Announcements

- Homework 2 online due today
- No Homework next week
- Exam 1 Next Friday!

Exam Details

- In class
- Randomized assigned seats
- You may use 6 one-sided pages of notes
- No textbook or electronic aids
- 3 Questions in 45 minutes
  - 1^{st} straightforward implementation of algorithm
  - 2^{nd} requires some thought
  - 3^{rd} can be quite tricky

Exam Topics

- Chapter 3
  - Graph basics
  - Explore/DFS
  - Connected components
  - Pre/Post orderings
  - DAGs
  - Topological sort
  - Strongly connected components
- Chapter 4
  - Shortest path definitions
  - BFS
  - Dijkstra
  - Priority queues
  - Bellman-Ford
  - Negative weight cycles
  - Shortest paths in DAGs

Review Options

- I will produce a brief review video
- Lecture podcasts / slides
- Textbook
- OH questions
- Old exams from problem archive

Last Time

- Shortest paths with negative edge weights
- Negative weight cycles
- Computing paths with bounded numbers of edges

Negative Edge Weights

- So far we have talked about the case of non-negative edge weights.
  - The usual case (distance & time usually cannot be negative).
  - However, if “lengths” represent other kinds of costs, sometimes they can be negative.
- Problem statement same. Find path with smallest sum of edge weights.

Negative Weight Cycles

Definition: A negative weight cycle is a cycle where the total weight of edges is negative.

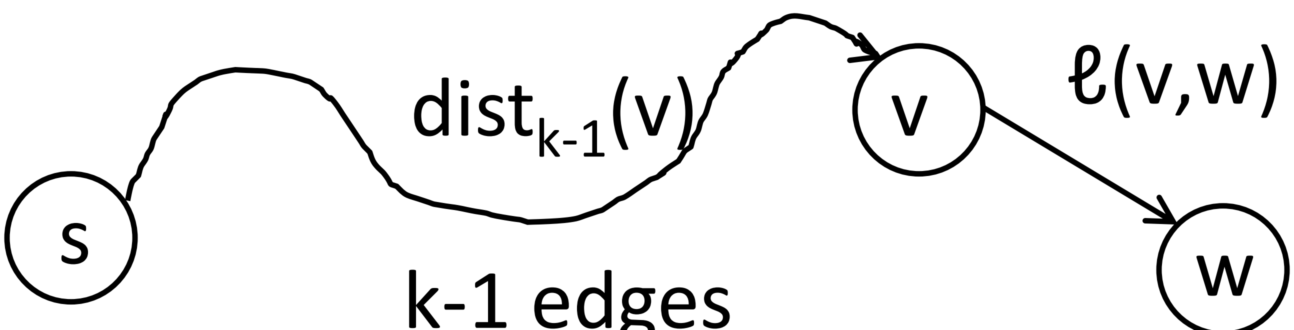
- If G has a negative weight cycle, then there are probably no shortest paths.
  - Go around the cycle over and over.
- **Note:** For undirected G, a single negative weight edge gives a negative weight cycle by going back and forth on it.

Algorithm Idea

Instead of finding shortest paths (which may not exist), find shortest paths of length at most k.

For $w \neq s$,

$$\mathrm{dist}_k(w) = \min_{(v,w)\in E} \mathrm{dist}_{k-1}(v) + \ell(v,w).$$



Algorithm

```
Bellman-Ford(G,s,ℓ)
  dist_0(v) ← ∞ for all v
    //cant reach
  dist_0(s) ← 0
  For k = 1 to n
    For w ∈ V
      dist_k(w)←min(dist_{k-1}(v)+ℓ(v,w))
    dist_k(s) ← min(dist_k(s),0)
      // s has the trivial path
```

Today

- Bellman-Ford
  - Computing shortest paths
  - Detecting negative weight cycles
- Shortest paths in DAGs
- Introduction to divide & conquer

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```

O(|E|)

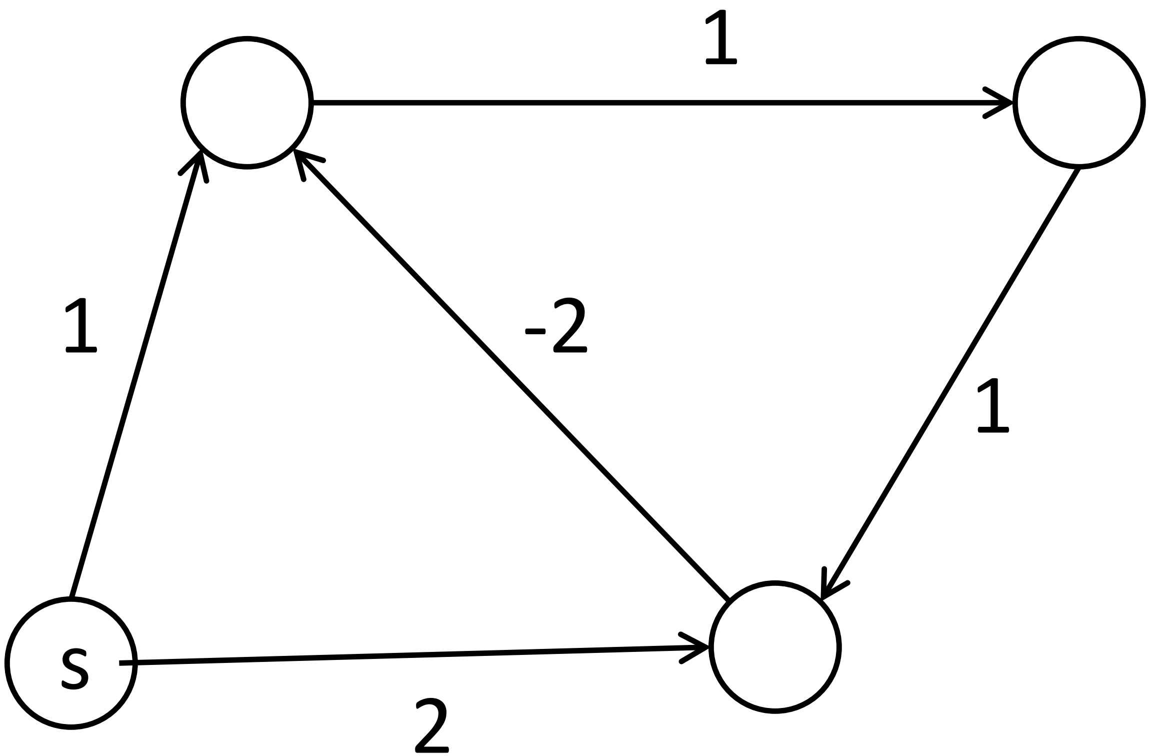
Algorithm

```
Bellman-Ford(G,s,ℓ)
  dist₀(v) ← ∞ for all v
    //cant reach
  dist₀(s) ← 0
  For k = 1 to n
    For w ∈ V
      distₖ(w)←min(distₖ₋₁(v)+ℓ(v,w))
    distₖ(s) ← min(distₖ(s),0)
      // s has the trivial path
```

What value of k do we use?

O(|E|)

Example



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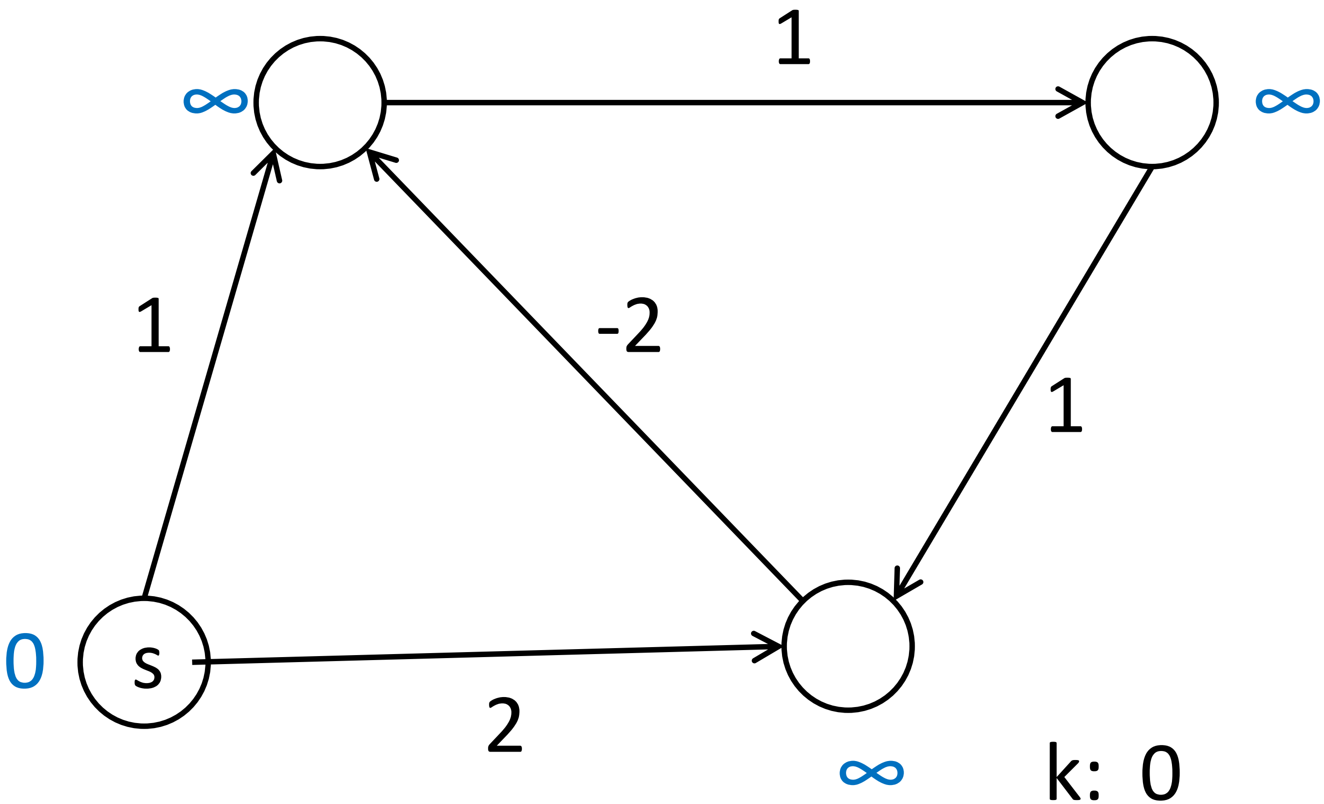

