Announcements

- Homework 2 online due Friday

Last Time

- Shortest Paths in Graphs with Edge Lengths
- Priority Queues

Edge Lengths

The number of edges in a path is not always the right measure of distance. Sometimes, taking several shorter steps is preferable to taking a few longer ones.

We assign each edge (u,v) a non-negative <u>length</u> $\ell(u,v)$. The length of a path is the sum of the lengths of its edges.

Problem: Shortest Paths

Problem: Given a Graph G with vertices s and t and a length function ℓ, find the shortest path from s to t.

Algorithm

```
Distances(G,s,ℓ)
  dist(s) ← 0
  While(not all distances found)
    Find minimum over (v,w) ∈ E
      with v discovered w not
      of dist(v)+ℓ(v,w)
    dist(w) ← dist(v)+ℓ(v,w)
    prev(w) ← v
```

Still too Slow

- Repeatedly ask for smallest vertex
  - Even though not much is changing from round to round, the algorithm is computing the minimum from scratch every time
- Use a data structure!
  - Data structures help answer a bunch of similar questions faster than answering each question individually
- For this kind of question, want a priority queue.

Priority Queue

A <u>Priority Queue</u> is a datastructure that stores elements sorted by a <u>key</u> value.

<u>Operations:</u>

- Insert – adds a new element to the PQ.
- DecreaseKey – Changes the key of an element of the PQ to a specified *smaller* value.
- DeleteMin – Finds the element with the smallest key and removes it from the PQ.

