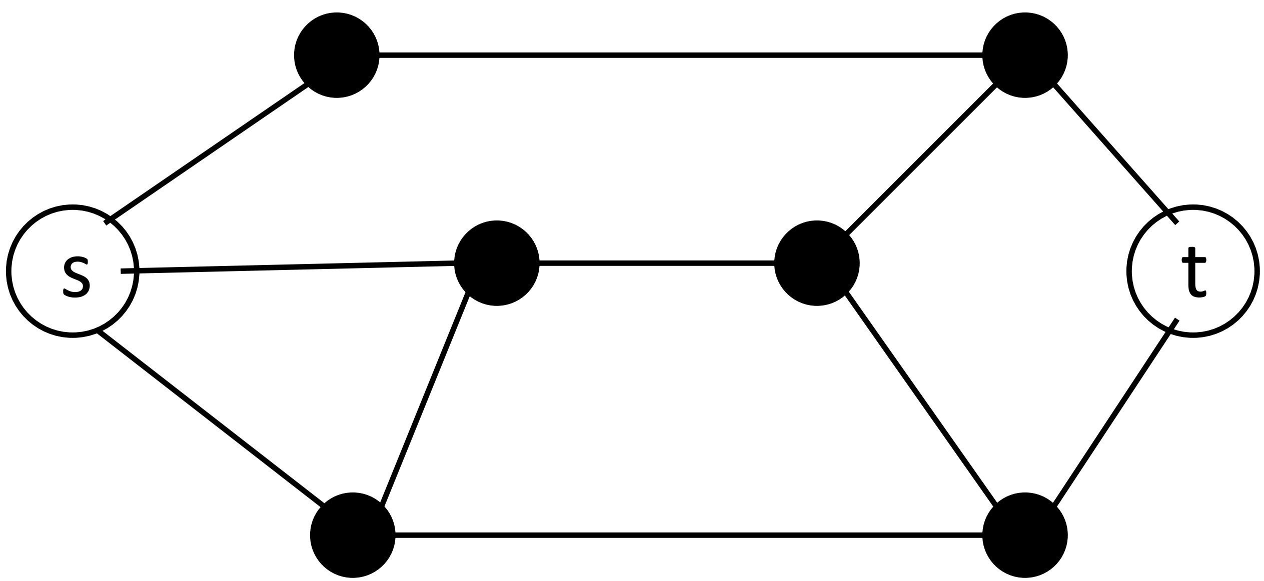


How do you *know*?

It is not hard to convince yourself that the shortest s-t path below has 3 edges, but how do we *know* there is nothing better?



Observation

If there is a length $\leq d$ s-v path, then there is some w adjacent to v with a length $\leq(d-1)$ s-w path.

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Proof: w is the next to last vertex on the path.

