CSE 20 Spring 2022
Homework 4 solutions
Due date: Thursday, Apr 28, 2022 at 11:59pm

In this assignment,

You will evaluate predicates at different values and analyze the meaning of quantified statements. You will practice some basic proof techniques.

In this class, unless the instructions explicitly say otherwise, you are required to justify all your answers.

1. (**20 points**) For each statement, translate it to the form:

$$\forall x\ (\_\_\_\_\_\ )$$

or

$$\exists x\ (\_\_\_\_\_\ )$$

using the predicates given and state the domain. (note, you are expected to write your answer without a $\neg$ before the quantifier.)

For example: "All dinosaurs are extinct"

- $E(x)$ is "x is extinct"

Then my answer would be:

$$\forall x(E(x))$$

with the domain defined to be the set of all dinosaurs.

If you are using more than one quantifier (and more than one variable,) then define a single domain for both variables.

(a) All positive real numbers have a multiplicative inverse.

- $M(x)$ is "x has a multiplicative inverse."

Solution:
$\forall x M(x)$
with the domain defined to be the set of all positive real numbers.

(b) All positive real numbers have a multiplicative inverse.

- $T(x, y)$ is "$xy = 1$."

Solution:
$\forall x \exists y T(x, y)$
with the domain defined to be the set of all positive real numbers.

(c) Some cars are electric.

- $E(x)$ is "x is electric."

Solution:
$\exists x E(x)$
with the domain defined to be the set of all positive real numbers.

(d) The number 33 is not even.

- $A(x)$ is "x is even."

Solution:
$\neg A(33)$
OR
$\forall x \neg A(x)$
with the domain defined to be the set $\{33\}$
OR
$\exists x \neg A(x)$
with the domain defined to be the set $\{33\}$

(e) The number 0 is less than or equal to all natural numbers.

- $M(x, y)$ is "$x \leq y$."

  Solution:
  $\forall x M(0, x)$
  with the domain defined to be the set of all natural numbers.

(f) Every month I pay the rent or my roommate pays the rent (we never both pay rent during a month.)

- $IP(x)$ is "I pay rent on month x."
- $RP(x)$ is "my roommate pays rent on month x."

  Solution:
  $\forall x(IP(x) \oplus RP(x))$
  with the domain defined to be the set of all months.

(g) There not a rational number that is the square root of 2.

- $R(x)$ is "$x^2 = 2$."

  Solution:
  $\forall x(\neg R(x))$
  with the domain defined to be the set of all rational numbers.

(h) The square of any rational number is a rational number.

- $Q(x)$ is "x is rational."

  Solution:
  $\forall x Q(x^2)$
  with the domain defined to be the set of all rational numbers.

(i) Not all integers are prime.

- $P(x)$ is "x is prime"

  Solution:
  $\exists x \neg P(x)$
  with the domain defined to be the set of all integers.

(j) For any natural number x there exists a natural number y such that $x < y$.

- $LT(x, y)$ is "$x < y$"

  Solution:
  $\forall x \exists y(LT(x, y))$
  with the domain defined to be the set of all natural numbers.

2. (**16 points**) For each quantified statement and each domain, determine if it is true or false. If it is true then just say true. If it is false, give a reason why it is false.

(a) $\forall x \exists y(x > y)$

i. Domain: $\mathbb{Z}$ (the set of all integers.)

   Solution: True, for each integer x, the integer $x-1$ is an integer such that $x > x-1$.

ii. Domain: $\mathbb{Z}^+$ (the set of all positive integers.)

   Solution: False, as a counterexample, consider 1, then there is no positive integer y such that $1 > y$.

iii. Domain: $\mathbb{Z}^-$ (the set of all negative integers.)

   Solution: True, for each negative integer x, the negative integer $x - 1$ is a negative integer such that $x > x - 1$.

iv. Domain: $\mathbb{Q}^+$ (the set of all positive rational numbers.)

Solution: True, for each positive rational number x, the the positive rational number $x/2$ is a positive rational number such that $x > x/2$.

(b) $\forall x \forall y(\exists z(x < z < y) \to (x < y - 1))$

i. Domain: $\mathbb{Z}$ (the set of all integers.)

Solution: True, for each integer x and y, if there exists an integer z such that $x < z < y$ then $x \leq z - 1$ and $z \leq y - 1$ so $x \leq y - 2$ and it follows that $x < y - 1$.

ii. Domain: $\mathbb{Z}^+$ (the set of all positive integers.)

Solution: True, for each positive integer x and y, if there exists a positive integer z such that $x < z < y$ then $x \leq z - 1$ and $z \leq y - 1$ so $x \leq y - 2$ and it follows that $x < y - 1$.

iii. Domain: $\mathbb{Q}$ (the set of all rational numbers.)

Solution: False, as a counterexample, consider $x = 0$ and $y = 1$, then it is true that there is a rational number z between x and y, namely, $z = 1/2$. But $x \not< y - 1$. So, the implication is False.

iv. Domain: $\mathbb{R}^*$ (the set of all non-zero real numbers.)

