ECE 101 – Linear Systems Fundamentals

Problem Set #4A Solutions

(send questions/comments to psiegel@ucsd.edu)

Problem 1. Note that $X(j\omega)$ is a complex number, so it can be written as $re^{j\theta}$, where r is the radius and θ is the angle, i.e., $r = |X(j\omega)|$ and $\theta = \angle X(j\omega)$:

$$X(j\omega) = |X(j\omega)|e^{\angle X(j\omega)}$$

We thus have an expression for $X(j\omega)$ for each signal, so to find the time domain signal, we simply plug $X(j\omega)$ into the Inverse Fourier Transform (eq. 4.8):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|e^{\angle X(j\omega)}e^{j\omega t}\,d\omega$$

For the first signal,

$$x_1(t) = \frac{1}{2\pi}\int_{-\omega_o}^{\omega_o} e^{-j\omega t_o}e^{j\omega t}\,d\omega = \frac{1}{2\pi}\frac{1}{j(t-t_o)}\left[e^{j\omega_o(t-t_0)} - e^{-j\omega_o(t-t_o)}\right] = \boxed{\frac{\sin(\omega_o(t-t_o))}{\pi(t-t_o)}}$$

In a similar fashion, the second signal is:

$$x_2(t) = \frac{1}{2\pi}\left(\int_{-\omega_o}^{0} e^{j\pi/2}e^{j\omega t}\,d\omega + \int_{0}^{\omega_o} e^{-j\pi/2}e^{j\omega t}\,d\omega\right) = \frac{1}{2\pi}\left(\int_{-\omega_o}^{0} je^{j\omega t}\,d\omega + \int_{0}^{\omega_o} -je^{j\omega t}\,d\omega\right)$$

$$= \frac{1}{2\pi}\frac{1}{t}\left[2-(e^{j\omega_o t} - e^{-j\omega_o t})\right] = \frac{1}{\pi t}(1-\cos(\omega_o t)) = \boxed{\frac{2\sin^2(\omega_o t/2)}{\pi t}}$$

where in the last step we used the trig identity of Problem 1.51(d) in the text.

So, we can see that simply changing the phase spectrum of the Fourier Transform can have a big effect on the corresponding time-domain signal: here, the first signal is a decaying sine function, while the second is a decaying sine-squared.

Problem 2.

(a) The Fourier Transform of x(t) can be solved by just plugging into the analysis equation (4.9):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{0}^{1} te^{-j\omega t}\,dt = \left(\frac{t}{-j\omega} - \frac{1}{(-j\omega)^2}\right)e^{-j\omega t}\Big|_0^1 = \boxed{\frac{1}{\omega^2}\left((1+j\omega)e^{-j\omega}-1\right)}$$

(b) For the remaining signals, we wish to use the properties of the CTFT. For $x_1(t)$, note that this triangular pulse is really two copies of x(t)—a shifted copy, and a shifted time reversal:

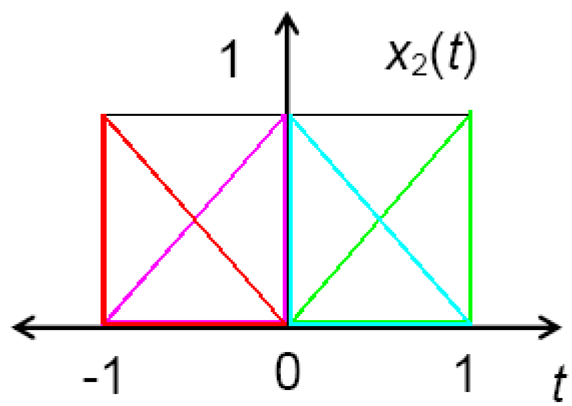
$$x_1(t) = x(t+1) + x(-t+1)$$

To find the Fourier Transform in terms of $X(j\omega)$, take the FT of both sides of the equation, and use the Time Shifting and Time Reversal properties (see Table 4.1) on the right hand side:

$$X_1(j\omega) = \boxed{e^{j\omega}X(j\omega) + e^{-j\omega}X(-j\omega)}$$

Note that when we took $X(-j\omega)$ (for the time-reversed signal), the negative sign gets applied to each ω in the expression, including the ω in the exponential, which is why the second term has $e^{-j\omega}$ instead of $e^{j\omega}$.

