Lecture 18

Sampling and reconstruction

Preview of today's lecture

- Sampling theorem
  - Establish the fundamental connection between continuous-time bandlimited signals and discrete-time signals
  - Illustrate the impact of sampling in the time and frequency domains

- Reconstruction theorem
  - Define the reconstruction formula
  - Explain the role of the sinc function in reconstruction
  - Illustrate reconstruction in time and frequency domains

- Important example
  - Be able to illustrate spectra with and without aliasing

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 16, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 15 Fourier transform

Fourier

Sampling theorem

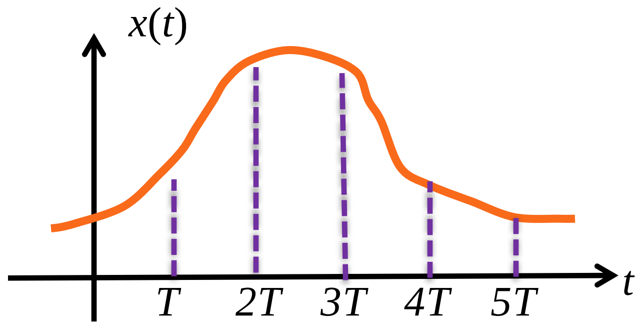
Key points

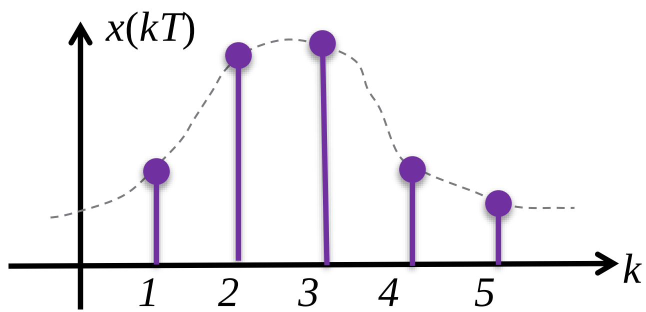
- Establish the fundamental connection between continuous-time bandlimited signals and discrete-time signals
- Illustrate the impact of sampling in the time and frequency domains

What is sampling?

- For a given CT signal $x(t)$

Kronecker delta functions

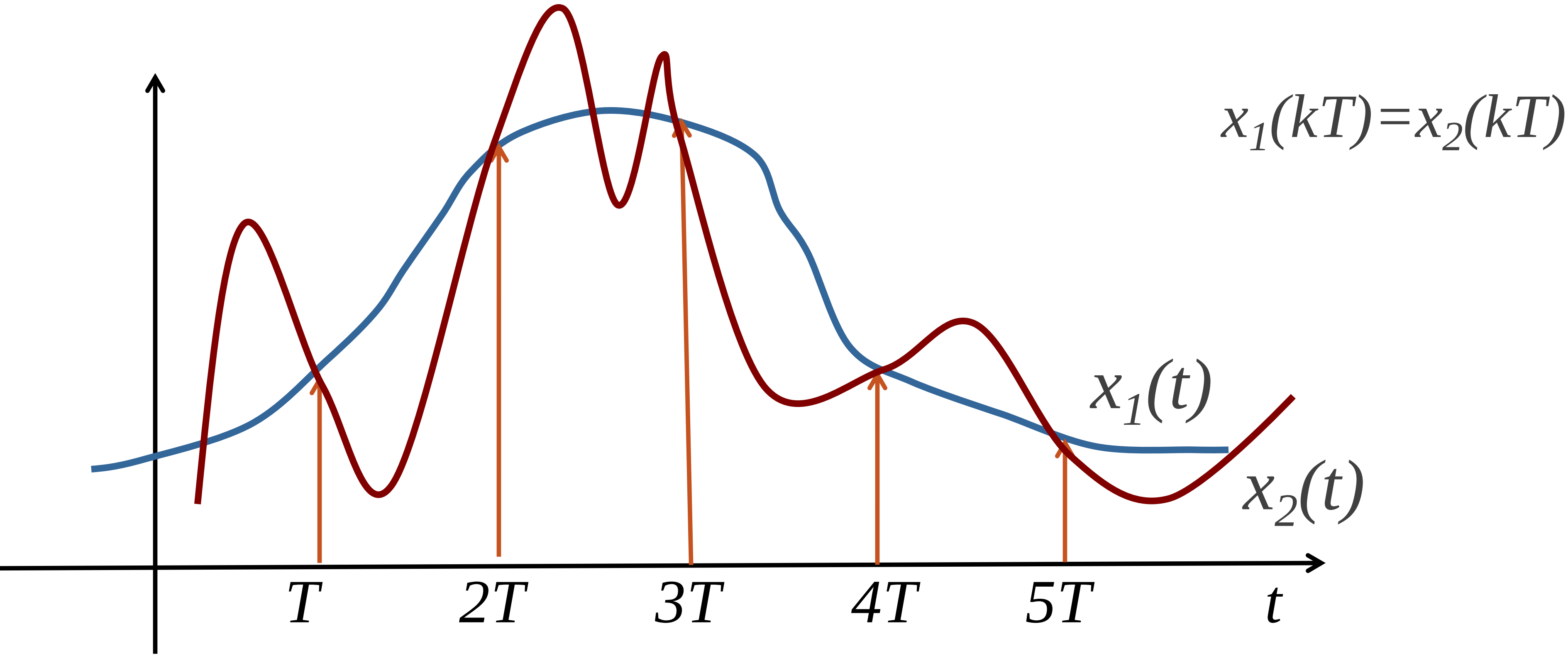




- The signal $x(kT)$ is called a sampled version of $x(t)$
  - The sampled signal is a discrete-time signal, written as $x[k]$

- The critical question related to sampling
  - Is it possible to recover $x(t)$ from $x(kT)$?

