Learning Outcomes:

For the second midterm, students are expected to be able to ...

1. Apply the method of Variation of Parameters and the Principle of Superposition to obtain an appropriate "guess" for the particular solution of a non-homogeneous 2nd-order linear equation.

2. Solve Cauchy-Euler equations $at^2y'' + bty' + cy = f(t)$ (a special class of non-homogeneous 2nd-order linear ODE with variable-coefficients).

3. Apply the method of Reduction of Order to solve general non-homogeneous 2nd-order linear ODE with variable-coefficients.

4. Solve linear homogeneous systems $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix with distinct, real eigenvalues, complex eigenvalues, and repeated eigenvalues.

5. Solve non-homogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$ for 2×2 matrix A.

6. Visualize the phase portrait and analyze the behavior and stability of the origin under different scenarios of the 2×2 linear homogeneous systems $\mathbf{x}' = A\mathbf{x}$.

The **same set of integration formulas from the first study guide will be provided on the exam** and you may apply any of those results directly without showing work/proof.

The following formulas for Variations of Parameter (Section 4.6) and Reduction of Order (Section 4.7) below will also be provided:

Variation of Parameters. Consider the ODE

$$y''(t) + p(t)y'(t) + q(t)y = g(t)$$

and suppose that the corresponding homogeneous equation has the solution

$$y_H(t) = C_1 y_1(t) + C_2 y_2(t).$$

Then a particular solution to the non-homogeneous equation can be found in the form

$$Y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = \int \frac{-g(t)y_2(t)}{W_{[y_1,y_2]}(t)}\, dt \quad \text{and} \quad u_2 = \int \frac{g(t)y_1(t)}{W_{[y_1,y_2]}(t)}\, dt.$$

Here, $W_{[y_1,y_2](t)}$ is the Wronskian of y_1, y_2.

Reduction of Order. Consider the ODE

$$y''(t) + p(t)y'(t) + q(t)y = g(t)$$

and suppose that we are given one solution y_1 (not identically zero) to the corresponding homogeneous part. We can obtain a second solution y_2 for the homogeneous part (independent from y_1) of the form

$$y_2(t) = v(t)y_1(t),$$

where

$$v(t) = \int \frac{1}{y_1^2} \cdot \exp\left(-\int p(t)dt\right) dt$$

1 Solving non-homogeneous ODEs

Consider the non-homogeneous second-order linear ODE with variable-coefficients:

$$y''(t) + p(t)y'(t) + q(t)y = g(t).$$

According to the **Superposition Principle**, the solution to this ODE is given by

$$y(t) = y_H(t) + y_P(y)$$

where

- y_H is the solution to the corresponding homogeneous part $y''(t) + p(t)y'(t) + q(t)y = g(t)$
- y_P is any particular solution to the given non-homogenous equation.

That is,

General solution to the non-hom. equation/system	=	General solution to the corresponding homogeneous part	+	One particular solution to the given non-hom. equation/system

If we can obtain the general solution to the corresponding homogeneous part $y''(t) + p(t)y'(t) + q(t)y = 0$ (i.e. having two independent solutions y_1, y_2 to the homogeneous part), then we can use **variation of parameters** to find a particular solution y_P of the form

$$y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = \int \frac{-g(t)y_2(t)}{W_{[y_1,y_2]}(t)}\, dt \quad \text{and} \quad u_2 = \int \frac{g(t)y_1(t)}{W_{[y_1,y_2]}(t)}\, dt.$$

Here, $W_{[y_1,y_2](t)}$ is the Wronskian of y_1, y_2, defined by $W_{[y_1,y_2](t)} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$.

Remark: Method of Undetermined Coefficients is another way to solve non-homogeneous equations; however, this is quite a limited technique, applicable to only the case where the LHS has **constant coefficients** and the RHS contains polynomials; exponentials; sines and/or cosines; and sum and product of the three.

In general, we should not expect to always be able to find two linearly independent solutions to the corresponding homogeneous equation, as this problem is intractable. However, there are two scenarios that we can manage.

