CSE 20 Spring 2022
Midterm Practice Problem Solutions

This practice exam may help you study for the midterm on Thursday, May 5. I recommend that you also review each HW assignment (by looking at the feedback you received as well as solutions posted on Piazza), examples from class and the Canvas review quizzes (including the solutions posted on Piazza).

1. Number systems and base expansion

(a) Compute the ternary (base 3) expansion of 38.

We can express 38 in terms of powers of 3 as

$$38 = (1 \times 27) + (1 \times 9) + 2 = 1 \times 3^3 + 1 \times 3^2 + 0 \times 3^1 + 2 \times 3^0 = (1102)_3.$$

Therefore, by definition of base 3 expansion, $38 = (1102)_3$.

(b) Convert 41/16 to binary by using the binary point (the binary version of a decimal point.) Compute 4 bits after the binary point.

We can simplify the fraction above to a decimal value of 2.5625. We can break it down like so, 2+0.5+0.0625 = 2.5625. Then we convert to binary:

$$2 = (1 \times 2^1 + 0 \times 2^0) = (10)_2$$

for the binary point it's as follows:

$$\begin{gathered} 0.5 = (1 \times 2^{-1}) \\ 0.25 = (0{\times}2^{-2}) \\ 0.125 = (0{\times}2^{-3}) \\ 0.0625 = (1{\times}2^{-4}) \end{gathered}$$

So then putting it all together we have $41/16 = (10.1001)_2$

(c) Compute the product of $(6A)_{16}$ and $(11)_{16}$, without converting either number to another base.

We use the usual algorithm for multiplication, except using hexadecimal symbols.

$$\begin{array}{r} 6A \\ \underline{\cdot\ 11} \\ 6A \\ \underline{+6\ A0} \\ 70A \end{array}$$

Thus, the product of $(6A)_{16}$ and $(11)_{16}$ is $(70A)_{16}$.

(d) Confirm your answer for part (b) by converting $(6A)_{16}$ and $(11)_{16}$ to decimal, multiplying them, and converting the product back to base 16.

$(6A)_{16} = 6 * 16 + 10 = 106$
$(11)_{16} = 1 * 16 + 1 = 17$
$106 * 17 = 1802$
$n = 1802$, $b = 16$,

q	k	a_k
1802	0	1802 **mod** 16 = 10
1802 **div** 16 = 112	1	112 **mod** 16 = 0
112 **div** 16 = 7	2	7 **mod** 16 = 7
7 **div** 16 = 0		

So $1802 = (70A)_{16}$

(e) How many bits will there be in the binary (base 2) expansion of 2020?

$$\lfloor \log_2(2020) + 1 \rfloor = 11$$

(f) Convert the straight binary number: $(110011001100)_2$ into BCD.

This number is 3276 in base 10.
In BCD, the digit 3 is 0011
In BCD, the digit 2 is 0010
In BCD, the digit 7 is 0111
In BCD, the digit 6 is 0110
Answer: 0011001001110110

2. Recursively Defined Sets and Functions

(a) Evaluate G on the following strings in S_4: (no justification necessary)

$G(3210) = 1$
$G(2000) = 3$
$G(10101) = 0$

(b) Prove the statement:

$$\exists x(G(x) = 1)$$

Proof by witness: x = 10

G(10) = G(s0) = G(s) + 1 = G(1) + 1 = 0 + 1

This statement is proved true as we show that x= 10 a valid witness

(c) Disprove the statement:

$$\forall x(G(x) = 1)$$

We can disprove this statement with a counterexample.