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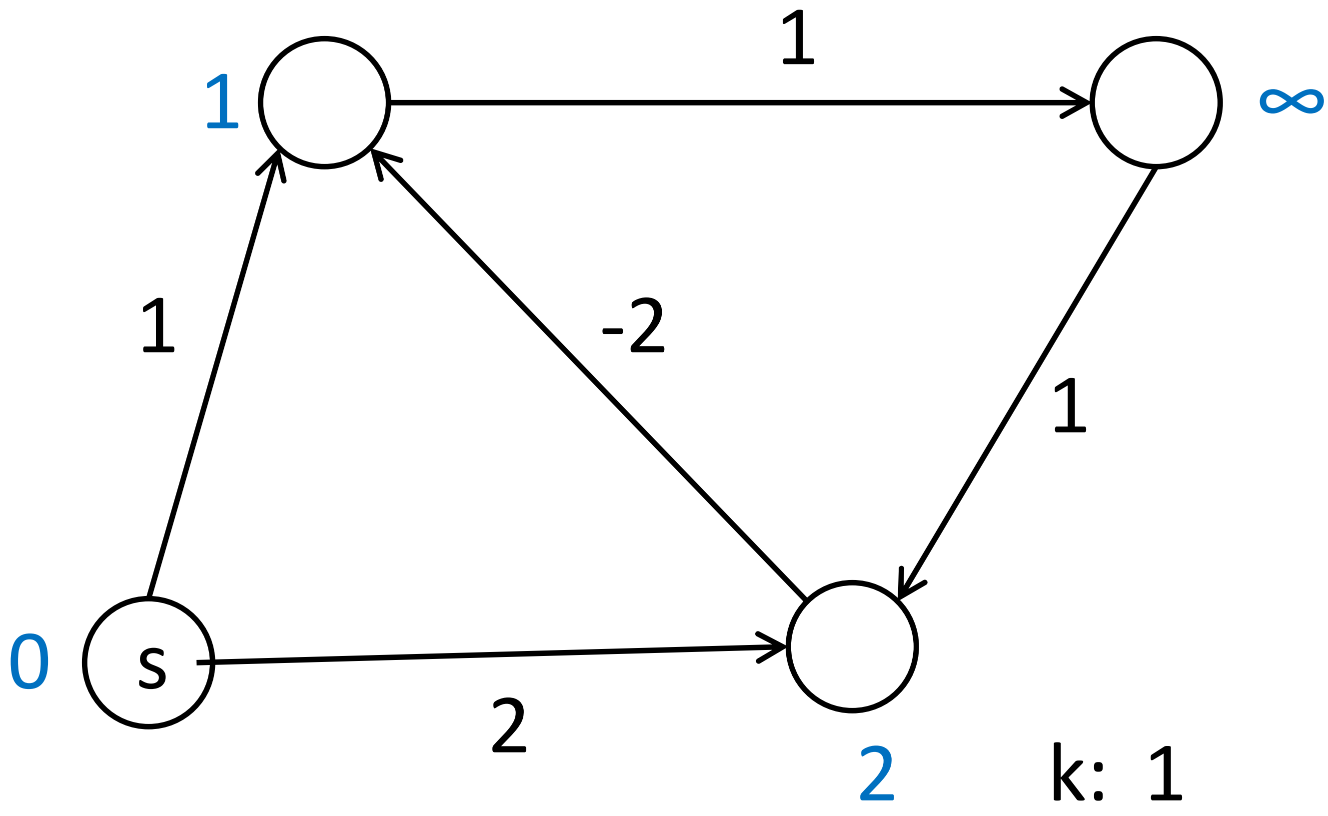

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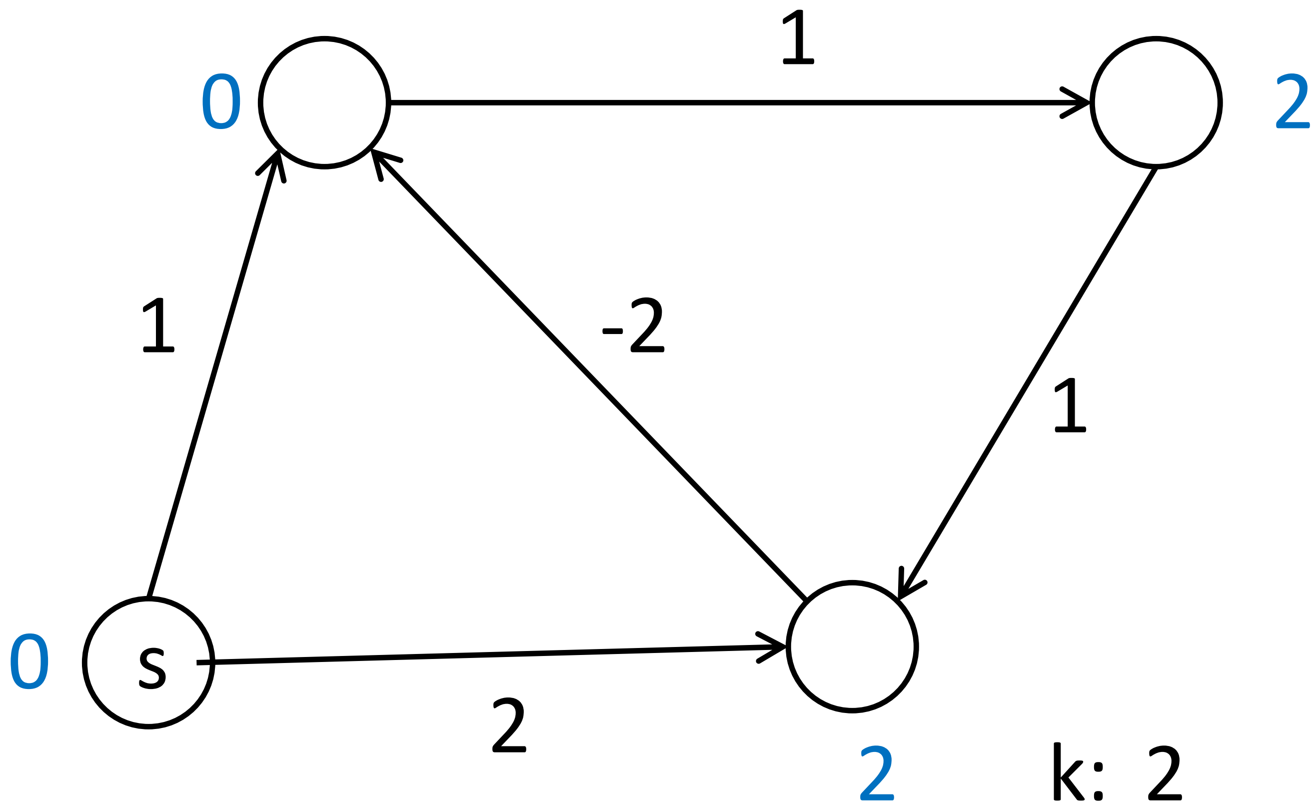

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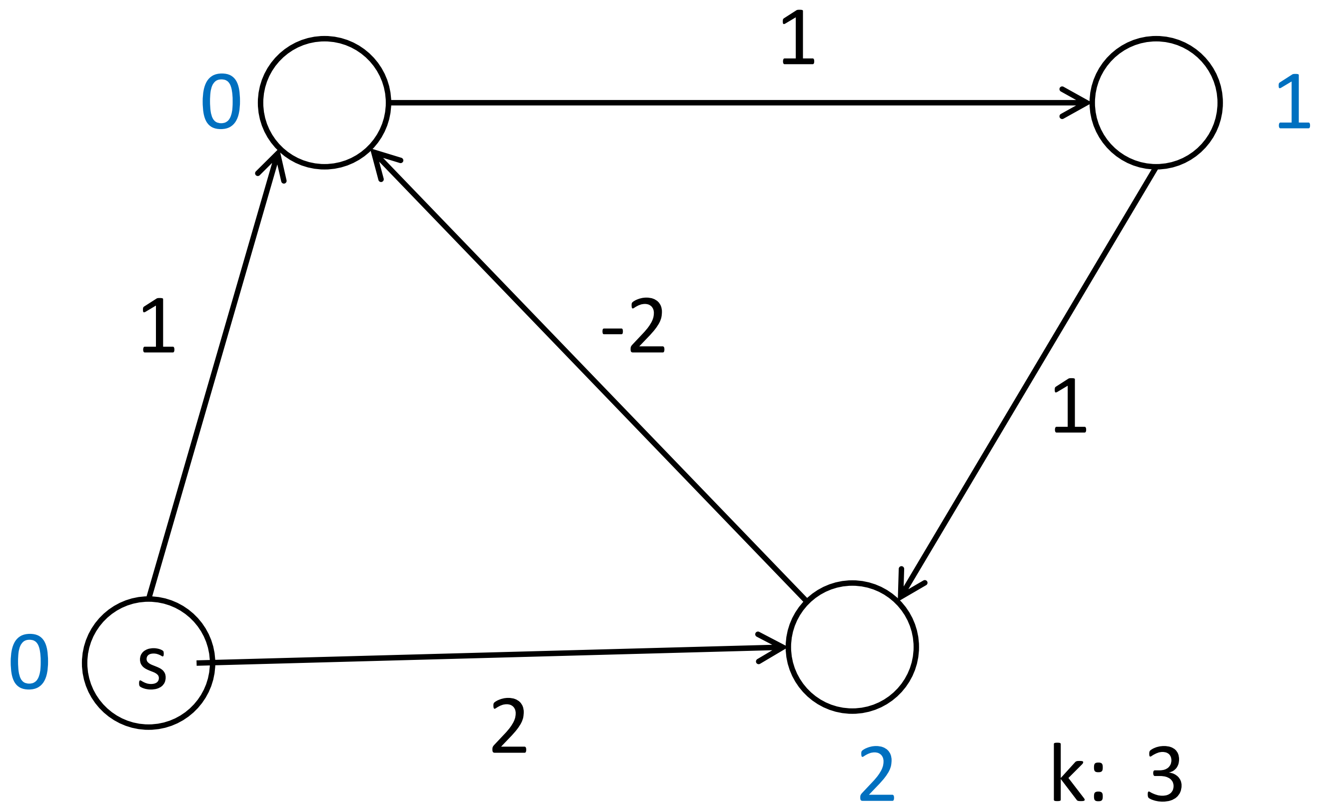

