UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2022

Linear Systems Fundamentals

SOLUTIONS TO FINAL EXAM

Preface

- I recognize that final exams are stressful in the best of times, and the past 2 years have been unusually hard times.
- Please remember that despite the stress, I am counting on you to uphold academic integrity while you complete your final exam.
- Posting or seeking exam questions or answers online, or by consulting unauthorized resources, is a gross violation of our principles of integrity and engineering ethics.
- To be fair to all students, any integrity violations discovered during the final exam will be reported to the Dean of Engineering and to the office of Academic Integrity.
- Please make sure you understand and follow the academic integrity guidelines for the exam. If you are not sure, ask me.
- An honest effort, no matter what the outcome, is something to be proud of, especially in these challenging times.
- I am very proud of the commitment and resilience you have displayed. You should feel proud, too.
- **Good luck on the exam!**

PRINT YOUR NAME Harry Nyquist

SIGNATURE ↑↑↑↑↑↑↑↑↑↑↑↑

Your signature confirms that you have completed this exam on your own and in accordance with the Academic Integrity Agreement.

Student ID Number WS>2WMAX

Instructions

- No electronics allowed for problem solving.
- Four 2-sided sheets of your own notes are allowed.
- Time allowed: 3 hours (including Gradescope upload time).
- Write your solutions in the designated spaces in the exam.
- **Justify all of your answers. Credit requires proper justification.**
- If extra pages are needed to show all of your work, use the scratch pages at the end of the exam and submit them.
- **Upload your solutions by 6pm via Gradescope.**

Problem	Weight	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
6	20	20
Total	120	120

You've got this!

Name/Student ID: ____________________

Problem 1 [CTFT Properties] (20 points, 10 points each part)

Let $x(t) = u(t+1) - u(t-2)$. Let $X(j\omega)$ denote its Fourier transform.

(a) Determine $\int_{-\infty}^{\infty} X(j\omega) d\omega$.

(b) Determine $Y(j\frac{\pi}{2})$ where $Y(j\omega)$ is the Fourier transform of $y(t) = \frac{dx(t)}{dt}$.

Write answers to parts (a) and (b) on the following pages.

Justify your answers.

Name/Student ID: ____________________

Problem 1 (cont.)

(a) (10 points)

Determine $\int_{-\infty}^{\infty} X(j\omega)d\omega$.

The synthesis property states that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Setting $t = 0$, this implies

$$\begin{aligned} x(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega 0}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega \end{aligned}$$

So,

$$2\pi x(0) = \int_{-\infty}^{\infty} X(j\omega)d\omega.$$

For this given signal, we have $x(0) = 1$, so we get

$$\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi.$$

$$\boxed{\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi}$$

Name/Student ID: ____________________

Problem 1 (cont.)

(b) (10 points)

Determine $Y(j\frac{\pi}{2})$ where $Y(j\omega)$ is the Fourier transform of $y(t) = \frac{dx(t)}{dt}$.

Approach 1:
Referring to Table 4.2, we have the transform pair
$u(t) \leftrightarrow U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$.
Applying the Time Shifting property from Table 4.1, we get
$X(j\omega) = e^{j\omega}U(j\omega) - e^{-j2\omega}U(j\omega)$
(or $X(j\omega) = \int_{-1}^{2} e^{-j\omega t}dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{-1}^{2} = \frac{1}{j\omega}(e^{j\omega} - e^{-j2\omega})$)
Applying the Differentiation property, we get
$Y(j\omega) = j\omega X(j\omega) = (e^{j\omega} - e^{-j2\omega})(1 + j\omega\pi\delta(\omega))$
Evaluating at $\omega = \frac{\pi}{2}$, and noting that $\delta(\frac{\pi}{2}) = 0$, we get
$Y(j\frac{\pi}{2}) = e^{j\frac{\pi}{2}} - e^{-j\pi} = j + 1$.

Approach 2:
$\frac{dx(t)}{dt} = \delta(t+1) - \delta(t-2)$.
Referring to Table 4.2, we have the CTFT transform pair
$\delta(t - t_0) \leftrightarrow e^{-j\omega t_0}$
So, $Y(j\omega) = e^{j\omega} - e^{-2j\omega}$ and $Y(j\frac{\pi}{2}) = e^{j\frac{\pi}{2}} - e^{-j\pi} = j + 1$.

