ECE 101: Linear Systems Fundamentals

Summer Session II 2020 - Lecture 8

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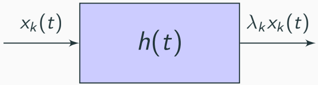
Today's topics

- Eigenfunctions of LTI systems
- Definition of continuous-time Fourier series (CTFS)
- The synthesis and analysis equations for CTFS
- Properties of continuous-time Fourier series

Signals and Systems (2th Edition): sections 3.2, 3.3, 3.4, and 3.5

Fourier series: motivation

- Suppose that we can find a rich family of signals $x_k(t)$ such that:

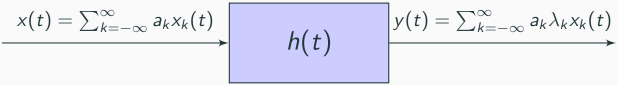


for a scalar $\lambda_k \in \mathbb{C}$

- And suppose that you can write a signal $x(t)$ as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k x_k(t).$$

- Then what would be the response to $x(t)$?



Eigenfunctions of LTI Systems

Eigenfunctions

Definition.

- In an LTI system, if the output signal is a scaled version of its input, then the input function is called an **eigenfunction** of the system. The scaling factor is called the **eigenvalue** of the system.

Eigenfunctions: CT LTI systems

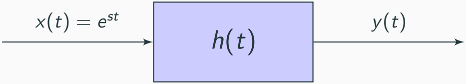
- Let's investigate the output of a CT LTI system to the input signal $x(t) = e^{st}$ where $s \in \mathbb{C}$

$$\xrightarrow{x(t)=e^{st}} \boxed{h(t)} \xrightarrow{y(t)}$$

- We have

$$\begin{aligned}
y(t) &= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau} d\tau \\
&= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau
\end{aligned}$$

Eigenfunctions: CT LTI systems



- We have

$$y(t) = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

- If we define $H(s)$ as

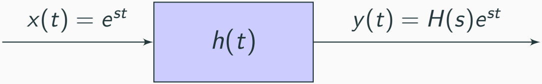
$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

  then

$$y(t) = H(s) e^{st} = H(s) x(t)$$

- $H(s)$ is called the **transfer function** of the system.
- $H(s)$ is determined by the impulse response, $h(t)$, and is independent of t.

Exponentials are Eigenfunctions



- Using the definition of eigenfunction, we have shown that

  1. e^{st} is an eigenfunction of any continuous-time LTI system, and
  2. $H(s)$ is the corresponding eigenvalue

Exponentials are Eigenfunctions

- Setting $s = j\omega$, we specialize to the subclass of periodic complex exponentials of the form

$$x(t) = e^{st} = e^{j\omega t}, \ \ \omega \in \mathbb{R}$$

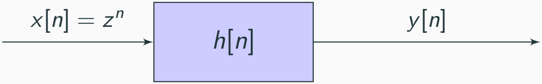
- For these eigenfunctions, the corresponding eigenvalue is

$$H(s)\Big|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

- $H(j\omega)$ is called the **frequency response** of the LTI system

Eigenfunctions: DT LTI systems

- Let's investigate the output of a DT LTI system to the input signal $x[n] = z^n$ where $z \in \mathbb{C}$



- We have

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \\ &= \sum_{k=-\infty}^{\infty} h[k] z^n z^{-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} \end{aligned}$$

Eigenfunctions: CT LTI systems

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- We have

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

- If we define $H(z)$ as

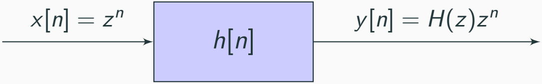
$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

  then

$$y[n] = H(z) z^n = H(z) x[n]$$

- $H(z)$ is called the **transfer function** of the system.
- $H(z)$ is determined by the impulse response, $h[n]$, and is independent of n.

