Problem 1: Multiple Choices with Justification

1. **Answer**: (a)
   Justification: the output $y[n]$ is computed as

$$\begin{aligned} y[n] &= h_2[n] * h_1[n] * x[n] \\ &= h_1[n] * h_2[n] * x[n] && \text{Commutatitve property} \\ &= h_1[n] * (\alpha^n u[n] * (\delta[n] - \alpha\delta[n-1])) \\ &= h_1[n] * (\alpha^n u[n] - \alpha * \alpha^{n-1} u[n-1]) \\ &= h_1[n] * \alpha^n (u[n] - u[n-1]) \\ &= h_1[n] * \alpha^n \delta[n] \\ &= h_1[n] * \alpha^0 \delta[n] && \text{Sampling property} \\ &= h_1[n] = \beta^n \cos(\frac{\pi n}{4}), \end{aligned}$$

   where we have used the commutative and associative properties of convolution, $\delta[n] = u[n] - u[n-1]$, and sampling property.

2. **Answer**: (b)
   Justification: 3^n is an eigenfunction of any discrete-time LTI systems. Then, the output is

$$y[n] = H(z)3^n,$$

   where $z = 3$ and $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$. Therefore, we have

$$H(3) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]3^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n 3^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n = \frac{1}{1-\frac{1}{6}} = \frac{6}{5}.$$

   The output is thus given by $y[n] = \frac{6}{5} \times 3^n$.

3. **Answer**: (c)
   Justification: The fundamental frequency is $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$. Using the synthesis equation, we have

$$x[n] = \sum_{k=<n>} a_k e^{jk\omega_0 n} = \sum_{k=-1}^{1} a_k e^{jk\omega_0 n} = a_{-1}e^{-j\frac{2\pi}{3}n} + a_0 + a_1 e^{j\frac{2\pi}{3}n}.$$

   Since the cutoff frequency for the LTI system is $\frac{5\pi}{6}$, we have

$$\begin{aligned} b_{-1} &= H(-\frac{2\pi}{3})a_{-1} = a_{-1} = -j, \qquad b_0 = H(0)a_0 = 1, \\ b_1 &= H(\frac{2\pi}{3})a_{-1} = a_1 = j. \end{aligned}$$

   The output of this LTI system is given by

$$y[n] = b_{-1}e^{-j\frac{2\pi}{3}n} + b_0 + b_1 e^{j\frac{2\pi}{3}n} = 1 + j(e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n}) = 1 + j * 2j\sin\left(\frac{2\pi}{3}n\right) = 1 - 2\sin\left(\frac{2\pi}{3}n\right).$$

4. **Answer**: (a)
   Justification: The sampling frequency is $\omega_s = 2\pi f_s = 2\pi \times 10^4$ rad per second, and the sampling period is $T = \frac{1}{f_s} = 10^{-4}$ seconds.
   We have the following relationship

$$X_d(e^{j\Omega}) = X_p(j\omega)|_{\omega=\frac{\Omega}{T}}.$$

   Then, if $\Omega = \pi$, the corresponding value of ω is

$$\omega = \frac{\Omega}{T} = 10^4\pi.$$

Problem 2: CT Fourier Transform

1. By definition of CTFT, we have

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} (u(t+4) - u(t-2))e^{-j\omega t}dt \\
&= \int_{-4}^{2} e^{-j\omega t}dt \\
&= \frac{1}{-j\omega}e^{-j\omega t}|_{t=-4}^{2} = \frac{1}{-j\omega}(e^{-j\omega 2} - e^{j4\omega}) = \frac{1}{-j\omega}e^{j\omega}(e^{-j3\omega} - e^{j3\omega}) \\
&= e^{j\omega}\frac{2\sin(3\omega)}{\omega}.
\end{aligned}$$

   Alternative method: note that the signal $y(t) = x(t-1)$ is a rectangular pulse of amplitude 1 on the interval $[-3, 3]$. From Table 4.2, we have

$$Y(j\omega) = \frac{2\sin(3\omega)}{\omega}$$

   By the Time Shifting property in Table 4.1, we have

$$Y(j\omega) = e^{-j\omega}X(j\omega) \qquad \Rightarrow \qquad X(j\omega) = e^{j\omega}\frac{2\sin(3\omega)}{\omega}.$$

2. By the synthesis equation, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

   Therefore, we have

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega \qquad \Rightarrow \qquad \int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 2\pi.$$