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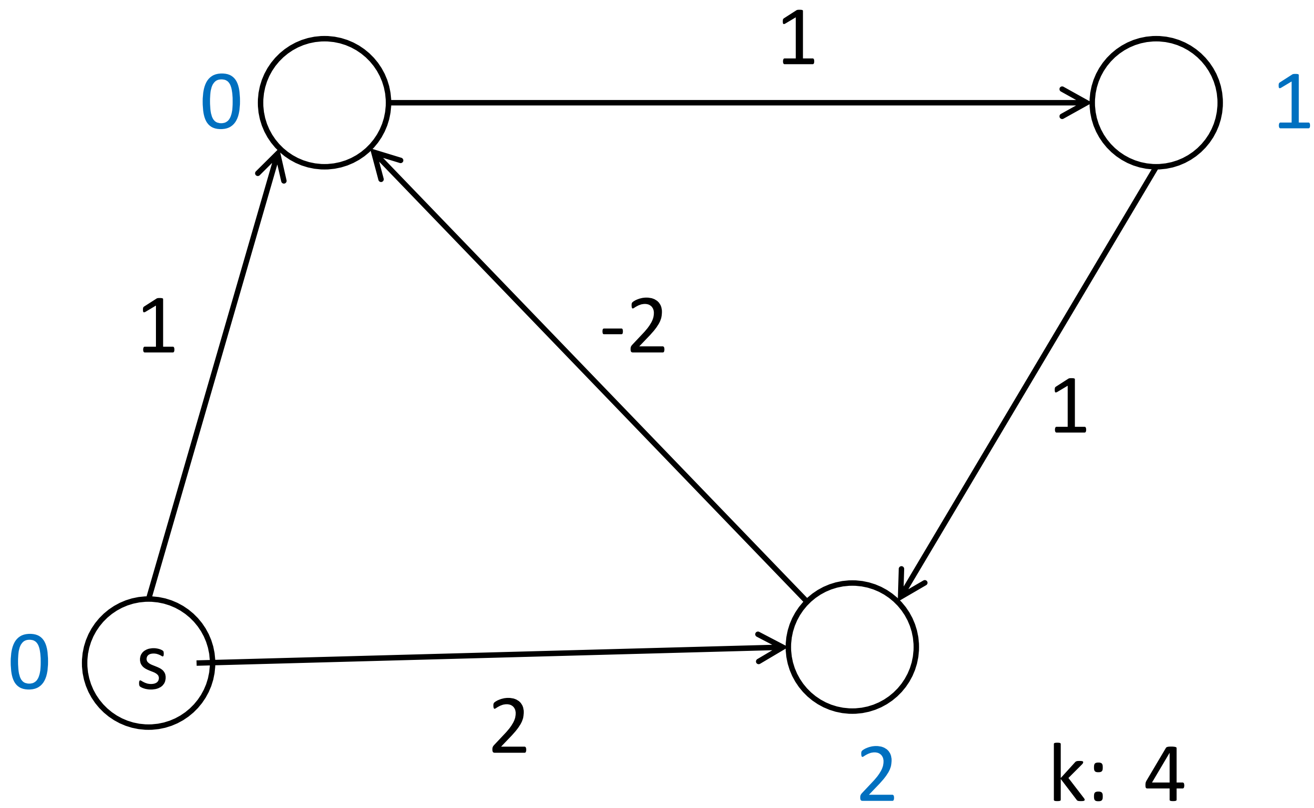

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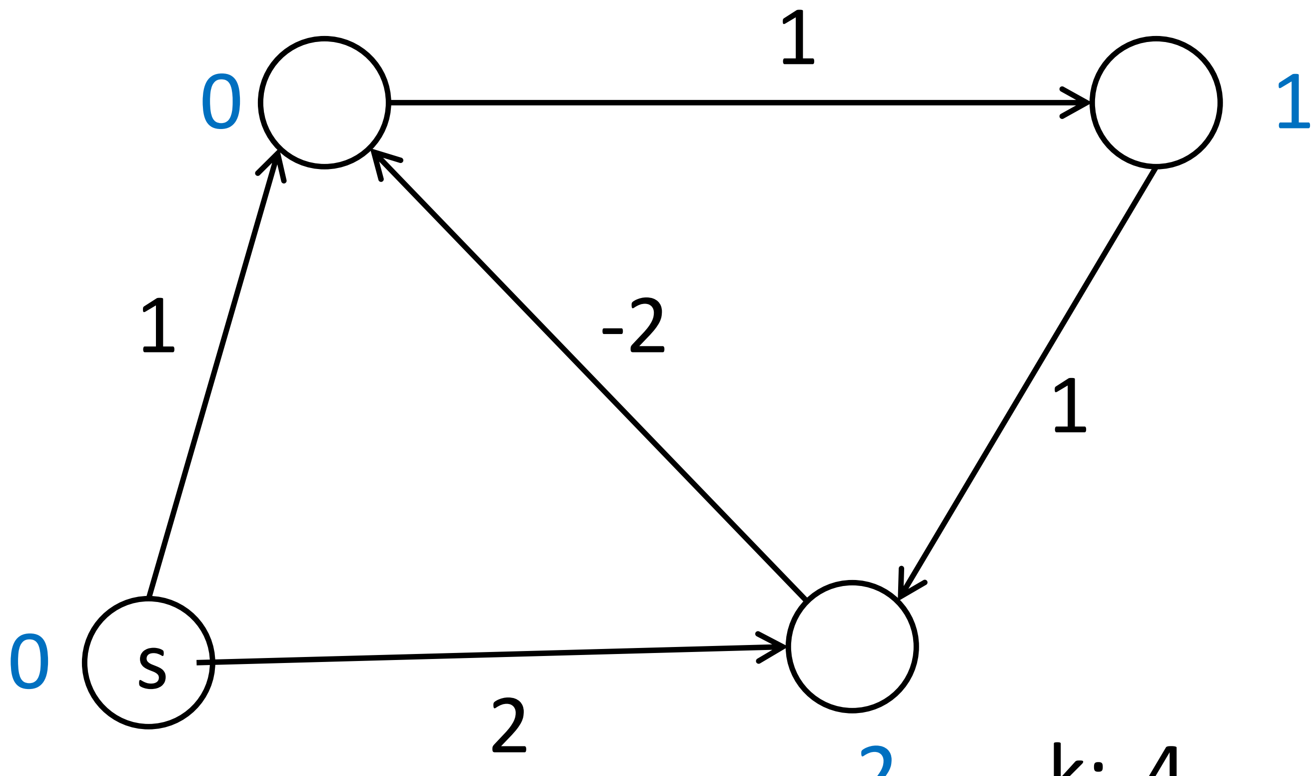

k: 4

Stabalizes

Analysis

Proposition: If $n \geq |V|-1$ and if G has no negative weight cycles, then for all v,

$$\text{dist}(v) = \text{dist}_n(v).$$

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Analysis

<u>Proposition:</u> If $n \geq |V|-1$ and if G has no negative weight cycles, then for all v,

$$\text{dist}(v) = \text{dist}_n(v).$$

- If there is a negative weight cycle, there probably is no shortest path.
- If not, we only need to run our algorithm for $|V|$ rounds, for a final runtime $O(|V||E|)$.

Proof

- We need to show that the shortest path has fewer than |V| edges.

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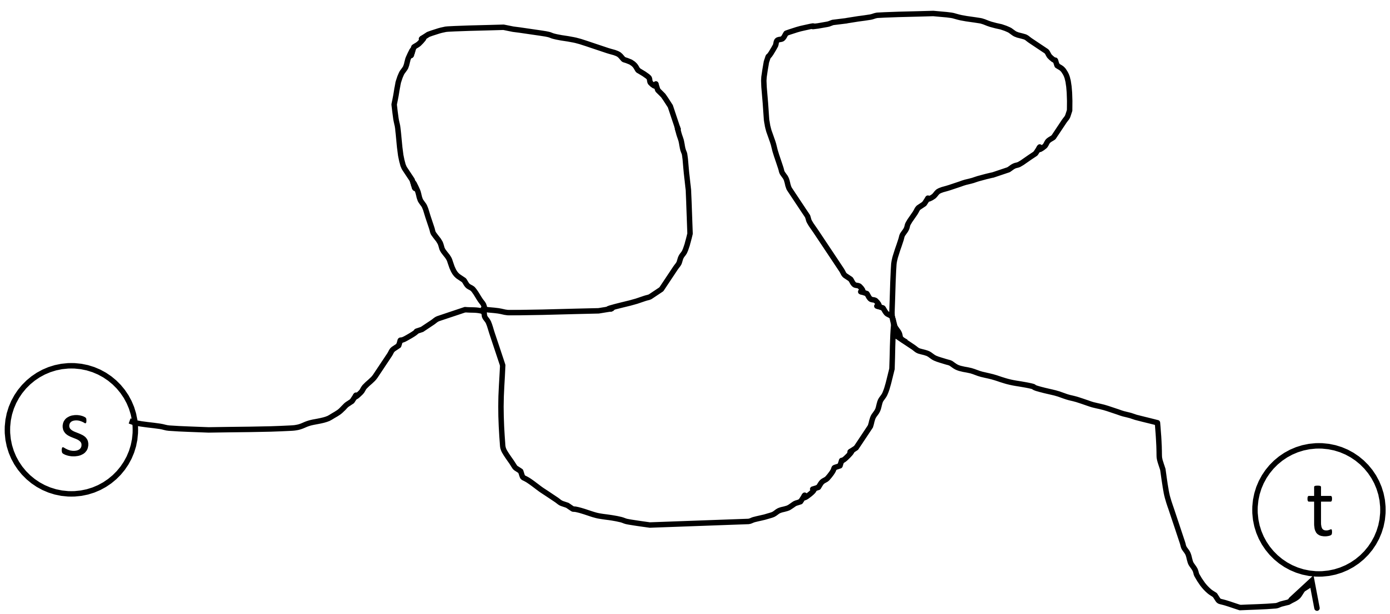
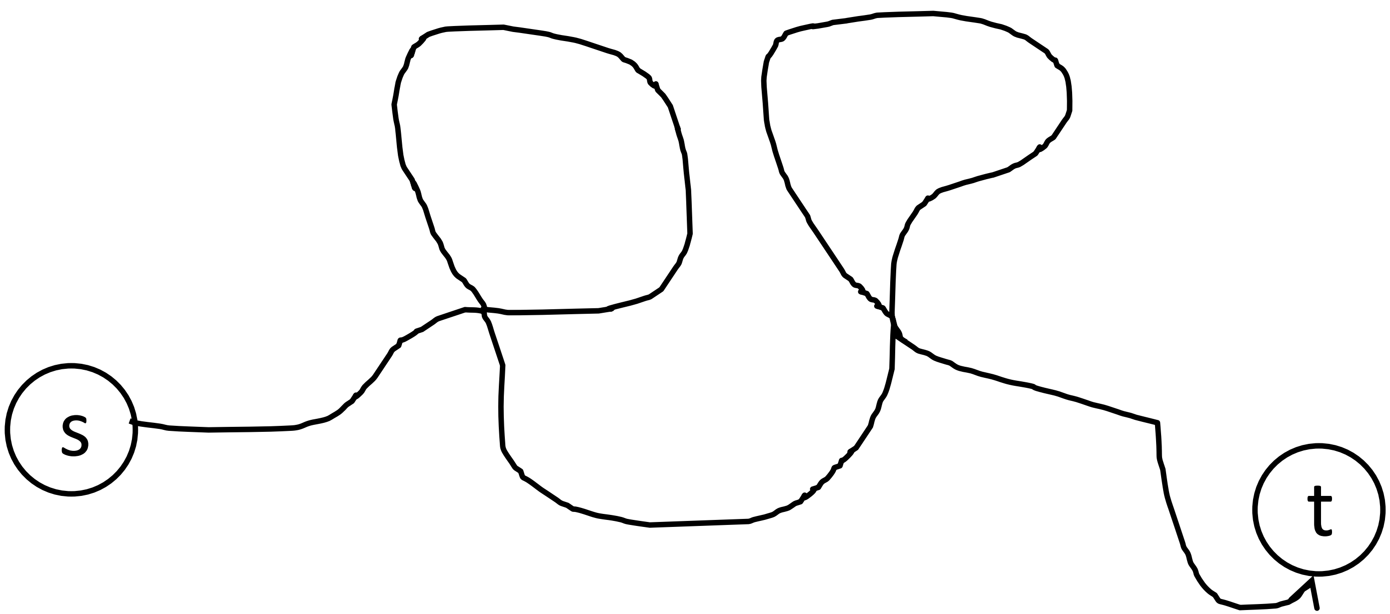
- If a path has at least |V| edges, it must contain the same vertex twice (by the pigeonhole principle).
- This means it has a loop.
- Removing the loop gives a shorter path.

