Problem 1: CT & DT signals

(a)   • **Impulse signals** are essential because they allow us to represent any continuous-time or discrete-time signal as a weighted sum or integral of shifted impulses (convolution with delta functions). Key equations are

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k], \qquad x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

   • **Complex exponentials** form the basis functions for Fourier series and Fourier transforms, which decompose signals into frequency components, providing a powerful analysis tool. For period signals $x[n]$ with fundamental period N (DT) or $x(t)$ with fundamental period T (CT), we have

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = \frac{2\pi}{N}$ (DT) or $\omega_0 = \frac{2\pi}{T}$ (CT).

(b) **True**. For convolution, we have the commutative and associative properties. Also, Convolution with $\delta(t-t_0)$ causes time shift by t_0. We have

$$\begin{aligned} \text{RHS} = x(t-4) * y(t) &= (x(t) * \delta(t-4)) * y(t) \\ &= x(t) * (\delta(t-4) * y(t)) \\ &= x(t) * y(t-4) = \text{LHS}. \end{aligned}$$

(c) **False**. For $y[n] = u[n-3] * x[n]$, we have

$$\begin{aligned} y[n] = u[n-3] * x[n] &= \sum_{k=-\infty}^{\infty} u[k-3]x[n-k] \\ &= \sum_{k=3}^{\infty} x[n-k]. \end{aligned}$$

Therefore, we have

$$y[0] = \sum_{k=3}^{\infty} x[-k] = \sum_{k=-\infty}^{-3} x[k].$$

(d) We write the DT signal as $x[n] = x_1[n] + x_2[n]$ where

$$x_1[n] = e^{j\frac{\pi}{6}n}, \qquad x_2[n] = \sum_{k=-\infty}^{\infty} (-1)^k \delta[n-5k]$$

   • $x_1[n]$ has a fundamental frequency $\frac{\pi}{6} = \frac{2\pi}{12}$ and thus the fundamental period $T_1 = 12$.
   • The fundamental period of $x_2[n] = \sum_{k=-\infty}^{\infty}(-1)^k\delta[n-5k]$ is $T_2 = 10$.
   • Therefore, the fundamental period of $x[n] = x_1[n] + x_2[n]$ is $\text{lcm}(T_1, T_2) = 60$. That is, the fundamental frequency is $\omega_0 = \frac{2\pi}{60} = \frac{\pi}{30}$.
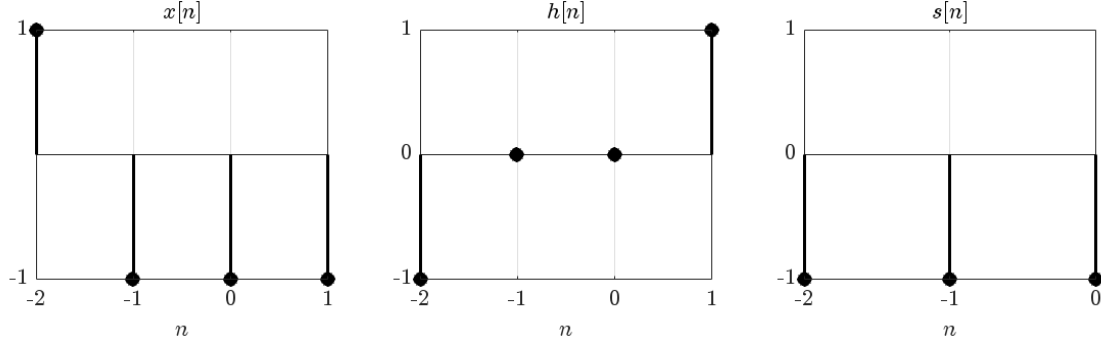
Fig. 1: Left: sketch for Problem 1(f); Middle: sketch for problem 2(a); Right: sketch for Problem 2(b).

(e) Using the same notation as in (d), the odd part of $x_2[n]$ is 0 since $x_2[n]$ is an even signal (this alternating polarity impulse train is even).