Today

- Priority Queues
- Dijkstra's Algorithm
- Negative Edge Weights

Attempt III

```
Distances(G,s,ℓ)

  For v ∈ V
    dist(v) ← ∞, done(v) ← false

  dist(s) ← 0
  While(not all vertices done)
    Find v not done with minimum dist(v)
    done(v) ← true
    For (v,w) ∈ E
      If dist(v)+ℓ(v,w) < dist(w)
        dist(w) ← dist(v)+ℓ(v,w)
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O(|V|) times

O(|V|) times

O(|E|) times

Runtime:
O(|V|) Inserts +
O(|V|) DelMins +
O(|E|) DecKeys

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We will discuss a few.

Unsorted List

Store n elements in an unsorted list.

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___ $\times O(|E|)$

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- Dijkstra – $O(|V|^2+|E|)$

Binary Heap

Store elements in a balanced binary tree with each element having smaller key than its children.

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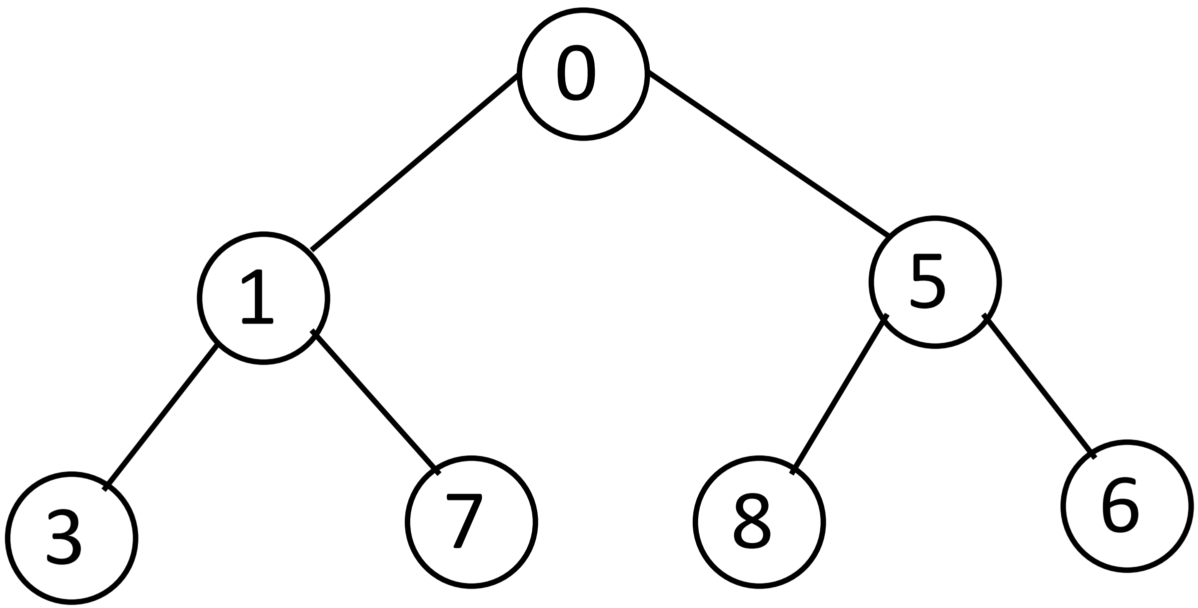

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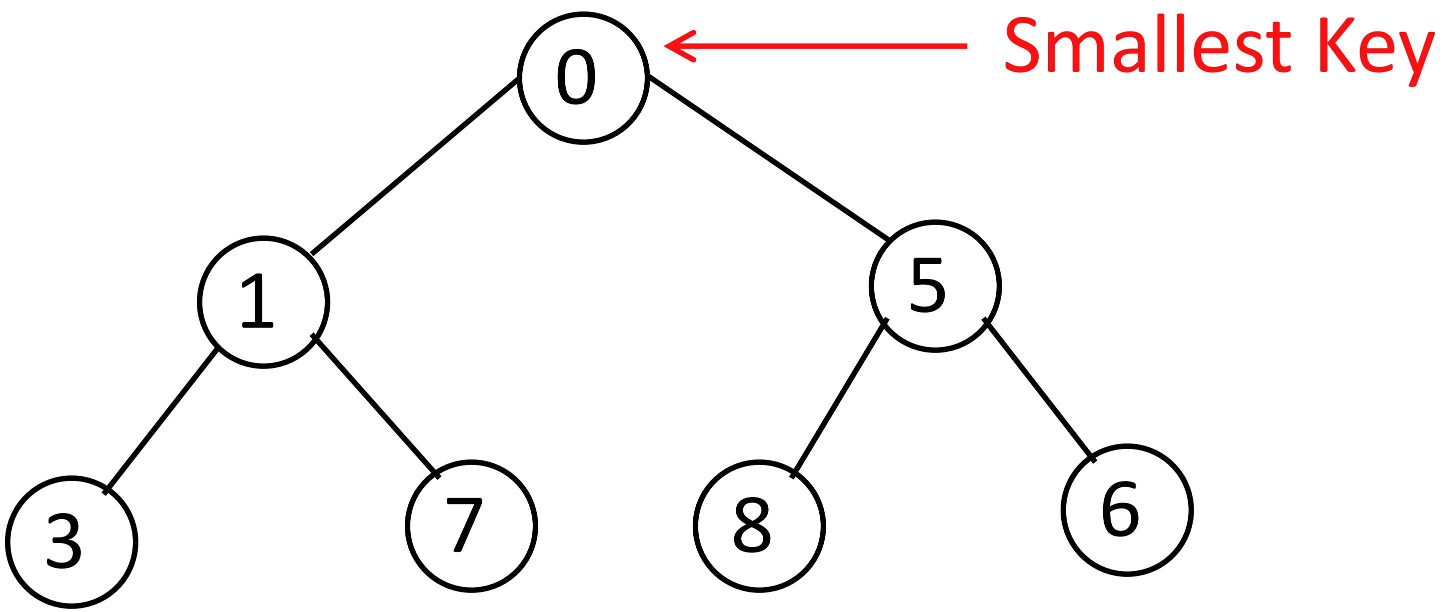

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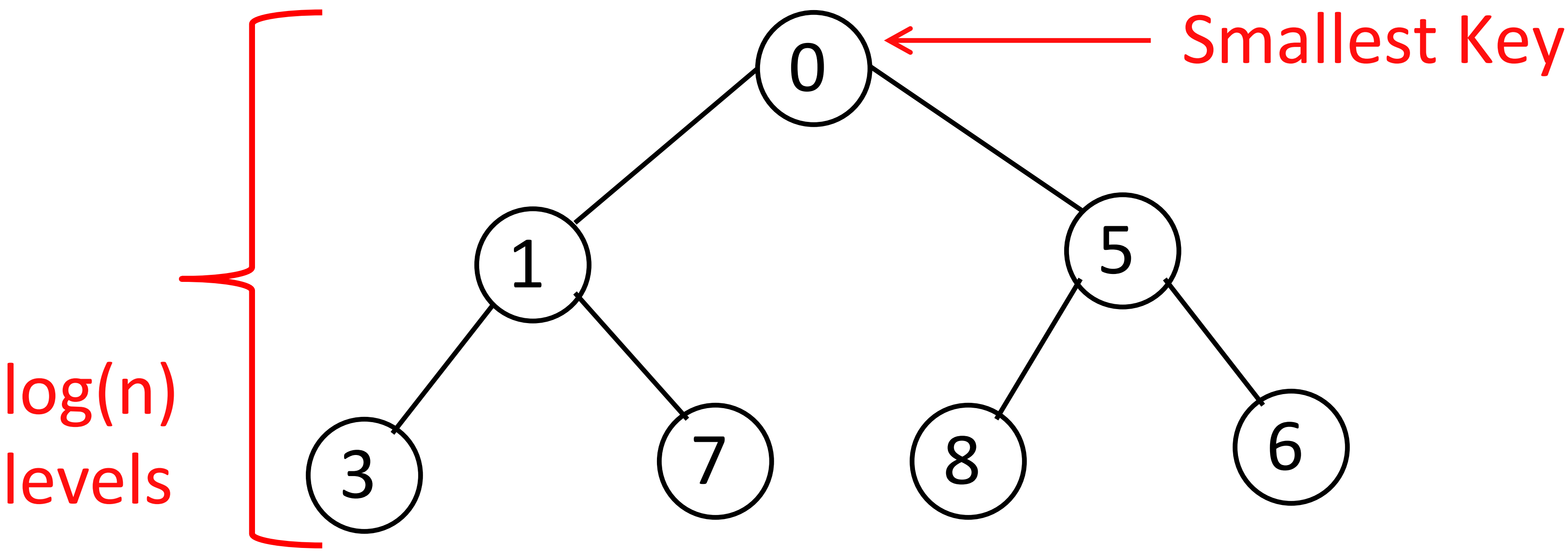
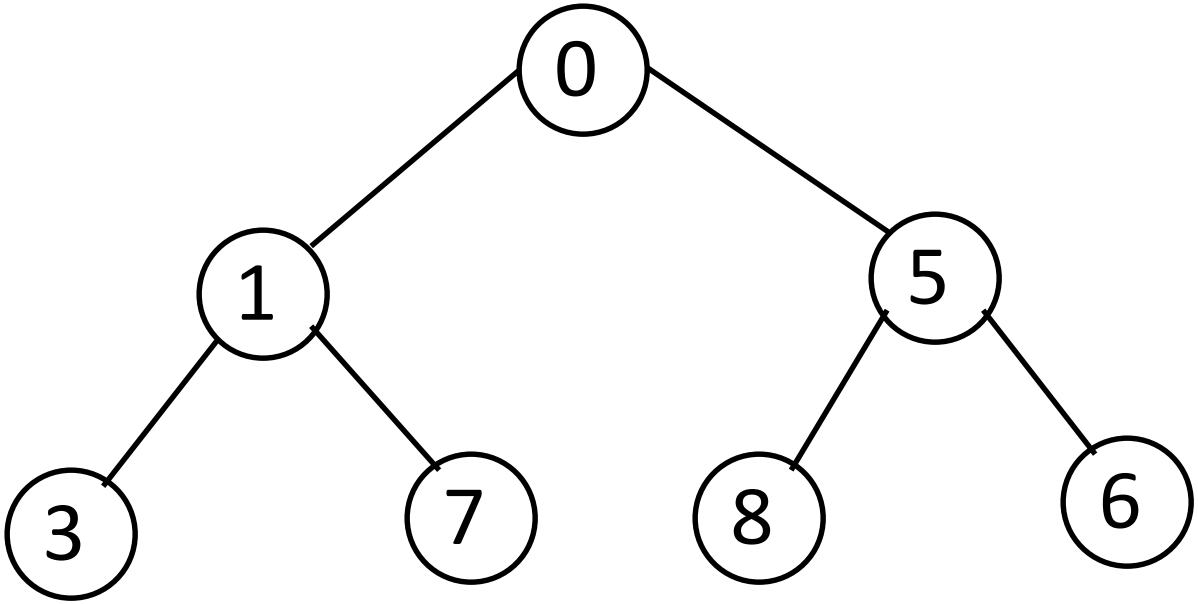


Insert



