Lecture 4

Exponential, sinusoids, complex exponentials, and the delta function

Preview of today's lecture

- CT sinusoids and exponentials
  - Determine the key parameters of a complex sinusoid and exponential
  - Sketch a complex exponential based on its form

- Unit-impulse function also known as the Dirac delta function
  - Explain the properties of delta unit impulse function
  - Exploit the sifting property to simplify expressions with deltas
  - Exploit the integration property to simplify expressions with deltas

Complex exponential signals

- General form of complex exponential

$$x(t) = Ce^{at}$$

- Simplifying with $C = |C|e^{j\theta}$ and $a = r + j\omega_0$

$$x(t) = |C|e^{j\theta}e^{rt+j\omega_0 t}$$

$$= |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

- Complex sinusoid is the special case $r = 0$

$$x(t) = |C|e^{j(\omega_0 t+\theta)}$$

Example

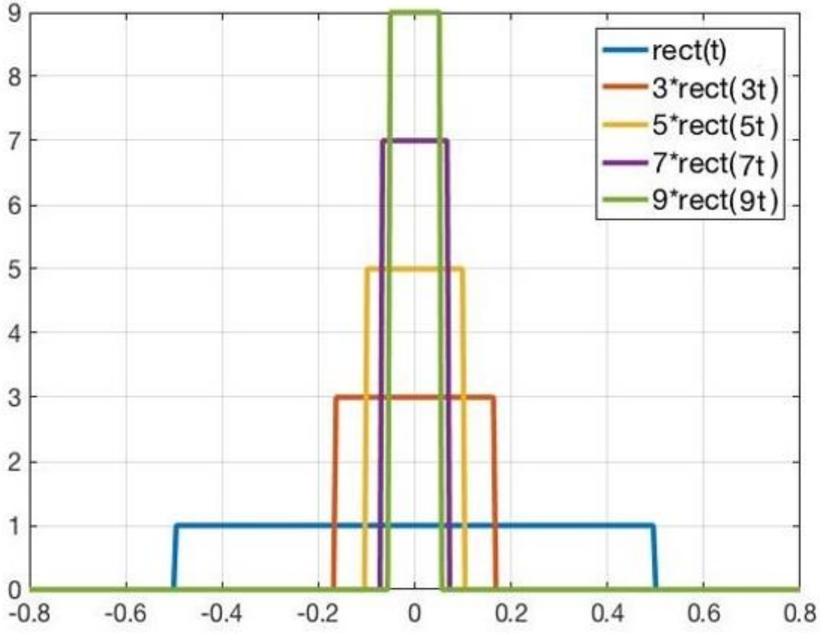
- Suppose $$C = 0.25\, e^{j0.2\pi}$$ $$a = 0.5 + j2\pi 1000$$

