MATH 20D: INTRO TO DIFFERENTIAL EQUATIONS
MIDTERM 2 PRACTICE EXERCISES – SOLUTIONS

Problem 1. Use the method of Variation of Parameters to solve the following equations and IVPs. For the last three items, check that the given two functions y_1, y_2 are indeed solutions to the corresponding homogeneous part.

(a) $y'' + 4y' + 4y = t^{-2}e^{-2t}$, for $t > 0$

(b) $y'' + 4y = 3\csc(2t)$, for $0 < t < \frac{\pi}{2}$

(c) $t^2y'' - 2y = 3t^2 - 1$ for $t > 0$; $y_1(t) = t^2$ and $y_2(t) = t^{-1}$

(d) $ty'' - (1+t)y' + y = t^2e^{2t}$ for $t > 0$; $y_1(t) = 1 + t$ and $y_2(t) = e^t$

(e) $(1-x)y'' + xy' - y = g(x)$ where $g(x)$ is a continuous function; $y_1(x) = e^x$ and $y_2(x) = x$

(f) Redo the last problem from the previous practice set using Variation of Parameters instead of Undetermined Coefficients

Solution.

(a) Consider the non-homogeneous differential equation $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0.$

The general solution to the homogeneous part is given by

$$y_H(t) = c_1e^{-2t} + c_2te^{-2t}.$$

The two functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solution with the Wronskian determinant

$$W(y_1, y_2) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-t} & e^{-2t} - 2te^{-t} \end{vmatrix} = e^{-4t} \neq 0.$$

We now use method of variation of parameters and look for a particular solution of the form

$$y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)}\,dt = -\int \frac{(te^{-2t})(t^{-2}e^{-2t})}{e^{-4t}}\,dt = -\ln(t), \quad \text{and}$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)}\,dt = \int \frac{(e^{-2t})(t^{-2}e^{-2t})}{e^{-4t}}\,dt = -\frac{1}{t}.$$

Thus, the particular solution is given by

$$y_P(t) = (-\ln(t))\,(e^{-2t}) + (-\frac{1}{t})(te^{-2t}) = -e^{-2t}\ln(t) - e^{-2t}.$$

The general solution to the given differential equation is

$$y(t) = y_H(t) + y_P(t) = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln(t).$$

Notice that the term $-e^{-2t}$ in the particular solution gets absorbed into c_1e^{-2t} in y_H.

(b) Consider the non-homogeneous differential equation $y'' + 4y = 3\csc(2t), \quad 0 < t < \frac{\pi}{2}$.

The general solution to the homogeneous part is given by

$$y_H(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

The two functions $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$ form a fundamental set of solution with the Wronskian determinant

$$W(y_1, y_2) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2 \neq 0.$$

We now use method of variation of parameters and look for a particular solution of the form

$$y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)}\,dt = -\int \frac{\sin(2t)(3\csc(2t))}{2}\,dt = -\frac{3t}{2}, \quad \text{and}$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)}\,dt = \int \frac{\cos(2t)(3\csc(2t))}{2}\,dt = \frac{3}{4}\ln(\sin(2t)).$$

Thus, the particular solution is given by

$$y_P(t) = \left(-\frac{3t}{2}\right)\cos(2t) + \frac{3}{4}\ln(\sin(2t))\sin(2t).$$

The general solution to the given differential equation is

$$y(t) = y_H(t) + y_P(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{2}t\cos(2t) + \frac{3}{4}\ln(\sin(2t))\sin(2t).$$

(c) For $t > 0$, consider the differential equation

$$t^2y'' - 2y = 3t^2 - 1 \Rightarrow y'' - 2t^{-2}y = 3 - t^{-2}.$$

It is easy to check, by substitution, that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the homogeneous part. Furthermore, they form a fundamental set of solutions with the Wronskian determinant $W(y_1, y_2) = -3$.

We now use method of variation of parameters and look for a particular solution of the form

$$y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)}\,dt = -\int \frac{t^{-1}(3 - t^{-2})}{-3}\,dt = \frac{1}{6t^2} + \ln(t), \quad \text{and}$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)}\,dt = \int \frac{t^2(3 - t^{-2})}{-3}\,dt = -\frac{t^3}{3} + \frac{t}{3}.$$

Thus, the particular solution is given by

$$y_P(t) = t^2\left(\frac{1}{6t^2} + \ln(t)\right) + t^{-1}\left(-\frac{t^3}{3} + \frac{t}{3}\right) = t^2\ln(t) - \frac{t^2}{3} + \frac{1}{2}.$$

(d) For $t > 0$, consider the differential equation

$$ty'' - (1+t)y' + y = t^2e^{2t} \Rightarrow y'' - \left(1 + \frac{1}{t}\right)y' + \frac{1}{t}y = te^{2t}.$$

It is easy to check, by substitution, that $y_1(t) = 1 + t$ and $y_2(t) = e^t$ are two solutions of the homogeneous part. Furthermore, they form a fundamental set of solutions with the Wronskian determinant $W(y_1, y_2) = te^t$.

We now use method of variation of parameters and look for a particular solution of the form

$$y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)}\, dt = -\int \frac{e^t(te^{2t})}{te^t}\, dt = -\frac{e^{2t}}{2}, \quad \text{and}$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)}\, dt = \int \frac{(1+t)(te^{2t})}{te^t}\, dt = te^t.$$

Thus, the particular solution is given by

$$y_P(t) = (1+t)\left(-\frac{e^{2t}}{2}\right) + e^t\left(te^t\right) = -\frac{e^{2t}}{2} + \frac{te^{2t}}{2}.$$

(e) Let $g(t)$ be an arbitrary continuous function. For $0 < x < 1$, consider the differential equation

$$(1-x)y'' + xy' - y = g(x) \Rightarrow y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = \frac{g(x)}{1-x}.$$

It is easy to check, by substitution, that $y_1(x) = e^x$ and $y_2(x) = x$ are two solutions of the homogeneous part. Furthermore, they form a fundamental set of solutions with the Wronskian determinant $W(y_1, y_2)(x) = (1-x)e^x$.