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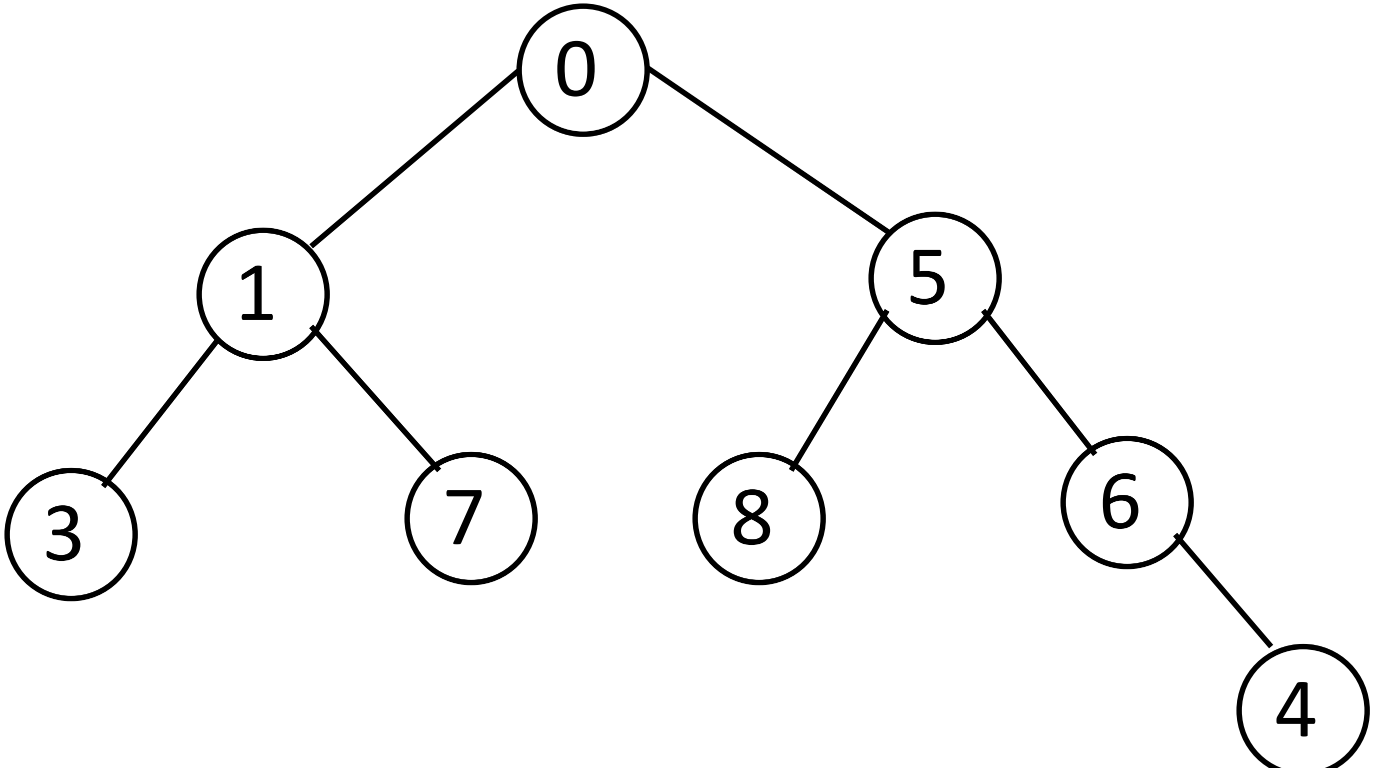
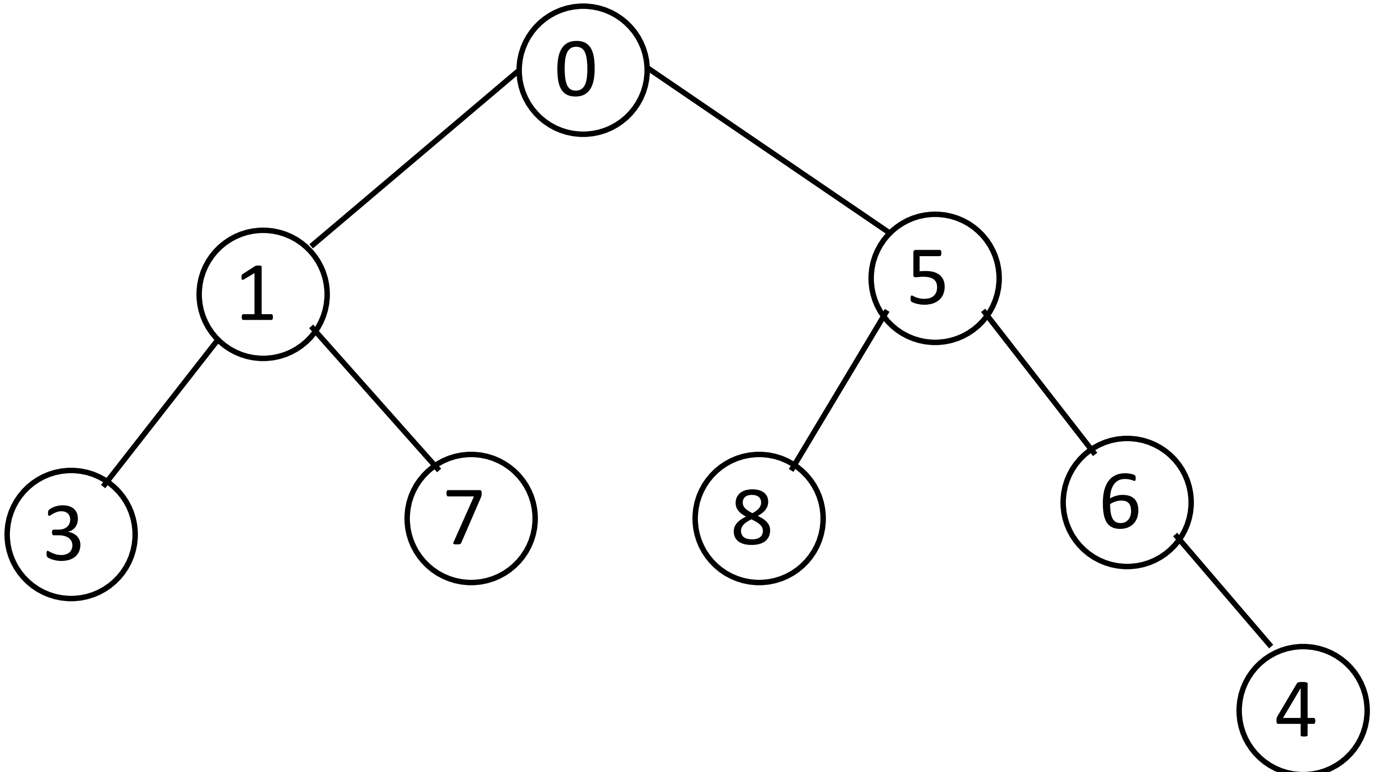


- Add key at bottom

Insert



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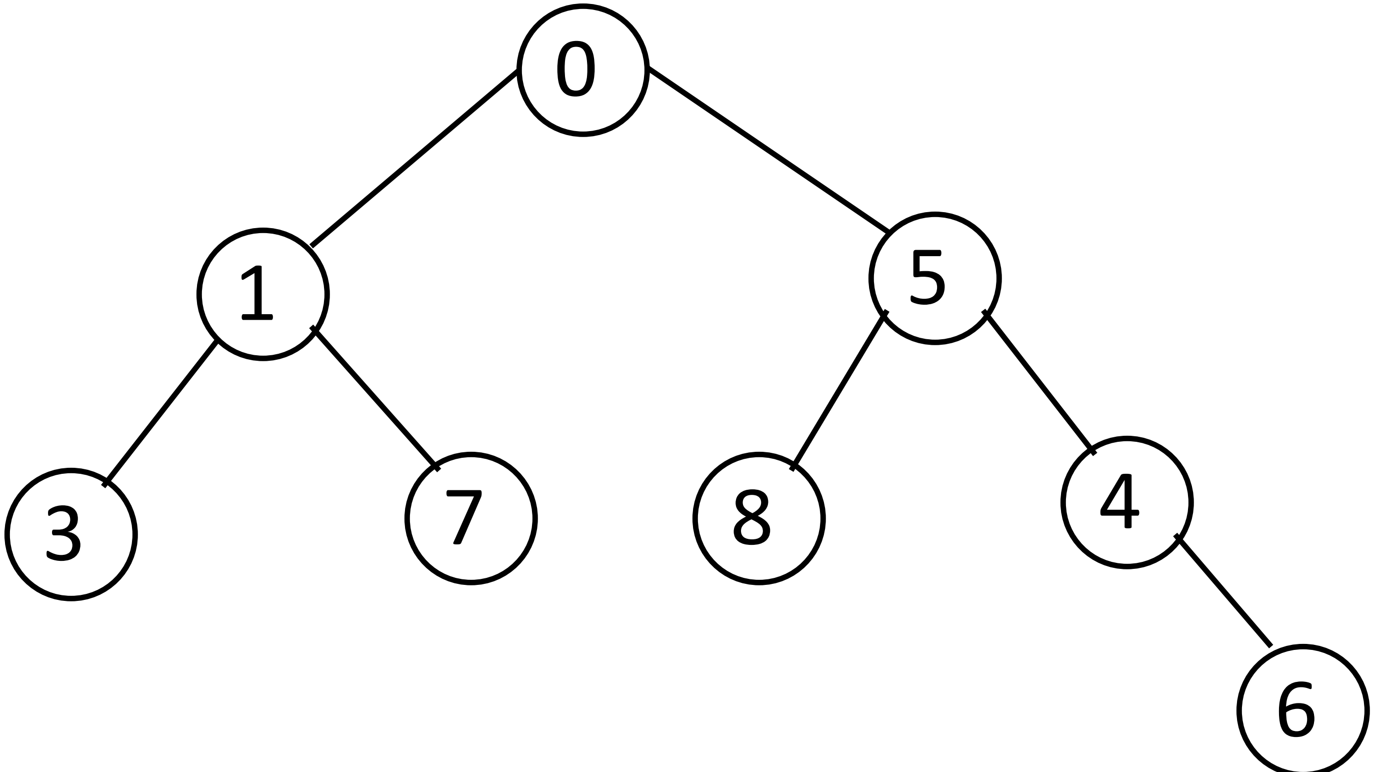
- Bubble up

Insert



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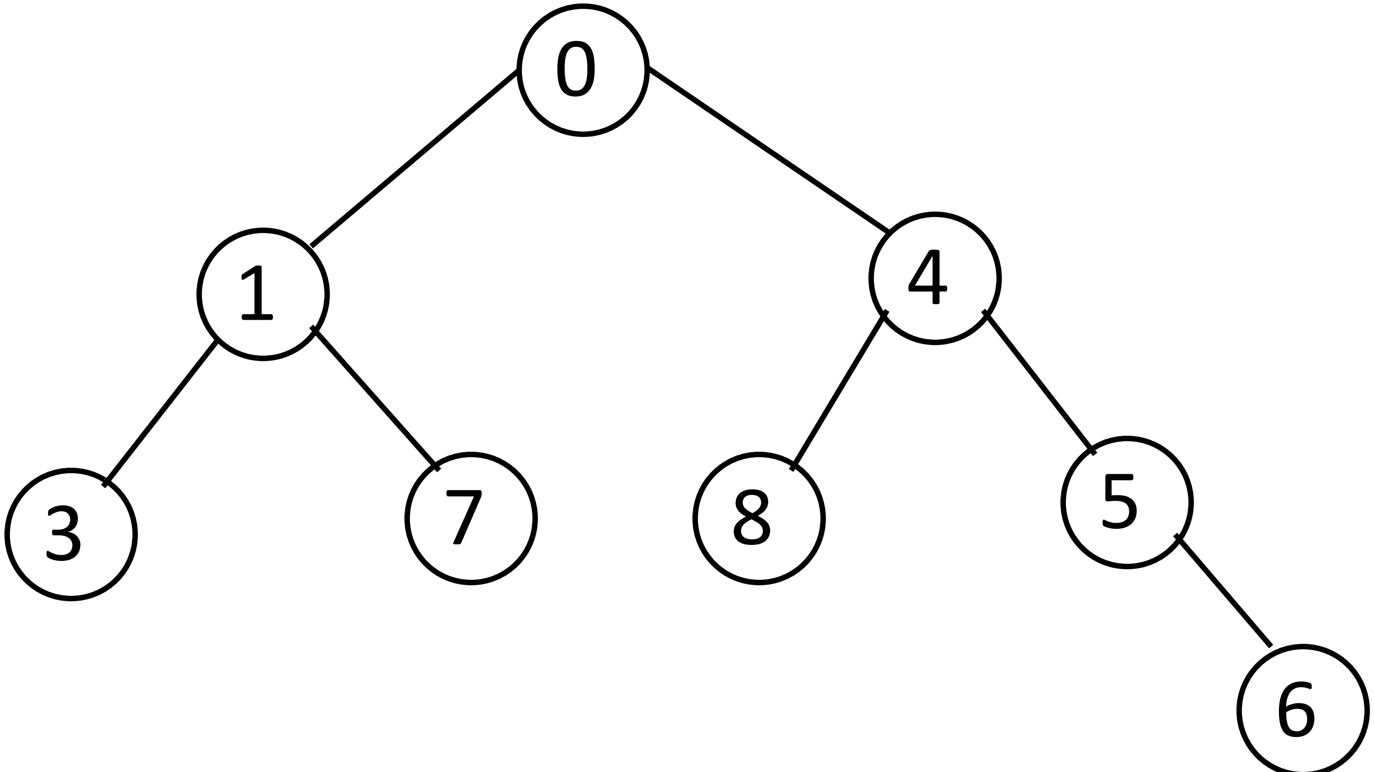

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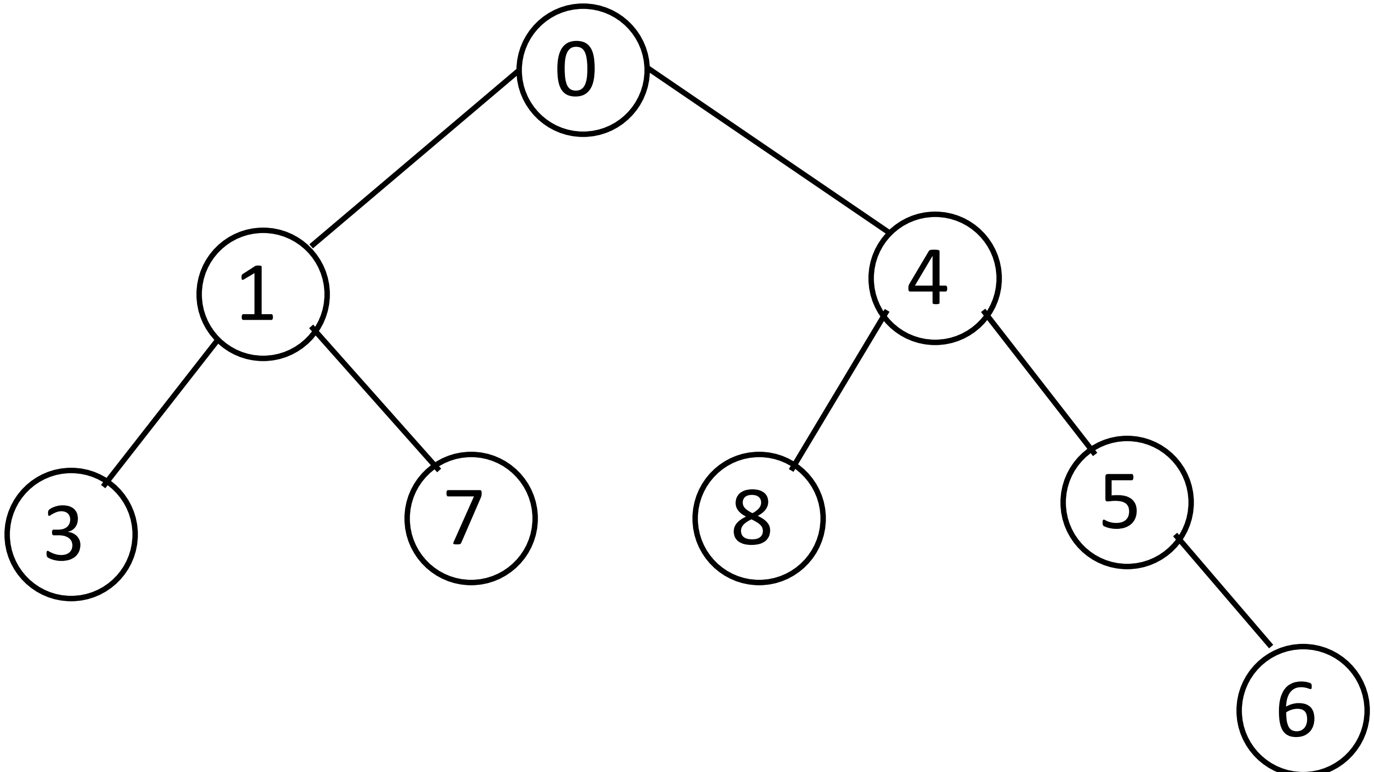

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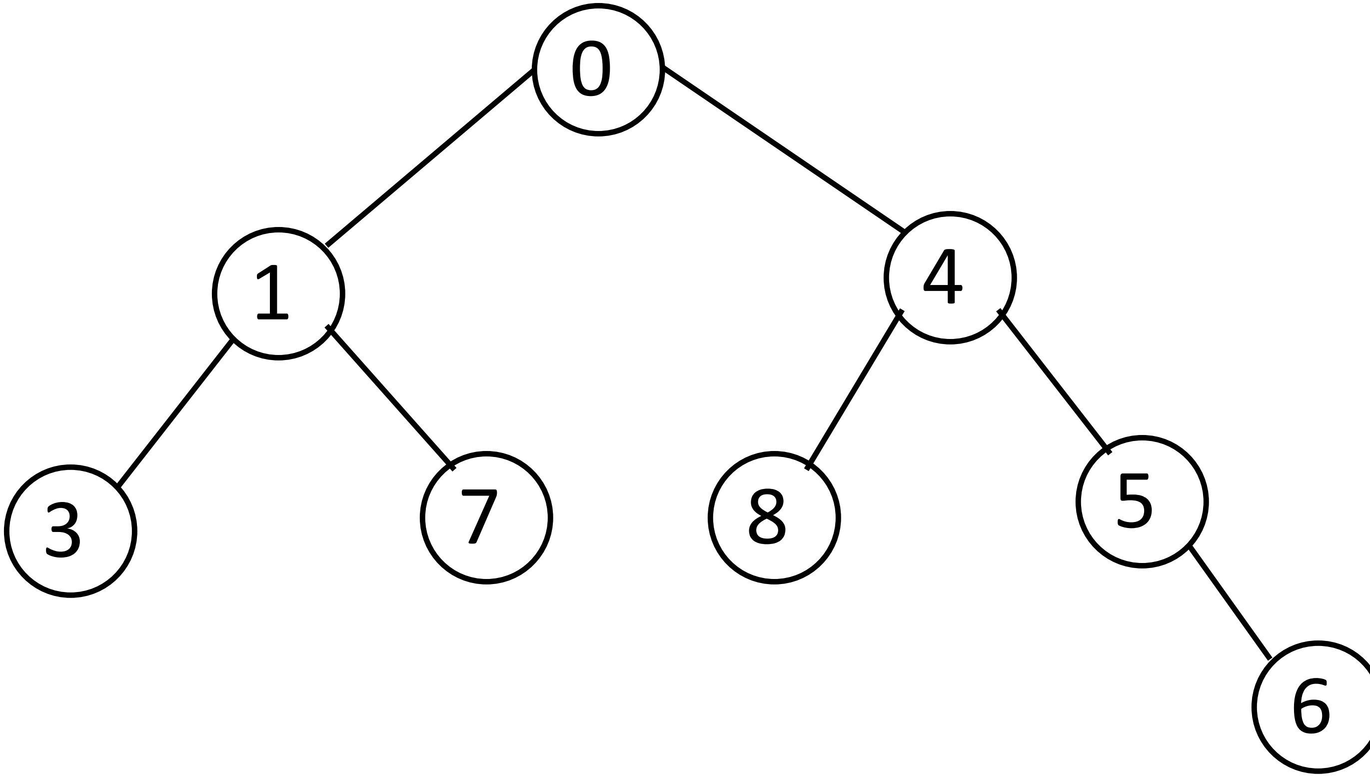
O(log(n)) time

Decrease Key



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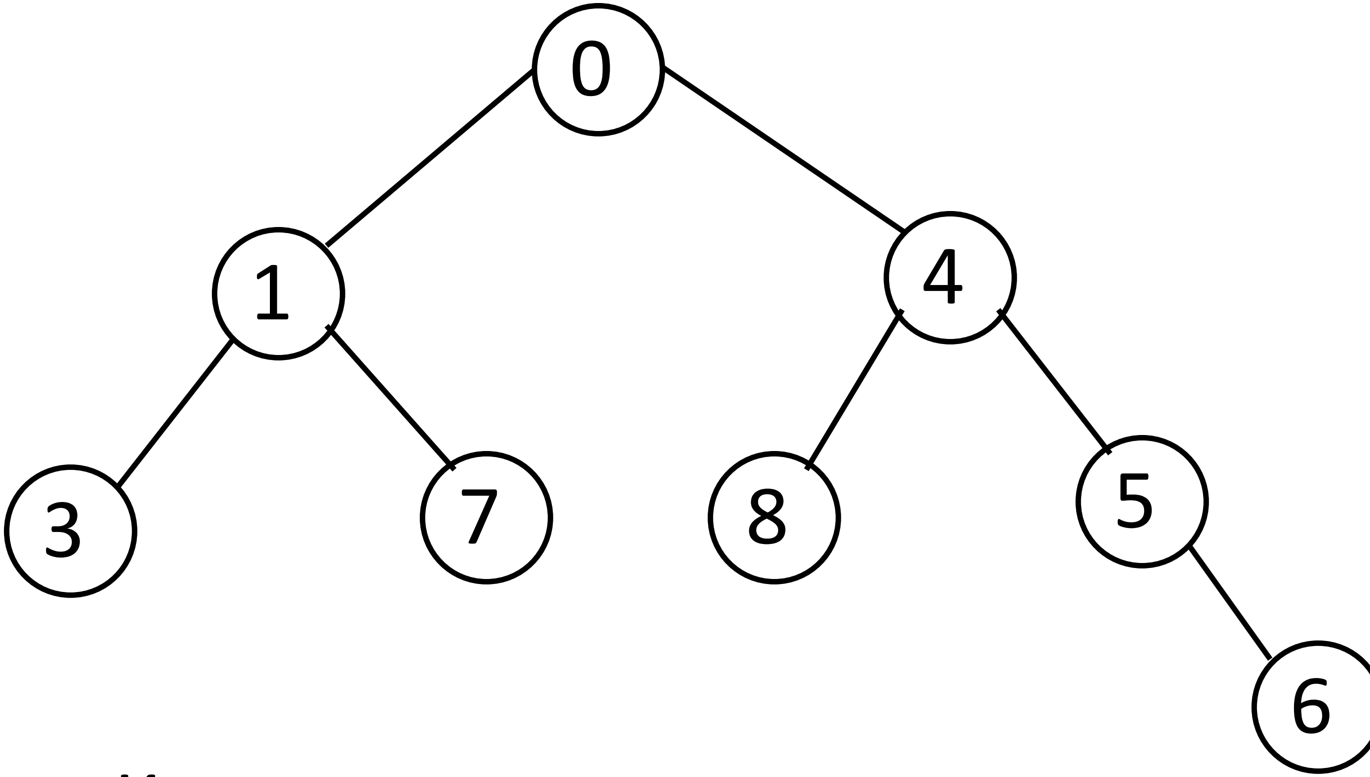
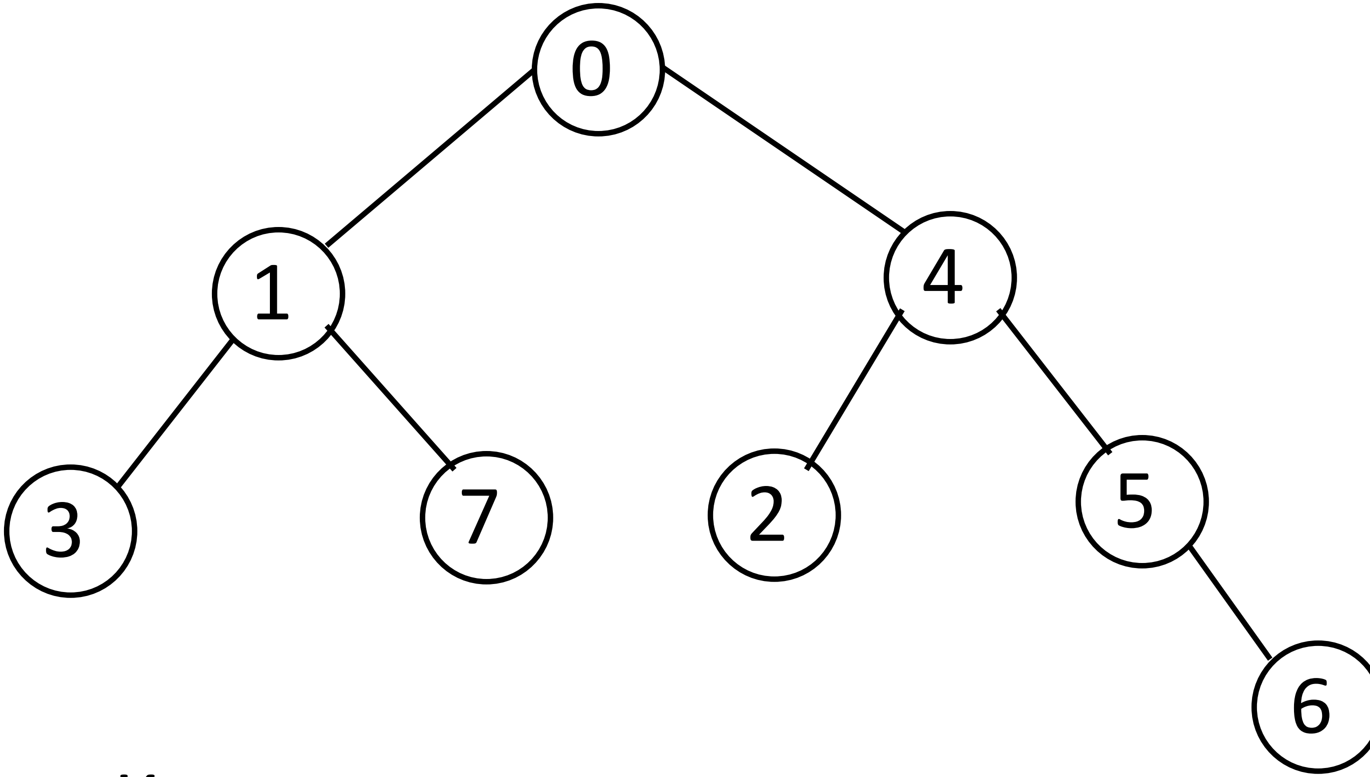


- Change Key

Decrease Key



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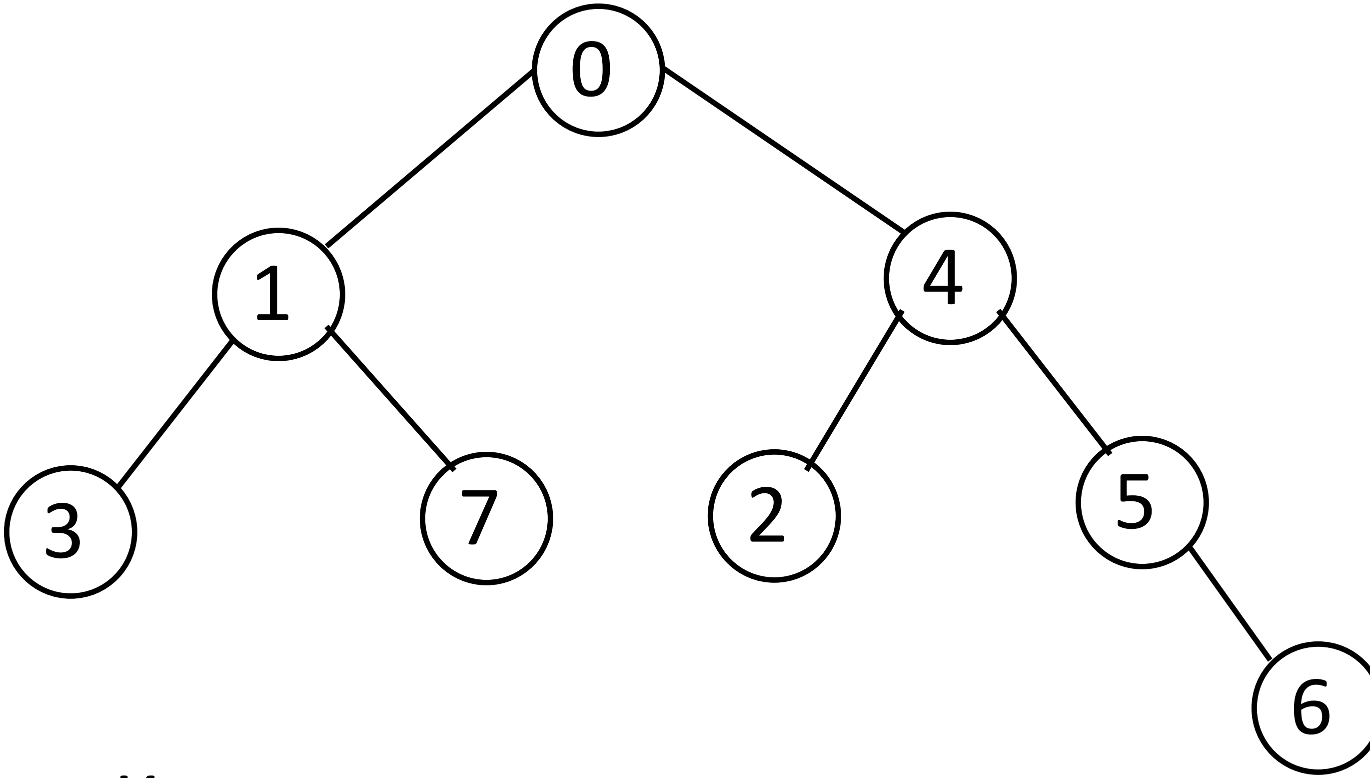

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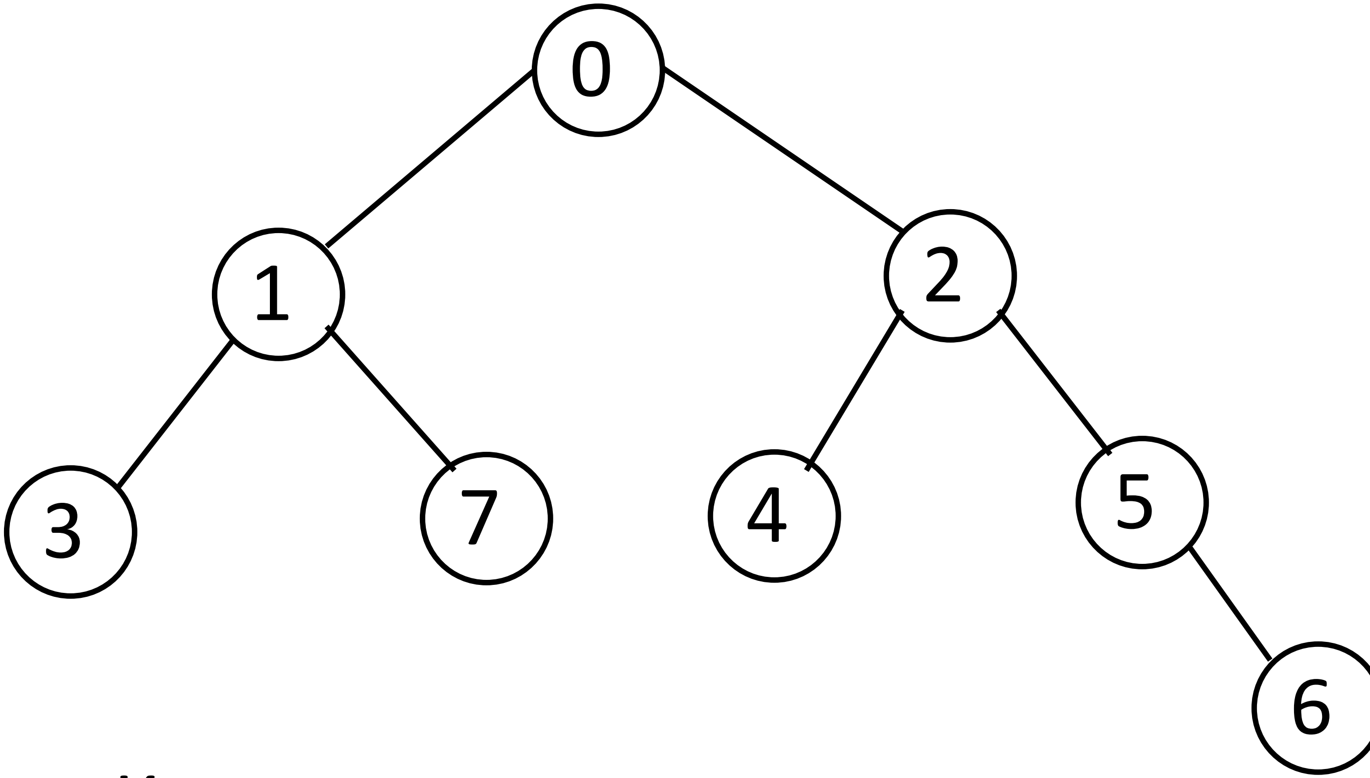

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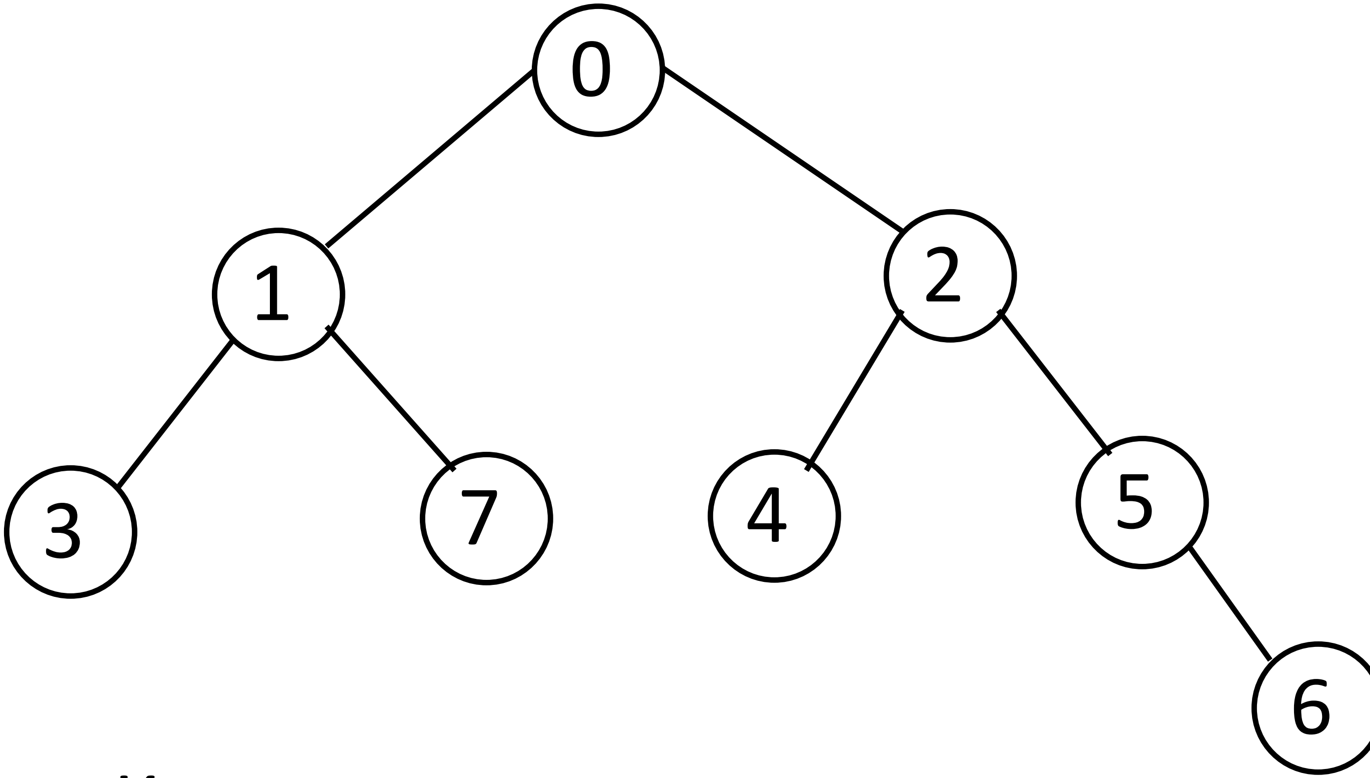

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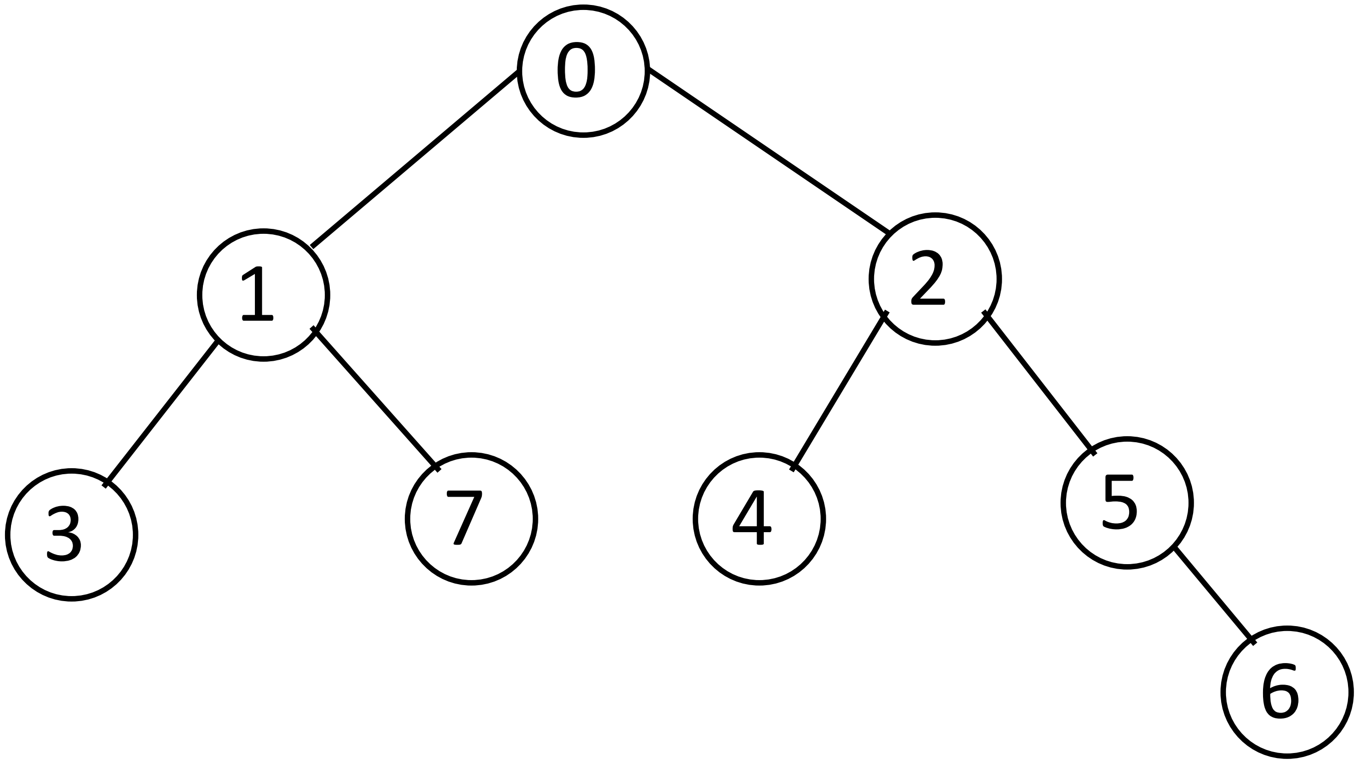
O(log(n)) time

DeleteMin



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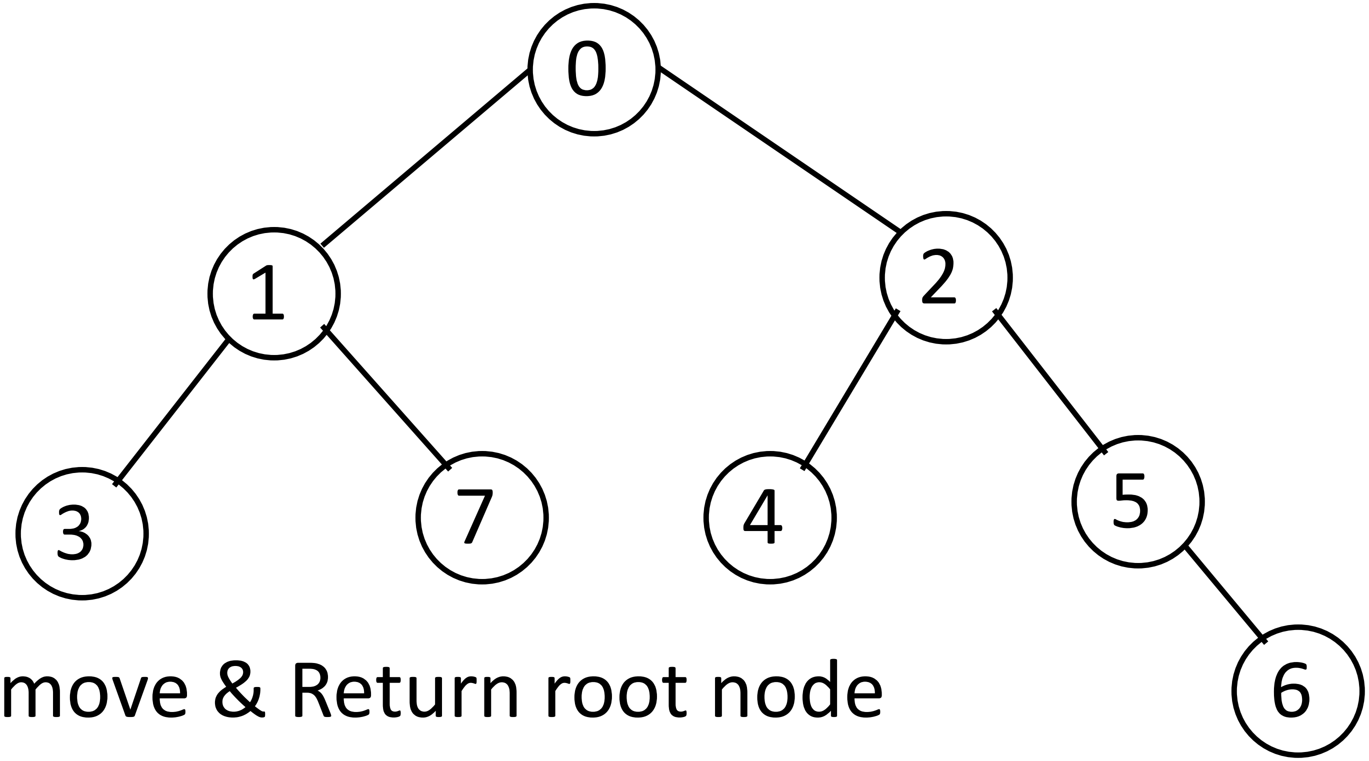
