Announcements

- Homework 0 Due today
- Homework 1 online due next Friday
- Discussion section notes/podcast online
- Remember FinAid survey
- No class on Monday

Last Time

- Graphs
- explore/DFS
- Connected components

Graph Definition

Definition: A *graph* G = (V,E) consists of two things:

- A collection V of *vertices*, or objects to be connected.
- A collection E of *edges*, each of which connects a pair of vertices.

Runtime of DFS

```
explore(v)
  v.visited ← true
  For each edge (v,w)
     If not w.visited
          explore(w)
```

Run once per vertex

O(|V|) total

Run once per neighboring vertex

O(|E|) total

```
DFS(G)
  Mark all v ∈ G as unvisited
  For v ∈ G
    If not v.visited, explore(v)
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O(|V|)

Final runtime: O(|V|+|E|)

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Only Counting work from Non-recursive calls!

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O(|V|)

Final runtime: O(|V|+|E|)

Connected Components

Theorem: The vertices of a graph G can be partitioned into *connected components* so that v is reachable from w if and only if they are in the same connected component.

Today

- Computing connected components
- Pre- / Post- orders
- Directed graphs
- Topological orderings

Problem: Computing Connected Components

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Problem: Given a graph G, compute its connected components.

Easy: For each v, run `explore(v)` to find vertices reachable from it. Group together into components.

Runtime: $O(|V|(|V|+|E|))$.

Better: Run `explore(v)` to find the component of v. Repeat on unclassified vertices.

DFS lets us do this!

```
ConnectedComponents(G)

  For v ∈ G
    v.visited ← false
  For v ∈ G
    If not v.visited

      explore(v)
```

```
explore(v)
  v.visited ← true

  For each edge (v,w)
    If not w.visited
      explore(w)
```

DFS lets us do this!

```
ConnectedComponents(G)
  CCNum ← 0
  For v ∈ G
    v.visited ← false
  For v ∈ G
    If not v.visited
      CCNum++
      explore(v)
```

```
explore(v)
  v.visited ← true

  For each edge (v,w)
    If not w.visited
      explore(w)
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DFS lets us do this!

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  For v ∈ G
    v.visited ← false
  For v ∈ G
    If not v.visited
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      explore(v)
```

```
explore(v)
  v.visited ← true
  v.CC ← CCNum
  For each edge (v,w)
    If not w.visited
      explore(w)
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DFS lets us do this!

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  For v ∈ G
    If not v.visited
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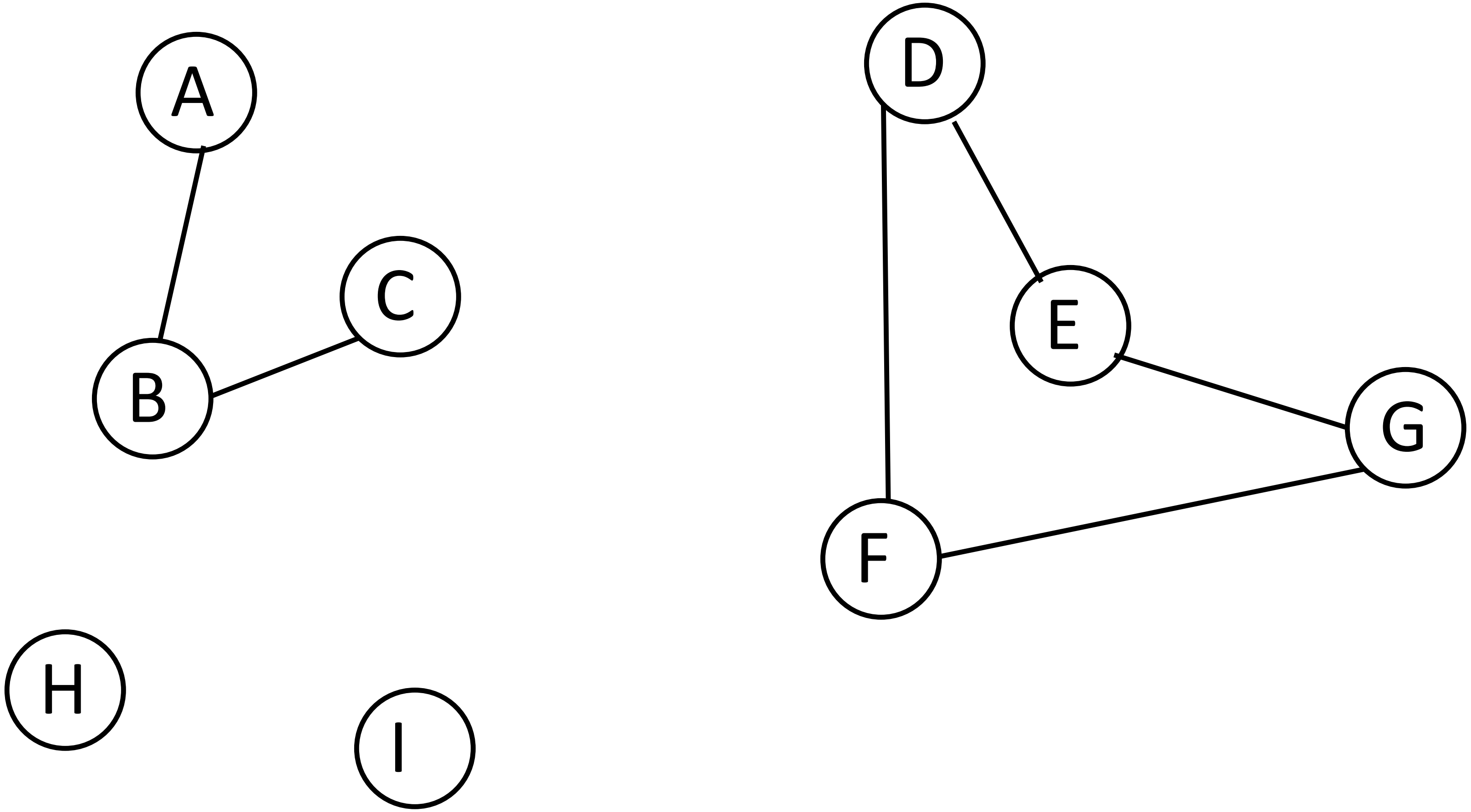
      explore(v)
```

