CSE 20 Spring 2022
Homework 6
Due date: Thursday, May 19, 2022 at 11:59pm

In this assignment,
You will practice proof by structural induction, regular induction and strong induction.
In this class, unless the instructions explicitly say otherwise, you are required to justify all your answers.

1. (8 points) Consider the recursively defined set of binary strings B defined by:

   Basis Step: $0 \in B, 1 \in B$
   Recursive Step: if $x \in B$ then $xx \in B$ where xx is the string concatenated with itself.

   Prove using structural induction that for all elements x of B, the length of x is an integer power of 2.

   Solution:
   proof by structural induction:
   Basis Step: Show that it is true for 0 and 1.

   - 0 has length 1 and 1 is a power of 2 because $1 = 2^0$.
   - 1 has length 1 and 1 is a power of 2 because $1 = 2^0$.

   Inductive Step: Let y be an arbitrary element of B. Assume that the length of y is a power of 2, i.e. that there exists an integer k such that the length of y is equal to 2^k.
   (We must show that the length of yy is a power of 2.)
   Then yy is y concatenated with itself so the length of yy is twice the length of y. The length of y is 2^k so the length of yy is $2(2^k) = 2^{k+1}$ and since $k+1$ is an integer then the length of yy is an ingeter power of 2.

2. (8 points) Recall the recursive definition of $\mathbb{N}$.

   Basis Step: $0 \in \mathbb{N}$
   Recursive Step: If $m \in \mathbb{N}$ then $m + 1 \in \mathbb{N}$.

   Consider the function $sumeven : \mathbb{N} \to \mathbb{N}$ defined recursively as:

   Basis Step: $sumeven(0) = 0$
   Recursive Step: If $m \in \mathbb{N}$ then $sumeven(m + 1) = sumeven(m) + (2m + 2)$

   Use structural induction to show that for all $n \in \mathbb{N}$, that $sumeven(n) = n(n + 1)$.

   Solution:
   proof by structural induction:
   Basis Step: Show that it is true for 0.

   - $sumeven(0) = 0$ and $0(0 + 1) = 0$.

   Inductive Step: Let k be an arbitrary element of $\mathbb{N}$. Assume that $sumeven(k) = k(k+1)$.
   (We must show that $sumeven(k + 1) = (k + 1)(k + 2)$.)
   Then $sumeven(k+1) = sumeven(k)+(2k+2)$. By the inductive hypothesis, $sumeven(k) = k(k + 1)$ so

   $$sumeven(k + 1) = k(k + 1) + 2k + 2 = (k + 2)(k + 1)$$

   as required.

3. (8 points) Use induction to show that
   For all integers $n \in \mathbb{Z}^+$, $4|(5^n - 1)$

Solution:
Proof by induction.
Base case: When $n = 1$, $5^1 - 1 = 4$ and $4|4$ as required.
Inductive step:
Let k be an arbitrary integer such that $k \geq 1$ and assume that $4|5^k - 1$.
(Want to show that $4|5^{k+1} - 1$.)
By the inductive hypothesis, $4|5^k - 1$ which means that there exists an integer c such that $5^k - 1 = 4c$ or in other words, $5^k = 4c + 1$.
Consider $5^{k+1} - 1$. We can rewrite it as:

$$5^{k+1} - 1 = 5(5^k) - 1 = 5(4c + 1) - 1 = 20c + 5 - 1 = 20c + 4 = 4(5c + 1)$$

and since $5c + 1$ is an integer, this means that $4|5^{k+1} - 1$ as required.

4. (10 points) Suppose you only have 5 cent coins and 9 cent coins, Fill in the blanks of the following strong induction proof that you can make change for any integer amount $n \geq 32$.

Solution: By strong induction on n.
Basis step: Show that it is true for:

- $n = 32$ $5(1) + 9(3)$
- $n = 33$ $5(3) + 9(2)$
- $n = 34$ $5(5) + 9(1)$
- $n = 35$ $5(7) + 9(0)$
- $n = 36$ $5(0) + 9(4)$

Inductive step: Let h be an arbitrary integer such that $h \geq$ 37 and assume that, for all m with 32 $\leq m < h$, we can make change for m cents.
WTS: we can make change for h cents.
Since 32 $\leq h - 5 < h$, by the Inductive Hypothesis
you can make change for $h - 5$ cents using 5 and 9 cent coins, i.e., $h - 5 = 5x + 9y$
Therefore we can make change for h cents using 5 cent coins and 9 cent coins by
adding one more 5 cent coin to get $h = 5(x + 1) + 9y$.

5. Let $A(n)$ be defined in the following way: $A(0) = 2$, $A(1) = 5$ and $A(n) = 5A(n-1) - 6A(n-2)$ for all $n \geq 2$.

Prove that $A(n) = 3^n + 2^n$ for all $n \geq 0$. (using strong induction.)

Solution:
Basis Step: $A(0) = 2$ and $3^0 + 2^0 = 2$.
$A(1) = 5$ and $3^1 + 2^1 = 5$.
Inductive step: Let h be an arbitrary integer such that $h \geq 2$.
Assume that $A(m) = 3^m + 2^m$ for all integers m in the range $0 \leq m < h$.
(Want to show that $A(h) = 3^h + 2^h$.)

$$\begin{aligned} A(h) &= 5A(h-1) - 6A(h-2) \\ &= 5\left(3^{h-1} + 2^{h-1}\right) - 6\left(3^{h-2} + 2^{h-2}\right) \\ &= 5 * 3^{h-1} - 6 * 3^{h-2} + 5 * 2^{h-1} - 6 * 2^{h-2} \\ &= (15 - 6) * 3^{h-2} + (10 - 6) * 2^{h-2} \\ &= 3^2 * 3^{h-2} + 2^2 * 2^{h-2} \\ &= 3^h + 2^h \end{aligned}$$