The following integration formulas will be provided on the exam and you may apply any of these results directly without showing work/proof:

$$\int (ax+b)^n \; dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + C$$

$$\int e^{ax+b} \; dx = \frac{1}{a} \cdot e^{ax+b} + C$$

$$\int u^{ax+b} \; dx = \frac{1}{a \ln u} \cdot u^{ax+b} + C \text{ for } u > 0, u \neq 1$$

$$\int \frac{1}{ax+b} \; dx = \frac{1}{a} \cdot \ln|ax+b| + C$$

$$\int \sin(ax+b) \; dx = -\frac{1}{a} \cdot \cos(ax+b) + C$$

$$\int \cos(ax+b) \; dx = \frac{1}{a} \cdot \sin(ax+b) + C$$

$$\int \sec^2(ax+b) \; dx = \frac{1}{a} \cdot \tan(ax+b) + C$$

$$\int \csc^2(ax+b) \; dx = \frac{1}{a} \cdot \cot(ax+b) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1}(x) + C$$

$$\int \frac{1}{1+x^2} \; dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \; dx = \sinh^{-1}(x) + C$$

The following formulas for Laplace transform will also be provided

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$	e^{at}	$\frac{1}{s-a}, \quad s > a$
$tf(t)$	$-F'(s)$	$e^{at}f(t)$	$F(s-a)$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$\sin(wt)$	$\frac{w}{s^2+w^2}, \quad s > 0$	$e^{at}\sin(wt)$	$\frac{w}{(s-a)^2+w^2}, \quad s > a$
$\cos(wt)$	$\frac{s}{s^2+w^2}, \quad s > 0$	$e^{at}\cos(wt)$	$\frac{s-a}{(s-a)^2+w^2}, \quad s > a$
$u(t-a), \ a \geq 0$	$\frac{e^{-as}}{s}, \quad s > 0$	$\delta(t-a), \ a \geq 0$	$e^{-as}, \quad s > 0$
$u(t-a)f(t)$	$e^{-as}\mathcal{L}\{f(t+a)\}$	$\delta(t-a)f(t)$	$f(a)e^{-as}$

Here, $u(t)$ is the unit step (Heaviside) function and $\delta(t)$ is the Dirac Delta function

Laplace transform of derivatives:

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\
\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\
\mathcal{L}\{f^{(n)}(t)\} &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)
\end{aligned}$$

Convolution: $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{(g * f)(t)\}$

1 First Order Linear Differential Equation

Differential equations of the form

$$\frac{dy}{dt} + p(t)\ y = g(t) \tag{1}$$

where t is the independent variable and $y(t)$ is the unknown function.

Solution steps:

0. Warning: If you are given an equation in the form $a(t)y' + b(t)y = c(t)$, where $a(t) \neq 0$, remember to transform it into the form given in (1) before doing the steps below.

1. Find the integrating factor $\mu(t)$ given by

$$\mu(t) = \exp\left(\int p(t)dt\right).$$

   Under the right integrating factor, we should have $\mu'(t) = \mu(t)p(t)$.

2. Multiplying $\mu(t)$ to both sides of (1) to obtain a product rule on the left-hand side, as follows.

$$\begin{aligned}\mu(t)\frac{dy}{dt} + \mu(t)p(t)\ y = \mu(t)g(t) &\Rightarrow \mu(t)\frac{dy}{dt} + \mu'(t)y = \mu(t)g(t)\\ &\Rightarrow (y \cdot \mu(t))' = \mu(t)g(t).\end{aligned}$$

   Again, one should be able to observe the Product Rule for derivative during this step is using the correct integrating factor.

3. Integrate both side of the result above with respect to (w.r.t) t to obtain the solution.

$$y \cdot \mu(t) = \int \mu(t)g(t)dt \Rightarrow y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$$

   where C is a constant.

4. Apply the initial condition (if any) to solve for the constant C.

2 Separable Differential Equation

Differential equations of the form

$$\frac{dy}{dt} = g(t)h(y). \tag{2}$$

Solution steps:

1. Solve $h(y) = 0$ for any constant solution.

2. Assuming that $h(y) \neq 0$, transform (2) into the following

$$\frac{dy}{h(y)} = g(t)dt.$$

3. Integrate both side (w.r.t y on the left and to t on the right) and solve for y in terms of t. Remember any constant that may resulted from the integration!