```
explore(v)
  v.visited ← true
  v.CC ← CCNum
  For each edge (v,w)
    If not w.visited
      explore(w)
```

Runtime $O(|V|+|E|)$.

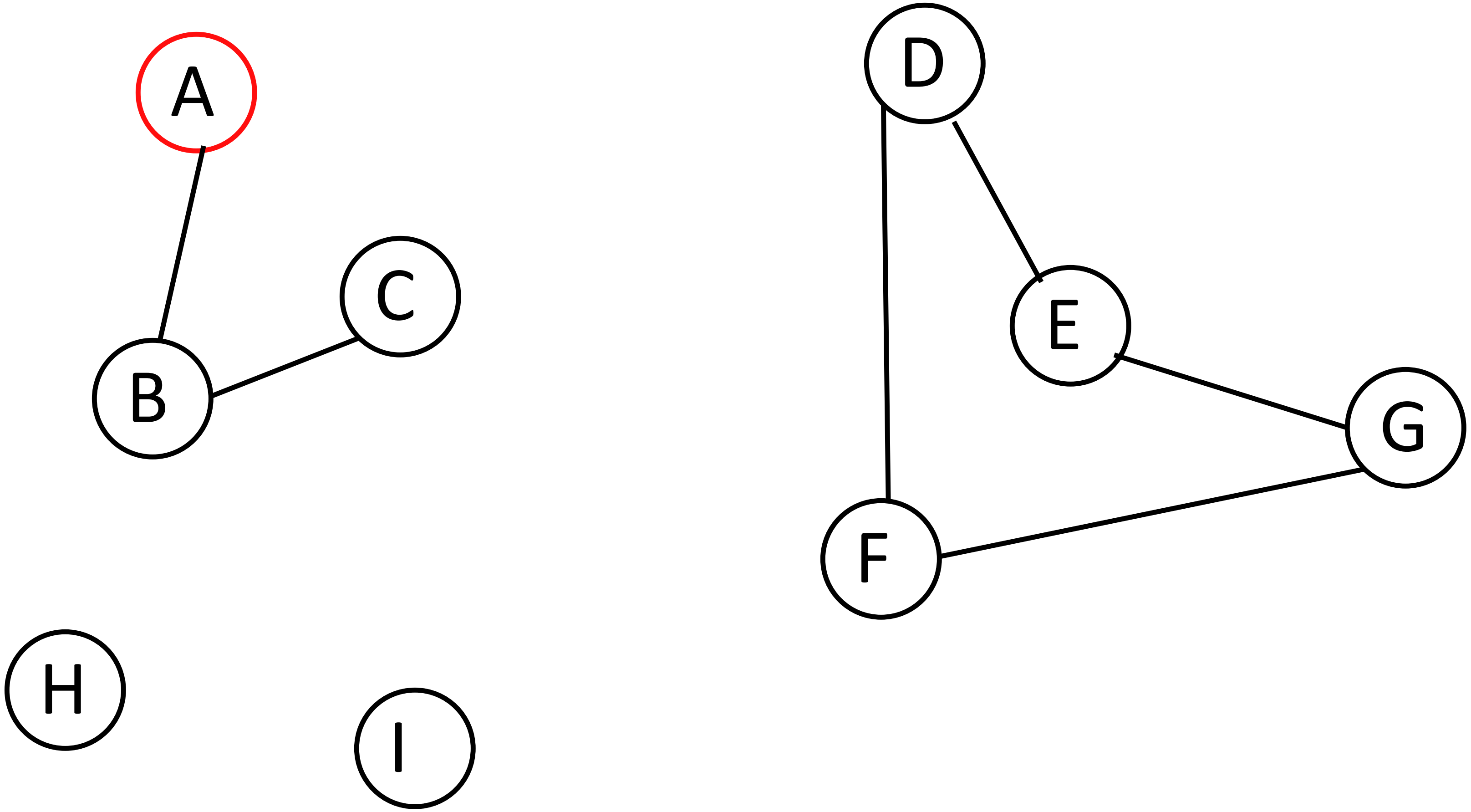
Example

`CCNum:`



Example

CCNum: 1



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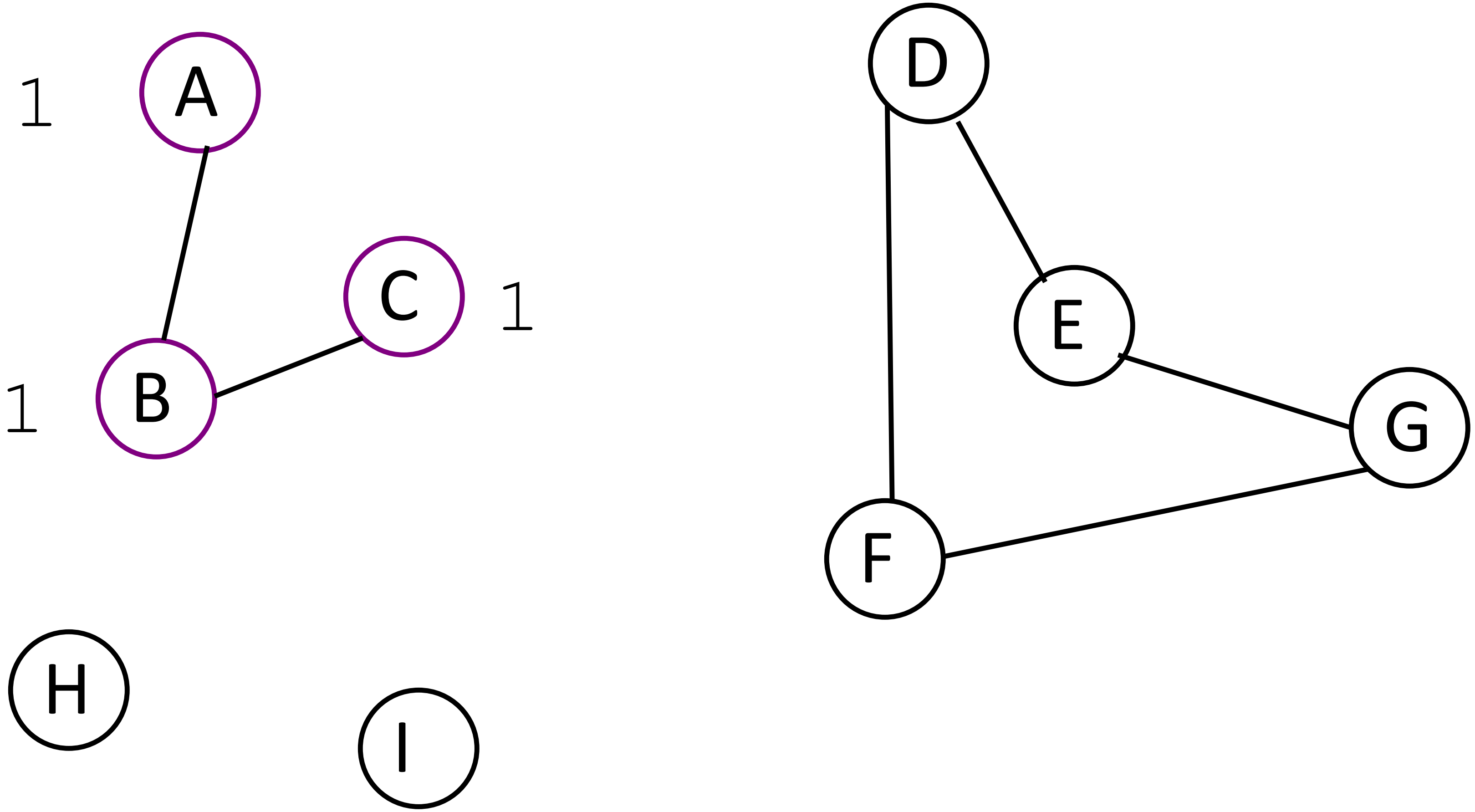
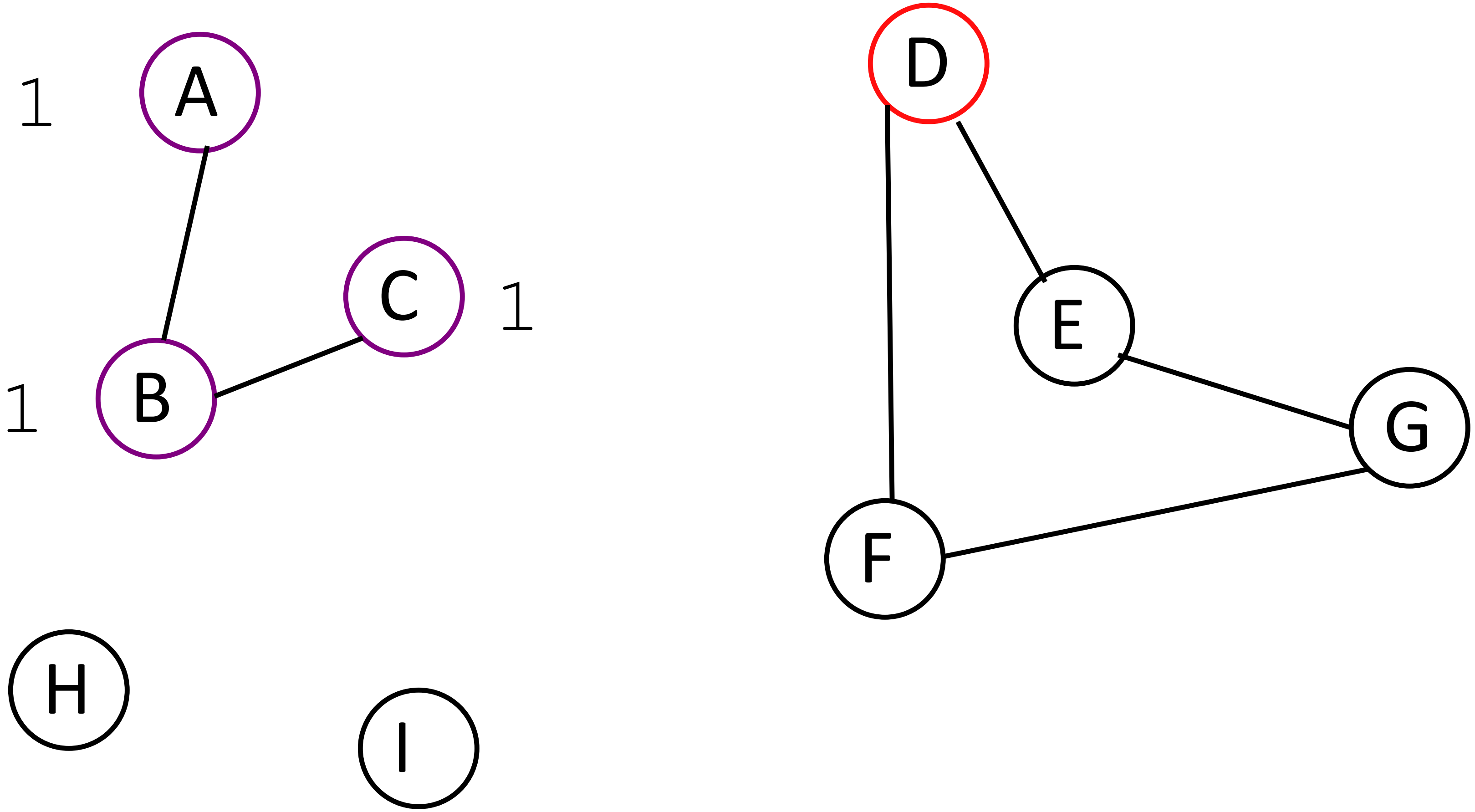


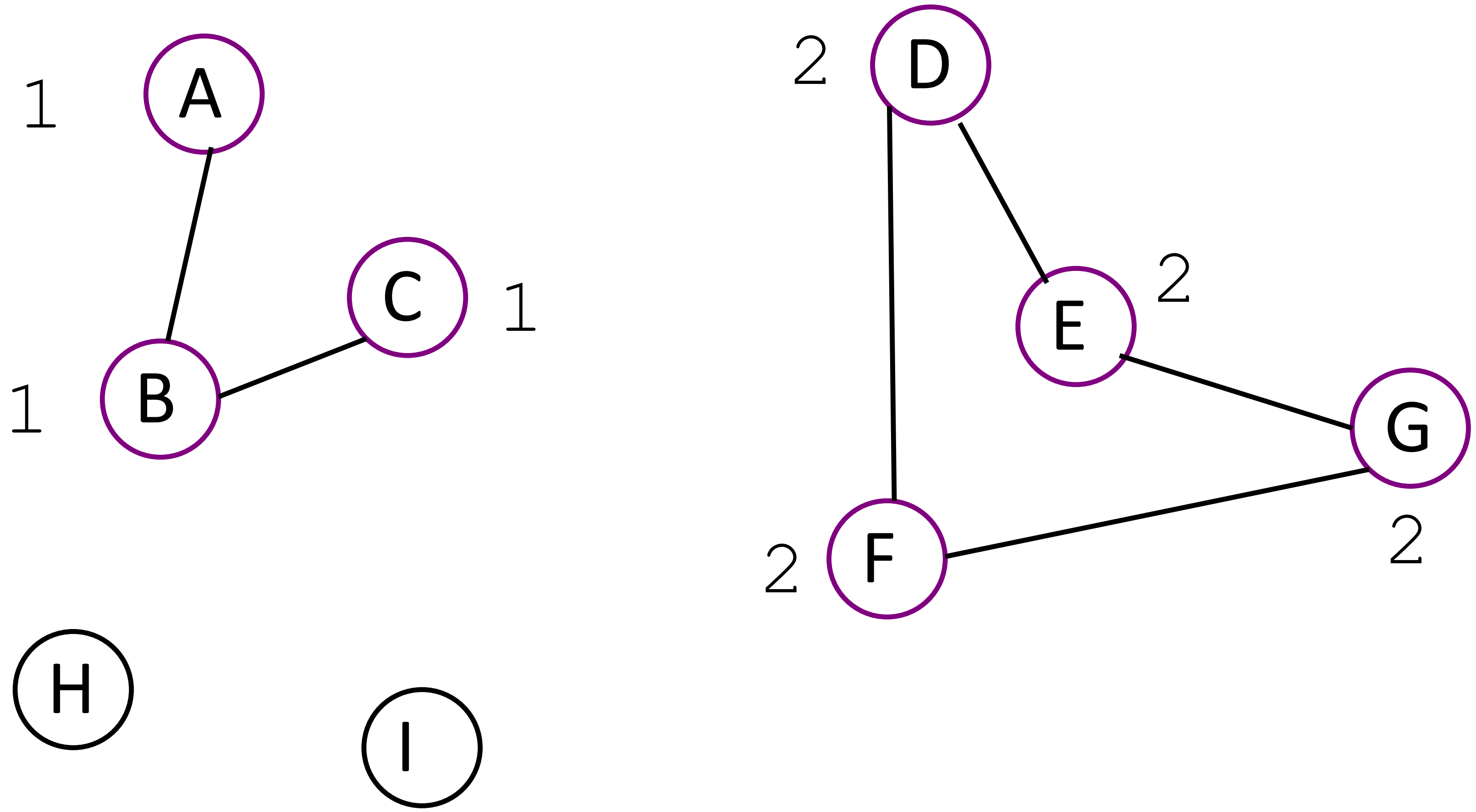
Example

CCNum: 2



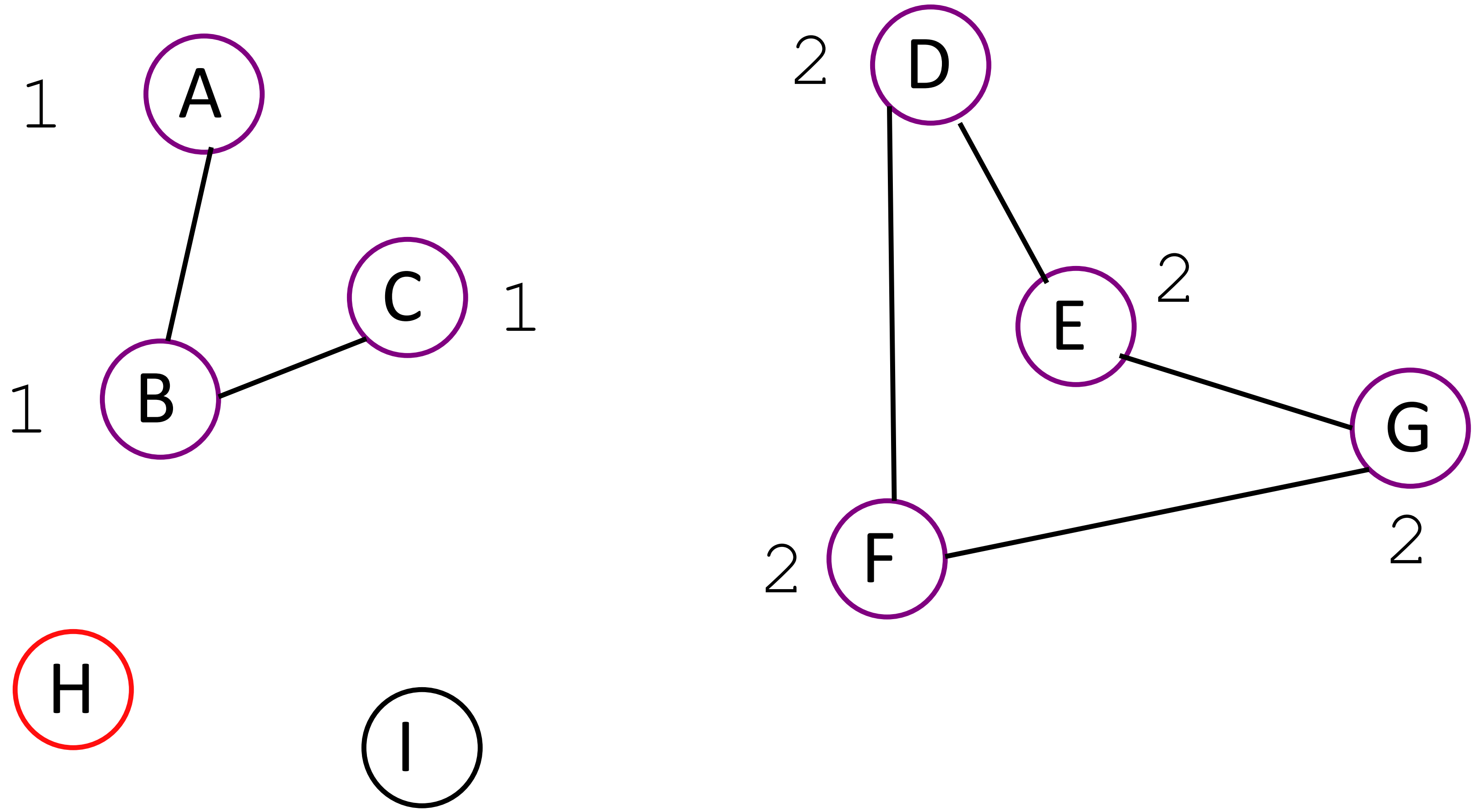
Example

CCNum: 2



Example

CCNum: 3



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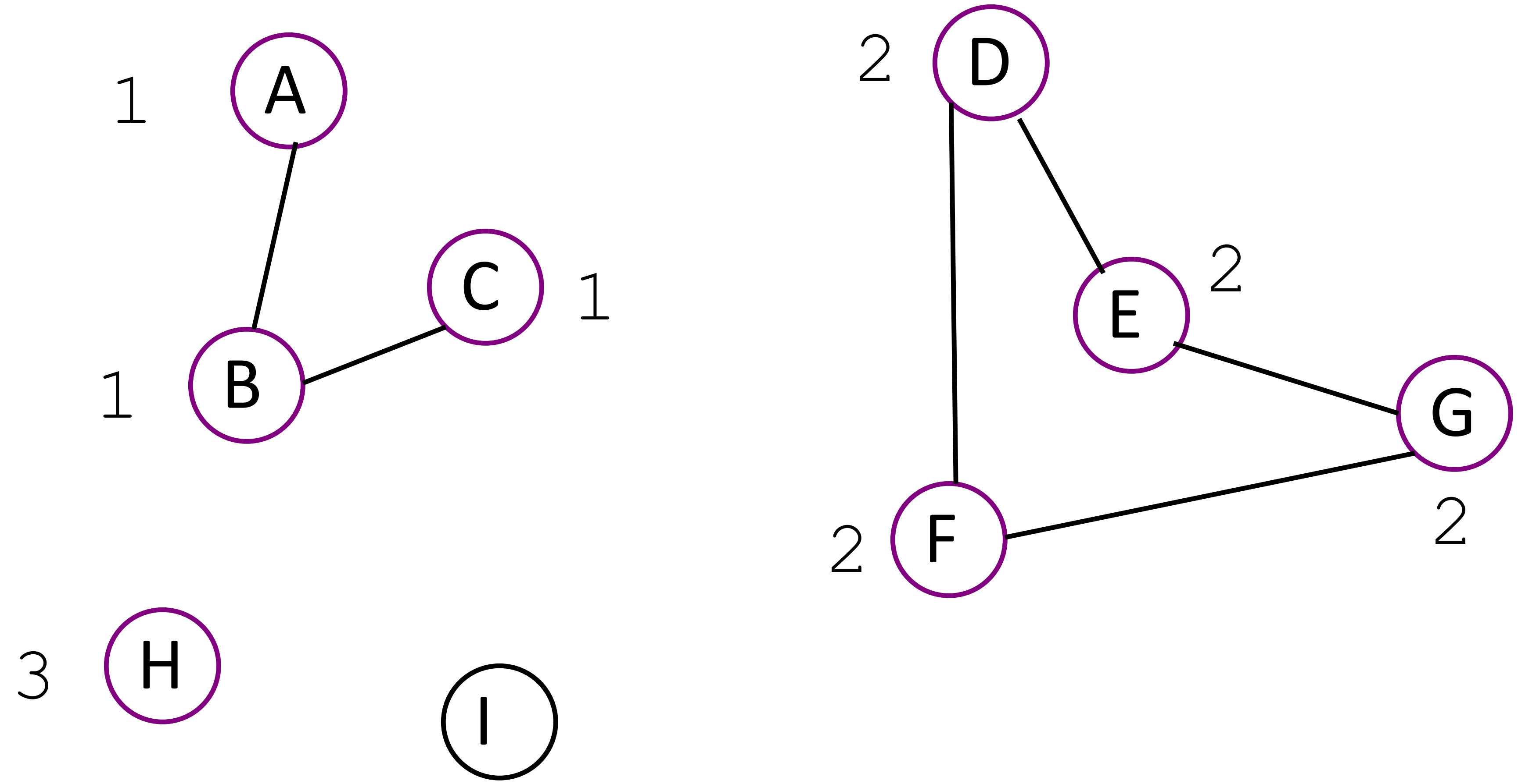
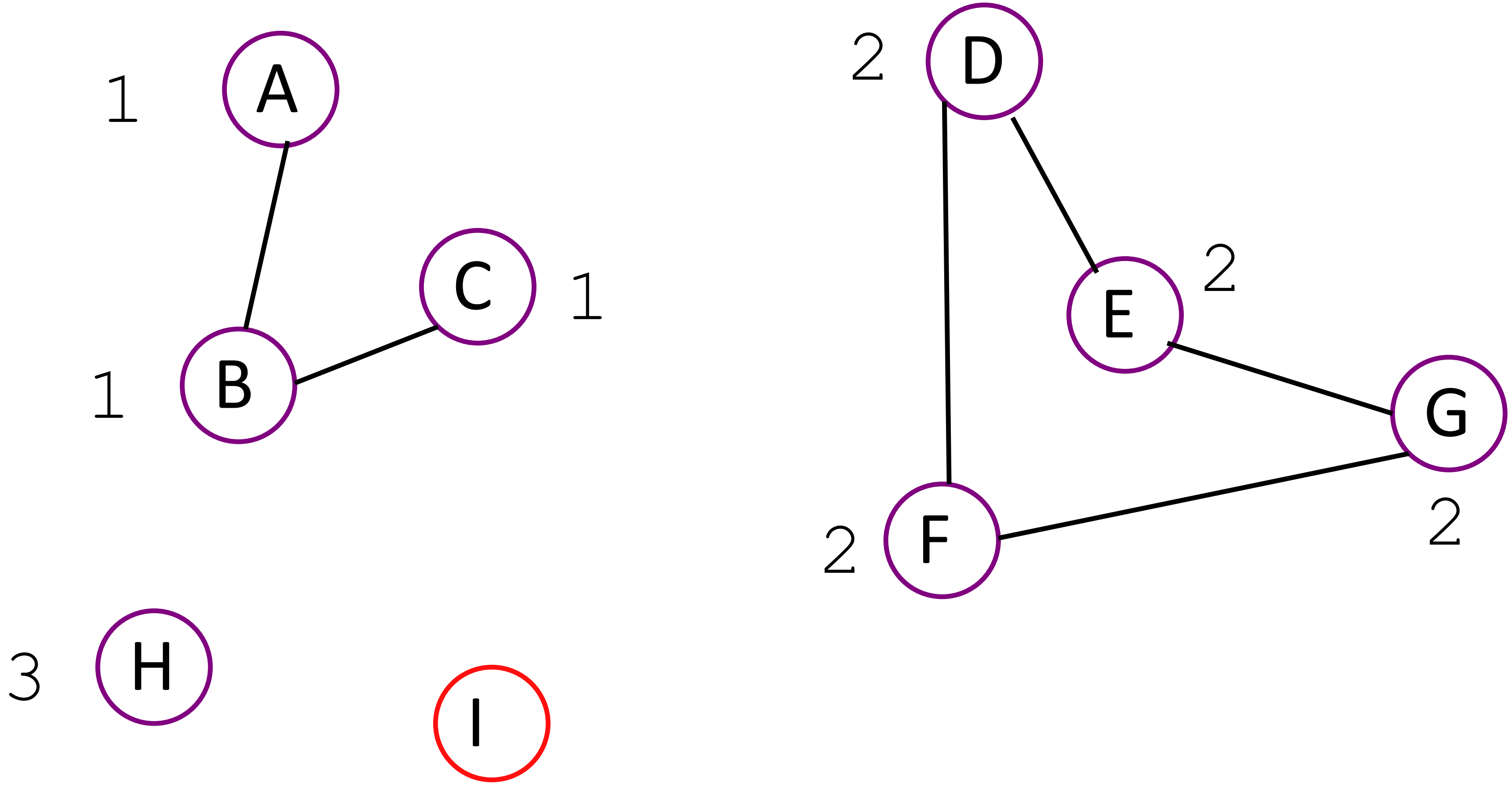


Example

CCNum: 4



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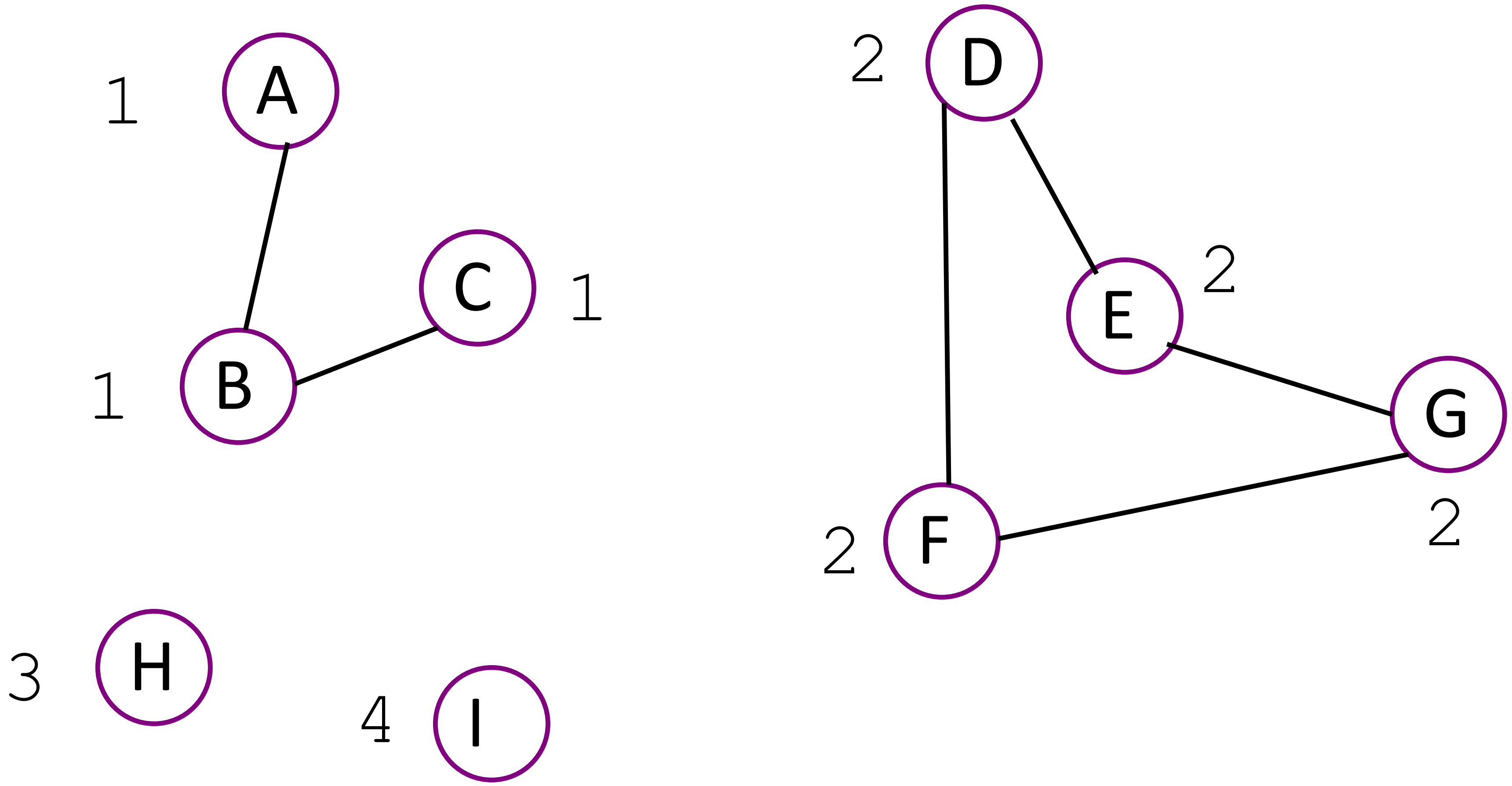

Discussion about DFS

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- No output.
- Marks all vertices as visited.
- Easier ways to do this.

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However, DFS also is a useful way to explore the graph. By *augmenting* the algorithm a bit (like we did with the connected components algorithm), we can learn useful things.

Pre- and Post- Orders

Augment how?

- Keep track of what algorithm does & in what order.
- Have a "clock" and note time whenever:
  - Algorithm visits a new vertex for the first time.
  - Algorithm finishes processing a vertex.
- Record values as `v.pre` and `v.post`.

Computing Pre- & Post- Orders