Therefore, the odd part of $x[n]$, $f[n] = \text{Odd}\{x[n]\}$, can be calculated as

$$f[n] = \frac{x_1[n] - x_1[-n]}{2} = \frac{e^{j\frac{\pi}{6}n} - e^{-j\frac{\pi}{6}n}}{2} = j\sin\left(\frac{\pi}{6}n\right).$$

(f) The signal can be simplified as follows

$$\begin{aligned} x[n] &= u[n+2] - 2u[n+1] + u[n-2] \\ &= (u[n+2] - u[n+1]) - (u[n+1] - u[n-2]) \\ &= \delta[n+2] - \delta[n+1] - \delta[n] - \delta[n-1]. \end{aligned}$$

A sketch shows impulses at $n = -2, n = -1, n = 0$, and $n = 1$ with values $1, -1, -1$, and -1, as shown in the left subfigure in Figure 1.

Problem 2: DT LTI systems

1) To find the impulse response $h[n]$, we set $x[n] = \delta[n]$ and the corresponding output is

$$h[n] = \delta[n-1] - \delta[n+2].$$

Sketch of $h[n]$: two impulses: one at $n = 1$ and another at $n = -2$, with amplitudes 1 and -1, respectively. The sketch is shown in the middle subfigure of Figure 1.

2) To determine the step response $s[n]$, set $x[n] = u[n]$ and the corresponding output is

$$s[n] = u[n-1] - u[n+2] = -\delta[n+2] - \delta[n+1] - \delta[n].$$

Sketch of $s[n]$: The step response is a sequence with values:
- $s[n] = 0$ for $n \leq -3$
- $s[n] = -1$ for $-2 \leq n \leq 0$
- $s[n] = 0$ for $n \geq 1$

The plot is shown in the right subfigure of Figure 1.

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3) To find the system function $H(z)$, we set $x[n] = z^n$ (the eigenfunction) and $y[n] = H(z)z^n$. The difference equation is then given by

$$H(z)z^n = z^{n-1} - z^{n+2},$$

and thus

$$H(z) = \frac{z^{n-1} - z^{n+2}}{z^n} = \frac{1}{z} - z^2.$$

Alternatively, we can use the formula below

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = z^{-1} - z^2.$$

4) To check if there exists a $z \neq 0$ such that $H(z) = 0$, we solve:

$$z^{-1} - z^2 = z^{-1}(1 - z^3) = 0.$$

Therefore, $z = 1$ makes $H(z) = 0$. So, $z = 1$ is a zero of $H(z)$.

5) (a) **Invertibility**: **False**. A system is invertible if and only if distinct system input signals produce distinct output signals.
From part 4), we know $H(1) = 0$, so the eigenvalue function property implies that the input $x[n] = 1^n = 1$ will produce $y[n] = H(1)x[n] = 0$. Indeed, we have

$$y[n] = x[n-1] - x[n+2] = 0, \qquad \text{if } x[n] = 1.$$

However, the inputs $x_2[n] = 0$ also produce the same zero output. In fact, any constant input $x[n] = c$ where c is a constant, will produce a zero output.

(b) **Stability**: **True**. The system is stable because

$$\sum_{k=-\infty}^{\infty} |h[k]| = 2 < \infty.$$

(c) **Causality**: **False**. The impulse response $h[n] = \delta[n-1] - \delta[n+2]$ has a component at $n = -2$, indicating that it is non-zero for negative n, which violates causality.
Alternatively, the value of $y[0] = x[-1] - x[+2]$ depends on the future input $x[2]$.

Problem 3: DTFS and DF Filter

1) (a) The fundamental frequency of $x[n]$, and therefore $y[n]$ is

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{2}.$$

The Time Shifting property (Table 3.2) says that the DTFS coefficients of $y[n] = x[n+1]$ are
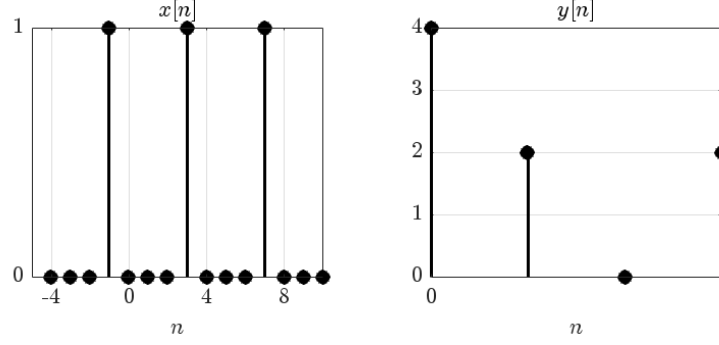
$$b_k = e^{jk\omega_0}a_k, \qquad \forall k$$