- Determine

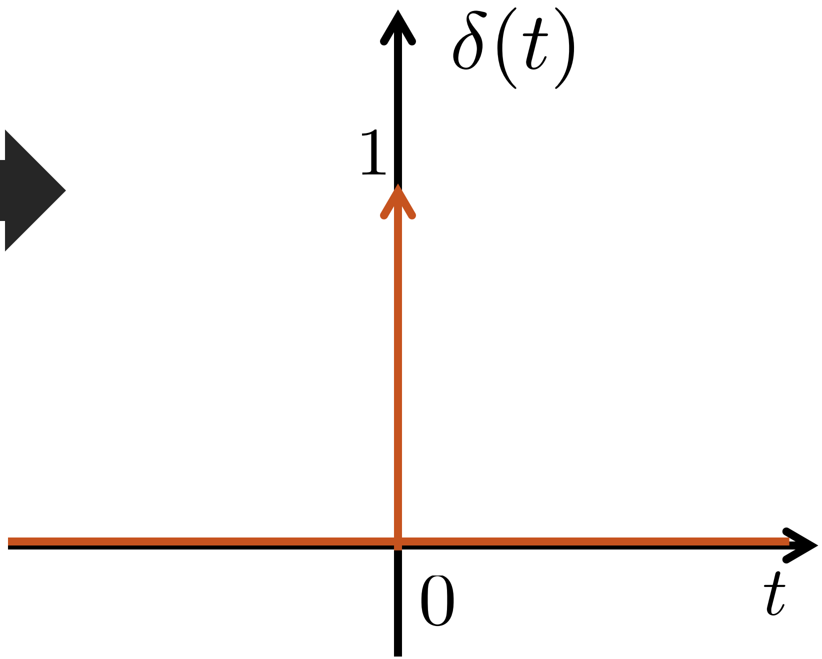
$$x(t)$$

$$\mathrm{Re}\{x(t)\}$$

Delta function



Dirac delta or unit-impulse function



Delta function as the limit of a sequence of every narrowing unit energy rectangles

Summary of the delta function

- Sifting with deltas pulls out the signal value but leaves the delta

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

- Integrating with deltas eliminates the delta and gives a value

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

- Other properties

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \delta(at) = \frac{1}{|a|}\delta(t)$$

Examples

- Simplify the following

$$(3r^2 + 2r + 1)\delta(r - 1)$$

$$\int_{-\infty}^{\infty} (3r^2 + 2r + 1)\delta(r - 1)dr$$

$$3r^2\delta(2r)$$

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

UC San Diego

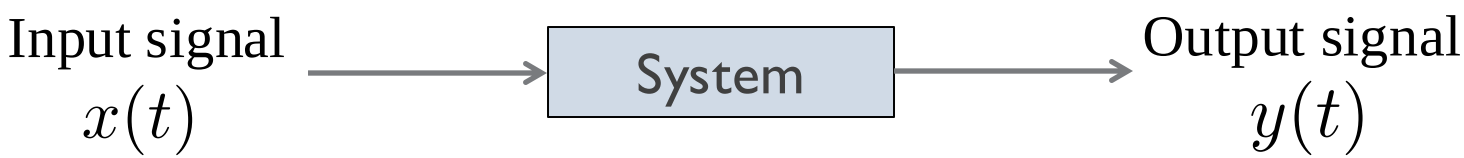
	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, Dirac Delta	1.3 - 1.4		
4	1/16	LTI in time	Systems, linearity, time invariance	1.6.5, 1.6.6	HW3	HW2
5	1/21	LTI in time	Impulse response and convolution	2.2		
6	1/23	LTI in time	Convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Introduction to Systems

Learning objectives

- Describe the output of the systems in terms of their inputs
- Give examples of continuous-time systems

Continuous-time systems

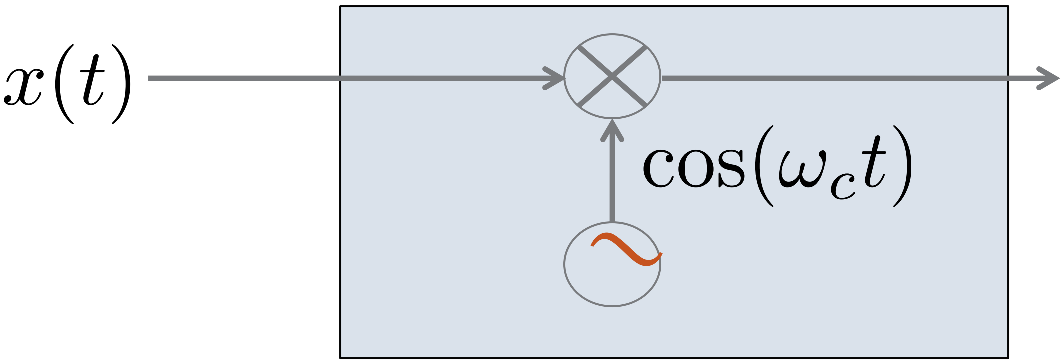


Amplifier (or all-pass amplifier)

$$x(t) \longrightarrow \boxed{A} \longrightarrow y(t) = Ax(t)$$

- A is the gain
- Passes all input frequencies equally (makes more sense w/ Fourier)