(c) Note that this rectangular pulse is really just the sum of four triangular pieces:



Thus, $x_2(t)$ can be represented as

$$x_2(t) = x(t) + x(-t) + x(t+1) + x(-t+1)$$

which has corresponding Fourier Transform (again, using the Time Shift and Time Reversal properties):

$$X_2(j\omega) = \boxed{X(j\omega) + X(-j\omega) + e^{j\omega}X(j\omega) + e^{-j\omega}X(-j\omega)}$$

(d) The shape of this signal is just the (scaled) rectangular pulse from part (c) minus the triangular pulse from part (b). Note that the signal is also time-scaled so that it ranges from [-2,2] instead of [-1,1]. So, we can represent $x_3(t)$ as:

$$x_3(t) = 2x_2(0.5t) - x_1(0.5t)$$
$$= [2x(0.5t) + 2x(-0.5t) + 2x(0.5t+1) + 2x(-0.5t+1)] - [x(0.5t+1) + x(-0.5t+1)]$$
$$= \mathbf{2x(0.5t) + 2x(-0.5t) + x(0.5t+1) + x(-0.5t+1)}$$

Applying the FT to both sides, and using the Time Shift, Time Reversal, and now also the Time Scaling properties, we have:

$$X_3(j\omega) = \boxed{4X(2j\omega) + 4X(-2j\omega) + 2e^{j2\omega}X(2j\omega) + 2e^{-j2\omega}X(-2j\omega)}$$

(e) This signal is the scaled triangle signal of part (b), minus a scaled rectangular pulse of part (c):

$$x_4(t) = (4/3)x_1(0.5t) - (1/3)x_2(0.5t)$$

Putting this in terms of x(t), we have:

$$x_4(t) = (4/3)[x(0.5t+1) + x(-0.5t+1)] - (1/3)[x(0.5t) + x(-0.5t) + x(0.5t+1) + x(-0.5t+1)]$$
$$= \mathbf{x(0.5t+1) + x(-0.5t+1) - (1/3)x(0.5t) - (1/3)x(-0.5t)}$$

Then $X_4(j\omega)$ is computed as in part (d), using the Time Shift, Time Reversal, and Time Scaling properties:

$$X_4(j\omega) = \boxed{2e^{j2\omega}X(2j\omega) + 2e^{-j2\omega}X(-2j\omega) - (2/3)X(2j\omega) - (2/3)X(-2j\omega)}$$

(f) This signal can be decomposed into a time-compressed rectangular pulse $x_2(t)$ (the middle portion) and two time-shifted copies of the original triangular signal x(t):

$$x_5(t) = x_2(2t) + x(t+1.5) + x(-t+1.5)$$
$$= \mathbf{[x(2t) + x(-2t) + x(2t+1) + x(-2t+1)] + x(t+1.5) + x(-t+1.5)}$$

Then the Fourier Transform is, using the Time Shifting, Time Reversal, and Time Scaling properties:

$$X_5(j\omega) = \boxed{0.5X(0.5j\omega) + 0.5X(-0.5j\omega) + 0.5e^{j0.5\omega}X(0.5j\omega) + 0.5e^{-j0.5\omega}X(-0.5j\omega) + e^{j1.5\omega}X(j\omega) + e^{-j1.5\omega}X(-j\omega)}$$

Problem 3. $y(t) = x_1(t)\cos(10\pi t)$, i.e., y(t) is the product of two signals, $x_1(t)$ and $\cos(10\pi t)$. So the Fourier Transform of the *product* of two signals in the time domain is $(1/2\pi)$ times the *convolution* of the two signals' FTs in the frequency domain. This, recall, is the Multiplication/Convolution property of the Fourier Transform.

