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Chapter 7
Sampling and reconstruction

Signals and Systems

Sampling theorem

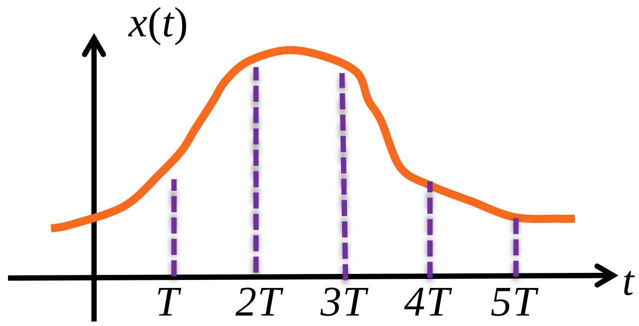
Key points

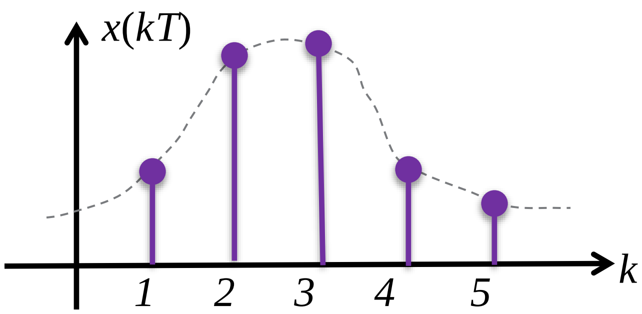
- Establish the fundamental connection between continuous-time bandlimited signals and discrete-time signals
- Illustrate the impact of sampling in the time and frequency domains

What is sampling?

- For a given CT signal $x(t)$

Kronecker delta functions

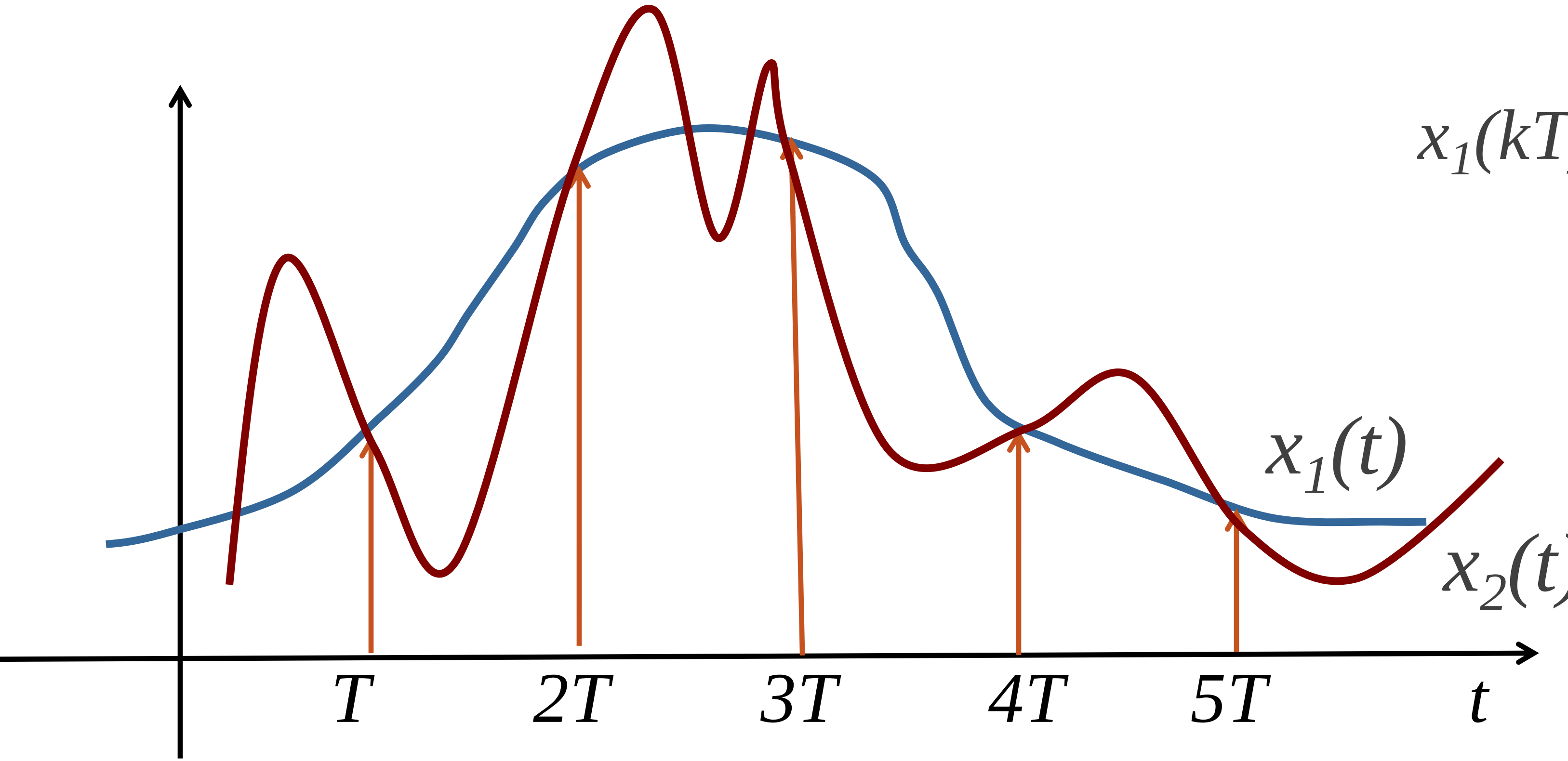




- The signal $x(kT)$ is called a sampled version of $x(t)$
  - The sampled signal is a discrete-time signal, written as *x[k]*

- The critical question related to sampling
  - Is it possible to recover *x(t)* from *x(kT)* ?

Why is there the potential for ambiguity?

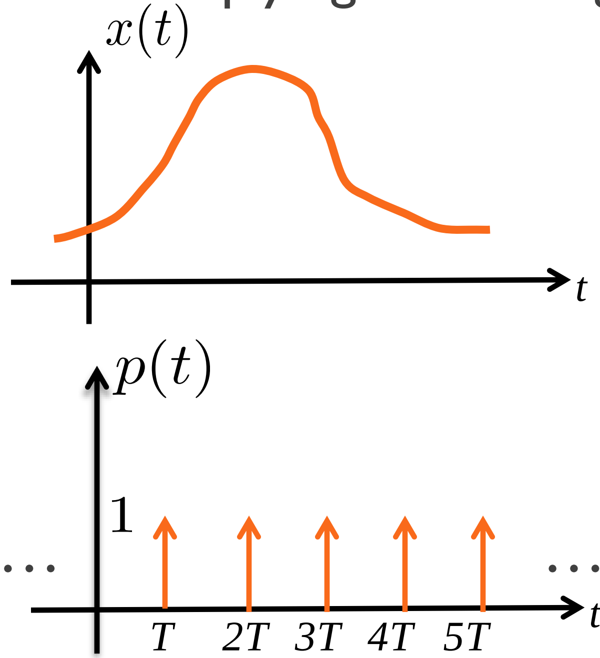


$x_1(kT)=x_2(kT)$

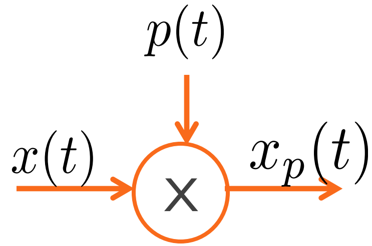
Two different signals can have the same samples

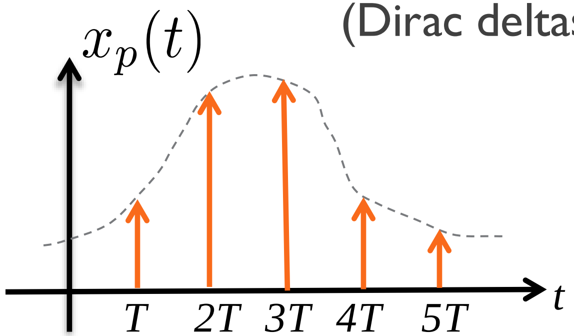
Impulse-train periodic sampling

- A convenient way to understand periodic sampling is through multiplying the CT signal by a periodic impulse train



Dirac delta functions

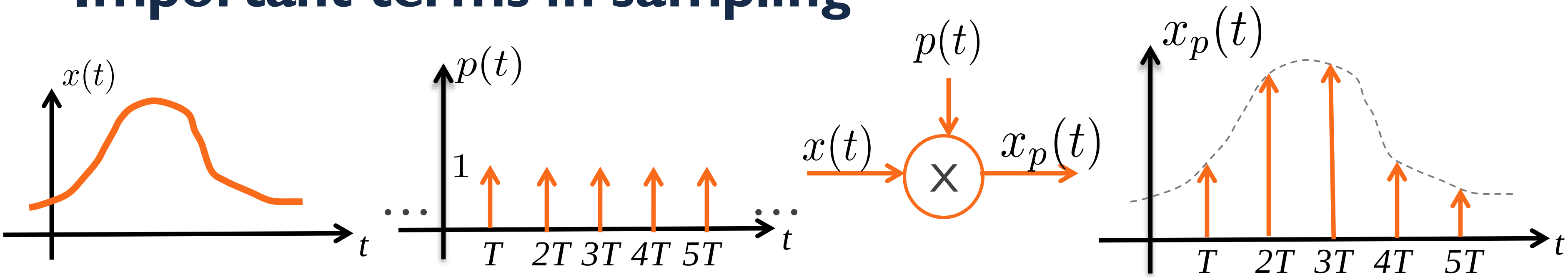




(Dirac deltas)

Contains the same values as the sampled signal

Important terms in sampling

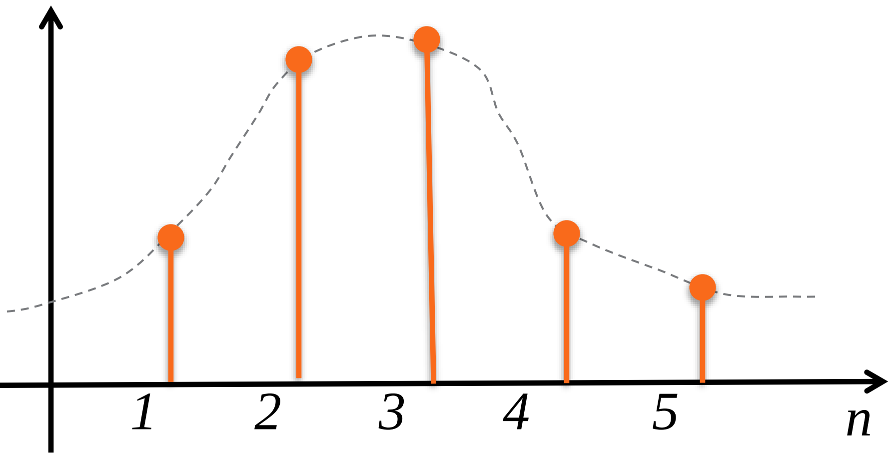


- The periodic impulse train $p(t)$ is the sampling function
- The period T is the sampling period
- The fundamental frequency of $p(t)$, $\omega_s = \dfrac{2\pi}{T}$ is the sampling frequency

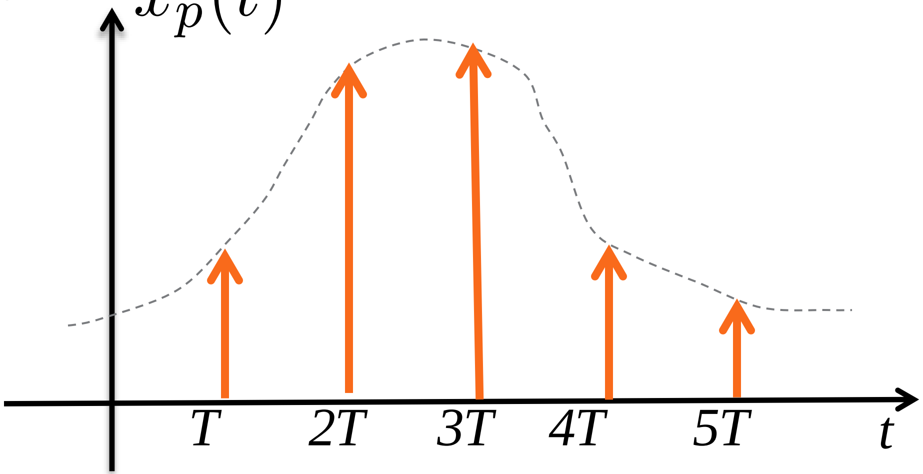
Comparing the impulse functions

We use $x_p(t)$ to understand the mathematical transformation from x(t) to x[n] in the frequency domain. The signal $x_p(t)$ is not generated in practice.

