ECE 101: Linear Systems Fundamentals

Spring 2020 - Lecture 2

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Today's topics

- Definition of periodic signals
- Definition of the energy and power of signals
- Definition of even and odd signals
- Decomposition theorem

Signals and Systems (2th Edition): sections 1.1.2 and 1.2

Periodic and Aperiodic Signals

- Continuous Time: $x(t)$ is *periodic* with period $T > 0$ if $x(t + T) = x(t)$ for all time t.
- **Fundamental period and frequency:**
  - smallest $T > 0$ that satisfies this is called the fundamental period
  - for the fundamental period T, $\omega_0 = \frac{2\pi}{T}$ is called the fundamental frequency.
- Discrete Time: $x[n]$ is *periodic* with period $N > 0$ if $x[n + N] = x[n]$ for all integer n.
- **Fundamental period and frequency:**
  - smallest $N > 0$ that satisfies this is called the fundamental period
  - for the fundamental period N_0, $\omega_0 = \frac{2\pi}{N_0}$ is called the fundamental frequency.

Periodic and Aperiodic Signals

- Example1: Determine the fundamental period of $x(t) = e^{j3\pi t/5}$.
- Solution: for periodicity, a period T should satisfy:

$$x(t) = x(t+T) \Rightarrow e^{j3\pi t/5} = e^{j3\pi(t+T)/5} = e^{j3\pi t/5} e^{j3\pi T/5}$$

$$e^{j3\pi t/5} = e^{j3\pi t/5} e^{j3\pi T/5}$$

- Therefore, $e^{j3\pi T/5} = 1$.

$$\begin{aligned}
e^{j2k\pi} &= \cos(2k\pi) + j\sin(2k\pi) = 1, \quad k \in \mathbb{Z} \\
e^{j3\pi T/5} &= 1 = e^{j2k\pi}, \quad k \in \mathbb{Z} \\
3\pi T/5 &= 2k\pi, \quad k \in \mathbb{Z} \\
T &= \frac{10}{3}k, \quad k \in \mathbb{Z}
\end{aligned}$$

Periodic and Aperiodic Signals

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- Solution: for periodicity, a period T should satisfy:

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$$e^{j3\pi t/5} = e^{j3\pi t/5}e^{j3\pi T/5}$$

- Therefore, $e^{j3\pi T/5} = 1$.
- This implies that $T = \frac{10}{3}k \text{ for some } k \in \mathbb{Z}$.
- The smallest such k that leads to $T > 0$ is $k = 1$. Therefore, the fundamental period is $T_0 = \frac{10}{3}$.

Periodic and Aperiodic Signals

- Example2: Determine the fundamental period of $x[n] = e^{j3\pi n/5}$.
- Solution: Again for periodicity, a period N should satisfy:

$$x[n] = x[n+N] \Rightarrow e^{j3\pi n/5} = e^{j3\pi(n+N)/5} = e^{j3\pi n/5}e^{j3\pi N/5}$$

$$e^{j3\pi n/5} = e^{j3\pi n/5}e^{j3\pi N/5}$$

- Therefore, $e^{j3\pi N/5} = 1$.

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Periodic and Aperiodic Signals

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- Solution: Again for periodicity, a period N should satisfy:

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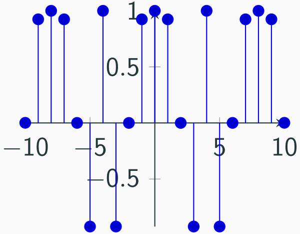
$$e^{j3\pi n/5} = e^{j3\pi n/5} e^{j3\pi N/5}$$

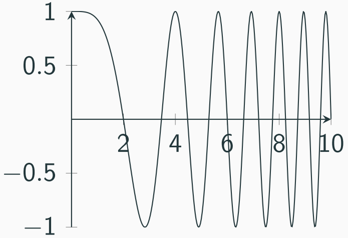
- Therefore, $e^{j3\pi N/5} = 1$.
- This implies that $N = \frac{10}{3} k \text{ for some } k \in \mathbb{Z}$.
- The smallest such k that leads to $N \geq 1$ is $k = 3$. Therefore, the fundamental period is $N_0 = 10$.

Periodic and Aperiodic Signals

- Trickier question: which one of the following signal is periodic?

$$x(t) = \cos\left(\frac{\pi t^2}{8}\right),$$
$$x[n] = \cos\left(\frac{\pi n^2}{8}\right).$$





Periodic and Aperiodic Signals

- Why is $x(t) = \cos\left(\frac{\pi t^2}{8}\right)$ not periodic?
- Periodicity implies $x(t+T) = x(t)$ for a period $T > 0$ and all time t.
- This means that

$$\cos\left(\frac{\pi(t+T)^2}{8}\right) = \cos\left(\frac{\pi t^2}{8}\right) \Rightarrow \left(\frac{\pi(t+T)^2}{8}\right) = \left(\frac{\pi t^2}{8}\right) = \left(\frac{\pi t^2}{8}\right) + 2k\pi$$

  for some integer k.
- Therefore, $(t+T)^2 = t^2 + 16k$ or $2tT + T^2 = 16k$ for some integer k and all real numbers t.

