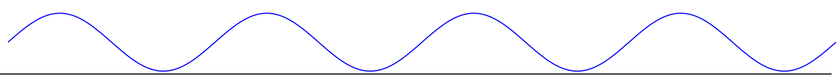
UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2019
Linear Systems Fundamentals

SOLUTIONS TO MIDTERM EXAM

You are allowed two 2-sided sheet of notes.

No books, no other notes, no electronics.

PRINT YOUR NAME J. B. Joseph Fourier

SIGNATURE 

Your signature confirms that you have completed this exam on your own and in accordance with the ECE 101 honor code.

Student ID Number Transform Student ID Number 101

Problem	Weight	Score
1	20 pts	20
2	30 pts	30
3	20 pts	20
4	30 pts	30
Total	100 pts	100

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Tables 3.1 and 3.2 from the textbook are attached to the exam.

Good luck!

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Problem 1 (20 pts)

(a) (8 pts)

Let $x(t) = 2^{-t}\,(u(t+1) - u(t-1))$. Define $y(t) = x((-2t) + 1)$.

To obtain $y(t)$ from $x(t)$ you can

First shift $x(t)$ by $\underline{\quad -1 \quad}$

Then scale the result by $\underline{\quad -2 \quad}$

or

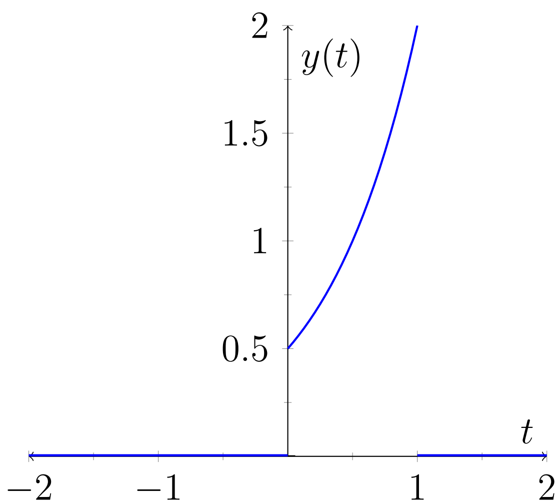
First scale $x(t)$ by $\underline{\quad -2 \quad}$

Then shift the result by $\underline{\quad 1/2 \quad}$

(b) (6 pts)

Let $x(t)$ and $y(t)$ be as in part (a) above.

Sketch precisely $y(t)$.



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Problem 1 (cont.)

(c) (6 pts)

Let $x(t)$ and $y(t)$ be as in part (a) above.

The signal $x(t)$ can be written as $x(t) = y(ct - d)$. Find c and d.

Confirm mathematically that your answer is correct.

From the second part of part (a), we see that to recover $x(t)$, we can shift $y(t)$ by $-1/2$ and scale the result by $-1/2$.

So,

$$x(t) = y((-{}^1\!/\!{}_2)t - (-{}^1\!/\!{}_2)) = y((-{}^1\!/\!{}_2)t + {}^1\!/\!{}_2)$$

To confirm mathematically, we evaluate $y((-{}^1\!/\!{}_2)t + {}^1\!/\!{}_2)$.

$$\begin{aligned} y((-{}^1\!/\!{}_2)t + {}^1\!/\!{}_2) &= x((-2)((-{}^1\!/\!{}_2)t + {}^1\!/\!{}_2) + 1) \\ &= x(t - 1 + 1) \\ &= x(t). \end{aligned}$$

$c =$ $-1/2$

$d =$ $-1/2$

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Problem 2 (30 pts)

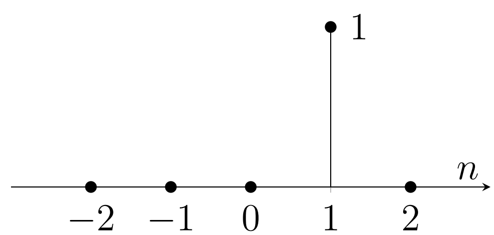
(a) (5 pts)

Consider the discrete-time system S_1 defined by

$$y[n] = \begin{cases} x[n-1], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1. \end{cases}$$

Determine and precisely sketch the impulse response $h[n]$.

$$h[n] = \begin{cases} \delta[n-1], & n \geq 1 \\ 0, & n = 0 \\ \delta[n], & n \leq -1. \end{cases} \qquad \text{That is, } h[n] = \delta[n-1].$$



(b) (5 pts)

Give the difference equation for the linear, time-invariant (LTI) system S_2 with the impulse response $h[n]$ you found in part (a).