$x[n]=x(nT)$ **Kronecker** delta functions



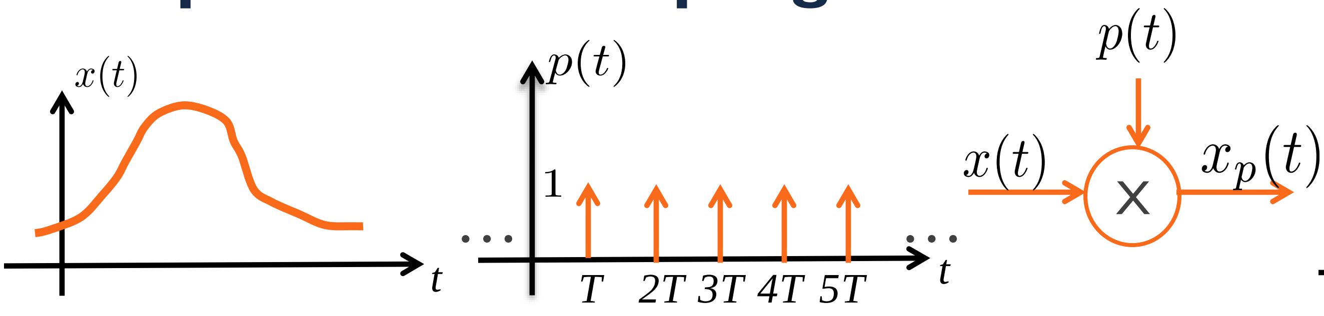
$x_p(t)$ **Dirac** delta functions

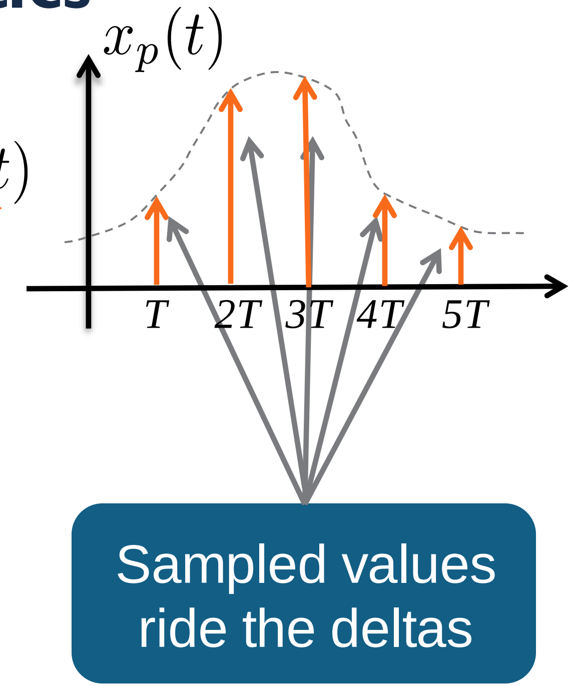


The sampled signal is a discrete-time sequence

The <u>impulse train of samples</u> is a continuous-time signal

Impulse-train sampling via mathematics





- The sampling function $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- The output signal $x_p(t) = x(t)p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Sampled signal is here via sifting property

Impulse train of samples in the frequency domain

- Multiplication in time domain → convolution in frequency domain

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

where

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

$$\boxed{\omega_s = \frac{2\pi}{T}}$$

- Then

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

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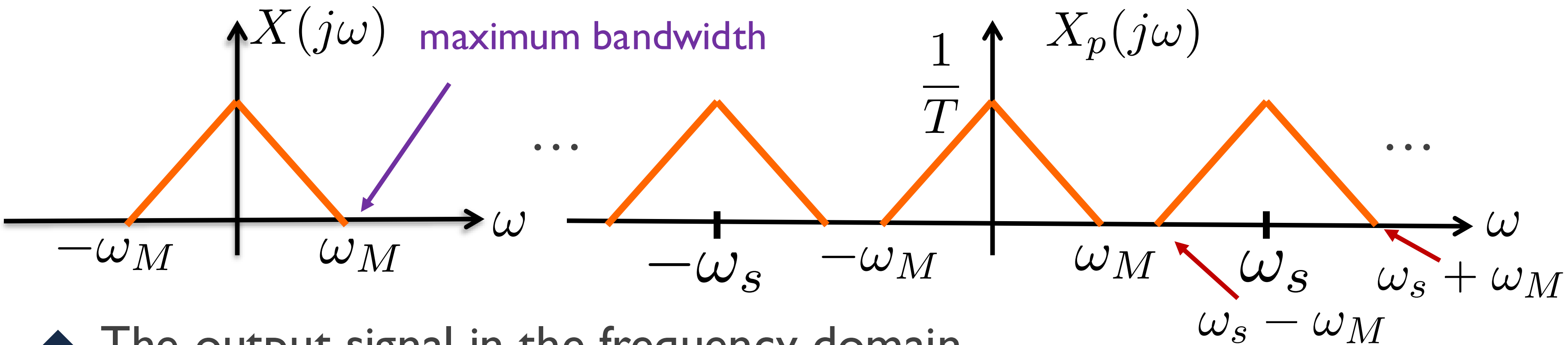
Understanding the frequency domain effect



- The output signal in the frequency domain

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

- A superposition of shifted versions of $X(j\omega)$ scaled by $\frac{1}{T}$

If $\omega_s - \omega_M > \omega_M$ ➡ $\omega_s > 2\omega_M$ then original spectrum is undistorted

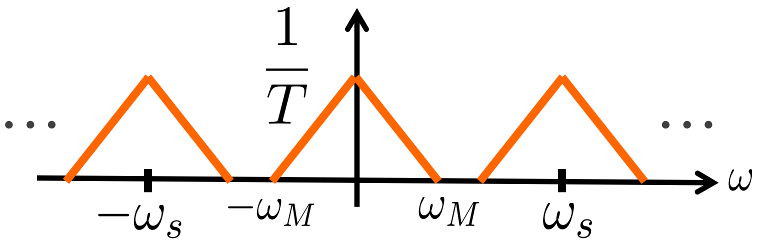
Sampling theorem

- When is $x(t)$ completely determined from $x_p(t)$?

- Sampling theorem
  - Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$.
  - Then, $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \ldots$

If $\boxed{\omega_s > 2\omega_M}$ $\qquad \omega_s = \frac{2\pi}{T}$

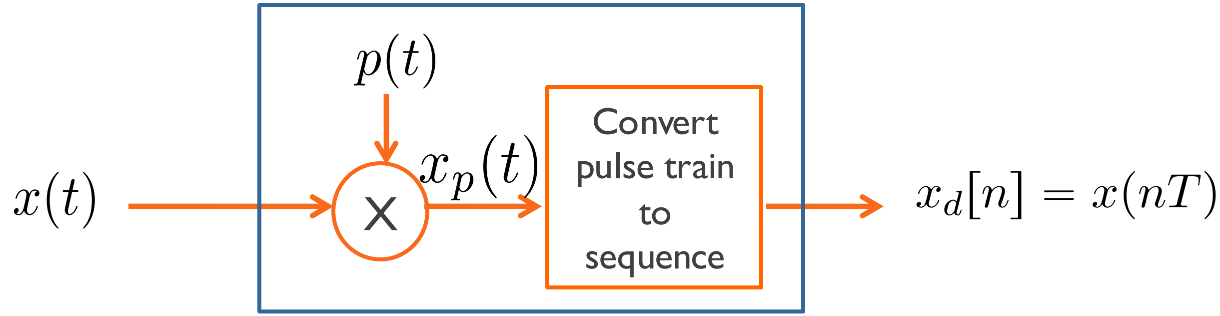
Nyquist frequency

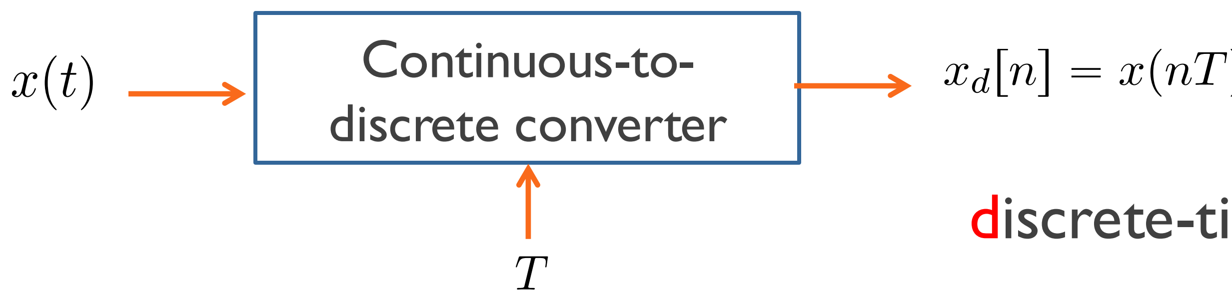


- The product $2\omega_M$ is called the "Nyquist rate"

Ideal continuous-to-discrete converter

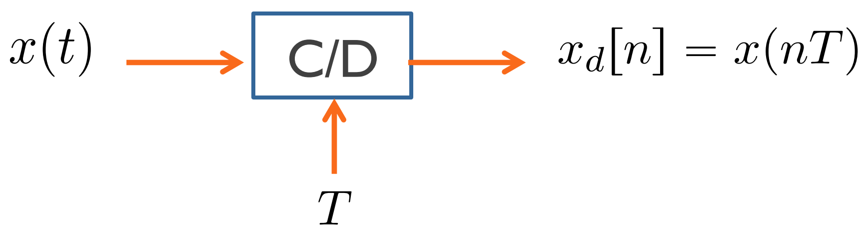
mathematical description





discrete-time

shorthand notation



Connecting the domains

	time domain	frequency domain
CT signal	$x(t)$	$X(j\omega)$
impulse train of samples	$x_p(t) = \sum_n x(nT)\delta(t - nT)$	$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$
DT signal	$x[n]$	$X_d(e^{j\Omega}) = X_p(j\Omega/T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\frac{\Omega}{T} - k\omega_s\right)\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\frac{\Omega}{T} - k\frac{2\pi}{T}\right)\right)$

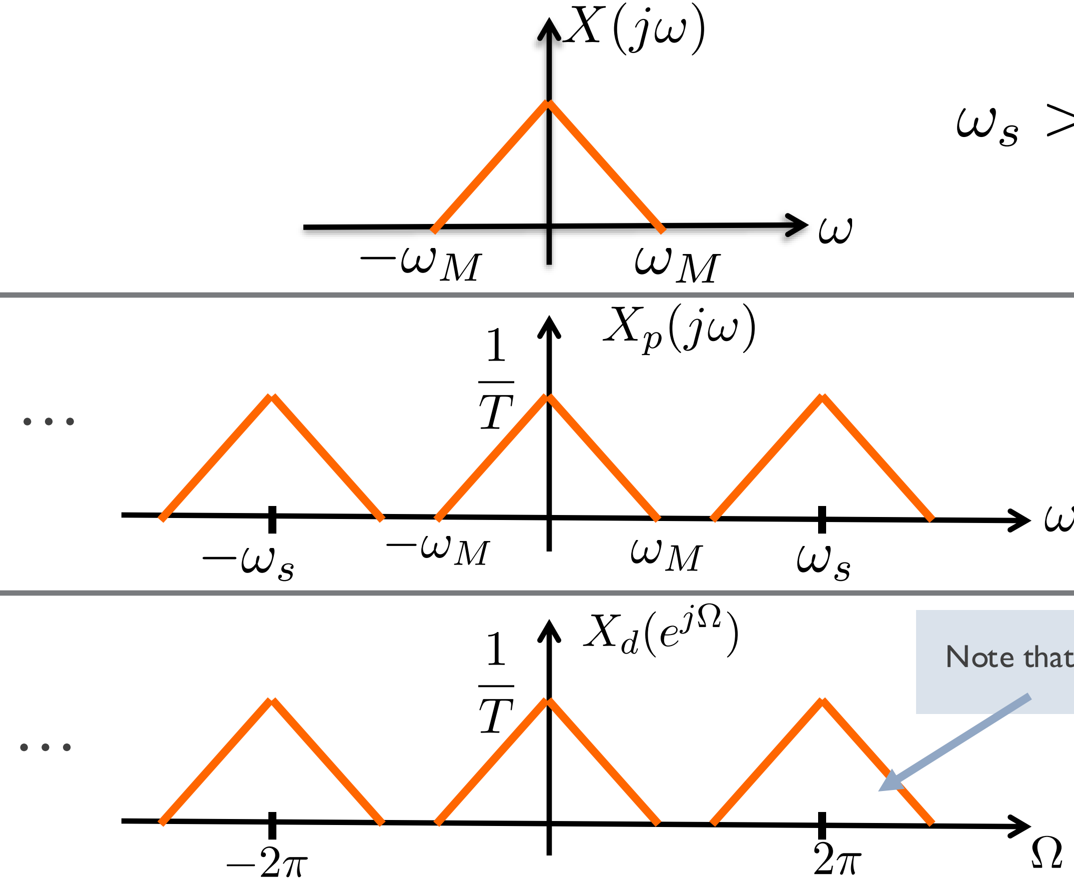
Sampling in the frequency domain – Nyquist OK

$$\omega_s > 2\omega_M$$

CT signal

impulse train of samples

DT signal



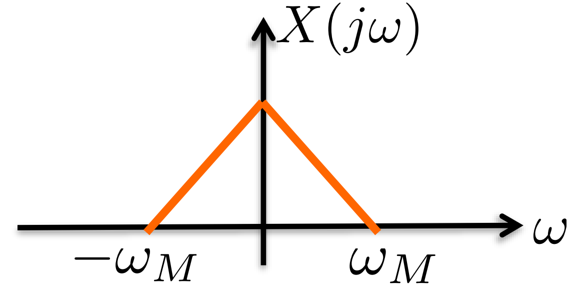
Note that $\frac{\omega_s}{T} = \frac{2\pi}{T}T = 2\pi$