1.1 When the coefficients have special forms – Cauchy-Euler equations

Cauchy-Euler equations have the form

$$at^2y'' + bty' + cy = f(t)$$

where a, b, c are constants. The solutions to the corresponding homogeneous equation

$$at^2y'' + bty' + cy = 0$$

can be found in the form t^r, for $t \neq 0$.

The homogeneous Cauchy-Euler equation $at^2y''+bty'+cy=0$ has the characteristic equation given by

$$ar^2 + (b-a)r + c = 0 \tag{1}$$

There are three cases:

1. If (1) has two distinct, real-valued solutions r_1, r_2. Then the general solution is given by

$$y = C_1 t^{r_1} + C_2 t^{r_2}$$

2. If (1) has complex solutions $r_{1,2} = \alpha \pm i\beta$. Then the general solution is given by

$$y = C_1 t^{\alpha} \cos(\beta \ln t) + C_2 t^{\alpha} \sin(\beta \ln t)$$

3. If (1) has a repeated/unique solution r then the general solution is given by

$$y = C_1 t^r + C_2 t^r \ln t$$

1.2 When we have one solution to the hom. part – Reduction of Order

Consider the ODE

$$y''(t) + p(t)y'(t) + q(t)y = g(t)$$

and suppose that we are given one solution y_1 (not identically zero) to the corresponding homogeneous part $y''(t)+p(t)y'(t)+q(t)y = 0$. We can obtain a second solution y_2 for the homogeneous part (independent from y_1) of the form

$$y_2(t) = v(t)y_1(t),$$

where

$$v(t) = \int \left[\frac{1}{y_1^2} \cdot \exp\left(-\int p(t)dt \right) \right] dt$$

2 Solving 2×2 homogeneous system $\mathbf{x}' = A\mathbf{x}$

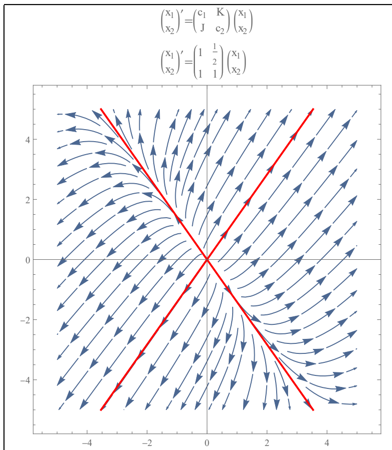
Here is a brief summary of different types of phase portrait for the linear system $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix. We assume that λ_1 and λ_2 are the eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. In all cases, the origin $(0, 0)$ is an equilibrium. As $t \to \infty$, the solution either converges to $(0, 0)$ (in the case of a stable equilibrium) or diverge to infinity (in the case of unstable equilibrium).

2.1 Distinct, real eigenvalues

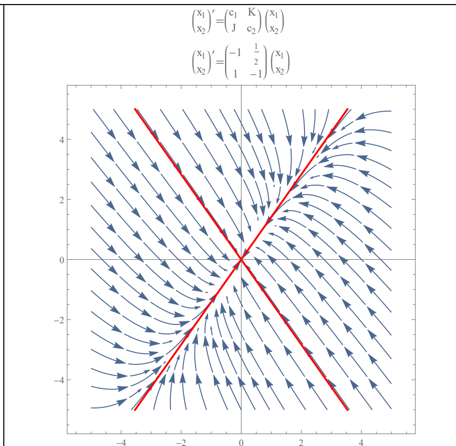
If λ_1 and λ_2 are distinct, real eigenvalues then the general solution is given by

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

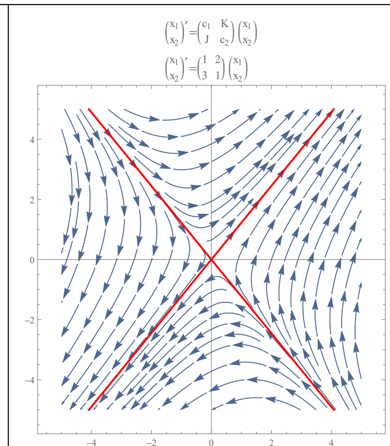
for some constants c_1, c_2. The phase portrait can be one of the following cases below. Here, the red lines are the eigenvectors.