Let x = 11 (11$\in S_4$),

by the definition provided for the recursive function G(11) = G(s1) = 0.
This statement is disproved as we use x= 11 as a valid witness

3. Implementing circuits A triangular number (or triangle number) counts the objects that can form an equilateral triangle, as in the diagram below. The n^{th} triangular number is the sum of the first n integers, as shown in the following figure illustrating the first four triangular numbers (what is the fifth one?):

Design a circuit that takes a 4-bit fixed width binary integer $x_3x_2x_1x_0$ as input, and outputs True (T or 1) if this integer is a triangular number, and False (F or 0) otherwise. You may assume that 0 is not a triangular number. (*Credit: UBC Department of Computer Science*)

$$\boxed{output = [x_1 \wedge (x_0 \oplus x_2) \oplus x_3] \vee [\neg x_3 \wedge \neg x_2 \wedge \neg x_1 \wedge x_0]}$$

4. Logical Equivalences

(a) Draw a logic circuit that uses **exactly three** gates and is logically equivalent to

$$q \leftrightarrow (p \wedge r)$$

You may (only) use AND, OR, NOT, and XOR gates.

$$\boxed{\neg(q \oplus (p \wedge r))}$$

(b) Write a compound proposition which is logically equivalent to

$$(p \oplus q) \leftrightarrow r$$

You may only use the logical operators negation ($\neg$), conjunction ($\wedge$), and disjunction ($\vee$).

$$\boxed{(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r)}$$

(c) Find a compound proposition that is in DNF (disjunctive normal form) and is logically equivalent to

$$(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r)$$

$$\boxed{(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)}$$

5. Logic

p is "The display is 13.3-inch"

q is "The processor is 2.2 GHz"

r is "There is at least 128GB of flash storage"

s is "There is at least 256GB of flash storage"

u is "There is at least 512GB of flash storage"

(a) Are the statements

$$p \rightarrow (r \vee s \vee u) \quad , \quad q \rightarrow (s \vee u) \quad , \quad p \leftrightarrow q \quad , \quad \neg u$$

consistent? If so, translate to English a possible assignment of truth values to the input propositions that makes all four statements true simultaneously.

These statements are consistent.
The statements are all true with the following assignments:

p	q	r	s	u
T	T	T	T	F
T	T	F	T	F
F	F	T	T	F
F	F	T	F	F
F	F	F	T	F
F	F	F	F	F

(b) Consider this statement in English:

It's not the case that both the display is 13.3-inch and the processor is 2.2 GHz.

Determine whether each of the compound propositions below is equivalent to the negation of that statement, and justify your answers using either truth tables or other equivalences.
Possible compound propositions:
(I) $\neg p \vee \neg q$ (II) $\neg(p \to \neg q)$ (III) $\neg(p \wedge q)$ (IV) $(\neg p \leftrightarrow \neg q) \wedge p$

The sentence can be translated to $\neg(p \wedge q)$. Its negation is then

$$\neg\neg(p \wedge q) \;\equiv\; (p \wedge q) \;\equiv\; \neg(p \to \neg q)$$

because $A \wedge \neg B \equiv \neg(A \to B)$ so $p \wedge q \equiv \neg(A \to \neg B)$.
This is also equivalent to $(\neg p \leftrightarrow \neg q) \wedge p$.

(c) Consider the compound proposition

$$(p \wedge q) \to (r \vee s \vee u)$$

Express the **contrapositive** of this conditional as a compound proposition.
Then, give an assignment of truth values to each of the input propositional variables for which the original compound proposition is True but its **converse** is False.

$$\neg(r \vee s \vee u) \to \neg(p \wedge q)$$

If we assign: $p = T, q = F, r = T, s = T, u = T$ then the original compound proposition:

$$(p \wedge q) \to (r \vee s \vee u)$$

is true because it is $F \to T$. The converse:

$$(r \vee s \vee u) \to (p \wedge q)$$

is not true because it is $T \to F$.

6. Logical equivalence

(a) Over the domain $\{1, 2, 3, 4, 5\}$ give an example of predicates $P(x), Q(x)$ which demonstrate that

$$\forall x P(x) \vee \forall x Q(x) \not\equiv \forall x(\ P(x) \vee Q(x)\ )$$

$P(x)$ is "$x = 1$" and $Q(x)$ is "x is an integer such that $2 \leq x \leq 5$"

Then $\forall x P(x) \vee \forall x Q(x)$ is false because each of $x = 2$ makes $P(x)$ false and $x = 1$ makes $Q(x)$ false.

$\forall x(\ P(x) \vee Q(x)\ )$ is true because either $x = 1$ or $2 \leq x \leq 5$.

