UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2023
Linear Systems Fundamentals

MIDTERM EXAM WITH SOLUTIONS

You are allowed one 2-sided sheet of notes.

No books, no other notes, no calculators.

PRINT YOUR NAME Leonhard Euler

Signature $e^{j\pi} + 1 = 0$

Student ID Number *A2.7182818*

Your signature confirms that you have completed this exam in accordance with the ECE 101 Academic Integrity Agreement.

Problem	Weight	Score
1	32 pts	32
2	36 pts	36
3	32 pts	32
Total	100 pts	100

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Tables 3.1 and 3.2 from the textbook are attached to the exam.
Scan, upload, and tag your solutions on Gradescope at the end of the exam.

Good luck!

Name/Student ID: ______________________________

Problem 1 [DT Signals] (32 points)

(a) (16 points, 8 points each part)

Indicate whether or not the following statements are True or False. As usual, $\delta[n]$ denotes the unit impulse signal, $u[n]$ denotes the unit step signal, and $*$ denotes the discrete-time (DT) convolution operator.

Justify your answers.

True False

☒ ☐ $x[n] * y[n-2] = x[n-2] * y[n]$, for all DT signals $x[n]$, $y[n]$.

Convolution with $\delta[n-n_0]$ causes time shift by n_0,
and DT convolution is commutative and associative:
$x[n] * y[n-2] = x[n] * (y[n] * \delta[n-2]) = x[n] * (\delta[n-2] * y[n])$
$= (x[n] * \delta[n-2]) * y[n] = x[n-2] * y[n]$.

Alternatively, use a change of variables in the convolution sum formula.

☐ ☒ Let $y[n] = x[n] * u[n-4]$, where $x[n]$ is any discrete-time signal. Then $y[0] = \sum_{n=-\infty}^{4} x[n]$.

Let $v[n] = u[n-4]$.
$y[n] = \sum_{k=-\infty}^{\infty} x[k]v[n-k]$ (Flip $v[k]$ and shift by n.)
$= \sum_{k=-\infty}^{\infty} x[k]u[(n-k)-4]$
$= \sum_{k=-\infty}^{\infty} x[k]u[(n-4)-k]$. (Flip $u[k]$ and shift by $n-4$).
So,
$y[0] = \sum_{k=-\infty}^{\infty} x[k]u[-4-k]$
$= \sum_{k=-\infty}^{-4} x[k]$. (Flip $u[k]$, shift by -4, multiply by $x[k]$, and sum)
$= \sum_{n=-\infty}^{-4} x[n]$.

Name/Student ID: ____________________

Problem 1 [DT Signals] (cont.)

(b) (16 points, 8 points each part)

Let $x[n] = e^{j\frac{\pi}{5}n} + \sum_{k=-\infty}^{\infty}(-1)^k \delta[n-4k]$.

(i) (8 points)

Determine the fundamental frequency Ω of $x[n]$. Check box below. **Justify your answer.**

- ☐ $\Omega = \frac{2\pi}{10}$
- ☐ $\Omega = \frac{2\pi}{20}$
- ☒ $\Omega = \frac{2\pi}{40}$
- ☐ $\Omega = \frac{2\pi}{80}$

$e^{j\frac{\pi}{5}n} = e^{j\frac{2\pi}{10}n}$ has fundamental period $N_1 = 10$.

$\sum_{k=-\infty}^{\infty}(-1)^k \delta[n-4k]$ has fundamental period $N_2 = 8$.

The fundamental period of the $x[n]$ is:

$N = \text{l.c.m.}\{N_1, N_2\} = \text{l.c.m.}\{10, 8\} = 40$.

The fundamental frequency is $\Omega = \frac{2\pi}{N} = \frac{2\pi}{40}$.

Problem 1 [DT Signals] (cont.)

(b) (cont.)

Let $x[n] = e^{j\frac{\pi}{5}n} + \sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k]$.

(ii) (8 points)

Determine the even part of $x[n]$, $f[n] = \mathcal{E}v\{x[n]\}$. Check box below. **Justify your answer.**

☐ $f[n] = 2\cos(\frac{\pi}{5}n) + 2\sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k]$

☒ $f[n] = \cos(\frac{\pi}{5}n) + \sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k]$

