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INSTRUCTIONS

These are practice problems that can be discussed freely with others.
KEY CONCEPTS Sum Rule, Product Rule, Power Rule, Permutations, Combinations, Quotient Rule, Pigeon-hole principle.
(Note: For this homework, you can leave your answers in terms of exponentials, factorials, binomial coefficients, etc.)

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*(Note: for justifying counting arguments, a good rule of thumb is to explain how you came up with every term and factor of your answer. In most cases, it is more important to have your answer in terms of exponentials, factorials and binomial coefficients rather than explicitly write out the numerical value.)*

1. A password for a particular website is a string over the alphabet that contains uppercase letters (26), lowercase letters (26), digits (10) and special characters $\{!, @, \#, \$, \%, \&, *, ?, =, +\}$ (10) (Explain your answers for each part.)

   (a) (3 points) How many 8-character passwords have at least one digit?

   **Solution:** There are $62^8$ 8-character passwords with no digits. Count with the complement and subtract it off from $72^8$ to get the number of passwords that have at least one digit.

   $$72^8 - 62^8$$

   (b) (3 points) How many 8-character passwords are there such that the first 4 characters are 4 distinct digits in decreasing order and the remaining 4 digits are all special characters
   e.g.: 8320$$!@ or 9643?%#*, or 3210&&&&

   **Solution:** There are $\binom{10}{4}$ ways of choosing the first 4 digits. Then there is only one way to put them in decreasing order. Then there are $10^4$ ways of choosing the last four special characters. All in all there are:

   $$\binom{10}{4}10^4$$

   (c) (3 points) How many 8 character passwords are do not start with a uppercase letter but have at least one uppercase letter
   e.g.: errTT68! or mAthMath or 1234567E

   **Solution:**
   Let's count with the complement but count the universe as passwords that don't start with an uppercase letter. There are: $46 * 72^7$ of these. Then there are $46^8$ passwords with no uppercase letters at all.
   All in all there are:
   $46 * 72^7 - 46^8$ passwords that do not start with an uppercase letter but have at least one uppercase letter.

   $$46 * 72^7 - 46^8$$

   (d) (3 points) Miles is not allowed to use his name as a substring. How many 8-character passwords avoid the word MILES as a substring (in any combination of uppercase and lowercase letters i.e. avoiding MILES, miles, Miles, mIlES,...)?
   (note: *substring* means consecutive subsequence)

**Solution:**
There are $2^5$ ways of spelling MILES with a mixture of uppercase and lowercase letters. There are 4 positions that MILES could occur as a substring (namely, starting at the 1st, 2nd, 3rd or 4th position.) And there are $72^3$ ways to fill in the remaining 3 positions. So there are $2^5 * 4 * 72^3$ ways that MILES could be a substring.
Count with the complement and subtract it off from $72^8$ to get the number of passwords that avoid MILES.

$$72^8 - 2^5 * 4 * 72^3$$

(e) (3 points) How many 8 character passwords have at least one uppercase letter, at least one lowercase letter, and at least one special character?

**Solution:** Let's say that $UC$ is the set of all passwords with at least one any uppercase letter, $LC$ is the set of all passwords with at least one lowercase letter, and $SC$ the set of all passwords with at least one special character.
Using counting with the complement, we are trying to count: $\left|\overline{UC \cap LC \cap SC}\right|$.
By DeMorgan's laws, this is the same as: $\left|\overline{\overline{UC} \cup \overline{LC} \cup \overline{SC}}\right| = \left|\overline{NU \cup NL \cup NS}\right|$ where $NU$ is the set of all passwords without uppercase letters, $NL$ is the set of all passwords without lowercase letters, and $NS$ the set of all passwords without special characters.
Then with $P$ being the universal set of all passwords,

$$\left|\overline{NU \cup NL \cup NS}\right| = |P| - |NU \cup NL \cup NS|$$

$$= |P| - (|NU| + |NL| + |NS| - |NU \cap NL| - |NU \cap NS| - |NL \cap NS| + |NU \cap NL \cap NS|)$$
$$= 72^8 - \left(46^8 + 46^8 + 62^8 - 20^8 - 36^8 - 36^8 + 10^8\right)$$

