CSE 20 Spring 2022
Homework 3 solution
Due date: Thursday, Apr 21, 2022 at 11:59pm

In this assignment,

You will consider how circuits and logic can be used to represent mathematical and technical claims. You will use propositional and predicate logic to evaluate these claims.

In this class, unless the instructions explicitly say otherwise, you are required to justify all your answers.

1. For each part of this question you will use the following input-output definition table with four inputs x_3, x_2, x_1, x_0

x_3	x_2	x_1	x_0	out
1	1	1	1	0
1	1	1	0	1
1	1	0	1	0
1	1	0	0	0
1	0	1	1	0
1	0	1	0	0
1	0	0	1	1
1	0	0	0	0
0	1	1	1	0
0	1	1	0	0
0	1	0	1	1
0	1	0	0	0
0	0	1	1	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

(a) (4 points) Construct a compound proposition that implements this input-output table.

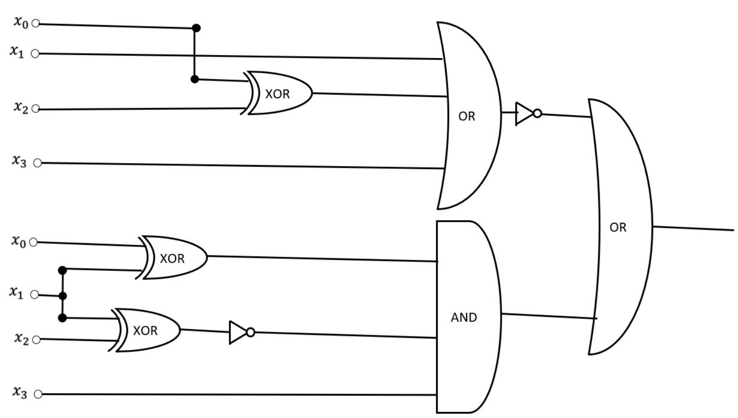
Solution:
In terms of DNF, a compound proposition that implements this truth table is:

$$(x_3 \wedge x_2 \wedge x_1 \wedge \neg x_0) \vee (x_3 \wedge \neg x_2 \wedge \neg x_1 \wedge x_0) \vee (\neg x_3 \wedge x_2 \wedge \neg x_1 \wedge x_0) \vee (\neg x_3 \wedge \neg x_2 \wedge \neg x_1 \wedge \neg x_0)$$

Or you could use a shortened version:

$$(x_3 \wedge \neg(x_2 \oplus x_1) \wedge (x_1 \oplus x_0)) \vee \neg(x_3 \vee x_1 \vee (x_2 \oplus x_0))$$

(b) (4 points) Draw a combinatorial circuit corresponding to this compound proposition.



(c) (9 points) For each of the following functions, consider whether this input-output table (and the logic circuit from part (a)) implements the rule for the function. If yes, explain why using the definitions of the operations involved and consider **all** possible inputs. if not, provide a specific example where the value of the function does not match the output of the definition table

i. $P_1 : \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \to \{0,1\}$ given by

$$P_1(\ (x_3, x_2, x_1, x_0)\ ) = \begin{cases} 1 & \text{when } x_3 + x_2 = x_1 + x_0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
This is incorrect because if $(x_3, x_2, x_1, x_0) = (1,1,1,0)$ then $out(1,1,1,0) = 1$ but $1 + 1 \neq 1 + 0$ so $P_1(1,1,1,0) = 0$.

ii. $P_2 : \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \to \{0,1\}$ given by

$$P_2(\ (x_3, x_2, x_1, x_0)\ ) = \begin{cases} 1 & \text{when } (x_3x_2x_1x_0)_{2,4} \textbf{ mod } 5 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
This is incorrect because if $(x_3, x_2, x_1, x_0) = (1,1,1,0)$ then $out(1,1,1,0) = 1$ but $(1110)_2$ **mod** $5 \neq 0$ so $P_2(1,1,1,0) = 0$.

iii. $P_3 : \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \to \{0,1\}$ given by

$$P_3(\ (x_3, x_2, x_1, x_0)\ ) = \begin{cases} 1 & \text{when } x_3 + x_2 = (x_1x_0)_{2,2} \\ 0 & \text{otherwise} \end{cases}$$

Solution:
This is correct.