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Fig. 2: Left: sketch for Problem 3(1b); Right: sketch for Problem 3(3):

where a_k are the DTFS coefficients of $x[n]$. Therefore, we have

$$a_k e^{jk\frac{\pi}{2}} = b_k = (-1)^k.$$

Solving this equation will give us the values of a_k:

$$a_0 = 1, \quad a_1 = -e^{-j\frac{\pi}{2}} = j, \quad a_2 = e^{-j\pi} = -1, \quad a_3 = -e^{-j\frac{3\pi}{2}} = -j.$$

(b) By the synthesis equation,

$$x[n] = \sum_{k=0}^{3} a_k e^{jk\frac{\pi}{2}n} = a_0 + a_1 \left(e^{j\frac{\pi}{2}}\right)^n + a_2 \left(e^{j\pi}\right)^n + a_3 \left(e^{j\frac{3\pi}{2}}\right)^n$$
$$= 1 + j^{n+1} + (-1)^{n+1} + (-j)^{n+1}.$$

The values of $x[n]$ at $n = 0, 1, 2, 3$ are given by $0, 0, 0, 4$. Thus, we have

$$x[n] = 4 \sum_{k=-\infty}^{\infty} \delta[n - 4k + 1].$$

The plot is shown in the left subfigure of Figure 2.

2) Given the signal

$$x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right),$$

we can expand this product using trigonometric identities:

$$x[n] = \frac{1}{2}\left(\cos\left(\frac{3\pi}{4}n\right) + \cos\left(\frac{\pi}{4}n\right)\right)$$
$$= \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n} + e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n})$$
$$= \frac{1}{4}(e^{-j3\frac{2\pi}{8}n} + e^{-j\frac{2\pi}{8}n} + e^{j\frac{2\pi}{8}n} + e^{j3\frac{2\pi}{8}n})$$

- The fundamental period N is determined by the smallest N such that both $\frac{3\pi}{4}N$ and $\frac{\pi}{4}N$ are integer multiples of 2π. Thus, $N = 8$.

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- From the above equation of $x[n]$, we can read the DTFS coefficients

$$a_{-3} = \frac{1}{4}, \quad a_{-2} = 0, \quad a_{-1} = \frac{1}{4}, \quad a_0 = 0, \quad a_1 = \frac{1}{4}, \quad a_2 = 0, \quad a_3 = \frac{1}{4}, \quad a_4 = 0.$$

The Time Reversal property says that the DTFS coefficients of $x[-n]$ are a_{-k}.
- If $x[n]$ is even, then $x[n] = x[-n]$, so $a_k = a_{-k}$.

- If $x[n]$ is odd, then $x[n] = -x[-n]$, so $a_k = -a_{-k}$.

The above properties show that $x[n]$ is even but not odd.

3) Let $\omega_0 = \frac{2\pi}{4}$. The output $y[n]$ is obtained by filtering $x[n]$ through the system $H(e^{j\omega})$. Only the components of $x[n]$ within the passband $|\omega| \leq \frac{2\pi}{3}$ will remain.
Hence, the frequency components $k\omega_0$ at $k = 0, 1, 3$ will remain, and $k = 2$ will fall outside the passband. Therefore, the output is given by

$$y[n] = a_0 + a_1 e^{j\frac{2\pi}{4}n} + a_3 e^{j3\frac{2\pi}{4}n}$$
$$= 2 + e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}$$
$$= 2 + 2\cos\left(\frac{\pi}{2}n\right).$$

Sketch of $y[n]$: $y[0] = 4$, $y[1] = 2$, $y[2] = 0$, and $y[3] = 2$, as shown in the right subfigure of Figure 2.