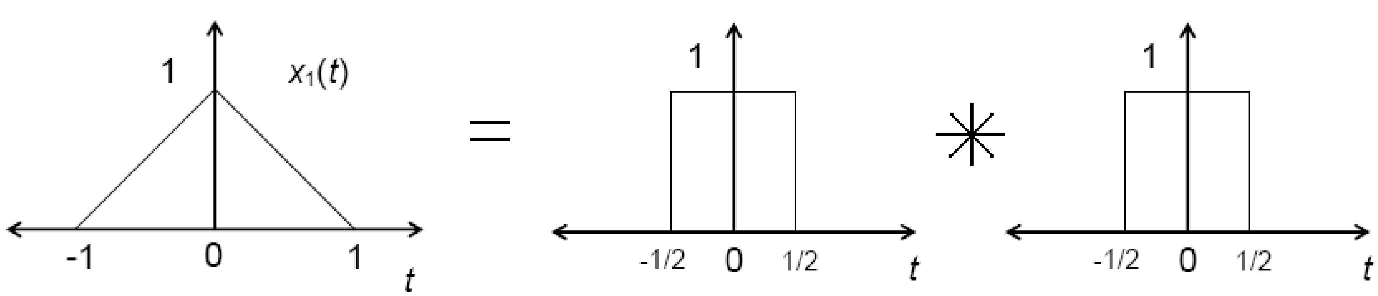
That is to say, if $X_1(j\omega)$ and $C(j\omega)$ are the FTs of $x_1(t)$ and $\cos(10\pi t)$, respectively, then

$$FT\{x_1(t)\cos(10\pi t)\} = (1/2\pi)[X_1(j\omega) * C(j\omega)]$$

Always remember, *multiplication in the time domain is convolution in the frequency domain, and vice-versa*.

So, to get the Fourier Transform of the product $x_1(t)\cos(10\pi t)$, we need the FTs of the two separate signals. $C(j\omega)$, the FT of $\cos(10\pi t)$, is $\boldsymbol{\pi[\delta(\omega - 10\pi) + \delta(\omega + 10\pi)]}$, as found in Table 4.2. $X_1(j\omega)$, the FT of $x_1(t)$, can be derived mathematically via Problem 2, but a perhaps easier way to do it is the following:

Note that $x_1(t)$, the triangular pulse signal, is just the convolution of two rectangular pulses:



Each rectangular pulse has Fourier Transform $\dfrac{\sin(\omega/2)}{\omega/2}$ = sinc(ω/2π) (see Example 4.4 and 4.5 in the text). So, the FT of the triangular pulse, which is the FT of the *convolution* of the two rectangular pulses, is the *product* of the rectangular pulse FTs, i.e.,

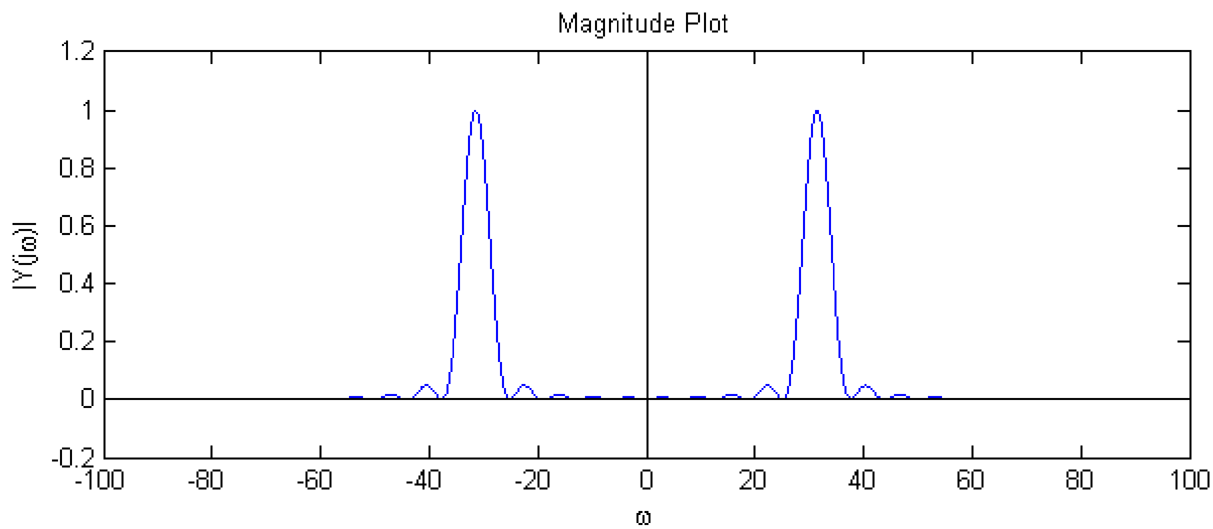
$$\text{FT}\{(\text{rectangular pulse}) * (\text{rectangular pulse})\} = \text{sinc}(\omega/2\pi) \times \text{sinc}(\omega/2\pi) = \text{sinc}^2(\omega/2\pi).$$