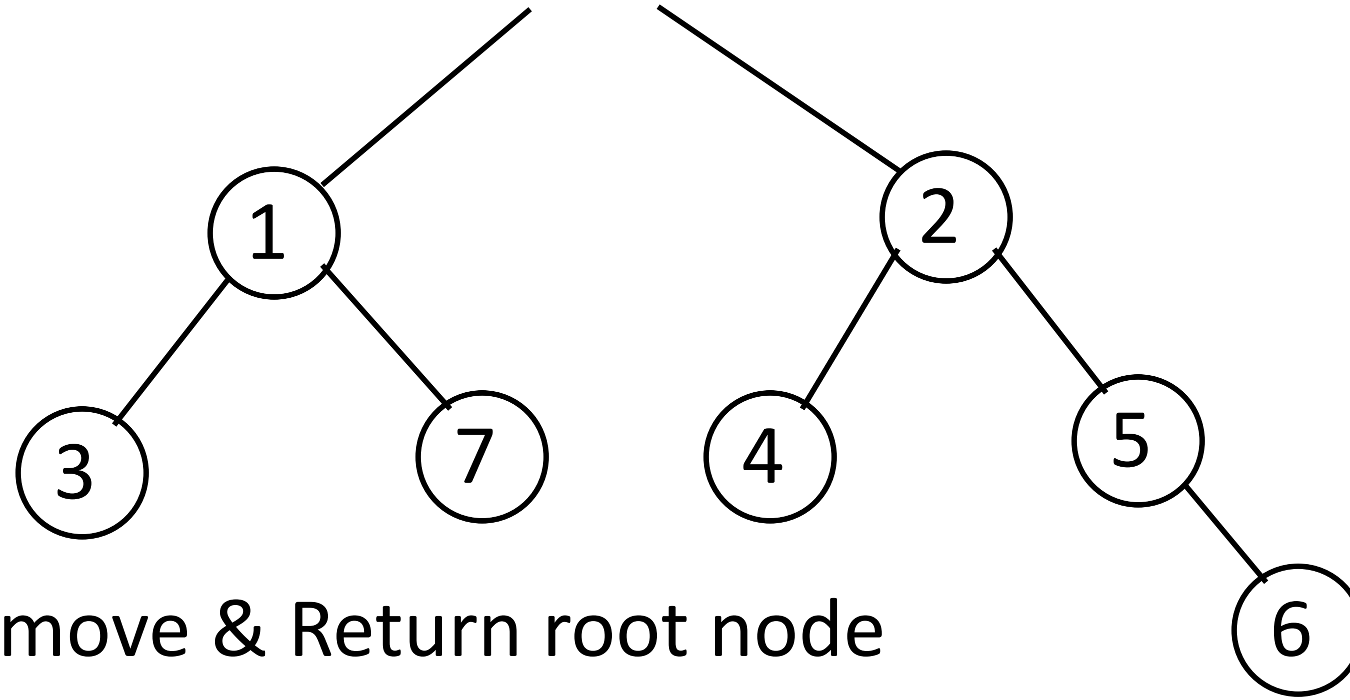


- Remove & Return root node

DeleteMin

Output:

0



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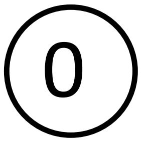

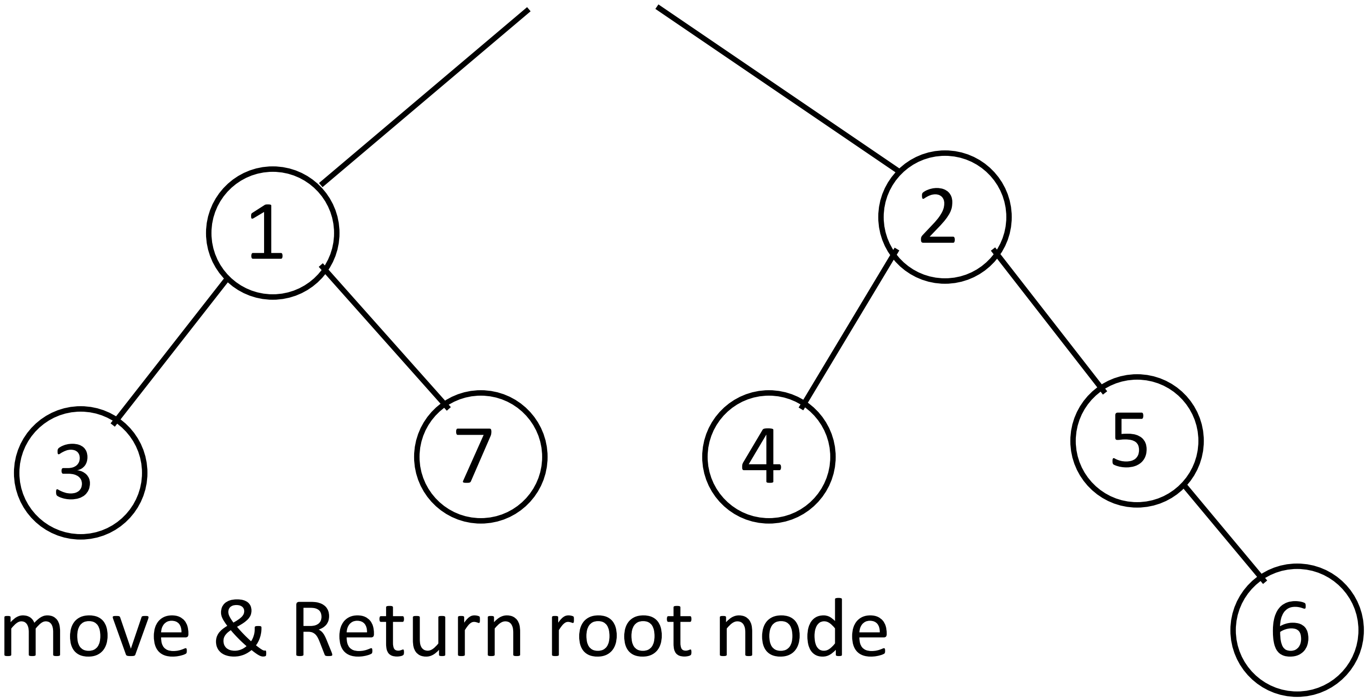

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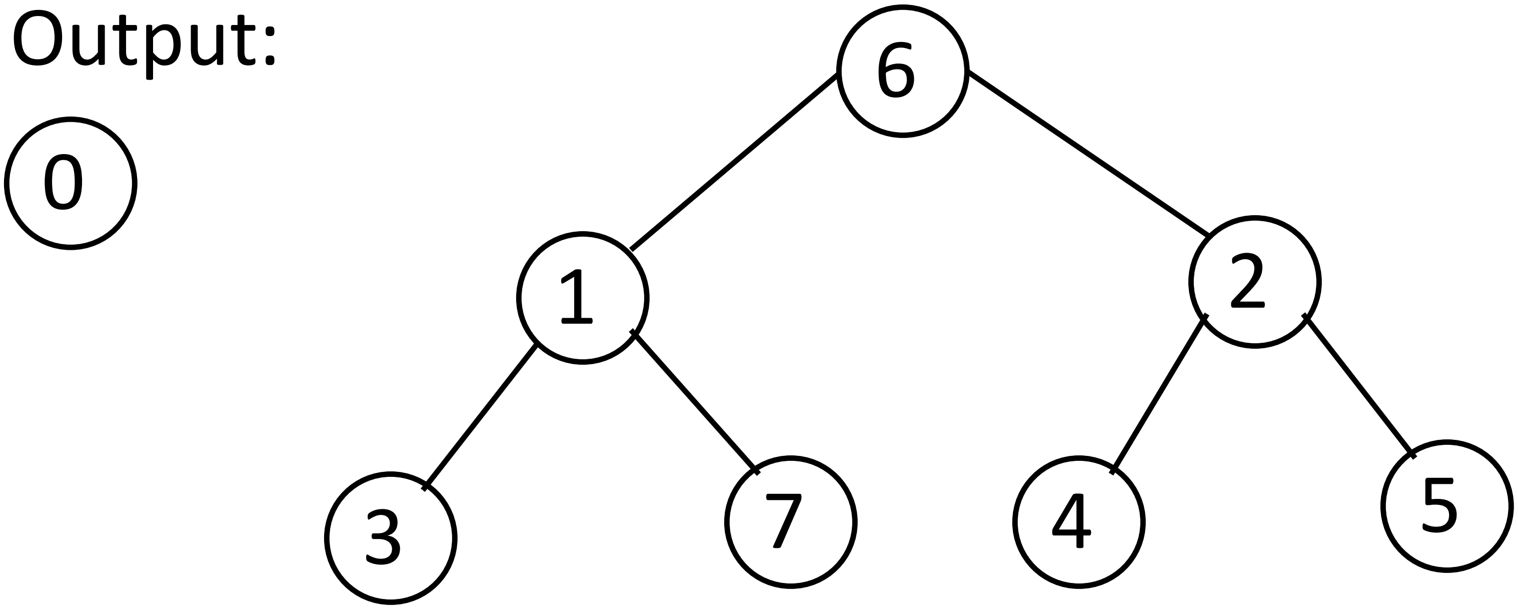

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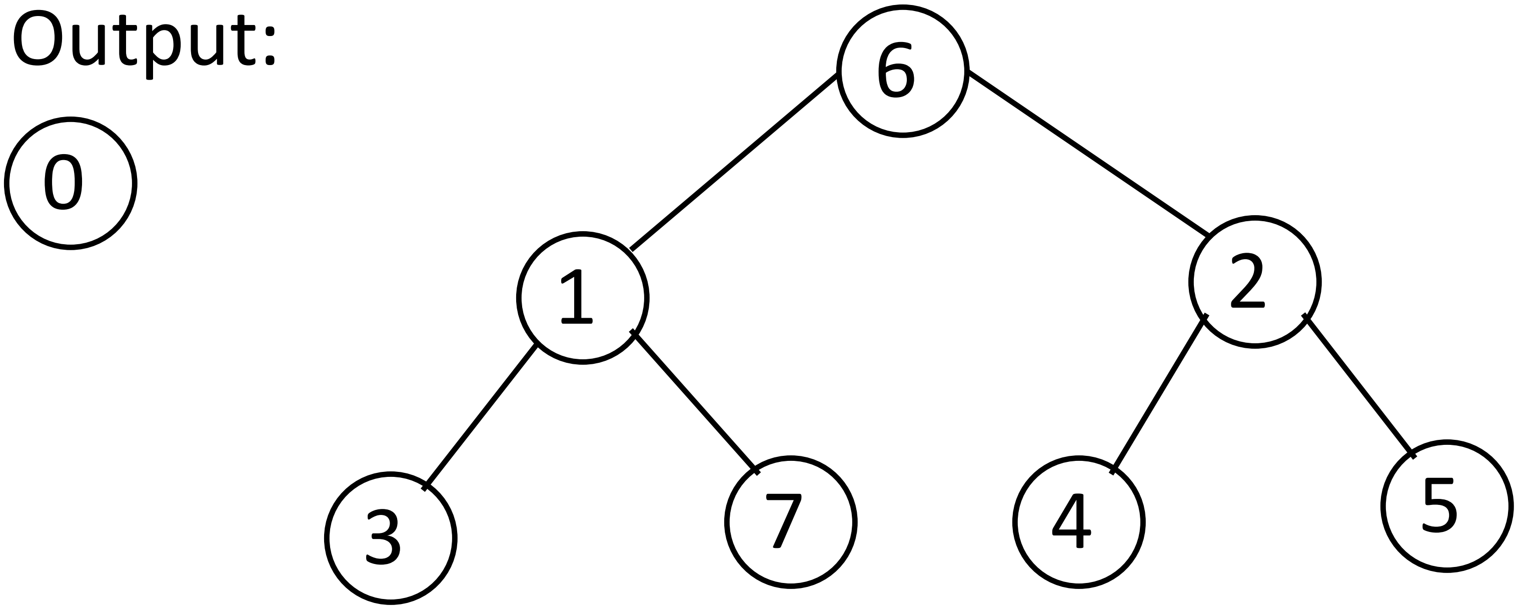

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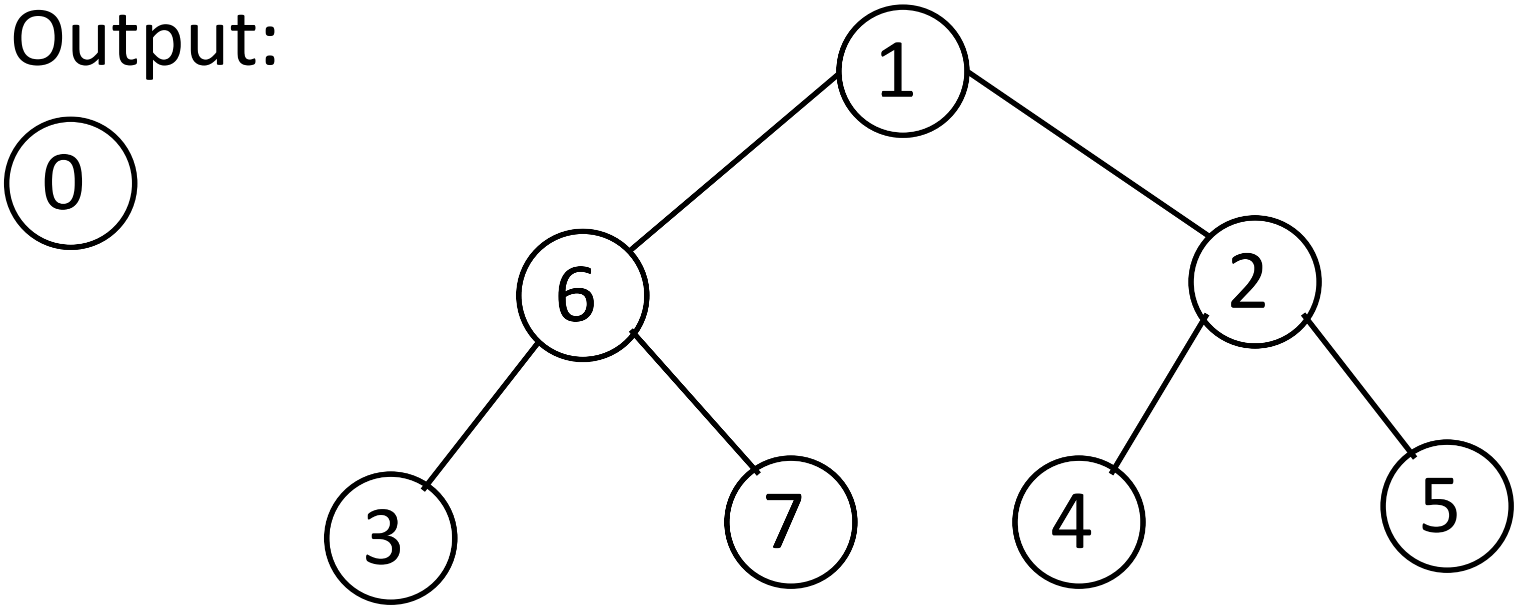

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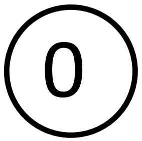

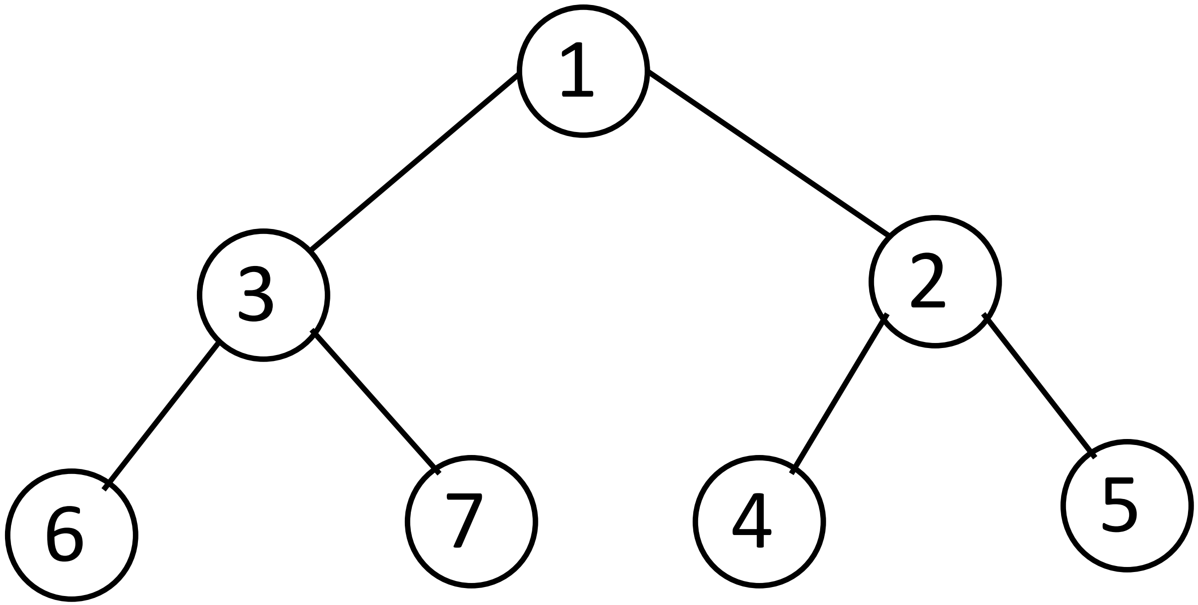

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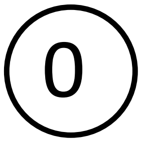

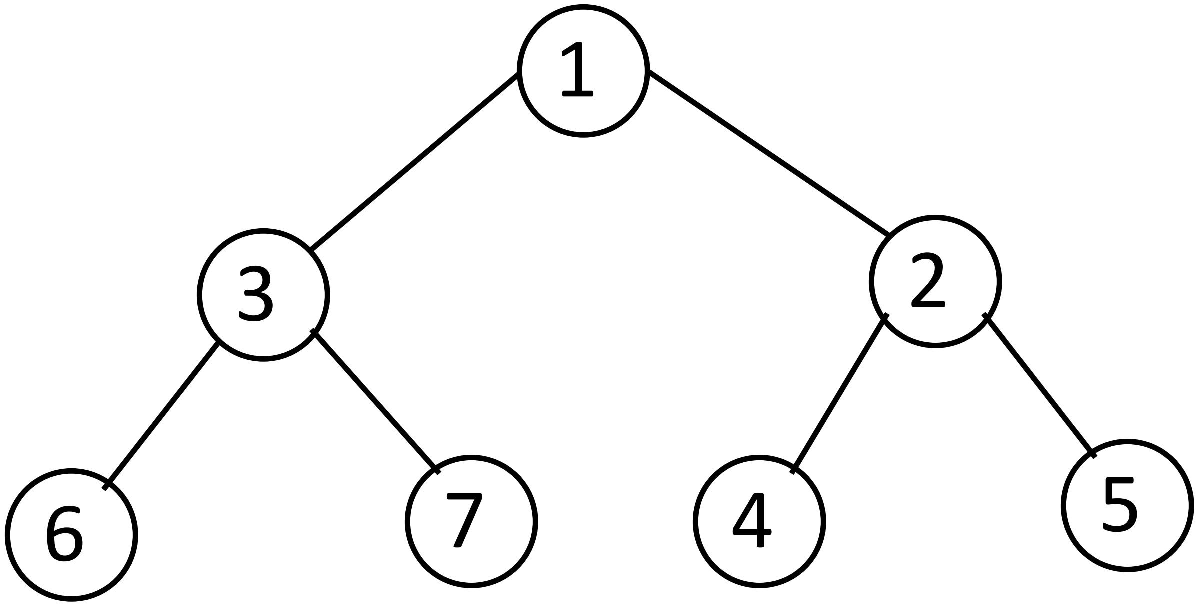

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Runtime O(log(n))

Summary

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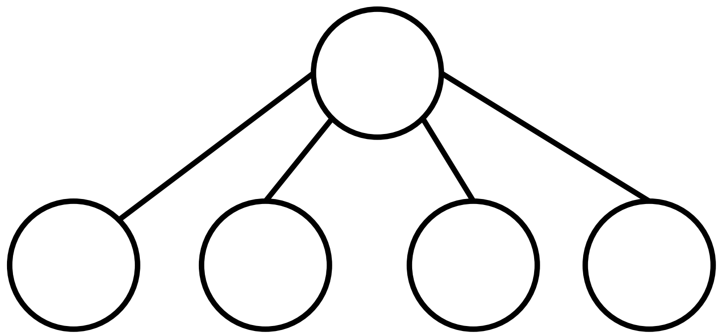
Almost linear!

d-ary Heap

- Like binary heap, but each node has d children



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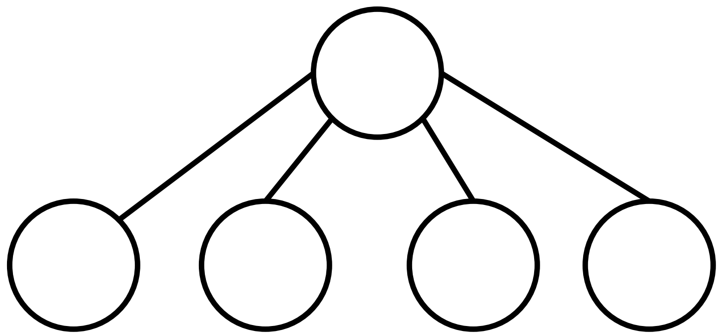

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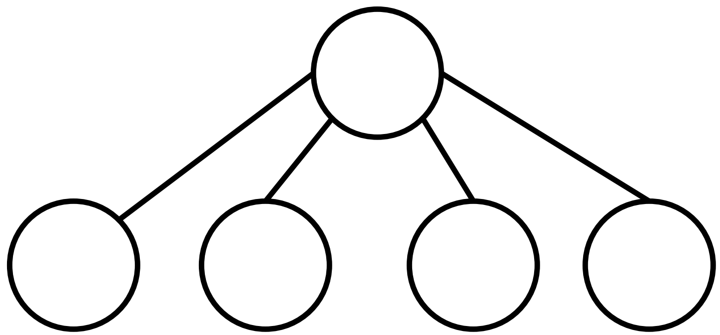

