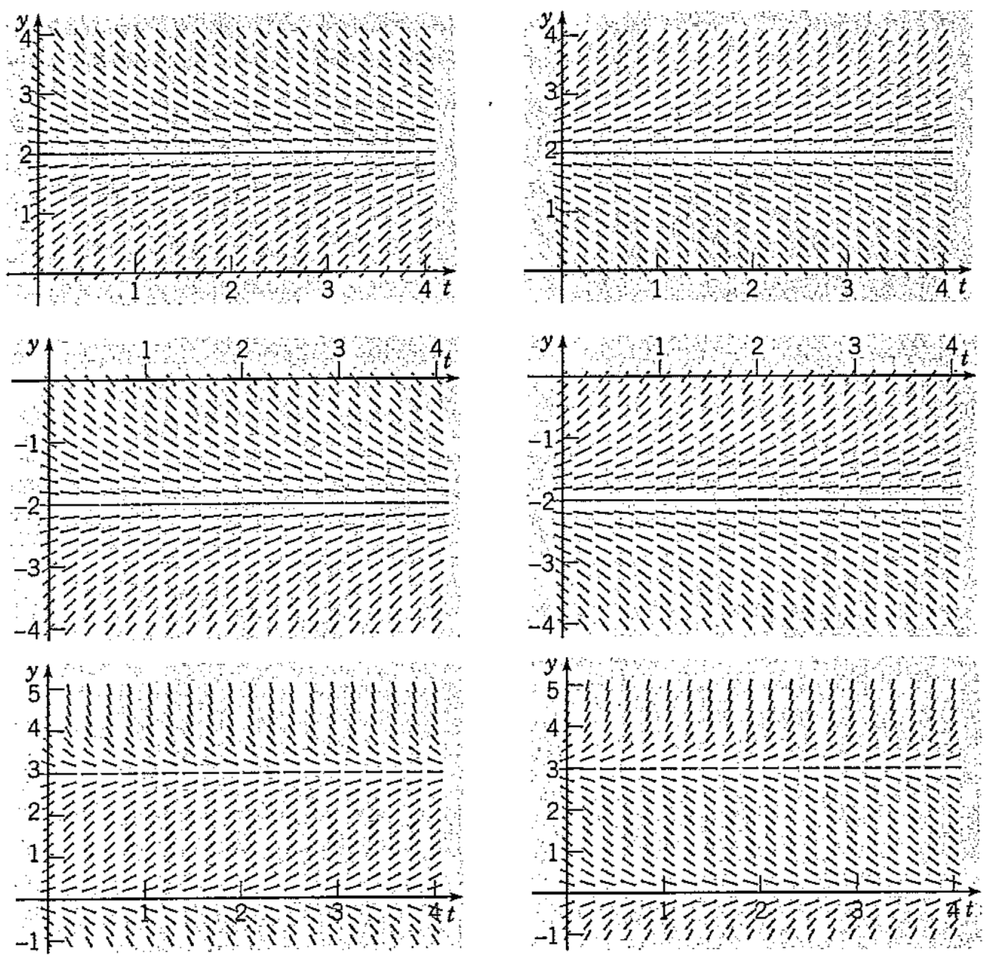
MATH 20D: INTRO TO DIFFERENTIAL EQUATIONS
MIDTERM 1 PRACTICE EXERCISES – SOLUTIONS

Problem 1. Consider the following list of autonomous differential equations:

$$
\begin{array}{llll}
(a)\ y' = 2y - 1 & (b)\ y' = y + 2 & (c)\ y' = y - 2 & (d)\ y' = y(y+3) \\
(e)\ y' = y(y-3) & (f)\ y' = 2y + 1 & (g)\ y' = -y - 2 & \\
(h)\ y' = y(3-y) & (i)\ y' = -2y + 1 & (j)\ y' = 2 - y &
\end{array}
$$

Match each of the following direction fields to its correct equation



Solution. In order of appearance as above:

$$
\begin{array}{ll}
y' = 2 - y & \qquad y' = y - 2 \\
y' = -2 - y & \qquad y' = y + 2 \\
y' = y(3-y) & \qquad y' = y(y-3)
\end{array}
$$

Problem 2.

(a) Solve the following IVP, sketch a few solutions and discuss their "long-term" behavior under different initial values of y_0

$$\frac{dy}{dt} = 5 - y; \quad y(0) = y_0$$

(b) Sketch the direction field and discuss the stability of each equilibrium in the following autonomous equation:

$$\frac{dy}{dt} = y(1-y^2), \text{ for } -\infty < y(0) < \infty$$

Solution.

(a) Consider the initial value problem (IVP) $\dfrac{dy}{dt} = -y+5,\ y(0) = y_0$. First observe that $y(t) \equiv 5$ is a solution of the IVP if and only if $y_0 = 5$. For $y \neq 5$,

$$\begin{aligned}
\frac{dy}{dt} = -y+5 \ &\Rightarrow \frac{dy/dt}{y-5} = -1 \\
&\Rightarrow \frac{d}{dt} \ln|y-5| = -1 \\
&\Rightarrow \ln|y-5| = -1 + c \\
&\Rightarrow |y-5| = e^{-t+c} = e^{-t}e^{c} \\
&\Rightarrow y = 5 \pm e^{c}e^{-t} = 5 + Ce^{-t}.
\end{aligned}$$

Thus, $y(t) = 5 + Ce^{-t}$. Using the initial condition $y(0) = y_0$, we obtain $C = 5 + (y_0 - 5)$. Hence, the solution to the IVP is

$$y(t) = 5 + (y_0 - 5)e^{-t}.$$

Figure 1 below displays the solution $y(t)$ under different initial values y_0.

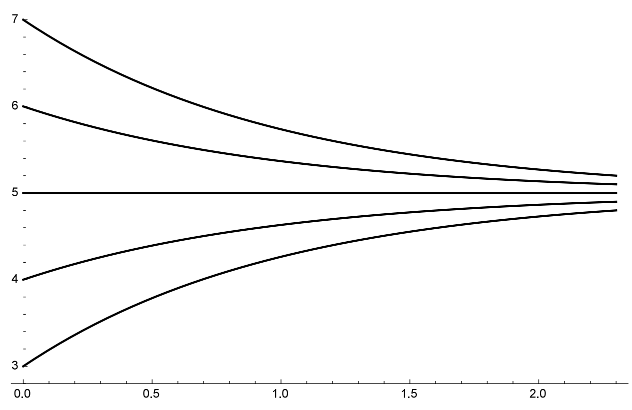


Figure 1: Solutions of the IVP for $y_0 = 3, 4, 5, 6, 7$

(b) The differential equation

$$\frac{dy}{dt} = y(1-y^2), -\infty < y_0 < \infty$$

has three equilibrium solutions: $y = 0$ (unstable), $y = -1$ and $y = 1$ (stable). The direction field is given in Figure 2 below.

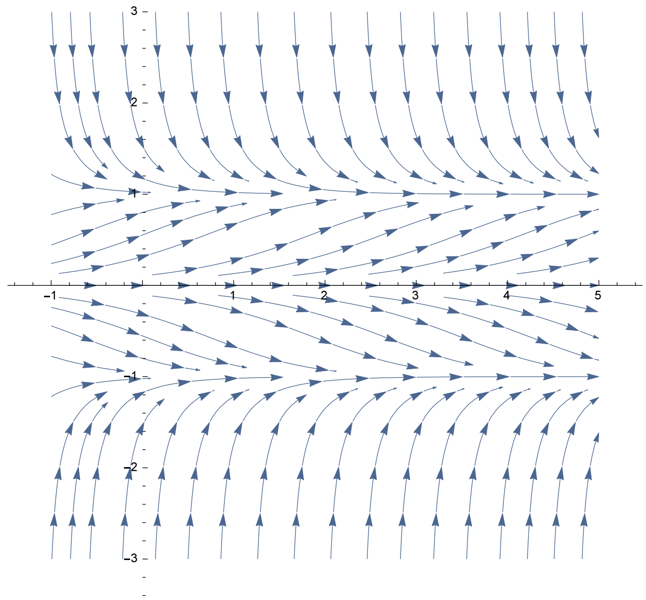


Figure 2: The direction field of $\frac{dy}{dt} = y(1 - y^2)$.

Problem 3. There are several owl (predators) and field mice (preys) occupy the same habitat. Suppose that the owls kill 450 mice per month (this is called the monthly predation rate). Thus, the population of filed mice satisfies the equation $\dfrac{dp}{dt} = 0.5p - 450$. Here t is measured in months.