3. By the Parseval's Relation for Aperiodic Signals in Table 4.2, we have

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

Therefore, we have

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt = 2\pi \int_{-4}^{2} 1 dt = 2\pi \times 6 = 12\pi.$$

4. Applying the Differentiation property, we get $Y(j\omega) = j\omega X(j\omega) = 2je^{j\omega} \sin(3\omega)$. Thus, we have

$$Y(j\frac{\pi}{4}) = 2je^{j\frac{\pi}{4}} \sin(3\frac{\pi}{4}) = 2j(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j)\frac{\sqrt{2}}{2} = j - 1.$$

Problem 3: Sampling Theory (18 pts; 6 pts each)

The sampling frequency is $\omega_{\mathrm{s}} = \frac{2\pi}{T} = 20\pi$.

1. Signal 1:

$$x_1(t) = \begin{cases} 1, & |t| < 1/5 \\ 0, & |t| \geq 1/5 \end{cases}$$

From Table 4.2, we find that

$$X_1(j\omega) = \frac{2\sin(\omega/5)}{\omega}.$$

This signal is not band-limited, so aliasing occurs. The signal cannot be reconstructed

2. Signal 2:

$$x_2(t) = \cos(4\pi t)e^{-j4\pi t}$$

Let $y_1(t) = \cos(4\pi t)$. From Table 4.2, its Fourier transform is given by

$$Y_1(j\omega) = \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi).$$

This signal has a maximum frequency 4π
Let $y_2(t) = e^{-j4\pi t}$. From Table 4.2, its Fourier transform is given by

$$Y_2(j\omega) = 2\pi\delta(\omega + 4\pi).$$

Applying the Multiplication Property of CTFT, we have

$$X_2(j\omega) = \frac{1}{2\pi} Y_1(j\omega) * Y_2(j\omega) = \pi\delta(\omega) + \pi\delta(\omega + 8\pi).$$

It follows that the maximum frequency of $X_2(j\omega)$ is $W = 8\pi$. Since $\omega_s > 2W$, no aliasing occurs. The signal can be reconstructed.

3. Signal 3:

$$x_3(t) = \frac{\sin^2(4\pi t)}{\pi t}$$

We let

$$z_1(t) = \frac{\sin(4\pi t)}{\pi t}, \qquad z_2(t) = \sin(4\pi t).$$

From Table 4.2, we have

$$Z_1(j\omega) = \begin{cases} 1, & \text{if } |\omega| < 4\pi \\ 0, & \text{if } |\omega| > 4\pi \end{cases}, \qquad Z_2(j\omega) = \frac{\pi}{j}\left(\delta(\omega - 4\pi) - \delta(\omega + 4\pi)\right)$$

Applying the Multiplication Property of CTFT, we have that

$$X_3(j\omega) = Z_1(j\omega) * Z_2(j\omega)$$

has a maximum frequency of $W = 8\pi$. Since $\omega_s > 2W$, no aliasing occurs. The signal can be reconstructed.

Problem 4: Amplitude Modulation (22 pts)

Consider the signals

$$x_1(t) = \frac{\sin(20t)}{\pi t}, \qquad \text{and} \qquad x_2(t) = \frac{\sin(10t)}{\pi t}$$

1. From Table 4.2, we have

$$X_1(j\omega) = \begin{cases} 1 & \text{if } |\omega| < 20 \\ 0 & \text{if } |\omega| > 20 \end{cases}, \qquad X_2(j\omega) = \begin{cases} 1 & \text{if } |\omega| < 10 \\ 0 & \text{if } |\omega| > 10 \end{cases}$$

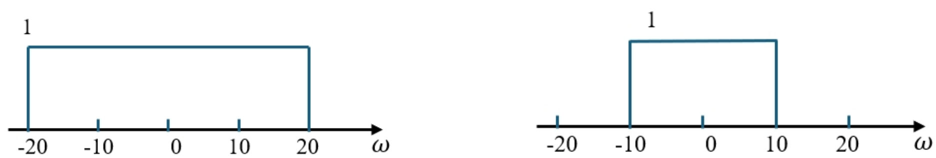

Fig. 1: Left: $X_1(j\omega)$; Right $X_2(j\omega)$

2. Let $Y_1(j\omega)$ be the Fourier transform of $y_1(t)$ and $Y_2(j\omega)$ be the Fourier transform of $y_2(t)$. Then $Y(j\omega) = Y_1(j\omega) + Y_2(j\omega)$, where

$$Y_1(j\omega) = \frac{1}{2}(X_1(j(\omega - 40)) + X_1(j(\omega + 40))),$$
$$Y_2(j\omega) = \frac{1}{2}(X_2(j(\omega - 80)) + X_1(j(\omega + 80))).$$