Amplitude modulation



$$y(t) = x(t)\cos(\omega_c t)$$

Information "rides" the carrier

Amplitude Modulation transmitter (AM radio)

- At the receiver:

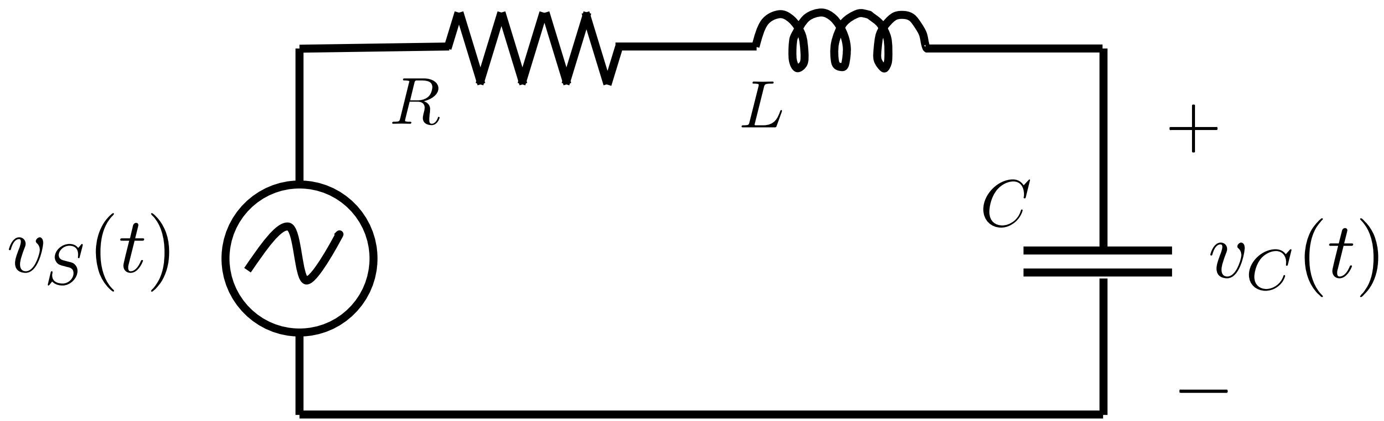
$$\begin{aligned}(x(t)\cos(\omega_c t))\cos(\omega_c t) &= \cos^2(\omega_c t)x(t) \\ &= \frac{1}{2}(1 + \cancel{\cos(2\omega_c t)})x(t) \\ &\longrightarrow x(t)\end{aligned}$$

Filter out high frequencies

Note: The filter operation only works for certain bandlimited signals x(t), not true in general (we will cover concepts of filtering and bandlimited in subsequent lectures)

