Name: ______________________ **PID:** ______________

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CSE 21 - FALL 2022

Practice Final Exam

1 Loop Invariant and Iterative Algorithm

Consider the following algorithm that takes as input a list of integers $(A[1], \ldots, A[n])$ and an integer $1 \leq k \leq n$ and outputs the array of sums of windows of length k, $S[I] = \sum_{i=I}^{i=I+k-1} A[i]$ for $1 \leq I \leq n-k+1$. For example if the input is: $(53, 10, 4, 7, 12, 18, 72, 71)$ and $k = 3$, then the output should be $S[1..6] = 67, 21, 23, 37, 102, 161$.

procedure `SumWindows`$(A[1], \ldots, A[n]; k)$: $1 \leq k \leq n$.)

1. $S = A[1]$
2. For $J = 2$ to k do: $S = S + A[J]$
3. $S[1] = S$
4. For $I = 2$ to $n - k + 1$ do:
5. $\quad S = S - A[I-1] + A[I+k-1]$
6. $\quad S[I] = S$.
7. Return $S[1..n-k+1]$.

Consider the following loop invariant:

For $1 \leq j \leq n-k+1$, *after the iteration of the second loop when* $I = j$, $S = \sum_{i=j}^{i=j+k-1} A[i]$

Fill in the blanks of the proof that the loop invariant is correct and proves the correctness of the algorithm.

(a) (3 points) **Base Case:** At $j = 1$,before the loop starts, state why the loop invariant is true :

Inductive Hypothesis: Suppose that for some $1 \leq j \leq n-k$, after the iteration when $I = j$, $S = \sum_{i=j}^{i=j+k-1} A[i]$.

Inductive Step: Show that after the next iteration. $S = \sum_{i=j+1}^{i=j+k} A[i]$.

(b) Use the loop invariant to show that all values of the output array $S[I]$ are correct.

Scratch work only. This page will not be graded.

2 Recursive Algorithm

Consider the following *recursive* algorithm that solves the same problem as above, i.e. it takes as input a list of integers $(A[1], \ldots, A[n])$ and a $1 \leq k \leq n$ and returns the array $S[1..n-k+1]$ where $S[I] = \sum_{i=I}^{i=I+k-1} A[i]$.

procedure `WindowSumRec`$(A[1\ldots, n], k)$

if $n = 1$ return the array $(A[1])$.

if $n = k$:

    $S[1] = WindowSumRec(A[1..n-1], k-1)$

    Return the array $(S[1] + A[n])$.

$S[1..n-k] = WindowSumRec(A[1..n-1], k)$

$S[n-k+1] = S[n] - A[n-k-1] + A[n]$ **return** $S[1..n-k+1]$

(a) Write the recursion for the runtime $T(n)$ for this algorithm assuming arithmetic takes constant time.

(b) Solve the recursion in the previous part to give the order of the runtime $T(n)$ for this algorithm assuming arithmetic takes constant time.

3 Counting

(15 points total) A license plate consists of 7 characters (each character can be any of the 10 digits or the 26 letters of the alphabet.)
For each problem, you can leave your answer in terms of exponentials, factorials, or binomial coefficients. No justification necessary.

(a) If the first three characters must be numbers and the last four characters must be letters then how many different license plates are there?

(b) If the first three characters must be different numbers in strictly increasing order and the last four characters must be different letters in alphabetical order then how many different license plates are there?

(c) If all 7 characters must be digits in non-decreasing order, then how many different license plates are there?

(d) Suppose that all 7 characters must be letters. There is a very bad 4-letter word (all letters are distinct, let's say the bad word is "MATH") that you cannot have on the license plate. How many license plates avoid this 4-letter word as a consecutive substring?

(e) Suppose that any character can be a digit or a letter but each license plate must have at least one digit and at least one character.

Scratch work only. This page will not be graded.