(a) Find the time at which the population becomes extinct if $p(0) = 850$

(b) Find the time of extinction if $p(0) = p_0$ where $0 < p_0 < 900$

(c) Find the initial population p_0 if the population is to become extinct in 1 year

Solution. First observe that $y(t) \equiv 900$ an equilibrium. For $y \neq 900$,

$$\begin{aligned}
\frac{dy}{dt} = 0.5y - 450 \ &\Rightarrow \frac{dy/dt}{y - 900} = \frac{1}{2} \\
&\Rightarrow \frac{d}{dt} \ln|y - 900| = \frac{1}{2} \\
&\Rightarrow \ln|y - 900| = \frac{1}{2} + c \\
&\Rightarrow |y - 900| = e^{t/2+c} = e^{t/2}e^{c} \\
&\Rightarrow y = 900 \pm e^{c}e^{t/2} = 900 + Ce^{t/2}.
\end{aligned}$$

Thus, the solution is given by

$$p(t) = 900 + Ce^{t/2}.$$

(a) If $p(0) = 850$ then $C = -50$. Thus, the solution to the IVP is $p(t) = 900 - 50e^{t/2}$. Setting $p(t) = 0$ and solve for t, we obtain $t = 2\ln(18) \approx 5.78$. Thus, the mice population becomes extinct in approximately 5.78 months.

(b) In general, if $p(0) = p_0$, for $0 < p_0 < 900$, then $C = p_0 - 900$. Thus, the solution to the IVP is $p(t) = 900 + (p_0 - 900)e^{t/2}$. The time of extinction then becomes

$$t = 2\ln\left(\frac{900}{900 - p_0}\right) \text{ months.}$$

(c) Suppose the population extincts in 1 year then $p(12) = 0$. So

$$900 + (p_0 - 900)e^6 = 0 \Rightarrow p_0 = 900(1 - e^{-6}).$$

Problem 4. Thorium-234 is a radioactive material which disintegrates at a rate proportional to the amount currently present. Let $Q(t)$ be the amount present at time t, then $\dfrac{dQ}{dt} = -rQ$ where $r > 0$ is the decay rate. Here, t is measured in days.

(a) If 100mg of Th-234 decays down to 82.04 mg in 1 week, then what is the decay rate r?

(b) Find an expression for the amount of Th-234 present at any time t

(c) Fund the time required for the Th-234 to decay to one-half of its original amount. This is called the half-life of a radioactive material.

Solution. Following the same technique as the previous two exercises, we can see that the solution to $dQ/dt = -rQ$ is given by

$$Q(t) = Q(0)\ e^{-rt}$$

a. If 100 mg of Th-234 decays to 82.04 mg in 1 week then

$$82.04 = 100e^{-7r} \Rightarrow r \approx 0.02828.$$

Hence, the decay rate is $r \approx 0.02828$ mg per day.

b. The amount of Th-234 present at any time t is given by $Q(t) = q_0 e^{-rt}$ where q_0 is the initial amount of Th-234.

c. Using $r \approx 0.02828$ from part (a), we obtain the half-life of Th-234 as $T \approx 24.5$ days.

Problem 5. Suppose a certain fish population obeys the logistic equation

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right),\ y(0) = y_0.$$

Here $r > 0$ is a constant that measures the annual growth rate of the population.

(a) Suppose $y_0 = K/3$. Find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ (per year).

(b) Suppose $\dfrac{y_0}{K} = \alpha$. Find the time T at which $\dfrac{y(T)}{K} = \beta$ where $0 < \alpha, \beta < 1$. Also find the value of T for $r = 0.025$ (per year), $\alpha = 0.1$, and $\beta = 0.9$

Solution. We first solve the given IVP.

$$\begin{aligned}
\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) &\Rightarrow \frac{K dy}{y(K-y)} = r dt \Rightarrow \frac{dy}{y} + \frac{dy}{(K-y)} = r dt \\
&\Rightarrow \int \frac{dy}{y} + \int \frac{dy}{(K-y)} = r \int dt \\
&\Rightarrow \ln|y| - \ln|K-y| = rt + c \Rightarrow \ln\left|\frac{y}{K-y}\right| = rt + c \\
&\Rightarrow \frac{y}{K-y} = Ce^{rt}
\end{aligned}$$

At $t = 0$, we have $\dfrac{y_0}{K - y_0} = C$ so the above identity becomes $\dfrac{y}{K-y} = \dfrac{y_0}{K-y_0}e^{rt}$, which can then be further simplified into $y(t) = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$.

(a) If $y_0 = K/3$, let τ be the time at which the initial population double, we then have

$$\frac{2K}{3} = \frac{K^2/3}{K/3 + (2K/3)e^{-r\tau}} \Rightarrow \tau = \frac{1}{r}\ln(4) = \frac{\ln(4)}{0.025} \approx 55.4518 \text{ (years)}.$$

(b) Let $y_0/K = \alpha$ and suppose $y(T)/K = \beta$ for $0 < \alpha, \beta < 1$. Then,

$$\beta K = y(T) = \frac{\alpha K^2}{\alpha K + (K - \alpha K)e^{-rT}} \Rightarrow T = \frac{1}{r}\ln\left[\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right].$$

When $\alpha = 0.1, \beta = 0.9$, and $r = 0.025$,

$$T = \frac{1}{r}\ln\left[\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right] \approx 175.778 \text{ (years)}.$$

Problem 6. Follow up with the previous exercise, we now assume that the rate at which fish are caught depends on the population y. That is, the more fish there are, the easier to catch them. Therefore, we now assume the rate at which fish are caught is given by Ey where $E > 0$ is a constant (with unit of 1/time) which measures the total effort made to harvest the given species of fish. To include this harvesting component, we now modify the equation from the previous exercise into

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) - Ey$$

This equation is known as the **Schaefer model**.

(a) Show that if $E < r$ then there are two equilibrium $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$. Here, $y = y_1$ is unstable while $y = y_2$ is asymptotically stable.

(b) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Determine E so as to maximize Y. This maximum value, Y_{max}, is called the **maximum sustainable yield**.

Solution. Consider the Schaefer model given by

$$\frac{dy}{dt} = f(y) = ry\left(1 - \frac{y}{K}\right) - Ey, \quad \text{for } r, E > 0.$$

.

(a) The equilibrium solution of this equation is given by

$$ry\left(1 - \frac{y}{K}\right) - Ey = 0 \Rightarrow y_1 = 0 \text{ or } y_2 = K\left(1 - \frac{E}{r}\right).$$

If $E < r$ then it is easy to see that $y_2 > 0$.

For the equilibrium $y_1 = 0$, we have $f(y) < 0$ on the left of 0 and $f(y) > 0$ on the right of 0 so $y_1 = 0$ is an unstable equilibrium.

For the equilibrium $y_2 = K\left(1 - \frac{E}{r}\right) = y_s$, we have $f(y) > 0$ on the left of y_s and $f(y) < 0$ on the right of y_s so $y_2 = K\left(1 - \frac{E}{r}\right)$ is an asymptotically stable equilibrium.

(b) $Y = Ey_2 = EK\left(1 - \dfrac{E}{r}\right) = -\dfrac{KE^2}{r} + EK.$

Through simple manipulation of the terms in Y,

$$\begin{aligned} Y = -\frac{KE^2}{r} + EK &= -\frac{K}{r}\left(E^2 - rE\right) \\ &= -\frac{K}{r}\left(E^2 - 2\,\frac{r}{2}\,E + \frac{r^2}{4}\right) + \frac{Kr}{4} \\ &= -\frac{K}{r}\left(E - \frac{r}{2}\right)^2 + \frac{Kr}{4} \leq \frac{Kr}{4}. \end{aligned}$$

So the maximum sustainable yield $Y_m = \dfrac{Kr}{4}$ when $E = \dfrac{r}{2}$.

Problem 7. For each item, verify that both $y_1(t)$ and $y_2(t)$ are solutions to the given equation:

(a) $y'' - y = 0$ with $y_1(t) = e^t,\ y_2(t) = \cosh(t)$

(b) $t^2y'' + 5ty' + 4y = 0$ with $y_1(t) = t^{-2},\ y_2(t) = t^{-2}\ln(t)$

Solution.

(a) Consider the differential equation $y'' - y = 0$.

- Let $y_1(t) = e^t$ then $y_1'(t) = y_1''(t) = e^t$. Hence, $y_1(t) = e^t$ satisfies the given differential equation and thus is a solution.

– Let $y_2(t) = \cosh t$ then $y_2'(t) = \sinh t$ and $y_2''(t) = \cosh t$. Hence, $y_2(t) = \cosh t$ satisfies the given differential equation and thus is also a solution.

(b) Consider the differential equation $t^2y'' + 5ty' + 4y = 0$ for $t > 0$.

– Let $y_1(t) = t^{-2}$ then $y_1'(t) = -2t^{-3}$ and $y_1''(t) = 6t^{-4}$. Hence,

$$t^2y_1''(t) + 5ty_1'(t) + 4y_1(t) = 6t^{-2} - 10t^{-2} + 4t^{-2} = 0.$$

Thus, $y_1(t) = t^{-2}$ is a solution.