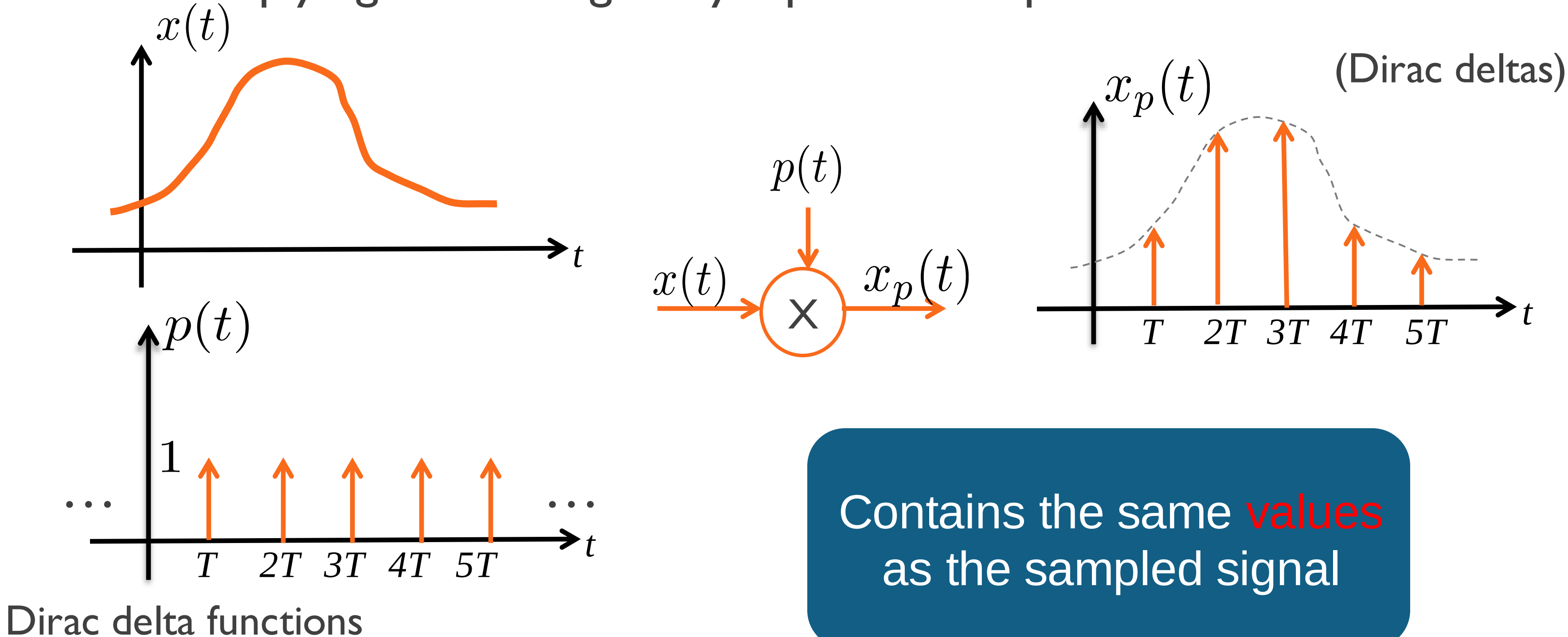
Why is there the potential for ambiguity?



Two different signals can have the same samples

Impulse-train periodic sampling

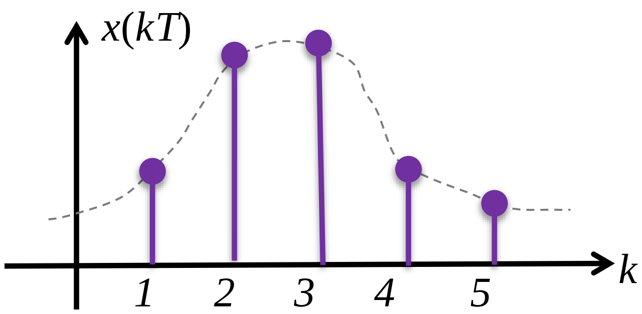
- A convenient way to understand periodic sampling is through multiplying the CT signal by a periodic impulse train



Contains the same values as the sampled signal

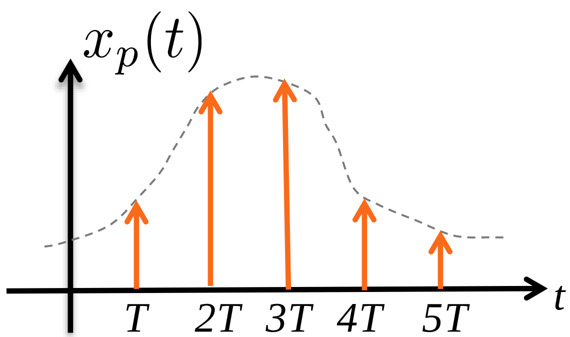
Our approach to sampling in this class

Discrete-time sampled signal



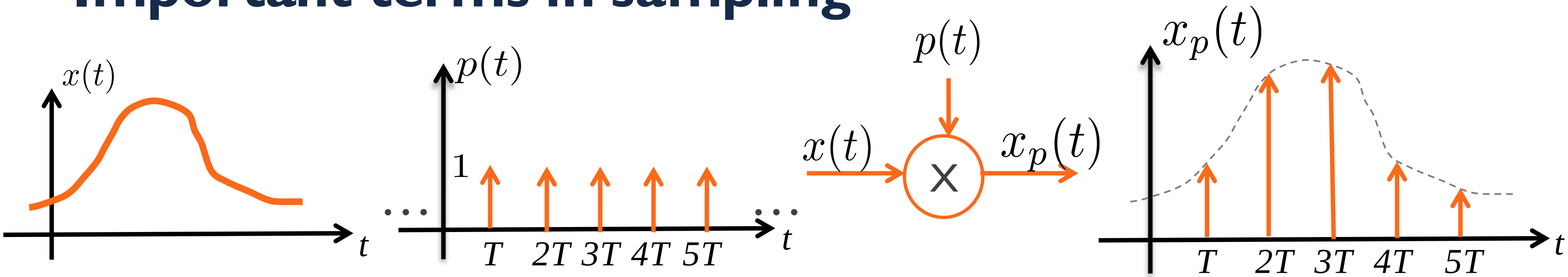
In digital signal processing, as explored in ECE 101, we care about the discrete-time signal

Impulse-train signal



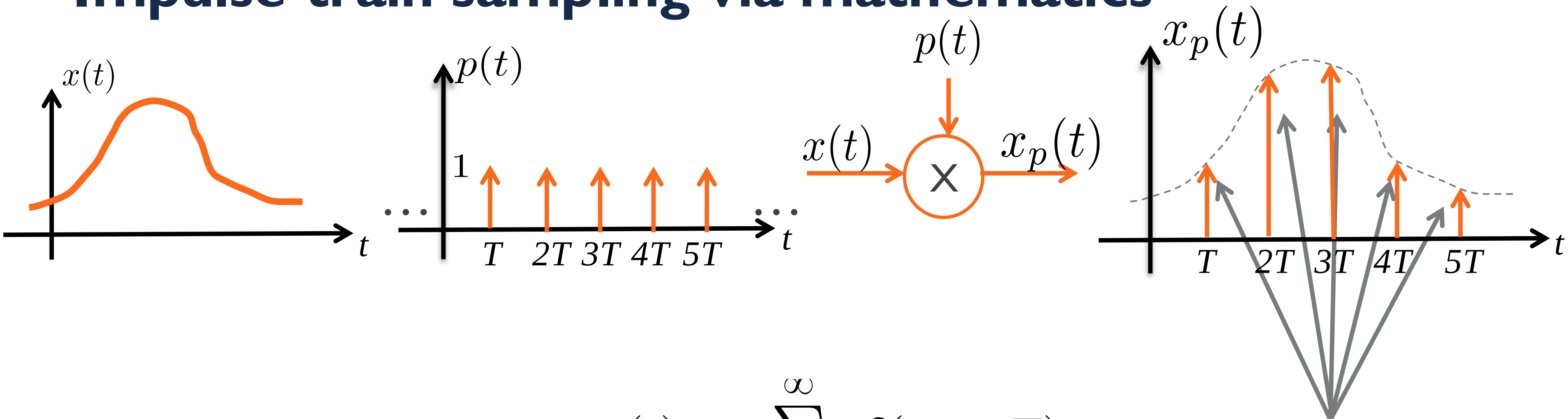
In this class, we will study sampling from the perspective of the impulse-train signal

Important terms in sampling



- The periodic impulse train $p(t)$ is the sampling function
- The period T is the sampling period
- The fundamental frequency of $p(t)$, $\omega_s = \dfrac{2\pi}{T}$ is the sampling frequency