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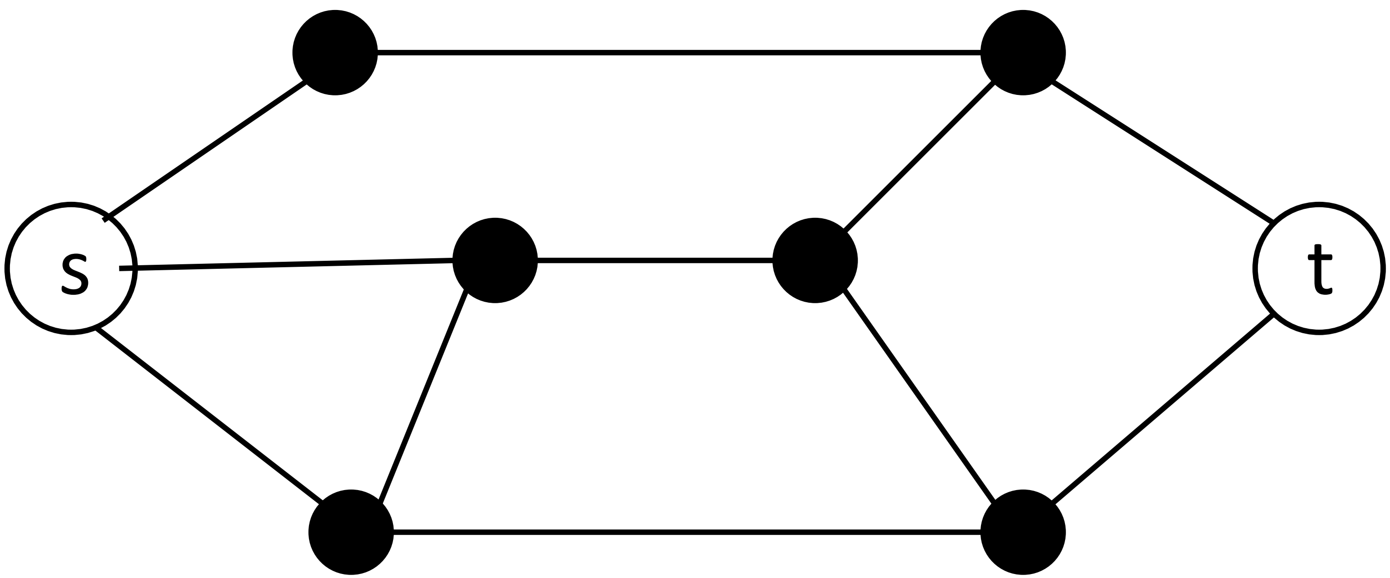
If there is a length ≤d s-v path, then there is some w adjacent to v with a length ≤(d-1) s-w path.

Proof: w is the next to last vertex on the path.

This means that if we know all of the vertices at distance ≤(d-1), we can find all of the vertices at distance ≤d.

Example



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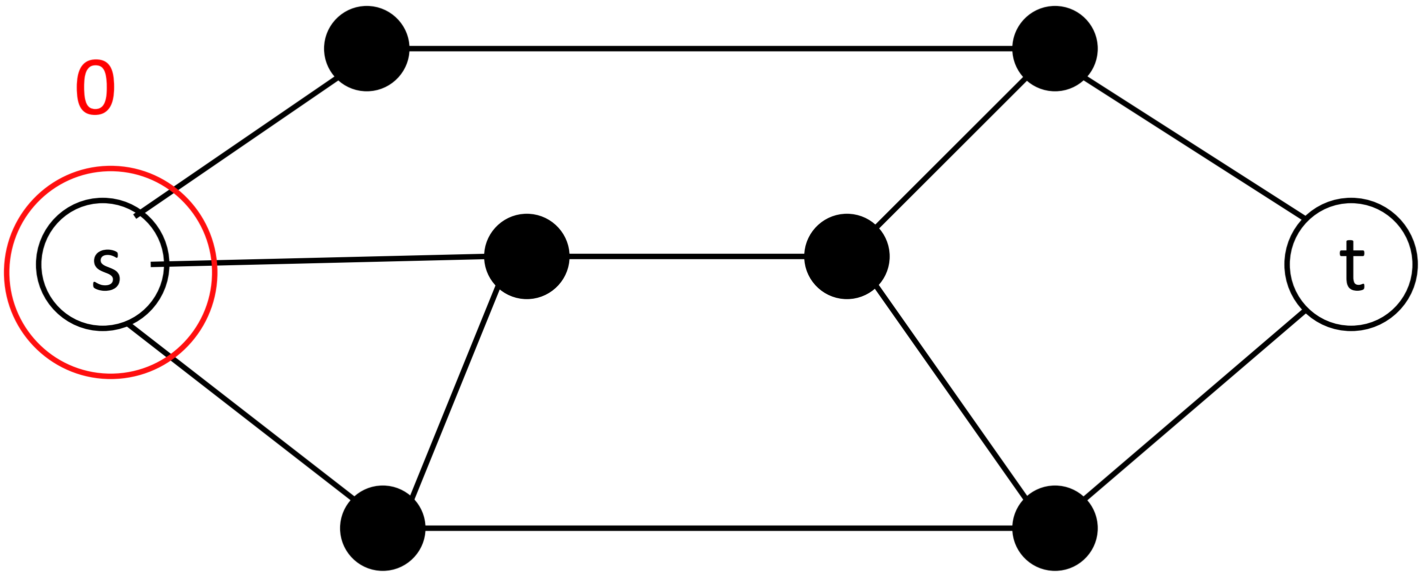