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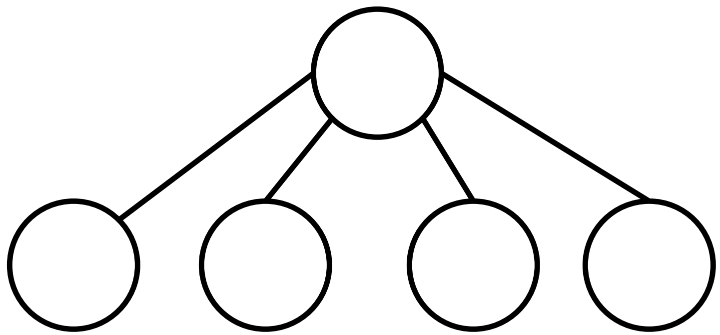

- Only log(n)/log(d) levels.
- Bubble up faster!
- Bubble down slower – need to compare to more children.

Summary

Runtime:

- Insert – O(log(n)/log(d))
- DecreaseKey – O(log(n)/log(d))
- DeleteMin – O(dlog(n)/log(d))

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Fibonacci Heap

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Runtime:

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- DeleteMin – $O(\log(n))$
- Dijkstra – $O(|V|\log|V|+|E|)$

Summary of Priority Queues

	Insert/DecreaseKey	DeleteMin	Dijkstra
List	$O(1)$	$O(n)$	$O(\|V\|^2 + \|E\|)$
Binary Heap	$O(\log(n))$	$O(\log(n))$	$O(\log\|V\|(\|V\| + \|E\|))$
d-ary Heap	$O\left(\frac{\log(n)}{\log(d)}\right)$	$O\left(\frac{d\log(n)}{\log(d)}\right)$	$O\left(\frac{\log\|V\|}{\log(d)}(d\|V\| + \|E\|)\right)$
Fibonacci Heap	$O(1)^*$	$O(\log(n))^*$	$O(\|V\|\log\|V\| + \|E\|)$

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  - The usual case (distance & time usually cannot be negative).
  - However, if “lengths” represent other kinds of costs, sometimes they can be negative.
- Problem statement same. Find path with smallest sum of edge weights.