Proof



Proof

Non-negative total weight
(no negative weight cycles)



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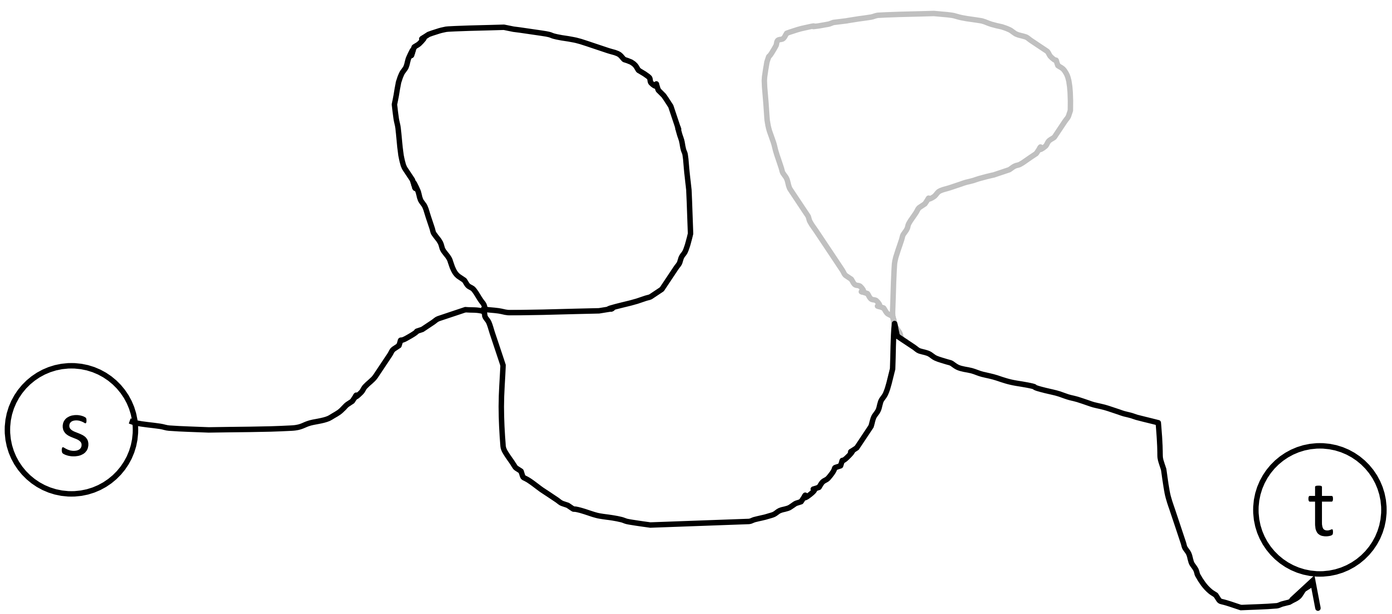

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Remove other cycles until none left

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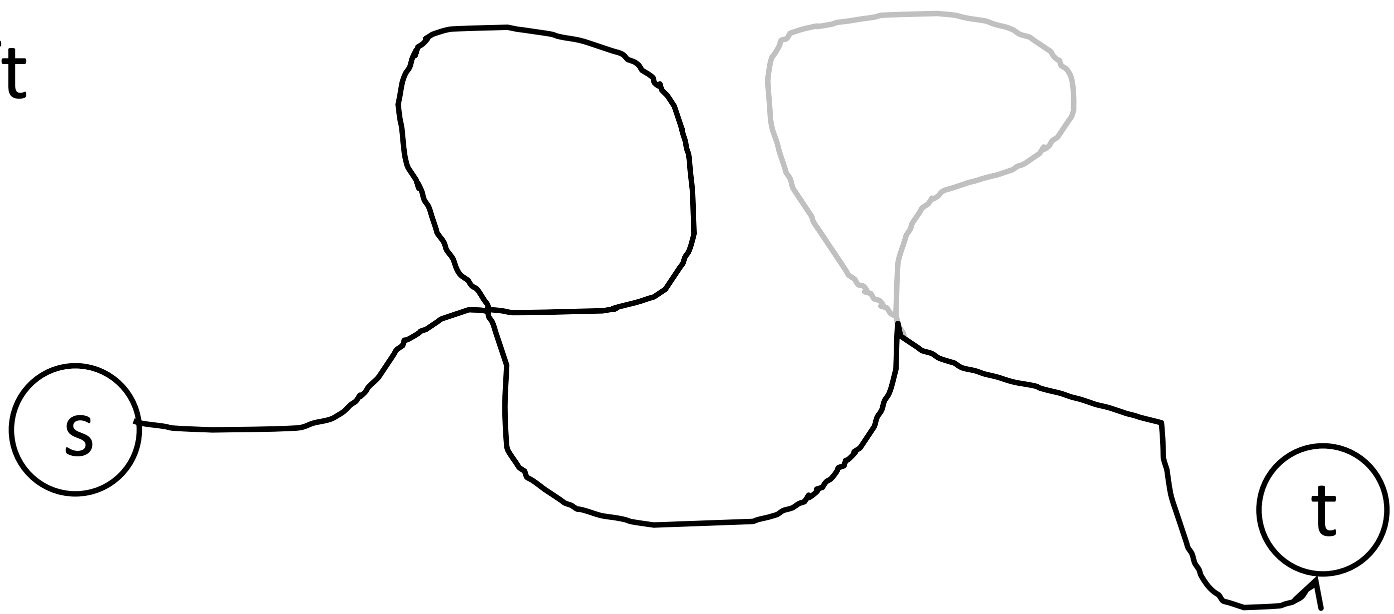

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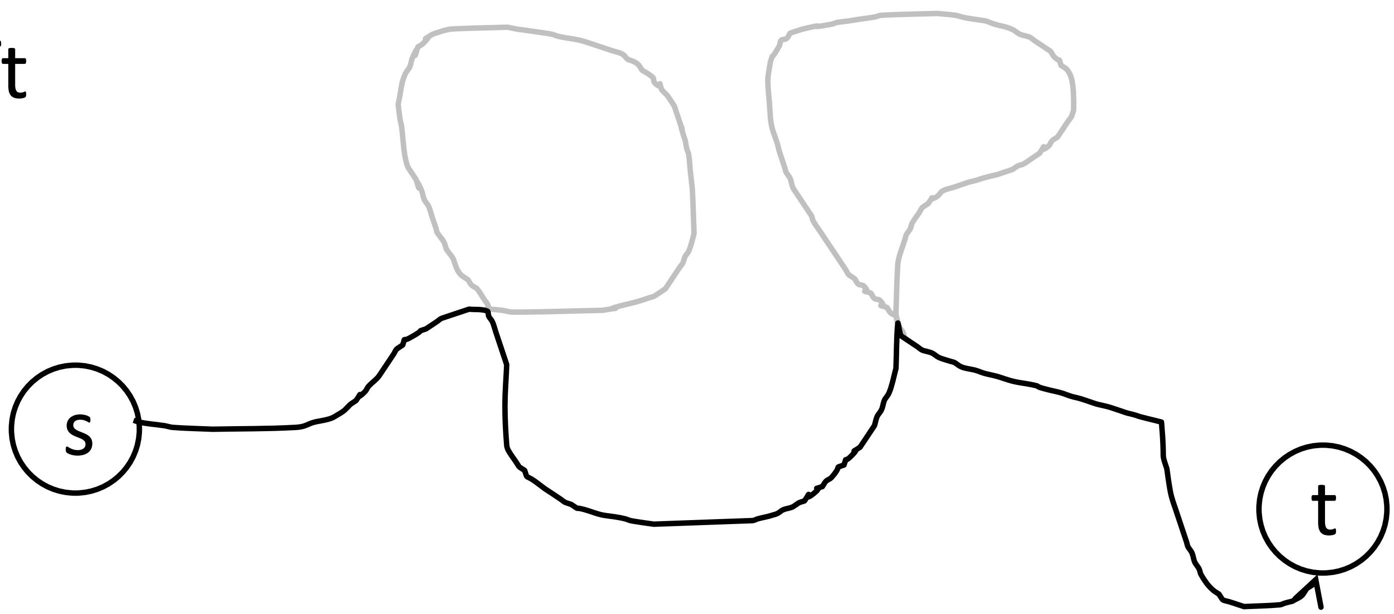

