CSE 20 Spring 2022
Homework 5 solutions
Due date: Thursday, May 12, 2022 at 11:59pm

In this assignment,

You will practice proof techniques, basic set theory and basic number theory.

In this class, unless the instructions explicitly say otherwise, you are required to justify all your answers.

1. (20 points) For each pair of sets, determine whether they are **disjoint, equal, proper subset** or **none of the above.** Give a justification for each answer.
   - In order to justify that A and B are equal, you will need to show that any arbitrary element of A is in B and any arbitrary element of B is in A.
   - In order to justify that A is a proper subset of B, you will need to show that any arbitrary element of A is in B and there exists an element in B that is not in A.
   - In order to justify that A and B are disjoint then you will need to show that any arbitrary element of A is not in B (or vice versa)
   - In order to justify **none of the above** then you will need to show that there exists an element that is in both sets A and B and that there exists an element of A that is not in B and there exists an element in B that is not in A.

   (a) $\{x \in \mathbb{Z} \mid x^2 \leq 2x\}$ and $\{x \in \mathbb{Z} \mid x \leq 2\}$

   Solution: $A \subsetneq B$

   $A \subseteq B$.

   Let x be an arbitrary element of A. Then $x^2 \leq 2x$. Since $0 \leq x^2$ and $x^2 \leq 2x$, then $0 \leq 2x$ and so $0 \leq x$.

   Case 1: $x = 0$. Then $0 \in A$ since $0^2 \leq 2(0)$ and $0 \in B$ since $0 \leq 2$.

   Case 2: $x > 0$ and $x \in A$. Then $x^2 \leq 2x$. Divide both sides by x and get that $x \leq 2$. Therefore $x \in B$.

   $A \neq B$.

   $-4 \notin A$ because $(-4)^2 \nleq 2(-4)$ and $-4 \in B$ because $-4 \leq 2$.

   (b) $\{x \in \mathbb{R} \mid x^2 \in \mathbb{Z}\}$ and $\mathbb{Q}$

   Solution: none of the above

   $A \nsubseteq B$:

   $\sqrt{2} \in A$ and $\sqrt{2} \notin B$.

   $B \nsubseteq A$:

   $1/2 \notin A$ and $1/2 \in B$.

   A and B are not disjoint:

   $1 \in A$ and $1 \in B$.

   (c) $\{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$ and $\{x \in \mathbb{R} \mid x \notin \mathbb{Z}\}$

   Solution: $A \subsetneq B$

   $A \subseteq B$.

   Let x be an arbitrary element of A. Then $x \in \mathbb{R}$ and $x \notin \mathbb{Q}$. Therefore $x \notin \mathbb{Z}$ because $\mathbb{Z} \subseteq \mathbb{Q}$. Therefore $x \in B$.

   $A \neq B$.

   $1/2 \notin A$ because $1/2 \in \mathbb{Q}$ and $1/2 \in B$ because $1/2 \notin \mathbb{Z}$.

   (d) $\mathbb{Z} \times \mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Z}$

   Solution: none of the above

   $A \nsubseteq B$:

$(1, 1/2) \in A$ and $(1, 1/2) \notin B$.

$B \not\subseteq A$:

$(1/2, 1) \notin A$ and $(1/2, 1) \in B$.

A and B are not disjoint:

$(1, 1) \in A$ and $(1, 1) \in B$.

(e) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y \text{ is even}\}$ and $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \text{ is even and } y \text{ is even}\}$

Solution: $B \subsetneq A$.

$B \subseteq A$:

Let (x, y) be an arbitrary element of B. Then x is even and y is even. Therefore $x + y$ is even so $(x, y) \in A$.

$A \neq B$:

$(1, 1) \in A$ and $(1, 1) \notin B$.

2. (24 points) For each statement, Determine if it is True. If it is true then prove it. If it is False then disprove it or prove its negation.

(a) For all $n \in \mathbb{Z}^{\geq 2}$, if n is composite, then there exists an integer d such that $d|n$ and $1 < d \leq \sqrt{n}$.

Solution: TRUE.

Proof 1: Proof by contradiction: Assume the negation of the statement, i.e., that there exists an integer $n \geq 2$ such that n is composite and for all integers d, if $d|n$ then either $d = 1$ or $d > sqrtn$.