Approach 3:
Let $v(t) = x(t+\frac{1}{2})$. Referring to Table 4.2, we have $V(j\omega) = 2\frac{\sin(\frac{3}{2}\omega)}{\omega}$ and using the Time Shifting property in Table 4.1, we get
$X(j\omega) = e^{-j\omega(\frac{1}{2})}2\frac{\sin(\frac{3}{2}\omega)}{\omega}$
Using the Differentiation in Time property in Table 4.1, we get
$Y(j\omega) = j\omega e^{-j\omega(\frac{1}{2})}2\frac{\sin(\frac{3}{2}\omega)}{\omega} = 2je^{-j\omega/2}\sin(3\omega/2)$
Evaluating at $\omega = \frac{\pi}{2}$, we get
$Y(j\omega) = 2je^{-j\frac{\pi}{4}}\sin(3\pi/4) = 2j(\sqrt{2}/2 - j\sqrt{2}/2)(\sqrt{2}/2) = j(1-j) = j + 1$

$$\boxed{Y(j\frac{\pi}{2}) = j + 1}$$

Name/Student ID: ______________________________

Problem 1 (cont.)

Check the box next to the solution you found above, or check **None of the above** if it does not appear on the list.

Problem 1(a):

- ☐ 1
- ☐ 2
- ☒ 2π
- ☐ $\frac{1}{2\pi}$
- ☐ None of the above.

Problem 1(b):

- ☐ 0
- ☒ $j+1$
- ☐ $j\frac{\pi}{2}$
- ☐ $\frac{2}{\pi}$
- ☐ None of the above.

Name/Student ID: ______________________________

Problem 2 [DTFT] (20 points; 10 points each part)

Let $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Let $y[n] = \delta[n] + \frac{1}{2}\delta[n-1]$.

(a) Determine the frequency response $H(e^{j\omega})$ of the LTI system that produces output $y[n]$ when $x[n]$ is the input.

(b) Determine the impulse response $h[n]$ of the LTI system that produces output $y[n]$ when $x[n]$ is the input.

Write answers to parts (a) and (b) on the following pages.

Justify your answers.

Name/Student ID: ______________________________

Problem 2 (cont.)

Let $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Let $y[n] = \delta[n] + \frac{1}{2}\delta[n-1]$.

(a) (10 points)

Determine the frequency response $H(e^{j\omega})$ of the LTI system that produces output $y[n]$ when $x[n]$ is the input.

Referrring to Table 5.2, we have DTFT transform pairs

$a^n u[n],\ |a| < 1 \leftrightarrow \frac{1}{1-ae^{-j\omega}}$

$\delta[n-n_0] \leftrightarrow e^{-j\omega n_0}$

So,

$x[n] = \left(\frac{1}{2}\right)^n u[n] \leftrightarrow X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$.

$y[n] = \delta[n] + \frac{1}{2}\delta[n-1] \leftrightarrow Y(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}$.

The frequency response satisfies

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1+\frac{1}{2}e^{-j\omega}}{\frac{1}{1-\frac{1}{2}e^{-j\omega}}} \\ &= (1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega}) \\ &= 1-\frac{1}{4}e^{-j2\omega} \end{aligned}$$

$$\boxed{H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j2\omega}}$$

Name/Student ID: ____________________

Problem 2 (cont.)

Let $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Let $y[n] = \delta[n] + \frac{1}{2}\delta[n-1]$.

(b) (10 points)

Determine the impulse response $h[n]$ of the LTI system that produces output $y[n]$ when $x[n]$ is the input

$h[n]$ is the inverse DTFT of the frequency response $H(e^{j\omega})$.