– Let $y_2(t) = t^{-2}\ln t$ then $y_2'(t) = t^{-3}(1 - 2\ln t)$ and $y_2''(t) = t^{-4}(-5 + 6\ln t)$. Hence,

$$t^2y_1''(t) + 5ty_2'(t) + 4y_2(t) = (-5 + 6\ln t)t^{-2} + (5 - 10\ln t)t^{-2} + (4\ln t)t^{-2} = 0.$$

Thus, $y_2(t) = t^{-2}\ln t$ is also a solution.

Problem 8.

(a) Determine the value of r for which the equation $y''' - 3y'' + 2y' = 0$ has solutions of the form $y = e^{rt}$.

(b) Determine the value of r for which the equation $t^2y'' - 4ty' + 4y = 0$ has solutions of the form $y = t^r$.

Solution.

(a) Consider the differential equation $y''' - 3y'' + 2y' = 0$. Let $y = e^{rt}$ then $y' = re^{rt}$, $y'' = r^2e^{rt}$, and $y''' = r^3e^{rt}$. Thus,

$$\begin{aligned} y''' - 3y'' + 2y' = 0 &\Rightarrow (r^3 - 3r^2 + 2r)e^{rt} = 0 \\ &\Rightarrow r^3 - 3r^2 + 2r = 0 \\ &\Rightarrow r = 0, r = 1, \text{ or } r = 2. \end{aligned}$$

Hence, when $r \in \{0, 1, 2\}$, the given differential equation has solutions of the form $y = e^{rt}$.

(b) Consider the differential equation $t^2y'' - 4ty' + 4y = 0$. Let $y = t^r$ then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Thus,

$$\begin{aligned} t^2y'' - 4ty' + 4y = 0 &\Rightarrow (r(r-1) - 4r + 4)t^r = 0 \\ &\Rightarrow r^2 - 5r + 4 = 0 \\ &\Rightarrow r = 1, \text{ or } r = 4. \end{aligned}$$

Hence, when $r \in \{1, 4\}$, the given differential equation has solutions of the form $y = t^r$.

Problem 9. Consider the equation $y' + 3y = t + e^{-2t}$

(a) Draw the direction field for the given equation

(b) Describe how the solutions behave for large t

(c) Find the general solution to the gen differential equation, and use it to determine how solutions behave as $t \to \infty$

Solution.

a. The direction field of this differential equation is displayed in Figure 3 below.

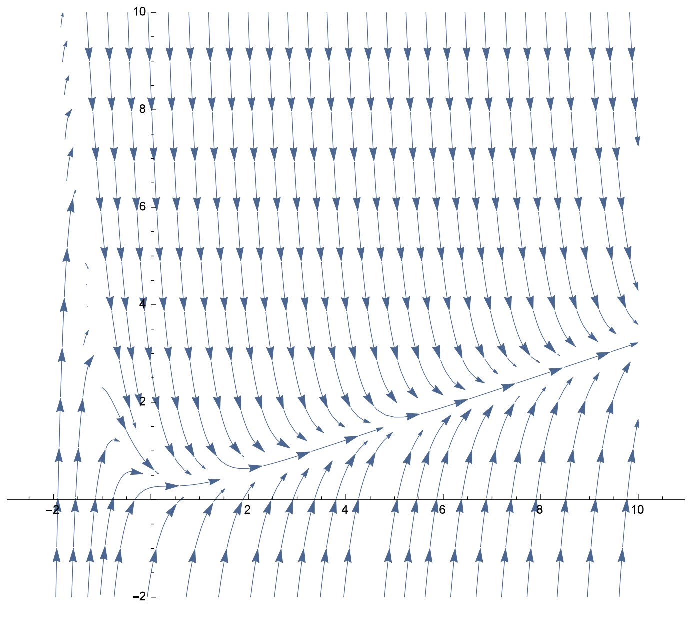


Figure 3: The direction field for the equation $y' + 3y = t + e^{-2t}$.

b. According to Figure 3, as t grows, the solution converges to some straight lines.

c. In this problem, $p(t) = 3$ and $g(t) = t + e^{-2t}$. Then

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int 3dt\right) = ce^{3t}$$

So the solution to the differential equation is given by

$$\begin{aligned} y(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s)g(s)ds + c\right] &= \frac{1}{e^{3t}}\left[\int_0^t e^{3s}(s + e^{-2s})ds\right] + \frac{c}{e^{3t}} \\ &= \frac{1}{e^{3t}}\left[\int_0^t se^{3s}ds + \int_0^t e^{s}ds\right] + \frac{c}{e^{3t}} \\ &= \frac{1 - e^{3t} + 3te^{3t}}{9e^{3t}} + \frac{e^t - 1}{e^{3t}} + \frac{c}{e^{3t}} \\ &= \frac{c}{e^{3t}} + \frac{1}{e^{2t}} + \frac{t}{3} - \frac{1}{9}. \end{aligned}$$

It is easy to see that as $t \to \infty$, $y(t)$ asymptotically approaches $y(t) = \dfrac{t}{3} - \dfrac{1}{9}$. This is the straight line we observed from part (b).

Problem 10. Solve the following first-order linear IVP:

(a) $y' + 2y = te^{-2t}$ with $y(1) = 0$.

(b) $ty' + 2y = t^2 - t + 1$ with $y(1) = \frac{1}{2}$ and $t > 0$.

(c) $y' + \dfrac{2y}{t} = \dfrac{\cos(t)}{t}$ with $y(\pi) = 0$ and $t > 0$.

(d) $t^3y' + 4t^2y = e^{-t}$ with $y(-1) = 0$ and $t < 0$.

Solution.

(a) Consider the IVP: $y' + 2y = te^{-2t}$ with $y(1) = 0$. Here, $p(t) = 2$ so the integrating factor is

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int 2dt\right) = e^{2t}$$

Therefore,

$$\begin{aligned} y' + 2y = te^{-2t} &\Rightarrow e^{2t}\,\frac{dy}{dt} + 2e^{2t}y = t \\ &\Rightarrow (e^{2t})\left(\frac{dy}{dt}\right) + (y)\left(\frac{d}{dt}e^{2t}\right) = t \\ &\Rightarrow \frac{d}{dt}(ye^{2t}) = t \\ &\Rightarrow ye^{2t} = \int tdt = \frac{t^2}{2} + c \\ &\Rightarrow y(t) = \frac{1}{2}t^2e^{-2t} + ce^{-2t}. \end{aligned}$$

Under the initial condition,

$$0 = y(1) = \frac{1}{2}e^{-2} + ce^{-2t} \Rightarrow c = -\frac{1}{2}.$$

Hence, the solution to the IVP is $y(t) = \frac{1}{2}t^2e^{-2t} - \frac{1}{2}e^{-2t}$

(b) Consider the IVP: $ty' + 2y = t^2 - t + 1$ with $y(1) = \frac{1}{2}$ and $t > 0$.

Observe that for $t > 0$, we can divide both sides by t to obtain the standard form $y' + \dfrac{2}{t}y = \dfrac{t^2 - t + 1}{t}$

Here, $p(t) = 2/y$ so the integrating factor is

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int \frac{2dt}{t}\right) = e^{\ln(2t)} = t^2$$

Therefore,

$$\begin{aligned}
y' + \frac{2}{t}y = \frac{t^2-t+1}{t} &\Rightarrow t^2y' + 2ty = t^3 - t^2 + t \\
&\Rightarrow \frac{d}{dt}(t^2y) = t^3 - t^2 + t \\
&\Rightarrow t^2y = \int (t^3 - t^2 + t)dt = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + c \\
&\Rightarrow y(t) = \frac{3t^4 - 4t^3 + 6t^2 + c}{12t^2}.
\end{aligned}$$

Under the initial condition,

$$\frac{1}{2} = y(1) = \frac{c+5}{12} \Rightarrow c = 1.$$

Hence, the solution to the IVP is

$$y(t) = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}.$$

(c) Consider the IVP: $y' + \frac{2y}{t} = \frac{\cos(t)}{t}$ with $y(\pi) = 0$ and $t > 0$. The integrating factor is the same as part (b). So

$$\begin{aligned}
y' + \frac{2y}{t} = \frac{\cos(t)}{t} &\Rightarrow t^2y' + 2ty = t\cos(t) \\
&\Rightarrow \frac{d}{dt}(t^2y) = t\cos(t) \\
&\Rightarrow t^2y = \int t\cos(t)dt = \cos(t) + t\sin(t) + c \\
&\Rightarrow y(t) = \frac{\cos(t) + t\sin(t) + c}{t^2}.
\end{aligned}$$

Under the initial condition,

$$0 = y(\pi) = \frac{c-1}{12} \Rightarrow c = 1.$$

Hence, the solution to the IVP is

$$y(t) = \frac{\cos(t) + t\sin(t) + 1}{t^2}.$$

(d) Consider the IVP: $t^3y' + 4t^2y = e^{-t}$ with $y(-1) = 0$ and $t < 0$. Observe that for $t < 0$, we can divide both sides by t^3 to obtain the standard form $y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$

Here, $p(t) = 4/y$ so the integrating factor is t^4, similar to part (b). Hence,

$$\begin{aligned}
y' + \frac{4}{t}y = \frac{e^{-t}}{t^3} &\Rightarrow t^4y' + 4t^3y = te^{-t} \\
&\Rightarrow \frac{d}{dt}(t^4y) = te^{-t} \\
&\Rightarrow t^4y = \int te^{-t}dt = -e^{-t}(1+t) + c \\
&\Rightarrow y(t) = \frac{-(1+t) + c}{e^tt^4}.
\end{aligned}$$

Under the initial condition,

$$0 = y(-1) = c \Rightarrow c = 0.$$

Hence, the solution to the IVP is

$$y(t) = -\frac{(1+t)}{e^t t^4}.$$

Problem 11. Consider the IVP $y' - \frac{3}{2}y = 3t + 2e^t,\ y(0) = y_0$.