```
ConnectedComponents(G)

  For v ∈ G
    v.visited ← false
  For v ∈ G
    If not v.visited
      explore(v)
```

```
explore(v)
  v.visited ← true


  For each edge (v,w)
    If not w.visited
      explore(w)
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    If not v.visited
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```

```
explore(v)
  v.visited ← true
  v.pre ← clock
  clock++
  For each edge (v,w)
    If not w.visited
      explore(w)
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Computing Pre- & Post- Orders

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explore(v)
  v.visited ← true
  v.pre ← clock
  clock++
  For each edge (v,w)
    If not w.visited
      explore(w)
  v.post ← clock
  clock++
```

Computing Pre- & Post- Orders

```
ConnectedComponents(G)
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  v.visited ← true
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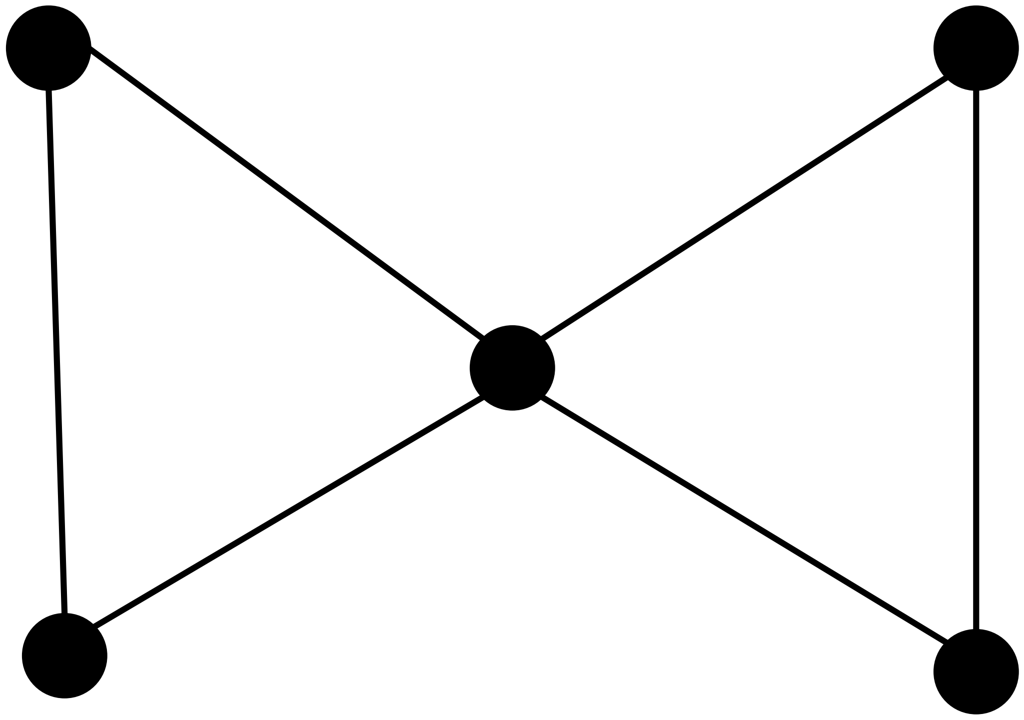
  clock++
  For each edge (v,w)
    If not w.visited
      explore(w)
  v.post ← clock
  clock++
```

Runtime O(|V|+|E|).

Example



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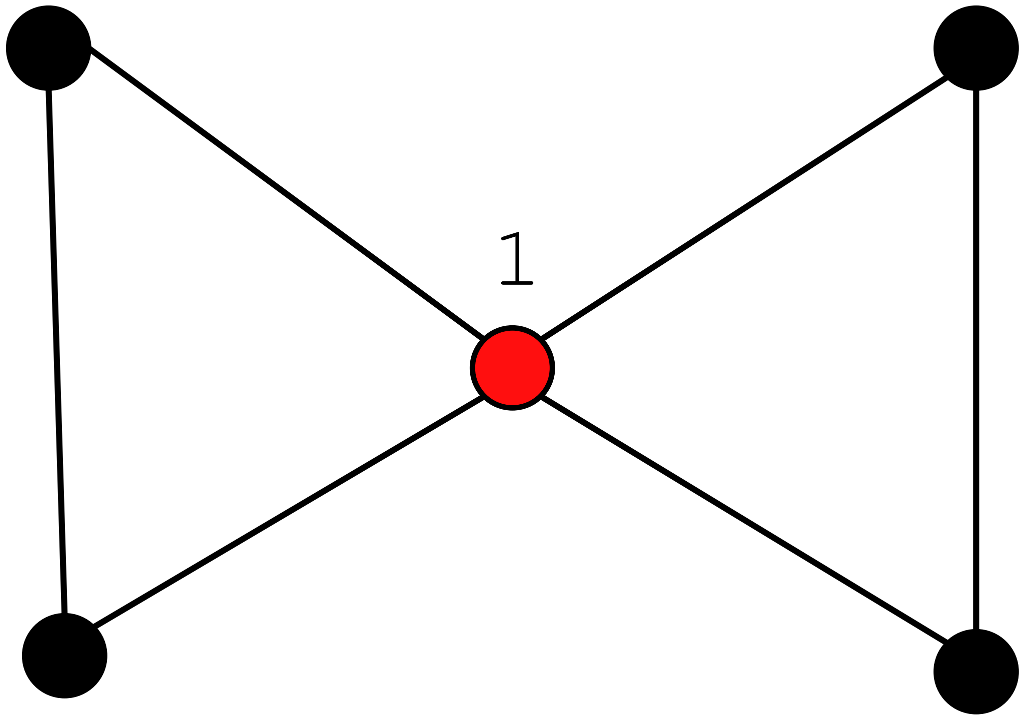

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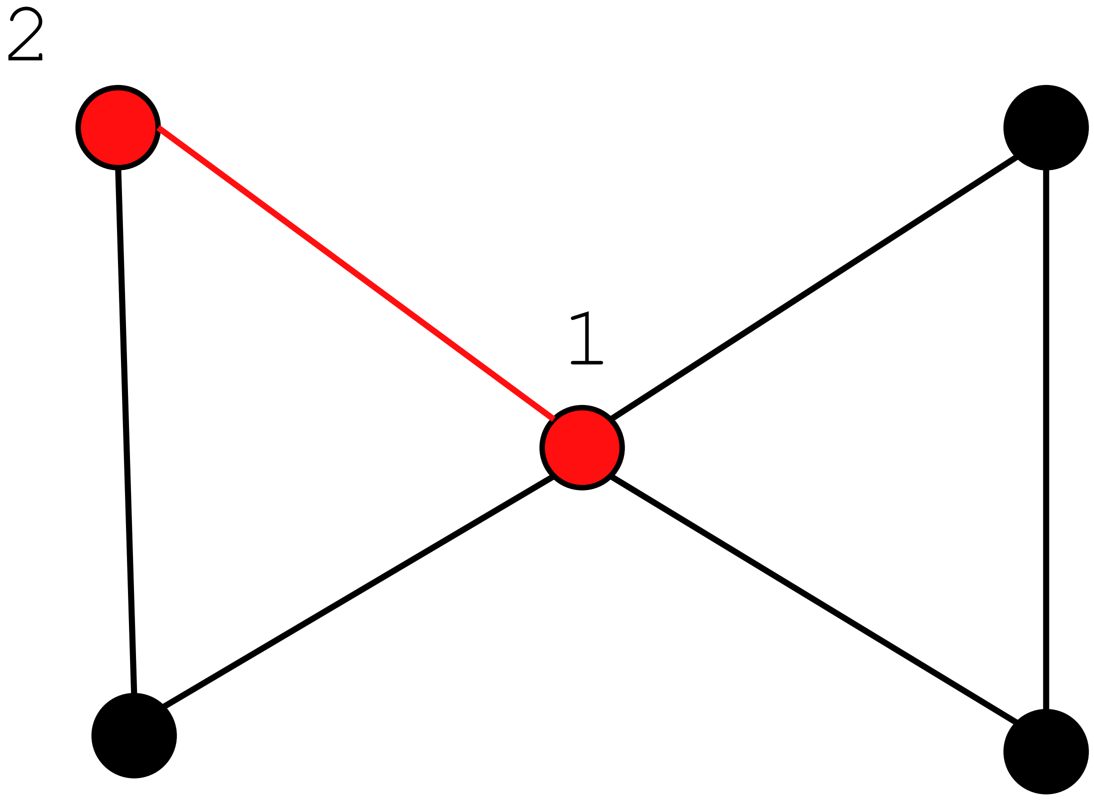