Impulse-train sampling via mathematics



Sampled values ride the deltas

- The sampling function $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- The output signal $x_p(t) = x(t)p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Sampled signal is here via sifting property

Impulse train of samples in the frequency domain

- Multiplication in time domain → convolution in frequency domain

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

where

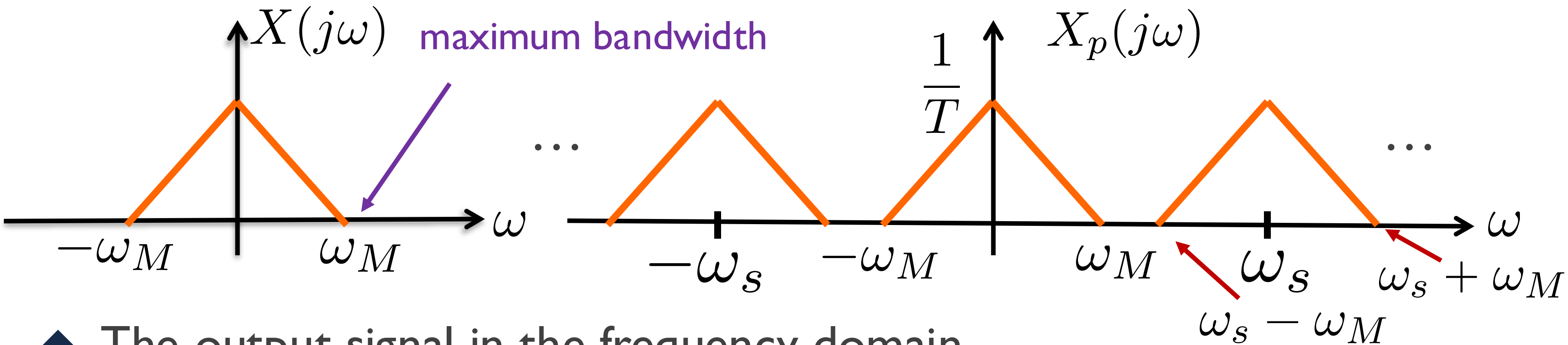
$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

$$\boxed{\omega_s = \frac{2\pi}{T}}$$

- Then

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Understanding the frequency domain effect



- The output signal in the frequency domain

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

- A superposition of shifted versions of $X(j\omega)$ scaled by $\frac{1}{T}$

If $\omega_s - \omega_M > \omega_M$ ➡ $\omega_s > 2\omega_M$ then original spectrum is undistorted

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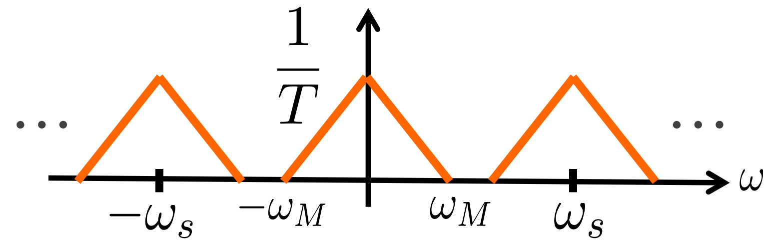
Sampling theorem

- When is $x(t)$ completely determined from $x_p(t)$?

- Sampling theorem
  - Let $x(t)$ be a band-limited signal with $X(j\omega) = 0 \text{ for } |\omega| > \omega_M$.
  - Then, $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \ldots$

If $\boxed{\omega_s > 2\omega_M}$ $\quad \omega_s = \dfrac{2\pi}{T}$

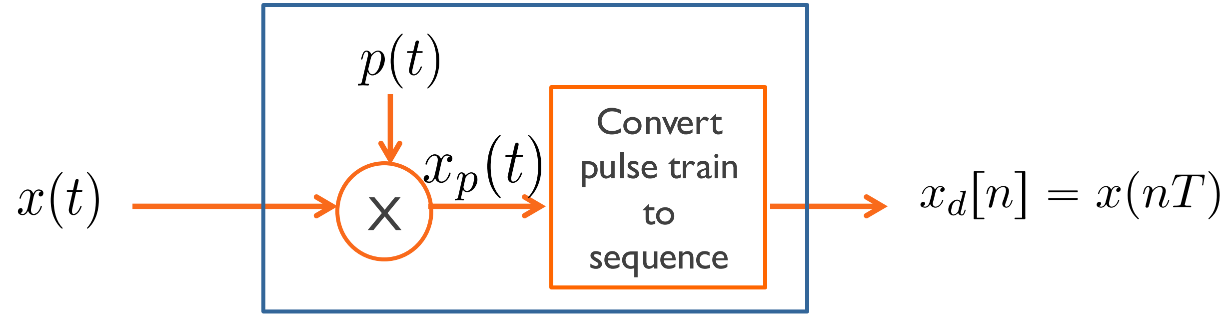
Nyquist frequency

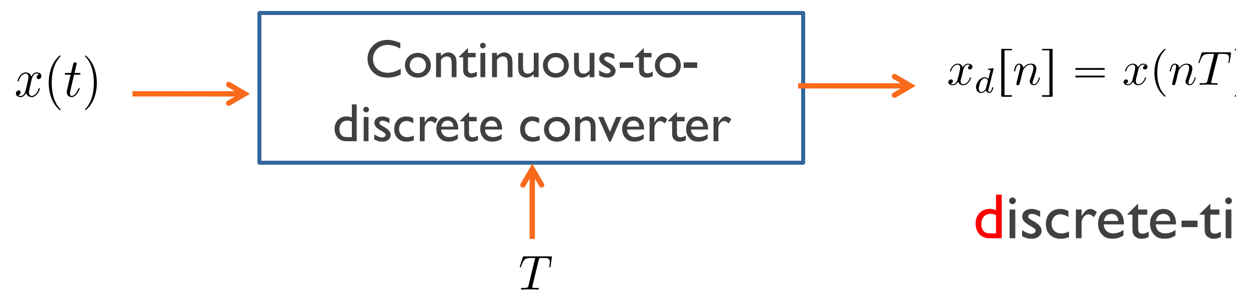


- The product $2\omega_M$ is called the "Nyquist rate"

Ideal continuous-to-discrete converter

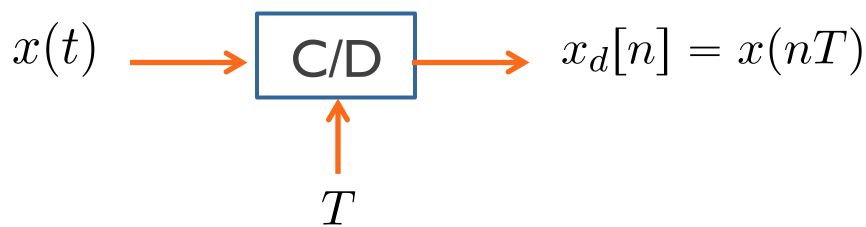
mathematical description





discrete-time

shorthand notation

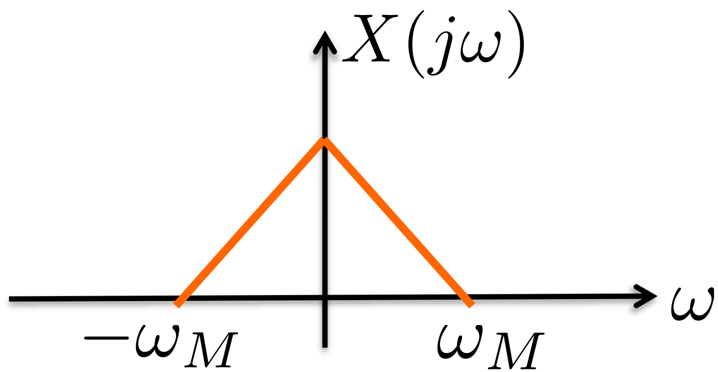


Connecting the domains

	time domain	frequency domain
CT signal	$x(t)$	$X(j\omega)$
impulse train of samples	$x_p(t) = \sum_n x(nT)\delta(t - nT)$	$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