4 Graphs

Suppose you are renting your apartment out for D days and you are using Airbnb to find people to rent it. Airbnb finds n people to rent it and gives you the start and end days of each person. You do not want your apartment to be empty on any day that you are gone. You cannot rent to two people at the same time and checkout is in the morning and checkin is in the afternoon so you could have a guest arrive on the same day that the previous guest left. Design a graph that models this problem and express what graph problem you must solve in order to decide if it is possible to rent your apartment with no vacant days.

(a) (2 points) Specify the vertices for your graph

(b) (2 points) Specify the edges for your graph

(c) (2 points) Draw your graph for the following example where you want to rent your apartment continuously from day 1 to day 10 and the renters are interested in the following intervals:

$$(1,5),(2,3),(8,10),(3,8),(1,2),(6,8),(4,10)$$

(d) (4 points) What graph problem would solve this problem and what graph algorithm would you use?

Scratch work only. This page will not be graded.

5 Trees and Recursive counting

(9 points) Let T_1 be the rooted tree consisting of a single vertex and let $T_2 = T_1$.
For $n \geq 3$, let T_n be the rooted tree whose left subtree is T_{n-2} and whose right subtree is T_{n-1}.
Give a recurrence for each of the following. No explanation needed.

(a) $E(n)$, the number of edges in T_n

Base case(s):

(b) $V(n)$, the number of vertices in T_n

Base case(s):

(c) $L(n)$, the number of leaves in T_n

Base case(s):

6 Expected Value

(10 points) Given a sequence of n integers: $(A[1], \ldots, A[n])$ such that each integer is in the range $(1, \ldots, n)$, a fixed point is any entry of the sequence where $A[i] = i$.

(a) If each entry of the sequence is selected independently from the range $(1, \ldots, n)$, compute the expected number of fixed points.

For example: $(2, 2, 3, 5, 1)$ then $A[2] = 2, A[3] = 3$ so this has 2 fixed points.

(b) A sequence of this sort is called a permutation if all values $(1, \ldots, n)$ each appear exactly once. If A is a permutation selected uniformly at random, compute the expected number of fixed points.

For example: $(4, 3, 2, 5, 1)$ has no fixed points.

Scratch work only. This page will not be graded.

7 Probability

(12 points) In the game of craps, you roll 2 fair six-sided dice.

(a) What is the probability that after rolling two dice, the sum of the two dice is 7 or 11?

(b) What is the probability that after rolling two dice, the sum of the two dice is 7 or eleven if you know that at least one of the dice is a six?

(c) What is the probability that after rolling two dice, the sum of the two dice is 7 or eleven if you know that at least one of the dice is greater than 3?

(d) Suppose you roll the pair of dice over and over until you get a 7 or a 10. What is the probability that you roll a 10 before you roll a 7?

Scratch work only. This page will not be graded.

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8 Multiple Choices

(18 points) No justification necessary, please write the *letter* clearly in the box to the right of the problem.

1. True or false: If $n \in \Omega(f(n))$ then $\sqrt{n} \in \Omega(f(n))$.

   (A) True (B) False

   1.

2. True or false: If $\lim_{n\to\infty} f(n)/g(n) = 1000$ then $f(n) \in O(g(n))$

   (A) True (B) False

   2.

3. True or false: $n^2 \log(n) \in \Theta(n \log(n^2))$.

   (A) True (B) False

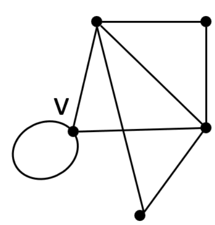
   3.

4. True or false: If $T(n) = T(n-1) + T(n-2) + O(1)$ and $T(1) = c$ then $T(n) \in O(2^n)$.

   (A) True (B) False

   4.

5. The degree of vertex **v** in the graph below is



   (A) 1 (B) 2 (C) 3 (D) 4 (E) None of the above

   5.