We now use method of variation of parameters and look for a particular solution of the form

$$y_P(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\, dt = -\int_{x_0}^{x} \frac{tg(t)}{(1-t)^2\, e^t}\, dt, \quad \text{and}$$

$$u_2(t) = \int_{x_0}^{x} \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\, dt = \int_{x_0}^{x} \frac{e^tg(t)}{(1-t)^2\, e^t}\, dt.$$

Thus, the particular solution is given by

$$y_P(t) = -e^x\int_{x_0}^{x} \frac{tg(t)}{(1-t)^2\, e^t}\, dt + x\int_{x_0}^{x} \frac{e^tg(t)}{(1-t)^2e^t}\, dt = \int_{x_0}^{x} \frac{(xe^t - te^x)g(t)}{(1-t)^2\, e^t}\, dt.$$

Problem 2. For each item below, you are given a differential equation and one solution y_1. Use Reduction of Order to find a second solution to the given equation.

(a) $t^2y'' - 4ty' + 6y = 0,\ t > 0,\quad y_1(t) = t^2$

(b) $t^2y'' + 2ty' - 2y = 0,\ t > 0,\quad y_1(t) = t$

(c) $t^2y'' + 3ty' + y = 0,\ t > 0,\quad y_1(t) = t^{-1}$

(d) $t^2y'' - t(t+2)y' + (t+2)y = 0,\ t > 0,\quad y_1(t) = t$

(e) $xy'' - y' + 4x^3y = 0,\ x > 0,\quad y_1(t) = \sin(x^2)$

(f) $(x-1)y'' - xy' + y = 0,\ x > 1,\quad y_1(t) = e^x$

(g) $x^2y'' - (x - 0.1875)y = 0,\ x > 0,\ y_1(x) = x^{1/4}e^{2\sqrt{x}}$

(h) $x^2y" + xy' + (x^2 - 0.25)y = 0,\ x > 0,\ y_1(x) = x^{-1/2}\sin(x)$

Solution. We simply ignore all the "$+C$" terms in every intermediate integration.

(a) Let $t > 0$, consider the differential equation $t^2y'' - 4ty' + 6y = 0 \Rightarrow y'' - \dfrac{4}{t}y' + \dfrac{6}{t^2}y = 0$.

With $y_1(t) = t^2$, we shall look for a second solution of the form $y_2(t) = v(t)y_1(t)$ where

$$v(t) = \int \frac{1}{y_1^2}\ \exp\left(-\int p(t)dt\right)dt.$$

Here, $p(t) = -\dfrac{4}{t}$ so $\exp\left(-\displaystyle\int p(t)dt\right) = \exp\left(\displaystyle\int \frac{4}{t}dt\right) = e^{4\ln(t)} = t^4$.

So $\displaystyle\int \frac{1}{y_1^2}\ \exp\left(-\int p(t)dt\right)dt = \int \frac{t^4}{(t^2)^2}dt = \int dt = t$

Therefore, $y_2(t) = v(t)y_1(t) = t^3$.

(b) Let $t > 0$, consider the differential equation $t^2y'' + 2ty' - 2y = 0 \Rightarrow y'' + \dfrac{2}{t}y' - \dfrac{2}{t^2}y = 0$.

With $y_1(t) = t$, we shall look for a second solution of the form $y_2(t) = v(t)y_1(t)$ where

$$v(t) = \int \frac{1}{y_1^2}\ \exp\left(-\int p(t)dt\right)dt.$$

Here, $p(t) = \dfrac{2}{t}$ so $\exp\left(-\displaystyle\int p(t)dt\right) = \exp\left(-\displaystyle\int \frac{2}{t}dt\right) = e^{-2\ln(t)} = t^{-2}$.

So $\displaystyle\int \frac{1}{y_1^2}\ \exp\left(-\int p(t)dt\right)dt = \int \frac{t^{-2}}{t^2}dt = \int t^{-4}dt = -\frac{1}{3}t^{-3}$. Since the coefficient $-1/3$ is ignorable, we can take $v(t) = t^{-3}$

Therefore, $y_2(t) = v(t)y_1(t) = t^{-2}$.

(c) Let $t > 0$, consider the differential equation $t^2y'' + 3ty' + y = 0 \Rightarrow y'' + \dfrac{3}{t}y' - \dfrac{y}{t^2} = 0$.

With $y_1(t) = t^{-1}$, we shall look for a second solution of the form $y_2(t) = v(t)y_1(t)$ where

$$v(t) = \int \frac{1}{y_1^2}\ \exp\left(-\int p(t)dt\right)dt.$$

Here, $p(t) = \frac{3}{t}$ so $\exp\left(-\int p(t)dt\right) = \exp\left(-\int \frac{3}{t}dt\right) = e^{-3\ln(t)} = t^{-3}$.

So $\int \frac{1}{y_1^2} \exp\left(-\int p(t)dt\right) dt = \int \frac{t^{-3}}{(t^{-1})^2}dt = \int t^{-1}dt = \ln(t)$.

Therefore, $y_2(t) = v(t)y_1(t) = t^{-1}\ln(t)$.

(d) Let $t > 0$, consider the differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0 \Rightarrow y'' - \frac{t+2}{t}y' + \frac{(t+2)y}{t^2} = 0.$$

With $y_1(t) = t$, we shall look for a second solution of the form $y_2(t) = v(t)y_1(t)$ where

$$v(t) = \int \frac{1}{y_1^2} \exp\left(-\int p(t)dt\right) dt.$$

Here, $p(t) = -\frac{t+2}{t}$ so $\exp\left(-\int p(t)dt\right) = \exp\left(\int \frac{t+2}{t}dt\right) = e^{t+2\ln(t)} = t^2e^t$.