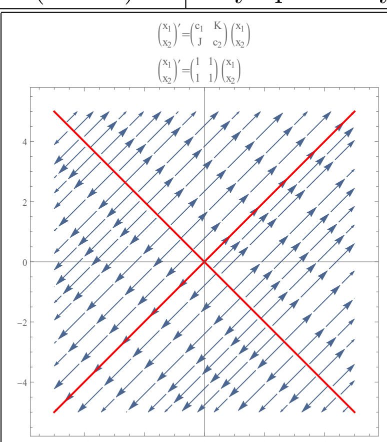
$0 < \lambda_1 < \lambda_2$
Unstable node (source)



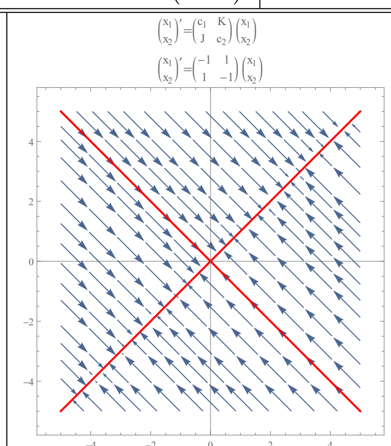
$\lambda_1 < \lambda_2 < 0$
Asymptotically stable node (sink)



$\lambda_1 < 0 < \lambda_2$
Saddle point



$\lambda_1 = 0 < \lambda_2$
Lines



$\lambda_1 < 0 = \lambda_2$
Lines

We call the eigenvector corresponding to the eigenvalue with **greater absolute value** the **fast direction**. The other eigenvector is called the **slow direction**.

- In the case the origin is a sink, the solution path first follows the fast direction, then once t is large enough, the solution then follows the slow direction toward the origin.
- In the case the origin is a source, the solution path first follows the slow direction, then once t is large enough, it then follows the fast direction to diverge to infinity.

The special case occurs when one of the eigenvalues is zero. In this case, the phase plane consists of straight lines/arrows that run parallel to the eigenvector of the non-zero eigenvalue, and pointing toward the direction of the zero eigenvalue. The vector direction depends on the sign of the non-zero eigenvalue.

2.2 Complex eigenvalues

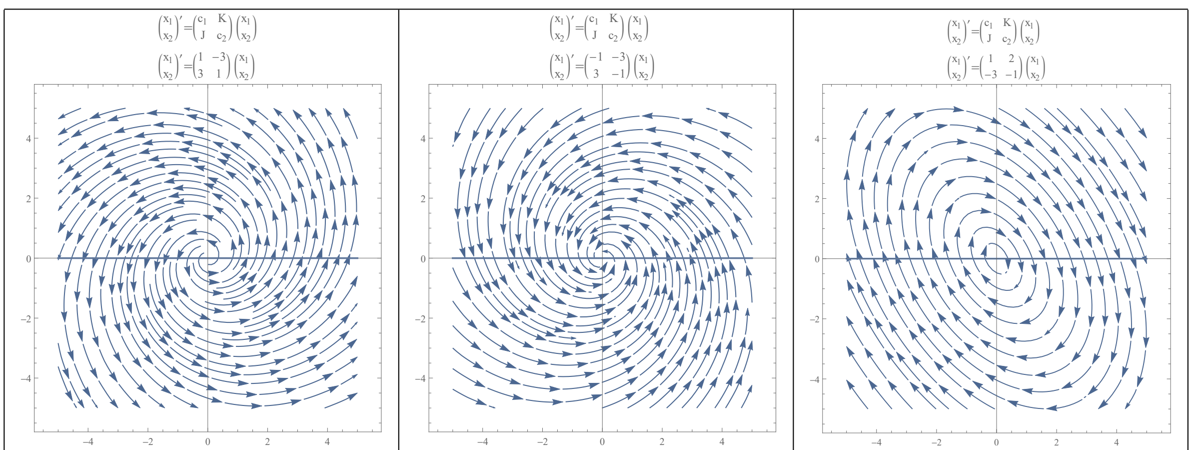
If $\lambda_{1,2} = a \pm ib$ are complex eigenvalues then the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ will occur as conjugate pairs. That is, we can decompose the eigenvectors as $\mathbf{v}_{1,2} = \mathbf{a} \pm i\mathbf{b}$. In this case, the two linearly independent solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ are given by