Sampling in the frequency domain – Nyquist Not OK

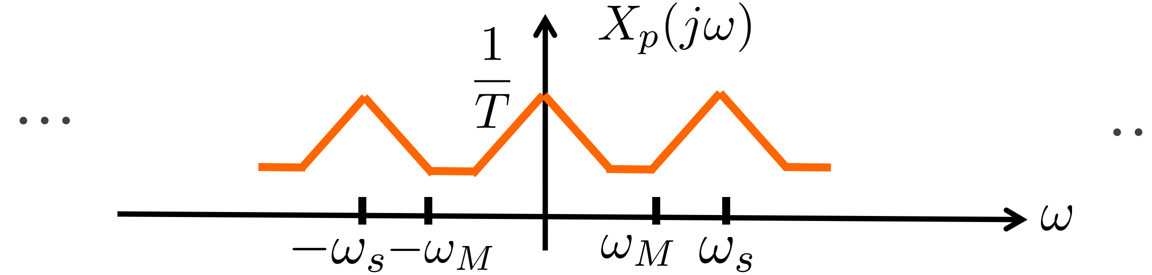
E.g.

$$\omega_M < \omega_s < 2\omega_M$$

CT signal



impulse train of samples

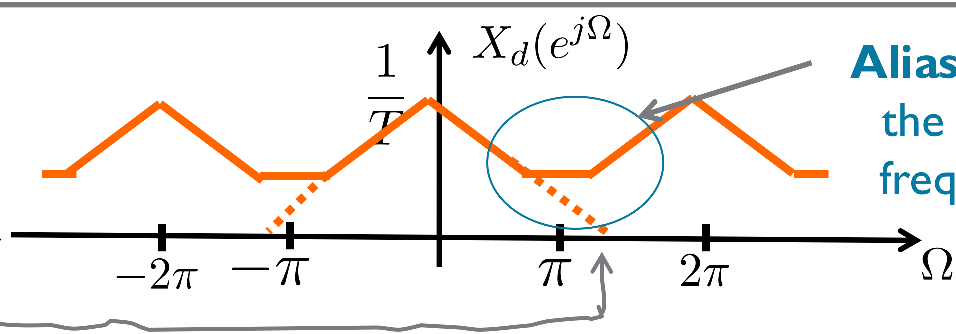


DT signal

$$\frac{\Omega}{T} = \omega_M$$

$$\Omega = \omega_M T$$

$$= 2\pi \frac{\omega_M}{\omega_s}$$



Aliasing! The shape of the original signal in frequency domain is distorted

Sampling summary

- The sampling theorem tells us when a continuous time signal may be periodically sampled with no loss
- The signal must be perfectly bandlimited and the sampling period must be small enough
- If the sampling theorem is not satisfied, it is still possible to sample the signal but aliasing will result

Classical example – the wagon wheel

Key points

- Explain the wagon wheel effect

The wagon wheel effect

- Classic demo
  - https://www.youtube.com/watch?v=VNftf5qLpiA
- Another demo that shows the effect of the number of spokes
  - https://www.youtube.com/watch?v=9MN5MF72PHs&t=34s
- Typical explanation on wikipedia
  - https://en.wikipedia.org/wiki/Wagon-wheel_effect

The reason that spoked wheels seem to go backwards is explained by the Nyquist sampling theorem

Sampling and aliasing

Key points

- Determine when there is aliasing
- Illustrate the effects of aliasing

Sampling theorem

- Let $x(t)$ be a continuous-time signal
- Then, $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \ldots$

Nyquist frequency

(1) If the signal is bandlimited, i.e. $X(j\omega) = 0 \text{ for } |\omega| > \omega_M$

(2) And the sampling frequency is chosen such that $\boxed{\omega_s > 2\omega_M}$

Nyquist rate

- The sampling frequency is $\omega_s = \dfrac{2\pi}{T}$
- The sampling period is $T = \dfrac{2\pi}{\omega_s} < \dfrac{2\pi}{2\omega_M} = \dfrac{\pi}{\omega_M}$

Sampling in the time and frequency domains

	time domain	frequency domain
CT signal	$x(t)$	$X(j\omega)$
impulse train of samples	$x_p(t) = \sum_n x(nT)\delta(t - nT)$	$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$
DT signal	$x_d[n] = x(nT)$	$X_d(e^{j\Omega}) = X_p(j\Omega/T)$ $= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\frac{\Omega}{T} - k\omega_s\right)\right)$ $= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\frac{\Omega}{T} - k\frac{2\pi}{T}\right)\right)$

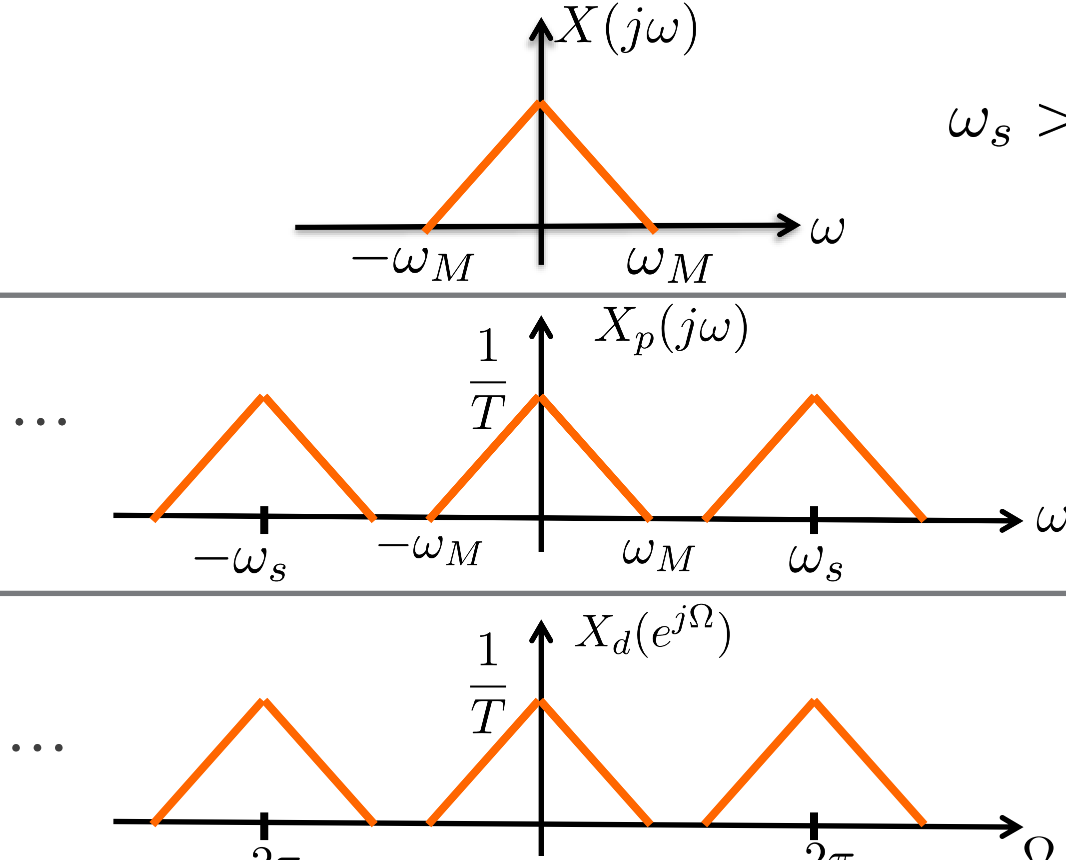
Sampling in the frequency domain – Nyquist OK

$$\omega_s > 2\omega_M$$

CT signal

impulse train of samples

DT signal

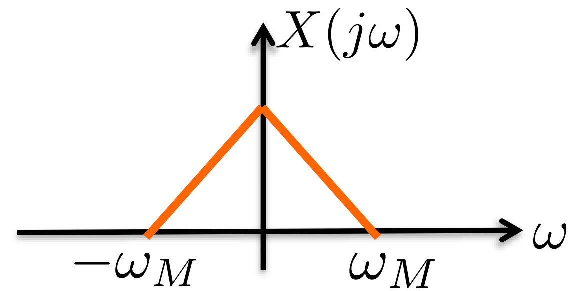


Sampling in the frequency domain – Nyquist Not OK

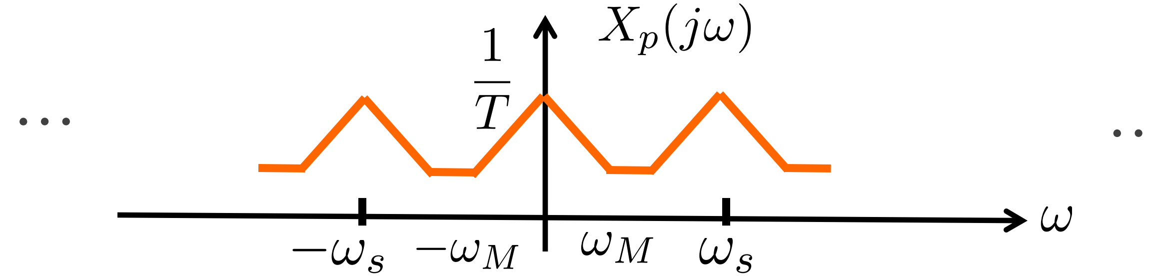
E.g.

$$\omega_M < \omega_s < 2\omega_M$$

CT signal



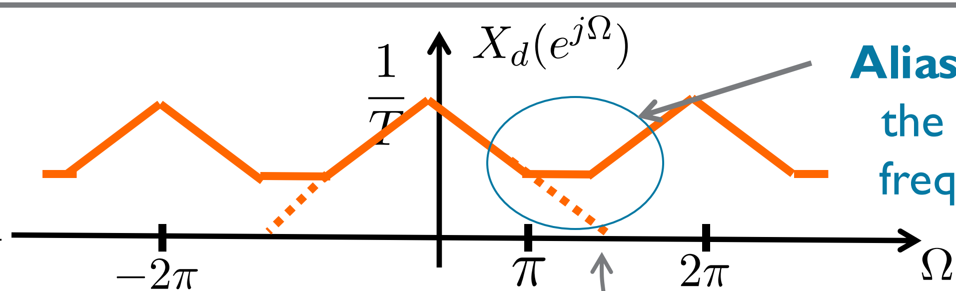
impulse train of samples



DT signal

$$\frac{\Omega}{T} = \omega_M$$

$$\Omega = \omega_M T = 2\pi \frac{\omega_M}{\omega_s}$$



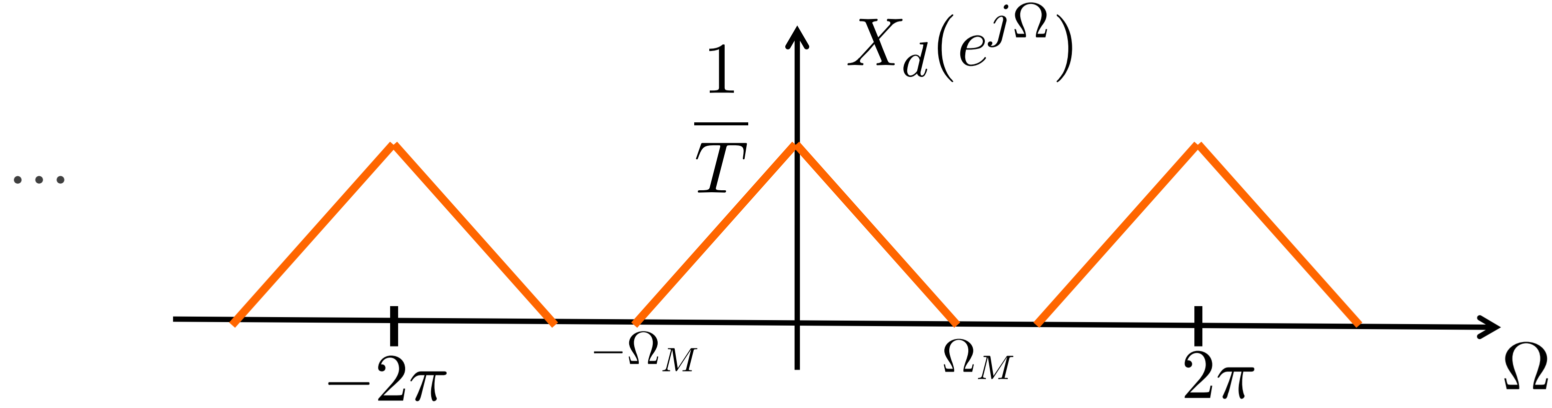
Aliasing! The shape of the original signal in frequency domain is distorted

Reconstruction of a signal from its samples

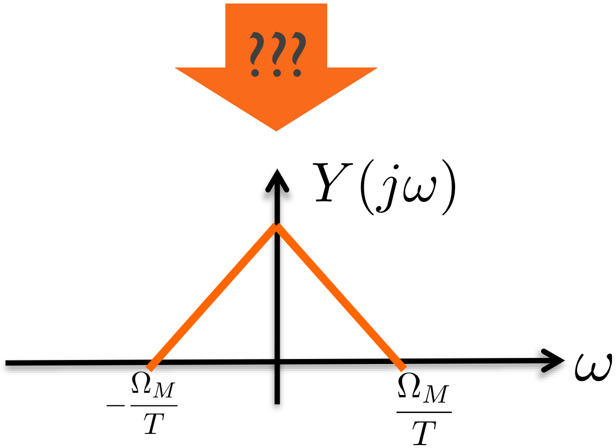
Key points

- Define the reconstruction formula
- Explain the role of the sinc function in reconstruction
- Illustrate reconstruction in time and frequency domains

How to extract the original signal?