Exponentials are Eigenfunctions



- Using the definition of eigenfunction, we have shown that

  1. z^n is an eigenfunction of any discrete-time LTI system, and
  2. $H(z)$ is the corresponding eigenvalue

Exponentials are Eigenfunctions

- Setting $z = e^{j\Omega}$, we specialize to the subclass of periodic complex exponentials of the form

$$x[n] = z^n = e^{j\Omega n}, \;\; \Omega \in \mathbb{R}$$

- For these eigenfunctions, the corresponding eigenvalue is

$$H(z)\Big|_{z=e^{j\Omega}} = H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

- $H(e^{j\Omega})$ is called the **frequency response** of the LTI system

Continuous-time Fourier Series (CTFS)

Decomposing Periodic Signals

How can we decompose signals into sum of exponentials (eigenfunctions of LTI systems)?

Decomposing Periodic Signals

- The question: which CT signals $x(t)$ can be expressed as a linear combination of complex exponentials?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

  where $\omega_0 \in \mathbb{R}^+$.

- Answer: The periodic signals with period $T = \frac{2\pi}{\omega_0}$ that have finite energy over one period or satisfy the Dirichlet conditions (see the textbook section 3.4)

Decomposing Periodic Signals

Decomposing Periodic Signals

For any periodic signal $x(t)$ with a period $T > 0$, that has finite energy over a period: we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{1}$$

where

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \tag{2}$$

and $\omega_0 = \frac{2\pi}{T}$.

- Eq. (1) is called the Fourier Series representation of $x(t)$.
- Coefficient (2) is called the kth Fourier coefficient of $x(t)$.
- a_k is unique

Key Equations for CT FS

- For a **periodic** signal $x(t)$ with *fundamental period* T and **fundamental frequency** $\omega_0 = \frac{2\pi}{T}$
- **Synthesis Equation:**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

- **Analysis Equation:**

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

- **Notation**

$$x(t) \xleftrightarrow{F.S.} a_k$$

CT Fourier Series: Example

$x(t+T) = x(t),\ T \in R$
$T > 0$

- What are the Fourier series coefficients for the signal $x(t) = 1 + \frac{1}{3}\cos(2\pi t) + \sin(3\pi t)$?

$x_1(t) = \cos(2\pi t) \longrightarrow T_{0_1} = \frac{2\pi}{\omega_{0_1}} = \frac{2\pi}{2\pi} = 1$

$\text{LCM}(1, \frac{2}{3}) = 2$

$x_2(t) = \sin(3\pi t) \longrightarrow T_{0_2} = \frac{2\pi}{\omega_{0_2}} = \frac{2\pi}{3\pi} = \frac{2}{3}$

$T_0 = 2,\quad \omega_0 = \frac{2\pi}{2} = \pi$

$$x(t) = 1 + \frac{1}{3}\left[\frac{1}{2} e^{j2\pi t} + \frac{1}{2} e^{-j2\pi t}\right] + \frac{e^{j3\pi t}}{2j} - \frac{e^{-j3\pi t}}{2j}$$

Note: $e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$\Longrightarrow \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

CT Fourier Series: Example

- What are the Fourier series coefficients for the signal $x(t) = 1 + \frac{1}{3}\cos(2\pi t) + \sin(3\pi t)$?

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\pi t} =$$

$$\cdots + a_{-4} e^{-j4\pi t} + a_{-3} e^{-j3\pi t} + a_{-2} e^{-j2\pi t} + a_{-1} e^{-j\pi t} + a_0 e^{j\times 0} + a_1 e^{j\pi t} + a_2 e^{j2\pi t} + a_3 e^{j3\pi t} + \cdots$$