Question: Dijkstra for Negative Edge Weights

Does Dijkstra’s algorithm work in graphs with negative edge weights?

A) Yes

B) No

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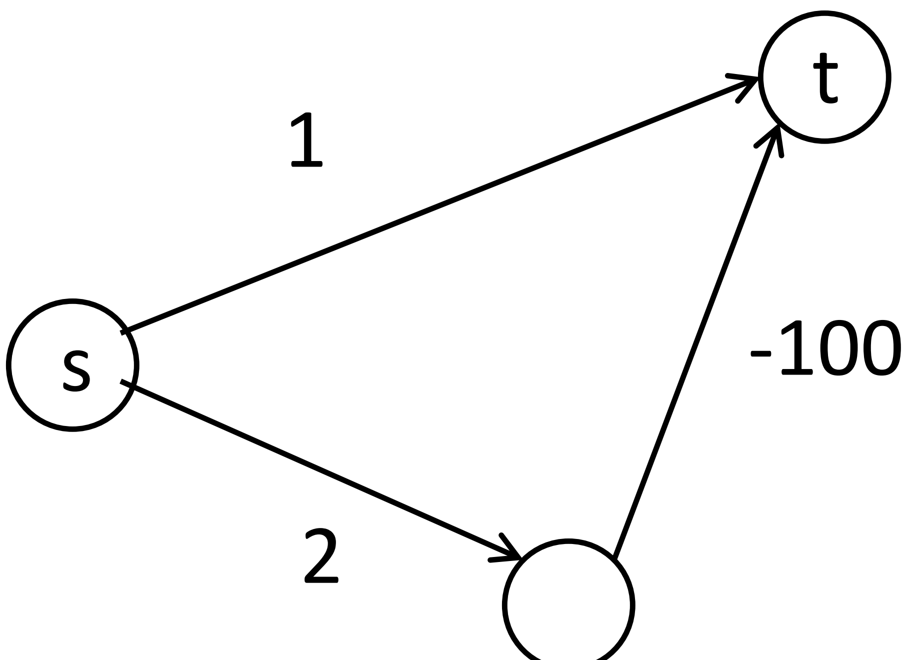

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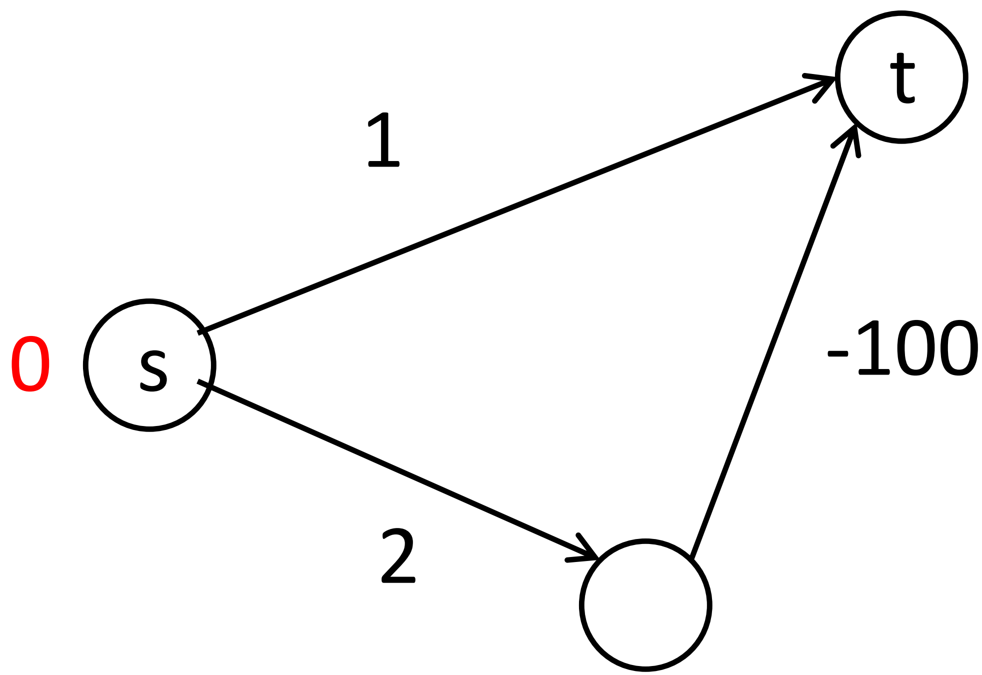

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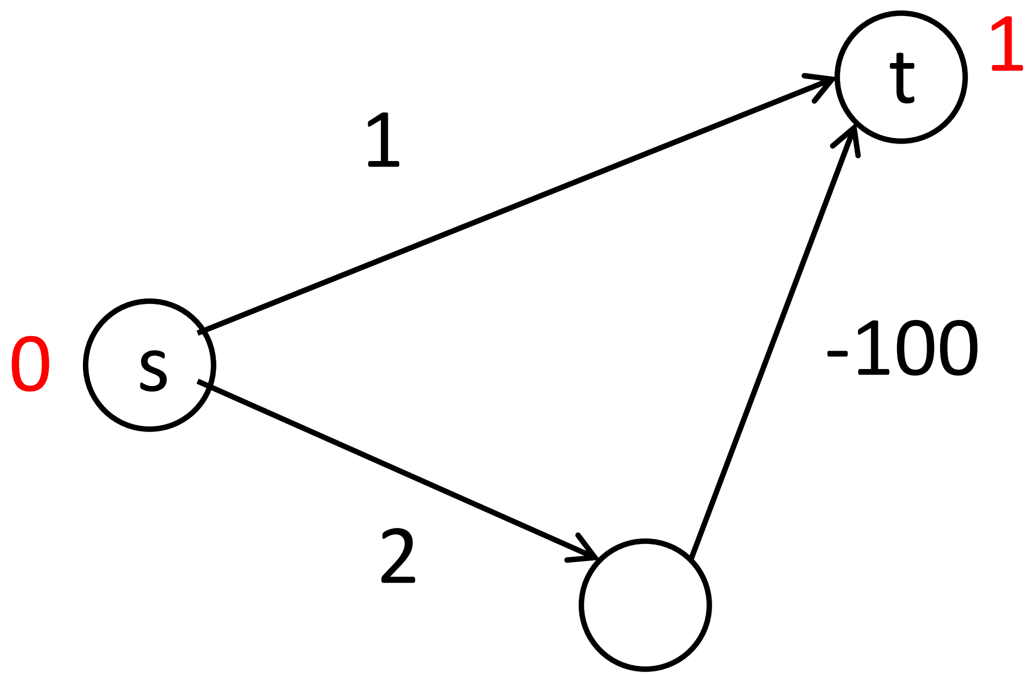

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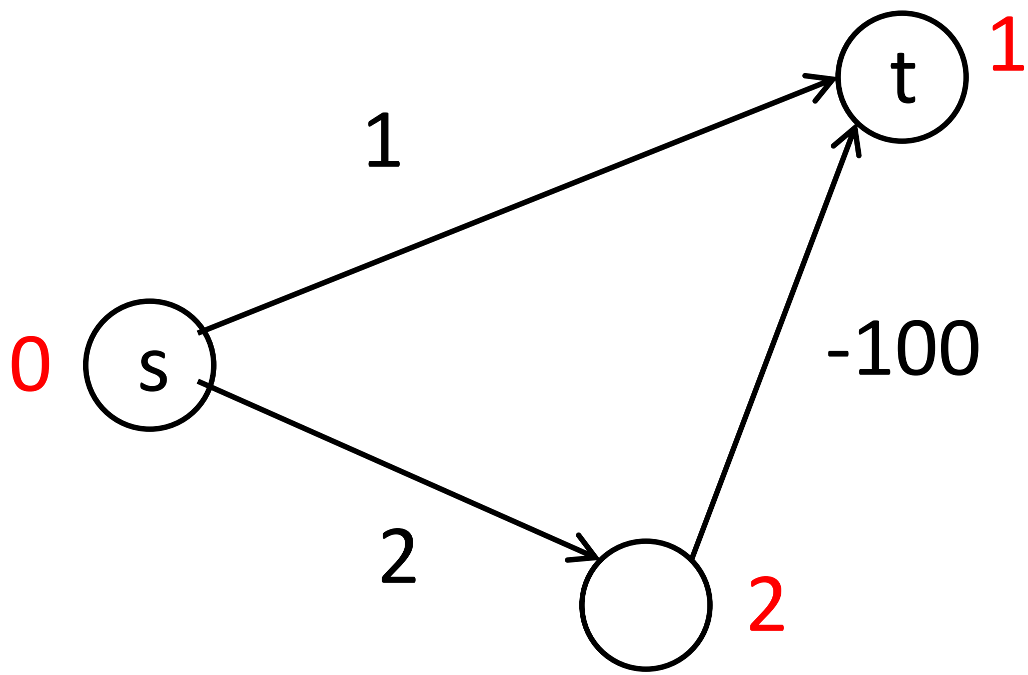

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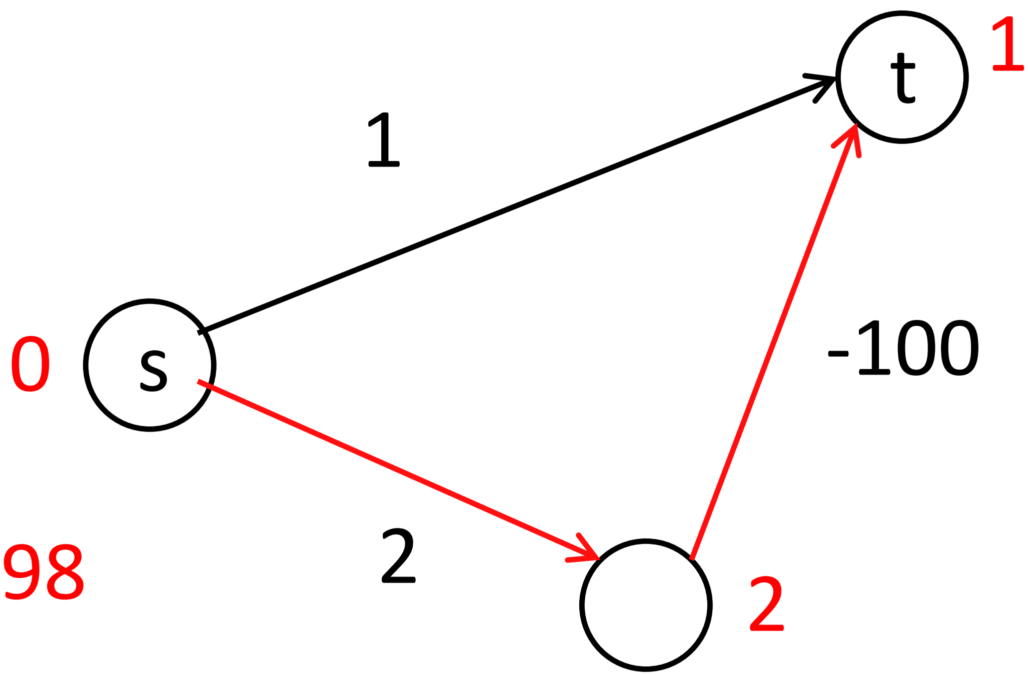
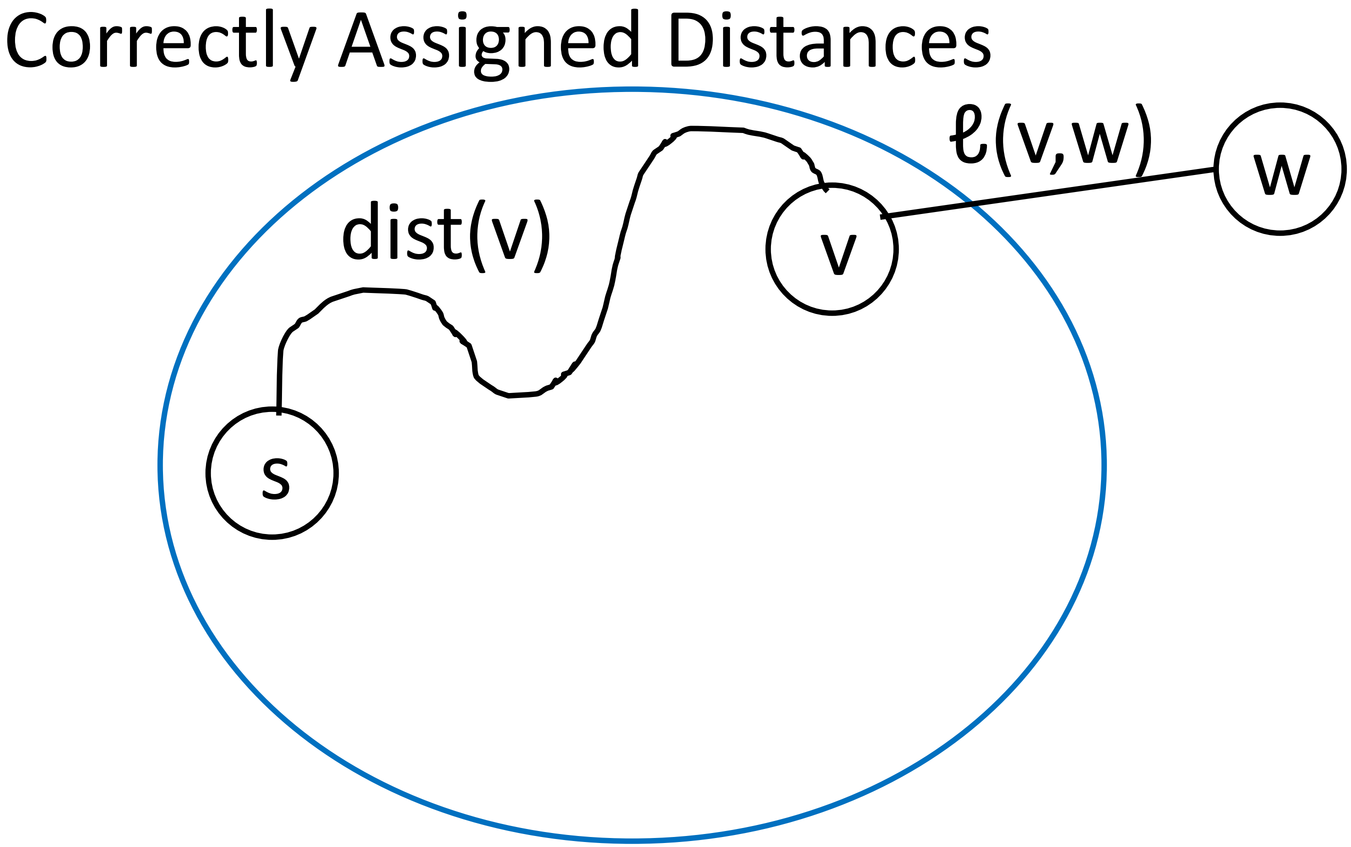


Length = -98

What Goes Wrong



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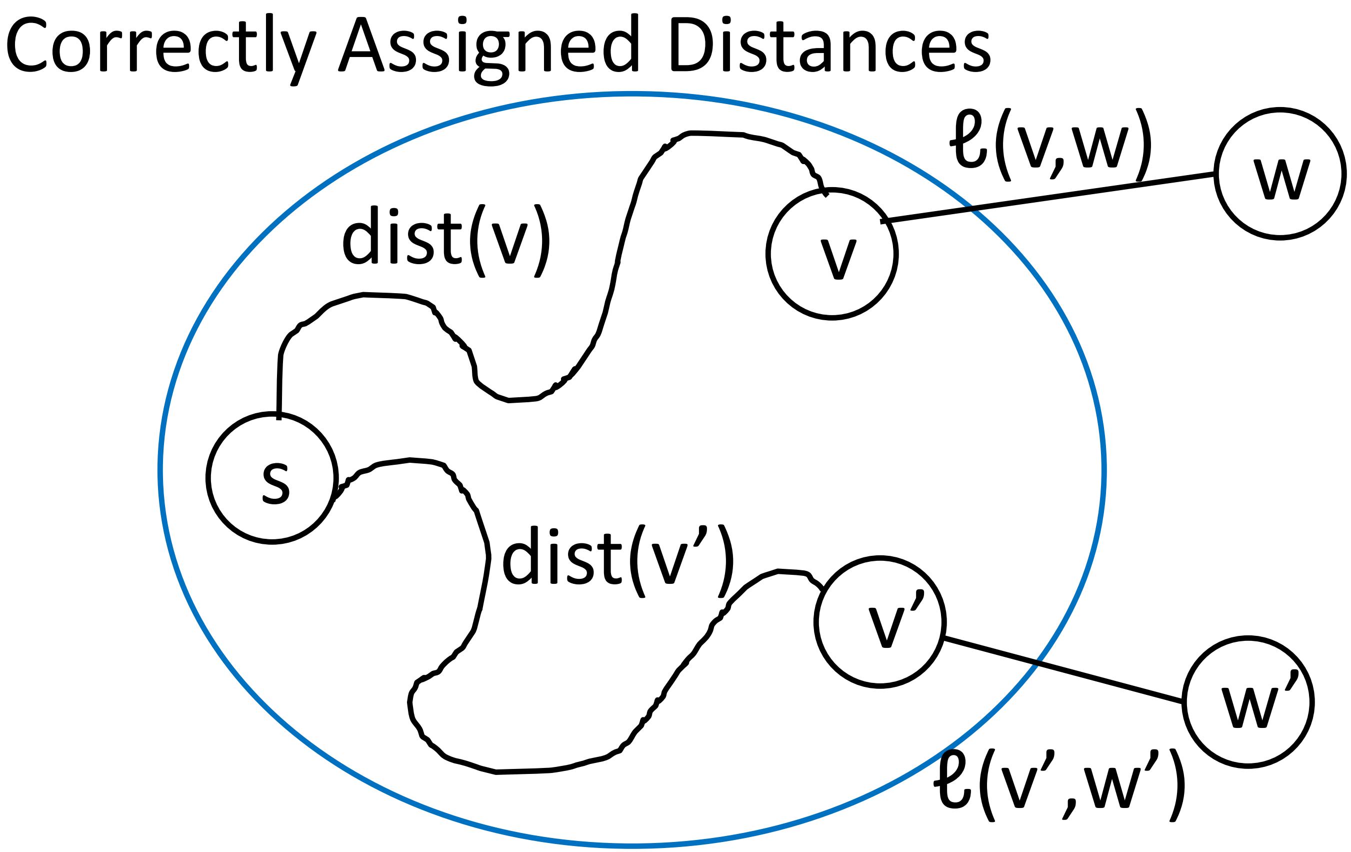

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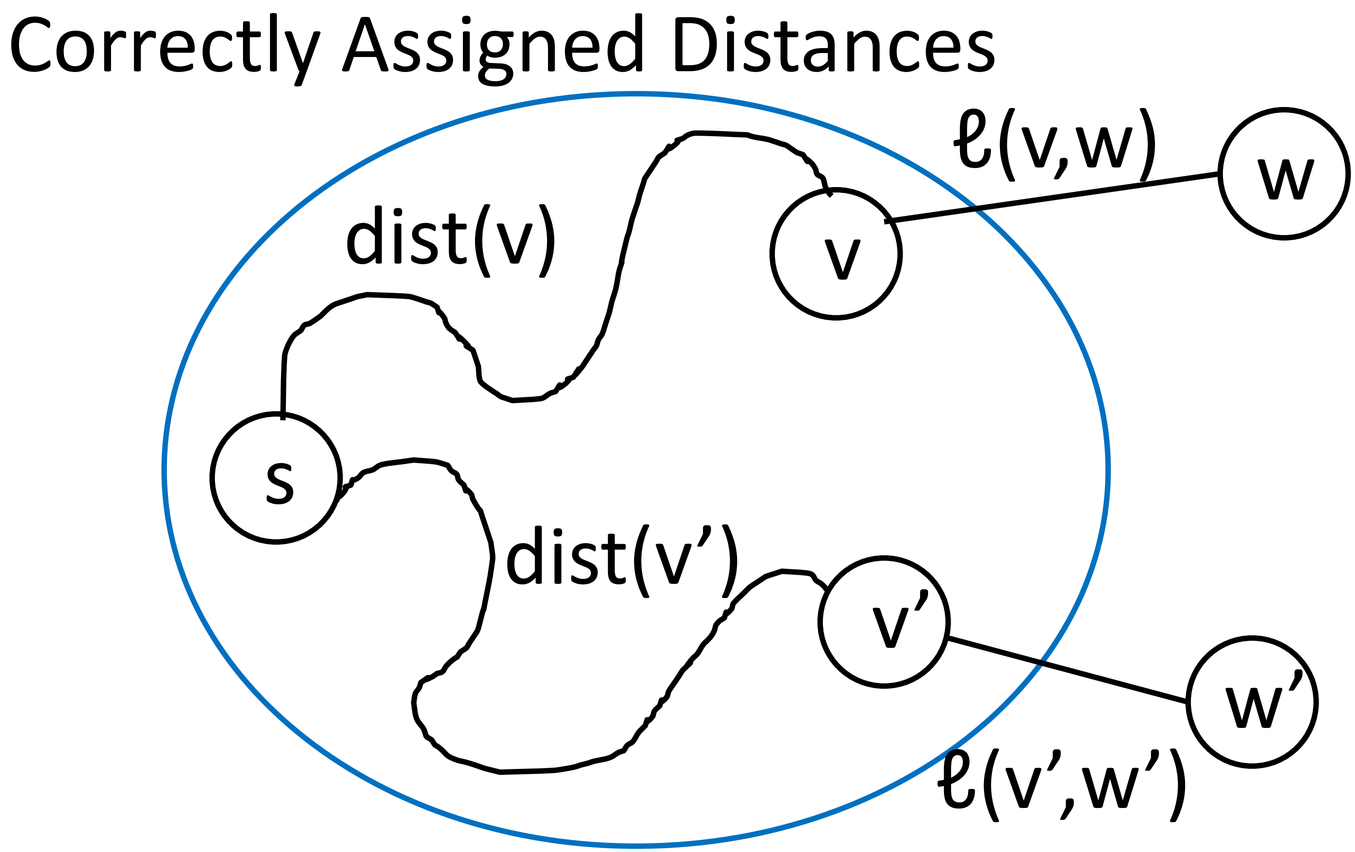

$$\text{dist}(v) + \ell(v,w) \leq \text{dist}(v') + \ell(v',w')$$

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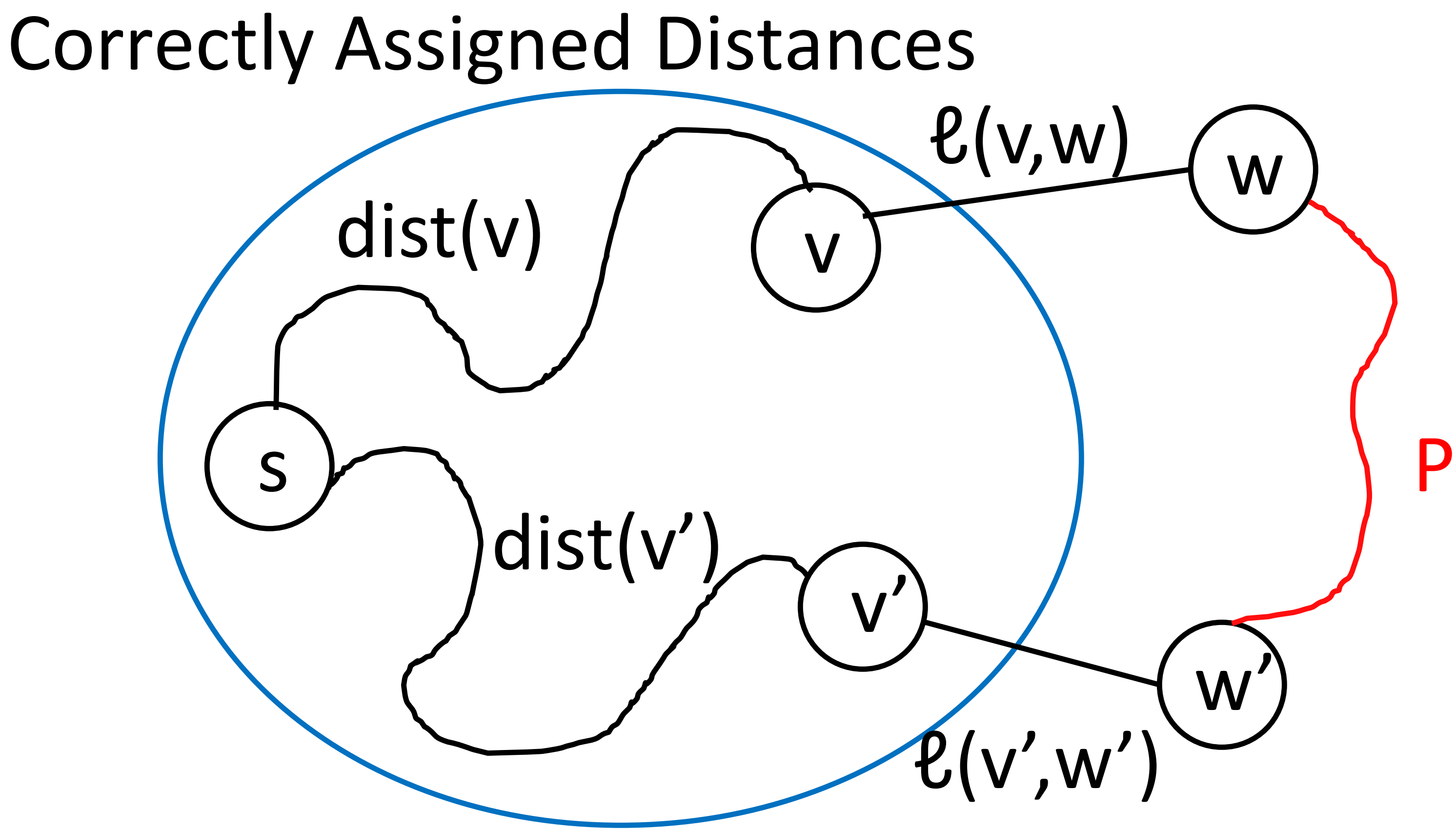

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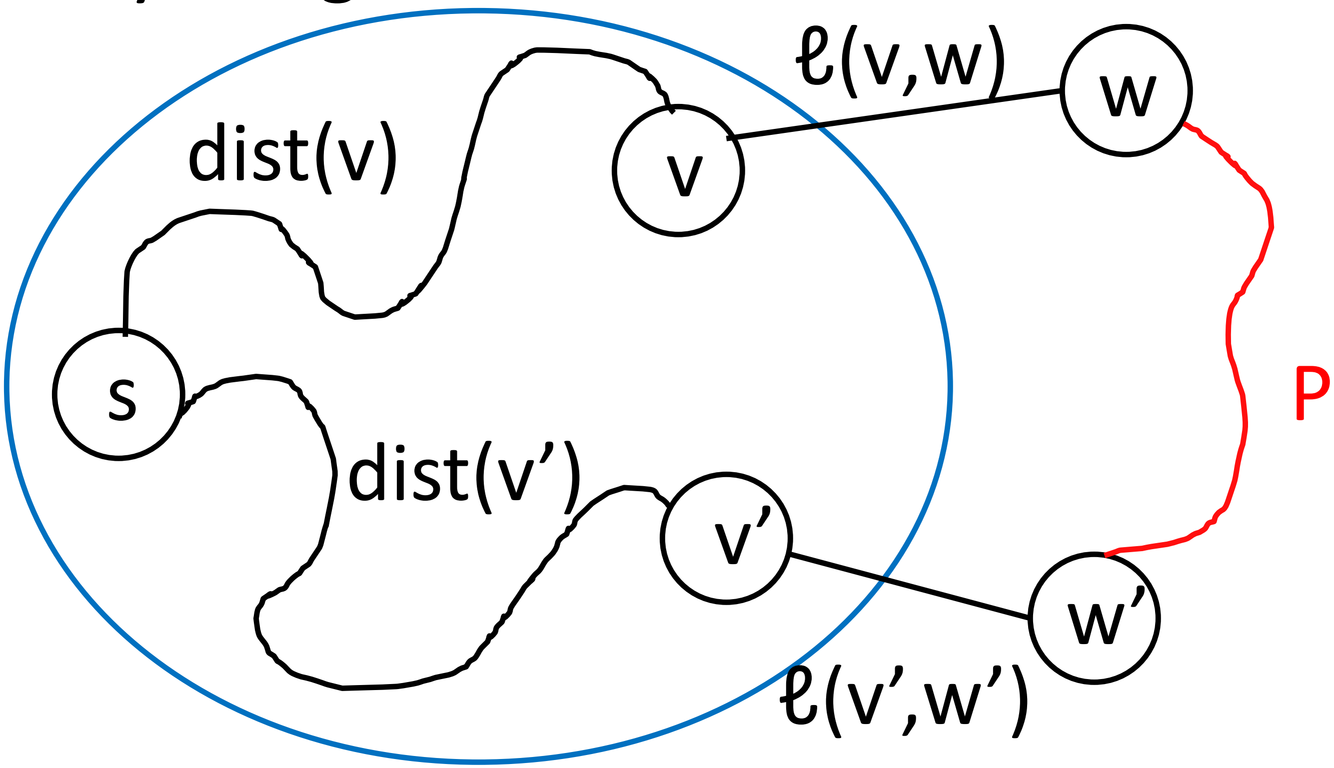
What Goes Wrong

Correctly Assigned Distances