Ω_M this is the bandwidth of the DT system

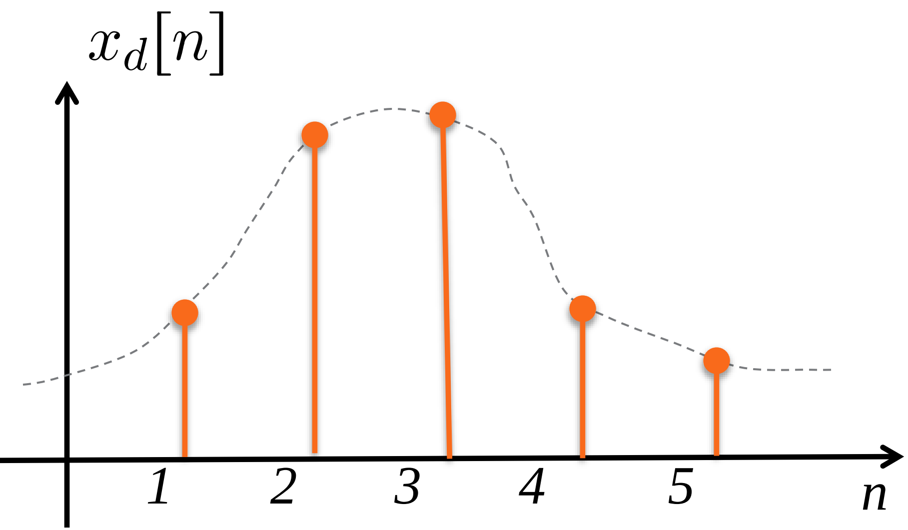


Ideally the original signal comes out if Nyquist was satisfied

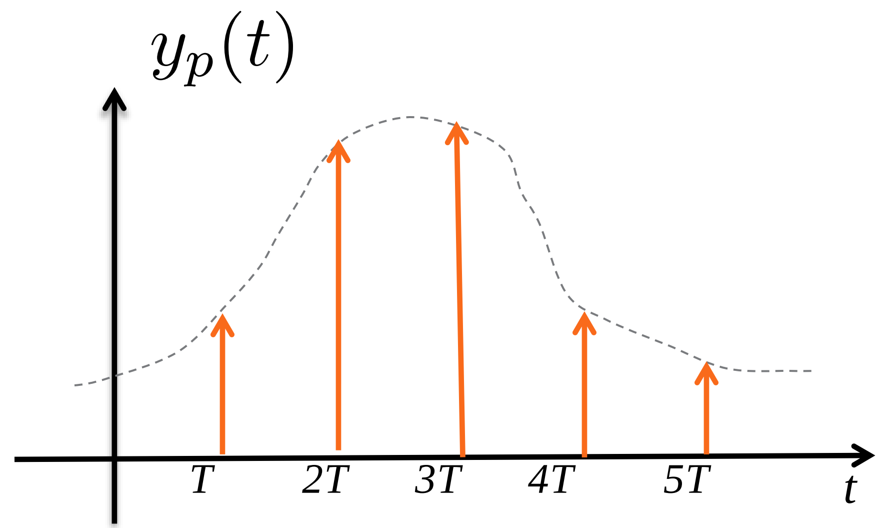
Step 1: convert to impulse train of samples

time domain

$$x_d[n] \longrightarrow \boxed{\text{Convert to pulse train of samples}} \longrightarrow y_p(t)$$



DT signal



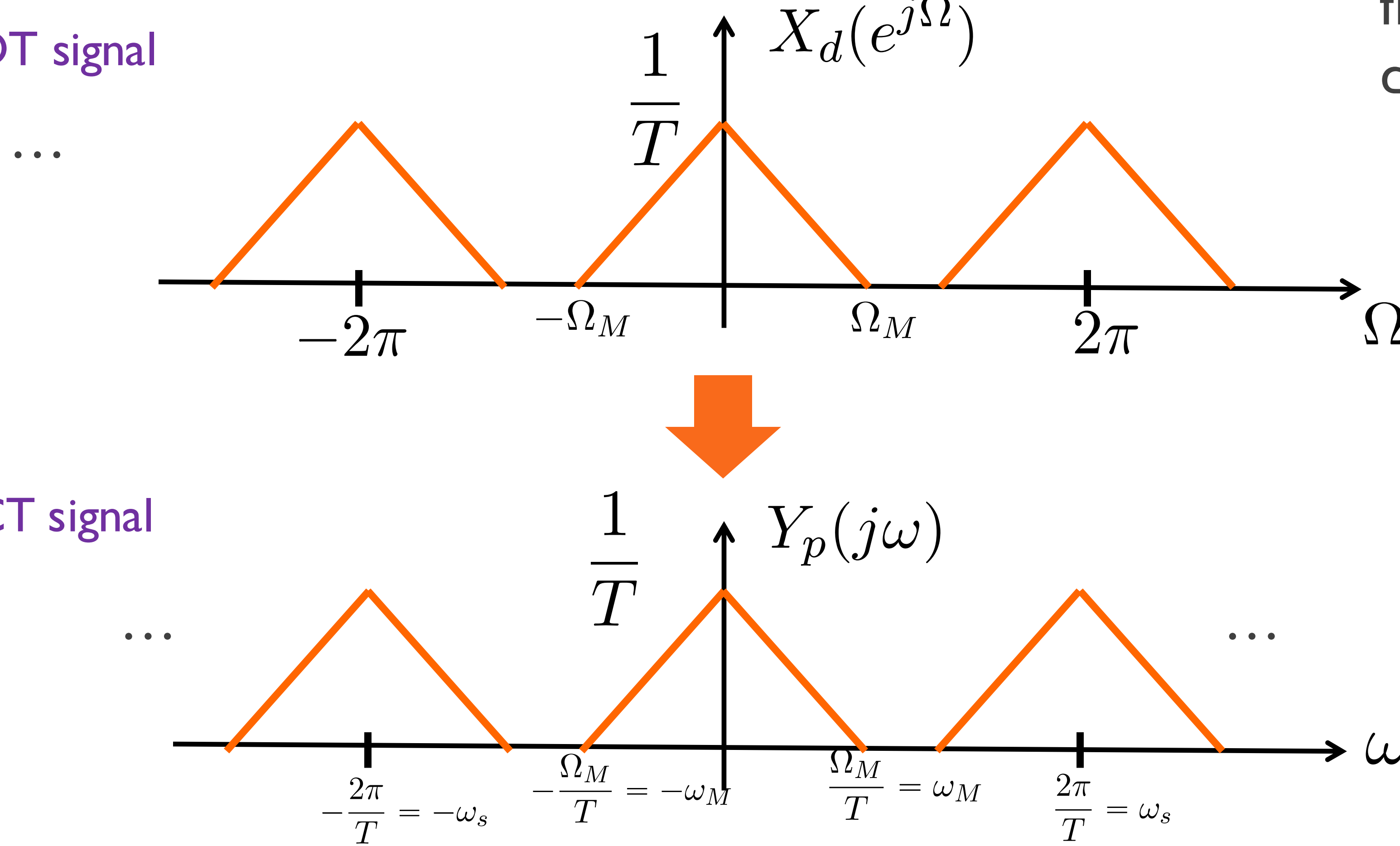
CT signal

Step 1: convert to impulse train of samples

DT signal

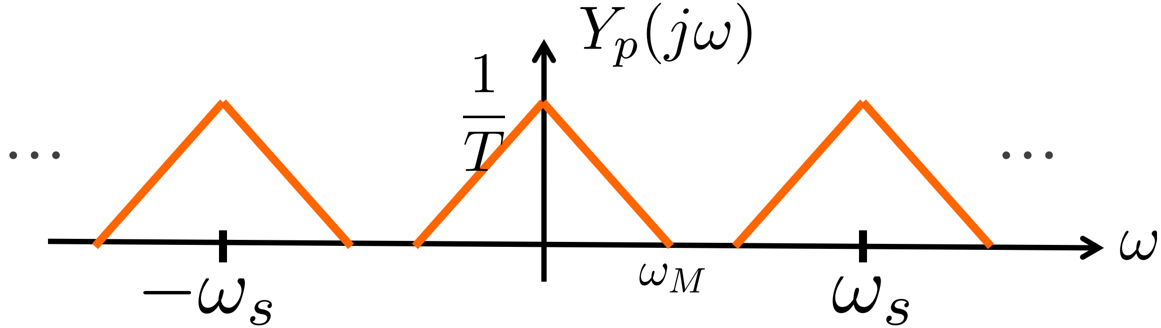
CT signal

frequency domain



Step 2: Filtering to reconstruct the signal

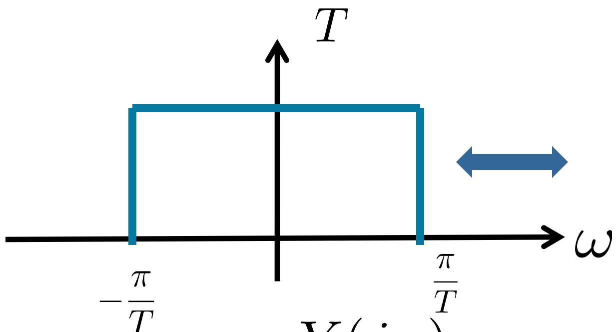
frequency domain



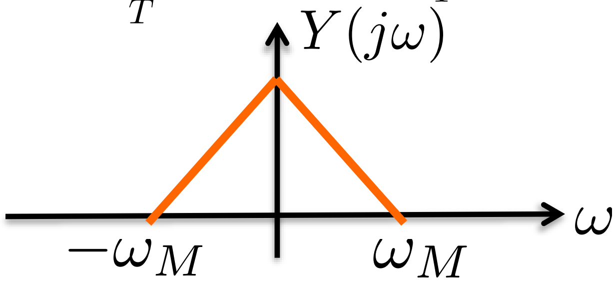
Ideal low pass filter

Cutoff $\frac{\pi}{T}$

Gain T

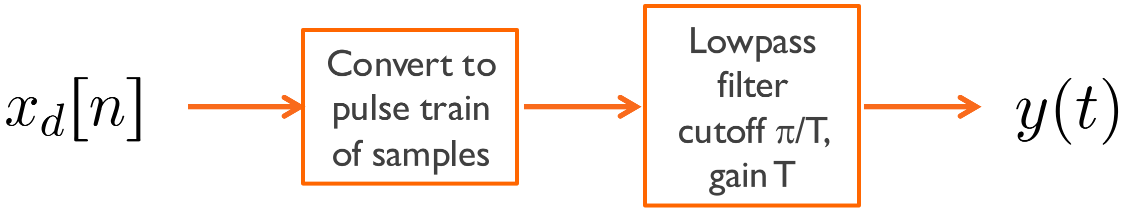


$$T\text{rect}\left(\frac{\omega}{2\pi/T}\right) \leftrightarrow \text{sinc}\left(\frac{t}{T}\right)$$