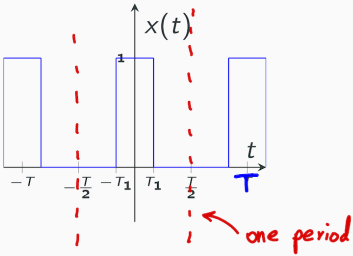
$$x(t) = 1 \times e^{j\times 0} + \frac{1}{3}\times\frac{1}{2} e^{j2\pi t} + \frac{1}{3}\times\frac{1}{2}\times e^{-j2\pi t} + \frac{1}{2j} e^{j3\pi t} - \frac{1}{2j} e^{-j3\pi t}$$

$a_0 = 1$, $a_2 = \frac{1}{6}$, $a_{-2} = \frac{1}{6}$, $a_3 = \frac{1}{2j}$, $a_{-3} = -\frac{1}{2j}$

CT Fourier Series: Example Periodic Pulse

- What are the Fourier series coefficients of the periodic rectangular wave (pulse)?

$\omega_0 = \frac{2\pi}{T}$



$$x(t) = \begin{cases} 1 & \text{if } |t| \leq T_1 \\ 0 & \text{if } T_1 < |t_1| \leq \frac{T}{2} \end{cases}.$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T_1}^{T_1} 1 \times e^{-jk\omega_0 t}\,dt$$

CT Fourier Series: Example Periodic Pulse

- What are the Fourier series coefficients of the periodic rectangular wave (pulse)?

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{-1}{Tjk\omega_0}\Big[e^{-jk\omega_0 t}\Big]_{-T_1}^{T_1}$$

$$= \frac{-1}{Tjk\omega_0}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right)$$

$$= \frac{-1}{Tjk\omega_0}\left(-2j\sin(k\omega_0 T_1)\right)$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

CT Fourier Series: Example Periodic Pulse

- What are the Fourier series coefficients of the periodic rectangular wave (pulse)?

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} x(t)\, e^{-j\times 0\times \omega_0 t}\, d(t) = \frac{1}{T}\int_{-T_1}^{T_1} 1\, dt = \frac{2T_1}{T}$$

or we can write : $a_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$

L'Hopital's rule:

$$a_0 = \lim_{k\to 0} \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \lim_{k\to 0} \frac{2\omega_0 T_1 \cos(k\omega_0 T_1)}{\omega_0 T} = \frac{2T_1}{T_1}$$

Properties of Continuous-time Fourier Series

Properties of Fourier Series (please read Section 3.5)

- Linearity: If $x(t) \xleftrightarrow{F.S.} \{a_k\}$ and $y(t) \xleftrightarrow{F.S.} \{b_k\}$, then $\alpha x(t) + \beta y(t) \xleftrightarrow{F.S.} \{\alpha a_k + \beta b_k\}$
- Time-shift: If $x(t) \xleftrightarrow{F.S.} \{a_k\}$ then $x(t - t_0) \xleftrightarrow{F.S.} \{a_k e^{-jk\omega_0 t_0}\}$
- Time-reversal: If $x(t) \xleftrightarrow{F.S.} \{a_k\}$ then $x(-t) \xleftrightarrow{F.S.} \{a_{-k}\}$
- Time-scaling: If $x(t) \xleftrightarrow{F.S.} \{a_k\}$ then $x(\alpha t) \xleftrightarrow{F.S.} \{a_k\}$!
- Multiplication: If $x(t) \xleftrightarrow{F.S.} \{a_k\}$ and $y(t) \xleftrightarrow{F.S.} \{b_k\}$, then $x(t)y(t) \xleftrightarrow{F.S.} \{a_k * b_k\}$
- Conjugation: If $x(t) \xleftrightarrow{F.S.} \{a_k\}$ then $x^*(t) \xleftrightarrow{F.S.} \{a^*_{-k}\}$
- PLEASE use **Table 3.1** of the book for more properties of Fourier series of CT-periodic signals.

Parseval's Relation

- Parseval's relation

Parseval's Relation

For a periodic signal $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, we have:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Note that $|a_k|^2 = \frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2$

- Parseval relation: Energy of a signal is equal to sum of the energy of its harmonic components!