So $\int \frac{1}{y_1^2} \exp\left(-\int p(t)dt\right) dt = \int \frac{t^2e^t}{t^2}dt = \int e^t dt = e^t$.

Therefore, $y_2(t) = v(t)y_1(t) = te^t$.

(e) Let $x > 0$, consider the differential equation $xy'' - y' + 4x^3y = 0 \Rightarrow y'' - \frac{1}{x}y' + 4x^2y = 0$.

With $y_1(x) = \sin(x^2)$, we shall look for a second solution of the form $y_2(x) = v(x)y_1(x)$ where

$$v(x) = \int \frac{1}{y_1^2} \exp\left(-\int p(x)dx\right) dx.$$

Here, $p(x) = -\frac{1}{x}$ so $\exp\left(-\int p(x)dx\right) = \exp\left(\int \frac{1}{x}dx\right) = e^{\ln|x|} = x$.

So

$$\begin{aligned}\int \frac{1}{y_1^2} \exp\left(-\int p(x)dx\right) dx &= \int \frac{x}{\sin^2(x^2)}dx = \int x\csc^2(x^2)dx \\ &= \frac{1}{2}\int \csc^2(x^2)d(x^2) = -\cot(x^2)\end{aligned}$$

Since the coefficient -1 is ignorable, we can take $v(t) = \cot(x^2)$

Therefore, $y_2(t) = v(t)y_1(t) = \cot(x^2)\sin(x^2) = \cos(x^2)$.

(f) Let $x > 1$, consider the differential equation $(x-1)y'' - xy' + y = 0 \Rightarrow y'' - \frac{x}{x-1}y' + \frac{y}{x-1} = 0$.

With $y_1(x) = e^x$, we shall look for a second solution of the form $y_2(x) = v(x)y_1(x)$ where

$$v(x) = \int \frac{1}{y_1^2} \exp\left(-\int p(x)dx\right) dx.$$

Here, $p(x) = -\dfrac{x}{x-1}$ so

$$\exp\left(-\int p(x)dx\right) = \exp\left(\int \frac{x}{x-1}dx\right) = \exp\left(\int dx + \int \frac{dx}{x-1}\right)$$
$$= \exp\left(x + \ln(x-1)\right) = (x-1)e^x.$$

So

$$\int \frac{1}{y_1^2}\ \exp\left(-\int p(x)dx\right) dx\ = \int \frac{(x-1)e^x}{e^{2x}}dx = \int (xe^{-x} - e^{-x})dx = -xe^{-x}$$

Again, the coefficient -1 is ignorable, we can take $v(t) = xe^{-x}$

Therefore, $y_2(t) = v(t)y_1(t) = x$.

(g) Let $x > 0$, consider the differential equation $x^2y'' - (x - 0.1875)y = 0 \Rightarrow y'' - \dfrac{(x-0.1875)y}{x^2} = 0.$

With $y_1(x) = x^{1/4}e^{2\sqrt{x}}$, we shall look for a second solution of the form $y_2(x) = v(x)y_1(x)$ where

$$v(x) = \int \frac{1}{y_1^2}\ \exp\left(-\int p(x)dx\right) dx.$$

Here, $p(x) = 0$ so $\exp\left(-\int p(x)dx\right) = e^0 = 1.$

So

$$\int \frac{1}{y_1^2}\ \exp\left(-\int p(x)dx\right) dx\ = \int \frac{1}{x^{1/2}e^{4\sqrt{x}}}dx = \int x^{-1/2}e^{-4\sqrt{x}}dx$$
$$= -\frac{1}{2}\int e^{-4\sqrt{x}}\left(-2x^{-1/2}dx\right)$$
$$= -\frac{1}{2}\int e^{-4\sqrt{x}}d(-4\sqrt{x})$$
$$= -\frac{1}{2}e^{-4\sqrt{x}}$$

Again, the coefficient $-1/2$ is ignorable, we can take $v(t) = e^{-4\sqrt{x}}$

Therefore, $y_2(t) = v(t)y_1(t) = x^{1/4}e^{-2\sqrt{x}}$.

(h) Let $x > 0$, consider the differential equation $x^2y" + xy' + (x^2 - 0.25)y = 0 \Rightarrow y'' + \dfrac{1}{x}y' + \dfrac{(x^2-0.25)y}{x^2} = 0.$

With $y_1(x) = x^{-1/2}\sin(x)$, we shall look for a second solution of the form $y_2(x) = v(x)y_1(x)$ where

$$v(x) = \int \frac{1}{y_1^2}\ \exp\left(-\int p(x)dx\right) dx.$$

Here, $p(x) = \dfrac{1}{x}$ so $\exp\left(-\int p(x)dx\right) = \exp\left(-\int \dfrac{dx}{x}\right) = e^{-\ln(x)} = x^{-1}.$

So $\displaystyle\int \frac{1}{y_1^2}\,\exp\left(-\int p(x)dx\right)dx = \int \frac{x^{-1}}{x^{-1}\sin^2(x)}dx = \int \csc^2(x)dx = -\cot(x)$. Again, the coefficient -1 is ignorable, we can take $v(t) = \cot(x)$

Therefore, $y_2(t) = v(t)y_1(t) = x^{-1/2}\sin(x)\cot(x) = x^{-1/2}\cos(x)$.

Problem 3.

(a) Let α, β be real-valued constants and consider the equation $t^2\dfrac{d^2y}{dt^2} + \alpha t\dfrac{dy}{dt} + \beta y = 0$. This is called **Cauchy-Euler equation.** Use the substitution $x = \ln(t)$ to transform the given equation into a second-order homogeneous linear equation with constant coefficients.