$h[n] \leftrightarrow H(e^{j\omega})$

Approach 1:

Using the result of part (a) and the second transform pair, we have

$h[n] = \delta[n] - \frac{1}{4}\delta[n-2]$

Approach 2:

$y[n] = x[n] * h[n]$

We saw in class that

$x[n] * (\delta[n] - \frac{1}{2}\delta[n-1]) = \delta[n]$

So,

$x[n] * (\delta[n-1] - \frac{1}{2}\delta[n-2]) = \delta[n-1]$

Therefore,

$$\begin{aligned} x[n] &* (\delta[n] - \frac{1}{2}\delta[n-1]) + \frac{1}{2}(\delta[n-1] - \frac{1}{2}\delta[n-2]) \\ &= x[n] * (\delta[n] - \frac{1}{4}\delta[n-2]) \\ &= \delta[n] + \frac{1}{2}\delta[n-1] = y[n] \end{aligned}$$

This implies that $h[n] = \delta[n] - \frac{1}{4}\delta[n-2]$.

$$\boxed{h[n] = \delta[n] - \frac{1}{4}\delta[n-2]}$$

Name/Student ID: ____

Problem 2 (cont.)

Check the box next to the solution you found above, or check **None of the above** if it does not appear on the list.

Problem 2(a):

- ☐ $\frac{1-\frac{1}{2}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$
- ☐ $\frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$
- ☐ $\frac{1}{1-\frac{1}{4}e^{-j2\omega}}$
- ☒ $1 - \frac{1}{4}e^{-j2\omega}$
- ☐ None of the above.

Problem 2(b):

- ☒ $\delta[n] - \frac{1}{4}\delta[n-2]$
- ☐ $\delta[n] - \frac{1}{2}\delta[n-2]$
- ☐ $\left(\frac{1}{4}\right)^n u[n]$
- ☐ $\left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n-1]$
- ☐ None of the above.

Name/Student ID: ____________________

Problem 3 [Modulation] (20 points, 10 points each part)

Let $x(t) = \frac{\sin(20t)}{\pi t}$.

The signal $x(t)$ is modulated with a carrier signal $\cos(30t)$ to produce the signal $y(t) = x(t)\cos(30t)$. The signal $y(t)$ is passed through a bandpass filter with frequency response

$$H_{BP}(j\omega) = \begin{cases} 1, & 30 < |\omega| < 50 \\ 0, & \text{otherwise.} \end{cases}$$

Let $w(t)$ be the output of the bandpass filter. The signal $w(t)$ is modulated with a carrier signal $\cos(40t)$ to produce $z(t) = w(t)\cos(40t)$. The signal $z(t)$ is then passed through an ideal low-pass filter with gain 2 and cut-off frequency $\omega_c = 20$. Let $r(t)$ be the output of the low-pass filter.

(a) Determine the Fourier transform $R(j\omega)$ of $r(t)$, expressed in terms of $X(j\omega)$.

(b) Determine the output signal $r(t)$.

Write answers to parts (a) and (b) on the following pages.

Justify your answers.

Name/Student ID: __________________________________

Problem 3 (cont.)

(a) (10 points)

Determine the Fourier transform $R(j\omega)$ of $r(t)$, expressed in terms of $X(j\omega)$.

From Table 4.2, we have the CTFT transform pair

$$x[(t) \leftrightarrow X(j\omega) = \begin{cases} 1, & |\omega| < 20 \\ 0, & |\omega| > 20 \end{cases}$$

$Y(j\omega) = \frac{1}{2}\left(X(j(\omega+30)) + X(j(\omega-30))\right)$

Let $\tilde{X}(j\omega) = X(j2\omega) = \begin{cases} 1, & |\omega| < 10 \\ 0, & |\omega| > 10 \end{cases}$

Then $W(j\omega) = \frac{1}{2}\left(\tilde{X}(j(\omega+40)) + \tilde{X}(j(\omega-40))\right)$, and

$$\begin{aligned} Z(j\omega) &= \frac{1}{2}\left(W(j(\omega+40)) + W(j(\omega-40))\right) \\ &= \frac{1}{4}\tilde{X}(j(\omega+80)) + \frac{1}{4}\tilde{X}(j(\omega)) + \frac{1}{4}\tilde{X}(j(\omega)) + \tilde{X}(j(\omega-80)) \\ &= \frac{1}{4}\tilde{X}(j(\omega+80)) + \frac{1}{2}\tilde{X}(j(\omega)) + \frac{1}{4}\tilde{X}(j(\omega-80)) \end{aligned}$$

So $R(j\omega) = 2(\frac{1}{2}\tilde{X}(j(\omega))) = X(j2\omega)$.

$$\boxed{R(j\omega) = \tilde{X}(j(\omega)) = X(j2\omega)}$$

Name/Student ID: ____

Problem 3 (cont.)