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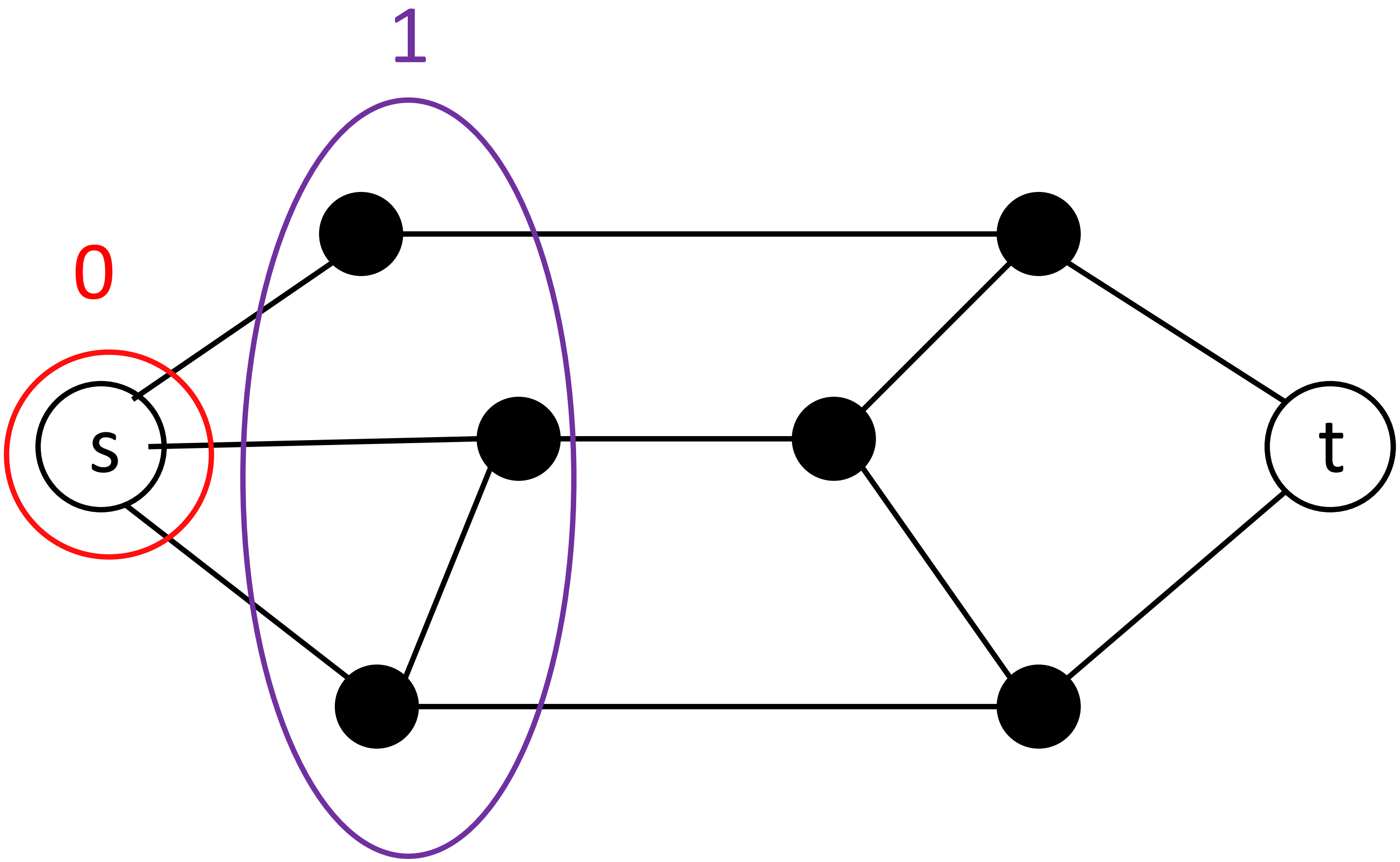