- $P_3((1,1,1,1)) = 0$ because $1 + 1 \neq (11)_{2,2}$
- $P_3((1,1,1,0)) = 1$ because $1 + 1 = (10)_{2,2}$
- $P_3((1,1,0,1)) = 0$ because $1 + 1 \neq (01)_{2,2}$
- $P_3((1,1,0,0)) = 0$ because $1 + 1 \neq (00)_{2,2}$
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- $P_3((0,0,1,0)) = 0$ because $0 + 0 \neq (10)_{2,2}$
- $P_3((0,0,0,1)) = 0$ because $0 + 0 \neq (01)_{2,2}$
- $P_3((0,0,0,0)) = 1$ because $0 + 0 = (00)_{2,2}$

2. Suppose you want to create a digital 6-sided die that will display the values 1,2,3,4,5,6 as they look on a die.

The display is a 3 by 3 grid of cells.

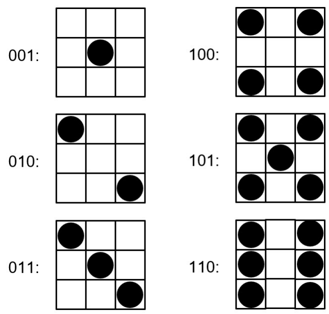
a	b	c
d	e	f
g	h	i

The input a 3-bit string: $(x_2 x_1 x_0)$.

The only allowable inputs are the numbers 1,2,3,4,5,6 in binary:

$$\{001, 010, 011, 100, 101, 110\}$$

For each input, certain combinations of cells will illuminate: (illuminate means that the cell will show a dot.)



(a) (7 points) Fill in the "truth table" for a circuit that displays the dice numbers for the allowable inputs.
(You do not have to fill in the rows for 000 and 111)

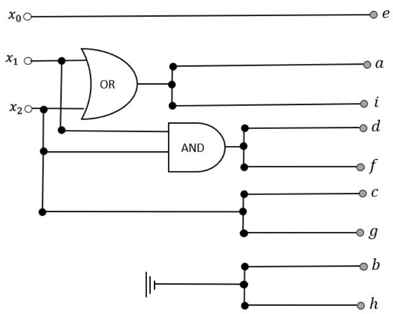
x_2	x_1	x_0	a	b	c	d	e	f	g	h	i
0	0	0									
0	0	1	0	0	0	0	1	0	0	0	0
0	1	0	1	0	0	0	0	0	0	0	1
0	1	1	1	0	0	0	1	0	0	0	1
1	0	0	1	0	1	0	0	0	1	0	1
1	0	1	1	0	1	0	1	0	1	0	1
1	1	0	1	0	1	1	0	1	1	0	1
1	1	1									

(b) (7 points) Draw the circuit. (Since 000 and 111 will never be inputs, these strings can map to any combination of illuminated cells.)

Solution:

- $a = x_2 \vee x_1$
- $b = F$

- $c = x_2$
- $d = x_2 \wedge x_1$
- $e = x_0$
- $f = x_2 \wedge x_1$
- $g = x_2$
- $h = F$
- $i = x_2 \vee x_1$



(you may rearrange your outputs circuits and duplicate your inputs to make your circuit neater. Your circuit may be a drawing.)

3. (a) Use the following rules to show that $((q \wedge r) \rightarrow p)$ is logically equivalent to $((\neg p \wedge q) \rightarrow \neg r)$:

$$
\begin{aligned}
((q \wedge r) \rightarrow p) &= \neg(q \wedge r) \vee p && \text{(using definition of implication)}\\
&= (\neg q \vee \neg r) \vee p && \text{(using DeMorgan's law)}\\
&= (p \vee \neg q) \vee r && \text{(associativity and commutativity of } \vee)\\
&= \neg(p \vee \neg q) \rightarrow \neg r && \text{(implication in terms of not and or)}\\
&= (\neg p \wedge q) \rightarrow \neg r && \text{(using DeMorgan's law)}\\
&= ((\neg p \wedge q) \rightarrow \neg r)
\end{aligned}
$$

(b) Use Truth Tables to show that $((q \wedge r) \rightarrow p)$ is logically equivalent to $((\neg p \wedge q) \rightarrow \neg r)$.

p	q	r	$q \wedge r$	$(q \wedge r) \rightarrow p$	$\neg p \wedge q$	$((\neg p \wedge q) \rightarrow \neg r)$
T	T	T	T	T	F	T
T	T	F	F	T	F	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	F
F	T	F	F	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

(c) Use the following rules to show that $q \rightarrow (\neg p \vee \neg r)$ is logically equivalent to $\neg((p \vee r) \wedge q)$:

$$
\begin{aligned}
q \rightarrow (\neg p \wedge \neg r) &= \neg(\neg p \wedge \neg r) \rightarrow \neg q && \text{(using contrapositive)}\\
&= (p \vee r) \rightarrow \neg q && \text{(using DeMorgan's law)}\\
&= \neg(p \vee r) \vee \neg q && \text{(implication in terms of not and or)}\\
&= \neg((p \vee r) \wedge q) && \text{(using DeMorgan's law)}\\
&= \neg((p \vee r) \wedge q)
\end{aligned}
$$