RLC circuit

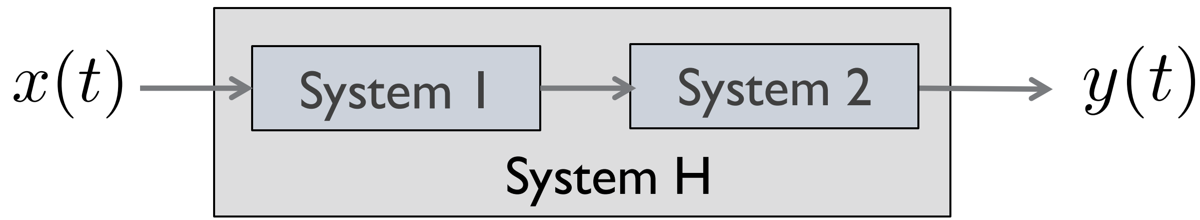
Example RLC circuit



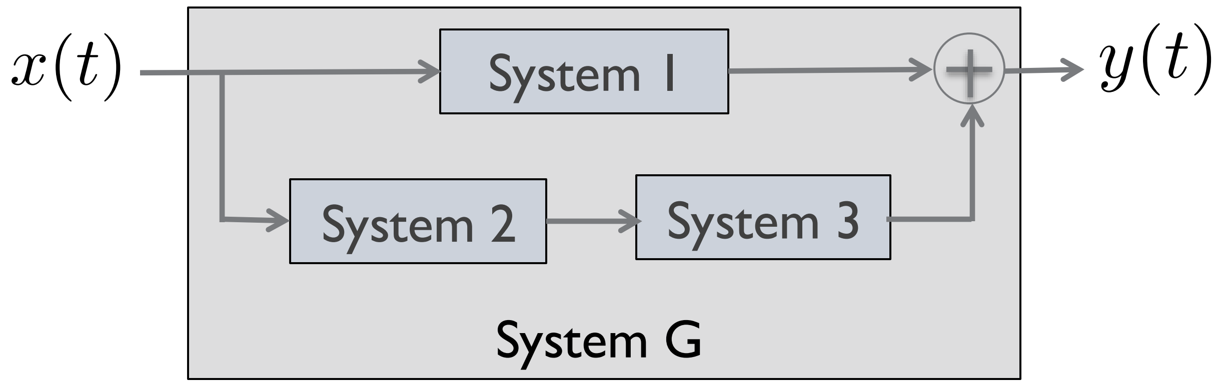
$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

Inter-connected systems – "systems-of-systems"

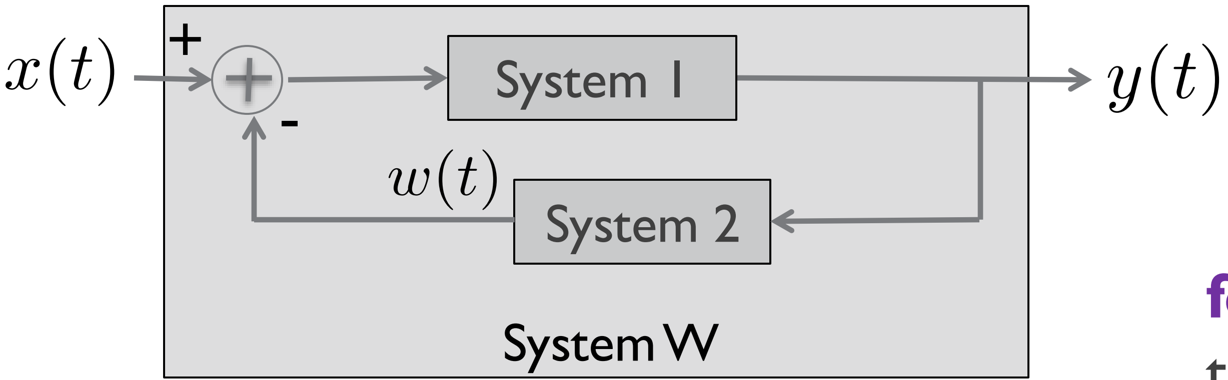
Serial



Parallel



Interconnected systems with feedback



feedback is when the output goes back into the input

Feedback is used in control systems

System introduction in summary

- A system is a functional unit that relates an input signal to an output signal
- Systems can be described in many ways including mathematically or via a block diagram
- Continuous-time systems have continuous inputs and outputs

Time invariance

Learning objectives

- Determine if a system is time invariant or time varying

Time invariant (TI)

A system is time invariant if behaves in the same way regardless of the current time

Formally: consider the system

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If

$$x(t - t_o) \longrightarrow \boxed{\text{System}} \longrightarrow y(t - t_o)$$

For all t_0 then the system is time invariant otherwise it is time varying.

How to check if the system is time-invariant?

- Method #1 (direct approach)
  - Consider the system

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

  - Apply the shifted input $x_2(t) = x_1(t - t_0)$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

  - Does the $y_2(t) = y_1(t - t_0)$? If yes, the system is TI

How to check if the system is time-invariant?