(b) Use the method from part (a) to solve the following equations:

(i) $t^2y'' - 4ty' - 6y = 0$

(ii) $t^2y'' - 3ty' + 4y = 0$

(c) Determine whether each equation from the previous problem (#2) is a Cauchy-Euler equation or not. If so, use the method from part (a) to solve it.

Solution.

(a) Consider the equation

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0 \tag{1}$$

for real constants α and β. Let $x = \ln(t)$ then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} \quad \text{and}$$
$$\frac{d^2y}{dt^2} = \frac{dy}{dx}\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}.$$

Now we have $t\dfrac{dy}{dt} = \dfrac{dy}{dx}$ and $t^2\dfrac{d^2y}{dt^2} = \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}$. Therefore, (1) becomes

$$0 = t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = \frac{d^2y}{dx^2} - \frac{dy}{dx} + \alpha\frac{dy}{dx} + \beta y = \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y.$$

(b) (i) For $t > 0$, consider the differential equation $t^2y'' - 4ty' - 6y = 0$.
Let $x = \ln(t)$ then we can transform the given equation into

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0. \tag{2}$$

The characteristic equation of (2) is $r^2 - 5r - 6 = 0$ which has two distinct real solutions $r_1 = -1$ and $r_2 = 6$. Thus, the general solution of (2) is $y(x) = c_1e^{-x} + c_2e^{6x}$. Hence, the general solution of the original equation is given by

$$y(t) = c_1e^{-\ln(t)} + c_2e^{6\ln(t)} = c_1t^{-1} + c_2t^6.$$

(ii) Let $t > 0$ and consider the equation $t^2y'' - 3ty' + 4y = 0$. Under the same substitution $x = \ln(t)$, this differential equation then becomes

$$\frac{d^2}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

The characteristic equation is given by $r^2 - 4r + 4 = 0$ which yields repeated root $r = 2$. Thus, the general solution is

$$y(x) = c_1e^{2x} + c_2xe^{2x} \Rightarrow y(t) = c_1t^2 + c_2t^2\ln(t).$$

(c) The first three items are Cauchy-Euler equations. The general solution, unsurprisingly, is given by $c_1y_1 + c_2y_2$ where y_1, y_2 are given/obtained from the previous problem.

Problem 4. Verify that the given vector $\mathbf{x}$ satisfies the given differential equation:

(a) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$ and $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t}$

(b) $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t$

Solution.

(a) Consider the differential equation $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$ and let $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t}$. Then

$$\mathbf{x}' = \begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}.$$

Thus, $\mathbf{x}$ is a solution to the given differential equation.

(b) Consider the differential equation $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ and let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t$. Then

$$\mathbf{x}' = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t.$$

On the other hand,

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t &= \left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) e^t + \left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) te^t \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t = \mathbf{x}'. \end{aligned}$$

Thus, $\mathbf{x}$ is a solution to the given differential equation.

Problem 5.

(a) Determine if the two vectors $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2\sin t \\ \sin t \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} \sin t \\ 2\sin t \end{bmatrix}$ are linearly independent for $-\infty < t < \infty$.

(b) Show that two vectors $\mathbf{x}^{(1)}(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$ are linearly dependent at every point t_0 in the interval $[0, 1]$. However, the two vectors are linearly independent on the same interval.

Solution.

(a) Let $-\infty < t < \infty$ and consider two vectors $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2\sin t \\ \sin t \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} \sin t \\ 2\sin t \end{bmatrix}$. We form the matrix $\mathbf{X}(t)$ whose columns are given by the vectors $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ as follows.

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} 2\sin t & \sin t \\ \sin t & 2\sin t \end{bmatrix}.$$

Since $\det(\mathbf{X}(t)) = 3\sin^2 t \neq 0$ for all $t \neq k\pi$ where $k \in \mathbb{Z}$, the two vectors $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are then linearly independent on the interval $(-\infty, \infty)$.

(b) Let $0 \leq t \leq 1$ and consider two vectors $\mathbf{x}^{(1)}(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$. To show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at every point in the interval $[0, 1]$, we first choose an arbitrary value $t_0 \in [0, 1]$. Under this value t_0, we can see that

$$\mathbf{x}^{(1)}(t_0) - e^{t_0}\mathbf{x}^{(2)}(t_0) = \mathbf{0}.$$

Thus, there exists constants $c_1 = 1$ and $c_2 = -e^{t_0}$ such that $c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0) = \mathbf{0}$. This shows that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent for each $t \in [0, 1]$.

However, when we consider the vector functions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ and attempt to find a two constants c_1 and c_2, not all zero, such that $c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = \mathbf{0}$, for all $t \in [0, 1]$, we can see that there is no such pair of constants exists. Thus, the vector functions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on the interval $[0, 1]$.

Remark: When considering the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$, we can see that

$$W\begin{bmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) \end{bmatrix} = \det\begin{bmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) \end{bmatrix} = \det\begin{bmatrix} e^t & 1 \\ te^t & t \end{bmatrix} = 0 \quad \text{for all } 0 \leq t \leq 1.$$

Thus, $W\begin{bmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) \end{bmatrix} = 0$ does **not** imply that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent. This exercise gives an example of two linearly independent functions with zero Wronskian.

Problem 6. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & i \\ -i & 1 \end{bmatrix}$

Solution.