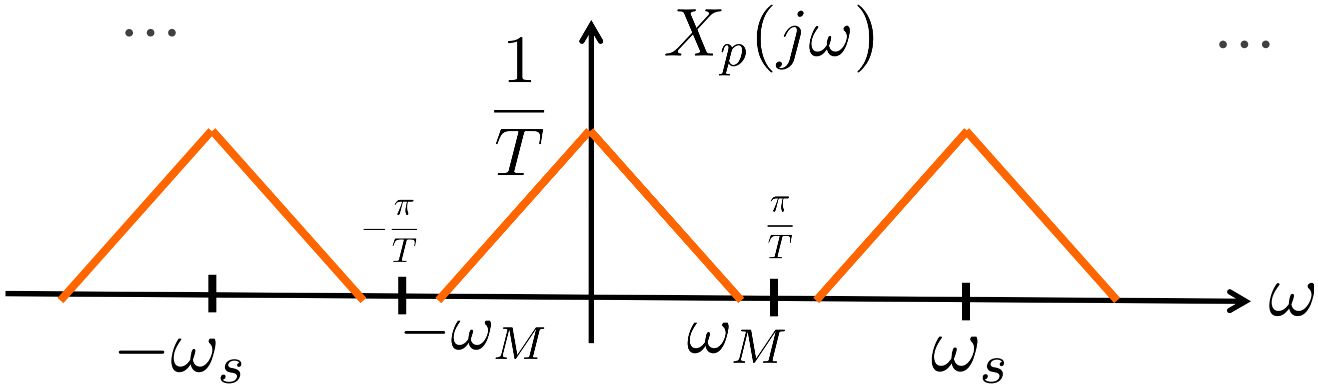
Sampling in the frequency domain – Nyquist OK

CT signal



$$\omega_s > 2\omega_M$$

impulse train of samples

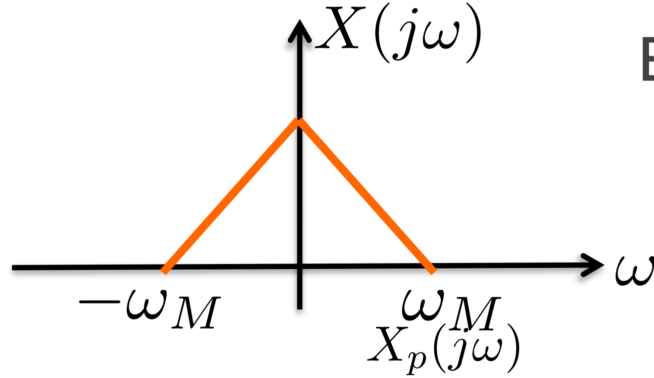


Sampling in the frequency domain – Nyquist Not OK

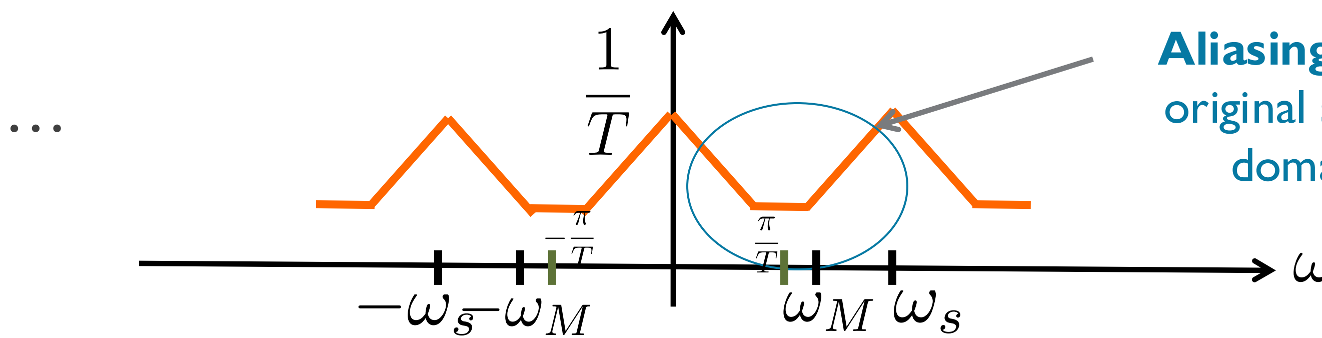
E.g.

$\omega_M < \omega_s < 2\omega_M$

CT signal



impulse train of samples



Aliasing! The shape original signal in freq domain is distort

...

Sampling summary

- The sampling theorem tells us when a continuous time signal may be periodically sampled with no loss
- The signal must be perfectly bandlimited and the sampling period must be small enough
- If the sampling theorem is not satisfied, it is still possible to sample the signal but aliasing will result

Classical example – the wagon wheel

Key points

- Explain the wagon wheel effect

The wagon wheel effect

- Classic demo
  - https://www.youtube.com/watch?v=VNftf5qLpiA
- Another demo that shows the effect of the number of spokes
  - https://www.youtube.com/watch?v=9MN5MF72PHs&t=34s
- Typical explanation on wikipedia
  - https://en.wikipedia.org/wiki/Wagon-wheel_effect

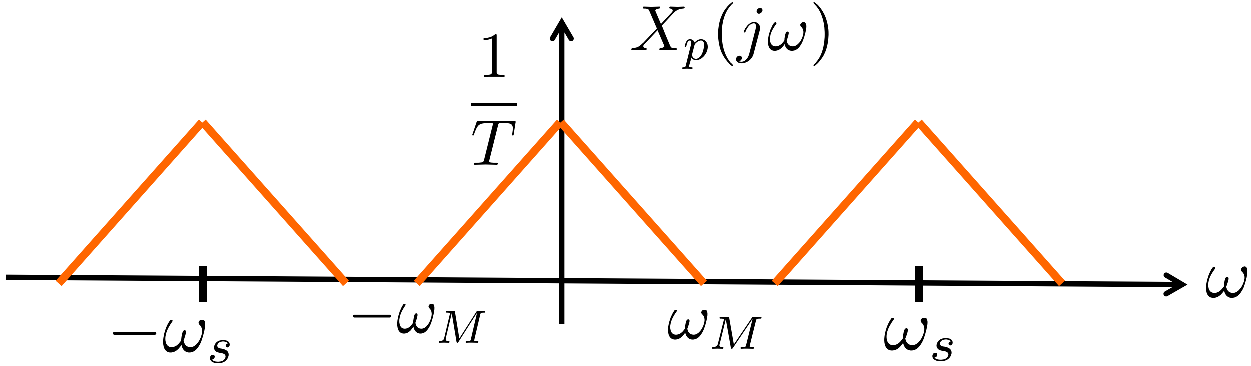
The reason that spoked wheels seem to go backwards is explained by the Nyquist sampling theorem

Reconstruction of a signal from its samples

Key points

- Define the reconstruction formula
- Explain the role of the sinc function in reconstruction
- Illustrate reconstruction in time and frequency domains

How to recover the original signal?