(f) (3 points)
How many 8 character passwords have 4 copies of two different characters?
e.g. 45454545 or tttTTtTT or $$e$e$ee

**Solution:**
Choose two different characters in $\binom{72}{2}$ ways. Then order the two characters in some order (maybe by ASCII code). Then choose 4 of the positions for the first character in the order. The second character you chose will go in the other 4 positions.

$$\binom{72}{2}\binom{8}{4}$$

(g) (3 points)
How many 8 character passwords three different characters with 3 copies of one 3 copies of another and 2 copies of the other?
e.g. 4ww54545 or tt0TtT0T or e$e$???e

**Solution:** There are 72 ways to choose the first character with 3 copies, 71 ways to choose the second character with 3 copies, and 70 ways to choose the character with 2 copies. Then there are $\binom{8}{3}$ ways to choose the positions of the first character, $\binom{5}{3}$ ways to choose the positions of the second character, and the last character goes in the last two positions. Then we must divide by two since we have counted each such password twice because each character can be the first or the second.

$$72 * 71 * 70 * \binom{8}{3}\binom{5}{3}/2$$

2. A *disjoint subset pair* of the set $[n] = \{1, 2, 3, \ldots, n\}$ is an *ordered* pair of subsets $(A, B)$ such that $A \subseteq [n], B \subseteq [n]$ and $A \cap B = \emptyset$.

For the set [8], here are a few examples of disjoint subset pairs:

$(\{1,3,5,7\},\{2,4,6,8\})$,
$(\{4,5,6\},\{8\})$
$(\emptyset,\emptyset)$
$(\emptyset,\{1,6\})$

(a) (3 points) List out all disjoint subset pairs of the set $\{1,2\}$.

**Solution:**
$\{\emptyset,\emptyset\}$
$\{\emptyset,\{1\}\}$
$\{\emptyset,\{2\}\}$
$\{\emptyset,\{1,2\}\}$
$\{\{1\},\emptyset\}$
$\{\{1\},\{2\}\}$
$\{\{2\},\emptyset\}$
$\{\{2\},\{1\}\}$
$\{\{1,2\},\emptyset\}$

(b) (3 points) For an arbitrary $n$, how many disjoint subset pairs are there of the set $[n]$?

**Solution:**
There are $3^n$ disjoint subset pair because for each element, it could go in the set $A$ or set $B$ or not in either set.

(c) (3 points) For an arbitrary $n$, how many disjoint subset pairs are there of the set $[n]$ such that each element of $[n]$ is either in $A$ or in $B$.

**Solution:** There are $2^n$ disjoint subset pairs such that each element is either in $A$ or in $B$ because for each element, it could go in either $A$ or in $B$.

(d) (3 points) For an arbitrary $n$, how many disjoint subset pairs are there of the set $[n]$ such that neither of the sets is empty?

**Solution** There are $2^n$ ways that the first set could be empty and there are $2^n$ ways that the second set could be empty. then subtract off 1 for the case that both sets are empty. Then subtract this from all disjoint subset pairs:

$$3^n - 2 * 2^n + 1$$

(e) (3 points) For an arbitrary $n$, how many disjoint subset pairs are there of the set $[n]$ such that neither of the sets is empty and the minimum of the first set is smaller than the minimum of the second set?

**Solution:**
It's just the part (d) divided by 2 since you can count all the sets from part (d) then arrange them so that the minimum of the first set is less than the minimum of the second set. Then you will have counted each one twice; once for when it was in the correct order and once for when it was out of order:

$$(3^n - 2 * 2^n + 1)/2$$

3. How many ways are there for $n$ children to sit on a bench with $n$ seats? How many ways are there for $n$ children to sit on a merry-go-round with $n$ identical horses?

Solution: The children could line up in any order on the bench, so there are $n!$ ways they could sit on the bench. If the merry-go-round is locked into place, where you identify which horse is where, there are also $n!$ ways the children could sit on the merry-go-round. However, each of the orders of the children on the merry-go-round can rotate in $n$ possible ways, meaning by the quotient rule, there are only $n!/n = (n-1)!$ distinct ways for the children to sit. Another way to see this, is that it doesn't matter where the first child sits, because all the horses are identical. But after that, the other $n-1$ children can be arbitrarily ordered with respect to the first, say clockwise.