Step 2: Filtering to reconstruct the signal

time domain

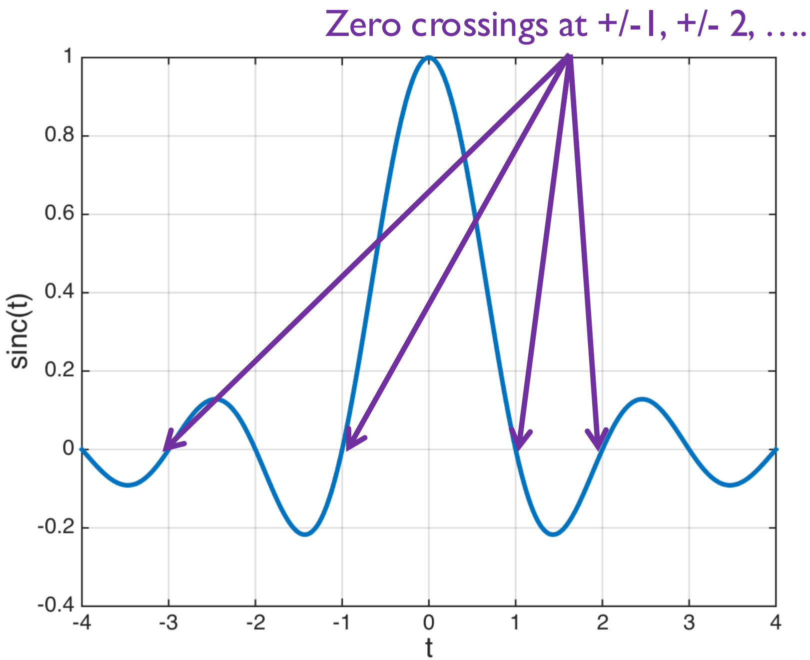


$$
\begin{aligned}
y(t) &= y_p(t) * h(t) \\
&= h(t) * \sum_{n=-\infty}^{\infty} x_d[n]\delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_d[n]h(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_d[n]\mathrm{sinc}\left(\frac{t - nT}{T}\right)
\end{aligned}
$$

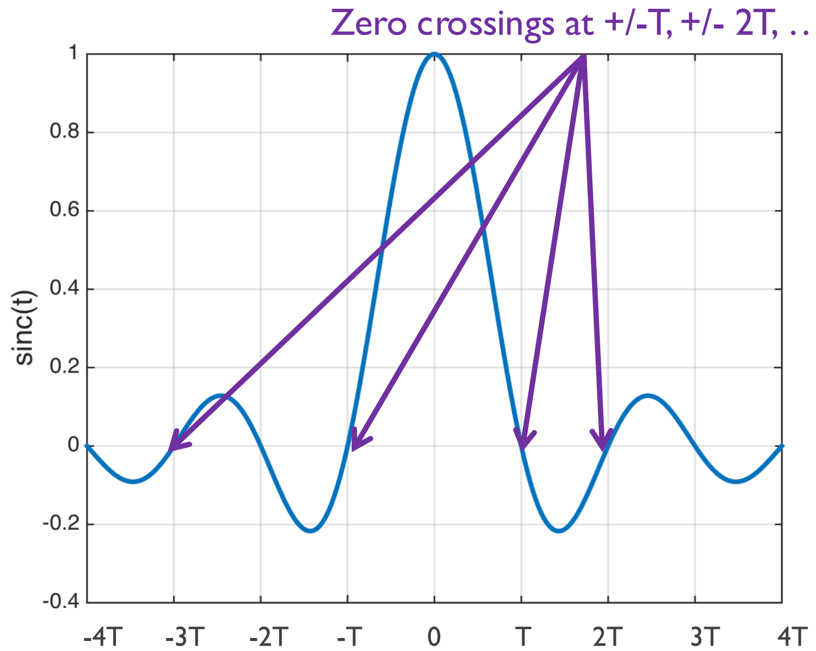
sinc interpolation

Reconstruction formula!

Step 2: Sinc is critical to reconstruction



$$\mathrm{sinc}\,(t)$$

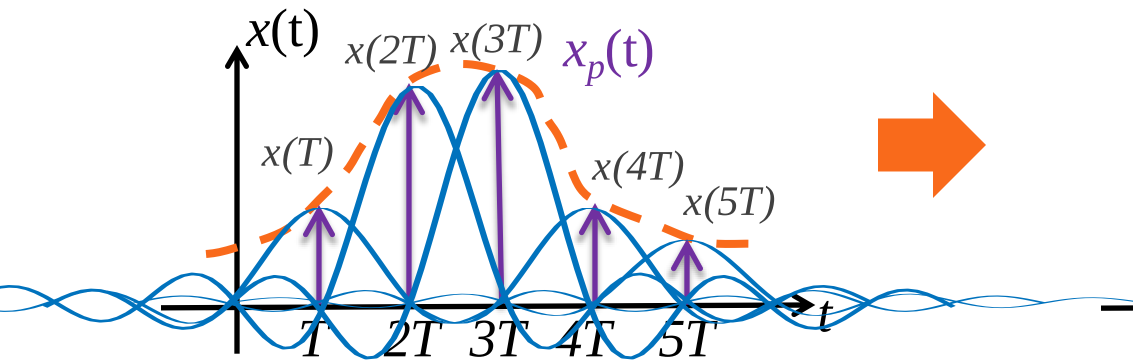


$$\mathrm{sinc}\left(\frac{t}{T}\right)$$

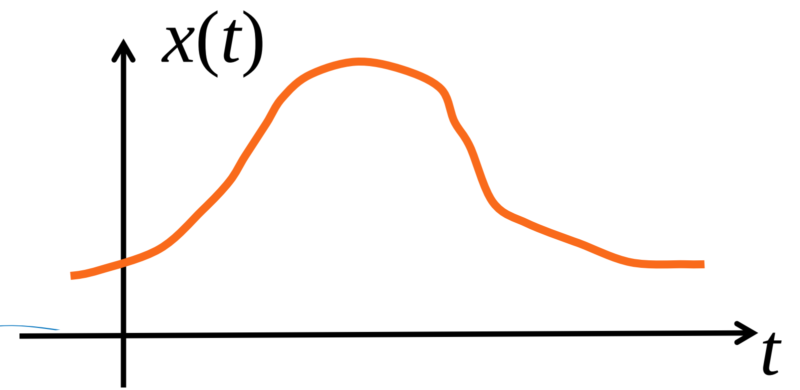
Zero crossings occur exactly at the sampling intervals

Step 2: Reconstructing using the sinc functions

time domain



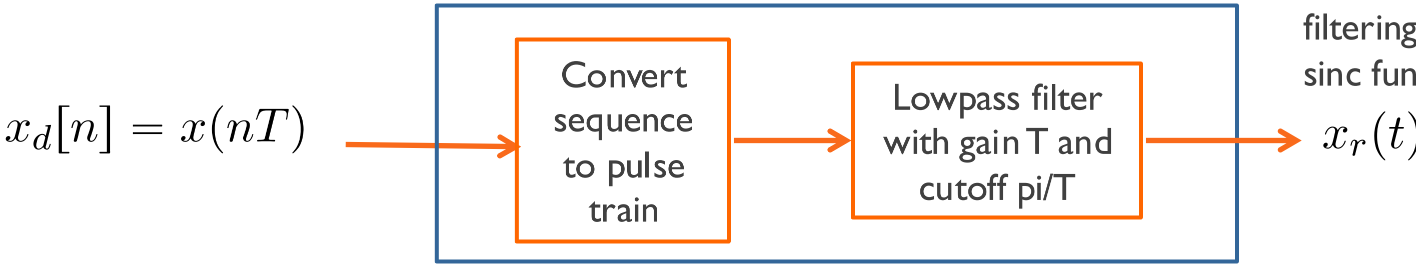
$$\sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\frac{t-nT}{T}\right)$$

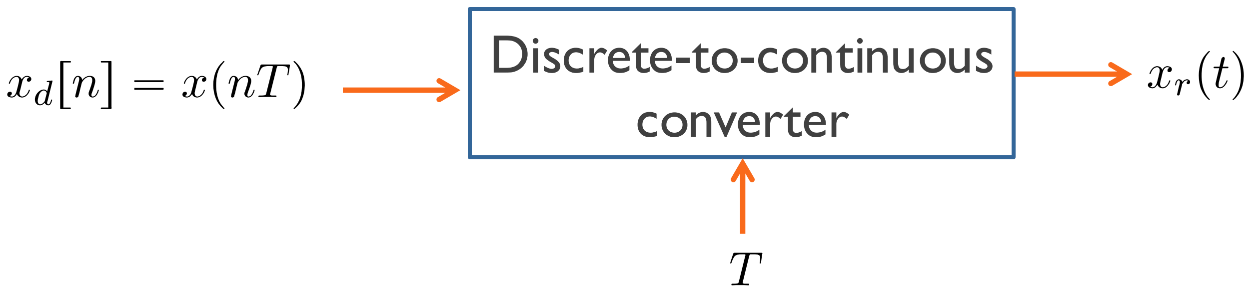


Reconstructed signal results from a superposition of sinc functions

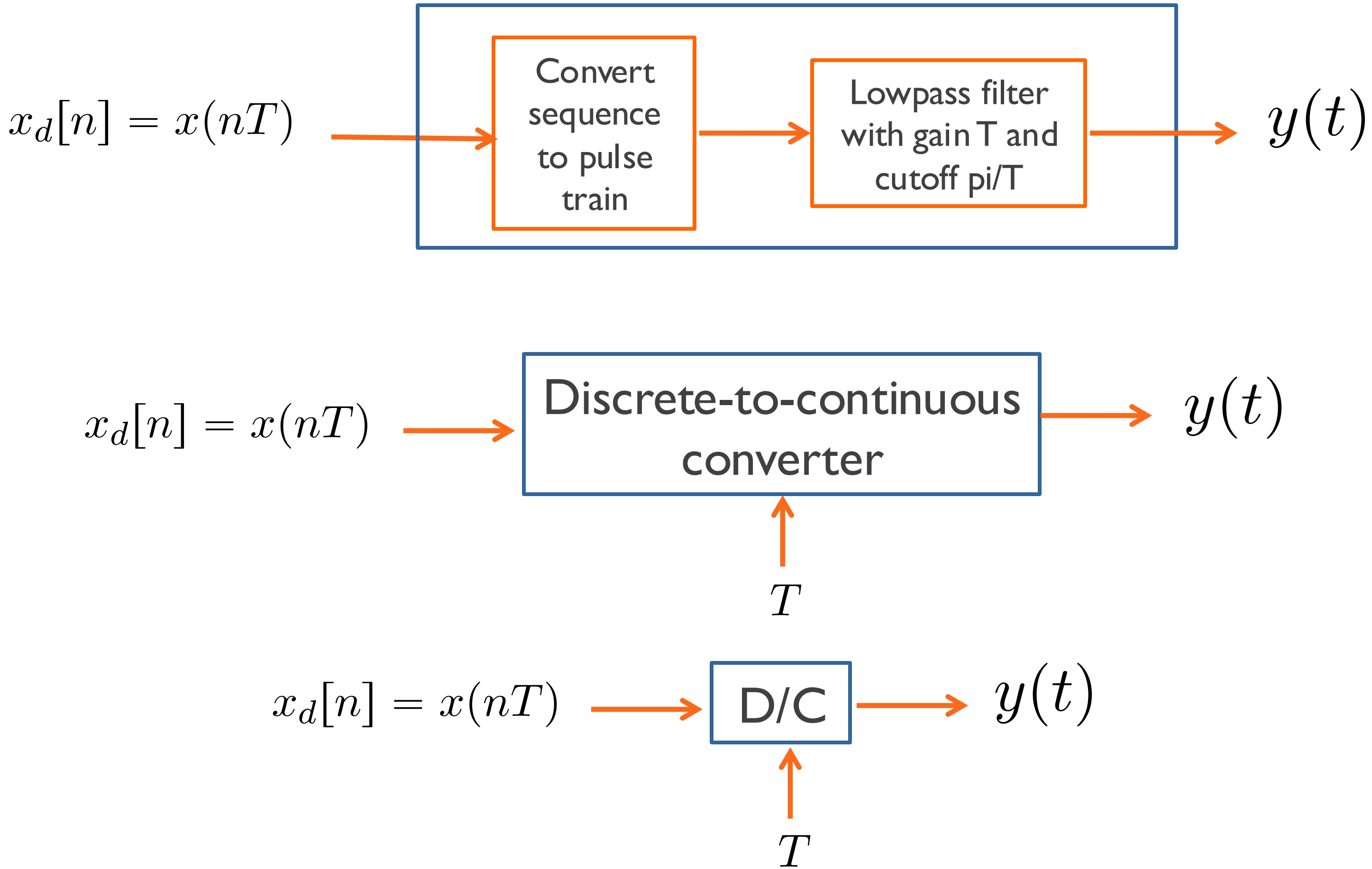
Ideal discrete-to-continuous converter

Remember that ideal lowpass filtering is convolution with a sinc function





Ideal discrete-to-continuous converter



Reconstruction in the time and frequency domains

	time domain	frequency domain
DT signal	$x_d[n]$	$X_d(e^{j\Omega})$
impulse train of samples	$y_p(t) = \sum_{n=-\infty}^{\infty} x_d[n]\delta(t - nT)$	$Y_p(j\omega) = X_d(e^{j\omega T})$
CT signal	$y(t) = \sum_{n=-\infty}^{\infty} x_d[n]\text{sinc}\left(\frac{t-nT}{T}\right)$	$Y(j\omega) = T\text{rect}\left(\frac{\omega}{2\pi/T}\right) Y_p(j\omega)$ $= T\text{rect}\left(\frac{\omega}{2\pi/T}\right) X_d(j\omega T)$