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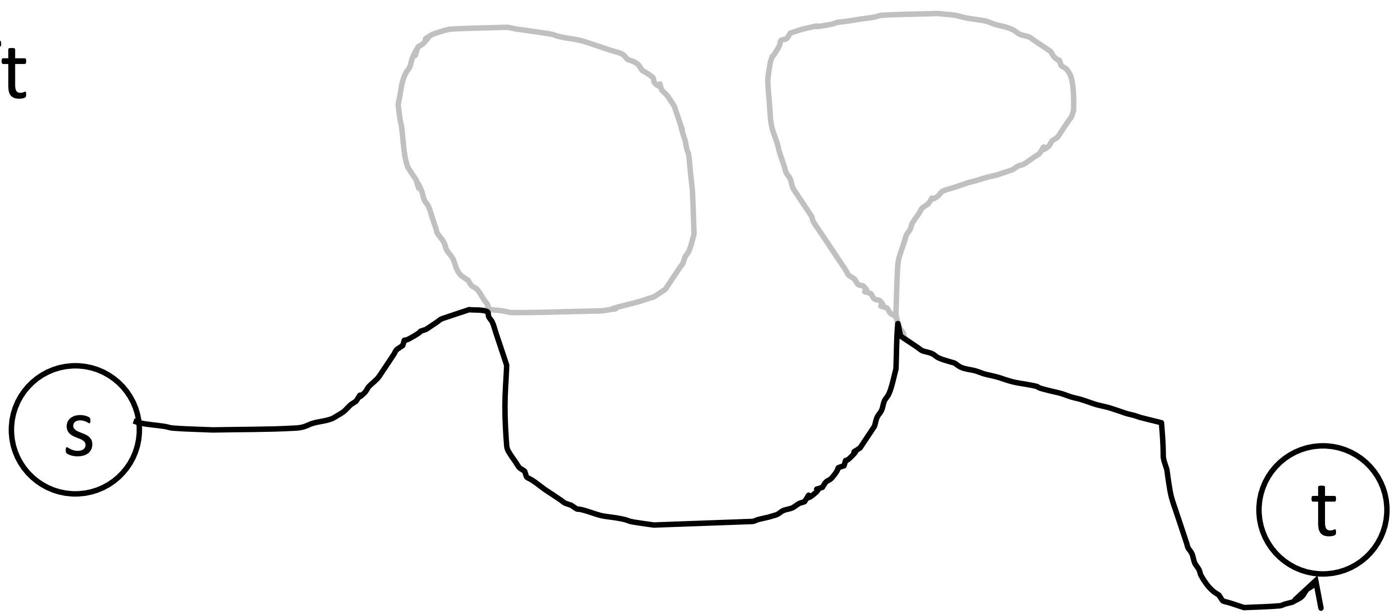

At most |V|-1 edges

New Algorithm

While Bellman-Ford computes shortest paths in time $O(|V||E|)$, it is possible to do better. A recent breakthrough gave an algorithm that runs in time

$$O(\log^8|V|\log(W)(|V|+|E|))$$

where W is the most negative edge weight.

Detecting Negative Cycles

If there are no negative weight cycles, Bellman-Ford computes shortest paths (and they might not exist otherwise).

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If there are no negative weight cycles, Bellman-Ford computes shortest paths (and they might not exist otherwise).

How do we know whether or not there are any?

Cycle Detection

Proposition: For any $n \geq |V| - 1$, there are no negative weight cycles reachable from s if and only if for every $v \in V$

$$\mathrm{dist}_n(v) = \mathrm{dist}_{n+1}(v)$$

Cycle Detection

Proposition: For any $n \geq |V| - 1$, there are no negative weight cycles reachable from s if and only if for every $v \in V$

$$dist_n(v) = dist_{n+1}(v)$$

- Detect by running one more round of Bellman-Ford.
- Need to see if *any* v's distance changes.

Proof of "Only If"

- Suppose no negative weight cycles.

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- So

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Suppose $dist_n(v) = dist_{n+1}(v)$ for all v.

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$$\begin{aligned}
\text{dist}_{n+2}(w) &= \min_{(v,w)\in E} (\text{dist}_{n+1}(v) + \ell(v,w)) \\
&= \min_{(v,w)\in E} (\text{dist}_{n}(v) + \ell(v,w)) \\
&= \text{dist}_{n+1}(w).
\end{aligned}$$

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But if there were a negative weight cycle, distances would decrease eventually.

Alternative Proof

- Assume $dist_n(v) = dist_{n+1}(v) = d(v)$ for all v.

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  - $\ell(v,w) \geq d(w) - d(v)$

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  - $\ell(v,w) \geq d(w) - d(v)$
- Given cycle $v_1,v_2,v_3,\ldots,v_m$ total length of cycle is

$$\ell(v_1, v_2) + \ell(v_2, v_3) + \ldots + \ell(v_m, v_1)$$

$$\geq \; -d(v_1) + d(v_2) - d(v_2) + d(v_3) - \ldots - d(v_m) + d(v_1) = 0.$$

Potential Function

- Let $\ell'(v,w) = \ell(v,w) - d(v) + d(w) \geq 0$

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- Let ℓ'(v,w) = ℓ(v,w)-d(v)+d(w) ≥ 0
  - Imagine somebody lending you d(w) when you arrive at w, but having to pay it back when you leave.
- For any s-t path P, s,v_1,v_2,…,t

$$\begin{aligned}
\ell'(P) &= \ell'(s,v_1) + \ell'(v_1,v_2) + \ldots + \ell'(v_m,t) \\
&= \ell(s,v_1) + \ell(v_1,v_2) + \ldots + \ell(v_m,t) \\
&\qquad - d(s) + d(v_1) - d(v_1) + d(v_2) + \ldots - d(v_m) + d(t) \\
&= \ell(P) - d(s) + d(t).
\end{aligned}$$

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- Shortest paths same. Non-negative edges.

Shortest Paths in DAGs

We saw that shortest paths is harder when we needed to deal with negative weight cycles. For general graphs, we needed to use Bellman-Ford, which is much slower than our other algorithms.

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One way to avoid this was to make edge weights non-negative. In this case, we could use Dijkstra.

Another way to get rid of negative weight cycles, is to get rid of cycles. If G is a DAG, there are better algorithms.

Fundamental Shortest Paths Formula

For $w \neq s$,

$$\mathrm{dist}(w) = \min_{(v,w) \in E} \mathrm{dist}(v) + \ell(v, w).$$

Hard to apply in general because there's no order to solve equations in.

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DAG gives topological order!

Algorithm

```
ShortestPathsInDAGs(G,s,ℓ)
  TopologicalSort(G)
  For w ∈ V in topological order
    If w = s, dist(w) ← 0
    Else
     dist(w)←min(dist(v)+ℓ(v,w))

\\ dist(v) for all upstream v
 already computed
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$O(|V|+|E|)$

$O(|V|)$ total

Algorithm

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O(|V|) total

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Algorithm

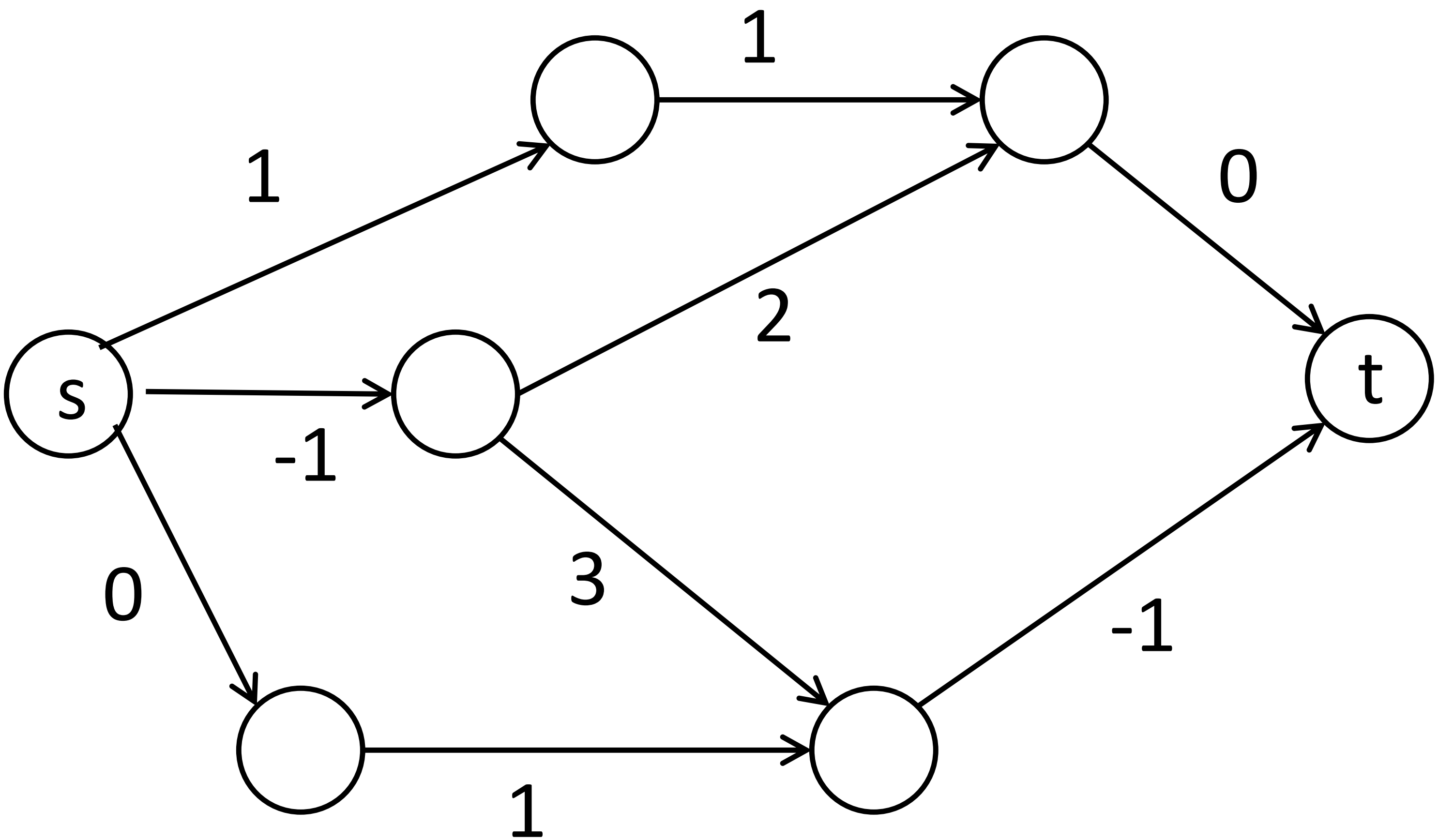
Runtime O(|V|+|E|)