4. If given an initial condition, use this to solve for the constants from step 3.

2.1 Homogeneous Differential Equation

This is not entirely a new kind of differential equation and can be transformed into a separable one. Let x be the independent variable and $y(x)$ be the unknown function. Homogeneous differential equations are of the form

$$\frac{dy}{dx} = f(x,y) \tag{3}$$

where $f(x,y)$ is homogeneous. That is, $f(tx,ty) = f(x,y)$ for any number t. An alternative definition of homogeneous is that the function $f(x,y)$, after some transformations, should depend only on the ratio y/x.

Solution steps:

1. Let $v = \dfrac{y}{x}$ so that $y = x \cdot v$. Differentiate (implicitly) both sides w.r.t x to transform the left-hand side of (3) into

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

   Notice that both y and v here are functions of x.

2. For the right-hand side of (3), apply the substitution $v = y/x$ to obtain $f(1, y/x) = f(1, v) = g(v)$.

3. Solve the equation

$$v + x\frac{dv}{dx} = g(v)$$

   using separable techniques for v. Then go back to y by replacing v with y/x.

4. Use initial conditions to solve for the constants.

2.2 Autonomous Differential Equation & Equilibrium Solutions

Autonomous equations mostly can be solved using separable techniques. They are differential equations of the form

$$\frac{dy}{dt} = f(y)$$

where the right-hand side depends only on y.

If y_0 is a solution of $f(y) = 0$ then the constant solution $y \equiv y_0$ is the *equilibrium solution.*

We can easily classify the behavior of equilibrium solutions by simply looking at the direction field. However, if the direction field is unavailable, then we need to consider the sign of $f(y)$ for y "near" y_0. Specifically, we can classify the equilibrium into three types as follows:

- If $f(y) < 0$ for $y < y_0$ and $f(y) > 0$ for $y > y_0$ then y_0 is an **unstable** equilibrium or a source.
- If $f(y) > 0$ for $y < y_0$ and $f(y) < 0$ for $y > y_0$ then y_0 is an **asymptotically stable** equilibrium or a sink.
- If the sign of $f(y)$ is **unchanged as we pass through** y_0 then y_0 is a **semi-stable** equilibrium or a node.

3 Exact Differential Equation

Differential equations of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \text{ or equivalently } M(x,y)dx + N(x,y)dy = 0$$

where

$$M_y(x,y) := \frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial y}(x,x) =: N_x(x,y).$$

Solution steps:

0. Check that $M_y(x,y) = N_x(x,y)$ to confirm that we indeed have an exact equation. If not, multiply the integrating factor to both sides to obtain an exact differential equation. (Note: obtaining the integrating factor for exact equation will not be asked on this exam.)

1. Compute the function $F(x,y)$ using the following two formulas

$$F(x,y) = \int M(x,y)dx = F_M(x,y) + c_1(y),$$
$$F(x,y) = \int N(x,y)dy = F_N(x,y) + c_2(x).$$

   Remark: the constant term coming from the above integrations is not entirely a constant, but a function depending on the remaining variable.

2. Set $F_M(x,y) + c_1(y) = F_N(x,y) + c_2(x)$ and solve for $F(x,y)$. The general solution will be given by

$$F(x,y) = C$$

   for some constant C.

3. Use the initial conditions to solve for C, if needed.

Note: the textbook actually presents another method to solve exact equations. This is equivalent to ours so you may use either methods.

4 Second Order Linear Differential Equation

We consider the homogeneous second order linear differential equation with constant coefficients given by

$$ay'' + by' + cy = 0 \tag{4}$$

where a, b, and c are constants. The characteristic equation of (4) is given by

$$ar^2 + br + c = 0. \tag{5}$$

There are three possibilities:

- If (5) has two distinct real solutions r_1 and r_2 then the general solution of (4) is given by

  $$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  for any constants c_1 and c_2.

- If (5) has repeated root $r_1 = r_2 = r$ then the general solution of (4) is given by

  $$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

  for any constants c_1 and c_2.

- If (5) has complex roots $r_{1,2} = u \pm iv$ then the general solution of (4) is given by

  $$y(t) = c_1 e^{ut} \cos(vt) + c_2 e^{ut} \sin(vt)$$

  for any constants c_1 and c_2.