(b) (10 points)

Determine the output signal $r(t)$.

From the result of part (a), and referring to Table 4.2,

$$r(t) = \frac{\sin(10t)}{\pi t}.$$

$$\boxed{r(t) = \frac{\sin(10t)}{\pi t}}$$

Name/Student ID: ______________________________

Problem 3 (cont.)

Check the box next to the solution you found above, or check **None of the above** if it does not appear on the list.

Problem 3(a):

- ☐ 0
- ☐ $X(j(\omega - 10)) + X(j(\omega + 10))$
- ☐ $\frac{1}{2}X(j\omega)$
- ☒ $X(j2\omega)$
- ☐ None of the above.

Problem 3(b):

- ☐ 0
- ☒ $\frac{\sin(10t)}{\pi t}$
- ☐ $\frac{\sin(10t)}{2\pi t}$
- ☐ $\frac{\sin(20t)}{\pi t}\cos(10t)$
- ☐ None of the above.

Name/Student ID: ______________________________

Problem 4 [Sampling Theory] (20 points, 10 points each part)

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$ satisfying

$$X(j\omega) = 0, \text{ for } |\omega| > W.$$

Determine the minimum sampling frequency ω_s required to avoid aliasing and allow reconstruction of $y(t)$ (defined below) from its samples, or write "Reconstruction not possible for any ω_s" if no such ω_s exists. Refer specifically to the formula for $Y(j\omega)$.

(a) $y(t) = (x(3t-1))^2$

(b) $y(t) = x(t) * \frac{\sin(2Wt)}{\pi t}$

Write answers to parts (a) and (b) on the following pages.

Justify your answers.

Name/Student ID: ____________________

Problem 4 (cont.)

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$ satisfying $X(j\omega) = 0$, for $|\omega| > W$. Determine the minimum sampling frequency ω_s required to avoid aliasing and allow reconstruction of $y(t)$ from its samples, or write "Reconstruction not possible for any ω_s" if no such ω_s exists. Refer to $Y(j\omega)$.

(a) (10 points)
$y(t) = (x(3t-1))^2$

The signal $x(3t-1)$ is obtained from $x(t)$ by:
(1) shifting by 1, then (2) scaling the result by 3.

Let $w(t) = x(t-1)$.

By the Time Shifting property in Table 4.1, we get
$W(j\omega) = e^{-j\omega}X(j\omega)$.

Let $z(t) = w(3t) = x(3t-1)$.
By the Scaling property in Table 4.1, we get
$Z(j\omega) = \frac{1}{3}W(\frac{j\omega}{3}) = \frac{1}{3}e^{-j\omega/3}X(\frac{j\omega}{3})$.
So, $|Z(j\omega)| = \frac{1}{3}|e^{-j\omega/3}X(\frac{j\omega}{3})| = \frac{1}{3}|X(\frac{j\omega}{3})|$.
Thus $Z(j\omega) = 0,\ |\omega| > 3W$.
By the Multiplication property of the CTFT in Table 4.1,we get

$$\begin{aligned} Y(j\omega) &= \frac{1}{2\pi}Z(j\omega) * Z(j\omega) \\ &= \frac{1}{2\pi}\left(\frac{1}{3}e^{-j\omega/3}X(\frac{j\omega}{3})\right) * \left(\frac{1}{3}e^{-j\omega/3}X(\frac{j\omega}{3})\right) \end{aligned}$$

The convolution of $Z(j\omega)$ with itself doubles the maximum frequency, so $Y(j\omega) = 0,\ |\omega| > 6W$.

We must have $\omega_s > 2\omega_{\max}$, so it follows that $\omega_s > 2(6W) = 12W$.

$\boxed{\omega_s > 12W}$

Name/Student ID: ______________________________

Problem 4 (cont.)

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$ satisfying $X(j\omega) = 0$, for $|\omega| > W$. Determine the minimum sampling frequency ω_s required to avoid aliasing and allow reconstruction of $y(t)$ from its samples, or write "Reconstruction not possible for any ω_s" if no such ω_s exists. Refer to $Y(j\omega)$.