Thus, we have

$$Y(j\omega) = \frac{1}{2}(X_1(j(\omega-40)) + X_1(j(\omega+40))) + \frac{1}{2}(X_2(j(\omega-80)) + X_1(j(\omega+80))).$$

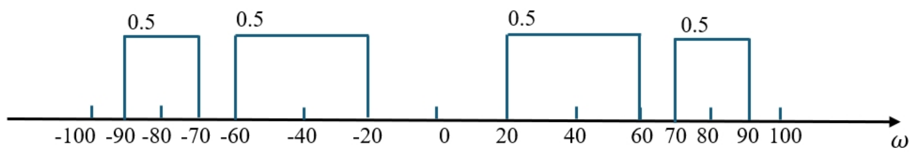

Fig. 2: Sketch of $Y(j\omega)$

3. We have

$$\begin{aligned} Z(j\omega) &= \frac{1}{2}(Y(j(\omega-40)) + Y(j(\omega+40))) \\ &= \frac{1}{2}(Y_1(j(\omega-40)) + Y_1(j(\omega+40))) + \frac{1}{2}(Y_2(j(\omega-40)) + Y_2(j(\omega+40))). \end{aligned}$$

Then, we have

$$\begin{aligned} Z(j\omega) =& \frac{1}{4}X_1(j(\omega-80) + \frac{1}{2}X_1(j(\omega)) + \frac{1}{4}X_1(j(\omega+80)) \\ &+ \frac{1}{4}X_2(j(\omega-120)) + \frac{1}{4}X_2(j(\omega-40)) \\ &+ \frac{1}{4}X_2(j(\omega+40)) + \frac{1}{4}X_2(j(\omega+120)) \end{aligned}$$

The sketch is shown below

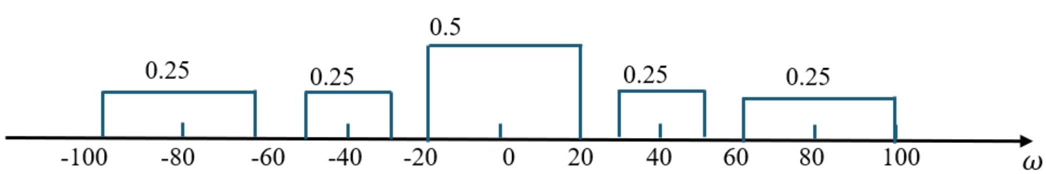

Fig. 3: Sketch of $Z(j\omega)$ for frequencies between $[-100, 100]$

4. First demodulate by multiplying $y(t)$ by $\cos(80t)$.
   Then apply a low-pass filter with cut-off frequency $10 < \omega_{co} < 20$ and gain 2.

Problem 5: Laplace Transform (22 pts)

Let

$$H(s) = \frac{s}{s^2 + 3s + 2} \tag{1}$$

1. We first write

$$H(s) = \frac{s}{s^2 + 3s + 2} = \frac{s}{(s+1)(s+2)}.$$

   It has two poles $s_1 = -1$ and $s_2 = -2$. Thus, it has 3 distinct signals that have Laplace transform expressed as $H(s)$ in their region of convergence:
   - Signal 1 with ROC $\mathrm{Re}\{s\} < -2$.
   - Signal 2 with ROC $-2 < \mathrm{Re}\{s\} < -1$.
   - Signal 3 with ROC $-1 < \mathrm{Re}\{s\}$.

2. If the signal $h(t)$ has a Fourier transform, then its ROC contains the imaginary axis. Thus, its ROC becomes $-1 < \mathrm{Re}\{s\}$.
   Expand $X(s)$ using Partial Fraction Expansion

$$\begin{aligned} X(s) &= \frac{A}{s+1} + \frac{B}{s+2} \\ &= \frac{-1}{s+1} + \frac{2}{s+2} \end{aligned}$$

   where we have used

$$\begin{aligned} A = (s+1)X(s)|_{s=-1} &= \frac{s}{s+2}|_{s=-1} = -1, \\ B = (s+2)X(s)|_{s=-2} &= \frac{s}{s+1}|_{s=-2} = 2. \end{aligned}$$

   By Table 9.2, we have the basic Laplace transform pairs

$$\frac{1}{s+a}, \mathrm{Re}\{s\} > -a, \quad \leftrightarrow \quad e^{-at}u(t)$$

   We conclude that

$$h(t) = -e^{-t}u(t) + 2e^{-2t}u(t) = (2e^{-2t} - e^{-t})u(t).$$

3. When $\omega = 0$, we have

$$|H(j\omega)| = 0.$$

   When $\omega = L \to \infty$, we have

$$|H(j\omega)| = \frac{|j\omega|}{|j\omega + 1||j\omega + 2|} \approx \frac{L}{L^2} = \frac{1}{L} \to 0.$$

   The system is band-pass in nature.