Hence, the Fourier Transform $X_1(j\omega)$ of the triangular pulse is $\mathbf{X_1(j\omega) = sinc^2(\omega/2\pi)} = \dfrac{\sin^2(\omega/2)}{(\omega/2)^2}$.

Now that we have $X_1(j\omega)$ and $C(j\omega)$, the overall FT of $x_1(t)\cos(10\pi t)$ is:

$$\text{FT}\{x_1(t)\cos(10\pi t)\} = (1/2\pi)[X_1(j\omega) * C(j\omega)] = (1/2\pi)[\text{sinc}^2(\omega/2\pi) * \pi(\delta(\omega - 10\pi) + \delta(\omega + 10\pi))]$$
$$= \boxed{\tfrac{1}{2}\,\mathbf{sinc^2((\omega - 10\pi)/2\pi)} + \tfrac{1}{2}\,\mathbf{sinc^2((\omega + 10\pi)/2\pi)}}$$

So we essentially just get the sum of two copies of the sinc-squared—one centered at 10π, the other centered at –10π. The magnitude plot is below:



Note that $\tfrac{1}{2}\,\text{sinc}^2((\omega - 10\pi)/2\pi) + \tfrac{1}{2}\,\text{sinc}^2((\omega + 10\pi)/2\pi)$ is entirely real-valued and non-negative. Thus, the magnitude spectrum captures all of $Y(j\omega)$, i.e., $Y(j\omega) = |Y(j\omega)|$. As a result, there is no phase, i.e., the phase spectrum is **0** for all ω.

Problem 4. (a) $y(t) = x(t) * h(t) = x(t) - x(t-2T)$. Thus, h(t) must be $\mathbf{\delta(t) - \delta(t-2T)}$. The FT of h(t) is then

$$H(j\omega) = \text{FT}\{\delta(t) - \delta(t-2T)\} = \mathbf{1 - e^{-j\omega 2T}}$$

where the exponential term is obtained by using the Time Shifting property on δ(t-2T).

You can also derive H(jω) by taking the Fourier Transform of both sides and dividing:

$$y(t) = x(t) * h(t) = x(t) - x(t-2T) \quad \leftrightarrow \quad Y(j\omega) = X(j\omega)H(j\omega) = X(j\omega) - e^{-j\omega 2T}X(j\omega)$$
$$\Rightarrow Y(j\omega)/X(j\omega) = H(j\omega) = 1 - e^{-j\omega 2T}$$