4.1 Principle of Superposition for homogeneous second order linear ODE

In general, given the second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0. \tag{6}$$

If $y_1(t)$ and $y_2(t)$ are two solution of (6) such that the Wronskian determinant

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0$$

for some value of $t \in I$ then we say that $y_1(t)$ and $y_2(t)$ are linearly independent on I and they form a *fundamental set of solutions*. In this case, any linear combination of $y_1(t)$ and $y_2(t)$ will be a solution to (6). That is, the general solution of (6) is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for any c_1 and c_2.

5 Solving non-homogeneous ODEs

Consider the non-homogeneous second-order linear ODE with variable-coefficients:

$$y''(t) + p(t)y'(t) + q(t)y = g(t). \tag{7}$$

According to the **Superposition Principle**, the solution to this ODE is given by

$$y(t) = y_H(t) + y_P(y)$$

where

- y_H is the solution to the corresponding homogeneous part $y''(t) + p(t)y'(t) + q(t)y = g(t)$
- y_P is any particular solution to the given non-homogenous equation.

That is,

General solution to the non-hom. equation/system	=	General solution to the corresponding homogeneous part	+	One particular solution to the given non-hom. equation/system

5.1 Method of Undetermined Coefficients

In general, it is not easy to find the particular solution y_P. However, if the homogeneous part has constant coefficients and depending on the choice of the right-hand side $g(t)$, we may be able to "guess" the general form/type of y_P. Below are several cases for $g(t)$ that we can apply the method to obtain a "first guess" for y_P.

$g(t)$	First guess for y_P
ke^{rt}	Ae^{rt}
$k\cos(\omega t)$ or $k\sin(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
$P_n(t)$	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$
$P_n(t)e^{rt}$	$\left(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0\right)e^{rt}$
$P_n(t)e^{rt}\cos(\omega t)$ or $P_n(t)e^{rt}\sin(\omega t)$	$\left(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0\right)e^{rt}\cos(\omega t) + \left(B_n t^n + B_{n-1}t^{n-1} + \cdots + B_1 t + B_0\right)e^{rt}\sin(\omega t)$

Here, $P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ represents an n-th degree polynomial.

If any term in the first guess has been included in the solution to the corresponding homogeneous equation, we need to **multiply the entire guess** by t. If there is still repeated term in the second guess, multiply by t again and keep doing this until there is no repeated term.

Finally, substitute the general form of y_P into (7) and find solve for the coefficients.

Remark: This is quite a limited technique, applicable to only the case where the LHS has **constant coefficients** and the RHS contains polynomials; exponentials; sines and/or cosines; and sum and product of the three. A more general technique is represented in the next part.

5.2 Variation of Parameters

If we can obtain the general solution to the corresponding homogeneous part $y''(t)+p(t)y'(t)+q(t)y=0$ (i.e. having two independent solutions y_1, y_2 to the homogeneous part), then we can use **variation of parameters** to find a particular solution y_P of the form

$$y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = \int \frac{-g(t)y_2(t)}{W_{[y_1,y_2]}(t)}\, dt \quad \text{and} \quad u_2 = \int \frac{g(t)y_1(t)}{W_{[y_1,y_2]}(t)}\, dt.$$

Here, $W_{[y_1,y_2](t)}$ is the Wronskian of y_1, y_2, defined earlier.

Remark: In general, we should not expect to always be able to find two linearly independent solutions to the corresponding homogeneous equation, as this problem is intractable. However, there are two scenarios that we can manage.

5.3 When the coefficients have special forms – Cauchy-Euler equations

Cauchy-Euler equations have the form

$$at^2y'' + bty' + cy = f(t) \quad \text{or equivalently} \quad ay'' + \frac{b}{t}y' + \frac{c}{t^2}y = g(t)$$

where a, b, c are constants and $t \neq 0$.