(a) Let $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$. The eigenvalues of A are the roots of the equation

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 2 \text{ and } \lambda_2 = 4.$$

For $\lambda_1 = 2$, we have

$$\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

For $\lambda_2 = 4$, we have

$$\begin{bmatrix} 5-4 & -1 \\ 3 & 1-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Let $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$. The eigenvalues and eigenvectors of A are

$$\lambda_1 = 1 + 2i \text{ with the corresponding eigenvector } \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \text{ and}$$

$$\lambda_2 = 1 - 2i \text{ with the corresponding eigenvector } \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}.$$

(c) Let $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$. The eigenvalues and eigenvectors of A are

$$\lambda_1 = -3 \text{ with the corresponding eigenvector } \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and}$$

$$\lambda_2 = -1 \text{ with the corresponding eigenvector } \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(d) Let $A = \begin{bmatrix} 3 & i \\ -i & 1 \end{bmatrix}$. The eigenvalues and eigenvectors of A are

$$\lambda_1 = 0 \text{ with the corresponding eigenvector } \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and}$$

$$\lambda_2 = 2 \text{ with the corresponding eigenvector } \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Problem 7. For each of the following matrix A,

i. Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$;

ii. Identify the stability of the origin, and classify the origin as a saddle, node, degenerate node, center, spiral point, or star point;

iii. Discuss the limit $\lim_{t\to\infty} \mathbf{x}(t)$ under several different initial conditions. Make sure to analyze all different possible behavior of the solutions.

(a) $\begin{bmatrix} 2 & 7 \\ -5 & -10 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & -4 \\ 1 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 6 \\ -3 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & 8 \\ 2 & 6 \end{bmatrix}$ (f) $\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$ (h) $\begin{bmatrix} 6 & -5 \\ 0 & 6 \end{bmatrix}$

Solution.

(a) $A = \begin{bmatrix} 2 & 7 \\ -5 & -10 \end{bmatrix}$ has two eigenvalues $\lambda_1 = -5, \lambda_2 = -3$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$\mathbf{x} = C_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

Since $\lambda_1 < \lambda_2 < 0$, the center is a node that is asymptotically stable (i.e. a sink).

$\lim_{t \to \infty} \mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for **all** initial conditions.

(b) $A = \begin{bmatrix} -3 & 6 \\ -3 & 3 \end{bmatrix}$ has complex eigenvalues $\lambda_{1,2} = \pm 3i$ with eigenvectors $\mathbf{x}_{1,2} = \begin{bmatrix} 1 \mp i \\ 1 \end{bmatrix}$

$$\begin{aligned} \mathbf{x} &= C_1 \left(\cos(3t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(3t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + C_2 \left(\sin(3t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(3t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} (C_1 - C_2)\cos(3t) + (C_1 + C_2)\sin(3t) \\ C_1 \cos(3t) + C_2 \sin(3t) \end{bmatrix} \end{aligned}$$

Since the eigenvalues are complex with $Re(\lambda_{1,2}) = 0$ (purely imaginary), the origin is a center. Solutions stay "trapped" on their circular orbit.

(c) $A = \begin{bmatrix} 8 & -4 \\ 1 & 4 \end{bmatrix}$ has only one eigenvalue $\lambda = 6$ and one eigenvector $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\mathbf{x} = C_1 e^{6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{6t} \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$

Since there is one eigenvector, the center is a degenerate/improper node. Since the eigenvalue is positive, the degenerate node is unstable.

The solution diverges to infinity for **all non-zero** initial values. If the initial condition is $\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then the solution stays at the origin.

(d) $A = \begin{bmatrix} 6 & 8 \\ 2 & 6 \end{bmatrix}$ has two eigenvalues $\lambda_1 = 10, \lambda_2 = 2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\mathbf{x} = C_1 e^{10t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Since $\lambda_1 > \lambda_2 > 0$, the center is a node that is unstable (i.e. a source).

$\lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix}\infty\\ \infty\end{bmatrix}$ for **all non-zero** initial conditions. If the initial condition is $\mathbf{x}(0) = \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ then the solution stays at the origin.

(e) $A = \begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}$ has complex eigenvalues $\lambda_{1,2} = -1 \pm i$ with eigenvectors $\mathbf{x}_{1,2} = \begin{bmatrix}\pm i\\ 1\end{bmatrix}$

$$\begin{aligned}\mathbf{x} &= C_1 e^{-t}\left(\cos(t)\begin{bmatrix}0\\ 1\end{bmatrix} - \sin(t)\begin{bmatrix}1\\ 0\end{bmatrix}\right) + C_2 e^{-t}\left(\sin(t)\begin{bmatrix}0\\ 1\end{bmatrix} + \cos(t)\begin{bmatrix}1\\ 0\end{bmatrix}\right)\\ &= e^{-t}\begin{bmatrix}-C_1\sin(t) + C_2\cos(t)\\ C_1\cos(t) + C_2\sin(t)\end{bmatrix}\end{aligned}$$

The origin is a spiral point that is asymptotically stable (complex eigenvalue with negative real part). $\lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix}0\\ 0\end{bmatrix}$ for **all** initial conditions. The solutions approach the origin along a spiral orbit.

(f) $A = \begin{bmatrix}-4 & 0\\ 0 & -4\end{bmatrix}$ has one eigenvalue $\lambda = -4$ but there are 2 eigenvectors $\mathbf{x}_1 = \begin{bmatrix}1\\ 0\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\ 1\end{bmatrix}$

$\mathbf{x} = C_1 e^{-4t}\begin{bmatrix}1\\ 0\end{bmatrix} + C_2 e^{-4t}\begin{bmatrix}0\\ 1\end{bmatrix}$

The origin is a star point (1 eigenvalue with 2 independent eigenvectors) that is asymptotically stable ($\lambda < 0$)

$\lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix}0\\ 0\end{bmatrix}$ for **all** initial conditions.

(g) $A = \begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$ has two eigenvalues $\lambda_1 = 5, \lambda_2 = -1$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix}1\\ 1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}-2\\ 1\end{bmatrix}$

$\mathbf{x} = C_1 e^{5t}\begin{bmatrix}1\\ 1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}-2\\ 1\end{bmatrix}$

Since there are two eigenvalues of opposite signs, the origin is a saddle point. Saddle point is always unstable.