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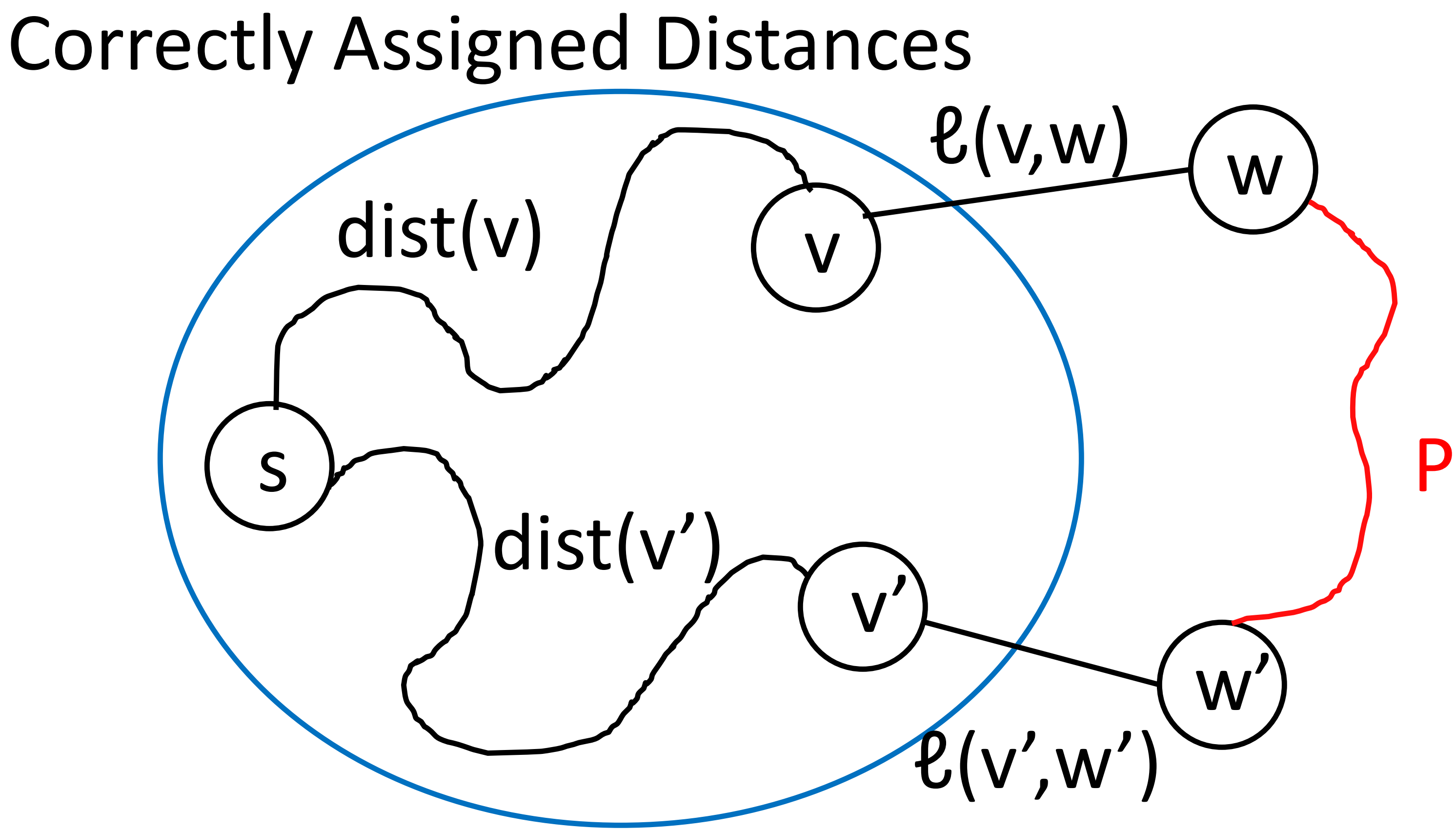
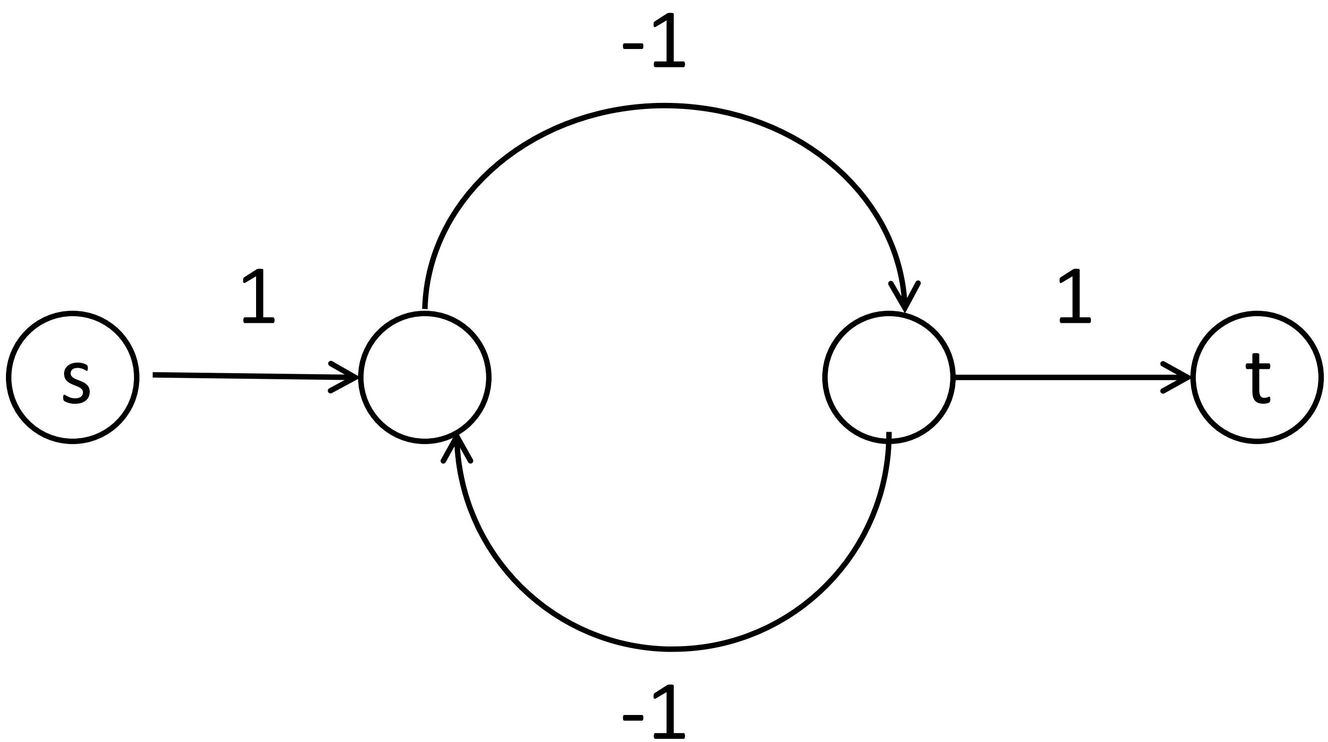


$$\text{dist}(v) + \ell(v,w) \leq \text{dist}(v') + \ell(v',w') + \ell(P)$$

Doesn’t work if $\ell(P)$ is negative!

Problem

What is the shortest path length from s to t?



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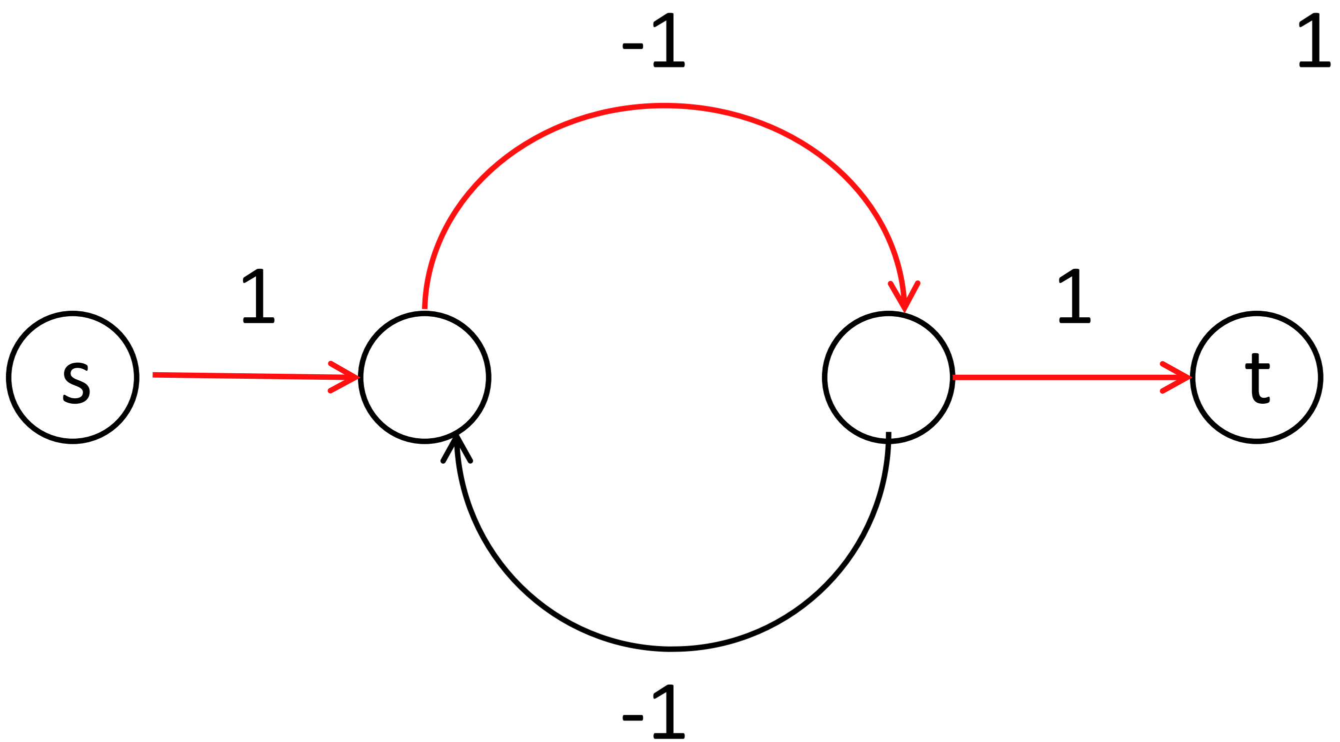

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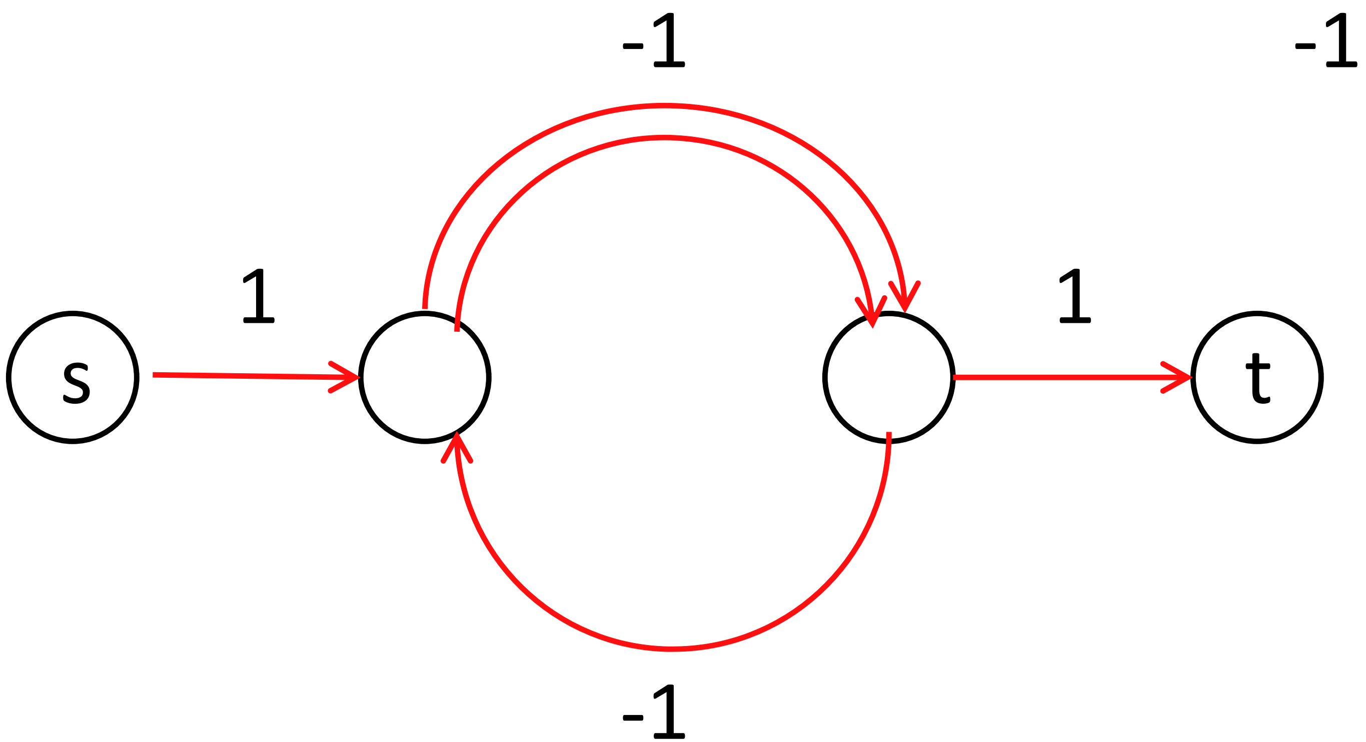

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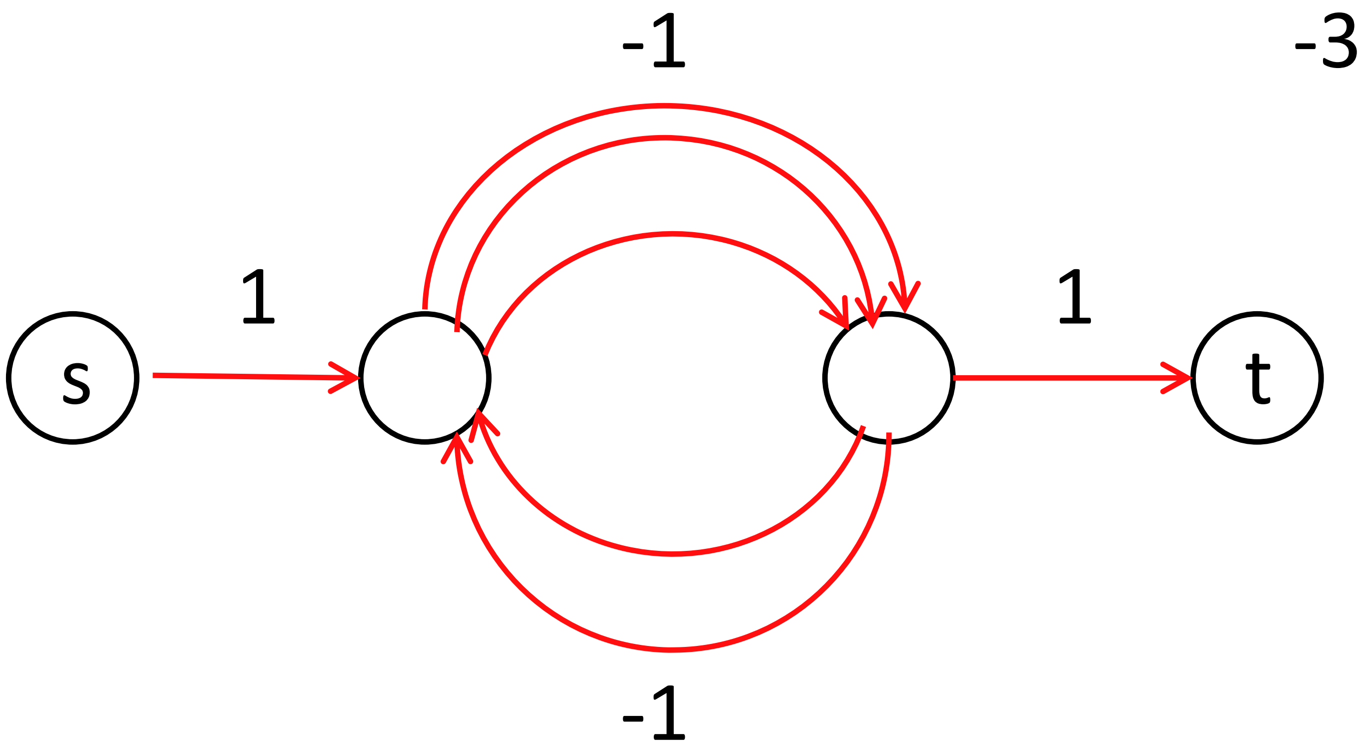
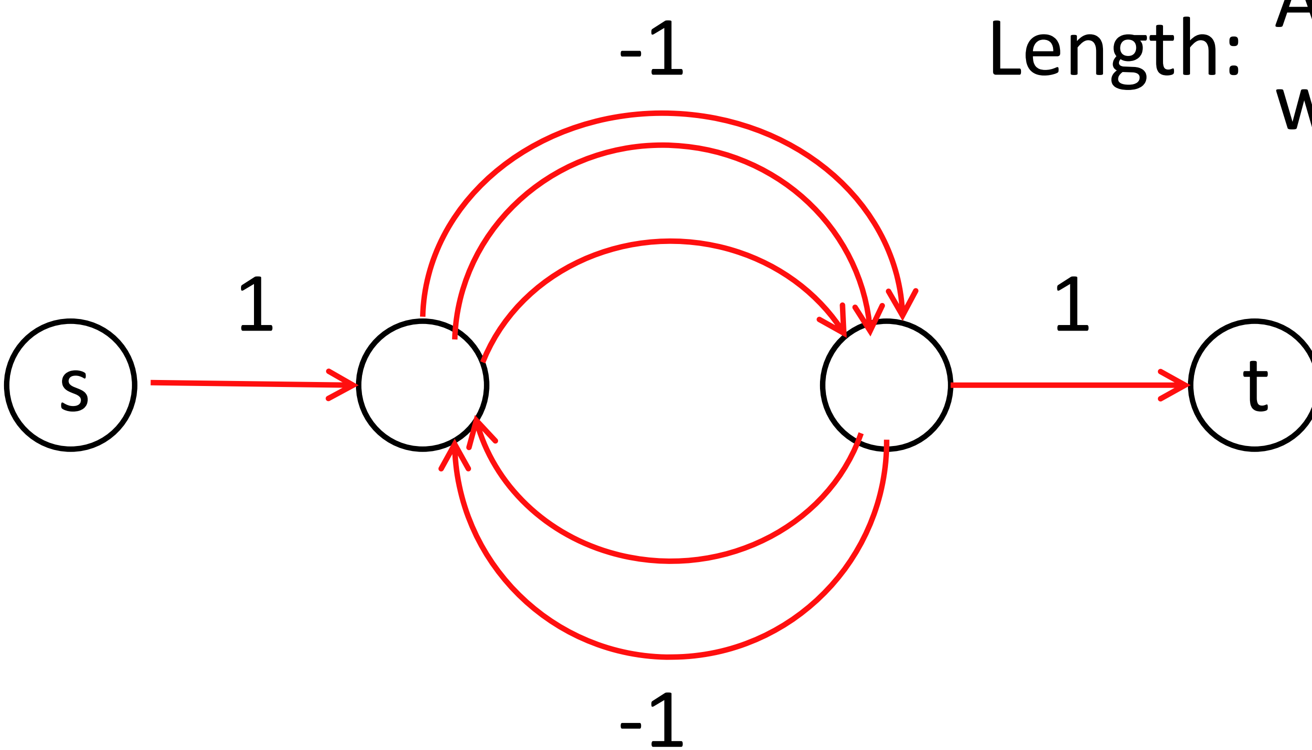


Problem

What is the shortest path length from s to t?

Length: As small as we like



Negative Weight Cycles

Definition: A negative weight cycle is a cycle where the total weight of edges is negative.

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- If G has a negative weight cycle, then there are probably no shortest paths.
  - Go around the cycle over and over.

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- If G has a negative weight cycle, then there are probably no shortest paths.
  - Go around the cycle over and over.
- **Note:** For undirected G, a single negative weight edge gives a negative weight cycle by going back and forth on it.