The solutions to the corresponding homogeneous equation can be found in the form t^r. The homogeneous Cauchy-Euler equation $at^2y'' + bty' + cy = 0$ has the characteristic equation

$$ar^2 + (b-a)r + c = 0 \tag{8}$$

There are three cases:

1. If (8) has two distinct, real-valued solutions r_1, r_2. Then the general solution is given by
$$y = C_1t^{r_1} + C_2t^{r_2}$$
2. If (8) has complex solutions $r_{1,2} = \alpha \pm i\beta$. Then the general solution is given by
$$y = C_1t^{\alpha}\cos(\beta \ln|t|) + C_2t^{\alpha}\sin(\beta \ln|t|)$$
3. If (8) has a repeated/unique solution r then the general solution is given by
$$y = C_1t^r + C_2t^r \ln|t|$$

5.4 When we have one solution to the hom. part – Reduction of Order

Consider the ODE

$$y''(t) + p(t)y'(t) + q(t)y = g(t)$$

and suppose that we are given one solution y_1 (not identically zero) to the corresponding homogeneous part $y''(t)+p(t)y'(t)+q(t)y=0$. We can obtain a second solution y_2 for the homogeneous part (independent from y_1) of the form

$$y_2(t) = v(t)y_1(t),$$

where

$$v(t) = \int \left[\frac{1}{y_1^2} \cdot \exp\left(-\int p(t)dt\right)\right] dt$$

6 Solving 2×2 homogeneous system $\mathbf{x}' = A\mathbf{x}$

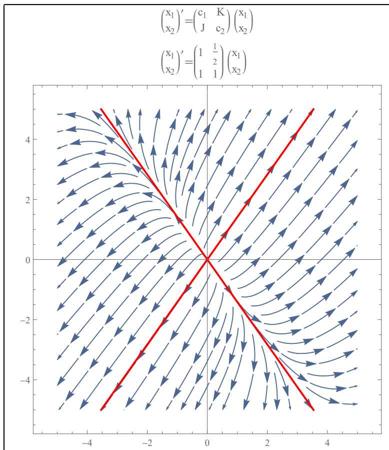
Here is a brief summary of different types of phase portrait for the linear system $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix. We assume that λ_1 and λ_2 are the eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. In all cases, the origin $(0, 0)$ is an equilibrium. As $t \to \infty$, the solution either converges to $(0, 0)$ (in the case of a stable equilibrium) or diverge to infinity (in the case of unstable equilibrium).

6.1 Distinct, real eigenvalues

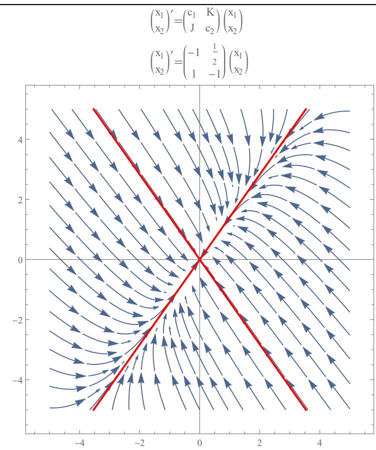
If λ_1 and λ_2 are distinct, real eigenvalues then the general solution is given by

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

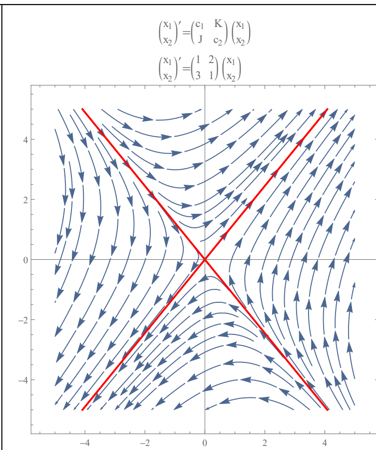
for some constants c_1, c_2. The phase portrait can be one of the following cases below. Here, the red lines are the eigenvectors.



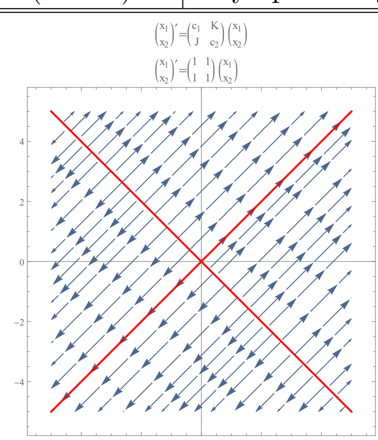
$0 < \lambda_1 < \lambda_2$
Unstable node (source)