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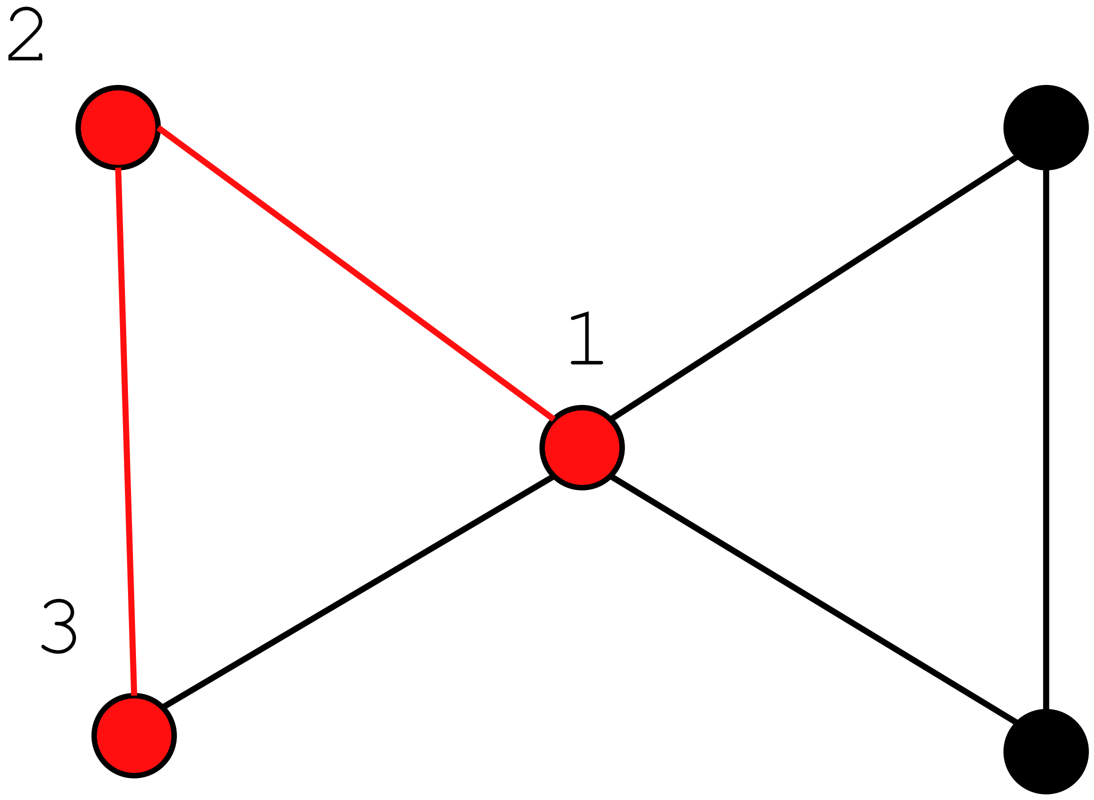

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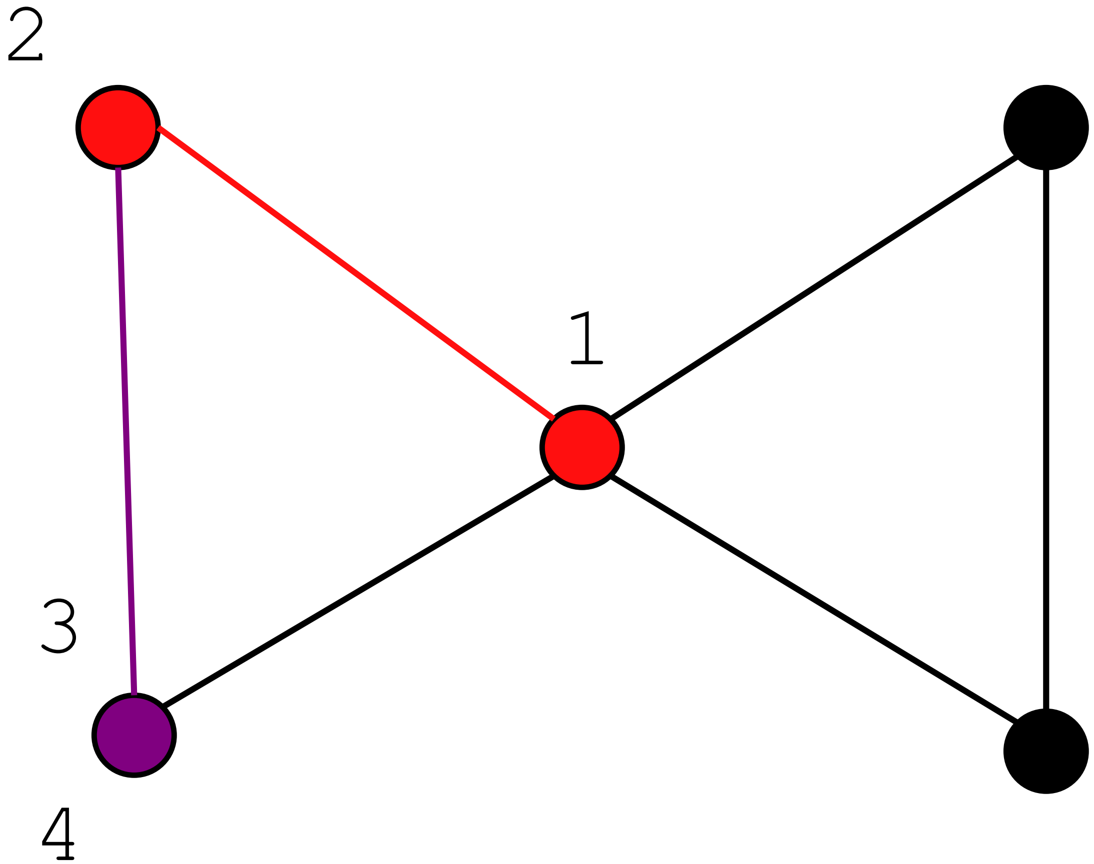

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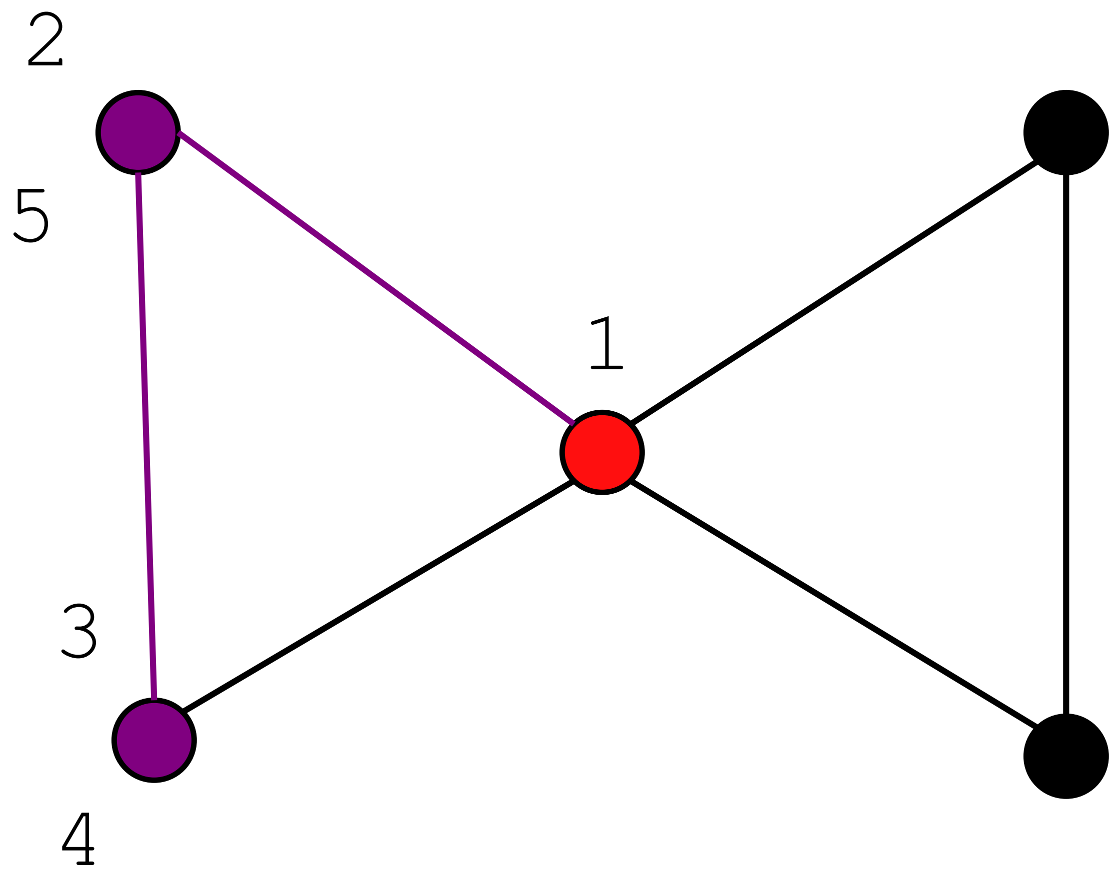

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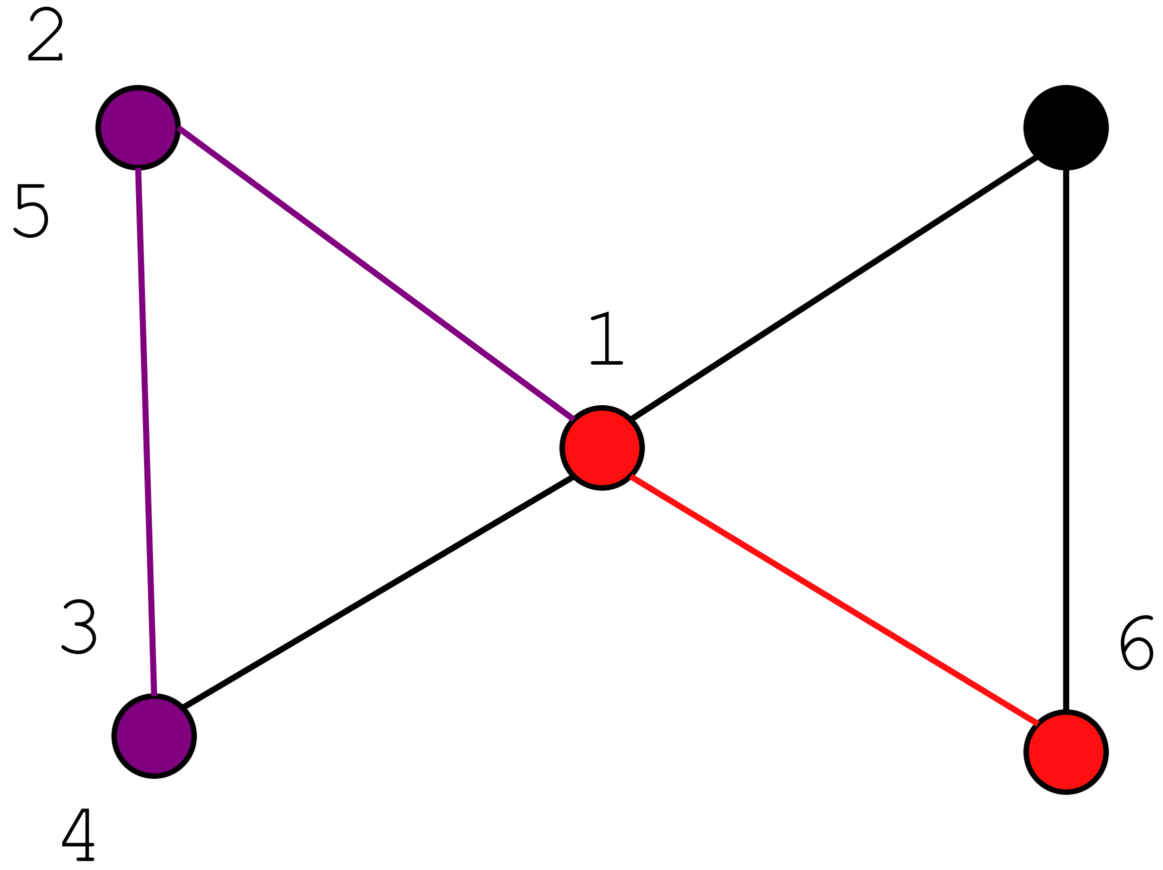