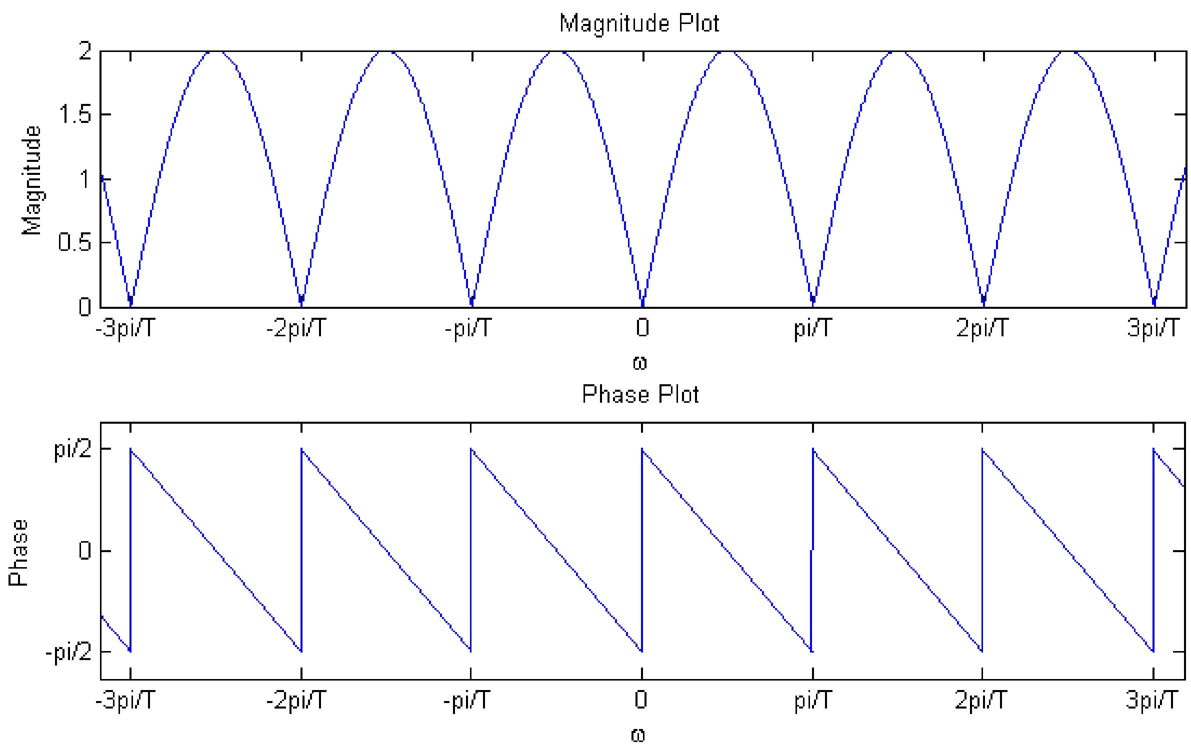
To find the magnitude and phase, note that

$$H(j\omega) = 1 - e^{-j\omega 2T} = e^{-j\omega T}(e^{j\omega T} - e^{-j\omega T}) = 2je^{-j\omega T}\sin(\omega T)$$

The magnitude is then $|H(j\omega)| = 2|\sin(\omega T)|$. For the phase, note that the sine function goes through a 180-degree phase shift every π radians (i.e., each time the argument ωT is a multiple of π). So, the phase is computed by:

$$\begin{aligned} \angle H(j\omega) &= \angle 2j + \angle e^{-j\omega T} + \angle \sin(\omega T) \\ &= \frac{\pi}{2} - \omega T + \pi\left\lfloor \frac{\omega T}{\pi} \right\rfloor \end{aligned}$$

The last term is using the floor function, and is saying that we should add π radians to the phase each time ω crosses a multiple of π/T. The magnitude and phase plots are below:



(b) Looking at the magnitude plot, the system provides maximum gain (i.e. magnitude) when

$$\boxed{\omega = \frac{1}{T}\left(\frac{\pi}{2} + \pi k\right),\ k \text{ an integer}}$$

i.e., when the phase is 0.

(c) Looking at the magnitude plot again, the system provides zero gain when

$$\boxed{\omega = \frac{\pi k}{T},\ k \text{ an integer}}$$

i.e., when the phase is maximal.