(b) (10 points)

$y(t) = x(t) * \frac{\sin(2Wt)}{\pi t}$

Let $z(t) = \frac{\sin(2Wt)}{\pi t}$.

From Table 4.2, we have

$z(t) = \frac{\sin(2Wt)}{\pi t} \leftrightarrow Z(j\omega) = \begin{cases} 1, & |\omega| < 2W \\ 0, & |\omega| > 2W \end{cases}$

By the Convolution property of CTFT in Table 4.1, we have

$$\begin{aligned} Y(j\omega) &= X(j\omega)Z(j\omega) \\ &= X(j\omega) \end{aligned}$$

where we have used the fact that $X(j\omega) = 0$, for $|\omega| > W$.

It follows that $Y(j\omega) = 0, \ |\omega| > W$.

We must have $\omega_s > 2\omega_{\max}$, so $\omega_s > 2W$.

$\boxed{\omega_s > 2W}$

Name/Student ID: ______________________________

Problem 4 (cont.)

Check the box next to the solutions you found above, or check **"Reconstruction not possible for any** ω_s**"** if no such ω_s exists.

Problem 4(a):

- ☐ $3W$
- ☐ $6W$
- ☐ $9W$
- ☒ $12W$
- ☐ Reconstruction not possible for any ω_s.

Problem 4(b):

- ☐ $\frac{W}{2}$
- ☐ W
- ☒ $2W$
- ☐ $4W$
- ☐ Reconstruction not possible for any ω_s.

Name/Student ID: ______________________________

Problem 5 [Laplace Transform] (20 points)

Let

$$X(s) = \frac{s+2}{(s^2 - s - 2)} = \frac{s+2}{(s+1)(s-2)}.$$

(a) Determine the number N of distinct signals that have Laplace transform expressed as $X(s)$ in their region of convergence.

(b) Determine the signal $x(t)$ with Laplace transform $X(s)$ that has a Fourier transform.

(c) Determine an explicit formula for the magnitude $|X(j\omega)|$ of the Fourier transform of the signal $x(t)$ in part (b). Express it without using any imaginary numbers.

Write your answers to parts (a), (b), and (c) on the following pages.

Justify your answers.

Name/Student ID: ______________________________

Problem 5 (cont.)

Let $X(s) = \frac{s+2}{(s^2-s-2)} = \frac{s+2}{(s+1)(s-2)}$.

(a) (5 points)

Determine the number N of distinct signals that have Laplace transform expressed as $X(s)$ in their region of convergence.

The Laplace transform has first-order poles at $s = -1$ and $s = 2$.

The possible regions of convergence are:

ROC I: $\mathcal{R}e\{s\} < -1$

ROC II: $-1 < \mathcal{R}e\{s\} < 2$

ROC III: $\mathcal{R}e\{s\} > 2$

Each ROC corresponds to a distinct signal, so $N = 3$.

$$\boxed{N = 3}$$

Name/Student ID: ____________________

Problem 5 (cont.)

Let $X(s) = \frac{s+2}{(s^2-s-2)} = \frac{s+2}{(s+1)(s-2)}$.

(b) (10 points)

Determine the signal $x(t)$ with Laplace transform $X(s)$ that has a Fourier transform.

Expand $X(s)$ using Partial Fraction Expansion:

$$\begin{aligned} X(s) &= \frac{A}{s+1} + \frac{B}{s-2} \\ &= \frac{-1/3}{s+1} + \frac{4/3}{s-2} \end{aligned}$$

where we have used

$$\begin{aligned} A &= (s+1)X(s)\Big|_{s=-1} = \frac{s+2}{s-2}\Big|_{s=-1} = -\frac{1}{3} \\ B &= (s-2)X(s)\Big|_{s=2} = \frac{s+2}{s+1}\Big|_{s=2} = \frac{4}{3} \end{aligned}$$

The existence of the Fourier transform means that the ROC contains the $j\omega$-axis, so it must be ROC II: $-1 < \mathcal{R}e\{s\} < 2$.