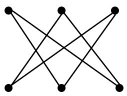
6. Suppose an undirected graph has eight vertices: two vertices of degree 3, three vertices of degree 4, one vertex of degree 2, and two isolated vertices. The number of edges in the graph is

   (A) 9 (B) 15 (C) 20 (D) 10 (E) 30

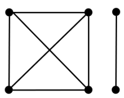
   6.

Scratch work only. This page will not be graded.

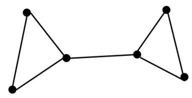
7. Which of the following graphs has an Eulerian circuit? (note: the vertices are the bold dark nodes, there are not necessarily vertices at the intersection of edges.



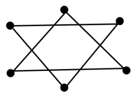
(A)



(B)



(C)



(D)

(E) None of the above

7.

8. The number of simple, undirected graphs with 3 (labelled) vertices is

$(A)\ 2^3$ $(B)\ 2^8$ $(C)\ 2^9$ $(D)\ 2^{27}$ (E) None of the above

8.

9. Consider the following algorithm

procedure *PrintSomething* (n: a power of 2)

```
for i = 1 to n
    j = n
    while j > 1
        print (i, j)
        j = j - 1
```

The runtime of this algorithm is

$(A)\ \Theta(\log n)$ $(B)\ \Theta(n)$ $(C)\ \Theta(n\log(n))$ $(D)\ \Theta(n^2)$ (E) None of the above

9.

10. Given a set of n^2 objects, under a **fixed-length** encoding scheme, what is the minimum number of bits required to encode each element in this set?

$(A)\ 2^n$ $(B)\ \binom{n}{2}$ $(C)\ 4^n$ $(D)\ \lceil 2\log(n)\rceil$ (E) none of the above

12.

11. Let T_n be the set of ternary strings (strings with 0, 1, and 2 bits) of length n that never have the same symbol appearing twice in a row. For example, 01212010 is such a string of length $n = 8$. Under a **fixed-length** encoding scheme, what is the minimum number of bits required to encode each element in T?

$(A)\ n$ $(B)\ n+1$ $(C)\ 3^n$ $(D)\ \lceil n\log(3)\rceil$ (E) none of the above

13.

(A) 0 $\quad (B)$ $\frac{1}{3}$ $\quad (C)$ $\frac{4}{9}$ $\quad (D)$ $\frac{2}{3}$ $\quad (E)$ 1 $\quad (F)$ $\frac{7}{6}$ $\quad (G)$ not enough information

14.

13. Suppose that A and B are events with probabilities $P(A) = 2/3$ and $P(B) = 2/3$. What is the smallest $P(A \cap B)$ can be?

(A) 0 $\quad (B)$ $\frac{1}{3}$ $\quad (C)$ $\frac{4}{9}$ $\quad (D)$ $\frac{2}{3}$ $\quad (E)$ 1 $\quad (F)$ $\frac{7}{6}$ $\quad (G)$ not enough information

15.

14. Suppose that events E and F are disjoint with $P(E) \neq 0$ and $P(F) \neq 0$. Which of the following can never be true:

(A) $P(E) = P(F)$ $\quad (B)$ $P(E) + P(F) < 1$ $\quad (C)$ $P(E|F) = P(E)$

(D) $P(E \cup F) = P(E) + P(F)$ $\quad (E)$ none of the above

16.

15. Suppose that events E and F are independent and $P(E) \neq 0$ and $P(F) \neq 0$. Then which of the following is false:

(A) $P(E|F) = P(F|E)$ $\quad (B)$ $P(E|F) = \frac{P(E \cap F)}{P(F)}$ $\quad (C)$ $P(E \cap F) = P(E)P(F)$

(D) $P(F|E) = P(F)$ $\quad (E)$ all are true

17.

16. Suppose that X and Y are random variables (not necessarily independent.) Which of the following is false?

(A) $E(X+Y) = E(X) + E(Y)$ $\quad (B)$ $V(X) = E(X^2) - E(X)^2$ $\quad (C)$ $\sigma(X) \geq V(X)$

(D) $E(XY) = E(X)E(Y)$ $\quad (E)$ more than one of the above

18.

END