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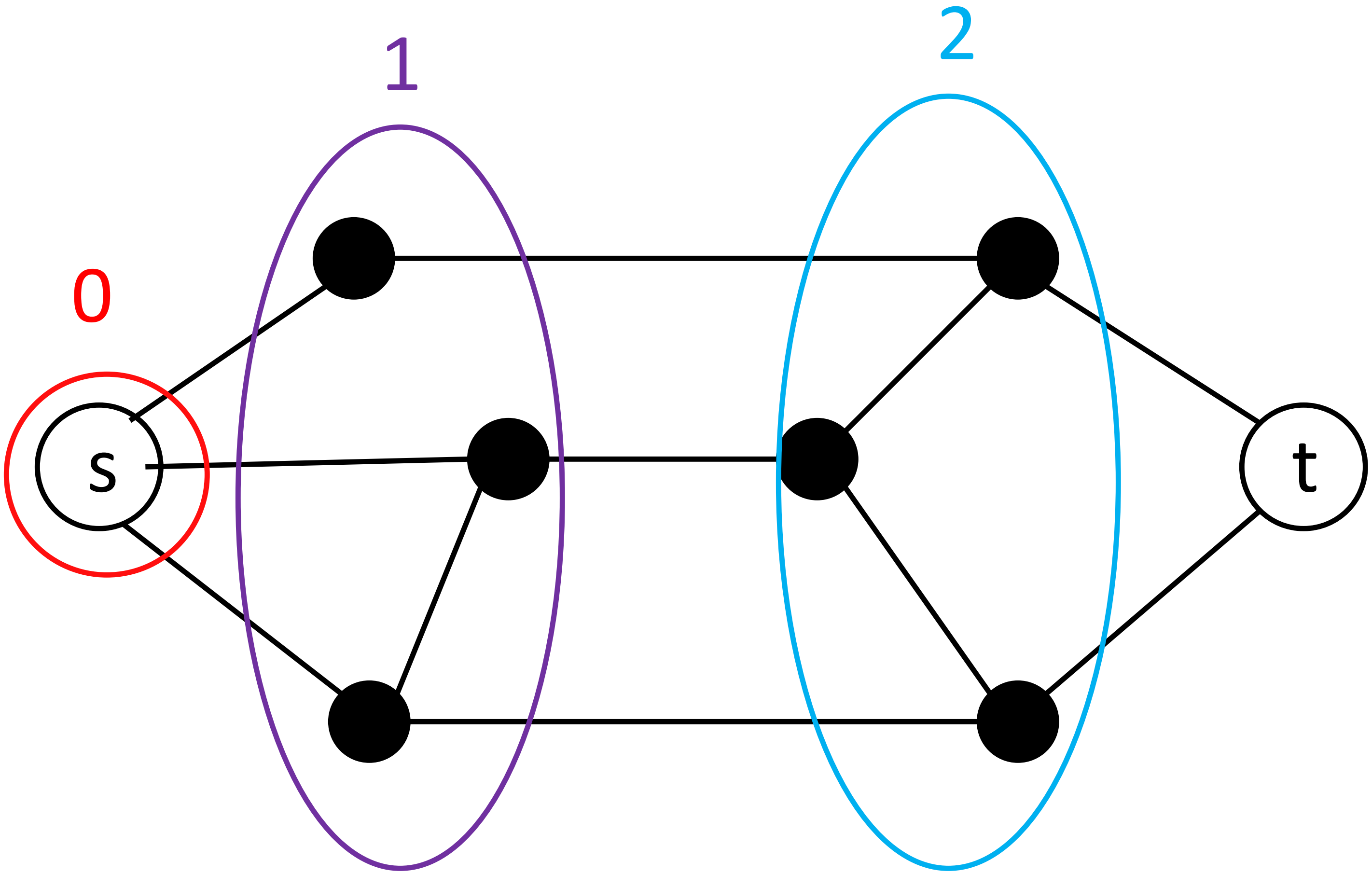