Output is always bandlimited no matter the input

If Nyquist is satisfied then

- In the frequency domain

$$Y(j\omega) = X(j\omega)$$

- In the time domain

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

Thoughts on reconstruction

- Optimal reconstruction involves interpolation of the samples with a sinc function, with the bandwidth determined by the reconstruction frequency

- The output of the discrete-to-continuous converter is always bandlimited

- Aliasing is created by sampling a signal with a sampling frequency less than the Nyquist rate and thus is the "fault" of the continuous-to-discrete conversion, reconstruction just operates on the samples already given

Important example involving the sampling and reconstruction of a sinusoid

Key points

- You should be able to determine the frequency of an undersampled sinusoid after reconstruction
- When Nyquist is not satisfied, aliasing is created

Sampling a sinusoid

- Consider the following signal

$$x(t) = \cos(37\pi t + \pi/4)$$

- Determine the following
  - Nyquist frequency $\omega_M = 37\pi$
  - Nyquist rate $2\omega_M = 74\pi$
  - Maximum sampling period $T < \frac{2\pi}{2\omega_M} = \frac{2\pi}{74\pi} = \frac{1}{37}$ seconds
- General form of sampled signal

$$x[n] = \cos(37\pi T n + \pi/4)$$

Suppose Nyquist is satisfied

$$T < \frac{2\pi}{\omega_s} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

- Suppose that $T = 1/74$ seconds
- Find the impulse train signal

$$\begin{aligned} x_p(t) &= \sum_n x(nT)\delta(t - n/74) \\ &= \sum_n \cos((37\pi/74)n + \pi/4)\delta(t - n/74) \\ &= \sum_n \cos((\pi/2)n + \pi/4)\delta(t - n/74) \end{aligned}$$

- Find the discrete-time signal

$$\begin{aligned} x[n] &= \cos((37\pi/74)n + \pi/4) \\ &= \cos((\pi/2)n + \pi/4) \end{aligned}$$

Find the CT transforms

$$T < \frac{2\pi}{\omega_s} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

- Suppose that $T = 1/74$

- Find the CTFT

$$X(j\omega) = \pi e^{j\pi/4}\delta(\omega - 37\pi/74) + \pi e^{-j\pi/4}\delta(\omega + 37\pi/74)$$

- Find the CTFT of the impulse train signal

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$= 74\sum_{k=-\infty}^{\infty}\left(\pi e^{j\pi/4}\delta(\omega - k148\pi - 37\pi) + \pi e^{-j\pi/4}\delta(\omega - k148\pi + 37\pi)\right)$$

Find the DT transforms

$$T < \frac{2\pi}{\omega_s} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

- Find the DTFT of the sampled signal

$$X_d(e^{j\Omega}) = X_p(j\Omega/T)$$ (from previous slide)

$$= 74 \sum_{k=-\infty}^{\infty} \left(\pi e^{j\pi/4} \delta(74\Omega - k148\pi - 37\pi) + \pi e^{-j\pi/4} \delta(74\Omega - k148\pi + 37\pi) \right)$$

$$= \sum_{k=-\infty}^{\infty} \left(\pi e^{j\pi/4} \delta(\Omega - k2\pi - \pi/2) + \pi e^{-j\pi/4} \delta(\Omega - k2\pi + \pi/2) \right)$$

Where we have used the fact that to pull out the 74 $\delta(at) = \frac{1}{|a|}\delta(t)$

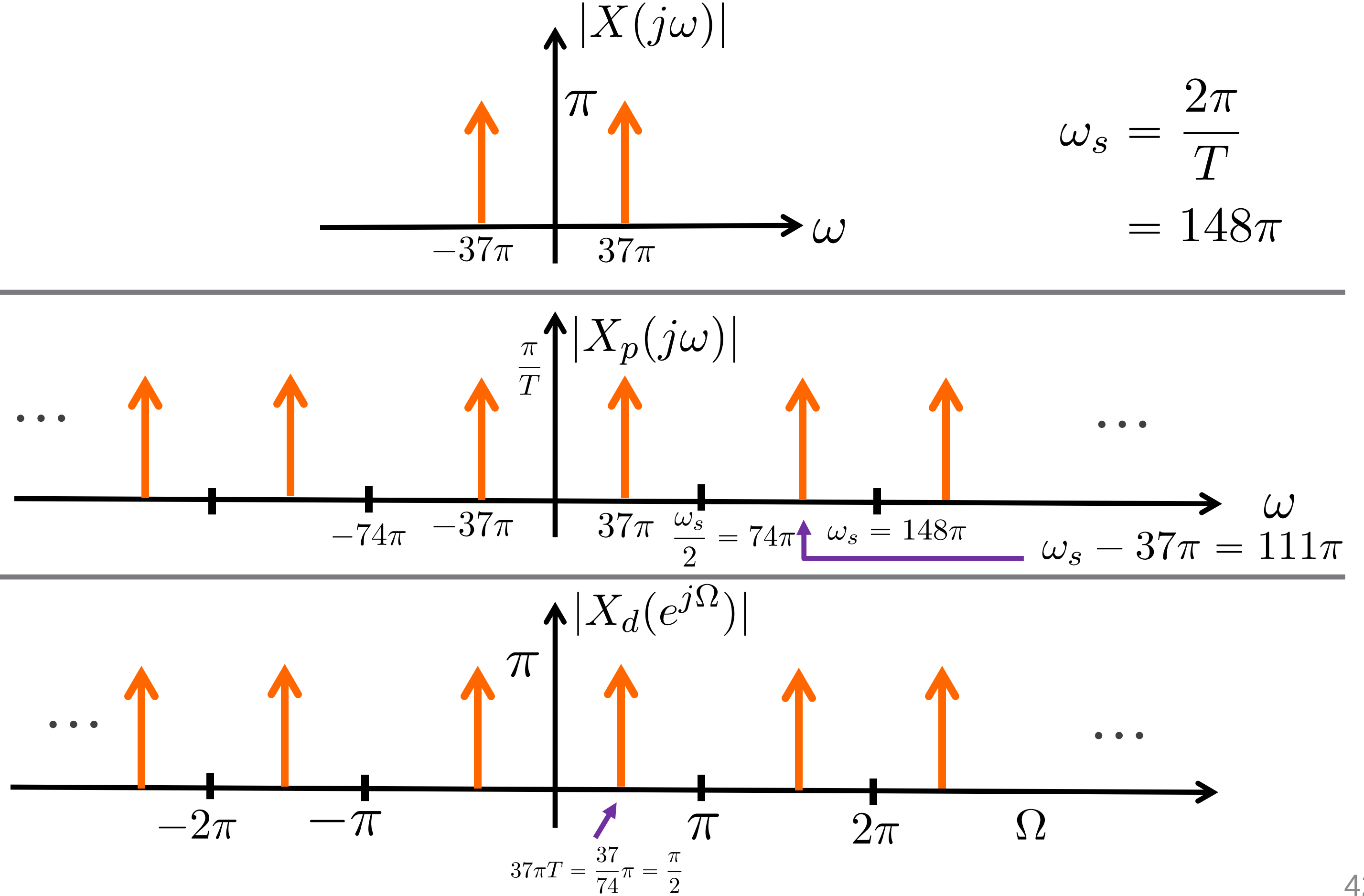
Oversampling (using a rate greater than Nyquist)

T=1/74

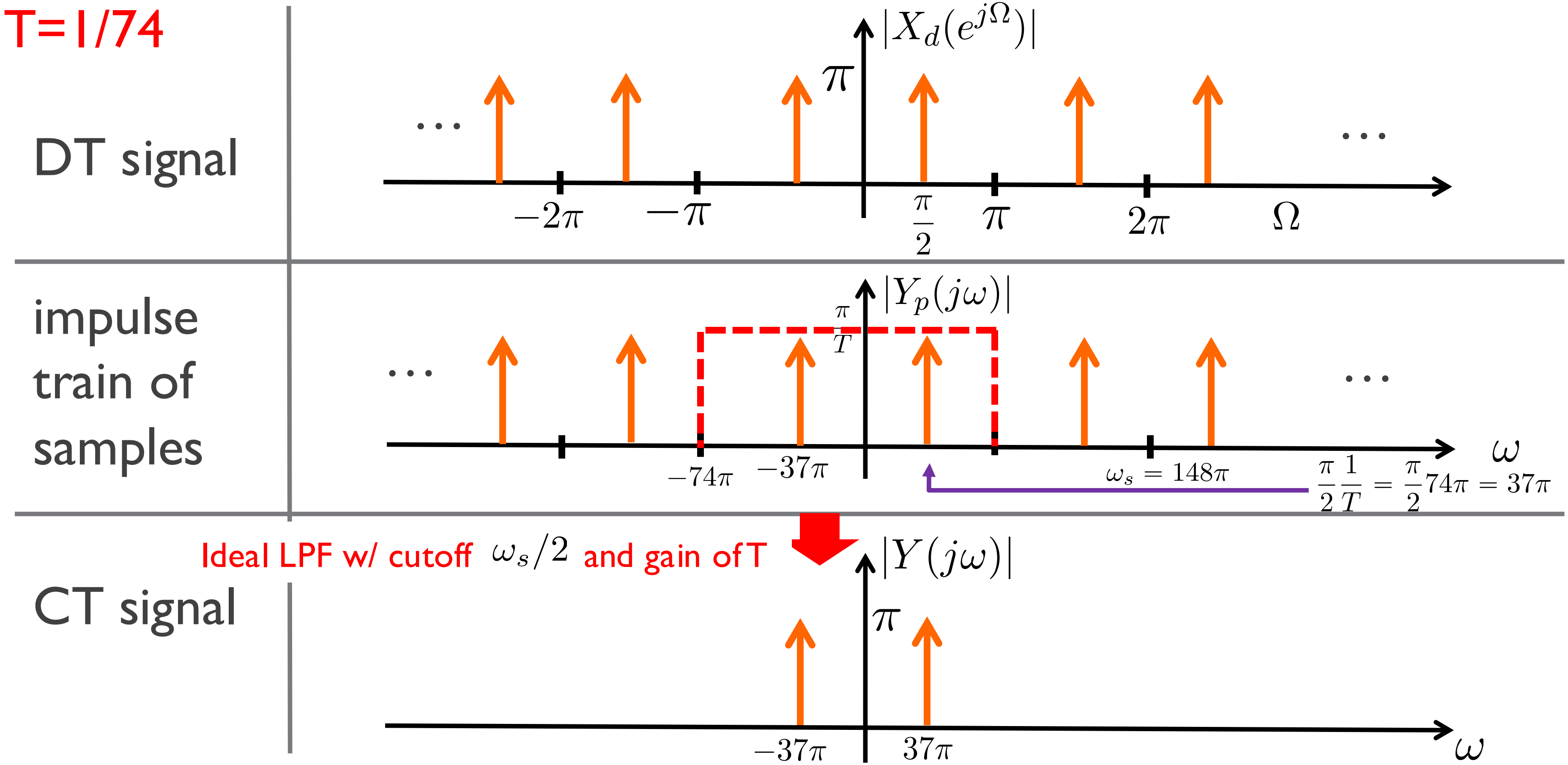
CT signal

impulse train of samples

DT signal



Reconstruction (using a rate greater than Nyquist)



What if Nyquist is not satisfied?