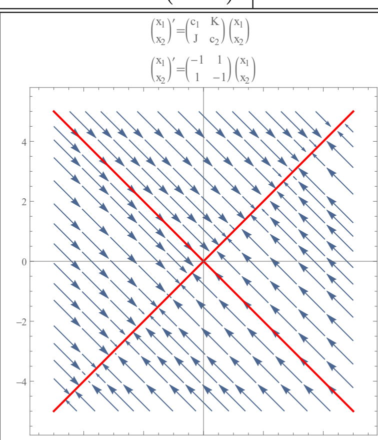
$\lambda_1 < \lambda_2 < 0$
Asymptotically stable node (sink)



$\lambda_1 < 0 < \lambda_2$
Saddle point



$\lambda_1 = 0 < \lambda_2$
Lines



$\lambda_1 < 0 = \lambda_2$
Lines

We call the eigenvector corresponding to the eigenvalue with **greater absolute value** the **fast direction**. The other eigenvector is called the **slow direction**.

- In the case the origin is a sink, the solution path first follows the fast direction, then once t is large enough, the solution then follows the slow direction toward the origin.
- In the case the origin is a source, the solution path first follows the slow direction, then once t is large enough, it then follows the fast direction to diverge to infinity.

The special case occurs when one of the eigenvalues is zero. In this case, the phase plane consists of straight lines/arrows that run parallel to the eigenvector of the non-zero eigenvalue, and pointing toward the direction of the zero eigenvalue. The vector direction depends on the sign of the non-zero eigenvalue.

6.2 Complex eigenvalues

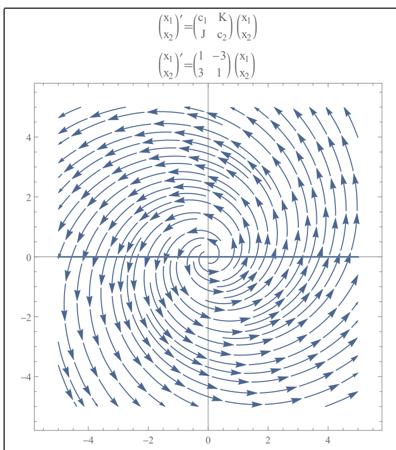
If $\lambda_{1,2} = a \pm ib$ are complex eigenvalues then the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ will occur as conjugate pairs. That is, we can decompose the eigenvectors as $\mathbf{v}_{1,2} = \mathbf{a} \pm i\mathbf{b}$. In this case, the two linearly independent solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ are given by

$$\mathbf{x}_1 = e^{at}\cos(bt)\mathbf{a} - e^{at}\sin(bt)\mathbf{b}$$
$$\mathbf{x}_2 = e^{at}\sin(bt)\mathbf{a} + e^{at}\cos(bt)\mathbf{b}$$

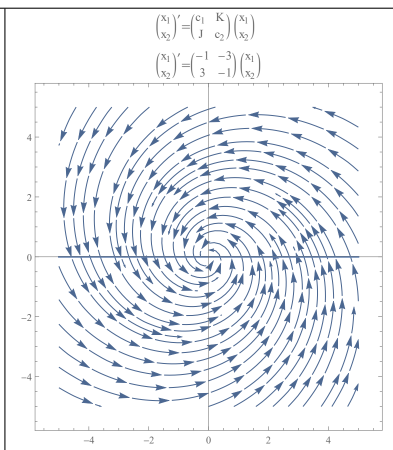
Thus, the general solution is given by

$$\mathbf{x} = C_1 e^{at}\left(\cos(bt)\mathbf{a} - \sin(bt)\mathbf{b}\right) + C_2 e^{at}\left(\sin(bt)\mathbf{a} + \cos(bt)\mathbf{b}\right)$$

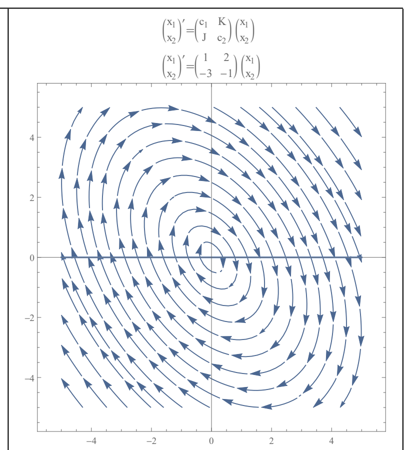
The phase portrait depends on the sign of a, the real part of the eigenvalues, and it can be one of the following cases:



$a > 0$
Unstable spiral (source)



$a < 0$
Stable spiral (sink)



$a = 0$
Circle

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the **orientation** depends on the sign of c. If $c > 0$ then the rotation is counter-clockwise (see the pictures above), if $c < 0$ then the rotation is clockwise. The case $c = 0$ can never occur when A has complex eigenvalues.