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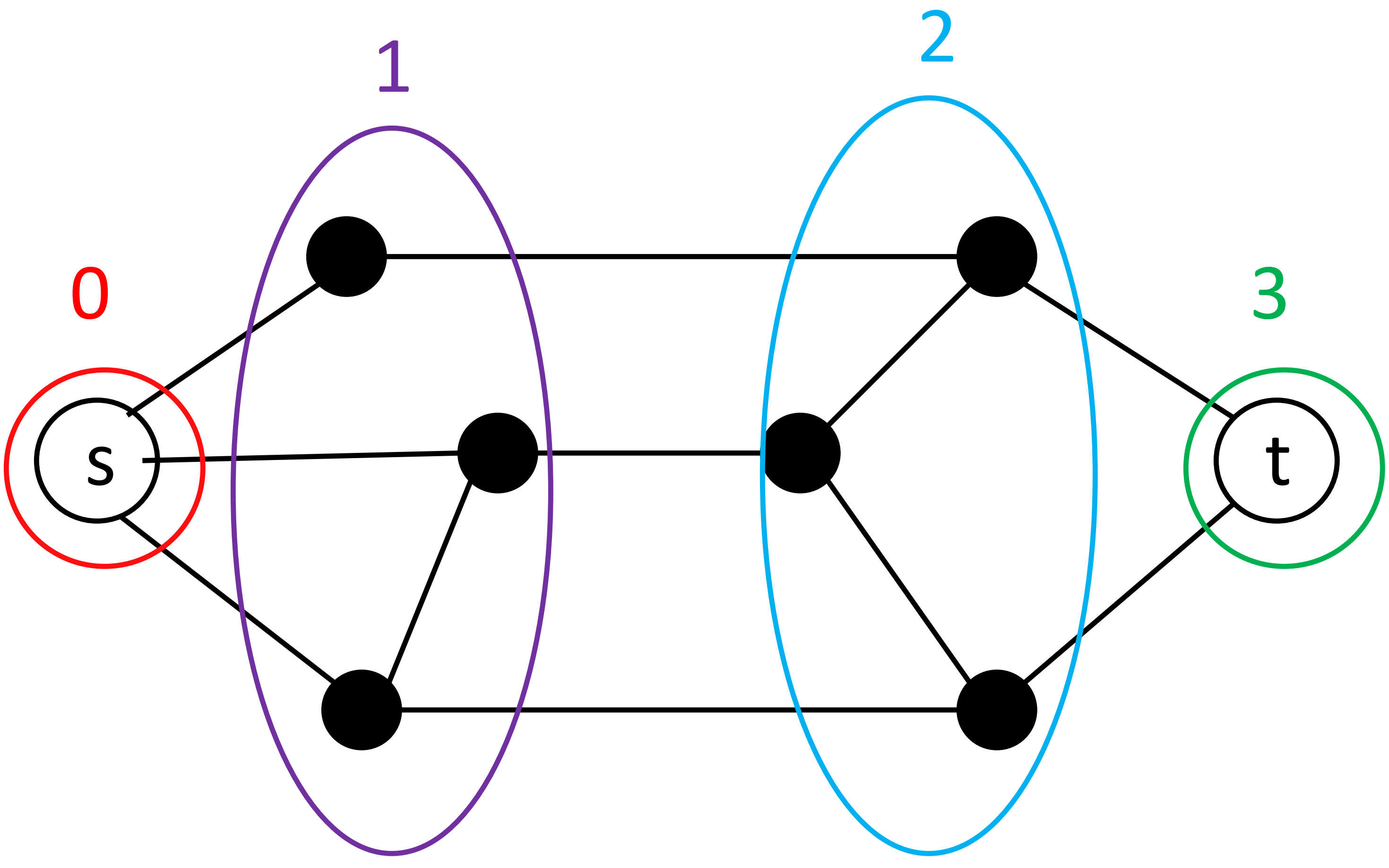

Algorithm Idea

For each d create a list of all vertices at distance d from s.

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For each d create a list of all vertices at distance d from s.

- For d=0, this list is just {s}.
- For larger d, we want all new vertices adjacent to vertices at distance d-1.