Periodic and Aperiodic Signals

- Why is $x[n] = \cos\left(\frac{\pi n^2}{8}\right)$ periodic?

$x[n+N] = x[n]$ for $n \in \mathbb{Z}$

$\cos\left(\frac{\pi}{8}(n+N)^2\right) = \cos\left(\frac{\pi}{8}n^2\right)$

$\cos\left(\frac{\pi}{8}n^2 + \underbrace{\frac{\pi}{8}(N^2 + 2nN)}_{2k\pi,\ k \in \mathbb{Z}}\right) = \cos\left(\frac{\pi}{8}n^2\right)$

$\frac{\pi}{8}(N^2 + 2nN) = 2k\pi \longrightarrow N^2 + 2nN = 16k \quad , \quad k \in \mathbb{Z}$

$\left.\begin{array}{l} N=1 \longrightarrow 1 + 2n = 16k \\ N=2 \longrightarrow 4 + 4n = 16k \end{array}\right\}$ not valid for all $n \in \mathbb{Z}$

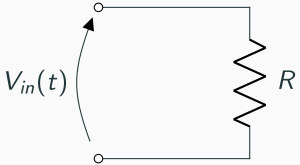
$\vdots$

$N = 8 \longrightarrow 64 + 16n = 16k$ valid for all $n \in \mathbb{Z}$

$x[n]$ is periodic and $N_0 = 8$

Energy and Power of a Signal

- Remember: Instantaneous power of a resistor = $P(t) = V(t)I(t) = \frac{1}{R}V^2(t)$



- Energy: power consumption over time = $\int_{t_1}^{t_2} P(t)dt$

Energy and Power of a Signal

- Motivated by these: We **DEFINE** the energy of a CT signal $x(t)$ over $[t_1, t_2]$ interval by:

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt.$$

- We define the energy of a DT signal $x[n]$ over $[n_1, n_2]$ interval by:

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2.$$

- We define *average power* of a signal:
  - Continuous-time: $x(t)$ over continuous interval (t_1, t_2) is:

$$P = \frac{1}{t_2 - t_1} E = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt.$$

  - Discrete-time: $x[n]$ over discrete interval $[n_1, n_2]$ is:

$$P = \frac{1}{n_2 - n_1 + 1} E = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2.$$

Total Energy of a Signal

- We define the total energy of
    - a continuous-time signal $x(t)$ to be

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

    - a discrete-time signal $x[n]$ to be

$$E_\infty = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2$$

- We say that a signal is finite energy if $E_\infty < \infty$.

Total Energy of a signal - Example

- Example: What is the total energy of the signal $x(t) = e^{-(1+j)t}$?
- Solution: First note that,

$$|x(t)| = |e^{-(1+j)t}| = |e^{-}t||e^{-jt}| = e^{-t}$$

Product rule for exponents:

$$e^{(\theta_1+\theta_2)} = e^{\theta_1}e^{\theta_2}$$

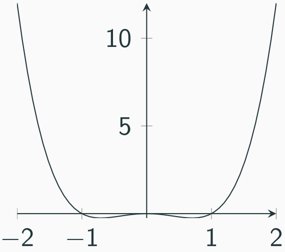
Even/Odd Signals

Definition

A CT (DT) signal $x(t)$ ($x[n]$) is *even* if

$$x(-t) = x(t) \qquad (x[-n] = x[n]),$$

for all t (n).

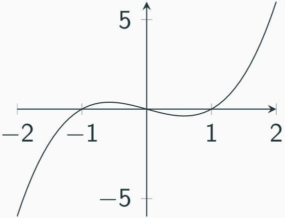


Even/Odd Signals

Definition

A CT (DT) signal $x(t)$ ($x[n]$) is *odd* if

$$x(-t) = -x(t) \qquad (x[-n] = -x[n]).$$



Even/Odd Signals

Decomposition Theorem

Every CT signal $x(t)$ we be expressed as

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t) = \frac{x(t)+x(-t)}{2}$ is even and $x_o(t) = \frac{x(t)-x(-t)}{2}$ is odd.

- The signal $x_e(t)$ is called the even part of $x(t)$, and is denoted by $\mathcal{E}v\{x(t)\}$
- The signal $x_o(t)$ is called the odd part of $x(t)$, and is denoted by $\mathcal{O}dd\{x(t)\}$

Even/Odd Signals

Decomposition Uniqueness

The decomposition is unique, i.e., if

$$x[n] = x_e[n] + x_o[n],$$

then $x_e[n]$ is even and $x_o[n]$ is odd, if and only if $x_e[n] = \mathcal{E}v\{x[n]\}$ and $x_o(t) = \mathcal{O}dd\{x[n]\}$

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- The same is valid for the continuous-time signals