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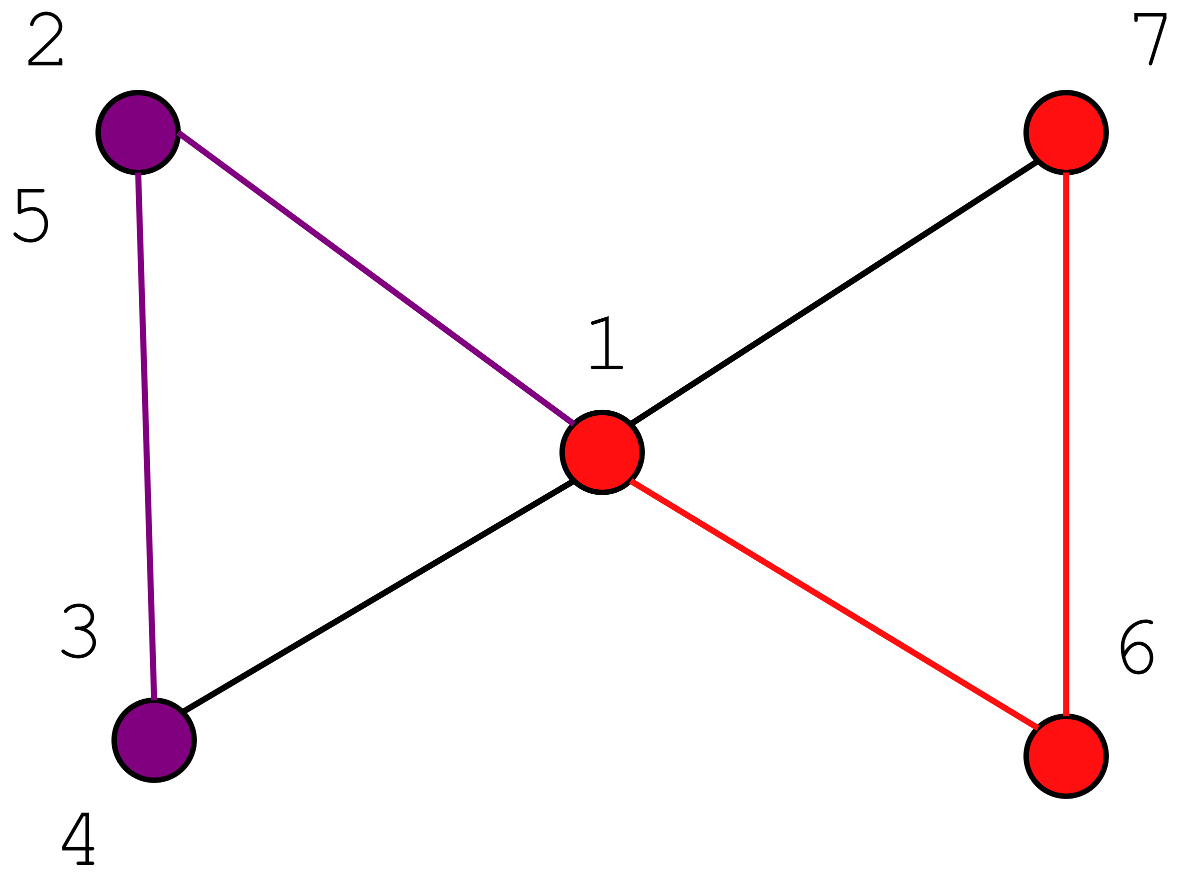

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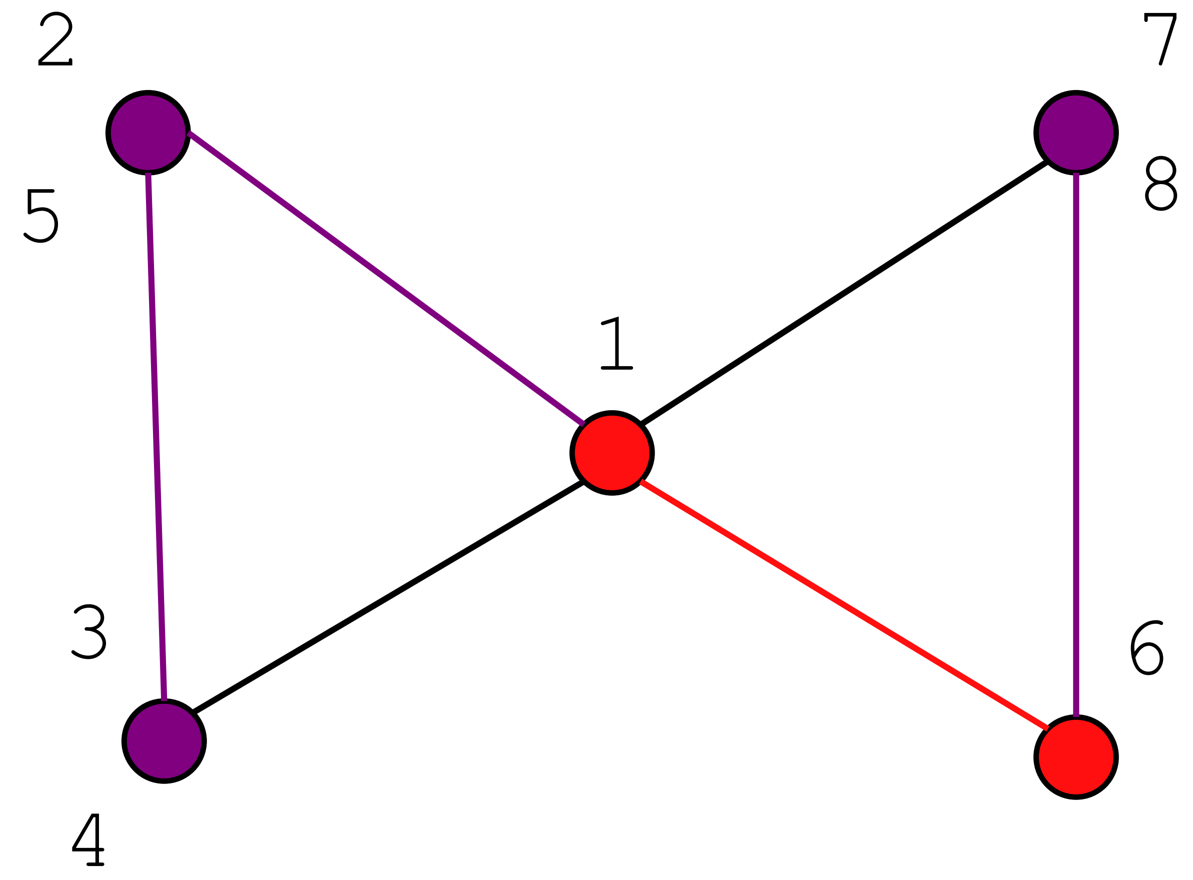

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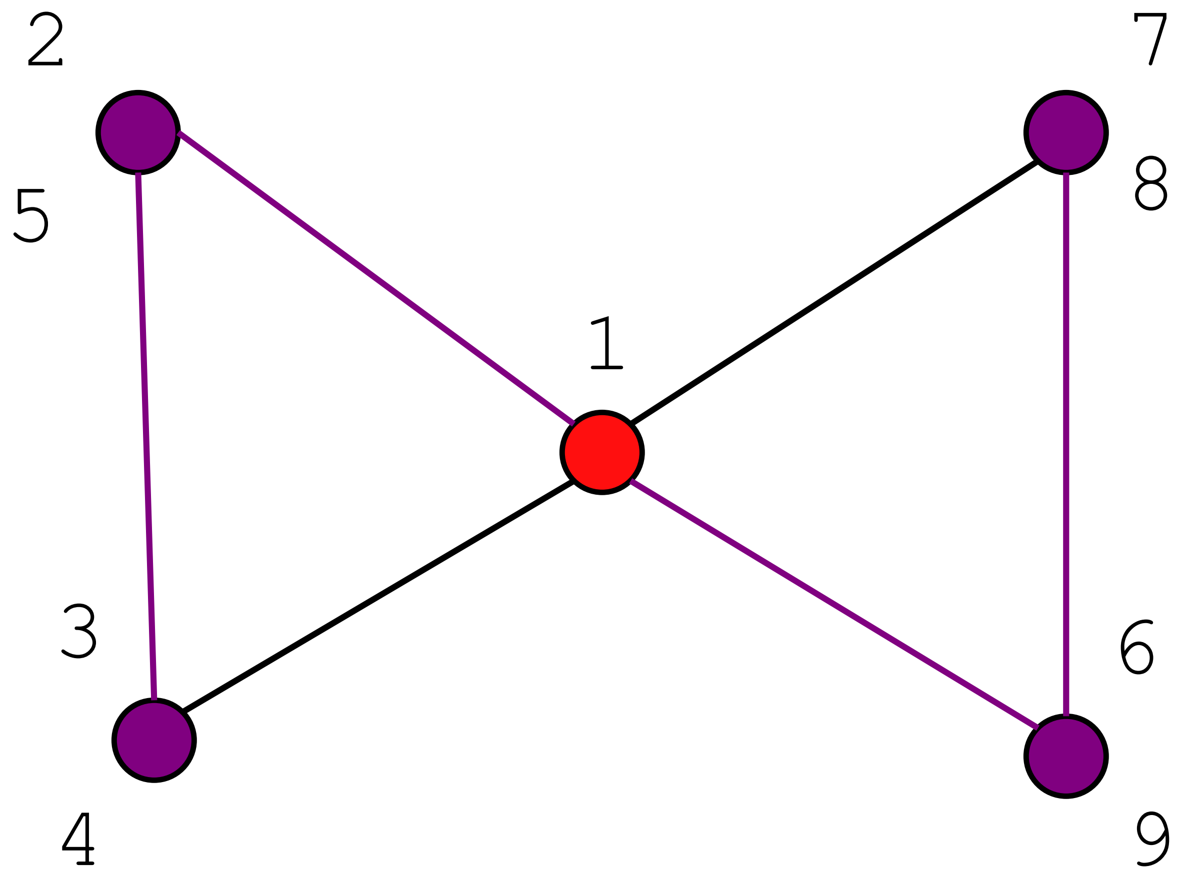

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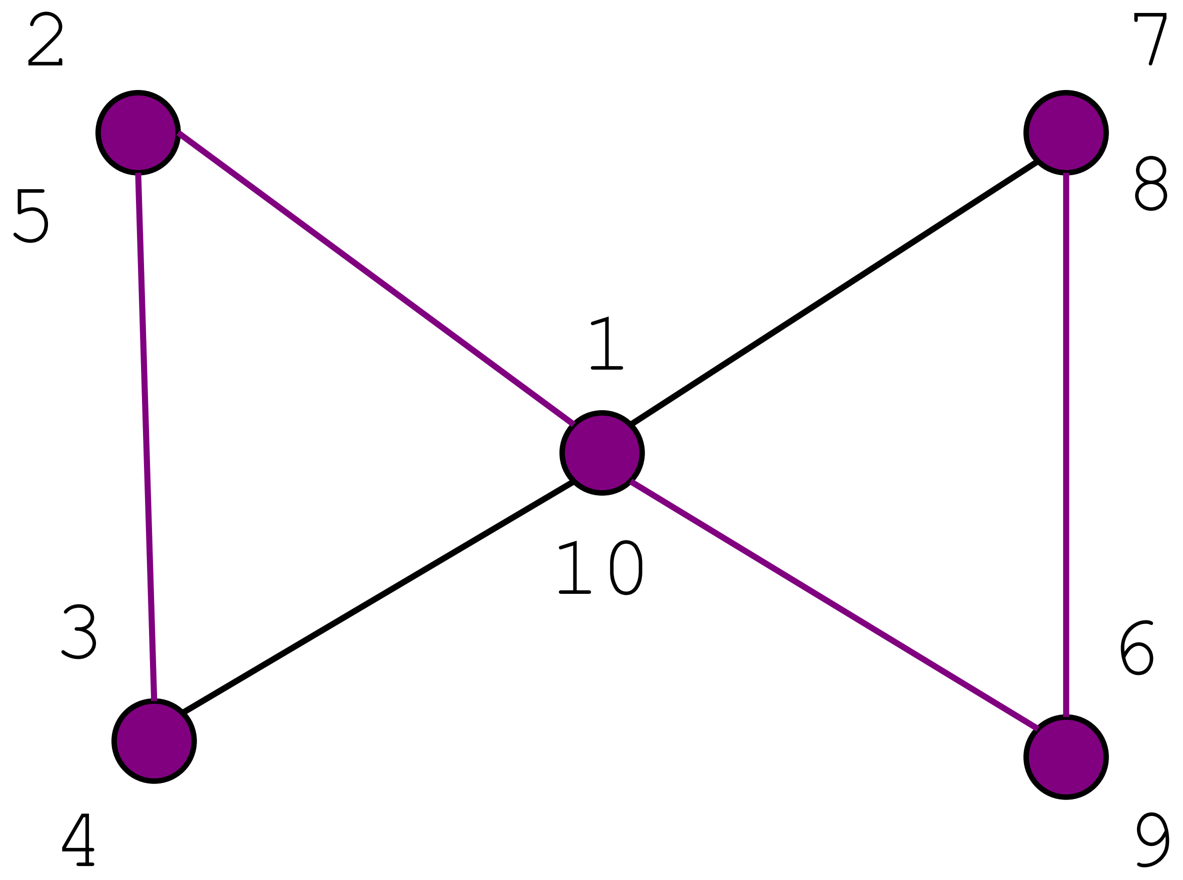
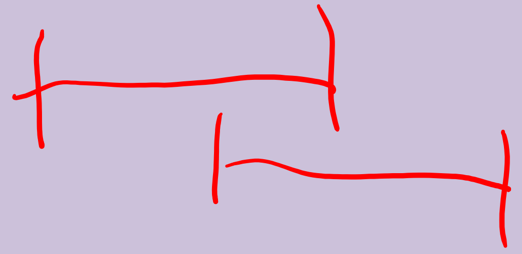


What do these orders tell us?

Prop: For vertices v, w consider intervals [`v.pre`, `v.post`] and [`w.pre`, `w.post`].

These intervals:



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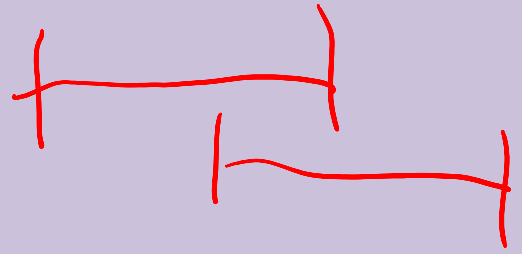
1. Contain each other if v is an ancestor/descendant of w in the DFS tree.



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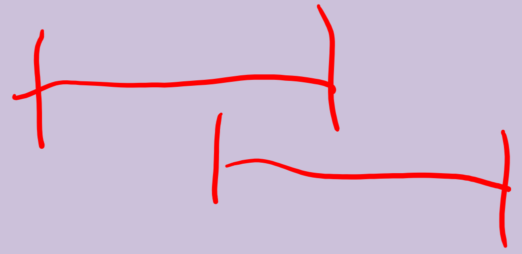
<u>Prop:</u> For vertices v, w consider intervals [`v.pre`, `v.post`] and [`w.pre`, `w.post`]. These intervals:

1. Contain each other if v is an ancestor/descendant of w in the DFS tree.
2. Are disjoint if v and w are cousins in the DFS tree.