Solution: False, as a counterexample, consider $x = 0$ and $y = 1$, then it is true that there is a real number z between x and y, namely, $z = 1/2$. But $x \not< y - 1$. So, the implication is False.

(c) $\forall x \exists y(x * y = 1)$

i. Domain: $\mathbb{Z}^*$ (the set of all non-zero integers.)

Solution: False, as a counterexample, consider $x = 2$ then there is no integer y such that $2y = 1$. The only number that satisfies this is $y = 1/2$ which is not an integer.

ii. Domain: $\mathbb{Q}^*$ (the set of all non-zero rational numbers.)

Solution: True, for any non-zero rational number x, consider as a witness the non-zero rational number $1/x$. Then $x(1/x) = 1$.

iii. Domain: $\mathbb{R}^*$ (the set of all non-zero real numbers.)

Solution: True, for any non-zero real number x, consider as a witness the non-zero real number $1/x$. Then $x(1/x) = 1$.

iv. Domain: $\mathbb{R} - \mathbb{Q}$ (the set of all irrational numbers.)

Solution: True, for any irrational number x, consider as a witness the irrational number $1/x$. Then $x(1/x) = 1$.

(d) $\forall x \exists y(x = y^2)$

i. Domain: $\mathbb{Z}^+$ (the set of all positive integers.)

Solution: False, consider $x = 2$, then there is no integer y such that $2 = y^2$. (the only two solutions to this equation are $y = \pm\sqrt{2}$ each of which are not integers.)

ii. Domain: $\mathbb{R} - \mathbb{Q}$ (the set of all irrational numbers.)

Solution: False, consider $x = -\pi$, then there is no irrational number y such that $2 = y^2$. (the only two solutions to this equation are $y = \pm\sqrt{2}$ each of which are not integers.)

iii. Domain: $\{1, 4, 9, 16, \ldots\}$ (the set of all positive integer squares.)

Solution: False, consider $x = 4$, then there is no positive integer square y such that $4 = y^2$. (the only two solutions to this equation are $y = \pm 2$ each of which are not positive integer squares.)

iv. Domain: $\mathbb{R}^+$ (the set of all positive real numbers.)

Solution: True, for any positive real number x, $\sqrt{x}$ is a positive real number and $(\sqrt{x})^2 = x$.

3. Consider the statements where the domain is the set $\{1, 2, 3, 4, 5\}$:

$$\forall x \exists y (P(x,y) \to Q(x))$$

and

$$\forall x (\exists y P(x,y) \to Q(x))$$

The predicates are defined to be: $P(x,y) =$ "$x > y$" and $Q(x) =$ "x is odd."

(a) Translate each statement into plain English.

Solution:

- For all x in the set $\{1,2,3,4,5\}$, there exists a y in the set $\{1,2,3,4,5\}$ such that if $x > y$ then x is odd.
- For all x in the set $\{1,2,3,4,5\}$, if there exists a y in the set $\{1,2,3,4,5\}$ such that $x > y$ then x is odd.

(b) Evaluate the truth value of each quantified statement using these predicates. Justify your answer.

Solution:
The first statement is true because $\exists y(P(x,y) \to Q(x))$ is true for all $x \in \{1,2,3,4,5\}$:

- $x = 1$.
  Let $y = 5$ be a witness. Then $(1 > 5) \to$ (1 is odd) is true because the hypothesis is false.
- $x = 2$.
  Let $y = 5$ be a witness. Then $(2 > 5) \to$ (2 is odd) is true because the hypothesis is false.
- $x = 3$.
  Let $y = 5$ be a witness. Then $(3 > 5) \to$ (3 is odd) is true because the hypothesis is false.
- $x = 4$.
  Let $y = 4$ be a witness. Then $(4 > 5) \to$ (4 is odd) is true because the hypothesis is false.
- $x = 5$.
  Let $y = 4$ be a witness. Then $(5 > 4) \to$ (5 is odd) is true because the hypothesis is True and the conclusion is true.

The second statement is False.
Consider the counterexample $x = 2$. Then $(\exists y(2 > y) \to$ (2 is odd) Is False since the hypothesis is true (based on the witness $y = 1$) but the conclusion is false since 2 is not odd.

(c) Does this prove that in general, these quantified statements are logically equivalent or not? Justify your answer.

Solution:
Yes, this is enough to show these are not logically equivalent. In order for them to be logically equivalent, they need to share the same truth value no matter what the predicates and domain are.

4. Let $A(x)$ and $B(x)$ be predicates with domain $\{0,1,2,3\}$ and let $C(x,y)$ and $D(x,y)$ be predicates with domain $\{0,1,2,3\} \times \{0,1,2,3\}$. They are defined according to the tables:

x	$A(x)$	$B(x)$
0	F	T
1	T	F
2	F	T
3	T	T

x	$C(x,y)$	$D(x,y)$
(0,0)	T	T
(0,1)	T	F
(0,2)	F	T
(0,3)	F	T
(1,0)	T	T
(1,1)	F	T
(1,2)	T	T
(1,3)	F	F
(2,0)	F	T
(2,1)	F	F
(2,2)	F	T
(2,3)	F	F
(3,0)	T	F
(3,1)	F	T
(3,2)	F	T
(3,3)	F	T

Evaluate each quantified statement and give an explanation for each one.