Find the value of y_0 that separate solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value behave as $t \to \infty$?

Solution. Consider the IVP $y' - \frac{3}{2}y = 3t + 2e^t$ with $y(0) = y_0$. Here, $p(t) = 3/2$ so the integrating factor is given by $\mu(t) = e^{3t/2}$. Therefore,

$$\begin{aligned}
y' - \frac{3}{2}y = 3t + 2e^t &\Rightarrow e^{-3t/2}y' - \frac{3}{2}e^{-3t/2}y = 3te^{-3t/2} + 2e^{-t/2} \\
&\Rightarrow \frac{d}{dt}(ye^{-3t/2}) = 3te^{-3t/2} + 2e^{-t/2} \\
&\Rightarrow ye^{-3t/2} = \int \left(3te^{-3t/2} + 2e^{-t/2}\right) dt = -\frac{2}{3}e^{-3t/2}\left(2 + 6e^t + 3t\right) + c \\
&\Rightarrow y(t) = -\frac{4}{3} - 2t - 4e^t + ce^{3t/2}.
\end{aligned}$$

Under the initial condition, $y_0 = y(0) = -\frac{4}{3} - 4 + c \Rightarrow c = y_0 + \frac{16}{3}$.
Thus, the solution to this IVP is

$$y(t) = -\frac{4}{3} - 2t - 4e^t + \left(y_0 + \frac{16}{3}\right)e^{3t/2}$$

Here, the fastest growing term is $e^{3t/2}$ so the behavior of the solutions as $t \to \infty$ is controlled by the sign of this term. Specifically, when $y_0 + 16/3 > 0$, the solution grows to positive infinity as $t\ to\infty$. On the other hand, if $y_0 + 16/3 < 0$, it goes to negative infinity as $t \to \infty$. Thus, $y_0 = -\frac{16}{3}$ is the critical value.

Under the initial condition $y(0) = y_0 = -\frac{16}{3}, y(t) \to -\infty$ as $t \to \infty$.

Problem 12. Find the implicit solution to the following separable equations:

$$(a)\ y' = \frac{x^2}{y} \qquad\qquad (b)\ y' = \frac{x^2}{y(1+x^3)}$$

Solution.

(a) For $y \neq 0$, we have

$$\begin{aligned}
\frac{dy}{dx} = \frac{x^2}{y} &\Rightarrow ydy = x^2dx \\
&\Rightarrow \int ydy = \int x^2dx \\
&\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + c \\
&\Rightarrow 3y^2 - 2x^3 = c.
\end{aligned}$$

(b) For $y \neq 0$ and $x \neq -1$, we have

$$\begin{aligned}
\frac{dy}{dx} = \frac{x^2}{y(1+x^3)} &\Rightarrow ydy = \frac{x^2}{1+x^3}\, dx \\
&\Rightarrow \int ydy = \int \frac{x^2}{1+x^3}\, dx \\
&\Rightarrow \frac{y^2}{2} = \frac{1}{3} \ln|1+x^3| + c \\
&\Rightarrow y^2 - \frac{2}{3}\ln|1+x^3| = c.
\end{aligned}$$

Problem 13. Find the explicit solution to the following IVP. Also determine the interval(s) in which the solution is defined.

(a) $y' = (1-2x)y^2,\ y(0) = -\dfrac{1}{6}$; (b) $y' = \dfrac{2x}{1+2y},\ y(2) = 0$; (c) $y' = 2y^2+2xy^2 = y^2(2+x),\ y(0) = 1$.

Solution.

(a) Observe that $y \equiv 0$ is *not* a solution of this IVP. When $y \neq 0$, we have

$$\begin{aligned}
\frac{dy}{dx} = (1-2x)y^2 &\Rightarrow y^{-2}dy = (1-2x)dx \\
&\Rightarrow \int y^{-2}dy = \int (1-2x)dx \\
&\Rightarrow -\frac{1}{y} = x - x^2 + c \\
&\Rightarrow y(x) = \frac{1}{x^2 - x - c}.
\end{aligned}$$

Under the initial condition, $-\dfrac{1}{6} = y(0) = -\dfrac{1}{c} \Rightarrow c = 6$.

Thus, the solution to the given IVP is

$$y(x) = \frac{1}{x^2 - x - 6}.$$

The function $y = 1/(x^2 - x - 6)$ is defined on three intervals $(-\infty, -2) \cup (-2, 3) \cup (3, +\infty)$. However, the initial value given at $x = 0$ limits the interval of our solution to $-2 < x < 3$.

(b) When $y \neq -\frac{1}{2}$, we have

$$\begin{aligned}\frac{dy}{dx} = \frac{2x}{1+2y} &\Rightarrow (1+2y)dy = 2xdx \\ &\Rightarrow \int (1+2y)dy = \int 2xdx \\ &\Rightarrow y + y^2 = x^2 + c \\ &\Rightarrow y(x) = \frac{-1 \pm \sqrt{4x^2+1+4c}}{2} = \frac{-1 \pm \sqrt{4x^2+K}}{2}. \qquad (1)\end{aligned}$$

Under the initial condition, we obtain

$$0 = y(2) = \frac{-1 \pm \sqrt{16+K}}{2} \Rightarrow K = -15.$$

This also means that only the function y with positive sign in (7) can be a solution. Thus, the solution to the given IVP is

$$y(x) = \frac{-1 + \sqrt{4x^2 - 15}}{2}.$$

The square root $\sqrt{4x^2-15}$ is defined for either $x > \frac{\sqrt{15}}{2}$ or $x < -\frac{\sqrt{15}}{2}$. However, the initial condition $y(2) = 0$ limits our solution to only the positive side. Hence, the solution is defined on the interval $x > \frac{\sqrt{15}}{2}$.

(c) Observe that $y \equiv 0$ is *not* a solution of this IVP. When $y \neq 0$, we have

$$\begin{aligned}\frac{dy}{dx} = y^2(2+x) &\Rightarrow y^{-2}dy = (2+x)dx \\ &\Rightarrow \int y^{-2}dy = \int (2+x)dx \\ &\Rightarrow -\frac{1}{y} = 2x + \frac{x^2}{2} + c \\ &\Rightarrow y(x) = -\frac{2}{x^2+4x+c}.\end{aligned}$$

Under the initial condition, $1 = y(0) = -\frac{2}{c} \Rightarrow c = -2$.

Thus, the solution to the given IVP is

$$y(x) = -\frac{2}{x^2+4x-2}.$$

Similar to part (a), the solution function is defined on three intervals $(-\infty, -2-\sqrt{6}) \cup (-2-\sqrt{6}, -2+\sqrt{6}) \cup (-2+\sqrt{6}, +\infty)$; however, initial value at $x = 0$ limits our solution to over the interval $(-2-\sqrt{6}, -2+\sqrt{6})$.

Problem 14. Convert the following homogeneous equations into separable equations and solve for their general solution. Be careful not to drop any solution!

$$(a)\ \frac{dy}{dx} = \frac{y-4x}{x-y}; \qquad (b)\ \frac{dy}{dx} = \frac{4y-3x}{2x-y}$$

Solution.

(a) Dividing the top and bottom of the RHS by $x \neq 0$ yields

$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{(y/x)-4}{1-(y/x)}.$$

Let $v = \dfrac{y}{x}$ or $v(x) = xv(x)$ so that $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$.
Therefore,

$$\begin{aligned} v + x\,\frac{dv}{dx} = \frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{v-4}{1-v} &\Rightarrow\ x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v^2-4}{1-v} \\ &\Rightarrow\ \frac{1-v}{v^2-4}\,dv = \frac{1}{x}\,dx \qquad (2) \end{aligned}$$

which is a separable differential equation.