$$\begin{aligned}\mathbf{x}_1 &= e^{at}\cos(bt)\mathbf{a} - e^{at}\sin(bt)\mathbf{b}\\ \mathbf{x}_2 &= e^{at}\sin(bt)\mathbf{a} + e^{at}\cos(bt)\mathbf{b}\end{aligned}$$

Thus, the general solution is given by

$$\mathbf{x} = C_1 e^{at}\left(\cos(bt)\mathbf{a} - \sin(bt)\mathbf{b}\right) + C_2 e^{at}\left(\sin(bt)\mathbf{a} + \cos(bt)\mathbf{b}\right)$$

The phase portrait depends on the sign of a, the real part of the eigenvalues, and it can be one of the following cases:



$a > 0$	$a < 0$	$a = 0$
Unstable spiral (source)	Stable spiral (sink)	Circle

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the **orientation** depends on the sign of c. If $c > 0$ then the rotation is counter-clockwise (see the pictures above), if $c < 0$ then the rotation is clockwise. The case $c = 0$ can never occur when A has complex eigenvalues.

2.3 Repeated eigenvalues

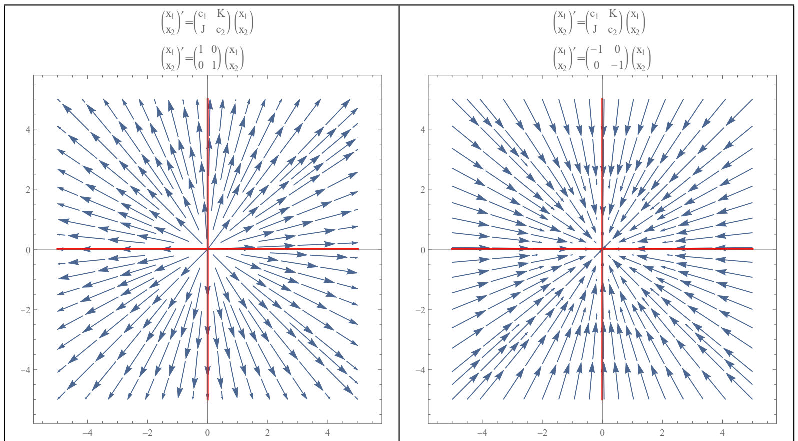
If $\lambda_1 = \lambda_2 = \lambda$ is the only eigenvalue then we now have two possibilities.

Two independent eigenvectors.

This sub-case only occurs when the matrix A is some scalar multiple of the identity matrix. In this sub-case, if λ has two independent eigenvectors then the general solution is given by

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 \mathbf{v}_2 e^{\lambda t}$$

for some constants c_1, c_2. The equilibrium is call a *proper node* (or star) and its stability depends on the sign of the (only) eigenvalue. There are two cases:



$a > 0$	$a < 0$
Unstable star point	Stable star point

One independent eigenvector.