6.3 Repeated eigenvalues

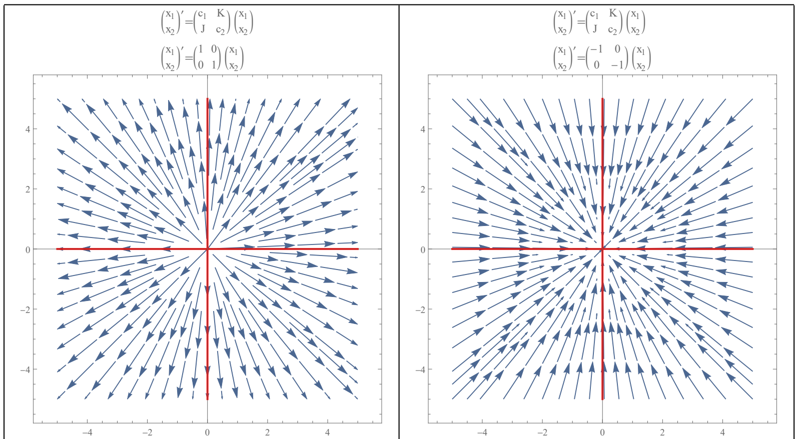
If $\lambda_1 = \lambda_2 = \lambda$ is the only eigenvalue then we now have two possibilities.

Two independent eigenvectors.

This sub-case only occurs when the matrix A is some scalar multiple of the identity matrix. In this sub-case, if λ has two independent eigenvectors then the general solution is given by

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 \mathbf{v}_2 e^{\lambda t}$$

for some constants c_1, c_2. The equilibrium is call a *proper node* (or star) and its stability depends on the sign of the (only) eigenvalue. There are two cases:



$a > 0$ Unstable star point	$a < 0$ Stable star point

One independent eigenvector.

In this case, if λ has only one independent eigenvector $\mathbf{v}$ then we know that one of the solution is given by $\mathbf{x}^{(1)} = \mathbf{v}e^{\lambda t}$. We now look for a second solution of the form

$$\mathbf{x}^{(2)} = \mathbf{v}te^{\lambda t} + \mathbf{u}e^{\lambda t}$$

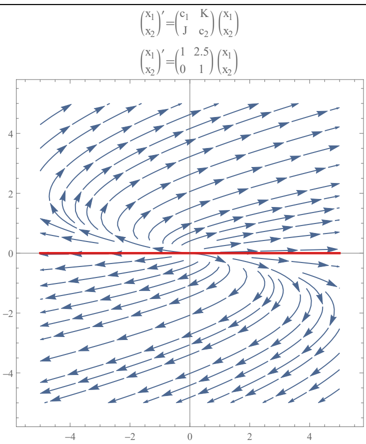
where $\mathbf{u}$ is obtained by solving the matrix-vector equation

$$(A - \lambda I)\mathbf{u} = \mathbf{v}.$$

The general solution is given by

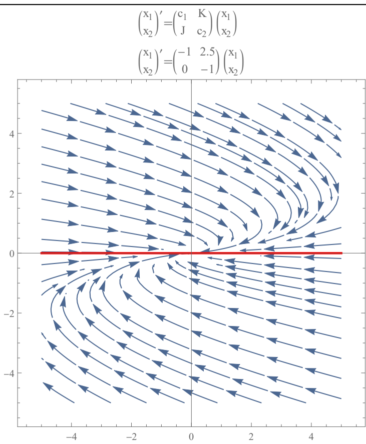
$$\mathbf{x} = c_1 \mathbf{v} e^{\lambda t} + c_2 \left(\mathbf{v} t e^{\lambda t} + \mathbf{u} e^{\lambda t} \right)$$

for some constants c_1, c_2. The equilibrium is called a *improper/degenerate node*. It can be seen as a transition from a (proper) node (in Case 1) into a spiral (in Case 2). Its stability depends on the sign of the (only) eigenvalue. There are three cases:



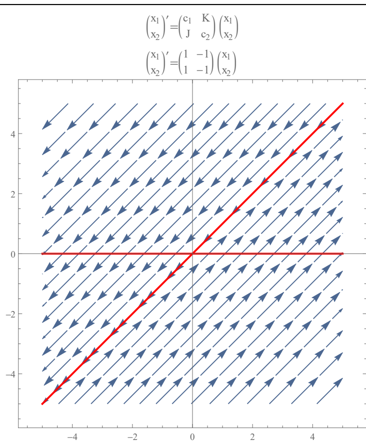
$\lambda > 0$

Unstable improper node



$\lambda < 0$

Stable improper node



$\lambda = 0$

Lines

7 Solving non-homogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$

A non-homogeneous linear system has the form $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$, where $\mathbf{g} \neq \mathbf{0}$. The general solution to this equation is given by

$$\mathbf{x} = \mathbf{x}_H + \mathbf{x}_P$$

where

- $\mathbf{x}_H$ is the general solution to the corresponding homogeneous equation $\mathbf{x}' = A\mathbf{x}$ and
- $\mathbf{x}_P$ is a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$

Let A be an $n \times n$ matrix and let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be n linearly independent solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$. Then the **fundamental matrix** for the system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\Psi(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{bmatrix}.$$

The general solution of $\mathbf{x}' = A\mathbf{x}$ can be expressed as

$$\mathbf{x}(t) = C_1\mathbf{x}_1 + \cdots + C_n\mathbf{x}_n = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} = \Psi(t) \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

For the method of **Variation of Parameters**, we shall look for a particular solution of the form

$$\mathbf{x}_P = \Psi(t)\mathbf{u} \quad \text{where} \quad \Psi(t)\mathbf{u}' = \mathbf{g}.$$

Thus, we can find the vector **u'** by multiplying the inverse of Ψ to the left of $\mathbf{g}$. That is, **u'** $= \Psi^{-1}\mathbf{g}$. Finally, simply integrate our answer for **u'** to obtain $\mathbf{u}$.

Notice that in the case A is a 2×2 then so is $\Psi(t)$. The inverse matrix of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

8 Lapace Transform

Let $f(t)$ be a function defined for $t \geq 0$. The **Laplace transform** of $f(t)$ denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$ is given by

$$\mathcal{L}\{f(t)\} := \int_0^\infty e^{st} f(t) dt,$$

provided that the improper integral exists.

Remark: Laplace transform is an operation that transform a function of $t \in [0, \infty)$ (i.e. the time domain) into a function s (i.e. the frequency domain).

Fact: Let $k, a > 0$. If f is piecewise continuous and $|f(t)| \leq ke^{at}$ for t "big enough" then the Laplace transform $\mathcal{L}\{f(t)\}$ exists for all $s > a$. In this case, the function f is said to be **of exponential order** as its growth rate is no faster than the growth rate of exponential functions.

8.1 Properties of Laplace Transform:

1. Laplace transform is a **linear operator**: for any functions f, g and scalar c:

   $$\mathcal{L}\{(f+g)(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \quad \text{and} \quad \mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}$$

   provided that both transformation $\mathcal{L}\{f(t)\}, \mathcal{L}\{g(t)\}$ exist.

2. Derivative of Laplace transform:

   $$\mathcal{L}\{tf(t)\} = -F'(s), \quad \text{or equivalently, } \mathcal{L}\{-tf(t)\} = F'(s).$$

   In general,

   $$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad \text{or equivalently, } \mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s).$$

   Here, $F^{(n)}(s)$ is the n-th derivative of $F(s)$.