Integrate the two sides of (2) gives

$$\begin{aligned} \int \frac{1-v}{v^2-4}\,dv = \int \frac{1}{x}\,dx \Rightarrow\ & \int \left(\frac{-1/4}{v-2} + \frac{-3/4}{v+2}\right) dv = \int \frac{1}{x}\,dx \\ \Rightarrow\ & -\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| + c \\ \Rightarrow\ & \ln|(v-2)(v+2)^3| = \ln|x^{-4}| + c \\ \Rightarrow\ & (v-2)(v+2)^3 = Cx^{-4}. \qquad (3) \end{aligned}$$

Lastly, by replacing v with y/x in (3) we obtain the solution to the given equation as

$$\left(\frac{y}{x} - 2\right)\left(\frac{y}{x} + 2\right)^3 = \frac{C}{x^4} \Rightarrow (y-2x)(y+2x)^3 = C.$$

Notice that in order to obtain the expression in (3), we implicitly assume that $v \neq \pm 2$. The cases give rise to the functions $y = \pm 2x$ which can be shown to be solutions to the given equation. However, both cases have been included in the general solution in (3) for $C = 0$. Therefore, we can conclude that $(y-2x)(y+2x)^3 = C$ is the general solution to the given equation, in implicit form.

(b) Assuming all the expressions are well-defined, using the same technique as in part (a), we have:

$$\begin{aligned}
\frac{dy}{dx} = \frac{4y-3x}{2x-y} &\Rightarrow \frac{dy}{dx} = \frac{4(y/x)-3}{2-(y/x)} \quad \text{for } x \neq 0 \\
&\Rightarrow v + x\frac{dv}{dx} = \frac{4v-3}{2-v} \quad \text{where } y(x) = xv(x) \\
&\Rightarrow x\frac{dv}{dx} = \frac{4v-3}{2-v} - v = \frac{(v-1)(v+3)}{2-v} \\
&\Rightarrow \frac{2-v}{(v-1)(v+3)}\, dv = \frac{1}{x}dx \\
&\Rightarrow \int \frac{2-v}{(v-1)(v+3)}\, dv = \int \frac{1}{x}dx \\
&\Rightarrow -\frac{5}{4}\ln|v+3| + \frac{1}{4}\ln|v-1| = \ln|x| + c \\
&\Rightarrow \ln\left|\frac{v-1}{(v+3)^5}\right| = \ln|x^4| + c \qquad (4) \\
&\Rightarrow \frac{v-1}{(v+3)^5} = Cx^4 \\
&\Rightarrow \frac{(y/x)-1}{((y/x)+3)^5} = Cx^4 \\
&\Rightarrow \frac{y-x}{(y+3x)^5} = C.
\end{aligned}$$

Therefore, the function y implicitly defined as $C(y+3x)^5 = y - x$, where C is a constant, is a solution to the given differential equation.

Notice that in order to obtain the expression in (4), we implicitly assume that $v \neq 1$ and $v \neq -3$. These cases give rise to the functions $y = x$ and $y = -3x$, respectively. It is then easy to check that they both are solutions to the given differential equation. While the former is included in the general solution when $C = 0$, the latter is not. Thus, in order to obtain the complete set of solutions, we need to include the case $y = -3x$.

Hence, the solutions to the given differential equation are $y = -3x$, and $C(y+3x)^5 = y - x$.

Problem 15. Determine if each the following equations and IVPs is exact or not. If so, also find the solution.

(a) $(2x+3) + (2y-2)y' = 0$

(b) $(e^x \sin y + 3y) - (3x - e^x \sin y)y' = 0$

(c) $\left(\dfrac{y}{x} + 6x\right) - (\ln x - 2)y' = 0$

(d) $(2x-y) + (2y-x)y' = 0$ with $y(1) = 3$

For part (d) also find the explicit form of the solution and determine the interval in which the solution is defined.

Solution.

(a) Consider the equation

$$(2x + 3) + (2y - 2)y' = M(x, y) + N(x, y)y' = 0.$$

We have $M_y(x, y) = 0 = N_x(x, y)$ so this is an exact equation. We want to find a function $\psi(x, y)$ such that $\psi_x(x, y) = M(x, y)$ and $\psi_y(x, y) = N(x, y)$. So,

$$\psi(x, y) = \int M(x, y)dx = \int (2x + 3)dx = x^2 + 3x + h(y).$$

Thus,

$$2y - 2 = N(x, y) = \psi_y(x, y) = h'(y) \Rightarrow h(y) = \int (2y - 2)dy = y^2 - 2y + c.$$

Therefore,

$$\psi(x, y) = x^2 + 3x + y^2 - 2y + c.$$

The given exact equation then has solutions given implicitly by

$$x^2 + 3x + y^2 - 2y = C,$$

for any constant C.

(b) Consider the equation

$$(e^x \sin y + 3y) - (3x - e^x \sin y)y' = M(x, y) + N(x, y)y' = 0.$$

We have $M_y(x, y) = e^x \cos y + 3 \neq -3 + e^x \sin y = N_x(x, y)$ so this is *not* an exact equation.

(c) Let $x > 0$, consider the equation

$$\left(\frac{y}{x} + 6x\right) - (\ln x - 2)y' = M(x, y) + N(x, y)y' = 0.$$

We have $M_y(x, y) = 1/x = N_x(x, y)$ so this is an exact equation. So,

$$\psi(x, y) = \int M(x, y)dx = \int \left(\frac{y}{x} + 6x\right) dx = y \ln x + 3x^2 + h(y).$$

Thus,

$$\ln x - 2 = N(x, y) = \psi_y(x, y) = \ln x + h'(y) \Rightarrow h'(y) = -2 \Rightarrow h(y) = \int -2dy = 2y + c.$$

Therefore, the given exact equation then has solutions given implicitly by

$$y \ln x + 3x^2 - 2y = C.$$

(d) Consider the equation

$$(2x - y) + (2y - x)y' = M(x, y) + N(x, y)y' = 0.$$

We have $M_y(x, y) = -1 = N_x(x, y)$ so this is an exact equation. So,

$$\psi(x,y)=\int M(x,y)dx=\int(2x-y)dx=x^2+xy+h(y).$$

Thus,

$$\begin{aligned}2y-x=N(x,y)=\psi_y(x,y)=x+h'(y)\Rightarrow h'(y)&=2y-2x\\ \Rightarrow h(y)&=\int(2y-2x)dy=y^2-2xy+c.\end{aligned}$$

Therefore, the given exact equation then has solutions given implicitly by $x^2+y^2-xy=C$. Under the initial condition $y(1)=3$, we obtain $C=7$. Thus, the solution to the IVP (implicitly) is $y^2-xy+x^2-7=0$.

Writing y in terms of x yields $y=\dfrac{x\pm\sqrt{28-3x^2}}{2}$. Under the initial conditions $x_0=1$ and $y_0=3$, only the solution with the positive sign remains and this solution is defined for $-\sqrt{28/3}\le x\le\sqrt{28/3}$. However, one can check that when $x=\pm\sqrt{28/3}$, we obtain $y=x/2$ which is *not* a solution to the given equation.

Hence, the solution to the IVP is

$$y(t)=\frac{x+\sqrt{28-3x^2}}{2},$$

defined for $x\in\left(-\sqrt{28/3},\sqrt{28/3}\right)$.

Problem 16. Consider the equation

$$(xy^2+bx^2y)dx+(x+y)x^2dy=0.$$

Find the value of b for which this equation is exact. Then solve the equation using that value of b.

Solution. Here,

$$(xy^2+bx^2y)+(x^3+x^2y)y'=M(x,y)+N(x,y)y'=0.$$

We have $M_y(x,y)=2xy+bx^2$ and $N_x(x,y)=3x^2+2xy$. This is an exact equation if and only if $b=3$. So, when $b=3$,

$$\psi(x,y)=\int M(x,y)dx=\int(xy^2+3x^2y)dx=\frac{1}{2}\,x^2y^2+x^3y+h(y).$$

Thus,

$$x^3+x^2y=\psi_y(x,y)=x^3+x^2y+h'(y)\Rightarrow h'(y)=0\Rightarrow h(y)=c.$$

Therefore, the solution to the exact equation is $x^2y^2+2x^3y=C$.

