UNIVERSITY OF CALIFORNIA SAN DIEGO
Dept. of Electrical and Computer Engineering

Midterm #1

Date: February 4, 2025 Course: ECE 45

Name: ____
Last, First

- The exam will last 75 minutes.
- Please highlight your final solution with a $\boxed{\text{box}}$ if you want it graded. The solutions must be indicated in the corresponding region for the problem and clearly indicated.
- Point values for a problem are *evenly divided among subproblems* unless otherwise noted.
- The exam must be turned in at the conclusion of class when the instructor says that time is up. Late exams will not be graded.

Problem	Point Value	Your Score	Topic
1	25		System properties
2	25		Picaso's problem
3	25		Convolution
4	25		Phantastic phasors
Total	100		

Some formulas that may be useful

$$\text{Euler's theorem:} \quad e^{ju} = \cos u + j \sin u$$

$$\cos u = \frac{1}{2}\left(e^{ju} + e^{-ju}\right)$$

$$\sin u = \frac{e^{ju} - e^{-ju}}{2j}$$

$$\sin^2 u + \cos^2 u = 1$$

$$\cos^2 u - \sin^2 u = \cos 2u$$

$$2 \sin u \cos u = \sin 2u$$

$$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$$

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u)$$

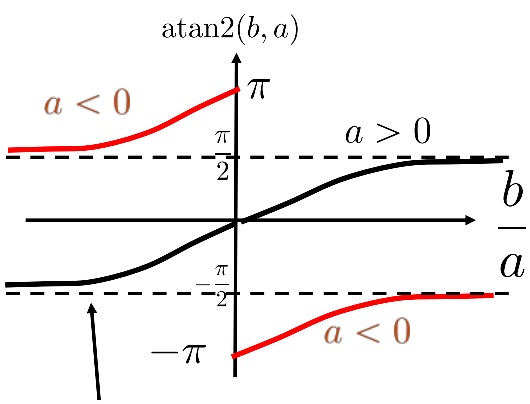
$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

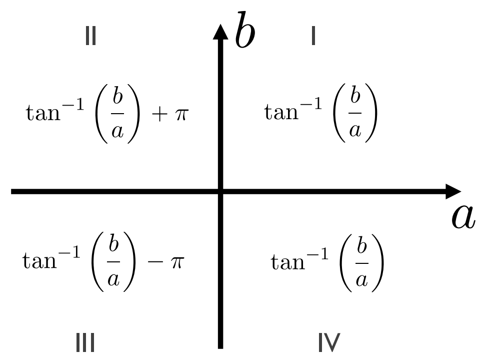
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u - v) + \sin(u + v)]$$





this is the typical arctangent computed from b/a

Problem 1.1 System properties (25 points)

Consider a system with the following input-output relationship. In this system, the input is $x(t)$, the output is $y(t)$, t is the time variable, and a is a real number

$$y(t) = 2x(t-2)\cos(at).$$

Please circle the correct answer or put your answer in the space provided along with the justification below. Be sure to comment on the dependence (if any) with a.

(a) Is this system linear? Yes No

Solution: Yes

(b) Is the system time invariant? Yes No

Solution: No unless $a = 0$.

(c) Compute the impulse response of the system. Simplify for the most credit.

Solution: We set $x(t) = \delta(t)$ and solve

$$\begin{aligned} h(t) &= 2\delta(t-2)\cos(at) \\ &= 2\delta(t-2)\cos(2a) \end{aligned}$$

where the second equality follows from the sifting property.

(d) If $x(t) = 2\cos(2at)$, determine the period of $y(t)$.

Solution: In this case

$$\begin{aligned} y(t) &= 2\cos(2a(t-2))\cos(at) \\ &= 2\cos(2at-4a)\cos(at) \\ &= \frac{1}{2}2\left[\cos(2at-4a-at)+\cos(2at-4a+at)\right] \\ &= \left[\cos(at-4a)+\cos(3at-4a)\right]. \end{aligned}$$

This is a sum of two sinusoids. Assuming $a \neq 0$, the first term is periodic with period $T_1 = 2\pi/a$ and the second term is periodic with period $T_2 = \frac{1}{3}2\pi/a$. The least common multiple of the two periods is $T = 2\pi/a$.