(b) Over the domain $\mathbb{R}$ give an example of predicates $P(x), Q(x)$ which demonstrate that

$$\exists x P(x) \land \exists x Q(x) \not\equiv \exists x(\ P(x) \land Q(x)\ )$$

> $P(x)$ is "$x = 1$" and $Q(x)$ is "$x = 2$"
>
> Then $\exists x P(x) \land \exists x Q(x)$ is true because $x = 1$ makes $P(x)$ true and $x = 2$ makes $Q(x)$ true.
>
> $\exists x(\ P(x) \land Q(x)\ )$ is false because there is no real number that is 1 and 2 at the same time.

7. Quantifiers

- Express the statement using quantifiers and the given predicates, along with any propositional logic connectives (e.g. $\land$, $\neg$, $\oplus$) and the arithmetic operations $+, -, *$.
- Form the negation of the statement, so that no negation operator is to the left of a quantifier.
- Express this negation you formed in simple and precise English.

a. Statement: There is a real number x, such that for all real numbers y, $x - 3y = 4$.
Allowed predicates: $Eq(a, b) =$ "$a = b$". Domain: Real numbers

- $$\exists x \forall y Eq(x - 3y, 4)$$
- $$\neg \exists x \forall y Eq(x - 3y, 4) \equiv \forall x \exists y (\neg Eq(x - 3y, 4))$$
- For all real numbers x, there exists a number y such that $x - 3y \neq 4$.

b. Statement: For all real numbers x, if $x^2 < 1$, then $x > 1$.
Allowed predicates: $G(a, b) =$ "$a > b$", $Eq(a, b) =$ "$a = b$". Domain: Real numbers

- $$\forall x (G(1, x^2) \to G(x, 1))$$
- $$\exists x (G(1, x^2) \land \neg G(x, 1))$$
- There exists a real number x such that $x^2 < 1$ and $x \leq 1$.

c. Statement: There is a real number with no reciprocal.
Reminder: the reciprocal of a real number a is a real number b such that $ab = 1$.
Allowed predicates: $Eq(a, b) =$ "$a = b$". Domain: Real numbers

- $$\exists x \forall y (\neg Eq(xy, 1))$$
- $$\forall x \exists y (Eq(xy, 1))$$
- All real numbers have a reciprocal.

d. Statement: For all integers a, b, if a is even and b is even, then $a + b$ is even.
Note: you may NOT use div/mod for this. Evenness can be expressed using an additional quantified variable.

Allowed predicate: $Eq(a, b) =$ "$a = b$". Domain: Integers

- $$\forall a \forall b((\exists i Eq(a, 2i) \land \exists j Eq(b, 2j)) \to \exists k(Eq(a + b, 2k)))$$
- $$\exists a \exists b(\exists i Eq(a, 2i) \land \exists j Eq(b, 2j) \land \forall k \neg Eq(a + b, 2k))$$
- There are integers a and b such that a is even, b is even, and $a + b$ is not even.

8. Proofs

Prove the following Claim:

For all integers n, if n^2 is odd then n is odd.

proof

Toward a universal generalization, let n be an arbitrary integer.

Toward a proof by contrapositive, assume that n is even.

By the definition of *even*, there exists an integer c such that $n = 2c$.

Squaring both sides yields $n^2 = (2c)^2 = 2(2c^2)$.

By the closure property of integers, $2c^2$ is an integer.

By the definition of *even*, n^2 is even (which is what was required.) QED

Note: We can write this claim as a universal quantified statement:

$$(\forall n \in \mathbb{Z})(n^2 \textit{ is odd} \to n \textit{ is odd})$$

The proof starts with a universal generalization to resolve the $\forall n$ part.
Then the proof uses the contrapositive which means that it changes the implication to its logically equivalent contrapositive:

$$(n^2 \textit{ is odd} \to n \textit{ is odd}) \equiv (n \textit{ is not odd} \to n^2 \textit{ is not odd})$$

But since every integer is even or odd but never both, we can rewrite this as the equivalent statement:

$$(n \textit{ is not odd} \to n^2 \textit{ is not odd}) \equiv (n \textit{ is even} \to n^2 \textit{ is even})$$