If the initial condition starts on the direction of the eigenvector corresponding to the negative eigenvalue (i.e. all initial conditions of the form $\mathbf{x}(0) = \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = k\begin{bmatrix}-2\\ 1\end{bmatrix}$ for any real number k), then the solution becomes $\mathbf{x} = ke^{-t}\begin{bmatrix}-2\\ 1\end{bmatrix}$. So $\lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix}0\\ 0\end{bmatrix}$.

The solution diverges to infinity for **all other** initial conditions outside of this line.

(h) $A = \begin{bmatrix}6 & -5\\ 0 & 6\end{bmatrix}$ has one eigenvalue $\lambda = 6$ and one eigenvector $\mathbf{x} = \begin{bmatrix}1\\ 0\end{bmatrix}$.

$\mathbf{x} = C_1 e^{6t}\begin{bmatrix}1\\ 0\end{bmatrix} + C_2 e^{6t}\left(t\begin{bmatrix}1\\ 0\end{bmatrix} + \begin{bmatrix}0\\ -1/5\end{bmatrix}\right)$

The center is a degenerate/improper node (1 eigenvalue with 1 eigenvector) that is unstable (positive eigenvalue).

The solution diverges to infinity for **all non-zero** initial conditions. If the initial condition is $\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then the solution stays at the origin.

Problem 8. For each pair of A and $\mathbf{x}_0$, solve the IVP $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$.

(a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \; \mathbf{x}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}, \; \mathbf{x}_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix}, \; \mathbf{x}_0 = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$

(d) $A = \begin{bmatrix} -5/2 & 3/2 \\ -3/2 & 1/2 \end{bmatrix}, \; \mathbf{x}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Solution.

(a) Consider the IVP

$$\mathbf{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The eigenvalues and eigenvectors of A are given by $r_1 = 2, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $r_2 = 4, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since the eigenvalues are distinct, the general solution is given by

$$\mathbf{x} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}.$$

Using the initial condition, we obtain

$$\begin{aligned} c_1 + c_2 &= 2 \\ 3c_1 + c_2 &= -1 \end{aligned}$$

which gives $c_1 = -3/2$ and $c_2 = 7/2$. Thus, the solution to the IVP is

$$\mathbf{x} = -\frac{3}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}.$$

(b) Consider the IVP

$$\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The eigenvalues and eigenvectors are given by $r_1 = -2+i, \mathbf{x}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ and $r_2 = -2-i, \mathbf{x}_2 =$

$\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$. The general solution is given by

$$\begin{aligned}
\mathbf{x} &= c_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} + c_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(-2-i)t} \\
&= c_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{-2t}(\cos t + i \sin t) + c_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{-2t}(\cos t - i \sin t) \\
&= C_1 e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -\cos t + \sin t \\ \sin t \end{bmatrix}.
\end{aligned}$$

Using the initial condition, we obtain

$$\begin{aligned}
C_1 - C_2 &= 1 \\
C_1 &= -2
\end{aligned}$$

which gives $C_1 = -2$ and $C_2 = -3$. Thus, the solution to the IVP is

$$\mathbf{x} = -2e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} - 3e^{-2t} \begin{bmatrix} -\cos t + \sin t \\ \sin t \end{bmatrix} = e^{-2t} \begin{bmatrix} \cos t - 5 \sin t \\ -2\cos t - 3 \sin t \end{bmatrix}.$$

This solution converges to $(0, 0)$ as $t \to \infty$.

(c) Consider the IVP

$$\mathbf{x}' = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$$

The eigenvalues and eigenvectors of A are given by $r_1 = -1, \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$; $r_2 = 1, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$; and $r_3 = 4, \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$. Since the eigenvalues are distinct, the general solution is given by

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix} e^{4t}.$$

Using the initial condition, we obtain

$$\begin{aligned}
c_1 + c_2 + 2c_3 &= 7 \\
-2c_1 + 2c_2 + c_3 &= 5 \\
c_1 - c_2 - 8c_3 &= 5
\end{aligned}$$

which gives $c_1 = 3, c_2 = 6$ and $c_3 = -1$. Thus, the solution to the IVP is

$$\mathbf{x} = 3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-t} + 6 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{t} - \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix} e^{4t}.$$

(d) Consider the IVP

$$\mathbf{x}' = \begin{bmatrix} -5/2 & 3/2 \\ -3/2 & 1/2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

There is a repeated eigenvalue $r = -1$ with only one independent eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, one solution to the system is

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

We now look for a second solution $\mathbf{x}^{(2)}(t)$ independent of $\mathbf{x}^{(1)}(t)$ in the form

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} e^{-t}$$

where $(\mathbf{A} - \mathbf{I}) \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In this case, we obtain the second solution

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t/2} + \begin{bmatrix} -2/3 \\ 0 \end{bmatrix} e^{-t/2}.$$

Therefore, the general solution to the system is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t/2} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2/3 \\ 0 \end{bmatrix} \right) e^{-t/2}.$$

Under the initial condition, we obtain

$$\begin{aligned} c_1 - \frac{2}{3}c_2 &= 3 \\ c_1 &= -1 \end{aligned}$$

which gives $c_1 = -1$ and $c_2 = -6$. Thus, the solution to the IVP is

$$\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t/2} - \begin{bmatrix} 6 \\ 6 \end{bmatrix} te^{-t/2}.$$

Problem 9. Consider the two vectors $\mathbf{x}^{(1)}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$.

(a) Compute their Wronskian and determine the interval(s) in which the two given vectors are linearly independent.

(b) Find a system of homogeneous differential equations $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ satisfied by $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)$

Solution. Consider the two vectors $\mathbf{x}^{(1)}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$.

(a) $W = W\begin{bmatrix}\mathbf{x}^{(1)} & \mathbf{x}^{(2)}\end{bmatrix} = \det\begin{bmatrix} t & t^2 \\ 1 & 2t\end{bmatrix} = t^2.$

Since $W\begin{bmatrix}\mathbf{x}^{(1)} & \mathbf{x}^{(2)}\end{bmatrix} \neq 0 \Leftrightarrow t \neq 0$, the two vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent at every point $t \neq 0$. Furthermore, they are linearly independent in every interval (see the remark after Problem 5).