If $a = 0$ then $y(t) = 2$ and the signal is a constant or DC. It is equivalently periodic with any period T.

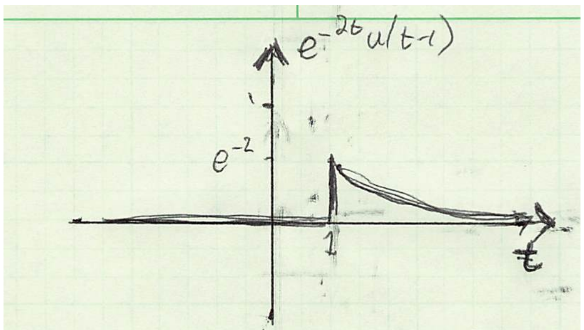
You could solve the problem in different ways. This is just one way to do it.

Problem 1.2 Picaso's problem (18 points)

Sketch the following functions

(a) $e^{-2t}u(t-1)$

Solution: Must label x and y axis and draw appropriately. Should label the starting point of the exponential of e^{-2} and should also indicate clearly the values that are zero.



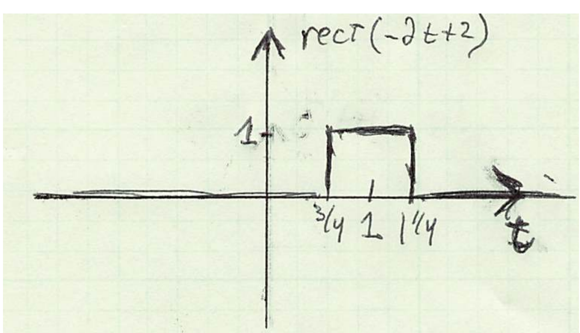
(b) $\text{rect}\,(-2t+2)$

Solution:

Must label x and y axis and draw appropriately. Note that

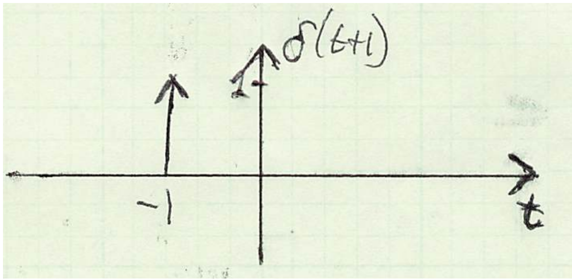
$$\begin{aligned}\text{rect}\,(-2t+2) &= \text{rect}\,(2t-2)\\ &= \text{rect}\,(2(t-1))\,.\end{aligned}$$

The signal $\text{rect}(2t)$ is a rectangle function with amplitude of 1 from $-1/4$ to $1/4$. Shifting this signal gives



(c) $\delta(t+1)$

Solution: This problem you need to plot the arrow that we use in the class to denote the delta function and label the axis as appropriate include the value of 1 on the y-axis.



Problem 1.3 Convolution (32 points)

Perform the following convolution (where we use $*$ as shorthand for the convolution operation as in the class)

$$y(t) = e^{-2t}u(t-1) * \mathrm{rect}\,(-2t+2) * \delta(t+1).$$

Solution: There are different ways to solve this problem. For example, you could solve $e^{-2t}u(t-1) * \mathrm{rect}\,(-2t+2)$ and then apply the shifting to the solution due to $\delta(t+1)$. This is the approach taken for example in the homework.

Or you could solve $\mathrm{rect}\,(-2t+2) * \delta(t+1)$ and then convolve the result with $e^{-2t}u(t-1)$. There are other ways as well.

We provide an example doing the second case. First note that

$$\begin{aligned}\mathrm{rect}\,(-2t+2) * \delta(t+1) &= \mathrm{rect}\,(-2(t+1)+2) \\ &= \mathrm{rect}\,(-2t)\end{aligned}$$

.

