CSE 20 Spring 2022
Homework 7 solutions
Due date: Thursday, May 26, 2022 at 11:59pm

In this assignment,

You will compare set cardinalities, construct functions and determine if functions are one-to-one, onto or bijections. You will identify uncountable sets and countable sets.

In this class, unless the instructions explicitly say otherwise, you are required to justify all your answers.

1. (20 points) In this problem, we will show that $\mathbb{N} \times \mathbb{N}$ is countably infinite.

   (a) i. Describe a function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ that is one-to-one.

   Solution:
   Consider the function $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with the rule: $\forall n \in \mathbb{N}, h(n) = (n, n)$.

   ii. Prove that it is one-to-one.

   Solution:
   (Want to prove the statement: $\forall x, y \in \mathbb{N}, (h(x) = h(y) \to x = y)$)
   Proof:
   Let x, y be arbitrary elements of $\mathbb{N}$. Assume that $h(x) = h(y)$. Then $(x, x) = (y, y)$. So since each entry must be equal, $x = y$ as required.

   iii. What does it mean for $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ to be one-to-one? (Choose only one answer. No justification necessary.)

   - b: $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$

   (b) Consider the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as: for any $(x, y) \in \mathbb{N} \times \mathbb{N}$, $g(x, y) = 2^x * 3^y$.

   i. Compute $g(2, 3)$, $g(3, 2)$, $g(0, 0)$.

   Solution:
   - $g(2, 3) = 2^2 * 3^3 = 108$
   - $g(3, 2) = 2^3 * 3^2 = 72$
   - $g(0, 0) = 2^0 * 3^0 = 1$

   ii. Prove that g is not onto.

   Solution:
   Proof by contradiction:
   Assume that for all $n \in \mathbb{N}$, that there exists $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $2^x 3^y = n$.
   Based on this assumption, there exists $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $5 = 2^x * 3^y$. But since 2 does not divide 5, then $x = 0$ and since 3 does not divide 5 then $y = 0$. Therefore $5 = 2^0 * 3^0 = 1$ and $5 \neq 1$ which is a contradiction therefore the assumption is false and the function is not onto.

   iii. Prove that g is one-to-one.

   Solution:
   (Want to prove the statement: $\forall (x, y), (a, b) \in \mathbb{N} \times \mathbb{N}, (g(x, y) = g(a, b) \to (x, y) = (a, b)))$
   Proof:
   Let $(x, y), (a, b)$ be arbitrary elements of $\mathbb{N} \times \mathbb{N}$. Assume that $g(x, y) = g(a, b)$. Then $2^x * 3^y = 2^a * 3^b$.
   If we divide both sides by 2^a we get:

   $$2^{x-a} 3^y = 3^b$$

   Then the right-hand side must be an odd number which means that the left-hand side is also an odd number. This means that $x - a = 0$ so $x = a$. Now the equation is $3^y = 3^b$ so $y = b$.
   Therefore $(x, y) = (a, b)$ as required.

iv. What does it mean for $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ to be one-to-one? (Choose only one answer. No justification necessary.)

- e: $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$

(c) What theorem can we use to establish that $\mathbb{N} \times \mathbb{N}$ is countably infinite?

Solution:
The Cantor-Bernstein Theorem

2. (15 points) For each of the following sets, state whether it is **finite**, **countably infinite**, or **uncountable**. With each statement, justify your answer.

- If you want to prove that a set S is countably infinite, you can describe a way to list out all of its elements in a particular way, you can show that $|S| = |\mathbb{N}|$ or you can show that S is infinite and $|S| \leq |\mathbb{N}|$ or $|\mathbb{N}| \geq |S|$.
- If you want to prove that a set S is uncountably infinite, you can use a diagonalization proof, or show that it has the same (or larger) cardinality than a well-known uncountable set.

(a) The set of all truth tables with three propositional variables p, q, r.

Solution:
There are eight rows for a truth table with p, q, r. Each row can either be true or false so there are 2^8 different truth tables. 2^8 is finite so this set is finite.