```
ShortestPathsInDAGs(G,s,ℓ)
   TopologicalSort(G)                      O(|V|+|E|)
   For w ∈ V in topological order
      If w = s, dist(w) ← 0
      Else                                 O(|E|) total
        dist(w)←min(dist(v)+ℓ(v,w))
```

O(|V|) total

\\ dist(v) for all upstream v already computed

Example



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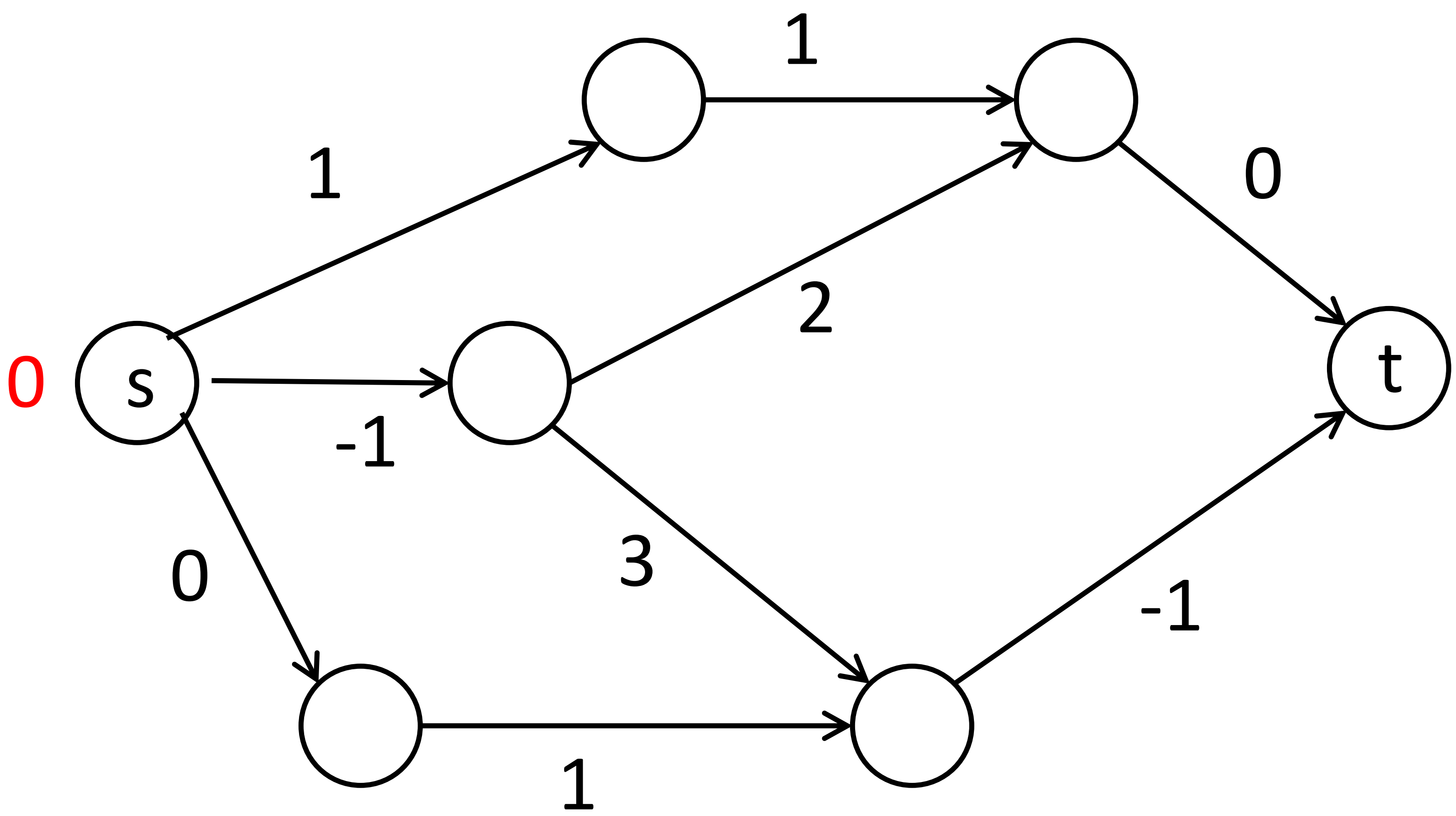

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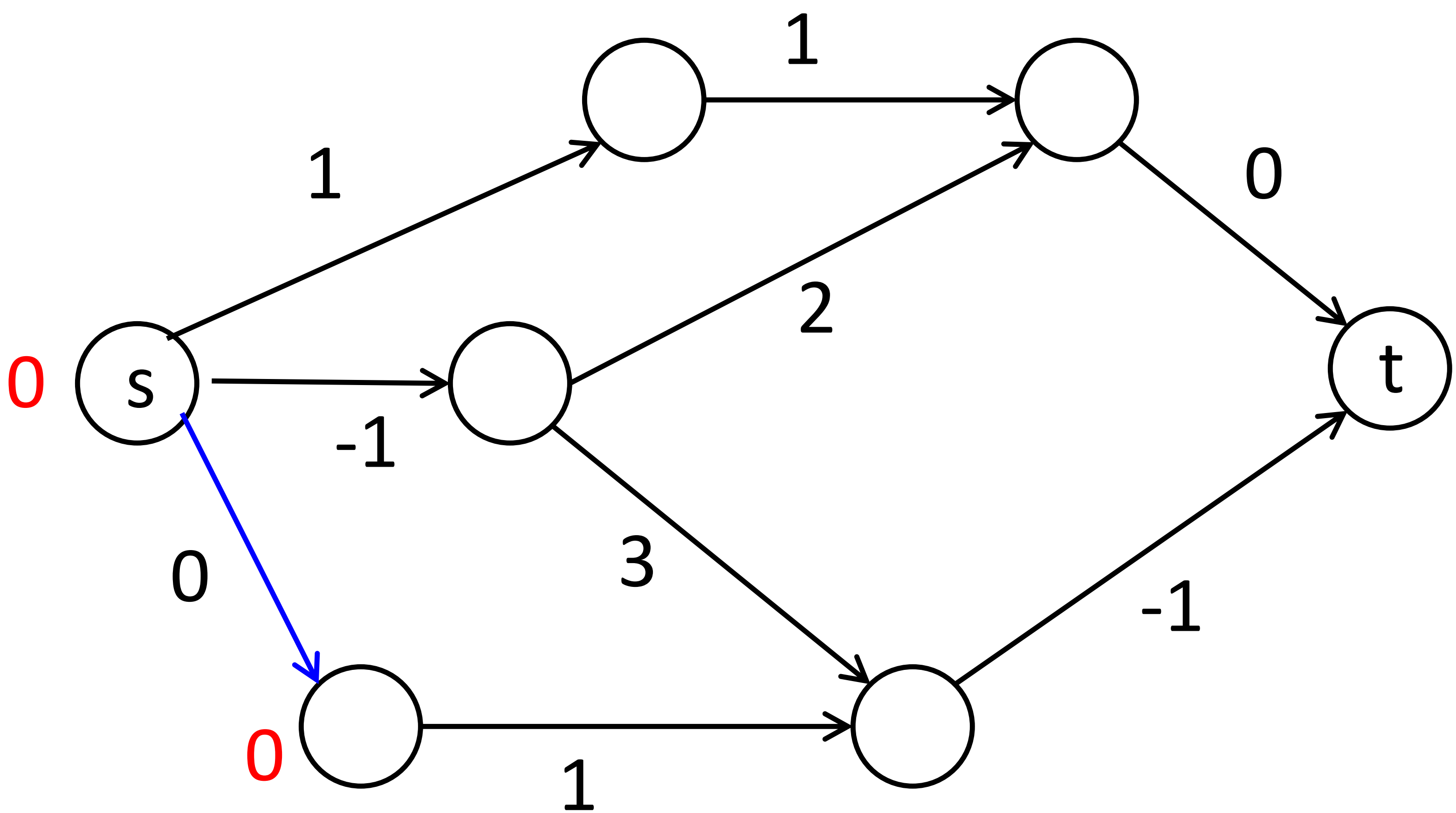

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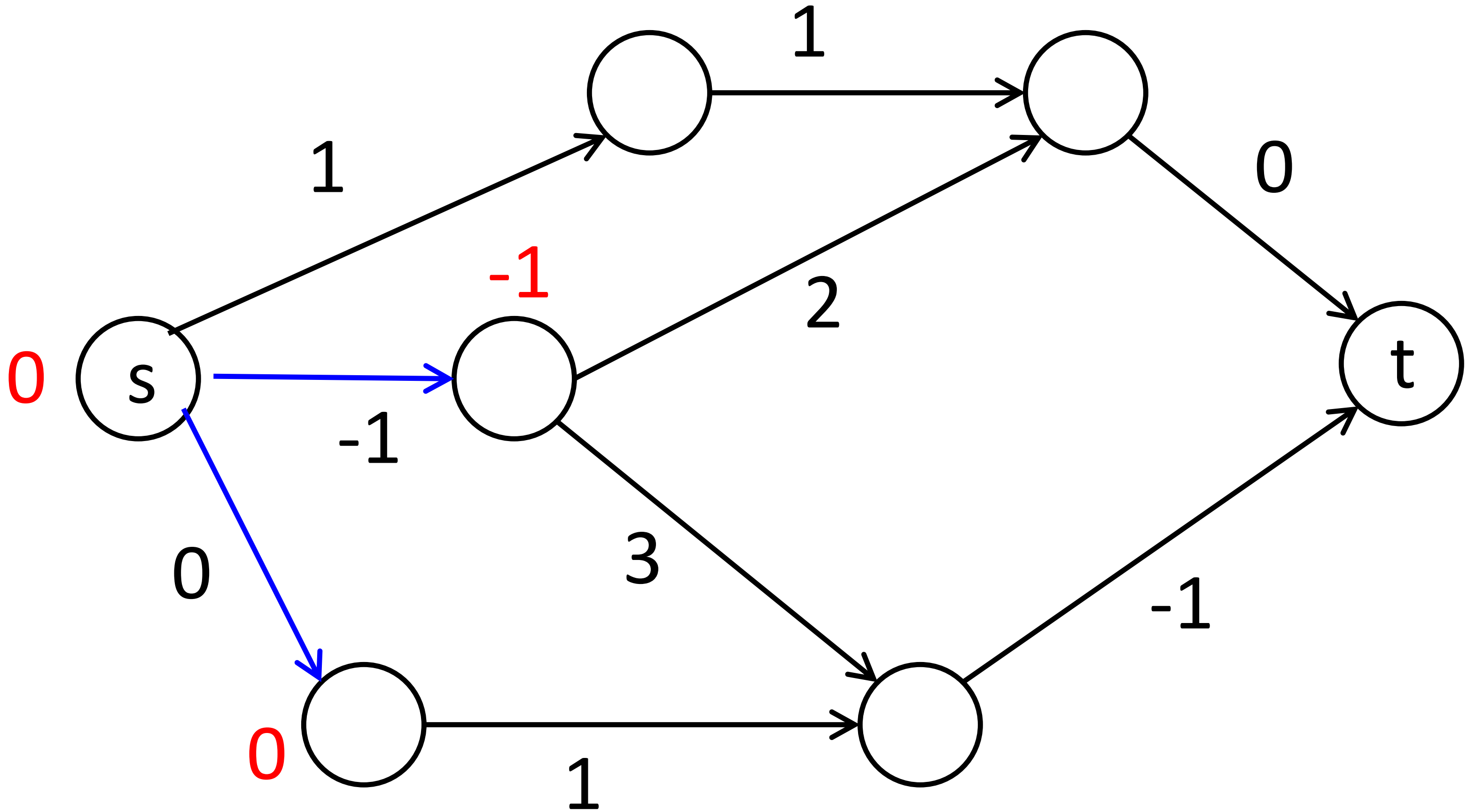

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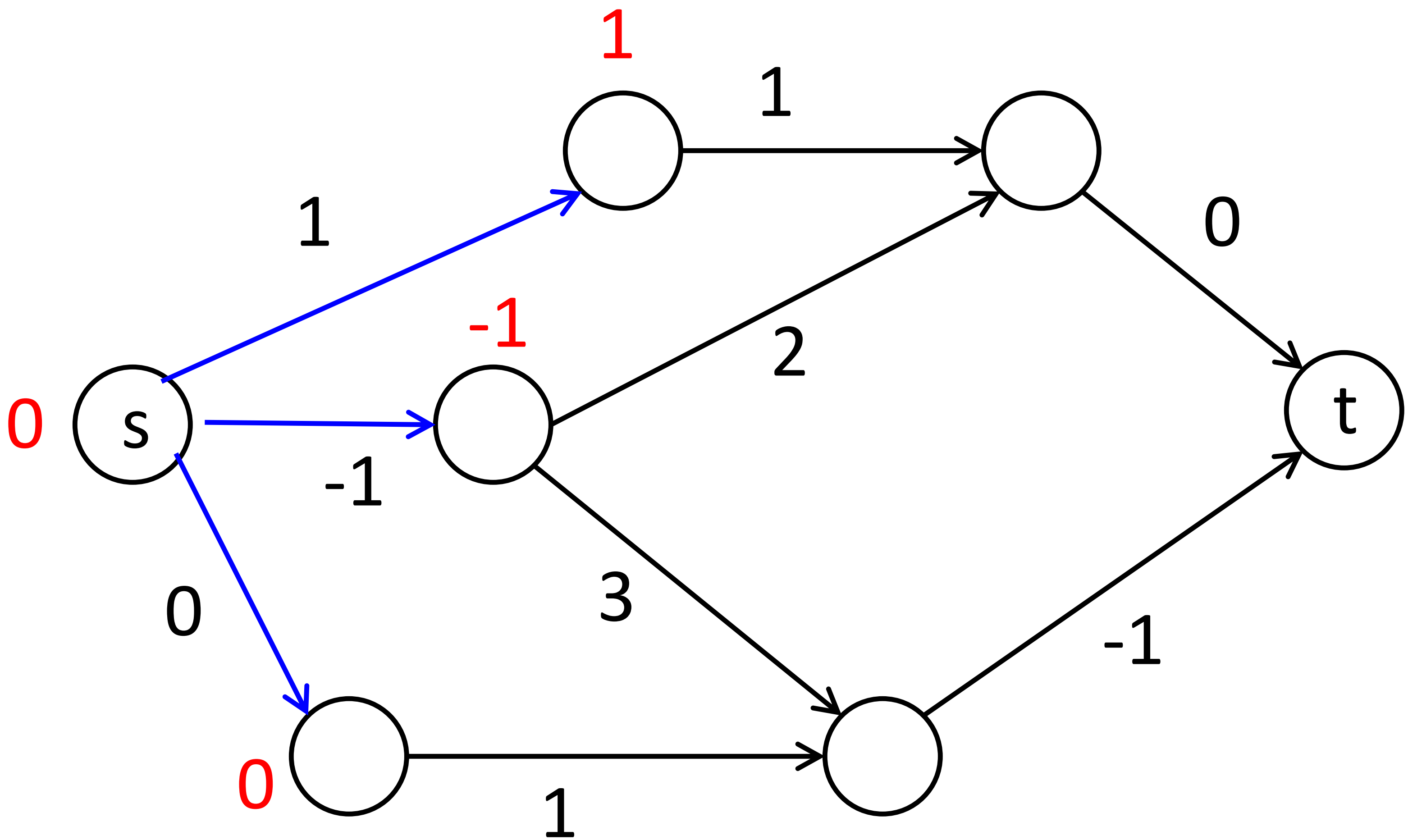

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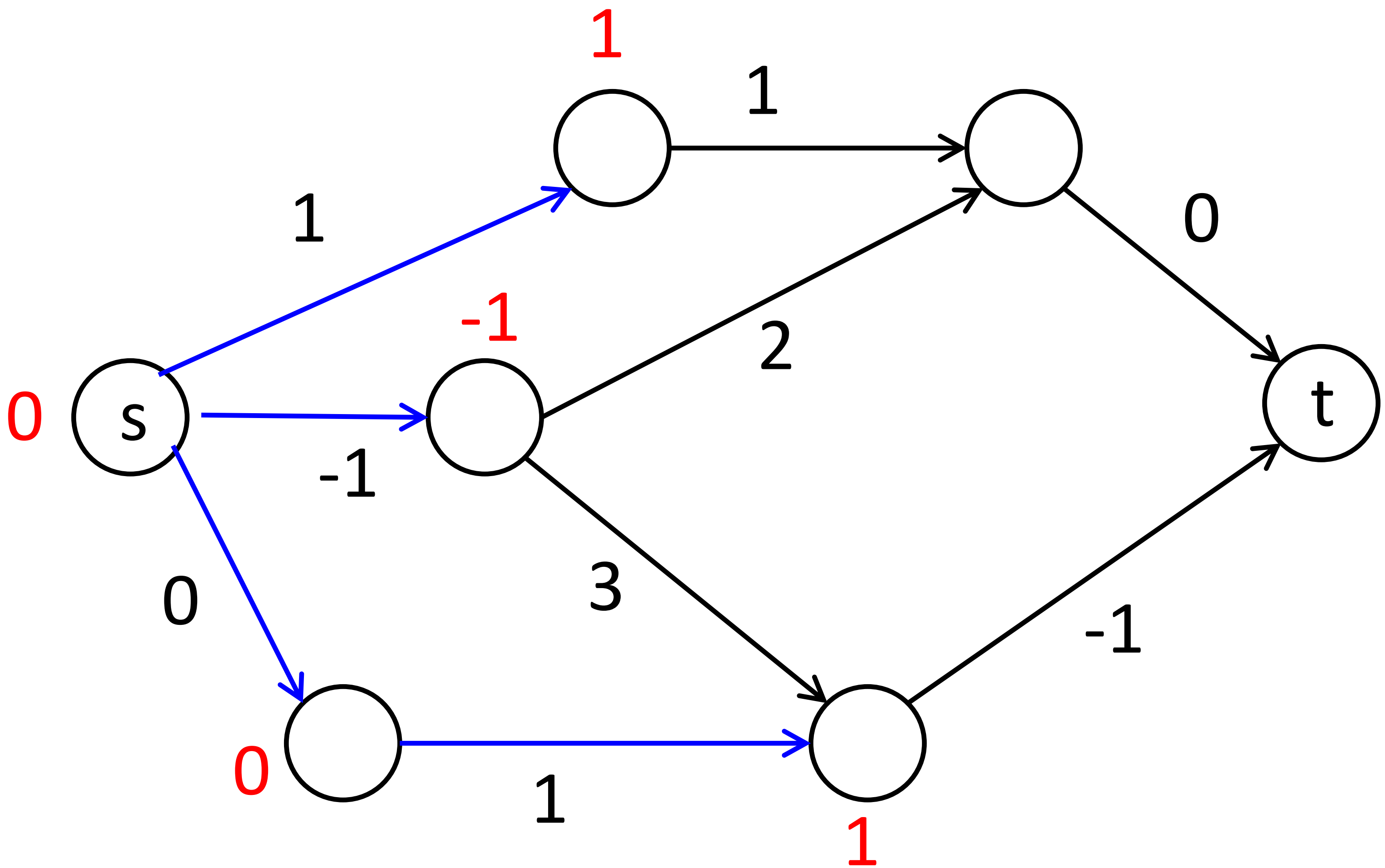

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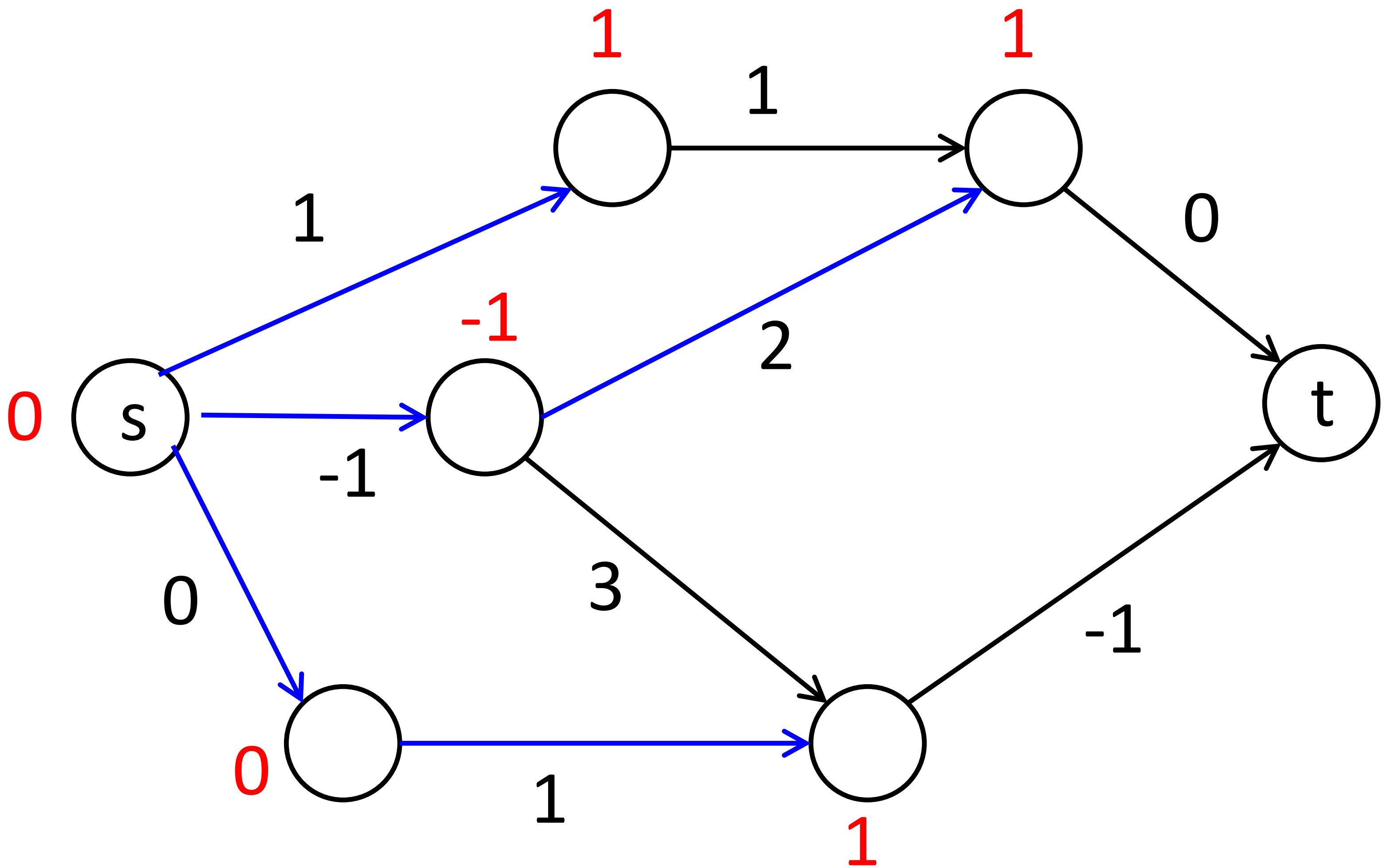

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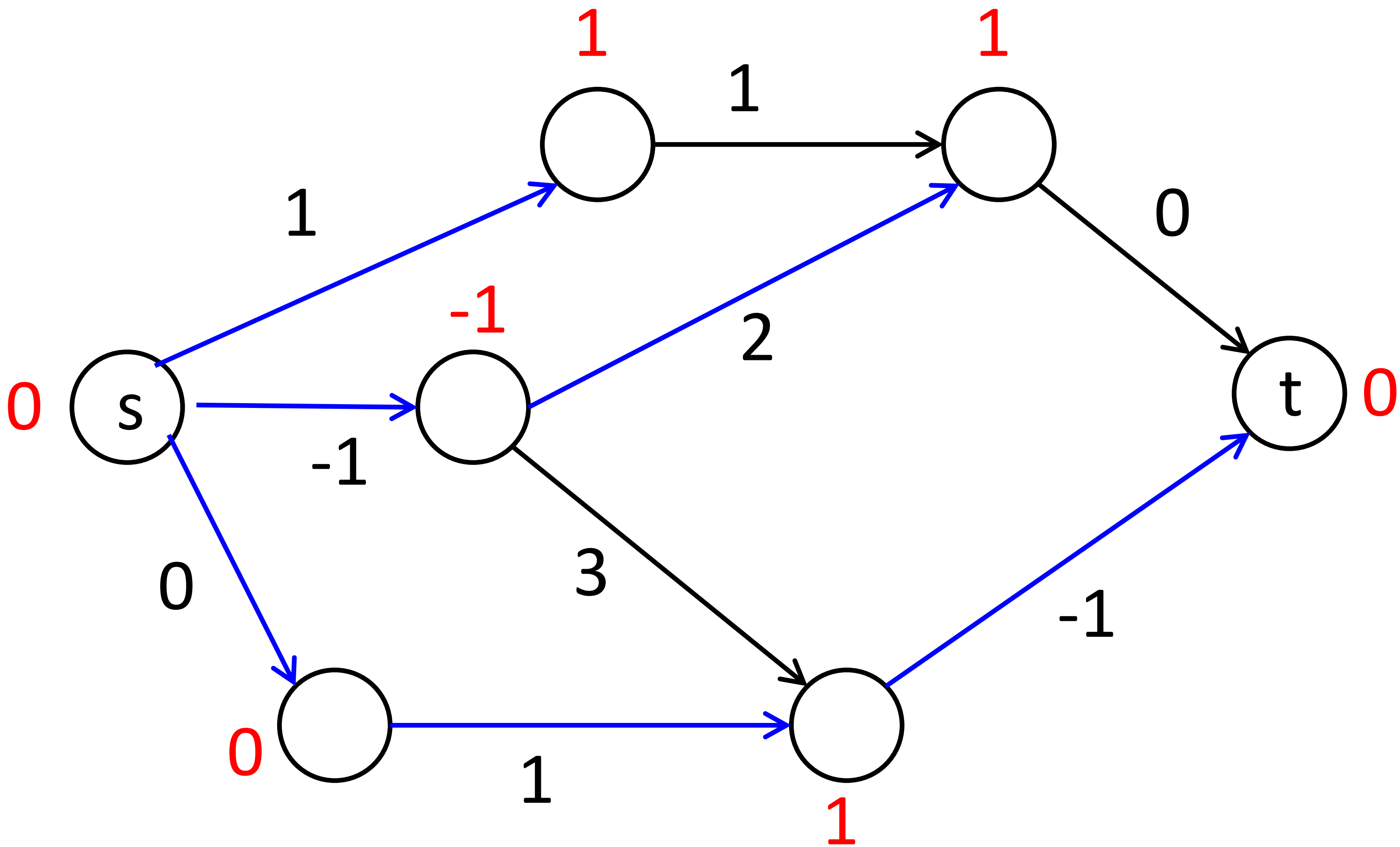

Shortest Path Algorithms Summary

Unit Weights: `Breadth First Search`
$O(|V|+|E|)$

Non-negative Weights: `Dijkstra`
$O(|V|\log|V|+|E|)$

Arbitrary Weights: `Bellman-Ford` $O(|V||E|)$

Arbitrary Weights, graph is a DAG:
`Shortest-Paths-In-DAGs` $O(|V|+|E|)$

Divide & Conquer (Ch 2)

- General Technique
- Master Theorem
- Karatsuba Multiplication
- Strassen’s Algorithm
- Merge Sort
- Order Statistics
- Binary Search
- Closest Pair of Points

Divide and Conquer

This is the first of our three major algorithmic techniques.

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1. Break problem into pieces
2. Solve pieces recursively
3. Recombine pieces to get answer

Example: Integer Multiplication

Problem: Given two n-bit numbers find their product.

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Naïve Algorithm: Schoolboy multiplication. The binary version of the technique that you probably learned in elementary school.

Schoolboy Multiplication

$$
\begin{array}{cccccc}
 & a_1 & a_2 & \dots & a_{n-1} & a_n \\
\text{x} & b_1 & b_2 & \dots & b_{n-1} & b_n \\
\hline
\end{array}
$$

Schoolboy Multiplication

$$
\begin{array}{cccccccc}
 & a_1 & a_2 & \ldots & a_{n-1} & a_n \\
x & b_1 & b_2 & \ldots & b_{n-1} & b_n \\
\hline
a_1 b_n & a_2 b_n & a_3 b_n & \ldots & a_{n-1} b_n & a_n b_n
\end{array}
$$

Schoolboy Multiplication

$$\begin{array}{ccccccc}
 & & a_1 & a_2 & \dots & a_{n-1} & a_n \\
 & \times & b_1 & b_2 & \dots & b_{n-1} & b_n \\