4. (4 points) A *three-of-a-kind* poker hand is a *subset* of 5 cards from a standard deck of 52 cards such that exactly three of the cards have the same rank and the other 2 cards have a different rank than the three of the same rank and different ranks than each other.

   Examples: $\{2\heartsuit, 2\clubsuit, 2\spadesuit, Q\clubsuit, 5\diamondsuit\}$, $\{A\diamondsuit, A\spadesuit, A\clubsuit, 2\clubsuit, 10\clubsuit\}$

   Find the error in the following incorrect argument that claims to count the number of *three-of-a-kind* poker hands:

   > Chose any rank in 13 ways; then choose three suits of that rank in $\binom{4}{3}$ ways; then choose two more cards of different values from the remaining 48 cards: (48 ways for the first card and 44 ways for the second card.)
   >
   > Therefore there are:
   > $$(13)\binom{4}{3}(48)(44) = 109824$$
   > different ways to get a *three-of-a-kind* poker hand.

   **Solution:**

   The error is that when you choose the last two cards, you are counting each pair twice so you need to divide the answer by 2 to get the correct result.

5. (4 points)

   Determine the correct number of *three-of-a-kind* poker hands. (justify your answer.)

   **Solution:**

   $$(13)\binom{4}{3}(48)(44)/2 = 54912$$

   There are 13 ways to choose the rank of the triple. Then there are $\binom{4}{3}$ ways to choose the suits of the triple. Then choose the next two cards in (48)(44) for the two other cards of different ranks and divide by two because you have counted each pair twice.

6. (4 points) A *full-house* poker hand is a *subset* of 5 cards from a standard deck of 52 cards that has 2 different ranks. One of the ranks occurs three times and the other occurs twice.

   Examples: $\{5\heartsuit, 5\spadesuit, 5\diamondsuit, K\clubsuit, K\spadesuit\}$, $\{3\spadesuit, 3\clubsuit, 3\heartsuit, Q\clubsuit, Q\heartsuit\}$

   Find the error in the following incorrect argument that claims to count the number of *full-house* poker hands:

   > Choose the ranks of the full-house in $\binom{13}{2}$ ways. Then choose the suits of the rank that occurs three times in $\binom{4}{3}$ ways. Then choose the suits of the rank that occurs twice in $\binom{4}{2}$ ways.
   >
   > Therefore there are:
   > $$\binom{13}{2}\binom{4}{3}\binom{4}{2} = 1872$$
   > different ways to get a *full-house* poker hand.

   **Solution:**

   The problem here is that we do not know which rank is designated as the triple and which rank is the pair because we chose them as a set rather than an ordered pair. The correct result is this multiplied by two because for each pair of ranks, you can have them be the pair and the triple either way.

7. (4 points)

   Determine the correct number of *full-house* poker hands. (justify your answer.)

**Solution:**
There are 13 ways to pick the triple. Then $\binom{4}{3}$ ways to pick the suits. Then there are 12 ways to pick the pair. Then there are $\binom{4}{2}$ ways to pick the suits.

$$(13)(12)\binom{4}{3}\binom{4}{2} = 3744$$

8. Suppose that you are on the bottom left corner (green) of a $8 \times 12$ grid of city blocks.

9. (5 points) How many different ways can you get from the bottom left corner (green) to the top right corner (orange) by using only North and East moves?

   **Solution:**
   $\binom{20}{8}$
   because there are 20 moves and you must "choose" 8 of them to be $N$ moves.

10. (6 points) Your house is located at the blue square (4 blocks to the East and 4 blocks North from the bottom left corner.) How many ways can you get from the bottom left corner (green) to the top right corner (orange) if your path must pass through the blue square? (using only North and East movements)

    **Solution:** There are $\binom{8}{4}$ ways to get from the green to the blue and $\binom{12}{4}$ ways to get from the blue to the orange.

    $$\binom{8}{4}\binom{12}{4}$$

11. (7 points) There are two intersections that are under construction and you cannot pass through them (red X's).