(d) Use Truth Tables to show that $q \to (\neg p \wedge \neg r)$ is logically equivalent to $\neg((p \vee r) \wedge q)$.

p	q	r	$(\neg p \wedge \neg r)$	$q \to (\neg p \wedge \neg r)$	$(p \vee r)$	$((p \vee r) \wedge q)$	$\neg((p \vee r) \wedge q)$
T	T	T	F	F	T	T	F
T	T	F	F	F	T	T	F
T	F	T	F	T	T	F	T
T	F	F	F	T	T	F	T
F	T	T	F	F	T	T	F
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	T	T	F	F	T

4. (12 points) A cat, a dog, a frog and a pig are all accused of robbing the bank.

The police have deduced the following:

- If the pig was involved in the robbery then so were the cat and the dog.
- Either the dog is innocent (not involved) or the frog is innocent (or both are innocent.)
- If the cat is innocent then either the frog is guilty or the pig is guilty (or both.)
- The frog is guilty if the pig is innocent.

(a) Determine if the police

- Have enough information to be certain who was involved in the robbery and who was not involved in the robbery.
- Do not have enough information to be certain who was involved in the robbery and who was not involved in the robbery.
- Have conflicting information (in other words, the statements are not consistent.)

(To show your work, either construct a truth table or use logical equivalences to justify your answer.)

Solution:

Define the propositional variables c, d, f, p as

- c means "the cat is guilty".
- d means "the dog is guilty".
- f means "the frog is guilty".
- p means "the pig is guilty".

Translating each of the statements:

- If the pig was involved in the robbery then so were the cat and the dog.

$$p \to (c \wedge d)$$

- Either the dog is innocent (not involved) or the frog is innocent (or both are innocent.)

$$\neg d \vee \neg f$$

- If the cat is innocent then either the frog is guilty or the pig is guilty (or both.)

$$\neg c \to (f \vee p)$$

- The frog is guilty if the pig is innocent.

$$\neg p \to f$$

c	d	f	p	$p \to (c \wedge d)$	$\neg d \vee \neg f$	$\neg c \to (f \vee p)$	$\neg p \to f$	
T	T	T	T	T	F	T	T	
T	T	T	F	T	F	T	T	
T	T	F	T	T	T	T	T	
T	T	F	F	T	T	T	F	
T	F	T	T	F	T	T	T	
T	F	T	F	T	T	T	T	
T	F	F	T	F	T	T	T	
T	F	F	F	T	T	T	F	
F	T	T	T	F	F	T	T	
F	T	T	F	T	F	T	T	
F	T	F	T	F	T	T	T	
F	T	F	F	T	T	F	F	
F	F	T	T	F	T	T	T	
F	F	T	F	T	T	T	T	
F	F	F	T	F	T	T	T	
F	F	F	F	T	T	F	F	

According to the statements, we know it could be 1 of 3 different possibilities highlighted in grey. We do not have enough information to be certain who was involved. The information is not conflicting.

(b) Now, suppose that the cat has an alibi (and can prove that she is not involved in the robbery.) (Same as before:)
With this new information, determine if the police

- Have enough information to be certain who was involved in the robbery and who was not involved in the robbery.
- Do not have enough information to be certain who was involved in the robbery and who was not involved in the robbery.
- Have conflicting information (in other words, the statements are not consistent.)

(To show your work, either construct a truth table or use logical equivalences to justify your answer.)

Solution:
The new information is that the cat is not involved so $\neg c$. Adding that to the truth table, we get:

c	d	f	p	$p \to (c \wedge d)$	$\neg d \vee \neg f$	$\neg c \to (f \vee p)$	$\neg p \to f$	$\neg c$
T	T	T	T	T	F	T	T	F
T	T	T	F	T	F	T	T	F
T	T	F	T	T	T	T	T	F
T	T	F	F	T	T	T	F	F
T	F	T	T	F	T	T	T	F
T	F	T	F	T	T	T	T	F
T	F	F	T	F	T	T	T	F
T	F	F	F	T	T	T	F	F
F	T	T	T	F	F	T	T	T
F	T	T	F	T	F	T	T	T
F	T	F	T	F	T	T	T	T
F	T	F	F	T	T	F	F	T
F	F	T	T	F	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	T	F	T	T	T	T
F	F	F	F	T	T	F	F	T

Now that we know the cat is not involved, with this extra information, there is only one row of all Trues. This means that the police can be certain that the frog is the only guilty animal.