☐ $f[n] = \cos(\frac{\pi}{5}n) + \sum_{k=-\infty}^{\infty}\delta[n-4k]$

☐ $f[n] = \cos(\frac{\pi}{5}n)$

$$
\begin{aligned}
\mathcal{E}v\{x[n]\} &= \mathcal{E}v\{e^{j\frac{\pi}{5}n}\} + \mathcal{E}v\{\sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k]\} \\
&= \frac{1}{2}\left(e^{j\frac{\pi}{5}n} + e^{-j\frac{\pi}{5}n}\right) + \\
&\quad \frac{1}{2}\left(\sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k] + \sum_{k=-\infty}^{\infty}(-1)^k\delta[-n-4k]\right) \\
&= \cos(\frac{\pi}{5}n) + \\
&\quad \frac{1}{2}\left(\sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k] + \sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k]\right) \\
&\quad \text{(i.e., the alternating polarity impulse train is even)} \\
&= \cos(\frac{\pi}{5}n) + \sum_{k=-\infty}^{\infty}(-1)^k\delta[n-4k]
\end{aligned}
$$

Name/Student ID: ________________________________

Problem 2 [DT LTI Systems] (36 pts)

Let S be the DT LTI system defined by the difference equation:

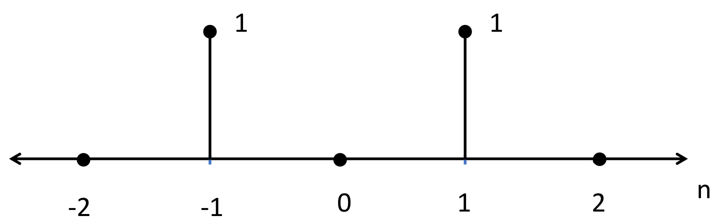
$$y[n] = x[n-1] + x[n+1].$$

(a) (6 points)

Determine the impulse response $h[n]$ of system S. Sketch $h[n]$ precisely.

Substitute $x[n] = \delta[n]$.

$h[n] = \delta[n-1] + \delta[n+1]$.



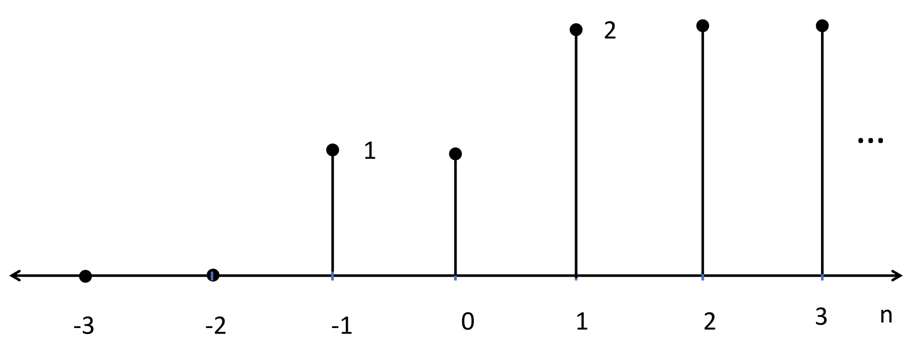
(b) (6 points)

Determine the step response $s[n]$ of system S. Sketch $s[n]$ precisely.

Either use $s[n] = h[n] * u[n] = \sum_{k=-\infty}^{n} h[k]$

or substitute $x[n] = u[n]$ to get $s[n] = u[n-1] + u[n+1]$.

Either expression yields: $s[n] = \begin{cases} 0 \;, & n \le -2 \\ 1 \;, & -1 \le n \le 0 \\ 2 \;, & n \ge 1. \end{cases}$



Name/Student ID: ______________________________

Problem 2 (cont.)

(c) (6 points)

Determine the system function $H(z)$ of system S.

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ &= z + z^{-1}. \end{aligned}$$

(d) (6 points)

Does there exist a complex number $z \neq 0$ such that $H(z) = 0$?

If not, why not? If so, give an example of such a number z.

$$\begin{aligned} H(z) &= z + z^{-1} \\ &= z^{-1}(1 + z^2) \end{aligned}$$

The equation $z^2 + 1 = 0$ can be written as $z^2 = -1$.

There are 2 solutions to this equation, namely $z = j$ and $z = -j$.

We can choose $z = j$.

We confirm that $H(j) = j + (j)^{-1} = j + (-j) = 0$.

Name/Student ID: ____________________

Problem 2 (cont.)

e) (12 points, 6 points each part)

Check the appropriate box indicating whether or not system S satisfies the specified properties. **Justify your answers.**

True	False	
☐	☒	Invertible

S is invertible if and only distinct system input signals produce distinct outputs signals.
From part(d), $H(j) = 0$, so the eigenfunction property implies that the input $x_1[n] = j^n$ produces the output $y_1[n] = H(j)j^n = 0$.
But, the input $x_2[n] = 0$ produces the output $y_2[n] = x_2[n-1] + x_2[n+1] = 0$, as well.
In general, any input signal satisfying $x[n-1] = -x[n+1]$ for all n produces an output signal 0.

Note: You can find $g[n]$ such that $h[n] * g[n] = \delta[n]$, namely: $g[n] = \sum_{k=0}^{\infty} (-1)^k \delta[n-(2k+1)]$
This means the systems are inverses for input signals that do not cause either system to blow up. But $x[n] = j^n$ causes the system with impulse response $g[n]$ to blow up.