Describe the action of the system S_2 in words.

The LTI system is defined by:

$$y[n] = x[n] * h[n] = x[n] * \delta[n-1] = x[n-1].$$

So, the difference equation is:

$$y[n] = x[n-1].$$

This is the system that shifts the input to the right by 1.

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Problem 2 (cont.)

(c) (8 pts)

Is the system S_1 of part (a) an LTI system?

More specifically, is S_1 linear? Is S_1 time-invariant?

Indicate below whether the system satisfies each listed property.

Justify your answers with specific reference to the system S_1.

True	False	
☐	☒	LTI. Since S_1 clearly does not simply shift every input signal to the right by 1, it is not equal to S_2. So it is not LTI.
☒	☐	Linear. Additivity and scalability both follow immediately from the definition of S_1. So S_1 is linear.
☐	☒	Time-invariant Let $n_0 = 1$. If $z[n] = y[n - n_0] = y[n-1]$, then $z[0] = x[-1]$. Let $x_1[n] = x[n - n_0] = x[n-1]$, with output $y_1[n]$. Then, $y_1[0] = 0$. If $x[-1] \neq 0$, then $z[0] \neq y_1[0]$. This means that S_1 is not time-invariant.

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Problem 2 (cont.)

(d) (12 pts)

Indicate below whether the system S_1 of part (a) satisfies each listed property.

Justify your answers with specific reference to the system S_1.

True	False	
☐	☒	Memoryless. Since $y[1] = x[0]$, the system in not memoryless.
☒	☐	Invertible (if so, give the inverse system) The inverse system producing $x[n]$ from $y[n]$ is $x[n] = \begin{cases} y[n+1], & n \geq 0 \\ y[n], & n \leq -1. \end{cases}$
☒	☐	Causal Note that $y[n]$ equals $x[n-1], x[n]$, or 0 for all n.
☒	☐	Stable Note that $\lvert y[n]\rvert \leq \max\{x[n-1], 0, x[n]\}$ for all n. If $\lvert x[n]\rvert \leq B$ for all n, then $\lvert y[n]\rvert \leq B$ for all n. So, bounded input produces bounded output.

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Problem 3 (20 pts)

(a) (6 pts)

Consider the continuous-time LTI system S defined by

$$y(t) = x(t) + \frac{1}{2}x(t-1).$$

Determine the impulse response $h(t)$ of the system.

By substituting $x(t) = \delta(t)$, we get

$$h(t) = \delta(t) + \frac{1}{2}\delta(t-1).$$

(b) (6 pts)

Determine the system function and frequency response of the system in part (a). **Justify your answers.**

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} \delta(t)e^{-st}dt + \int_{-\infty}^{\infty} \frac{1}{2}\delta(t-1)e^{-st}dt \\ &= 1 + \frac{1}{2}e^{-s}. \end{aligned}$$

Substituting $s = j\omega$ gives the frequency response:

$$H(j\omega) = 1 + \frac{1}{2}e^{-j\omega}.$$

$H(s) =$ $\underline{\quad 1 + \frac{1}{2}e^{-s} \quad}$

$H(j\omega) =$ $\underline{\quad 1 + \frac{1}{2}e^{-j\omega} \quad}$

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Problem 3 (cont.)

(c) (8 points)

Let $x(t) = \cos(\pi t)$ be the input to the system in part (a).

Determine the output $y(t)$. Express it in the form $y(t) = A\cos(\omega_0 t + \theta)$.

Justify your answer. Using Euler's relation:

$$\cos(\pi t) = \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t}.$$

Note that $H(j\pi) = H(-j\pi) = 1 + (-\frac{1}{2}) = \frac{1}{2}$.

From the eigenfunction property, we see that

$$\begin{aligned} y(t) &= \frac{1}{2}\left(\frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t}\right) \\ &= \frac{1}{2}\cos(\pi t). \end{aligned}$$

$A =$ $\frac{1}{2}$

$\omega_0 =$ π

$\theta =$ 0

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Problem 4 (30 pts)

(a) (16 points)

Let $x[n] = \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{4}n)$. The signal has fundamental period $N = 8$.
Is $x[n]$ even, odd, or neither?
Determine the Fourier series of $x[n]$.