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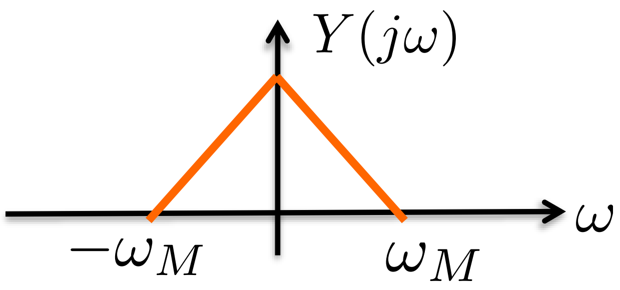
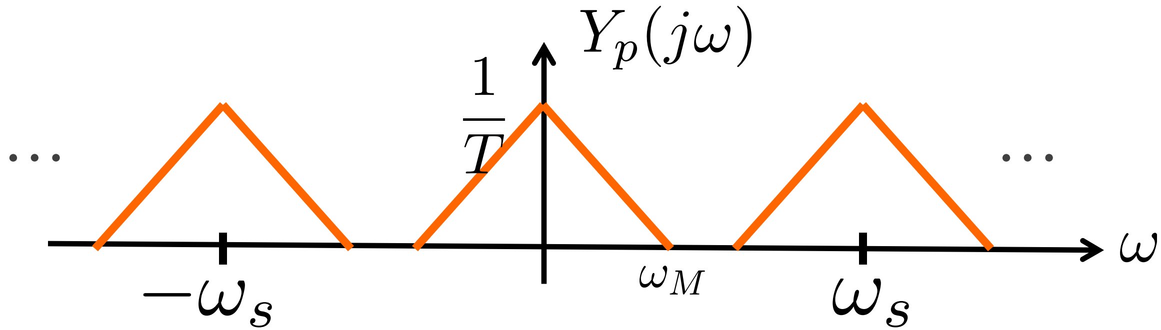


Ideally the original signal comes out if Nyquist was satisfied

Filter the reconstructed the signal

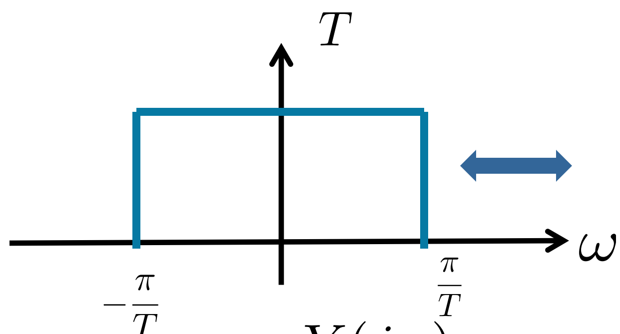
frequency domain



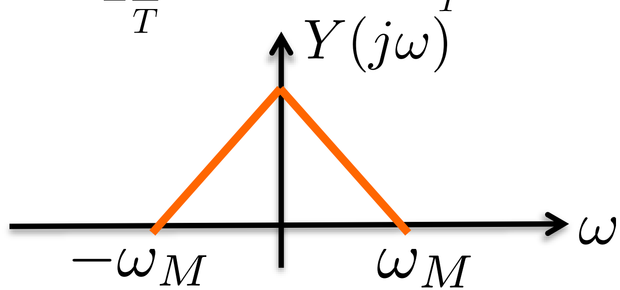
Ideal low pass filter

Cutoff $\frac{\pi}{T}$

Gain T



$$T\mathrm{rect}\left(\frac{\omega}{2\pi/T}\right) \leftrightarrow \mathrm{sinc}\left(\frac{t}{T}\right)$$



Filtering to reconstruct the signal

time domain

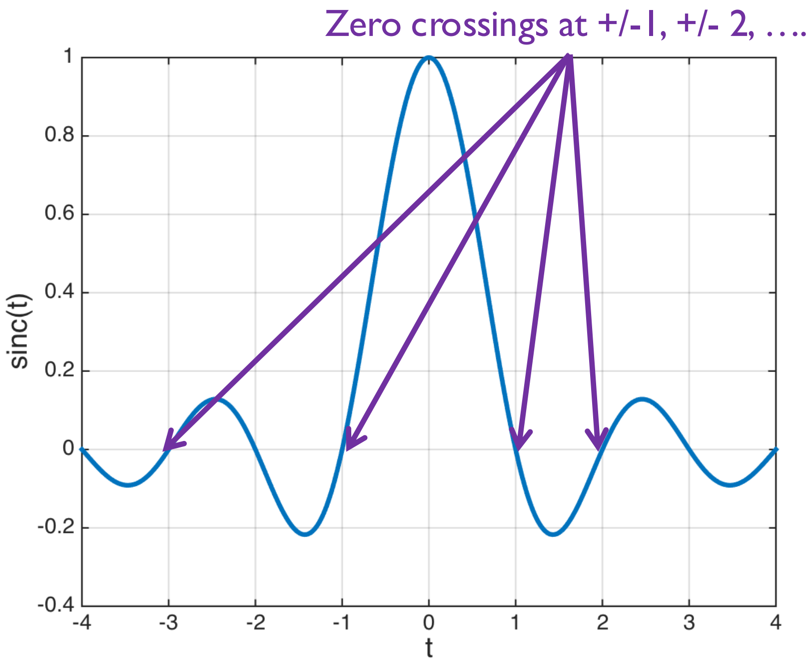
$x_d[n]$ → Convert to pulse train of samples → Lowpass filter cutoff π/T, gain T → $y(t)$

$$
\begin{aligned}
y(t) &= y_p(t) * h(t) \\
&= h(t) * \sum_{n=-\infty}^{\infty} x_d[n]\delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_d[n]h(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_d[n]\text{sinc}\left(\frac{t - nT}{T}\right)
\end{aligned}
$$

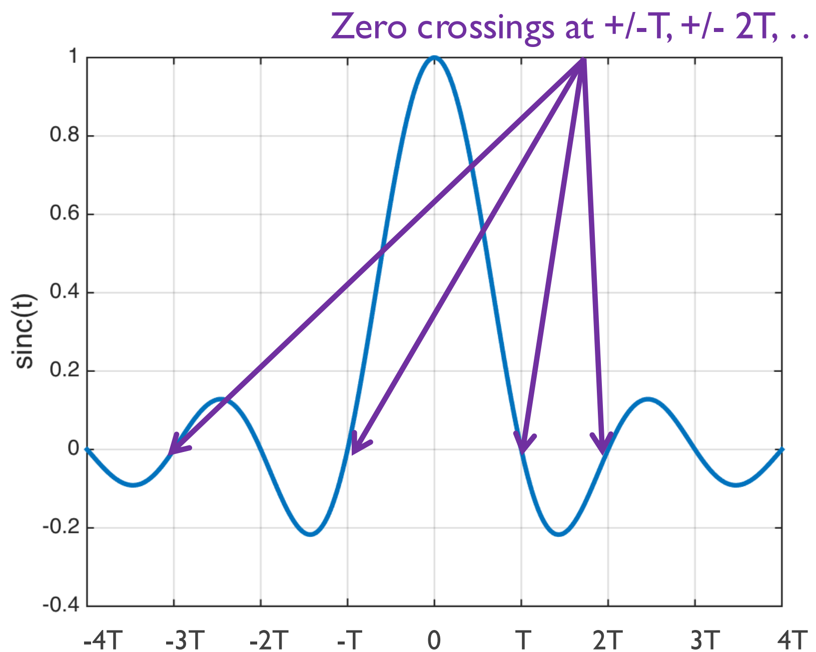
sinc interpolation

Reconstruction formula!