Now we compute

$$\begin{aligned}y(t) &= e^{-2t}u(t-1) * \mathrm{rect}\,(-2t) \\ &= \int_{-\infty}^{\infty} e^{-2\tau}u(\tau-1)\mathrm{rect}(-2(t-\tau))d\tau\end{aligned}$$

When $\frac{3}{4} \leqslant t < \frac{5}{4}$, $[1,\infty) \cap [-\frac{1}{4}+t, \frac{1}{4}+t] = [1, \frac{1}{4}+t]$, thus

$$\begin{aligned}y(t) &= \int_{1}^{\frac{1}{4}+t} e^{-2\tau} \cdot 1 d\tau \\ &= \left.\frac{e^{-2\tau}}{-2}\right|_{1}^{\frac{1}{4}+t} \\ &= \frac{e^{-2} - e^{-\frac{1}{2}-2t}}{2}\end{aligned}$$

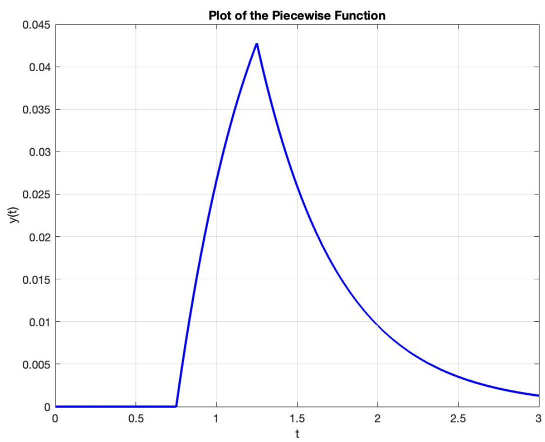
When $t \geqslant \frac{5}{4}$, $[1,\infty) \cap [-\frac{1}{4}+t, \frac{1}{4}+t] = [-\frac{1}{4}+t, \frac{1}{4}+t]$, thus

$$\begin{aligned}y(t) &= \int_{-\frac{1}{4}+t}^{\frac{1}{4}+t} e^{-2\tau} \cdot 1 d\tau \\ &= \left.\frac{e^{-2\tau}}{-2}\right|_{-\frac{1}{4}+t}^{\frac{1}{4}+t} \\ &= \frac{e^{\frac{1}{2}-2t} - e^{-\frac{1}{2}-2t}}{2} \\ &= \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2} \cdot e^{-2t}\end{aligned}$$

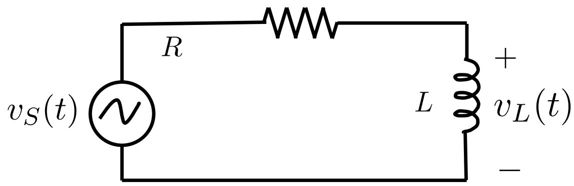
Therefore

$$y(t) = \begin{cases} \frac{e^{-2} - e^{-\frac{1}{2} - 2t}}{2}, & \text{if } \frac{3}{4} \leqslant t < \frac{5}{4} \\ \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2} \cdot e^{-2t}, & \text{if } t \geqslant \frac{5}{4} \\ 0, & \text{otherwise} \end{cases}$$

For reference, here is a plot of the solution, though a plt was not required to get the problem correct.



Problem 1.4 Phantastic phasors (25 points)



Consider the circuit shown above. Let $v_S(t)$ denote the input voltage signal and let $v_L(t)$ denote the output measured voltage signal. Assuming that $v_S(t) = A\cos(\omega t)$, find $v_L(t)$. You can assume that $A > 0$.

Solution: We use an approach based on phasors. The idea is to write $v_S(t)$ as a phasor $\mathbf{V}_S$ and $v_L(t)$ as a phasor $\mathbf{V}_L$ then to solve and go back into the time domain:

$$\begin{aligned}
\mathbf{V}_S &= A \\
\mathbf{V}_L &= \frac{j\omega L}{R + j\omega L}\mathbf{V}_S \\
&= A\frac{j\omega L}{R + j\omega L} \\
|\mathbf{V}_L| &= A\sqrt{\frac{\omega^2 L^2}{R^2 + \omega^2 L^2}} \\
\angle\mathbf{V}_L &= \angle(j\omega L) - \angle(R + j\omega L) \\
&= \pi/2 - \text{atan2}(\omega L, R) \\
v_L(t) &= A|\mathbf{V}_L|\cos(\omega t + \angle\mathbf{V}_L).
\end{aligned}$$