    How many ways can you get from the bottom left corner (green) to the top right corner (orange) if your path must *avoid* the red X's. (using only North and East movements.)

    **Solution:**
    The complement of this set is the set of all paths that go through either the first X or the second X or both. There are $\binom{6}{3}\binom{14}{5}$ ways to pass through the first X, $\binom{13}{5}\binom{7}{3}$ ways to pass through the second X and $\binom{6}{3}\binom{7}{2}\binom{7}{3}$ ways of passing through both X's.
    Therefore, using counting with the complement and inclusion-exclusion there are:

    $$\binom{20}{8} - \binom{6}{3}\binom{14}{5} - \binom{13}{5}\binom{7}{3} + \binom{6}{3}\binom{7}{2}\binom{7}{3}$$

1. How many different ways can you get from the bottom left corner (green) to the top right corner (orange) by using only North and East moves?

2. Your house is located 4 blocks to the East and 4 blocks North from the bottom left corner.) How many ways can you get from the bottom left corner to the top right corner if your path must pass through your house? (using only North and East movements)

3. (4 points) An anagram of a word is a rearrangement of the letters. For example, an anagram of *miles* is *slime* and an anagram of *dormitory* is *dirtyroom*

   How many anagrams can you make out of the word CALIFORNIA?

   **Solution:** There are two A's and two I's and one of every other letter. So there are 10! ways of arranging the letters if they were all different. But then you have to divide by (2!2!) because of the different ways you can arrange the I's and the A's.

   $$\frac{10!}{2!2!}$$

4. Let $p$ be a prime and $a$ be an integer $1 \leq a \leq p-1$. Consider the map $M(x) = (a * x) \mod p$ from $\{1, ...p-1\}$ to $\{1, ..p-1\}$. Show that $M$ is $1-1$, then use this to conclude that $a$ has a multiplicative inverse $\mod p$, i.e., there is a $b \in \{1...p-1\}$ so that $a * b \mod p = 1$.

   Solution: If $M(x)$ were not $1-1$, there would be $x \neq y \in \{1...p-1\}$, with $M(x) = M(y)$, so $ax \mod p = aymodp$. Without loss of generality $x > y$., or $a(x-y) \mod p = 0$. Then $p$ would divide $a(x-y)$. If a prime divides a product , it must divide one term or the other, but $0 < a < p$ and $0 < x - y < p - 1$, so $p$ cannot divide either term. From this contradiction, $M$ is $1-1$. Since the range and domain of $M$ are both size $p-1$, this means $M$ must be onto, and every element of the range is mapped to. In particular, 1 is mapped to, so there is a $b$ so that $M(b) = a * b \mod p = 1$.

5. How many pairs of sorted arrays $A[1..k]$ and $B[1..n]$ are there so that together, every number between 1 and $n+k$ appears in one of the two sorted arrays?

   Solution: Once we choose which of the numbers in the range $1..n+k$ are in $A$, the order they appear in is fixed (because $A$ is sorted) and the order the others appear in in $B$ is fixed. So there is one pair of sorted arrays for each way of picking $k$ of the $n+k$ numbers, or $\binom{n+k}{k}$ such arrays.

6. Use the above to show a lower bound on the number of comparisons that any algorithm that uses only swaps and comparison operations needs to merge two sorted arrays $A[1..k]$ and $B[1..n]$.

   Solution: Consider any sorting algorithm that uses $T$ comparisons to sort the arrays. If two pairs of arrays containing exactly the numbers $1...n+k$ get the same answers to all $T$ comparisons, the elements will all be swapped in the same way, and in particular, an starting position where the two arrays differ cannot be in the correct sorted position in both. Since there are $2^T$ possible sequences of answers to comparisons, and $\binom{n+k}{k}$ such pairs, we must have $2^T \geq \binom{n+k}{k}$ or $T \geq \log_2 \binom{n+k}{k}$. (If we want a more concrete bound, we can use the fact that $\binom{m}{k} \geq (m/k)^k$ to conclude $T \geq klog_2(n/k+1)$