By Table 9.2, we have the basic Laplace transform pairs:

$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$, $\mathcal{R}e\{s\} > -a$

$-e^{-at}u(-t) \leftrightarrow \frac{1}{s+a}$, $\mathcal{R}e\{s\} < -a$

We conclude that

$x(t) = -\frac{1}{3}e^{-t}u(t) - \frac{4}{3}e^{2t}u(-t)$.

$$\boxed{x(t) = -\frac{1}{3}e^{-t}u(t) - \frac{4}{3}e^{2t}u(-t)}$$

Name/Student ID: ______________________________

Problem 5 (cont.)

Let $X(s) = \frac{s+2}{(s^2-s-2)} = \frac{s+2}{(s+1)(s-2)}$.

(c) (5 points)

Determine an explicit formula for the magnitude $|X(j\omega)|$ of the Fourier transform of the signal $x(t)$ in part (b). Express it without using any imaginary numbers.

$$X(j\omega) = \frac{j\omega+2}{(j\omega+1)(j\omega-2)}.$$

$$|X(j\omega)| = \frac{|j\omega+2|}{|j\omega+1||j\omega-2|}.$$

Since $|j\omega+a| = \sqrt{\omega^2+a^2}$ for $a \in \mathbb{R}$, we have $|j\omega+2| = |j\omega-2|$, so

$$|X(j\omega)| = \frac{1}{|j\omega+1|} = \frac{1}{\sqrt{\omega^2+1}}.$$

$$\boxed{|X(j\omega)| = \frac{1}{\sqrt{\omega^2+1}}}$$

Name/Student ID: ____

Problem 5 (cont.)

Check the box next to the solutions you found above, or check **None of the above** if it does not appear on the list.

Problem 5(a):

- ☐ 1
- ☐ 2
- ☒ 3
- ☐ 4
- None of the above.

Problem 5(b):

- ☐ $-\frac{1}{3}e^{t}u(t) - \frac{4}{3}e^{-2t}u(-t)$
- ☐ $-\frac{1}{3}e^{-t}u(t) - \frac{4}{3}e^{2t}u(t)$
- ☐ $-\frac{4}{3}e^{t}u(t) - \frac{1}{3}e^{-2t}u(t)$
- ☐ $-\frac{4}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(t)$
- ☒ None of the above.

Problem 5(c):

- ☐ $|X(j\omega)| = \frac{1}{|w+1|}$
- ☐ $|X(j\omega)| = \frac{1}{|w-1|}$
- ☒ $|X(j\omega)| = \sqrt{\frac{1}{\omega^2+1}}$
- ☐ $|X(j\omega)| = \sqrt{\frac{\omega^2+4}{(\omega^2+1)(\omega^2-2)}}$
- ☐ None of the above.

Name/Student ID: ____________________

Problem 6 [LT and LTI Systems] (20 points)

Let $H(s)$ be given by

$$H(s) = \frac{s^2 - 1}{(s+2)(s-(-1+j))(s-(-1-j))}.$$

(a) Determine the region of convergence (ROC) of a causal, stable LTI system S_1 with transfer function $H_1(s) = H(s)$.

(b) Determine a differential equation relating the input $x(t)$ and corresponding output $y(t)$ of the system S_1 in part (a).

(c) The system S_1 is serially concatenated with another causal system S_2 that has impulse response $h_2(t) = \frac{3}{2}e^{t}u(t) - \frac{1}{2}e^{-t}u(t)$. Determine the transfer function $G(s)$ of the system obtained from this concatenation of S_1 and S_2.

Write your answers to parts (a), (b), and (c) on the following pages.

Justify your answers.

Name/Student ID: ____________________

Problem 6 (cont.)

Let $H(s) = \frac{s^2-1}{(s+2)(s-(-1+j))(s-(-1-j))}$.

(a) (5 points)

Determine the region of convergence (ROC) of a causal, stable LTI system S_1 with transfer function $H_1(s) = H(s)$.

The system has poles at $s = -2$, $s = -1 + j$, and $s = -1 - j$.

The ROC for a causal, stable system with a rational transfer function will be the half plane to the right of the rightmost pole (causal), and should contain the $j\omega$-axis (stable).

Therefore, the ROC is $\mathcal{R}e\{s\} > -1$.

$$\boxed{ROC = \mathcal{R}e\{s\} > -1}$$

Name/Student ID: ____________________

Problem 6 (cont.)