\hline
 & a_1 b_n & a_2 b_n & a_3 b_n & \dots & a_{n-1} b_n & a_n b_n \\
a_1 b_{n-1} & a_2 b_{n-1} & a_3 b_{n-1} & a_4 b_{n-1} & \dots & a_n b_{n-1} & 0
\end{array}$$

Schoolboy Multiplication

			a_1	a_2	…	a_{n-1}	a_n
			x b_1	b_2	…	b_{n-1}	b_n
			------------	------------	------------	------------	------------
		a_1b_n	a_2b_n	a_3b_n	…	$a_{n-1}b_n$	a_nb_n
	a_1b_{n-1}	a_2b_{n-1}	a_3b_{n-1}	a_4b_{n-1}	…	a_nb_{n-1}	0
	…	…	…	…	…	…	…
$+a_1b_n$	a_2b_n	a_3b_n	…	a_nb_n	0	0	0

Schoolboy Multiplication

$$
\begin{array}{rccccccc}
 & & & a_1 & a_2 & \ldots & a_{n-1} & a_n \\
 & & \mathrm{x} & b_1 & b_2 & \ldots & b_{n-1} & b_n \\
\hline
 & & a_1b_n & a_2b_n & a_3b_n & \ldots & a_{n-1}b_n & a_nb_n \\
 & a_1b_{n-1} & a_2b_{n-1} & a_3b_{n-1} & a_4b_{n-1} & \ldots & a_nb_{n-1} & 0 \\
 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
+a_1b_n & a_2b_n & a_3b_n & \ldots & a_nb_n & 0 & 0 & 0 \\
\hline
\end{array}
$$

ANSWER

Question: Runtime

What is the asymptotic runtime of the schoolboy algorithm?

A) $O(n)$
B) $O(n \log(n))$
C) $O(n^2)$
D) $O(n^3)$
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Need to write down $O(n^2)$ bits of numbers to add. Addition can be done in linear time.