Problem 17. Find an integrating factor and use it to solve the equation:

$$\left(\frac{4x^3}{y^2}+\frac{3}{y}\right)+\left(\frac{3x}{y^2}+4y\right)y'=0$$

Solution. Here,

$$\left(\frac{4x^3}{y^2}+\frac{3}{y}\right)+\left(\frac{3x}{y^2}+4y\right)y' = M(x,y)+N(x,y)y' = 0.$$

First observe that

$$\frac{N_x(x,y)-M_y(x,y)}{M(x,y)} = \frac{\frac{3}{y^2}-\left(-\frac{8x^3}{y^3}-\frac{3}{y^2}\right)}{\frac{4x^3}{y^2}+\frac{3}{y}} = \frac{2}{y}\ ,$$

so the integrating factor μ depends only on y. This integrating factor is then given by

$$\mu(y) = \exp\int\left(\frac{N_x(x,y)-M_y(x,y)}{M(x,y)}\right)dy = \exp\left(\int\frac{2}{y}dy\right) = e^{\ln(y^2)} = y^2.$$

Under the integrating factor $\mu(y)=y^2$, the given equation becomes

$$(4x^3+3y)+(3x+4y^3)y' = 0. \tag{5}$$

It is easy to check that (7) is an exact equation. So,

$$\psi(x,y) = \int(4x^3+3y)dx = x^4+3xy+h(y).$$

Thus,

$$3x+4y^3 = \psi_y(x,y) = 3x+h'(y) \Rightarrow h'(y) = 4y^3 \Rightarrow h(y) = y^4+c.$$

Hence, the solution is $x^4+3xy+y^4 = C$.

Problem 18. Solve the following second-order linear equation and IVP:

(a) $y''+2y'-3y=0$

(b) $6y''-5y'+y=0,\ y(0)=4,\ y'(0)=0$

(c) $y''-y'-2y=0,\ y(0)=\alpha,\ y'(0)=2$. Solve in terms of α then find the value of α such that the solution approaches zero as $t\to\infty$

(d) $y''-2y'+6y=0$

(e) $y''-2y'+5y=0,\ y(\pi/2)=0,\ y'(\pi/2)=2$

(f) $9y''+6y'+y=0$

(g) $y''-6y'+9y=0,\ y(0)=0,\ y'(0)=2$

(h) $y''-y'+0.25y=0,\ y(0)=0,\ y'(0)=\beta$. Solve in terms of β then find the value of β that separates the solutions that grow positively as $t\to\infty$ from those that grow negatively as $t\to\infty$.

(i) $9y''+12y'+4y=0\quad y(0)=\gamma>0,\ y'(0)=-1$. Solve in terms of γ then find the value of γ that separates the solutions that become negative from those that are always positive.

Solution.

(a) The characteristic equation is given by $r^2 + 2r - 3r = 0$ which gives solutions $r_1 = -1$ and $r_2 = 3$. Hence, the general solution of the given equation is $y(t) = c_1e^{-t} + c_2e^{3t}$.

(b) The characteristic equation is given by $6r^2 - 5r + r = 0$ which gives solutions $r_1 = 1/3$ and $r_2 = 1/2$. Hence, the general solution is $y(t) = c_1e^{t/3} + c_2e^{t/2}$.

Under the initial conditions, we have

$$\begin{aligned} 4 &= y(0) = c_1 + c_2 \\ 0 &= y'(0) = \frac{c_1}{3} + \frac{c_2}{2} \end{aligned}$$

which gives $c_1 = 12$ and $c_2 = -8$. Hence, the solution tp the IVP is

$$y(t) = 12e^{t/3} - 8e^{t/2}.$$

(c) The characteristic equation is given by $r^2 - r - 2r = 0$ which gives solutions $r_1 = 2$ and $r_2 = -1$. Hence, the general solution is $y(t) = c_1e^{2t} + c_2e^{-t}$.

Under the initial conditions, we have

$$\begin{aligned} \alpha &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = 2c_1 - c_2 \end{aligned}$$

which gives $c_1 = \dfrac{2+\alpha}{3}$ and $c_2 = \dfrac{2\alpha - 2}{3}$. Therefore, the solution of the given IVP, in terms of α is

$$y(t) = \frac{2+\alpha}{3}\, e^{2t} + \frac{2\alpha - 2}{3}\, e^{-t}. \tag{6}$$

Observe that if the coefficient of e^{2t} is nonzero then the long term behavior of the solution in (6) will be controlled by this term, which goes to $+\infty$ as $t \to \infty$. Therefore, in order for the solution in (6) to approach zero as $t \to \infty$, we must have $(\alpha + 2)/3 = 0$ which yields $\alpha = -2$.

(d) The characteristic equation is given by $r^2 - 2r + 6 = 0$ which has complex roots $r = 1 \pm i\sqrt{5}$. Hence, the general solution is

$$y(t) = c_1e^t \cos(\sqrt{5}t) + c_2e^t \sin(\sqrt{5}t).$$

(e) The characteristic equation is given by $r^2 - 2r + 5 = 0$ which has complex roots $r = 1 \pm 2i$. Hence, the general solution is

$$y(t) = c_1e^t \cos(2t) + c_2e^t \sin(2t).$$

Under the initial condition $y(\pi/2) = 0$, we have $0 = y(\pi/2) = c_1e^{\pi/2} \Rightarrow c_1 = 0$.

So $y(t) = c_2e^t \sin(2t)$ and thus, $y(t) = c_2e^t \sin(2t) + 2c_2e^t \cos(2t)$. Under the initial condition $y'(\pi/2) = 2$, we have

$$2 = y(\pi/2) = -2c_2e^{\pi/2} \Rightarrow c_2 = -e^{-\pi/2}.$$

Hence, the solution to the IVP is $y(t) = -e^{t-\pi/2}\sin(2t)$.

(f) Consider the differential equation $9y''+6y'+y=0$. The characteristic equation is $9r^2+6r+1=0$ which has repeated root $r=-1/3$. Thus, the general solution is given by

$$y(t)=c_1e^{-\frac{t}{3}}+c_2te^{-\frac{t}{3}}.$$

(g) The characteristic equation is $r^2-6r+9=0$ which has repeated root $r=3$. Thus, the general solution is given by

$$y(t)=c_1e^{3t}+c_2te^{3t}.$$

Under the initial condition $y(0)=0$, we have $c_1=0$.

Thus, $y(t)=c_2te^{3t}$ which gives $y'(t)=c_2e^{3t}+3c_2te^{3t}$. Under the initial condition $y'(0)=2$, we have $c_2=2$.

Hence, the solution to the IVP is $y(t)=2te^{3t}$.

(h) The characteristic equation is $r^2-r+1/4=0$ which has repeated root $r=1/2$. Thus, the general solution is given by

$$y(t)=c_1e^{\frac{t}{2}}+c_2te^{\frac{t}{2}}.$$

Under the initial condition $y(0)=2$, we have $c_1=2$.

Thus, $y(t)=2e^{t/2}+c_2te^{t/2}$ which gives $y'(t)=e^{t/2}+c_2e^{t/2}+\dfrac{c_2}{2}\,te^{t/2}$. Under the initial condition $y'(0)=\beta$, we have $\beta=1+c_2\Rightarrow c_2=\beta-1$.

Hence, the solution to the IVP is

$$y(t)=2e^{t/2}+(\beta-1)te^{t/2}.$$

Observe that when $\beta\neq 1$, the long-term behavior of y is controlled by the term $(\beta-1)te^{t/2}$. So if $\beta<1$ then $y\to-\infty$ as $t\to\infty$; whereas if $\beta\geq 1$ then $y\to\infty$ as $t\to\infty$. This shows that $\beta=1$ is the critical value.

(i) Consider the initial value problem (IVP) $9y''+12y'+4y=0\quad y(0)=a>0,\ y'(0)=-1$. The characteristic equation is given by $9r^2+12r+4=0$ which has repeated root $r=-2/3$. Thus, the general solution is

$$y(t)=c_1e^{-2t/3}+c_2te^{-2t/3}.$$

Under the initial conditions, we obtain

$$\begin{aligned}\gamma=y(0)&=c_1e^{-2(0)/3}+c_2(0)e^{-2(0)/3}=c_1\\ -1=y'(0)&=-\frac{2}{3}c_1e^{-2(0)/3}+c_2e^{-2(0)/3}-\frac{2c_2}{3}(0)e^{-2(0)/3}=c_2-\frac{2}{3}c_1.\end{aligned}$$

So $c_1=\gamma$ and $c_2=\dfrac{2a}{3}-1$ and thus, the solution to the IVP is

$$y(t)=\gamma e^{-2t/3}+\left(\frac{2a}{3}-1\right)te^{-2t/3}. \tag{7}$$

Observe that the long term behavior of the solution in (7) is controlled by $\left(\dfrac{2a}{3}-1\right)te^{-2t/3}$. So if $\gamma\geq\dfrac{3}{2}$, then $y(t)>0$ and $y(t)\to 0^+$ as $t\to\infty$; whereas if $0<\gamma<\dfrac{3}{2}$ then $y(t)\to 0^-$ as

$t \to \infty$. Hence, $\gamma = \frac{3}{2}$ is the critical value of γ that separates solutions that become negative from those that always positive.