(b) $\mathbb{N} \times \{0, 1\}$

Solution:
This set is countably infinite. We can list out all elements in the following way:

$$(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (3,0), (3,1), \ldots$$

alternating from 0 to 1 to 0 to 1 in the second entry. Each element will appear in this list.

(c) The set of all finite subsets of $\mathbb{N}$.

Solution:
This set is countably infinite. We can list out all elements by starting with the empty set then listing out all the sets based on their maximum element, i.e., all the sets with max element 0, all sets with max element 1, all sets with max element 2, and so on.

$$\emptyset, \{0\}, \{1\}, \{0,1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}, \{3\}, \ldots$$

(d) The set of all functions with domain $\mathbb{N}$ and codomain $\{\emptyset\}$.

Solution: This is a finite set.
There is only one function that has domain $\mathbb{N}$ and codomain $\{\emptyset\}$, namely the function $f : \mathbb{N} \to \{\emptyset\}$ given by the rule $f(n) = \emptyset$ for all $n \in \mathbb{N}$.

(e) The set of all functions with domain $\{\emptyset\}$ and codomain $\mathbb{N}$.

Solution:
This set is countably infinite.
We can make a sequence of functions indexed by i for $i \geq 0$ each with domain $\{emptyset\}$ and codomain $\mathbb{N}$ given by the rule:

$$f_i : \{\emptyset\} \to \mathbb{N}, \quad f_i(\emptyset) = i$$

We can list these functions : $f_0, f_1, f_2, \ldots$.
All such functions will appear in the list.

(f) The set of all infinites sequences of 0's and 1's.

Solution: this set is uncountable.
Consider the one-to-one function $w : \mathcal{P}(\mathbb{N}) \to \{0,1\}^\infty$ (where $\{0,1\}^\infty$ is the set of all infinite sequences of 0's and 1's.)
Given by the rule:
$w(S) = (b_0, b_1, b_2, \dots)$ such that $b_i = 0$ if $i \notin S$ and $b_i = 1$ if $i \in S$.
Then w is one-to-one:
Proof: let S, T be arbitrary elements of $\mathcal{P}(\mathbb{N})$. Then assume that $w(S) = w(T)$. Then $(b_0, b_1, \dots) = (d_0, d_1, \dots)$. Sequences are equal means that each entry is equal to the corresponding entry in the other sequence.
Therefore, for all elements $i \in S$, $b_i = 1$, $d_i = 1$ so $i \in T$ and for each $i \notin S$, $b_i = 0$, $d_i = 0$, so $i \notin T$. Therefore $S = T$.
So we have shown that $|\mathcal{P}(\mathbb{N})| \leq |\{0,1\}^\infty|$ and since $\mathcal{P}(\mathbb{N})$ is uncountable then $\{0,1\}^\infty$ is uncountable.

3. (15 points)

(a) Construct a one-to-one function from $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ (just give the rule of the function, you do not need to prove that it is one-to-one.)

Solution:
Consider the function: $g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ given by the rule $g(n) = \{0, 1, \dots, n\}$ for all $n \in \mathbb{N}$.

(b) Construct an onto function from $\mathcal{P}(\mathbb{N}) \to \mathbb{N}$ (just give the rule of the function, you do not need to prove that it is onto.)[Hint: use the well-ordering principle.]

Solution:
Consider the function: $h : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ given by the rule $h(S) = \min(S)$ for all $S \neq \emptyset$ and $h(\emptyset) = 0$.

(c) Are parts a and b enough to prove that $|\mathbb{N}| = |\mathcal{P}(\mathbb{N})|$? Why or why not?

Solution:
No. Because in order to show equality, you need $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$ and $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{N}|$. But we did not establish that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{N}|$.

(d) Recall from class (5/24) that in order to do the diagonalization argument, for each function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, we defined a set

$$D_f = \{x \in \mathbb{N} \mid x \notin f(x)\}$$

Describe D_f where f is the function you put from part a.

Solution:
$D_g = \emptyset$ because for each element $n \in \mathbb{N}$, $n \in g(n)$.