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For $w \neq s$,

$$\mathrm{dist}(w) = \min_{(v,w) \in E} \mathrm{dist}(v) + \ell(v,w).$$

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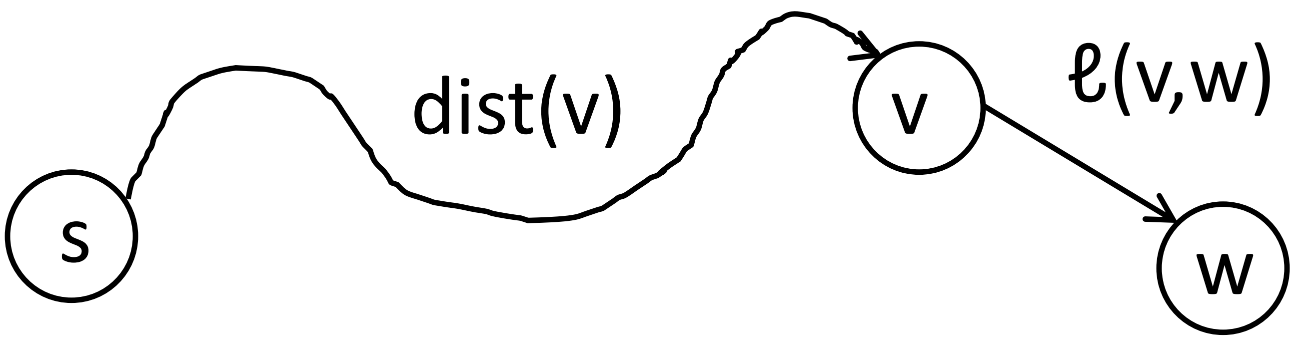

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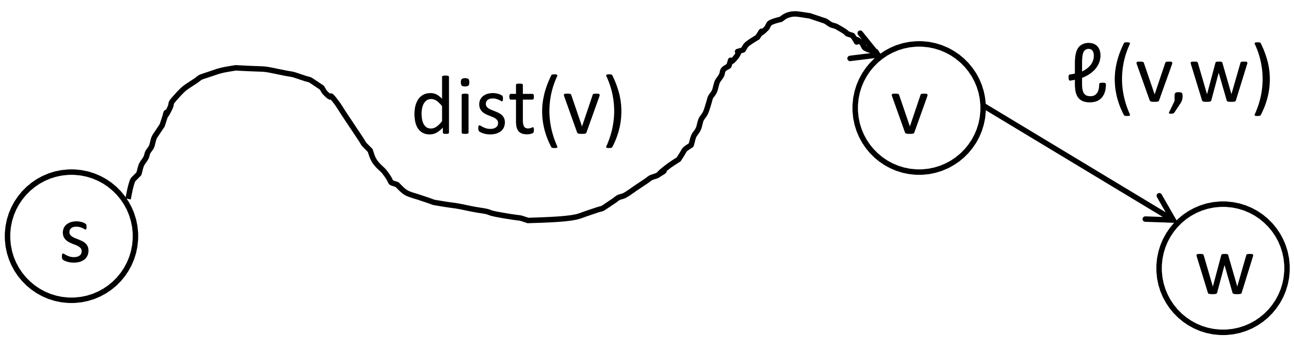

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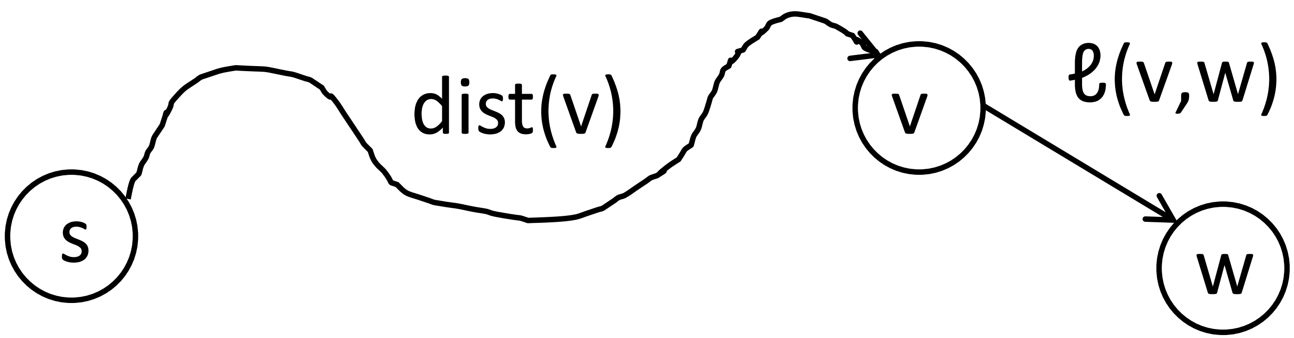

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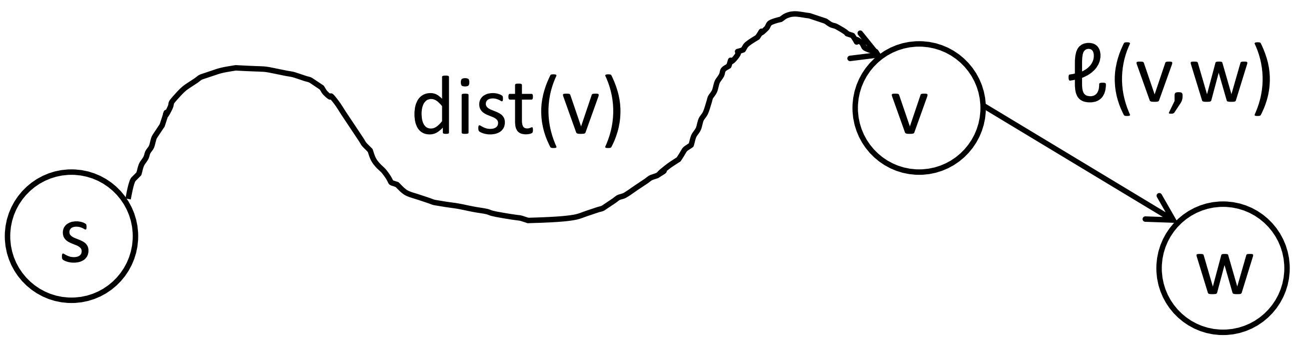

- System of equations to solve for distances.
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- With $\ell < 0$, might be no solution.

Algorithm Idea

Instead of finding shortest paths (which may not exist), find shortest paths of length at most k.

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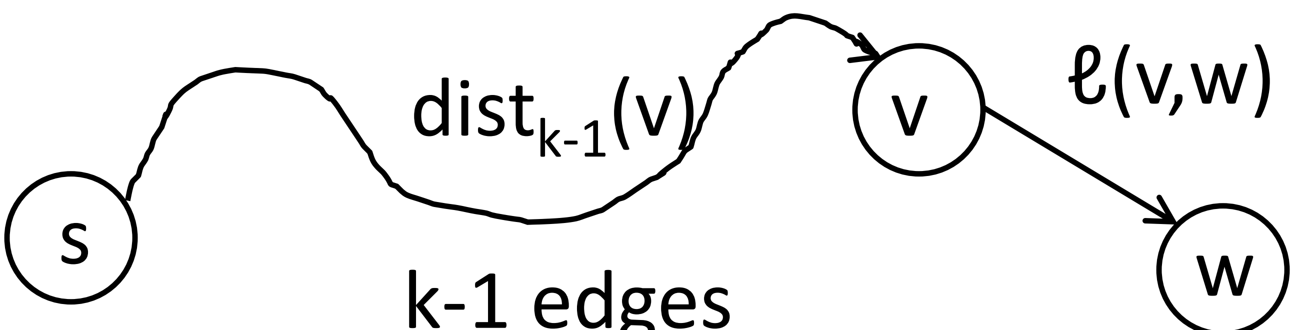
$$\text{dist}_k(w) = \min_{(v,w)\in E} \text{dist}_{k-1}(v) + \ell(v,w).$$



Algorithm

```
Bellman-Ford(G,s,ℓ)
  dist_0(v) ← ∞ for all v
    //cant reach
  dist_0(s) ← 0
  For k = 1 to n
    For w ∈ V
      dist_k(w)←min(dist_{k-1}(v)+ℓ(v,w))
    dist_k(s) ← min(dist_k(s),0)
      // s has the trivial path
```