Since $\cos(\frac{\pi}{2}n)$ is even and $\sin(\frac{\pi}{4}n)$ is odd, the product is odd.

Using Euler's relation, we get

$$\begin{aligned} x[n] &= \left(\frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n}\right)\left(\frac{1}{2j}e^{j\frac{\pi}{4}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}n}\right) \\ &= \frac{1}{4j}\left(e^{j\frac{3\pi}{4}n} - e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n}\right) \end{aligned}$$

The fundamental frequency is $\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$.
We can read off the non-zero Fourier series coefficients:
$a_3 = \frac{1}{4j} = -\frac{j}{4}, a_1 = -\frac{1}{4j} = \frac{j}{4}, a_{-1} = a_7 = \frac{1}{4j} = -\frac{j}{4}, a_{-3} = a_5 = -\frac{1}{4j} = \frac{j}{4}$,
where we have used the periodicity (with period 8) of the coefficients.
All other coefficients are 0.

$x[n]$ is even ______ odd ___X___ neither ______

$a_0 =$ ___0___ $a_4 =$ ___0___

$a_1 =$ ___$j/4$___ $a_5 =$ ___$j/4$___

$a_2 =$ ___0___ $a_6 =$ ___0___

$a_3 =$ ___$-j/4$___ $a_7 =$ ___$-j/4$___

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Problem 4 (30 pts)

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Let $x[n] = \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{4}n)$. The signal has fundamental period $N = 8$.

Is $x[n]$ even, odd, or neither?

Determine the Fourier series of $x[n]$.

Since $\cos(\frac{\pi}{2}n)$ is even and $\sin(\frac{\pi}{4}n)$ is odd, the product is odd.

We can also find the Fourier series coefficients of $x[n]$ directly.

By direct substitution, we find:

$x[0]{=}0, x[1]{=}0, x[2]{=}{-1}, x[3]{=}0, x[4]{=}0, x[5]{=}0, x[6]{=}1, x[7]{=}0.$

The analysis equation says

$$a_k = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-jk\frac{\pi}{4}n}.$$

Substituting in the values of $x[n]$ yields:

$$\begin{aligned} a_k &= \frac{1}{8}\left(-e^{-jk\frac{\pi}{2}} + e^{-jk\frac{3\pi}{2}}\right) \\ &= \frac{1}{8}\left(-e^{-jk\frac{\pi}{2}} + e^{jk\frac{\pi}{2}}\right) \\ &= \frac{1}{8}2j\sin(\frac{\pi}{2}k) \\ &= \frac{j}{4}\sin(\frac{\pi}{2}k) \end{aligned}$$

This agrees with the solution obtained by the other approach.

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Problem 4 (cont.)

(b) (7 points)

Let $x[n]$ be the signal of part (a). Determine $\sum_{n=0}^{7} x[n]$.

How does this sum relate to the Fourier series coefficient a_0?

From the analysis equation: $a_0 = \frac{1}{8}\sum_{n=0}^{7} x[n]$.

So, using the result of part (a), $\sum_{n=0}^{7} x[n] = 8a_0 = 0$.

Alternatively, note that $x[n]$ is real and odd, so the Fourier series are purely imaginary and odd. (See Table 3.2.)

That means $a_k = -a_{-k}$, for all k.

In particular, for $k = 0$, we get $a_0 = -a_0$, implying $a_0 = 0$.

(c) (7 points)

Let $y[n] = x[n-4]$, where $x[n]$ is the signal of part (a).

Is $y[n]$ even, odd, or neither?

Determine the relationship between the Fourier series b_k of $y[n]$ and the Fourier series a_k of $x[n]$.

From the Time Shifting property, $b_k = e^{-jk\frac{\pi}{4}4}a_k = e^{-jk\pi}a_k = (-1)^k a_k$.

The Fourier series are still purely imaginary and odd, so $y[n]$ is odd.

$y[n]$ is even ______ odd ___X___ neither ______

$b_k =$ ___$(-1)^k a_k$___

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