ShortestPaths(G,s)

```
Initialize Array A
A[0] ← {s}
dist(s) ← 0
For d = 0 to n
  For u ∈ A[d]
    For (u,v) ∈ E
      If dist(v) undefined
        dist(v) ← d+1
        add v to A[d+1]
```

ShortestPaths(G,s)

What if dist(v) undefined at end?

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ShortestPaths(G,s)

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For v ∈ V, dist(v) ← ∞
Initialize Array A
A[0] ← {s}
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For d = 0 to n
   For u ∈ A[d]
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Algorithm goes through A[0], A[1],… in order. Can just use a queue.

```
ShortestPaths(G,s)
  For v ∈ V, dist(v) ← ∞
  Initialize Queue Q
  Q.enqueue(s)
  dist(s) ← 0
  While Q not empty
    u ← front(Q)
      For (u,v) ∈ E
        If dist(v) = ∞
          dist(v) ← dist(u)+1
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Keep track of paths

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BreadthFirstSearch(G,s)
  For v ∈ V, dist(v) ← ∞
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Runtime

```
BFS(G,s)
   For v ∈ V, dist(v) ← ∞
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O(|V|)

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O(|V|)

O(|V|) iterations

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$O(|V|)$

$O(|V|)$ iterations

$O(|E|)$ total iterations

Total runtime:

$O(|V|+|E|)$

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- Processed vertices (visited, dist $< \infty$)
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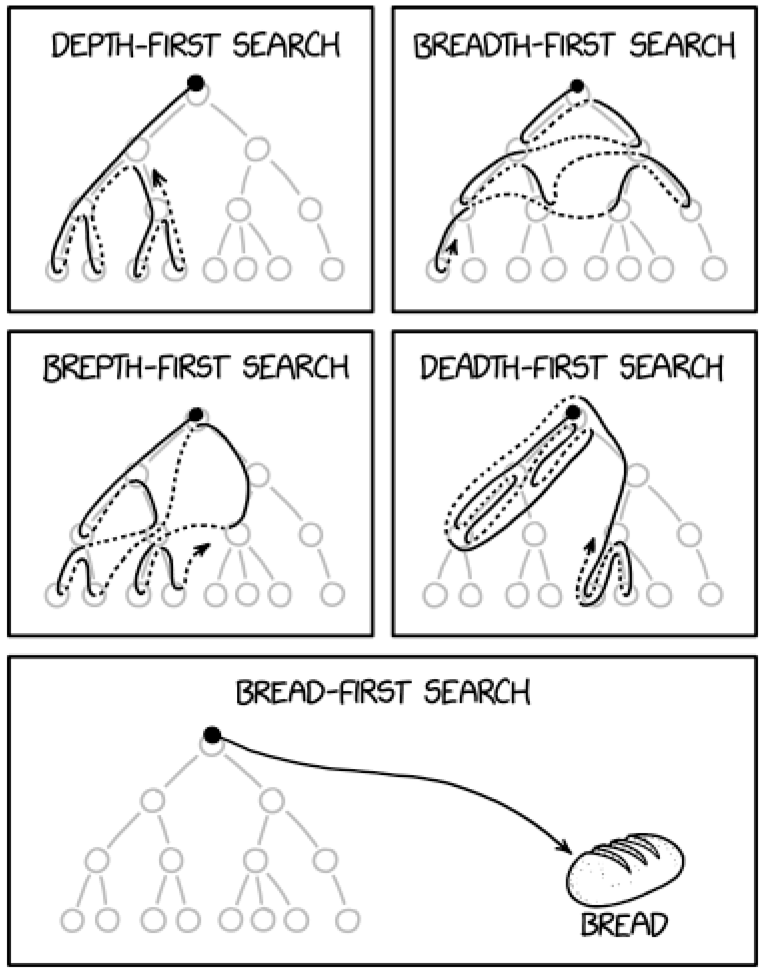
- Processed vertices (visited, dist < ∞)
- For each vertex, process all unprocessed neighbors
- Difference:
  - DFS uses a stack to store vertices waiting to be processed
  - BFS uses a queue
- Big effect
  - DFS goes depth first – very long path
  - BFS is breadth first – visits all side paths

DFS vs BFS vs Others?



https://xkcd.com/2407/