Step 2: Sinc is critical to reconstruction



$$\mathrm{sinc}\left(t\right)$$

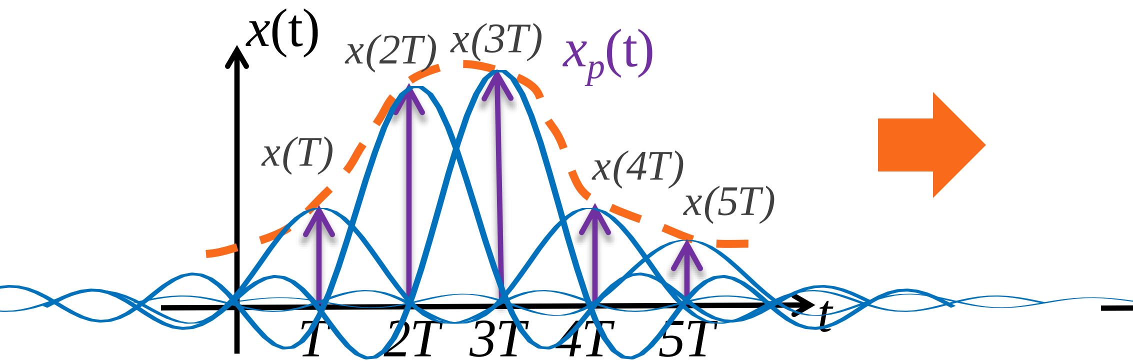


$$\mathrm{sinc}\left(\frac{t}{T}\right)$$

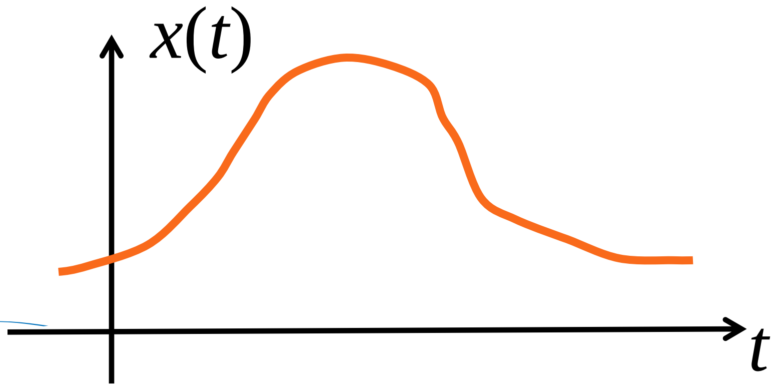
Zero crossings occur exactly at the sampling intervals

Reconstructing using the sinc functions

time domain

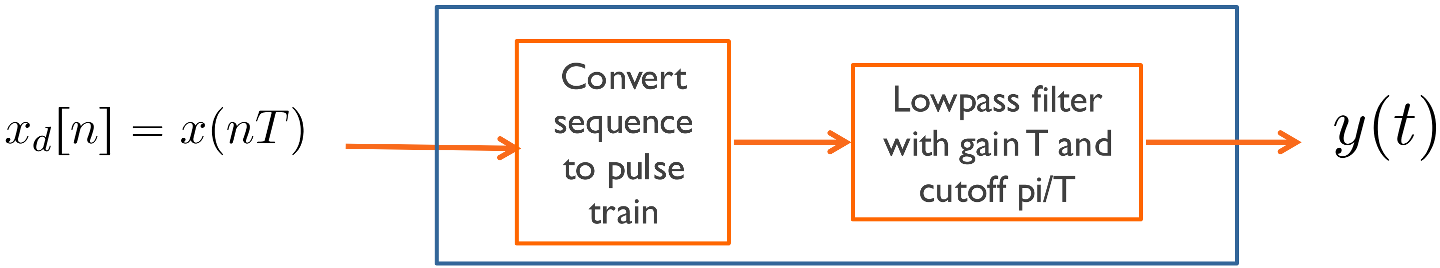


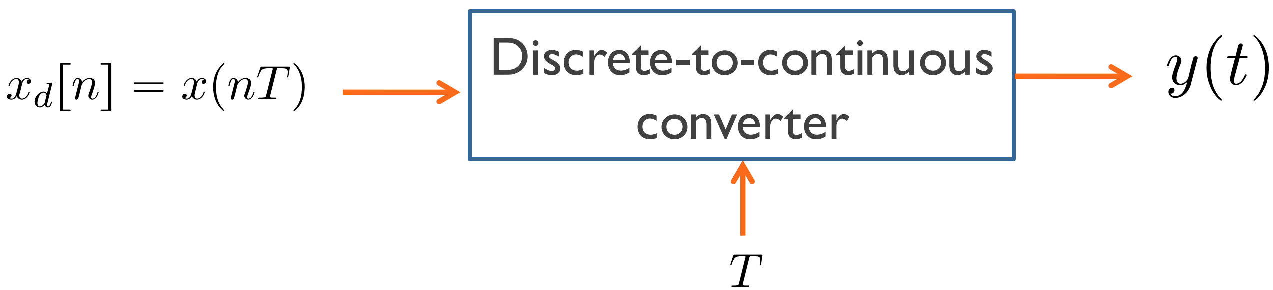
$$\sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

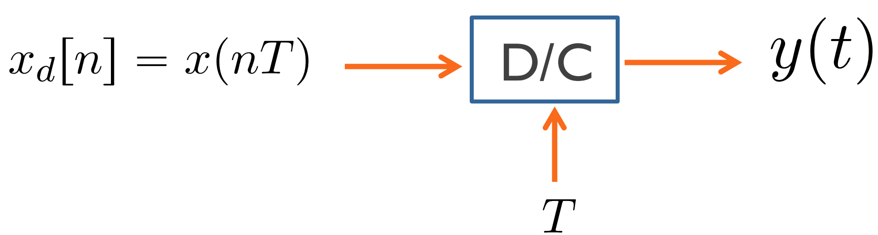


Reconstructed signal results from a superposition of sinc functions

Ideal discrete-to-continuous converter







Reconstruction in the time and frequency domains

	time domain	frequency domain
impulse train of samples	$y_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$	$Y_p(j\omega) = X_p(j\omega)$
CT signal	$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\frac{t-nT}{T}\right)$	$Y(j\omega) = T\text{rect}\left(\frac{\omega}{2\pi/T}\right)Y_p(j\omega)$ $= T\text{rect}\left(\frac{\omega}{2\pi/T}\right)X_p(j\omega)$

Output is always bandlimited no matter the input

If Nyquist is satisfied then

- In the frequency domain

$$Y(j\omega) = X(j\omega)$$

- In the time domain

$$y(t) = x(t)$$

Thoughts on reconstruction

- Optimal reconstruction involves interpolation of the samples with a sinc function, with the bandwidth determined by the reconstruction frequency