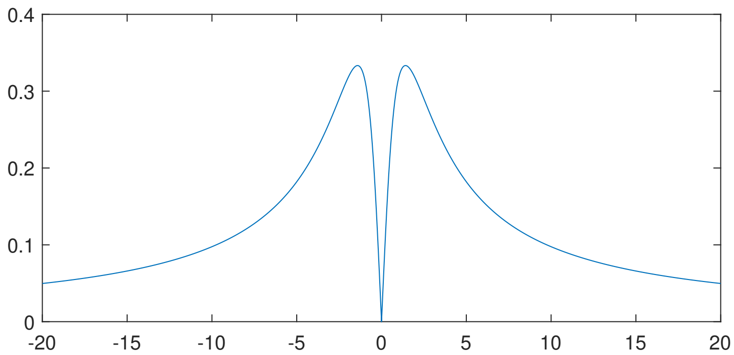

Fig. 4: $|H(j\omega)|$

Problem 6: Laplace Transform and LTI Systems (20 pts)

Consider an LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} \tag{2}$$

1. The differential equation

   $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

   corresponds to the system function

   $$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.$$

   In this case, the system function is

   $$H(s) = \frac{s}{s^2 + 3s + 2}.$$

   It has two poles $s_1 = -1, s_2 = -2$. Thus, we see that there are three possible regions of convergence:

   - ROC I: $\mathrm{Re}\{s\} < -2$
     The system is not causal because the ROC is a left half-plane.
     The system is not stable because the ROC does not contain the $j\omega$-axis
   - ROC II: $-2 < \mathrm{Re}\{s\} < -1$
     The system is not causal because the ROC is banded.
     The system is not stable because the ROC does not contain the $j\omega$-axis
   - ROC III: $-1 < \mathrm{Re}\{s\}$
     The system is causal because the ROC is a right half-plane.
     The system is stable because the ROC contains the $j\omega$-axis

2. It has a first-order zero $s = 0$, and two first-order poles $s_1 = -1, s_2 = -2$. The pole-zero plot is sketched below

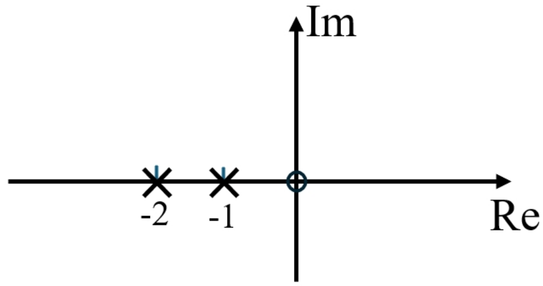

Fig. 5: Pole-zero plot

3. From Table 9.2, the transfer function $H_2(s)$ is given by

$$H_2(s) = \frac{3}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} = \frac{s+2}{s^2-1}$$

   with ROC given by $\mathrm{Re}\{s\} > 1$.
   The serial concatenation of S1 and S2 has an impulse response

$$h(t) = h_1(t) * h_2(t)$$

   and, by the Convolution property of LT in Table 9.1, the corresponding transfer function is

$$G(s) = H(s)H_2(s) = \frac{s}{s^2+3s+2}\frac{s+2}{s^2-1} = \frac{s}{(s+1)(s^2-1)}.$$

4. From Table 9.2, the Laplace Transform of the input is

$$X(s) = \frac{1}{s}.$$

   Thus, the output is

$$Y(s) = H(s)X(s) = \frac{s}{s^2+3s+2} \times \frac{1}{s} = \frac{1}{s^2+3s+2}.$$

   with a ROC containing the intersection of $\mathrm{Re}\{s\} > -1$ and $\mathrm{Re}\{s\} > 0$, which contains $\mathrm{Re}\{s\} > 0$.
   Through Partial Fraction Expansion, we have

$$Y(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} + \frac{-1}{s+2}.$$

   From Table 9.2, we know the step response is

$$y(t) = e^{-t}u(t) - e^{-2t}u(t) = (e^{-t} - e^{-2t})u(t).$$