Problem 19. Find a differential equation whose general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{-2t}$.

Solution. Let y be a function of t and suppose that the second order linear different equation $y'' + ay' + by = 0$ has general solution given by $y(t) = c_1 e^{-t/2} + c_2 e^{-2t}$.

The characteristic equation for this second order equation is $r^2 + ar + b = 0$. In addition, the powers of the exponentials in the solution tell us that $r_1 = -1/2$ and $r_2 = -2$ must be solutions to this characteristic equation. This gives

$$-a = r_1 + r_2 = -\frac{1}{2} - 2 = -\frac{5}{2} \text{ and } b = r_1 r_2 = \left(-\frac{1}{2}\right)(-2) = 1.$$

Thus, the required differential equation is

$$y'' + \frac{5}{2}y' + y = 0 \Rightarrow 2y'' + 5y' + 2y = 0.$$

Problem 20. Use the Wronskian to show that the following pairs of functions are linearly independent.

$(a)\ y_1(t) = e^{-2t},\ y_2(t) = te^{-2t}$ $\qquad$ $(b)\ y_1(t) = e^t \sin t,\ y_2(t) = e^t \cos t$

Solution.

(a) The Wronskian of $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ is given by

$$\begin{aligned} W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} &= \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} \\ &= (e^{-2t})(e^{-2t} - 2te^{-2t}) - (te^{-2t})(-2e^{-2t}) \\ &= e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}. \end{aligned}$$

Here the Wronskian is not uniformly zero, so the two given functions are linearly independent.

(b) The Wronskian of $y_1(t) = e^t \sin t$ and $y_2(t) = e^t \cos t$ is given by

$$\begin{aligned} W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} &= \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix} \\ &= (e^t \sin t)(e^t \cos t - e^t \sin t) - (e^t \cos t)(e^t \cos t + e^t \sin t) \\ &= e^{2t} \sin t \cos t - e^{2t} \sin^2 t - e^{2t} \cos^2 t - e^{2t} \cos t \sin t \\ &= -e^{2t}(\sin^2 t + \cos^2 t) = -e^{2t}. \end{aligned}$$

Again, $W \not\equiv 0$ so y_1 and y_2 are independent.

Problem 21.

(a) Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions to the differential equation $t^2y''-2y = 0$ for $t > 0$. Then show that $c_1t^2 + c_2t^2$ is also a solution to the same equation, for any constant c_1, c_2.

(b) Verify that $y_1(x) = x$ and $y_2(x) = \sin x$ are two solutions to the differential equation $(1 - x\cot x)y'' - xy' + y = 0$ for $0 < x < \pi$. Then show that $c_1x + c_2 \sin x$ is also a solution to the same equation, for any constant c_1, c_2.

(c) Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are two solutions to the differential equation $yy''+(y')^2 = 0$ for $t > 0$. Then show that $c_1 + c_2t^{1/2}$ in general is **not** a solution to the same equation. Does this contradict the result we saw in class?

Solution.

(a) For $t > 0$, consider the differential equation $t^2y'' - 2y = 0$ and two solutions $y_1(t) = t^2$ and $y_2(t) = t^{-1}$. We have,

- $y_1' = 2t, y_1'' = 2 \Rightarrow t^2y_1'' - 2y_1 = 2t^2 - 2t^2 = 0$. Thus, t^2 is a solution of the given equation.
- $y_2' = -t^{-2}, y_2'' = 2t^{-3} \Rightarrow t^2y_2'' - 2y_2 = 2t^2t^{-3} - 2t^{-1} = 2t^{-1} - 2t^{-1} = 0$. Thus, t^{-1} is also a solution of the given equation.

We can easily check that t^2 and t^{-1} are independent by evaluating their Wronskian. In class, we saw that if y_1 and y_2 are two independent solutions to a second order linear differential equation then any linear combination $c_1y_1 + c_2y_2$ is also a solution to the same equation. Hence, for any constants c_1 and c_2, $y(t) = c_1t^2+c_2t^{-1}$ is also a solution to the given differential equation. In fact, it is also easy to check that $y(t) = c_1t^2 + c_2t^{-1}$ also satisfies the equation above simply by plugging it in the given equation

(b) Consider the differential equation $(1 - x \cot x)y'' - xy' + y = 0$ for $0 < x < \pi$.

For $y_1(x) = x$, we have $y_1'(x) = 1$ and $y''(x) = 0$ so

$$(1 - x\cot x)y_1'' - xy_1' + y_1 = 0 - x(1) + x = 0.$$

Thus, $y_1(t) = x$ is a solution.

For $y_2(x) = \sin x$, we have $y_2'(x) = \cos x$ and $y_2''(x) = -\sin x$ so

$$\begin{aligned}(1 - x\cot x)y_2'' - xy_2' + y_2 &= (1 - x\cot x)(-\sin x) - x\cos x + \sin x \\ &= -\sin x + x\cos x - x\cos x + \sin x = 0.\end{aligned}$$

Thus, $y_2(t) = \sin x$ is also a solution.

The Wronskian is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y'1 & y'2 \end{vmatrix} = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = (x)(\cos x) - (1)(\sin x) \neq 0.$$

$W \neq 0$ shows that y_1 and y_2 form a fundamental set of solutions. Therefore, every function of the form $c_1y_1 + c_2y_2$ is also a solution to the given equation.

(c) For $t > 0$, consider the differential equation $yy'' + (y')^2 = 0$ and two solutions $y_1(t) = 1$ and $y_2(t) = t^{1/2}$. We have,

- $y_1' = 0, y_1'' = 0 \Rightarrow y_1 y_1'' + (y_1')^2 = (0)(1) + (0)^2 = 0$. Thus, y_1 is a solution of the given equation.
- $y_2' = \frac{1}{2}t^{-1/2}, y_2'' = -\frac{1}{4}t^{-3/2} \Rightarrow y_2 y_2'' + (y_2')^2 = t^{1/2}\left(-\frac{1}{4}t^{-3/2}\right) + \left(\frac{t^{-1/2}}{2}\right)^2 = \frac{-t^{-1}}{4} + \frac{t^{-1}}{4} = 0$. Thus, t^{-1} is also a solution of the given equation.

Now if $y_3 = c_1 + c_2 t^{1/2}$ then $y_3' = \frac{c_2}{2}t^{-1/2}$ and $y_3'' = -\frac{c_2}{4}t^{-3/2}$. Then

$$\begin{aligned} y_3 y_3'' + (y_3')^2 &= (c_1 + c_2 t^{1/2})\left(-\frac{c_2}{4}t^{-3/2}\right) + \left(\frac{c_2}{2}t^{-1/2}\right)^2 \\ &= -\frac{c_1 c_2}{4}t^{-3/2} - \frac{c_2^2}{4}t^{-1} + \frac{c_2^2}{4}t^{-1} = -\frac{c_1 c_2}{4}t^{-3/2} \neq 0 \end{aligned}$$

whenever both c_1, c_2 are non-zero.

This shows that $c_1 y_1(t) + c_2 y_2(t)$ in general is not a solution to the given solution, despite the fact that both $y_1(t)$ and $y_2(t)$ are solutions. However, this does not contradict the result we saw in class. Here, $y_1(t)$ and $y_2(t)$ are **not** linearly independent (one can easily check this through their Wronskian), and thus, there is no guarantee that their linear combination is still a solution.

Problem 22. Use the Method of Undetermined Coefficients to solve the following non-homogeneous equations and IVPs:

(a) $y'' + 2y' + 5y = 3\sin(2t)$

(b) $y'' - y' - 2y = 4t^2 - 2t$

(c) $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$

(d) $y'' - 2y' - 3y = -3te^{-t}$

(e) $y'' + 9y = t^2 e^{3t} + 6$

(f) $y'' - 2y' + y = te^t + 4$ and $y(0) = 1, y'(0) = 1$

(g) $y'' + 4y = 3\sin(2t)$ and $y(0) = 2,\ y'(0) = -1$

Solution.

(a) Consider the non-homogeneous differential equation $y'' + 2y' + 5y = 3\sin(2t)$.

The general solution to the homogeneous part is given by

$$y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t).$$

We now use method of undetermined coefficients to find a particular solution to the given differential equation. Observe that the right hand side is a function that involves only the sine function, so we will look for solutions of the form

$$y_P(t) = A\sin(2t) + B\cos(2t)$$

where A and B are the constants to be determined. Thus,

$$\begin{aligned} y_P'' + 2y_P' + 5y_P = 3\sin(2t) &\Rightarrow (-4A\sin(2t) - 4B\cos(2t)) + 2\,(2A\cos(2t) - 2B\sin(2t)) \\ &\qquad + 5\,(A\sin(2t) + B\cos(2t)) = 3\sin(2t) \\ &\Rightarrow (A - 4B)\sin(2t) + (4A + B)\cos(2t) = 3\sin(2t). \end{aligned}$$

So $A - 4B = 3$ and $4A + B = 0$ which mean $A = 3/17$ and $B = -12/17$. Hence, a particular solution to the given equation is

$$y_p(t) = \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t).$$

Observe that the particular solution y_P is independent of y_H. So, the general solution to the non-homogeneous equation is given by

$$y(t) = y_H(t) + y_P(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t) + \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t).$$

(b) Consider the non-homogeneous differential equation $y'' - y' - 2y = 4t^2 - 2t$.