- The output of the discrete-to-continuous converter is always bandlimited

- Aliasing is created by sampling a signal with a sampling frequency less than the Nyquist rate and thus is the "fault" of the continuous-to-discrete conversion, reconstruction just operates on the samples already given

Important example involving the sampling and reconstruction of a sinusoid

Key points

- You should be able to determine the frequency of an undersampled sinusoid after reconstruction
- When Nyquist is not satisfied, aliasing is created

Sampling a sinusoid

- Consider the following signal

$$x(t) = \cos(37\pi t + \pi/4)$$

- Determine the following
  - Nyquist frequency $\omega_M = 37\pi$
  - Nyquist rate $2\omega_M = 74\pi$
  - Maximum sampling period $T < \dfrac{2\pi}{2\omega_M} = \dfrac{2\pi}{74\pi} = \dfrac{1}{37}$ seconds

- General form of sampled signal

$$x(nT) = \cos(37\pi nT + \pi/4)$$

Suppose Nyquist is satisfied

$$T < \frac{2\pi}{2\omega_M} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

$$\omega_s = \frac{2\pi}{T} = 2\pi 74 = 148\pi$$

- Suppose that $T = 1/74$ seconds
- Find the impulse train signal

$$\begin{aligned} x_p(t) &= \sum_n x(nT)\delta(t - n/74) \\ &= \sum_n \cos((37\pi/74)n + \pi/4)\delta(t - n/74) \\ &= \sum_n \cos((\pi/2)n + \pi/4)\delta(t - n/74) \end{aligned}$$

Find the CT transforms

$$T < \frac{2\pi}{2\omega_M} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

$$\omega_s = \frac{2\pi}{T} = 2\pi 74 = 148\pi$$

- Suppose that $T = 1/74$

- Find the CTFT

$$X(j\omega) = \pi e^{j\pi/4}\delta(\omega - 37\pi) + \pi e^{-j\pi/4}\delta(\omega + 37\pi)$$

- Find the CTFT of the impulse train signal

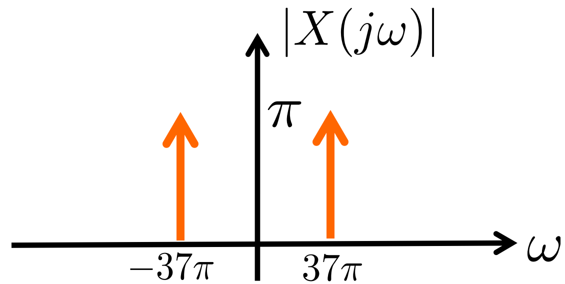
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$= 74\sum_{k=-\infty}^{\infty}\left(\pi e^{j\pi/4}\delta(\omega - k148\pi - 37\pi) + \pi e^{-j\pi/4}\delta(\omega - k148\pi + 37\pi)\right)$$

Oversampling (using a rate greater than Nyquist)

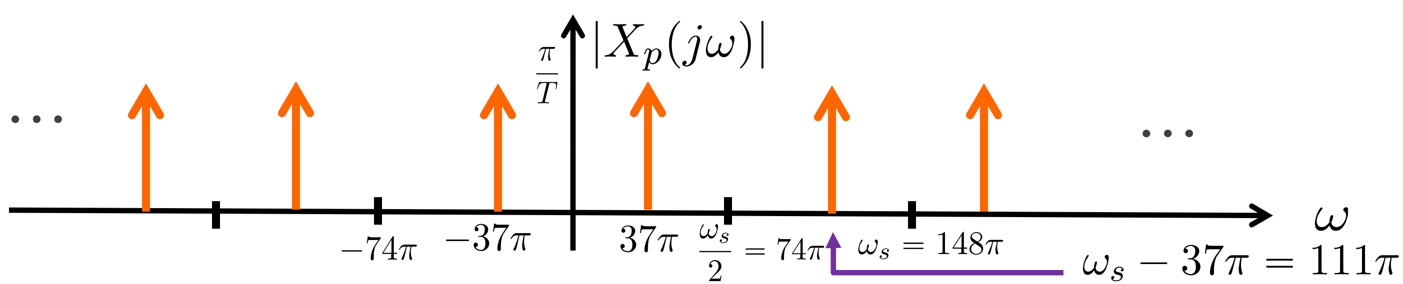
T=1/74

CT signal



$$\omega_s = \frac{2\pi}{T}$$
$$= 148\pi$$

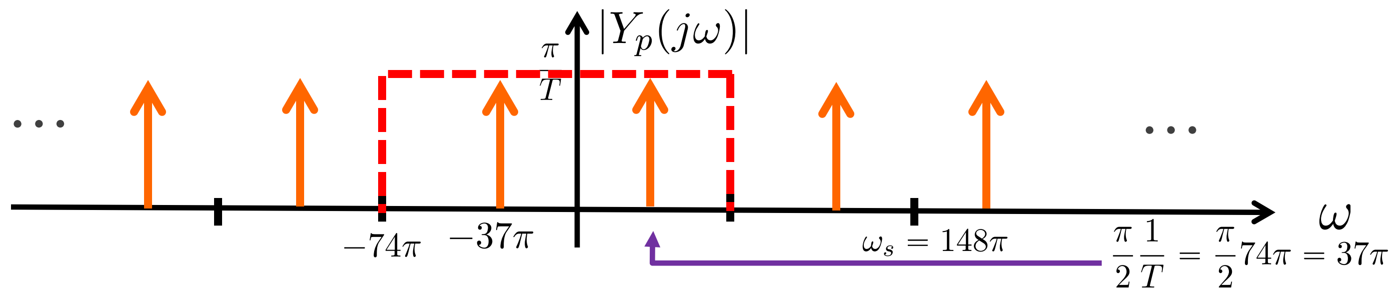
impulse train of samples



Reconstruction (using a rate greater than Nyquist)

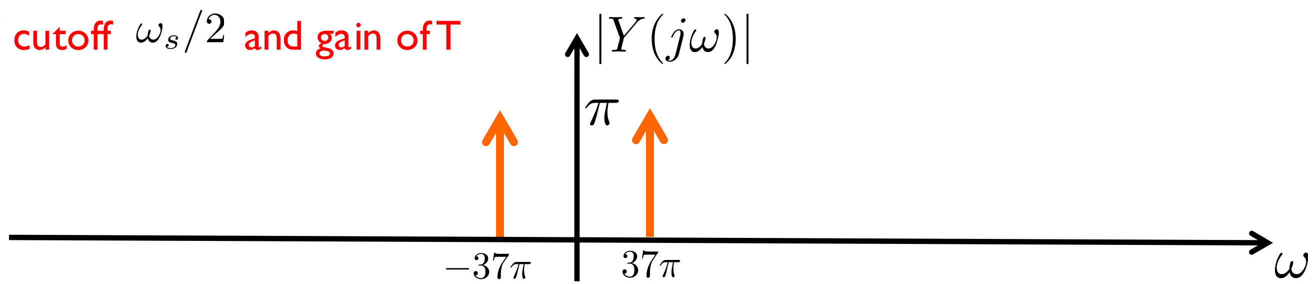
T=1/74

impulse train of samples



Ideal LPF w/ cutoff $\omega_s/2$ and gain of T

CT signal



What if Nyquist is not satisfied?