- To explain why a $\forall$ statement is true, show that the predicate is true for all elements in the domain.
- To explain why a $\forall$ statement is false, show that the predicate is false for one counterexample.
- To explain why a $\exists$ statement is true, show that the predicate is true for one witness in the domain.
- To explain why a $\exists$ statement is false, show that the predicate is false for all elements in the domain.

(a) $\forall x(A(x) \vee B(x))$

Solution:
True because:

- $A(0) \vee B(0) = F \vee T = T$
- $A(1) \vee B(1) = T \vee F = T$
- $A(2) \vee B(2) = F \vee T = T$
- $A(3) \vee B(3) = F \vee T = T$

(b) $\exists x(A(x) \leftrightarrow B(x))$

Solution:
True because as a witness $x = 3$:
$A(3) \leftrightarrow B(3) = T \leftrightarrow T = T$

(c) $\forall x \exists y(A(x) \vee C(x, y))$

Solution:
False. Consider $x = 2$ as a counterexample. Then

- When $y = 0$, $A(2) \vee C(2, 0) = F \vee F = F$
- When $y = 1$, $A(2) \vee C(2, 1) = F \vee F = F$
- When $y = 2$, $A(2) \vee C(2, 2) = F \vee F = F$
- When $y = 3$, $A(2) \vee C(2, 3) = F \vee F = F$

(d) $\exists x \forall y(B(x) \vee D(x, y))$

Solution:
True.
Consider $x = 0$ as a witness.

- When $y = 0$, $B(0) \vee D(0,0) = T \vee T = T$
- When $y = 1$, $B(0) \vee D(0,1) = T \vee F = T$
- When $y = 2$, $B(0) \vee D(0,2) = T \vee T = T$
- When $y = 3$, $B(0) \vee D(0,3) = T \vee T = T$

(e) $\forall x(A(x) \to D(x,x))$

Solution:
True:

- When $x = 0$, $A(0) \to D(0,0) = F \to T = T$
- When $y = 1$, $A(1) \to D(1,1) = T \to T = T$
- When $y = 2$, $A(2) \to D(2,2) = F \to T = T$
- When $y = 3$, $A(3) \to D(3,3) = T \to T = T$

(f) $\exists x \exists y \exists z(C(x,y) \wedge D(y,z))$

Solution:
True. Consider as witnesses: $x = 0, y = 0, z = 0$
Then $C(0,0) \wedge D(0,0) = T \wedge T = T$.

5. Consider the function $Divisors : \mathbb{Z}^+ \to \mathcal{P}(\mathbb{Z}^+)$ which is defined to be:
For any positive integer n,

$$Divisors(n) = \{k \in \mathbb{Z}^+ \mid k \text{ is a divisor of } n\}$$

(Assume that $\mathbb{Z}^+$ is the domain for the following statements.)

(a) Fill in the blanks of the proof of this statement:
$\exists x \forall y(x = y \vee Divisors(x) \subseteq Divisors(y))$

Consider $x = 1$ as a witness.
Let y be an arbitrary positive integer.
Case 1: y is equal to 1. In this case $x = y$ which satisfies $(x = y \vee Divisors(x) \subseteq Divisors(y))$
Case 2 y is not equal to 1. Then

[*explain why* $Divisors(1) \subseteq Divisors(y)$]

Solution:
$Divisors(1) = \{1\}$ since 1 is the only divisor of 1. Furthermore, for any positive integer y, 1 is a divisor of y because $y = y * 1$. Therefore $1 \in Divisors(y)$.

(b) Fill in the blanks of the proof of this statement:
$\forall x \forall y \exists z(Divisors(x) \subseteq Divisors(z) \wedge Divisors(y) \subseteq Divisors(z))$

Let x and y be arbitrary positive integers.
Consider $z = xy$ as a witness.

[*explain why* $Divisors(x) \subseteq Divisors(z)$]
(Hint: The argument should start by saying:
"Let a be an arbitrary element and assume that $a \in Divisors(x)$..."
Then show that a must also be an element of $Divisors(z)$.)

(Note: you do not have to show $Divisors(y) \subseteq Divisors(z)$ since the argument is essentially the same. Often when this happens, the writer of the proof will just say "$Divisors(y) \subseteq Divisors(z)$ is true by a similar argument.")

Solution:
Let a be an arbitrary element of $Divisors(x)$. Then $x = ac$ for some integer c. Multiply both sides of the equation by y and you get that $xy = acy$ and $xy = z$ so $z = acy$. Since cy is an integer, it follows that a is a divisor of z and so $a \in Divisors(z)$.