True	False	
☒	☐	Stable

S is stable if and only if $\sum_{n=-\infty}^{\infty} \big|h[n]\big| < \infty$.
Here $\sum_{n=-\infty}^{\infty} \big|h[n]\big| = |h[-1]| + |h[1]| = 1 + 1 = 2$.
Alternatively, $\big|y[n]\big| = \big|x[n-1] + x[n+1]\big| \leq \big|x[n-1]\big| + \big|x[n+1]\big|$.
Therefore, if $x[n]$ is bounded, so is $y[n]$.
So, system S satisfies the BIBO property.

Name/Student ID: ______________________

Problem 3 [DTFS and DT Filtering] (32 points)

(a) (10 points, 5 points each part)

Let $x[n]$ be the a discrete-time periodic signal with fundamental period $N = 2$, and Fourier Series coefficients a_k. Let $y[n] = x[n-1]$.

The Fourier Series coefficients b_k of $y[n]$ are $b_k = (-1)^k$, for all k.

(i) (5 points)

Determine the Fourier Series coefficients a_k of the signal $x[n]$.

(i) The fundamental frequency of $x[n]$, and therefore $y[n]$, is $\Omega_0 = \frac{2\pi}{N} = \pi$. Since $y[n]$ is obtained from $x[n]$ by a time shift of $n_0 = 1$, the time-shifting property of the CTFS (Table 3.2) implies that

$$b_k = e^{-jk(2\pi/N)n_0} a_k = e^{-jk\pi} a_k, \text{for all } k.$$

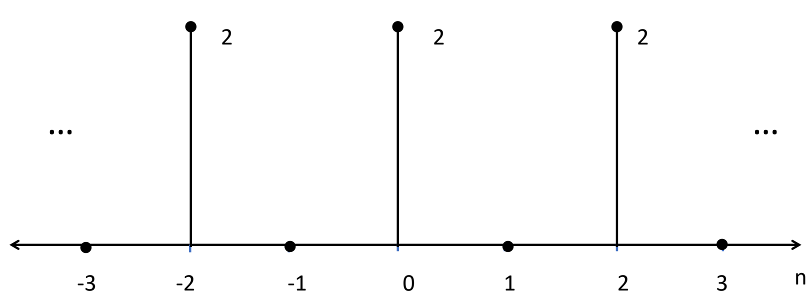
Noting that $e^{-jk\pi} = (-1)^k$, we conclude that $a_k = 1$, for all k.

(ii) (5 points)

Identify and sketch the signal $x[n]$ precisely.

Recalling that the Fourier Series coefficients of the unit impulse train with fundamental period N are $1/N$ for all k, we conclude that $x[n]$ is an impulse train with period $N = 2$ and amplitude 2:

$$x[n] = 2 \sum^{\infty} \delta[n - 2k].$$



Name/Student ID: ________________________________

Problem 3 (cont.)

(b) (10 points)

Find the fundamental period N and the DT Fourier Series coefficients a_k, $k = 0, \ldots, N-1$ of the signal $x[n] = \sin(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$. Use the DTFS coefficients to decide if the signal is even or odd.

Justify your answers.

Using Euler's formula, we get

$$\begin{aligned} x[n] &= \frac{1}{2j}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right)\frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) \\ &= \frac{1}{4j}\left(e^{j\frac{3\pi}{4}n} + e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n}\right) \\ &= \frac{1}{4j}\left(e^{j3(\frac{2\pi}{8})n} + e^{j\frac{2\pi}{8}n} - e^{-j\frac{2\pi}{8}n} - e^{-j3(\frac{2\pi}{8})n}\right) \end{aligned}$$

Alternatively, use $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

So, $x[n] = \frac{1}{2}\sin(\frac{3\pi}{4}n) + \frac{1}{2}\sin(\frac{\pi}{4}n)$. This yields the same exponential expansion.

The fundamental frequency is $\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$.

The non-zero Fourier series coefficients in the range $k = [-3, 4]$ are:

$a_3 = \frac{1}{4j}, a_1 = \frac{1}{4j}, a_{-1} = -\frac{1}{4j}, a_{-3} = -\frac{1}{4j}$,

and the other coefficients are 0: $a_4 = a_2 = a_0 = a_{-2} = 0$.

By periodicity of the DTFS coefficients, we have:

$a_5 = a_{-3} = -\frac{1}{4j}, a_6 = a_{-2} = 0, a_7 = a_{-1} = -\frac{1}{4j}$.