Since n is composite, then n has a non-trivial factor a (meaning that $a \neq 1$ and $a \neq n$.) Since $a|n$, then there exists an integer b such that $ab = n$. Therefore $b|n$ also. Since $a \neq n$, then $b \neq 1$. So, we have $a|n, a \neq 1$ and $b|n, b \neq 1$. According to the assumption, $a > \sqrt{n}$ and $b > \sqrt{n}$. So $ab > \sqrt{n}\sqrt{n} = n$.

This is a contradiction (since we showed $ab = n$ and $ab > n$.) Therefore the assumption is false and the original statement is true.

Proof 2: Direct proof: Let n be an arbitrary integer $n \geq 2$. Assume (by way of direct proof) that n is composite. Then n has a non-trivial factor a (meaning that $a \neq 1$ and $a \neq n$.) Then since $a|n$, there exists an integer b such that $ab = n$ so $b|n$ also. Without loss of generality, assume that $a \leq b$. Then it suffices to show that $a \leq \sqrt{n}$.

$a \leq b$. (multiply both sides by a)

$a^2 \leq ab = n$.

$a^2 \leq n$

$a \leq \sqrt{n}$ as required.

(b) There exists a pair of integers x and y such that $8x + 14y = 1$.

Solution: FALSE

Proof: by way of contradiction, assume the statement is true, i.e., that there exist integers x and y such that $8x + 14y = 1$. Then factor out a 2 and get: $2(4x + 7y) = 1$. By closure property of $\times, +$, $4x + 7y$ is an integer which means that 1 is an even number. But 1 is an odd number. (A number cannot be both even and odd. This is the contradiction.) Therefore the assumption is false and the statement is false.

(c) For all positive integers a, b, c if $a|c$ and $b|c$ then $ab|c$.

Solution: FALSE

Consider as a counterexample $a = 2$, $b = 2$ and $c = 2$, then $2|2$, $2|2$ but $2 * 2$ does not divide 2.

(d) For all sets A, B, C, if $A \cup C \subseteq B \cup C$ then $A \subseteq B$.

Solution: FALSE

Consider as a counterexample $A = \{1, 2\}, B = \{2, 3\}, C = \{1, 2, 3\}$. Then $A \cup C = \{1, 2, 3\}$ and $B \cup C = \{1, 2, 3\}$ so $A \cup C \subseteq B \cup C$ but A is not a subset of B because $1 \in A$ and $1 \notin B$.

(e) For all sets A, B, if $A \subseteq B$ then $\overline{B} \subseteq \overline{A}$.

(f) For all sets A, B, if $A \subsetneq B$ then $(A \cup B) - (A \cap B) \neq \emptyset$.

3. (10 points) Consider the statement:

"For all predicates P and Q with domain $\mathbb{N}$, $\forall n \in \mathbb{N}(P(n) \to Q(n)) \vee \forall n \in \mathbb{N}(Q(n) \to P(n))$."

And the proof of this statement.

proof: Let P and Q be arbitrary predicates and let n be an arbitrary natural number.

For each n, $P(n)$ is true or false and $Q(n)$ is true or false. So write out the truth table for $P(n) \to Q(n)$ and $Q(n) \to P(n)$ considering all possible combinations:

$P(n)$	$Q(n)$	$P(n) \to Q(n)$	$Q(n) \to P(n)$	$(P(n) \to Q(n)) \vee (Q(n) \to P(n))$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Since $(P(n) \to Q(n)) \vee (Q(n) \to P(n))$ is always true for each n, either $P(n) \to Q(n)$ or $Q(n) \to P(n)$.

(a) Show that this statement is not true by considering the witnesses: $P(n) =$ "n *is odd*" and $Q(n) =$ "n *is prime*".

Solution: $\forall n \in \mathbb{N}(P(n) \to Q(n))$ is false. Consider the counterexample $n = 9$ then n is odd and n is not prime.
$\forall n \in \mathbb{N}(Q(n) \to P(n))$ is also false. Consider the counterexample $n = 2$ then n is prime and n is not odd.

(b) Identify why this proof is wrong.

Solution:
This proof is proving the statement:

$$\forall n \in \mathbb{N}((P(n) \to Q(n)) \vee (Q(n) \to P(n)))$$

which actually is a true statement. The original claim is different than this since it has a universal quantifier for each of the implications.