Recall that Nyquist is satisfied if

$$\boxed{T < \frac{2\pi}{\omega_s} = \frac{2\pi}{74\pi} = \frac{1}{37}}$$

- Suppose that $T = \left(\frac{4}{3}\right)\frac{1}{37}$

- Find the discrete-time signal

$$\begin{aligned} x[n] &= x(nT) \\ &= \cos\left(\left(37\pi\left(\frac{4}{3}\right)\frac{1}{37}\right)n + \pi/4\right) \\ &= \cos\left(\frac{4}{3}\pi n + \pi/4\right) \\ &= \cos\left(-\frac{2}{3}\pi n + \pi/4\right) \\ &= \cos\left(\frac{2}{3}\pi n - \pi/4\right) \end{aligned}$$

Simplifying to get the discrete-time frequency in the interval [-pi,pi]

Find the CT transforms

$$T < \frac{2\pi}{\omega_s} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

- Recall that $T = \left(\frac{4}{3}\right)\frac{1}{37}$ which does not satisfy Nyquist

- Find the CTFT

$$X(j\omega) = \pi e^{j\pi/4}\delta(\omega - 37\pi) + \pi e^{-j\pi/4}\delta(\omega + 37\pi)$$

- Find the CTFT of the impulse train signal

$$\frac{1}{T} = 37\frac{3}{4} = 27.75$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$= 27.75\sum_{k=-\infty}^{\infty}\left(\pi e^{j\pi/4}\delta(\omega - k55.5\pi - 37\pi) + \pi e^{-j\pi/4}\delta(\omega - k55.5\pi + 37\pi)\right)$$

Find the DT transforms

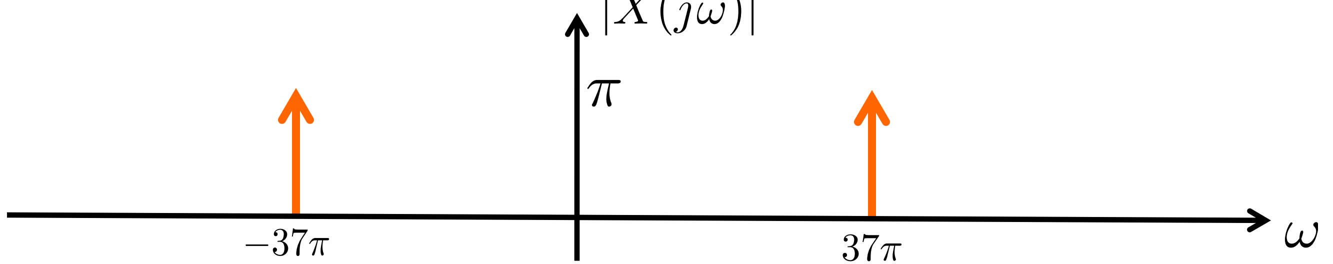
- Find the DTFT of the sampled signal

$$
\begin{aligned}
X_d(e^{j\Omega}) &= X_p(j\Omega/T) \quad \text{(from previous slide)} \\
&= 27.75 \sum_{k=-\infty}^{\infty} \left(\pi e^{j\pi/4} \delta(27.75\Omega - k55.5\pi - 37\pi) + \pi e^{-j\pi/4} \delta(27.75\Omega - k55.5\pi + 37\pi) \right) \\
&= \sum_{k=-\infty}^{\infty} \left(\pi e^{j\pi/4} \delta\left(\Omega - k2\pi - \frac{4}{3}\pi\right) + \pi e^{-j\pi/4} \delta\left(\Omega - k2\pi + \frac{4}{3}\pi\right) \right) \\
&= \pi e^{-j\pi/4} \delta\left(\Omega - \frac{2}{3}\pi\right) + \pi e^{j\pi/4} \delta\left(\Omega + \frac{2}{3}\pi\right) \text{ for } \Omega \in [-\pi, \pi]
\end{aligned}
$$

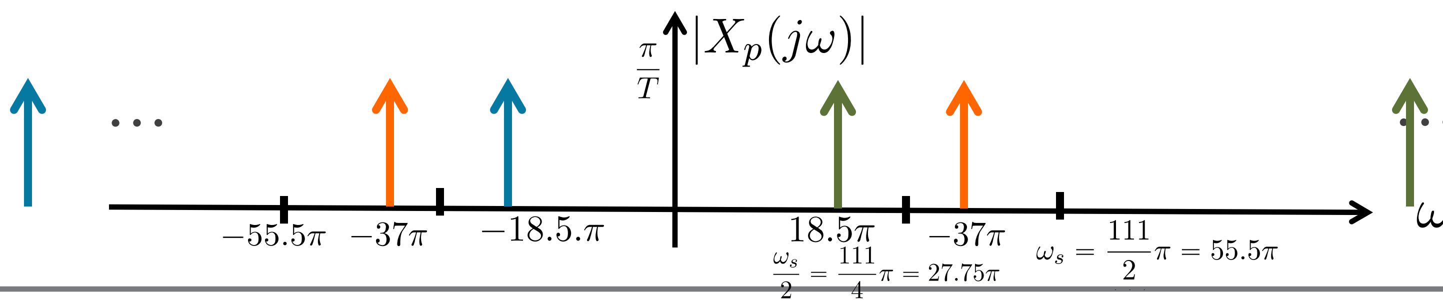
Undersampling (using a rate less than Nyquist)

$$T = \left(\frac{4}{3}\right)\frac{1}{37}$$

CT signal



impulse train of samples



Replica at $-\omega_s$

Original signal

Replica at ω_s

$$\omega_s - 37\pi = \frac{111 - 74}{2}\pi = 18.5\pi$$

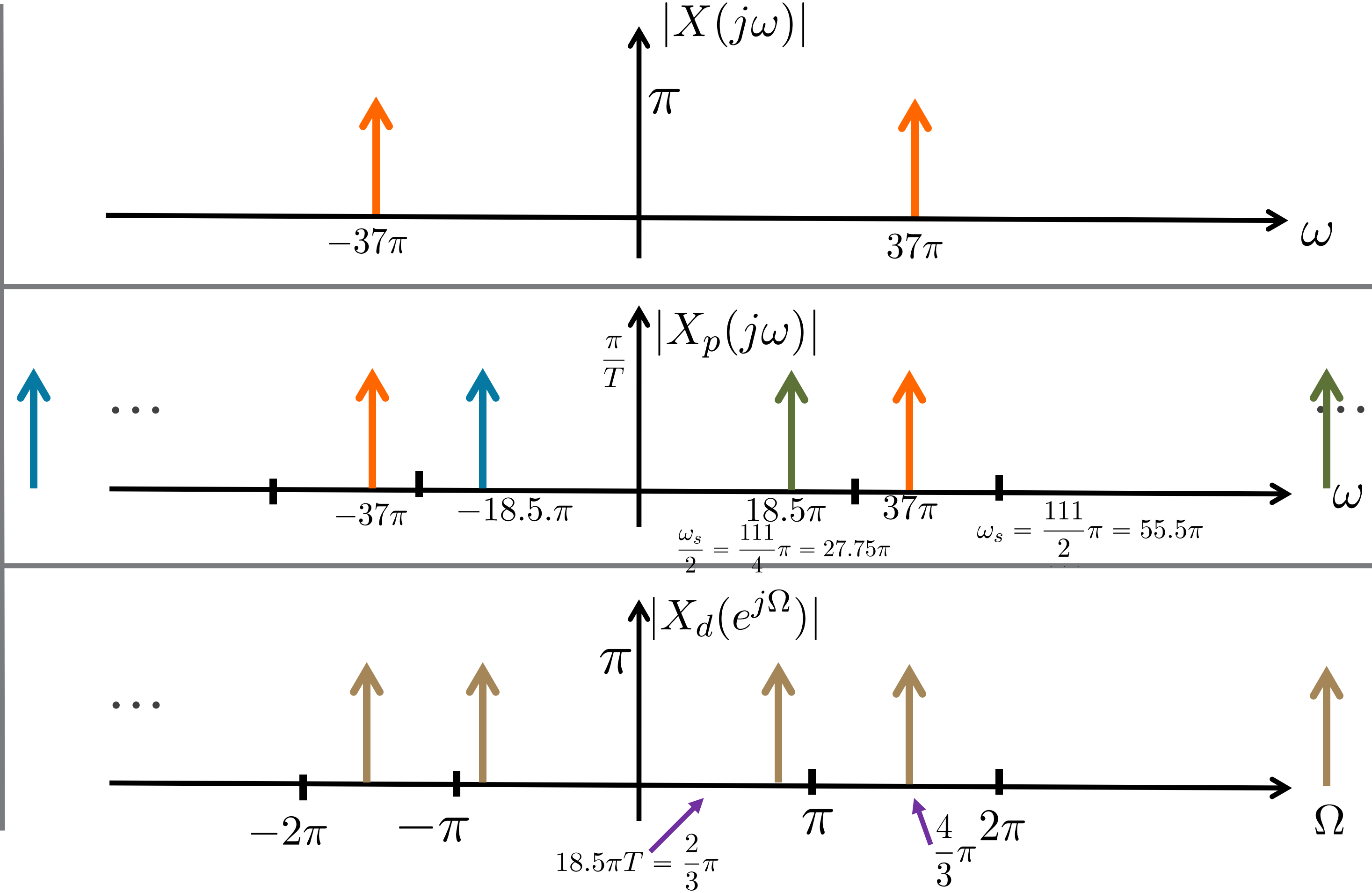
Undersampling (using a rate less than Nyquist)

$$T = \left(\frac{4}{3}\right)\frac{1}{37}$$

CT signal

impulse train of samples

DT signal



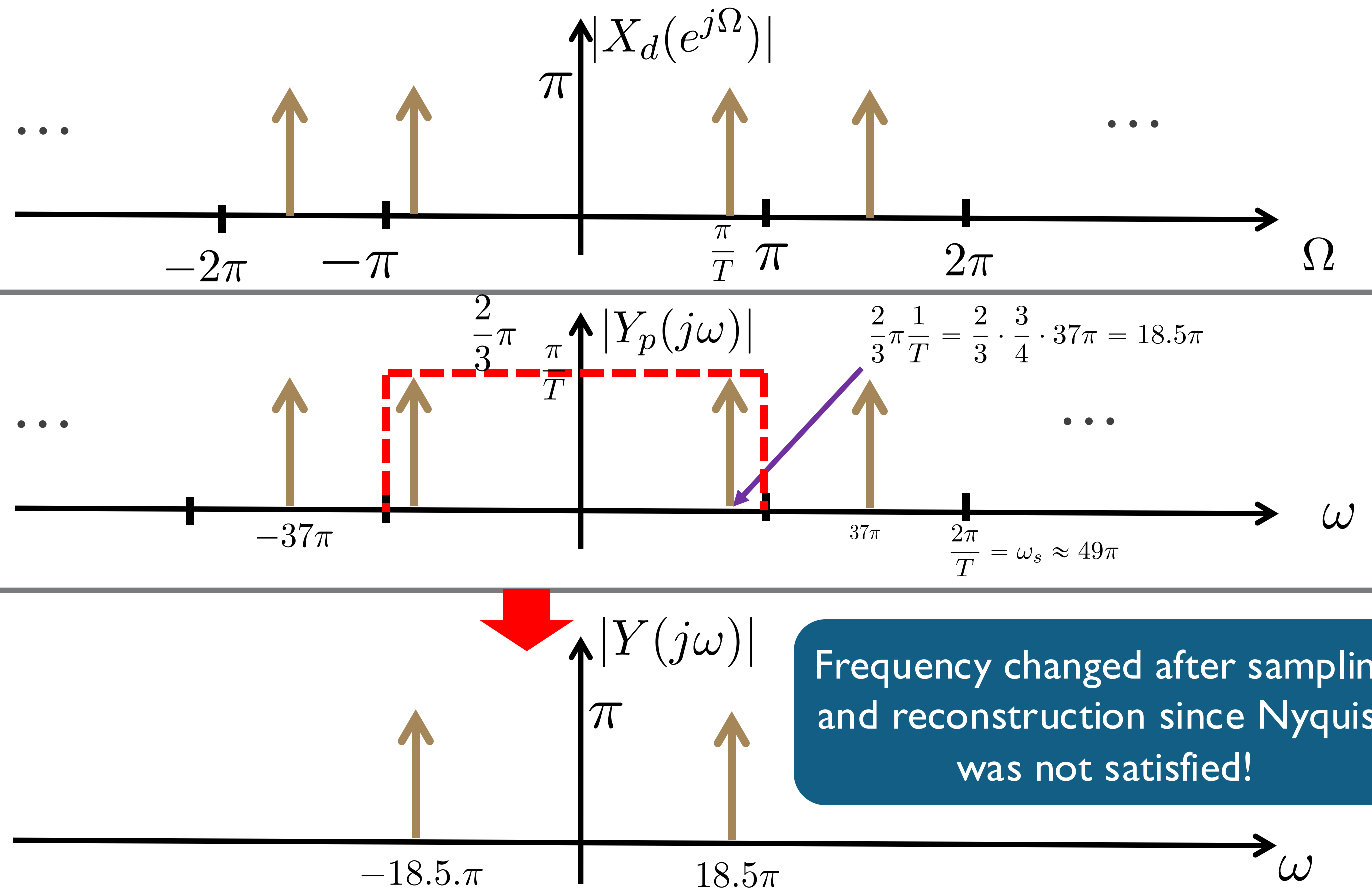
Reconstruction (using a rate less than Nyquist)