Let $H(s) = \frac{s^2-1}{(s+2)(s-(-1+j))(s-(-1-j))}$.

(b) (5 points) Determine a differential equation relating the input $x(t)$ and corresponding output $y(t)$ of the system S_1 in part (a).

$$\begin{aligned}(s+2)(s-(-1+j))(s-(-1-j)) &= (s+2)(s^2+2s+2)\\ &= s^3+4s^2+6s+4\end{aligned}$$

So,

$$H(s) = \frac{s^2-1}{s^3+4s^2+6s+4}$$

The transfer function

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

corresponds to the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

In this case, the differential equation is:

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = \frac{d^2x(t)}{dt^2} - x(t)$$

Alternatively, using $H(s) = \frac{Y(s)}{X(s)}$, cross multiply to get

$Y(s)(s^3+4s^2+6s+4) = X(s)(s^2-1)$

and apply the Differentiation in the Time Domain property of the Laplace Transform (Table 9.1).

Differential equation :

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = \frac{d^2x(t)}{dt^2} - x(t)$$

Name/Student ID: ______________________________

Problem 6 (cont.)

Let $H(s) = \frac{s^2-1}{(s+2)(s-(-1+j))(s-(-1-j))}$.

(c) (10 points)

The system S_1 is serially concatenated with another causal system S_2 that has impulse response $h_2(t) = \frac{3}{2}e^t u(t) - \frac{1}{2}e^{-t}u(t)$. Determine the transfer function $G(s)$ of the concatenation of S_1 and S_2.

Referring to Table 9.2, the transfer function of system S_2 is given by

$$\begin{aligned} H_2(s) &= \frac{3}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{1}{s+1}\right) \\ &= \frac{s+2}{s^2-1} \end{aligned}$$

with ROC given by $\mathcal{R}e\{s\} > 1$, the right half plane to the right of the rightmost pole at $s = 1$.

The serial concatentation of S_1 and S_2 has impulse response

$h(t) = h_1(t) * h_2(t)$

and, by the Convolution property of LT in Table 9.1, the corresponding transfer function is

$$\begin{aligned} G(s) &= H_1(s)H_2(s) \\ &= \left(\frac{s^2-1}{(s+2)(s^2+2s+2)}\right)\left(\frac{s+2}{s^2-1}\right) \\ &= \frac{1}{s^2+2s+2} \end{aligned}$$

$$\boxed{G(s) = \frac{1}{s^2+2s+2}}$$

Name/Student ID: ______________________________

Problem 6 (cont.)

Check the box next to the solutions you found above, or check **None of the above** if it does not appear on the list.

Problem 6(a):

- ☐ $-2 < \mathcal{R}e\{s\} < -1$
- ☒ $\mathcal{R}e\{s\} > 1$
- ☒ $\mathcal{R}e\{s\} > -1$
- ☐ $\mathcal{R}e\{s\} > -2$
- ☐ None of the above.

Problem 6(b):

- ☒ $\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = \frac{d^2x(t)}{dt^2} - x(t)$
- ☐ $\frac{d^3y(t)}{dt^3} - 4\frac{d^2y(t)}{dt^2} - 6\frac{dy(t)}{dt} - 4y(t) = \frac{d^2x(t)}{dt^2} - x(t)$
- ☐ $\frac{d^2y(t)}{dt^2} - y(t) = \frac{d^3x(t)}{dt^3} + 4\frac{d^2x(t)}{dt^2} + 6\frac{dx(t)}{dt} + 4x(t)$
- ☐ $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 6y(t) + 4 = \frac{dx(t)}{dt} - 1$
- ☐ None of the above.

Problem 6(c):

- ☐ $G(s) = s^2 + 2s + 2$
- ☒ $G(s) = \frac{1}{s^2+2s+2}$
- ☐ $G(s) = \frac{s^2-1}{s^2+2s+2}$
- ☐ $G(s) = \frac{(s^2-1)^2}{(s+2)^2(s^2+2s+2)}$
- ☐ None of the above.

Scratch page **Name/Student ID:** ______________________

Scratch page **Name/Student ID:** ______________________________

Scratch page **Name/Student ID:** ______________________________

Scratch page **Name/Student ID:** ____________________

Scratch page **Name/Student ID:** ______________________