$$T < \frac{2\pi}{\omega_s} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

- Suppose that $T = \left(\frac{4}{3}\right)\frac{1}{37}$ which does not satisfy Nyquist

$$\omega_s = \frac{2\pi}{T} = \frac{111\pi}{2} = 55.5\pi$$

- Find the CTFT

$$X(j\omega) = \pi e^{j\pi/4}\delta(\omega - 37\pi) + \pi e^{-j\pi/4}\delta(\omega + 37\pi)$$

- Find the CTFT of the impulse train signal

$$\frac{1}{T} = 37\frac{3}{4} = 27.75$$

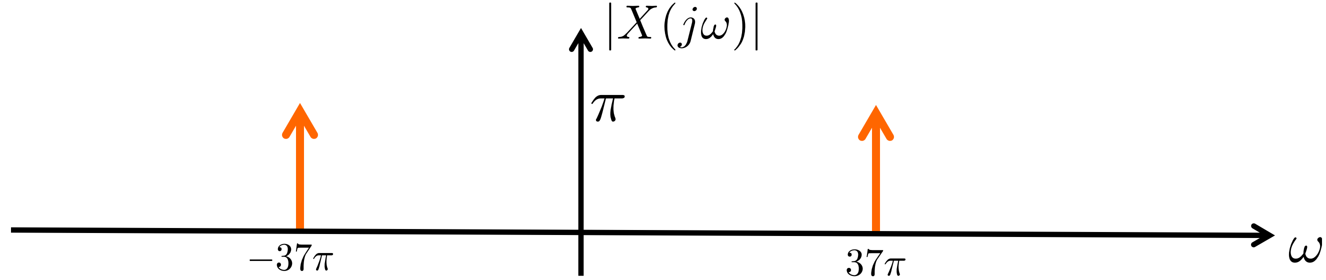
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$= 27.75 \sum_{k=-\infty}^{\infty} \left(\pi e^{j\pi/4}\delta(\omega - k55.5\pi - 37\pi) + \pi e^{-j\pi/4}\delta(\omega - k55.5\pi + 37\pi)\right)$$

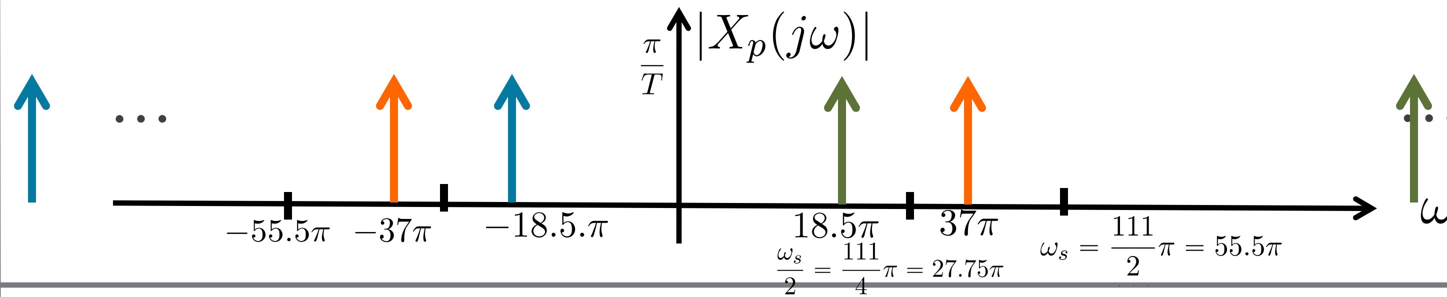
Undersampling (using a rate less than Nyquist)

$$T = \left(\frac{4}{3}\right)\frac{1}{37}$$

CT signal



impulse train of samples



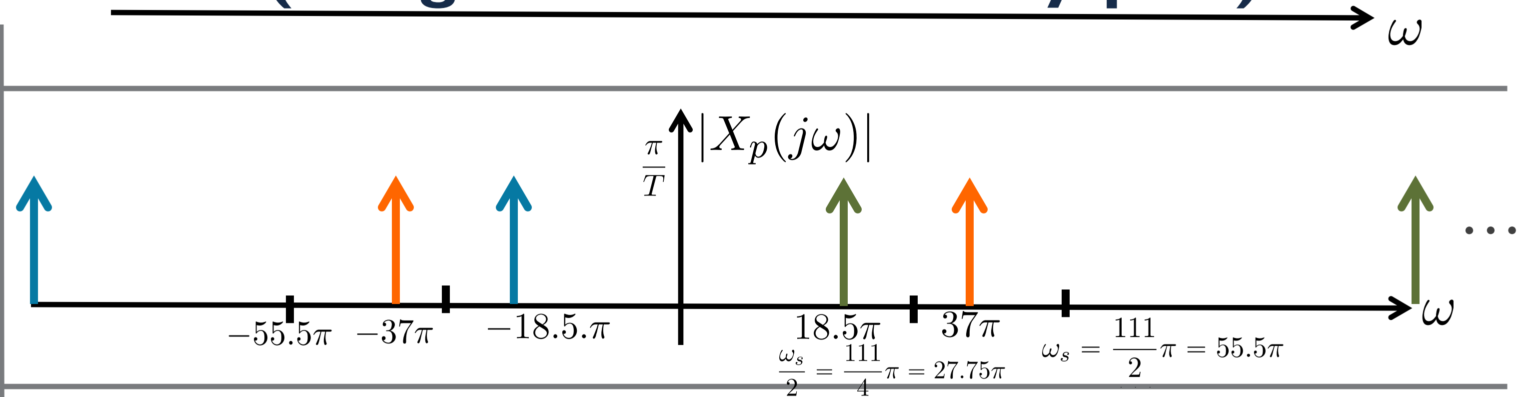
Replica at $-\omega_s$ Original signal Replica at ω_s

$$\omega_s - 37\pi = \frac{111 - 74}{2}\pi = 18.5\pi$$

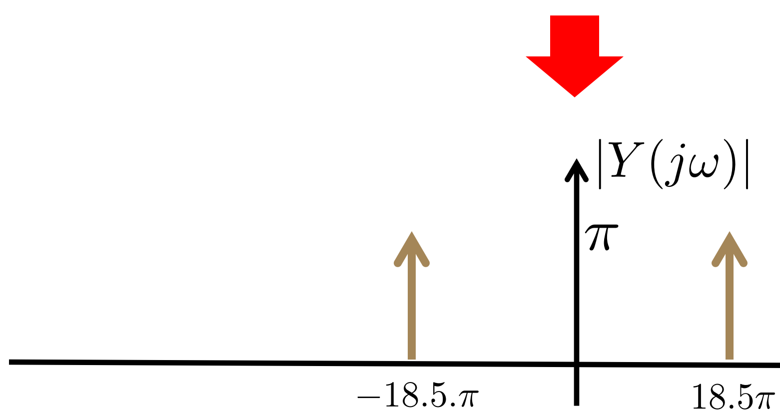
Reconstruction (using a rate less than Nyquist)

$$T = \left(\frac{4}{3}\right)\frac{1}{37}$$

impulse train of samples



CT signal



Frequency changed after sampling and reconstruction since Nyquist was not satisfied!