The general solution to the homogeneous part is given by

$$y_H(t) = c_1e^{-t} + c_2e^{2t}.$$

We now use method of undetermined coefficients to find a particular solution to the given differential equation. Observe that the right hand side is a polynomial of degree 2, so we will look for solutions of the form

$$y_P(t) = At^2 + Bt + C$$

where A, B and C are the constants to be determined. Thus,

$$\begin{aligned} y_P'' - y_P' - 2y_P = 4t^2 - 2t &\Rightarrow (2A) - (2At + B) - 2\left(At^2 + Bt + C\right) = 4t^2 - 2t \\ &\Rightarrow -2At^2 - 2(A + B)t + (2A - B - 2C) = 4t^2 - 2t. \end{aligned}$$

So $-2A = 4$, $-2(A + B) = -2$, and $(2A - B - 2C) = 0$ which mean $A = -2, B = 3$, and $C = -7/2$. Hence, a particular solution to the given equation is

$$y_p(t) = -2t^2 + 3t - \frac{7}{2}.$$

Observe that the particular solution y_P is independent of y_H. So, the general solution to the non-homogeneous equation is given by

$$y(t) = y_H(t) + y_P(t) = c_1e^{-t} + c_2e^{2t} - 2t^2 + 3t - \frac{7}{2}.$$

(c) Consider the non-homogeneous differential equation $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$.

The general solution to the homogeneous part is given by

$$y_H(t) = c_1e^{2t} + c_2e^{-3t}.$$

We now use method of undetermined coefficients to find a particular solution to the given differential equation. Observe that the right hand side is a sum of two exponential functions, so we will look for solutions of the form

$$y_P(t) = Ae^{3t} + Be^{-2t}$$

where A and B are the constants to be determined. Thus,

$$\begin{aligned} y_P'' + y_P' - 6y_P = 12e^{3t} + 12e^{-2t} &\Rightarrow \left(9Ae^{3t} + 4Be^{-2t}\right) + \left(3Ae^{3t} - 2Be^{-2t}\right) \\ &\qquad - 6\left(Ae^{3t} + Be^{-2t}\right) = 12e^{3t} + 12e^{-2t} \\ &\Rightarrow 6Ae^{3t} = 12e^{3t} + 12e^{-2t}. \end{aligned}$$

So $6A = 12$, and $-4B = 12$ which mean $A = 2$, and $B = -3$. Hence, a particular solution to the given equation is

$$y_p(t) = 2e^{3t} - 3e^{-2t}.$$

Observe that the particular solution y_P is independent of y_H. So, the general solution to the non-homogeneous equation is given by

$$y(t) = y_H(t) + y_P(t) = c_1e^{2t} + c_2e^{-3t} + 2e^{3t} - 3e^{-2t}.$$

(d) Consider the non-homogeneous differential equation $y'' - 2y' - 3y = -3te^{-t}$.

The general solution to the homogeneous part is given by

$$y_H(t) = c_1e^{3t} + c_2e^{-t}.$$

We now use method of undetermined coefficients to find a particular solution to the given differential equation. Observe that the right hand side is of the form te^{-t}, so we will look for solutions of the form

$$y_P(t) = (At + B)te^{-t} = At^2e^{-t} + Bte^{-t}$$

where A and B are the constants to be determined. Thus,

$$\begin{aligned} y_P'' - 2y_P' - 3y_P = -3te^{-t} &\Rightarrow \left(2Ae^{-t} - 2Be^{-t} - 4Ate^{-t} + Bte^{-t} + At^2e^{-t}\right) \\ &\quad - 2\left(Be^{-t} + 2Ate^{-t} - Bte^{-t} - At^2e^{-t}\right) - 3\left(At^2e^{-t} + Bte^{-t}\right) = -3te^{-t} \\ &\Rightarrow \left((2A - 4B) + -8At\right)e^{-t} = -3te^{-t}. \end{aligned}$$

So $2A - 4B = 0$, and $-8A = -3$ which mean $A = 3/8$, and $B = 3/16$. Hence, a particular solution to the given equation is

$$y_p(t) = \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}.$$

Observe that the particular solution y_P is independent of y_H. So, the general solution to the non-homogeneous equation is given by

$$y(t) = y_H(t) + y_P(t) = c_1e^{3t} + c_2e^{-t} + \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}.$$

(e) Consider the non-homogeneous differential equation $y'' + 9y = t^2 e^{3t} + 6$.

The general solution to the homogeneous part is given by

$$y_H(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

We now use method of undetermined coefficients to find a particular solution to the given differential equation. Due to the expression of the right hand side, we look for particular solutions of the form

$$y_P(t) = \left(At^2 + Bt + C\right) e^{3t} + D$$

where A, B, C and D are the constants to be determined. Thus,

$$\begin{aligned} y_P'' + 9y_P = -3te^{-t} \Rightarrow & \left(2A + 6B + 9C + (12A + 9B)t + 9At^2\right) e^{3t} \\ & + 9\left(\left(At^2 + Bt + C\right) e^{3t} + D\right) = t^2 e^{3t} + 4 \\ \Rightarrow & \left(18At^2 + (12A + 18B)t + 2A + 6B + 18C\right) e^{3t} + 9D = t^2 e^{3t} + 4. \end{aligned}$$

So $A = 1/18, B = -1/27, C = 1/162$, and $D = 2/3$. Hence, a particular solution to the given equation is

$$y_P(t) = \left(\frac{1}{18}t^2 - \frac{1}{27}t + \frac{1}{162}\right) e^{3t} + \frac{2}{3}.$$

Observe that the particular solution y_P is independent of y_H. So, the general solution to the non-homogeneous equation is given by

$$y(t) = y_H(t) + y_P(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac{1}{162}(9t^2 - 6t + 1)e^{3t} + \frac{2}{3}.$$

(f) Consider the initial value problem $y'' - 2y' + y = te^t + 4$ and $y(0) = 1, y'(0) = 1$.

The general solution to the homogeneous part is given by

$$y_H(t) = c_1 e^t + c_2 te^t.$$

We now use method of undetermined coefficients to find a particular solution to the given differential equation. We first break the right hand side into two parts te^t and 4. It is easy to see that the latter gives a particular solution of the form $y_{P_1}(t) = A$. For the former, we would normally look for solution of the form $y_{P_2}(t) = (Bt + C)e^t$. However, since both expressions here have been included in the solution of the homogeneous part, we need to multiply the assumed form of the particular solution y_{P_2} by some power of t, which is t^2 in this case. So, we are looking for a particular solution of the form

$$y_p(t) = A + Bt^3 e^t + Ct^2 e^t$$

where A, B and C are the constants to be determined.

Using the technique that has been discussed so far, we obtain the particular solution

$$y_P(t) = 4 + \frac{1}{6}t^3 e^t.$$

So the general solution is

$$y(t) = c_1 e^t + c_2 t e^t + \frac{1}{6} t^3 e^t + 4.$$

Under the initial conditions, we obtain the solution to the IVP

$$y(t) = -3e^t + 4te^t + \frac{1}{6} t^3 e^t + 4.$$

(g) Consider the initial value problem

$$y'' + 4y = 3\sin(2t) \quad \text{and} \quad y(0) = 2,\ y'(0) = -1.$$

The general solution to the homogeneous part is given by

$$y_H(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

We now use method of undetermined coefficients to find a particular solution to the given differential equation. The right hand side is $3\sin(2t)$ which normally would give a particular solution of the form $y_P(t) = A\cos(2t) + B\sin(2t)$. However, since both expressions here have been included in the solution of the homogeneous part, we need to multiply the assumed form of the particular solution y_{P_2} by some power of t, which is t in this case. So, we are looking for a particular solution of the form

$$y_p(t) = At\cos(2t) + Bt\sin(2t)$$

where A and B are the constants to be determined.

Using the technique that has been discussed so far, we obtain the particular solution

$$y_P(t) = -\frac{3}{4} t\cos(2t).$$

So the general solution is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{3}{4} t\cos(2t).$$

Under the initial conditions, we obtain the solution to the IVP

$$y(t) = 2\cos(2t) - \frac{1}{8}\sin(2t) - \frac{3}{4} t\cos(2t).$$