- Method #2 (counter example)

  If we suspect the system is time variant, find an example where time invariance fails "a counterexample"

  - This method is often quicker

- Note:
  - If you can not find a counter example, then you have to use Method 1
  - The counter example is just a simple way to **disprove** TI

- Hint: Usually, but not always, if the output includes any function of time other than x(t), it is time-varying

Time invariance example 1

System described by $y(t) = 3tx(t-3)$

$$y_1(t) = 3tx_1(t-3)$$ Output for generic input $x_1(t)$

$$y_2(t) = 3tx_2(t-3)$$ Output for generic input $x_2(t)$

Now, let: $x_2(t) = x_1(t-t_0)$

$$y_2(t) = 3tx_1(t-t_0-3)$$

$$y_1(t-t_0) = 3(t-t_0)x_1(t-t_0-3)$$

As $y_2(t)$ does not match $y_1(t-t_0)$ $\longrightarrow$ **time-variant system**

Time invariance example 2

System described by $y(t) = x(t)x(t-1)$

$$y_1(t) = x_1(t)x_1(t-1)$$

$$y_2(t) = x_2(t)x_2(t-1)$$

Let $x_2(t) = x_1(t - t_0)$

$$y_2(t) = x_1(t-t_0)x_1(t-t_0-1)$$

$$= y_1(t-t_0)$$

The system is **time invariant (TI)**

Time invariance example 3

System with AM modulation $y(t) = x(t)\cos(\omega_c t)$

Counterexample, let $x_1(t) = \delta(t), x_2(t) = \delta(t - \frac{\pi}{2\omega_c})$

$$y_1(t) = \delta(t)$$

$$y_2(t) = \delta\left(t - \frac{\pi}{2\omega_c}\right)\cos(\omega_c t) = 0$$

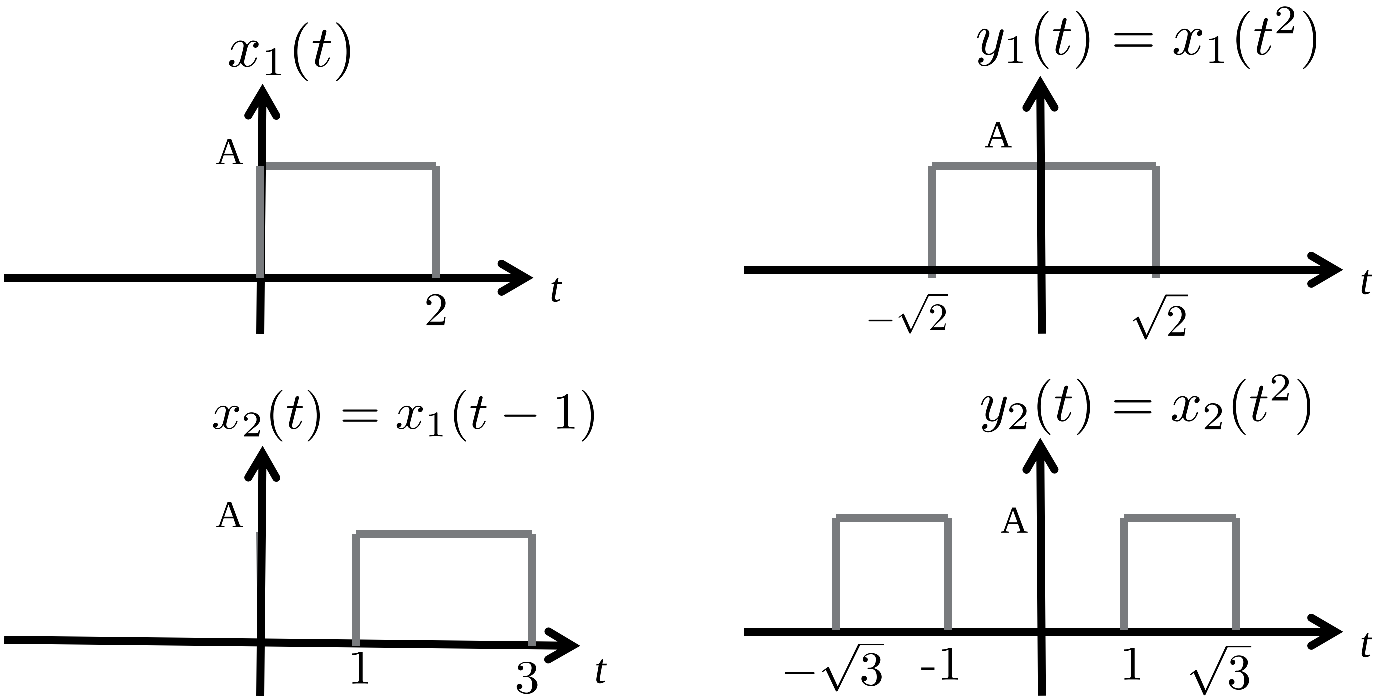
$$y_2(t) \neq y_1\left(t - \frac{\pi}{2\omega_c}\right)$$