In this case, if λ has only one independent eigenvector $\mathbf{v}$ then we know that one of the solution is given by $\mathbf{x}^{(1)} = \mathbf{v}e^{\lambda t}$. We now look for a second solution of the form

$$\mathbf{x}^{(2)} = \mathbf{v}te^{\lambda t} + \mathbf{u}e^{\lambda t}$$

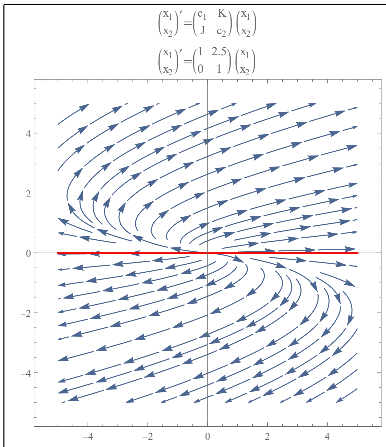
where $\mathbf{u}$ is obtained by solving the matrix-vector equation

$$(A - \lambda I)\mathbf{u} = \mathbf{v}.$$

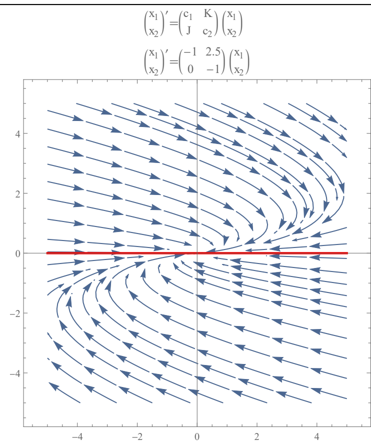
The general solution is given by

$$\mathbf{x} = c_1 \mathbf{v} e^{\lambda t} + c_2 \left(\mathbf{v}te^{\lambda t} + \mathbf{u}e^{\lambda t} \right)$$

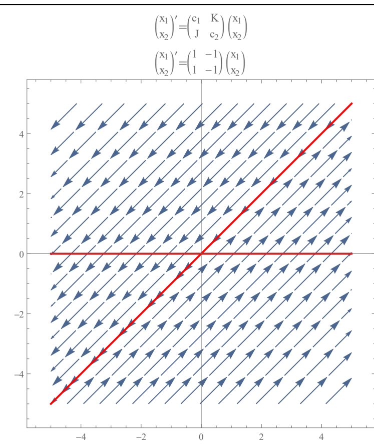
for some constants c_1, c_2. The equilibrium is called a *improper/degenerate node*. It can be seen as a transition from a (proper) node (in Case 1) into a spiral (in Case 2). Its stability depends on the sign of the (only) eigenvalue. There are three cases:



$\lambda > 0$
Unstable improper node



$\lambda < 0$
Stable improper node



$\lambda = 0$
Lines

3 Solving non-homogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$

A non-homogeneous linear system has the form $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$, where $\mathbf{g} \neq \mathbf{0}$. The general solution to this equation is given by

$$\mathbf{x} = \mathbf{x}_H + \mathbf{x}_P$$

where

- $\mathbf{x}_H$ is the general solution to the corresponding homogeneous equation $\mathbf{x}' = A\mathbf{x}$ and
- $\mathbf{x}_P$ is a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$

Let A be an $n \times n$ matrix and let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be n linearly independent solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$. Then the **fundamental matrix** for the system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\Psi(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{bmatrix}.$$

The general solution of $\mathbf{x}' = A\mathbf{x}$ can be expressed as

$$\mathbf{x}(t) = C_1\mathbf{x}_1 + \cdots + C_n\mathbf{x}_n = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} = \Psi(t) \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

For the method of **Variation of Parameters**, we shall look for a particular solution of the form

$$\mathbf{x}_P = \Psi(t)\mathbf{u} \quad \text{where} \quad \Psi(t)\mathbf{u}' = \mathbf{g}.$$

Thus, we can find the vector **u'** by multiplying the inverse of Ψ to the left of $\mathbf{g}$. That is, **u'** $= \Psi^{-1}\mathbf{g}$. Finally, simply integrate our answer for **u'** to obtain $\mathbf{u}$.

Notice that in the case A is a 2×2 then so is $\Psi(t)$. The inverse matrix of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$