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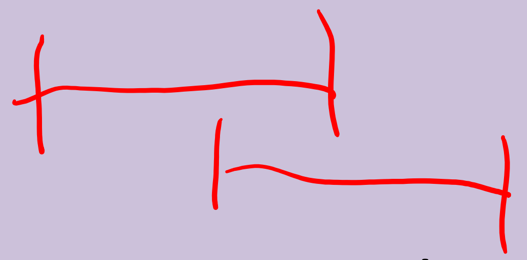
2. Are disjoint if v and w are cousins in the DFS tree.
3. Never interleave (`v.pre` < `w.pre` < `v.post` < `w.post`)



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  - v and w are cousins
- If algorithm discovers w *before* fully processing v:
  - Algorithm finishes processing w before it finishes v
  - `v.pre` < `w.pre` < `w.post` < `v.post`
  - Nested intervals
  - v is ancestor of w

Question: Possible Intervals

Which pairs of pre-post intervals are not possible for DFS? (multiple correct answers)

A) [1,2] & [3,4]

B) [1,3] & [2,4]

C) [1,4] & [2,3]

D) [1,5] & [2,4]

E) [1,6] & [2,5]

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Directed Graphs

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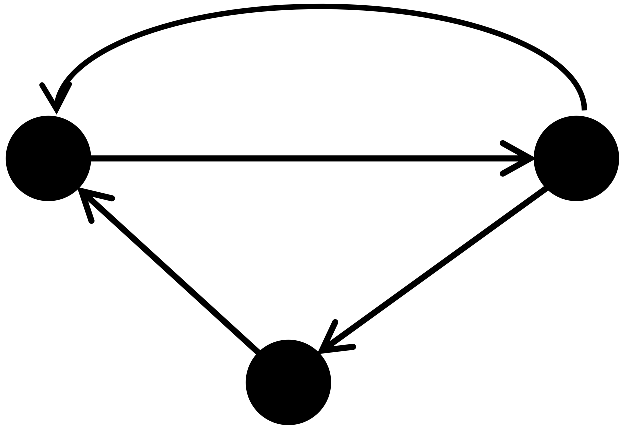
Draw edges with arrows to denote direction.



Question: Directed Graphs

Which of the following does NOT need to be modeled as a directed rather than undirected graph:

A) The Internet (links connecting webpages)

B) Facebook (friendships connecting people)

C) Twitter (followings connecting people)

D) Maps (roads connecting intersections)

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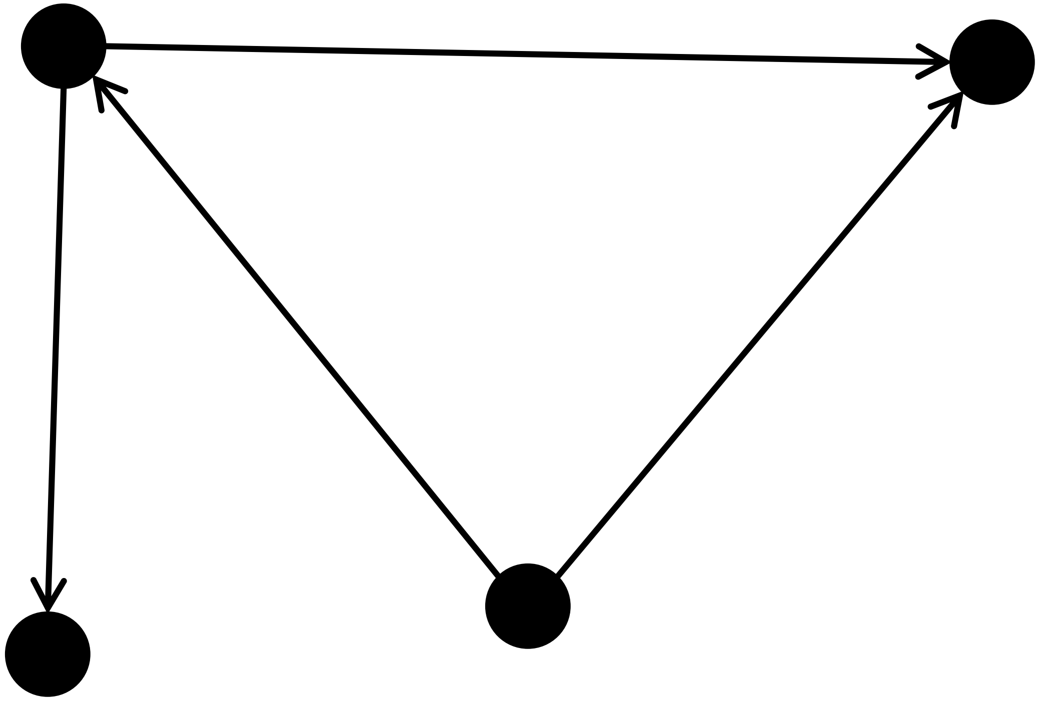
DFS on Directed Graphs

- Same code
- Only follow *directed* edges from v to w.
- Runtime still O(|V|+|E|)
- `explore(v)` discovers all vertices reachable from v following only directed edges.

Example



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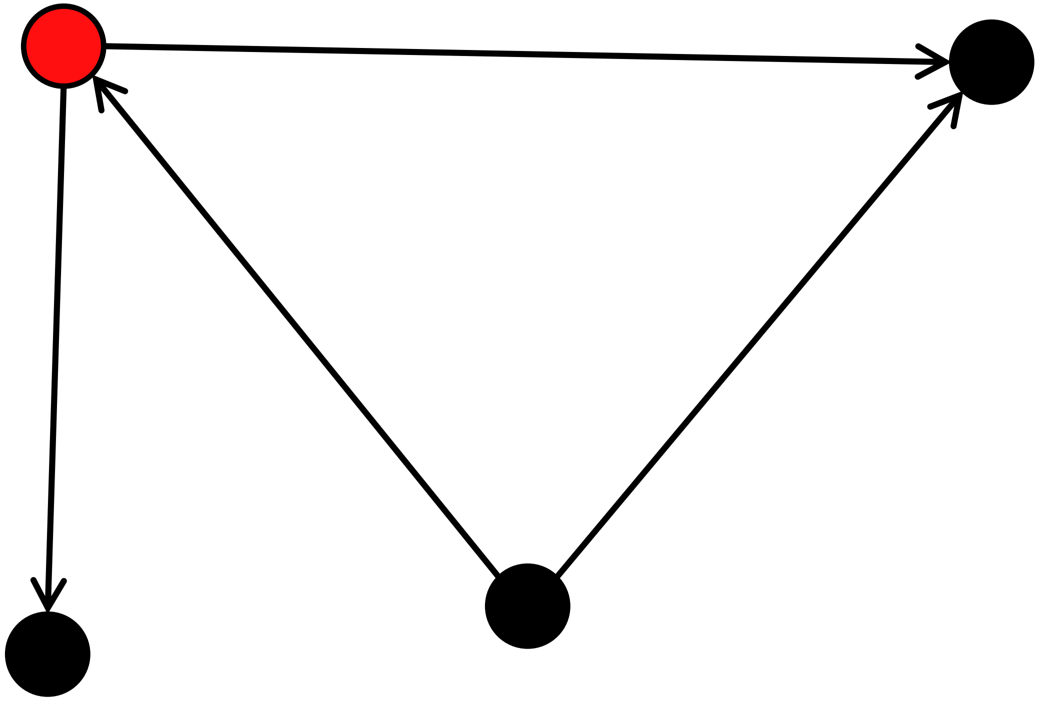

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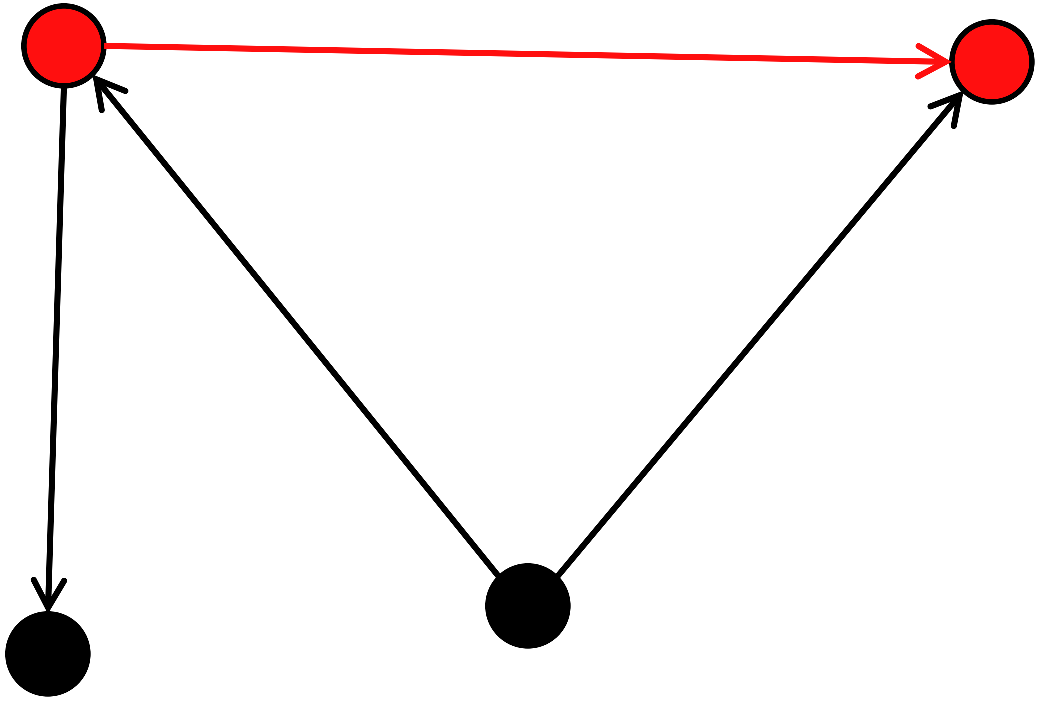

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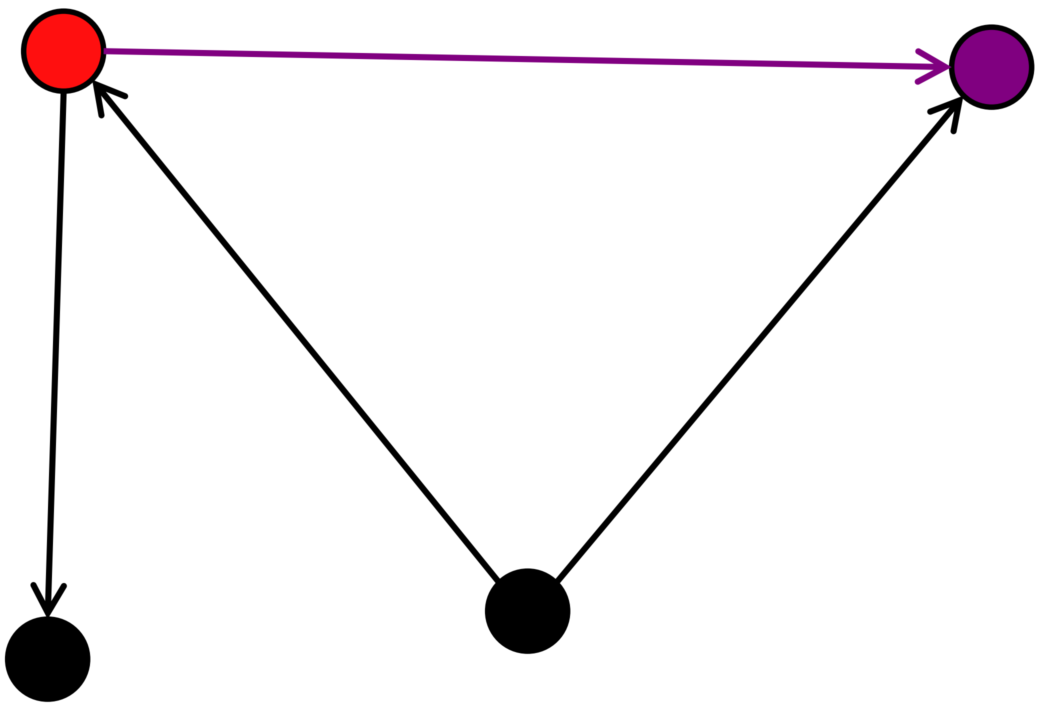

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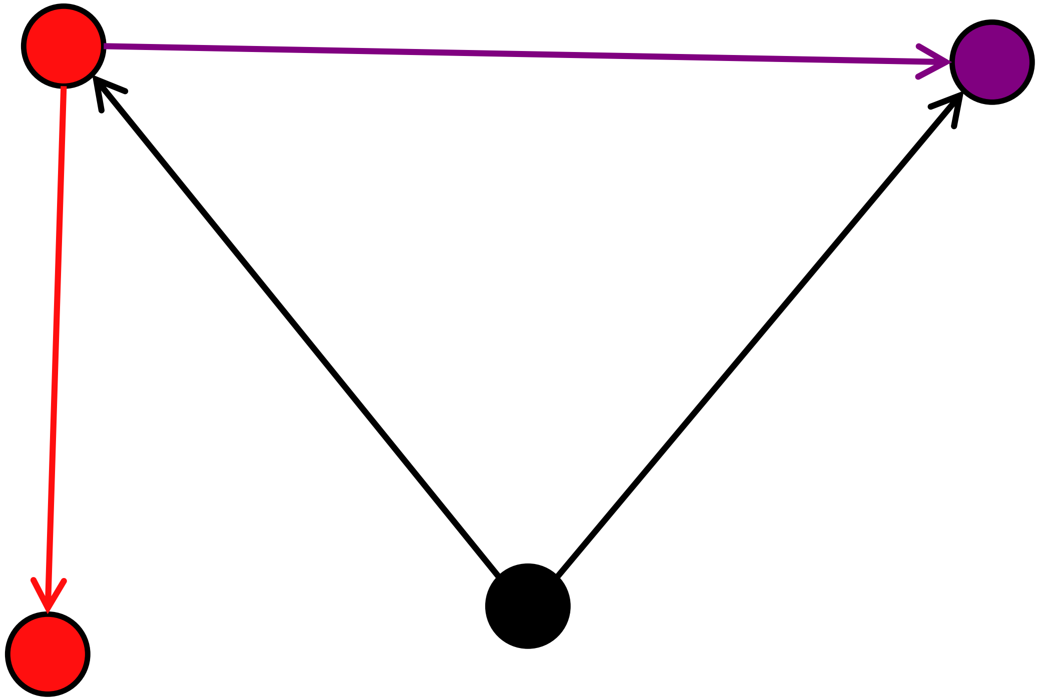