(b) To find $\mathbf{A}(t)$, we simply substitute the solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ into the equation $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ to obtain

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t)\end{bmatrix}\begin{bmatrix}t\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}2t\\2\end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t)\end{bmatrix}\begin{bmatrix}t^2\\2t\end{bmatrix}.$$

The above matrix-vector equations can be combined into

$$\begin{bmatrix} t & 1 & 0 & 0 \\ t^2 & 2t & 0 & 0 \\ 0 & 0 & t & 1 \\ 0 & 0 & t^2 & 2t \end{bmatrix}\begin{bmatrix} a_{11}(t) \\ a_{12}(t) \\ a_{21}(t) \\ a_{22}(t)\end{bmatrix} = \begin{bmatrix}1 \\ 2t \\ 0 \\ 2\end{bmatrix}$$

Using Gaussian elimination to solve the above system, we obtain $a_{11}(t) = 0$, $a_{12}(t) = 1$, $a_{21}(t) = -2/t^2$ and $a_{22}(t) = 2/t$.

Problem 10. Consider the system $\mathbf{x}' = \begin{bmatrix} 0 & -5 \\ 1 & \alpha\end{bmatrix}\mathbf{x}$ where α is a parameter.

In terms of α, discuss all possible behaviors of the solutions to this system. In each case, draw a phase portrait for a particular value of α.

Solution. Let α be a parameter. Consider the system $\mathbf{x}' = \begin{bmatrix} 0 & -5 \\ 1 & \alpha\end{bmatrix}\mathbf{x}$. Assume that $\mathbf{x} = \xi e^{rt}$, we then obtain the system

$$\begin{bmatrix} -r & -5 \\ 1 & \alpha - r\end{bmatrix}\begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

The characteristic equation is $r^2 - \alpha r + 5 = 0$ with the roots, given in terms of α, are

$$r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}.$$

When $-\sqrt{20} < \alpha, \sqrt{20}$, the roots obtained above are complex with the real part $\Re(r_{1,2}) = \alpha/2$. We further classify them into three following cases, based on the sign of this real part.

- When $\alpha = 0, \Re(r_{1,2}) = 0$ so the equilibrium point $(0,0)$ is a center. This is given in Figure 1.
- When $0 < \alpha < \sqrt{20}, \Re(r_{1,2}) > 0$ so the equilibrium point $(0,0)$ is an unstable spiral. An example of this case is given in Figure 2 with $\alpha = 3$.
- When $-\sqrt{20} < \alpha < 0, \Re(r_{1,2}) < 0$ so the equilibrium point $(0,0)$ is a stable spiral. An example of this case is given in Figure 3 with $\alpha = -2$.

When $\alpha^2 > 20$, the roots are real and distinct, with the product $r_1 r_2 = 5 > 0$. Thus, the two roots are of the same sign and the equilibrium point $(0,0)$ is a node.

- If both roots are positive, then $(0,0)$ is an unstable node. An example of this case is given in Figure 4 with $\alpha = 5.5$.
- If both roots are negative, then $(0,0)$ is an asymptotically stable node. An example of this case is given in Figure 5 with $\alpha = -5.5$.

Lastly, when $\alpha = \pm\sqrt{20}$, we obtain a repeated eigenvalue with only one independent eigenvector. So the origin is an improper node with stability depending on the sign of this repeated eigenvalue (refer back to the case where $\alpha^2 > 20$).

Pictures for each of the cases discussed in part (b) are illustrated in Figures 1-5 below.

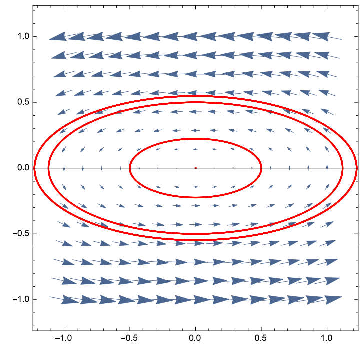

Figure 1: Center equilibrium with $\alpha = 0$.

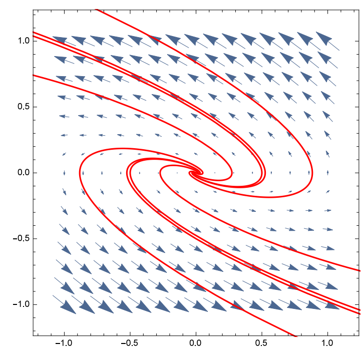

Figure 2: Unstable spiral equilibrium with $\alpha = 3$.

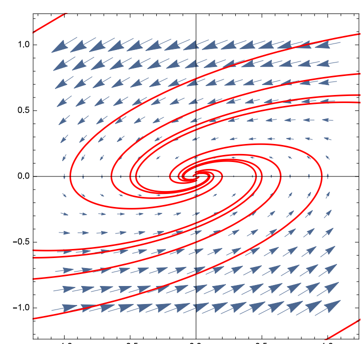

Figure 3: Stable spiral equilibrium with $\alpha = -2$.

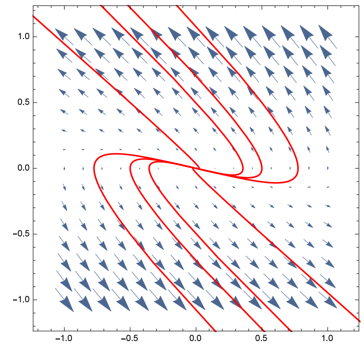

Figure 4: Unstable node equilibrium with $\alpha = 5.5$.