3. Laplace transform of derivatives: Suppose f is of exponential order (so that its Laplace transform exists). Further suppose that f is continuous and f' is piecewise continuous on $[0, +\infty)$. Then

   $$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

   As a consequence of this result, we also have the following results:

   $$\begin{aligned}
   \mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\
   \mathcal{L}\{f'''(t)\} &= s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0) \\
   &\vdots \\
   \mathcal{L}\{f^{(n)}(t)\} &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)
   \end{aligned}$$

4. Under Laplace transform, translation/shift is obtained by multiplication with exponential functions. That is, if $\mathcal{L}\{f(t)\} = F(s)$ then

   $$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

   Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$ then

   $$e^{at} f(t) = \mathcal{L}^{-1}\{F(s-a)\}.$$

5. Laplace transform is **not multiplicative**. That is, in general,

$$\mathcal{L}\{f(t) \cdot g(t)\} \neq \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}.$$

Let f, g be functions whose Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$. The **convolution** of f and g, denote $f * g$, is defined by

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau$$

Note: $(f * g)(t) = \displaystyle\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau = (g * f)(t)$.
Under Laplace transform: $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{(g * f)(t)\}$.

8.2 Solving Initial Value Problems using Laplace Transform

Remark: Properties (2) and (3) from the above section show that Laplace transform turns differentiation with respect to t into multiplication by s and differentiation with respect to s into multiplication by $-t$. Therefore, applying Laplace transform to an ODE will turn it into an algebraic equation (that is easier to solve).

Solution steps:

1. Take the Laplace transform of both sides of the given ODE whose unknown is $y(t)$.
2. Simplify the resultant algebraic equation and solve for $\mathcal{L}\{y\} = Y(s)$ in terms of s.
3. Find the inverse transform of $Y(s)$. This inverse transform $y(t)$ is the required solution to the given ODE.

This method solves the IVP **directly without going through the general solution**. It can also be applied to solve linear ODE of any order, and can work with both homogeneous and non-homogeneous cases.

8.3 Unit Step Functions

The **Heaviside step function** or **unit step function**, denoted $u(t)$, is a discontinuous function defined by

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Let $a \geq 0$. An alternative definition for unit step function is given by

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

The Laplace transform of the unit step function is given by

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \quad \text{and} \quad \mathcal{L}\{u_a(t)\} = \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

for $s > 0, a > 0$.

Unit step functions can be use as an "on-off switch" to control the behavior of a given function $f(t)$ over certain time intervals. Specifically, we have the following results:

$$u_a(t)f(t) = \begin{cases} 0, & t < a \\ f(t), & t \geq a \end{cases} \qquad \text{and} \qquad (1 - u_a(t))f(t) = \begin{cases} f(t), & t < a \\ 0, & t \geq a \end{cases}$$

$$(u_a(t) - u_b(t))\, f(t) = \begin{cases} 0, & t < a \\ f(t) & a \leq t < b \\ 0, & b \leq t \end{cases}$$

Here, $u_a(t) - u_b(t)$ is called the *window function.*

In general, if $g(t)$ is a piecewise function given by

$$g(t) = \begin{cases} f_1(t), & t < t_1 \\ f_2(t), & t_1 \leq t < t_2 \\ f_3(t), & t_2 \leq t < t_3 \\ \vdots & \\ f_n(t), & t_{n-1} \leq t \end{cases}$$

Then g can be expressed in terms of unit step functions as

$$g(t) = (1 - u_{t_1}(t))f_1(t) + (u_{t_1}(t) - u_{t_2}(t))f_2(t) + (u_{t_2}(t) - u_{t_3}(t))f_3(t) + \cdots + u_{t_{n-1}}f_n(t)$$

8.4 Unit Impulse Function

The **Dirac Delta function** or **unit impulse function**, denoted $\delta(t)$, is defined by two properties:

$$\delta(t) = 0, \text{ if } t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

Dirac Delta function can sometimes be seen as having ∞ value at $t = 0$. In addition, the impulse can occur at an arbitrary time $t = a$. In this case, we have

$$\delta(t-a) = 0, \text{ if } t \neq a \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a)dt = 1.$$

Properties of Dirac Delta function:

1. For $\epsilon > 0$, $\displaystyle\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)dt = 1$

2. For $\epsilon > 0$, and for any continuos function $f(t)$, $\displaystyle\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)f(t)dt = f(a)$

3. Laplace transforms for Dirac Delta functions:

$$\mathcal{L}\{\delta(t)\} = 1, \quad \mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad \text{and} \quad \mathcal{L}\{\delta(t-a)f(t)\} = f(a)e^{-as}, \text{ for } a > 0.$$