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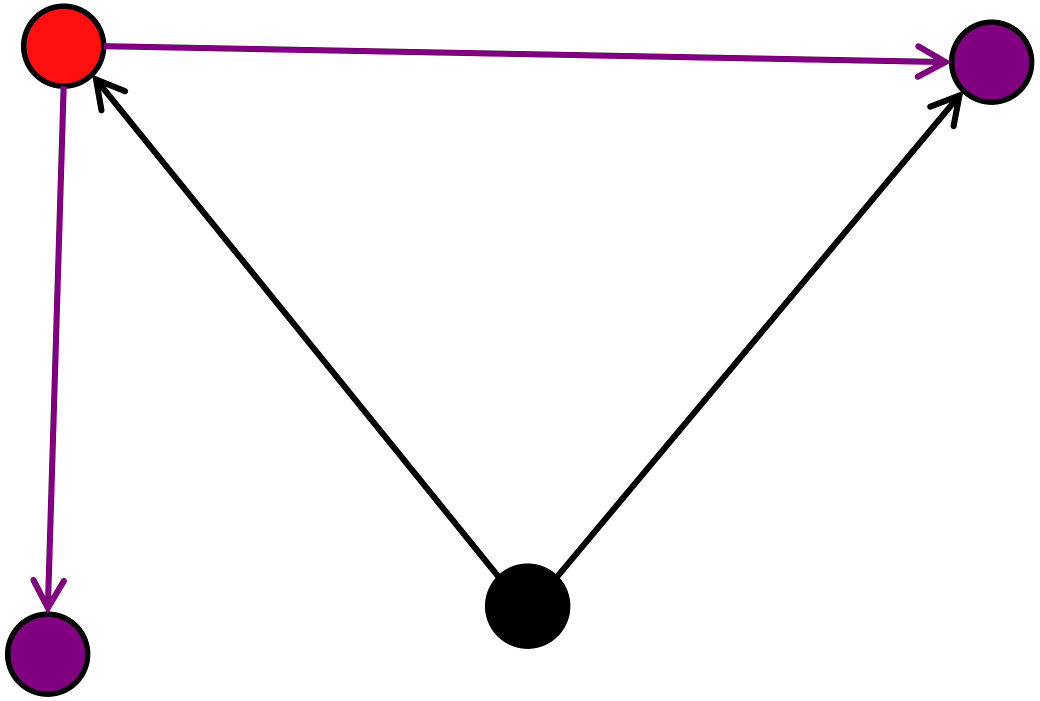

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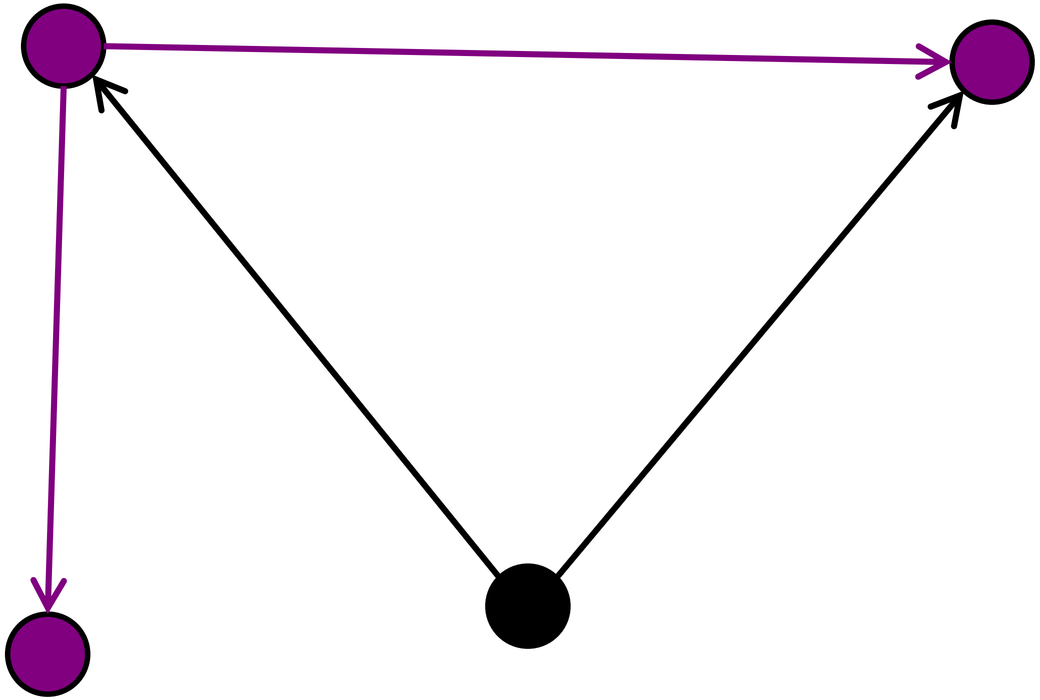
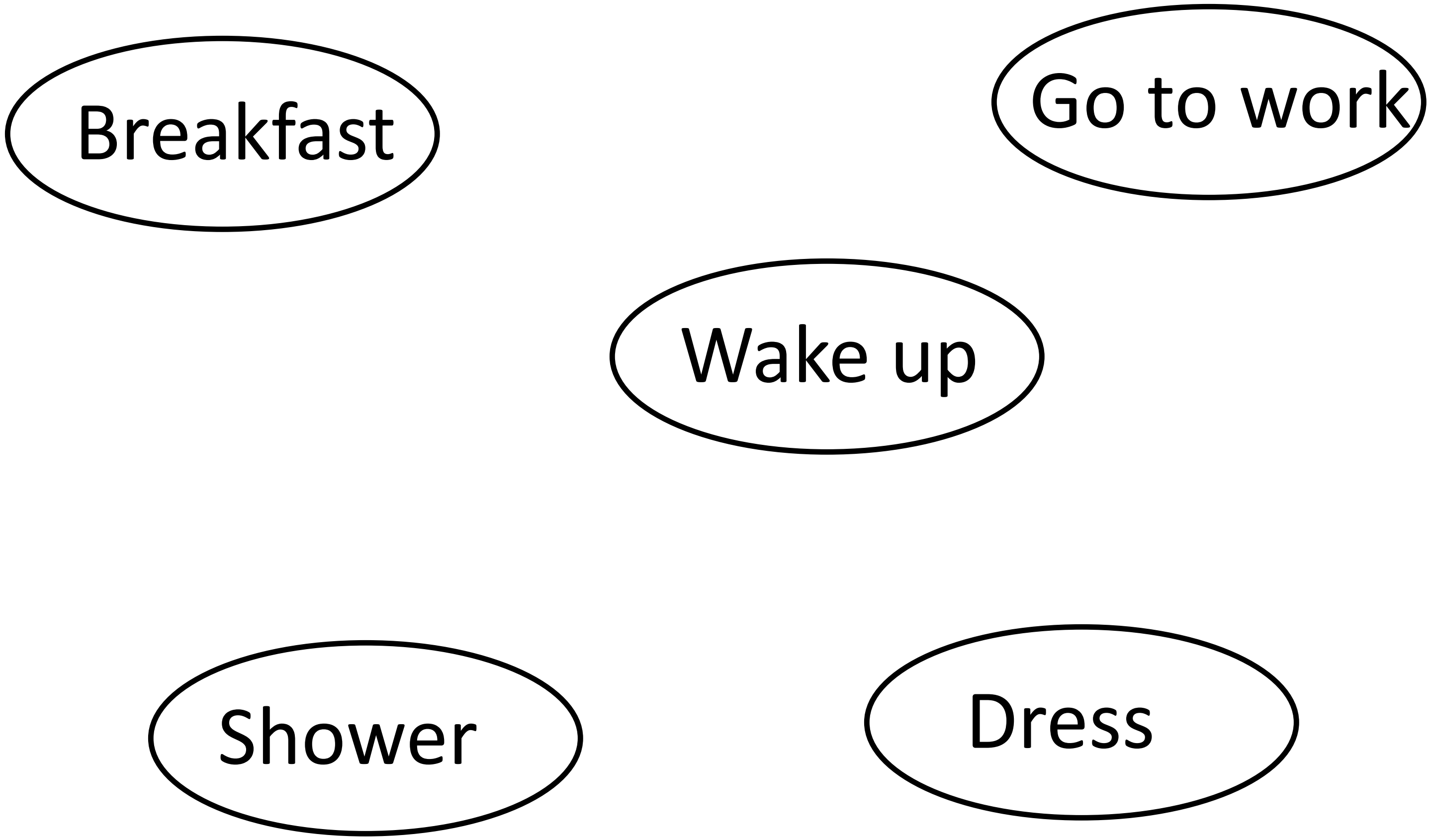


Directed Acyclic Graphs

- Directed graphs as dependencies
- Linear orderings
- DAGs definition
- Topological sort

Dependency Graphs



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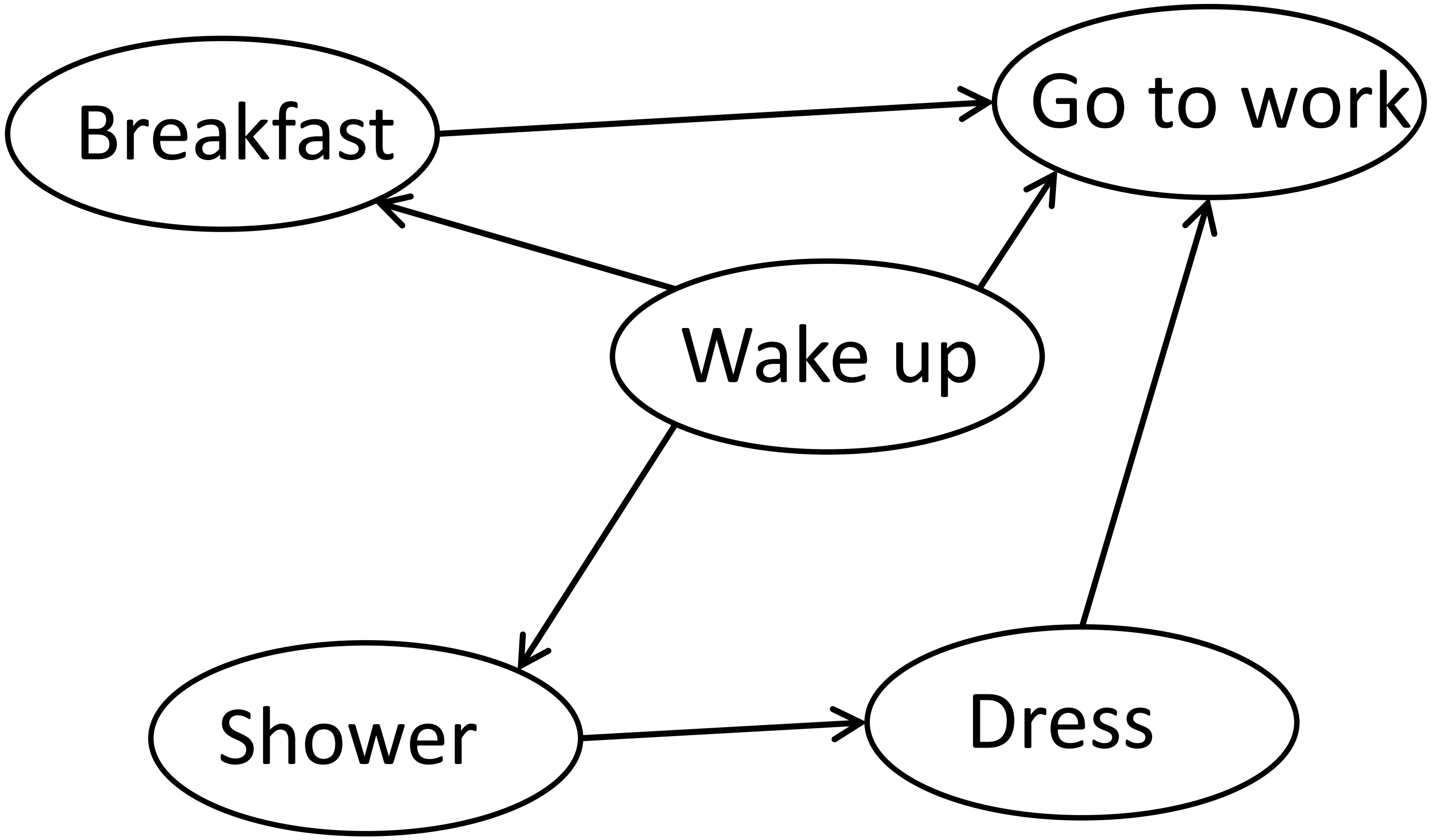

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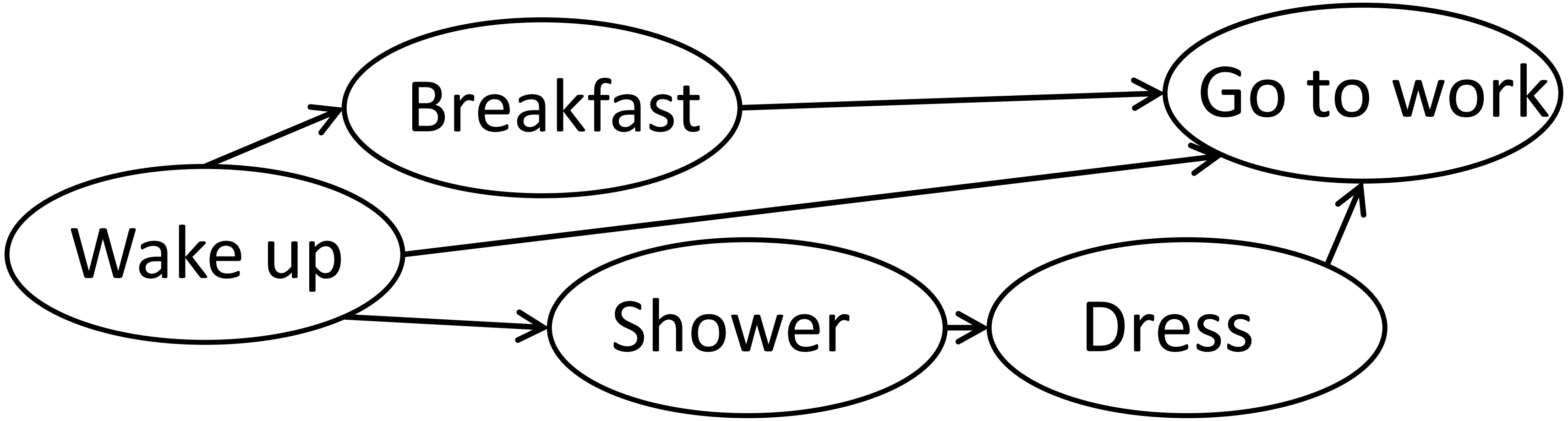

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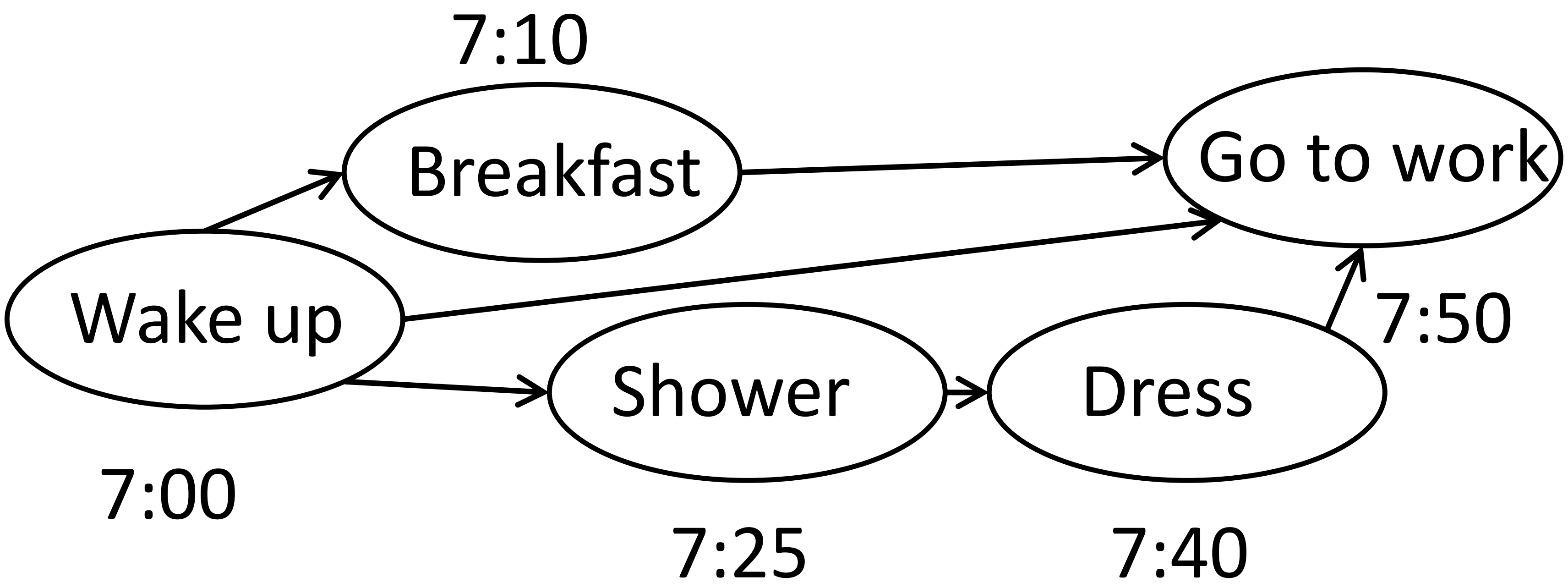

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Where an edge v→w means that v should come before w.