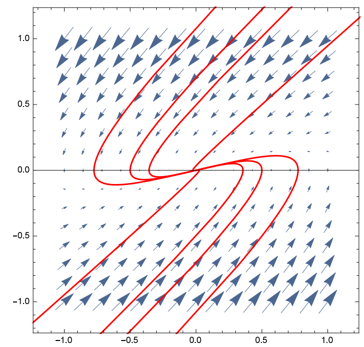

Figure 5: Stable node equilibrium with $\alpha = -5.5$.

Problem 11. Find the general solution to the given non-homogeneous systems:

(a) $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$

(b) $\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}, \; t > 0$

(c) $t\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2t \\ t^4 - 1 \end{bmatrix}, \; t > 0$

Solution.

(a) Consider the non-homogeneous linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}.$$

The general solution to the homogeneous part is given by

$$\mathbf{x}_H = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t.$$

Thus, the fundamental matrix and its inverse are given by

$$\psi(t) = \begin{bmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{bmatrix} \text{ and } \psi^{-1}(t) = -\frac{1}{2} \begin{bmatrix} e^t & -e^t \\ -3e^{-t} & e^{-t} \end{bmatrix}.$$

Using the variation of parameters method, we look for particular solution of the form

$$\mathbf{x}_P = \Psi(t)\mathbf{u}(t)$$

where $\Psi(t)\mathbf{u}'(t) = \mathbf{g}(t)$. So

$$\mathbf{u}'(t) = \psi^{-1}(t)\mathbf{g}(t) = -\frac{1}{2}\begin{bmatrix} e^{2t} - te^{t} \\ -3 + te^{-t} \end{bmatrix}.$$

This gives

$$\mathbf{u} = \begin{bmatrix} e^{2t}/4 + te^{t}/2 - e^{t}/2 \\ 3t/2 + te^{t}/2 + e^{t}/2 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \mathbf{x}_P(t) &= \begin{bmatrix} t - e^{t}/4 + 2t/3 \\ -1 + 2t - 3e^{t}/4 + 3te^{t}/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{t} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{t}. \end{aligned}$$

Therefore, the general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} e^{t} + \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} te^{t}.$$

(b) For $t > 0$, consider the non-homogeneous linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}.$$

The general solution to the homogeneous part is given by

$$\mathbf{x}_H = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t}.$$

Thus, the fundamental matrix and its inverse are given by

$$\psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5} \end{bmatrix} \text{ and } \psi^{-1}(t) = \frac{e^{5t}}{5}\begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ -2e^{5t}/5 & e^{5t}/5 \end{bmatrix}$$

Using the variation of parameters method, we look for particular solution of the form

$$\mathbf{x}_P = \Psi(t)\mathbf{u}(t)$$

where $\Psi(t)\mathbf{u}'(t) = \mathbf{g}(t)$. So

$$\mathbf{u}'(t) = \psi^{-1}(t)\mathbf{g}(t) = \begin{bmatrix} t^{-1} + 8t/5 \\ 4e^{5t}/5 \end{bmatrix}.$$

This gives

$$\mathbf{u} = \begin{bmatrix} \ln(t) + 8t/5 \\ 4e^{5t}/25 \end{bmatrix}.$$

Hence,

$$\begin{aligned}\mathbf{x}_P(t) &= \begin{bmatrix} \ln(t) + 8t/5 - 8/25 \\ 2\ln(t) + 16t/5 + 4/25 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ln(t) + \frac{8}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \frac{4}{25}\begin{bmatrix} -2 \\ 1 \end{bmatrix}\end{aligned}$$

Therefore, the general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ln(t) + \frac{8}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \frac{4}{25}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

(c) For $t > 0$, consider the non-homogeneous linear system

$$t\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2t \\ t^4 - 1 \end{bmatrix}.$$

It is easy to check that the general solution to the homogeneous part is given by

$$\mathbf{x}_H = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} t^{-1} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} t^2.$$

Thus, the fundamental matrix and its inverse are given by

$$\psi(t) = \begin{bmatrix} t^{-1} & 2t^2 \\ 2t^{-1} & t^2 \end{bmatrix} \text{ and } \psi^{-1}(t) = -\frac{1}{3t}\begin{bmatrix} t^2 & -2t^2 \\ -2t^{-1} & t^{-1} \end{bmatrix} = \begin{bmatrix} -t/3 & 2t \\ 2t^{-2} & -t^{-2} \end{bmatrix}$$

Notice that in this problem, $\mathbf{g}(t)$ is given by

$$\mathbf{g}(t) = \begin{bmatrix} -2 \\ t^3 - t^{-1} \end{bmatrix}.$$

Using the variation of parameters method, we look for particular solution of the form

$$\mathbf{x}_P = \Psi(t)\mathbf{u}(t)$$

where $\Psi(t)\mathbf{u}'(t) = \mathbf{g}(t)$. So

$$\mathbf{u}'(t) = \psi^{-1}(t)\mathbf{g}(t) = \begin{bmatrix} 2t^4/3 + \ 2t/3 - \ 2/3 \\ -t/3 - \ 4t^{-2}/3 + \ t^{-3}/3 \end{bmatrix}.$$

This gives

$$\mathbf{u} = \begin{bmatrix} 2t^5/12 + t^2/3 - 2t/3 \\ -t^2/6 + 4t^{-1}/3 - t^{-2}/6 \end{bmatrix}.$$

Hence,

$$\begin{aligned}\mathbf{x}_P(t) &= \begin{bmatrix} -t^4/5 + 3t - 1 \\ t^4/10 + 2t - 3/2 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} -2 \\ 1 \end{bmatrix} t^4 + \begin{bmatrix} 3 \\ 2 \end{bmatrix} t - \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}\end{aligned}$$

Therefore, the general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} t^{-1} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} t^2 + \frac{1}{10}\begin{bmatrix} -2 \\ 1 \end{bmatrix} t^4 + \begin{bmatrix} 3 \\ 2 \end{bmatrix} t - \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}.$$