$$= \delta\left(t - \frac{\pi}{2\omega_c}\right)$$

Time-variant

Time invariance example 4

$$y(t) = x(t^2)$$



Time-variant

(hint didn't work, Method 1 also tricky to see)

Time invariance summary

- A system is time invariant if behaves in the same way regardless of the current time
- Time invariant systems are much easier to design and analyze compared to time varying systems
- The main way to check time invariance is to shift the input and see if the output is always shifted
- A counter example is sufficient to show a system is not time invariant

Linearity

Learning objectives

- Determine if a system is linear or nonlinear
- Understand sub-properties of superposition and scaling

Scaling the input

If a system obeys the scaling property, then scaled inputs lead to scaled outputs

Consider the following system:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If the scaling property is satisfied then for any scalar value A

$$Ax(t) \longrightarrow \boxed{\text{System}} \longrightarrow Ay(t)$$

Superposition property

If a system obeys the superposition property, then system acts in the same way onto each system

If for two different inputs:

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

then superpositon holds if for input $x(t) = x_1(t) + x_2(t)$ then:

$$x_1(t) + x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t) + y_2(t)$$

Linear systems

A system is linear if scaling is preserved and superposition holds

Consider two different inputs and outputs

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

If the following holds for any scalar A and B then the system is linear

$$Ax_1(t) + Bx_2(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow Ay_1(t) + By_2(t)$$

How to check if the system linear?

- Direction approach
  - Check that scaling holds $Ax(t) \Rightarrow Ay(t)$
  - Check the superposition holds $x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)$
- Slightly faster direct approach
  - Check that $Ax_1(t) + Bx_2(t) \Rightarrow Ay_1(t) + By_2(t)$
- Find a counter example
  - One that may be useful (from the scaling property)

$$0 \longrightarrow \boxed{\text{System}} \longrightarrow 0$$

  - If a system generates a non-zero output to a signal that is zero for all time then it is non-linear

Linearity example I

$$y(t) = x(t)x(t-1)$$

Let us check if the scaling property holds $x_1(t) = Ax(t)$

$$\begin{aligned} y_1(t) &= x_1(t)Ax_1(t-1) \\ &= Ax(t)Ax(t-1) \\ &= A^2x(t)x(t-1) \\ &\neq Ay(t) \end{aligned}$$