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Definition: A topological ordering of a directed graph is an ordering of the vertices so that for each edge (v,w), v comes before w in the ordering.

Question: Existence of Orderings

Does every directed graph have a topological ordering?

A) Yes

B) No

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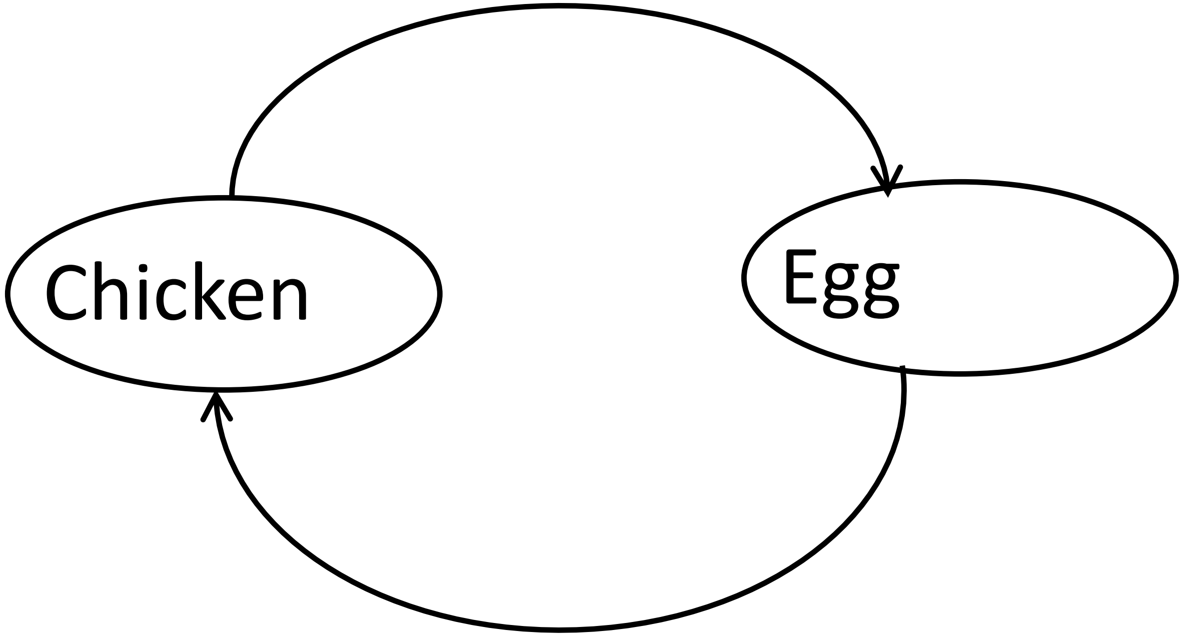
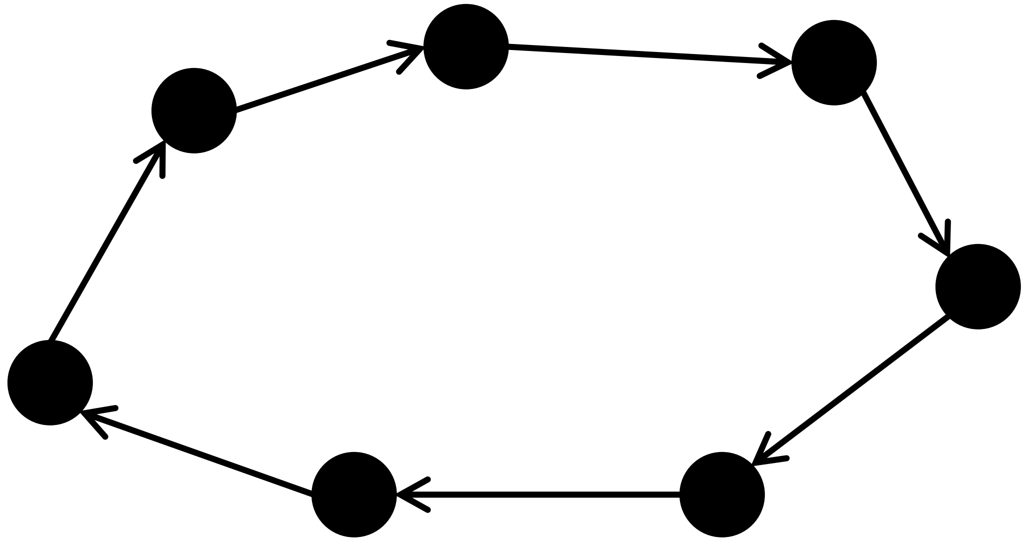
Counterexample



Cycles

<u>Definition:</u> A <u>cycle</u> in a directed graph is a sequence of vertices $v_1, v_2, v_3, \ldots, v_n$ so that there are edges $(v_1,v_2), (v_2,v_3), \ldots, (v_{n-1},v_n), (v_n,v_1)$



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- AFSOC we have an ordering.
- Find earliest v_i in the ordering.
- Note that v_i comes before v_{i-1}, in contradiction to the order property.

DAGs

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Surprisingly, yes.

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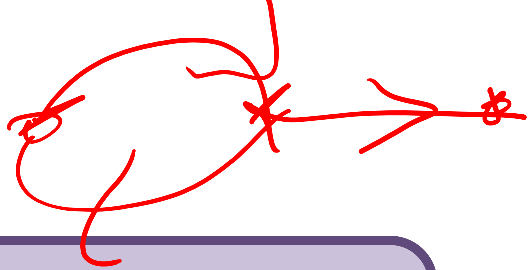
- Idea: find a sink, put at end, order remaining.
- Question: Does G have a sink?

Sinks



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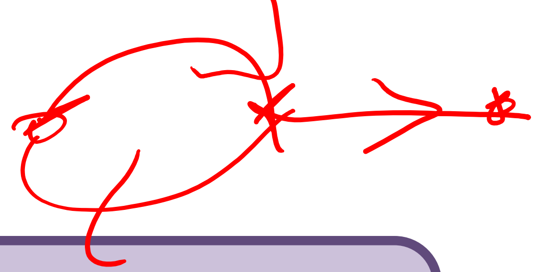

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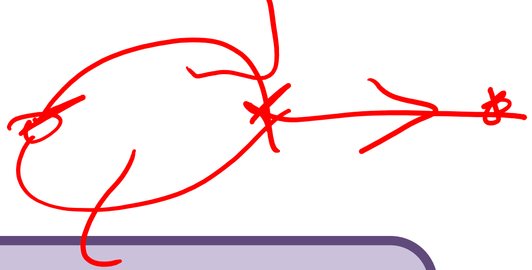

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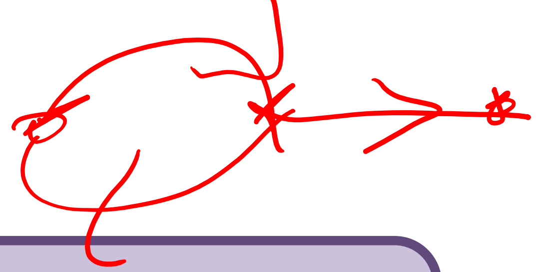

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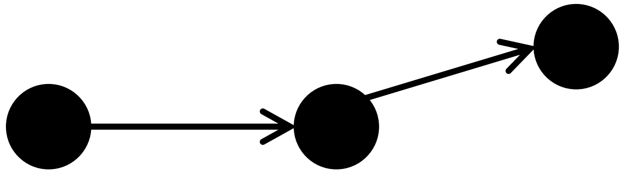

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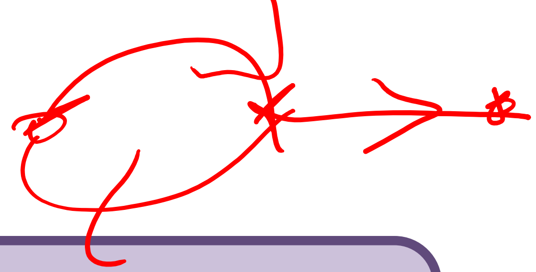

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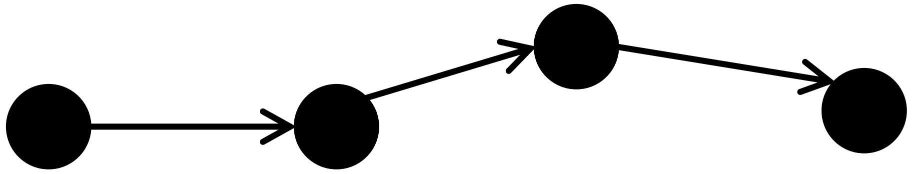

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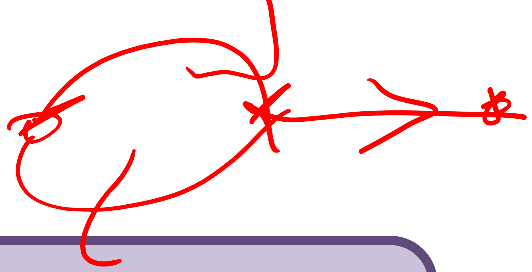

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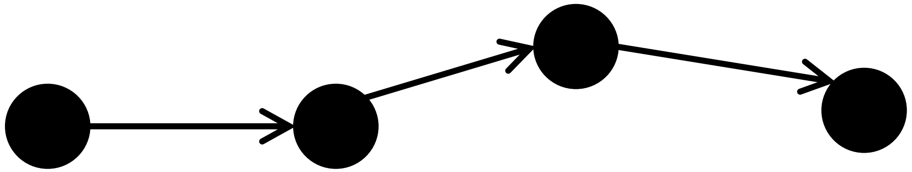

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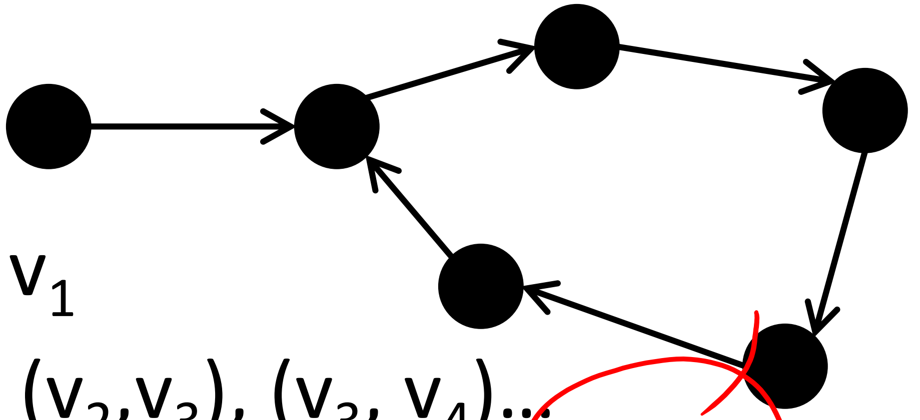

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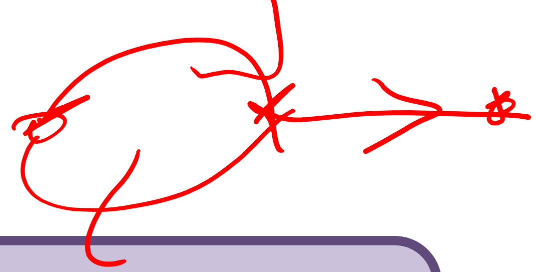

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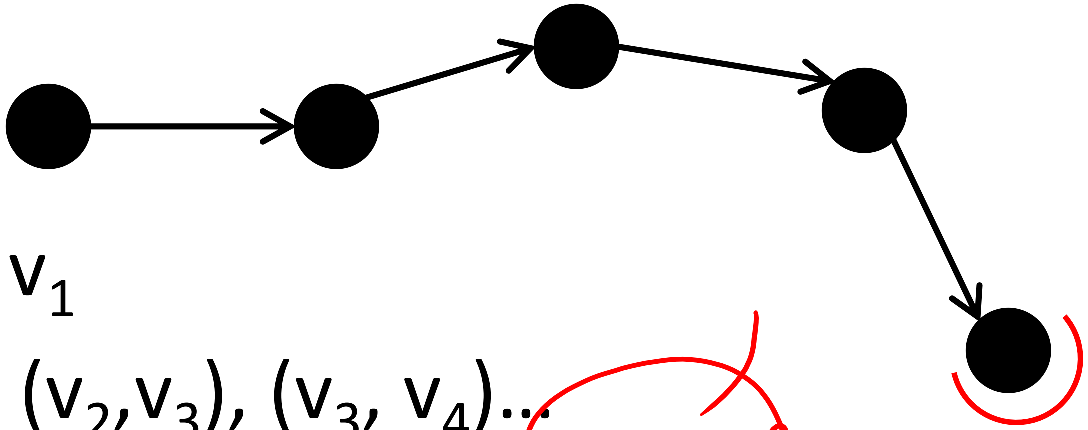

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- Start at vertex v = v_1
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  - Get stuck (found a sink)

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- Find sink v.
- Let G’ = G-v.
- Inductively order G’ (still a DAG).
- Add v to the end of the ordering.

Algorithm

Problem: Design an algorithm that given a DAG G computes a topological ordering on G.