$a_0 =$	0	$a_4 =$	0
$a_1 =$	$1/4j = -j/4$	$a_5 =$	$-1/4j = j/4$
$a_2 =$	0	$a_6 =$	0
$a_3 =$	$1/4j = -j/4$	$a_7 =$	$-1/4j = j/4$

The DTFS coefficients satisfy $a_k = -a_{-k}$, so they are odd, implying $x[n]$ is odd. Note also that since they are also purely imaginary, this confirms that $x[n]$ is real and odd.

Name/Student ID: ______________________________

Problem 3 (cont.)

(b) (8 points)

Let $x[n] = \sin(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$.

Find the fundamental period N and the DT Fourier Series coefficients a_k, $k = 0, \ldots, N$ of the signal $x[n] = \sin(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n) + 1$

ALTERNATIVE SOLUTION: We can also find the Fourier series coefficients of $x[n]$ directly.

By direct substitution, we find:

$x[0]{=}0, x[1]{=}\frac{\sqrt{2}}{2}, x[2]{=}0, x[3]{=}\frac{\sqrt{2}}{2}, x[4]{=}0, x[5]{=}-\frac{\sqrt{2}}{2}, x[6]{=}0, x[7]{=}-\frac{\sqrt{2}}{2}$.

The analysis equation says

$$a_k = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-jk\frac{\pi}{4}n}.$$

Substituting in the values of $x[n]$ yields:

$$\begin{aligned}
a_k &= \frac{1}{8}\frac{\sqrt{2}}{2}\left(e^{-jk\frac{\pi}{4}} + e^{-jk\frac{3\pi}{4}} - e^{-jk\frac{5\pi}{4}} + e^{-jk\frac{7\pi}{4}}\right)\\
&= \frac{1}{8}\frac{\sqrt{2}}{2}\left(e^{-jk\frac{\pi}{4}} + e^{-jk\frac{3\pi}{4}} - e^{jk\frac{3\pi}{4}} + e^{jk\frac{\pi}{4}}\right)\\
&= -\frac{\sqrt{2}}{16}\left(-2j\sin(\frac{\pi}{4}k) - 2j\sin(\frac{3\pi}{4}k)\right)\\
&= -\frac{\sqrt{2}}{8}j\left(\sin(\frac{\pi}{4}k) + \sin(\frac{3\pi}{4}k)\right)
\end{aligned}$$

Evaluating this for all values of $k = 0, \ldots, 7$ yields the same results as the other method.

Name/Student ID: ______________________________

Problem 3 (cont.)

(c) (12 points)

Let $x[n]$ be a DT signal with fundamental period $N = 4$ and DT Fourier series coefficients $a_0 = 3, a_1 = 1, a_2 = -1, a_3 = 1$. Consider the DT LTI system whose frequency response in the interval $[-\pi, \pi]$ is given by

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{2\pi}{3} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the signal $x[n]$ is the input to this system. Determine and sketch precisely the output $y[n]$ in the interval $[0, \ldots, 3]$.

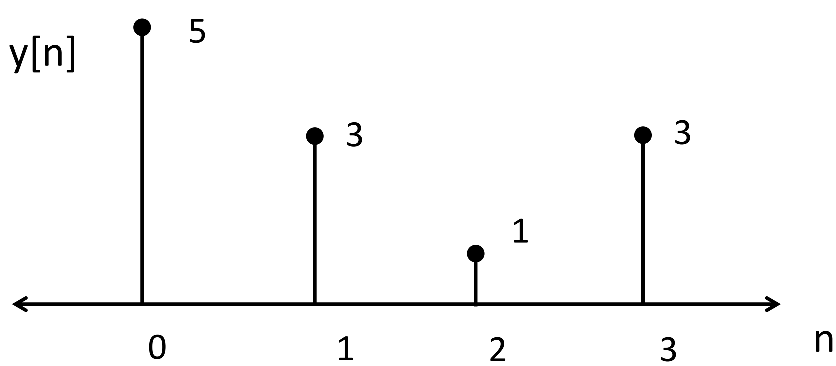
$\omega_0 = 2\pi/4 = \pi/2$

$y[n] = \sum_{k=0}^{3} a_k H(e^{jk\pi/2}) e^{jk(\pi/2)n}$

$H(e^{jk\pi/2}) = \begin{cases} 1 & \text{for } k = 0, 1, 3 \\ 0 & \text{for } k = 2. \end{cases}$

$$\begin{aligned} y[n] &= 3 \cdot e^{j0(\pi/2)n} + 1 \cdot e^{j(\pi/2)n} + 0 \cdot e^{j(2\pi/2)n} + 1 \cdot e^{j(3\pi/2)n} \\ &= 3 + j^n + (-j)^n \end{aligned}$$

So, $y[0] = 5, y[1] = 3, y[2] = 1, y[3] = 3$.



Scratch page **Name/Student ID:** ____________________________

Scratch page **Name/Student ID:** ______________________________