Scaling fails → Nonlinear

Linearity example I – alternative solution

Consider inputs

$$y(t) = x(t)x(t-1)$$

$$x_1(t) \to y_1(t) = x_1(t)x_1(t-1)$$

$$x_2(t) \to y_2(t) = x_2(t)x_2(t-1)$$

$$x_3(t) = Ax_1(t) + Bx_2(t)$$

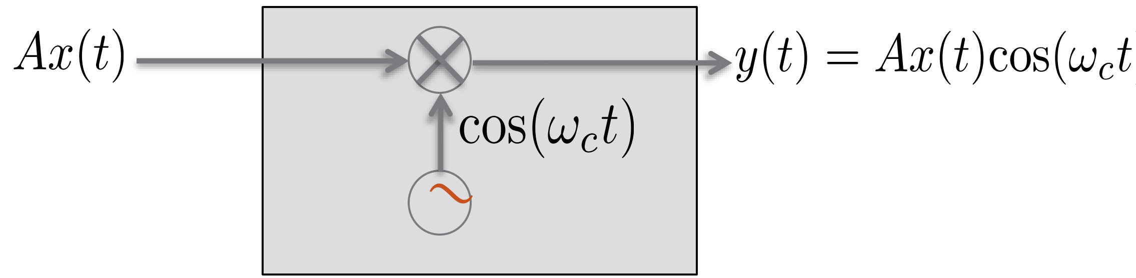
Nonlinear

Note that

$$\begin{aligned} y_3(t) &= x_3(t)x_3(t-1) \\ &= (Ax_1(t) + Bx_2(t))\,(Ax_1(t-1) + Bx_2(t-1)) \\ &= A^2x_1(t) + B^2x_2(t) + ABx_1(t)x_2(t-1) + ABx_1(t-1)x_2(t) \\ &\neq Ay_1(t) + By_t(t) = Ax_1(t)x_1(t-1) + Bx_2(t)x_2(t-1) \end{aligned}$$

Linearity example 2

System with AM modulation



Scaling is preserved

What about superposition?

$$
\begin{aligned}
y_1(t) &= x_1(t)\cos\omega_c t \\
y_2(t) &= x_2(t)\cos\omega_c t
\end{aligned}
$$

$$
\begin{aligned}
x(t) = x_1(t) + x_2(t) \longrightarrow y(t) &= (x_1(t) + x_2(t))\cos\omega_c t \\
&= x_1(t)\cos\omega_c t + x_2(t)\cos\omega_c t \\
&= y_1(t) + y_2(t)
\end{aligned}
$$

Linear system

Linearity example 3

- Consider the affine system

$$y[n] = 2x[n] + 1$$

- Suppose that

$$x_1[n] \rightarrow y_1[n]$$

$$x_2[n] \rightarrow y_2[n] \text{ where } x_2[n] = Ax_1[n]$$

- Now observe that

$$\begin{aligned} y_2[n] &= 2x_2[n] + 1 \\ &= 2Ax_1[n] + 1 \\ &\neq Ay_1[n] \end{aligned}$$

Scaling does not hold therefore is a **nonlinear system**

Linearity summary

- A system linear if *scaled inputs* lead to *scaled outputs* and the *sum of inputs* leads to a *sum of outputs* if the inputs were applied separately
- Linear systems are easier to design and analyze
- Many systems in practice are nonlinear but are designed to be as linear as possible, or are only used with inputs where they behave in a linear fashion
- Need to check both the scaling and superposition properties to prove that a system is linear

Reference example with details

- Is the following system linear? Time-invariant?

$$y(t) = t^2 x(t-1)$$

Reference example – checking time invariance

Consider the output to input $x_1(t)$

$$y_1(t) = t^2 x_1(t-1)$$

Define a new input

$$x_2(t) = x_1(t - t_0)$$

Compute the output

$$\begin{aligned} y_2(t) &= t^2 x_2(t-1) \\ &= t^2 x_1(t - 1 - t_0) \end{aligned}$$

Not time invariant!

Compare with a shifted version of the first output

$$y_1(t - t_0) = (t - t_0)^2 x_1(t - 1 - t_0) \neq y_2(t)$$

Reference example – checking linearity

Consider the inputs and outputs

$$x_1(t) \rightarrow y_1(t) = t^2 x_1(t-1)$$

$$x_2(t) \rightarrow y_2(t) = t^2 x_2(t-1)$$

Define a new input

$$x_3(t) = a x_1(t) + b x_2(t)$$

Compute the output

Linear!

$$\begin{aligned} y_3(t) &= t^2 x_3(t-1) \\ &= t^2 (a x_1(t-1) + b x_2(t-1)) \\ &= a y_1(t) + b y_2(t) \end{aligned}$$