$T = \left(\frac{4}{3}\right)\frac{1}{37}$

DT signal

impulse train of samples

CT signal

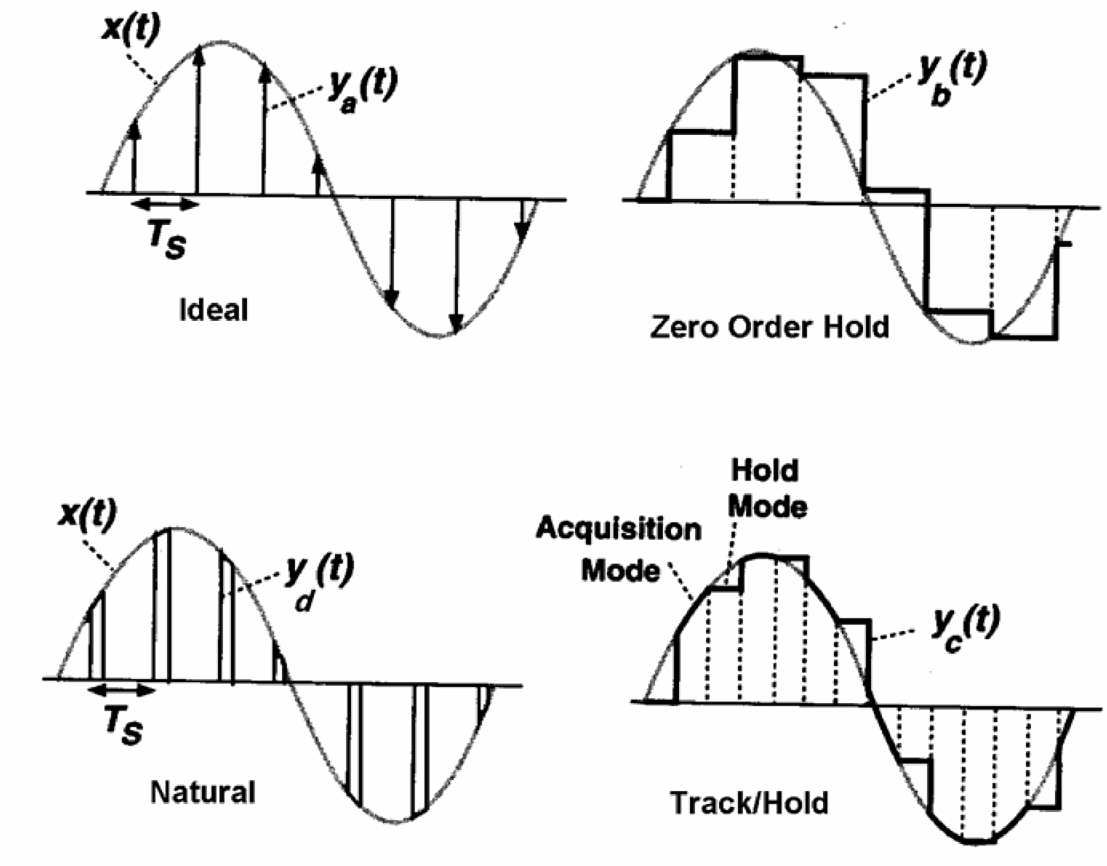


Sampling in practice

Key points

- Define the reconstruction formula
- Explain the role of the sinc function in reconstruction

Impulse sampling is not realistic

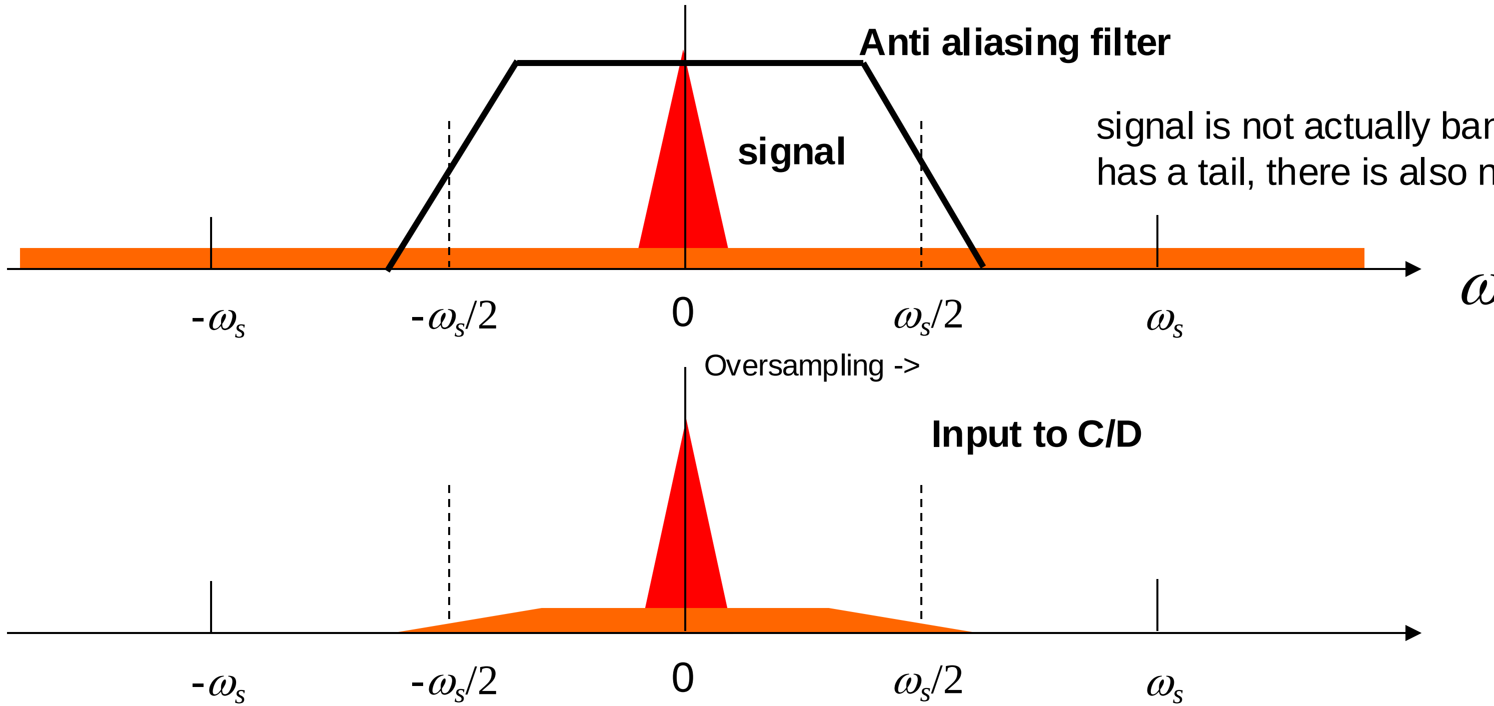


- Impulse Sampling (Theoretical – not implemented in practice)
- Natural Sampling (Theoretical - multiplier is a switch)
- Zero-order hold Sampling (Ideal Sample/Hold - instantaneous acquisition time is impractical)
- Track/Hold (Real Sample/Hold – Result is sampled and stored in a memory element)

Real device is an analog-to-digital converter ADC

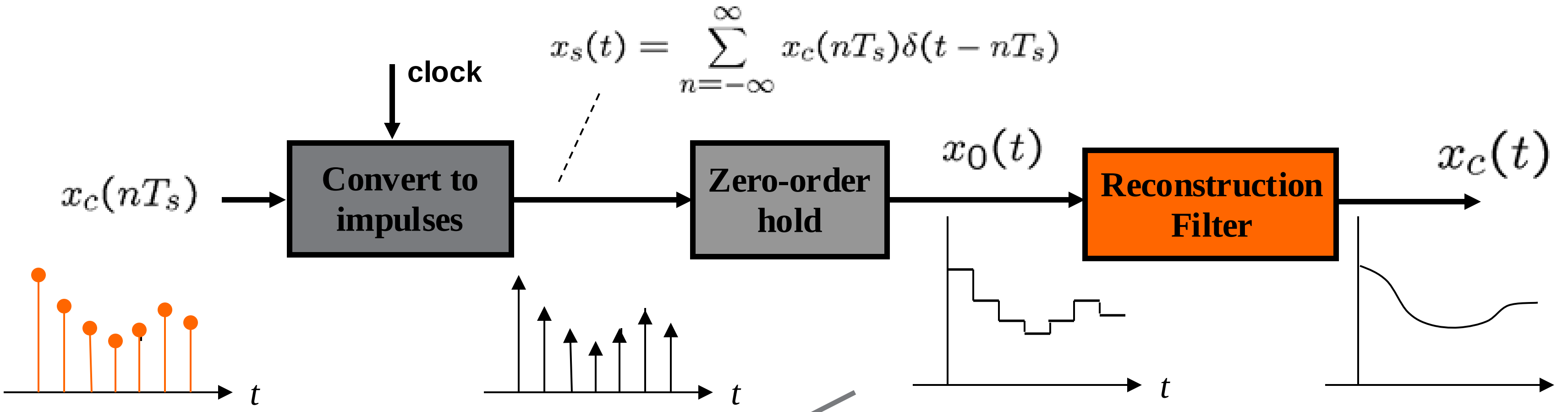
From notes by N. Prelcic, University of Vigo

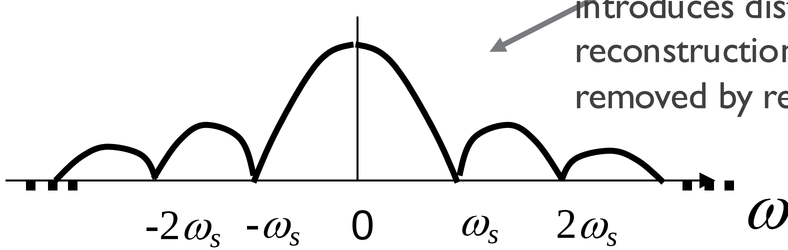
Oversampling is often used to avoid aliasing



Oversampling means sampling at a rate higher than Nyquist

Practical D/C model

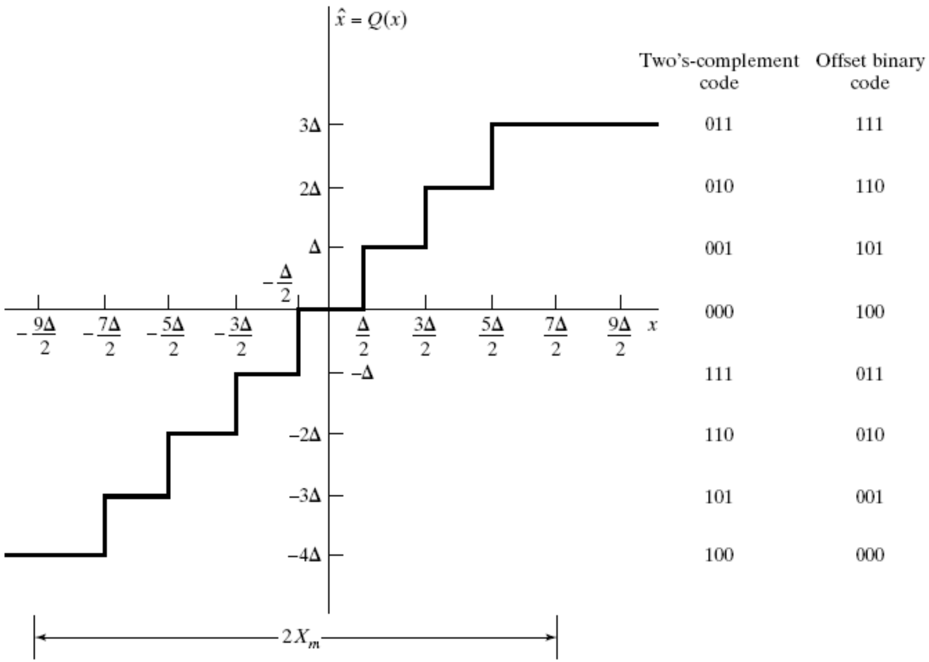


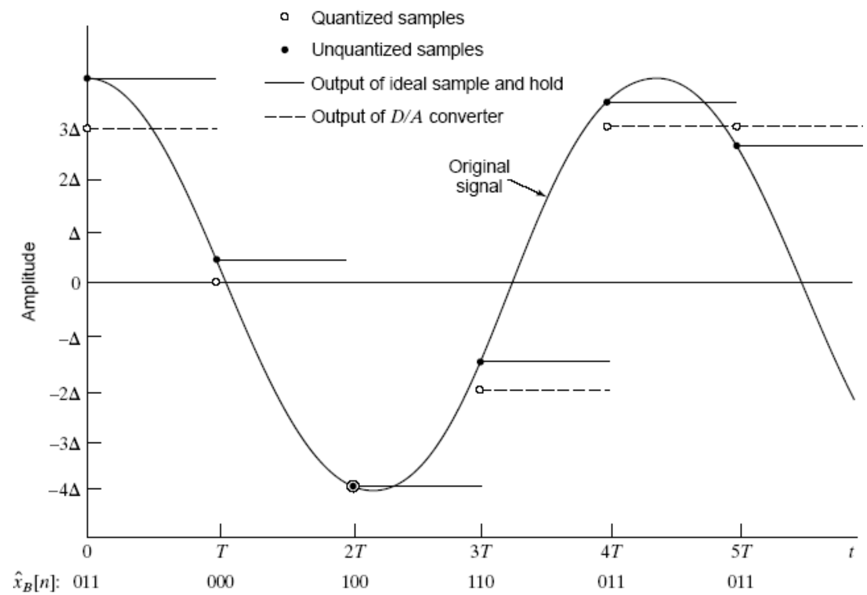


introduces distortion in reconstruction, which is removed by reconstruction

Real device is an digital-to-analog converter DAC

ADCs and DACs use quantized signals





N = total number of bits (including sign bit)
Full Scale level = FS
Quantization step Δ = weight of the LSB = (2·FS) / 2^N
2^N quantized levels, from –FS to (FS-LSB)
Weight of the MSB = FS/2

Thoughts on practical implications

- Realistic analog-to-digital converters and digital-to-analog converters approximate their ideal counterparts

- Oversampling – sampling at rates above Nyquist – relaxes some of the requirements of both sampling and reconstruction

- Developing practical converters is a major component of mixed-signal integrated circuit design