Lecture 10

Fourier series properties

Preview of today's lecture

- Fourier series properties
  - Proving that the properties hold
  - Using the properties in FS calculations

- More applications of properties (likely deferred to discussion sec.)

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

Working with the Fourier series

Learning objectives

- Compute the output of an LTI system to a periodic input
- Use the Fourier series coefficients of these signals in other problems

Basic signals

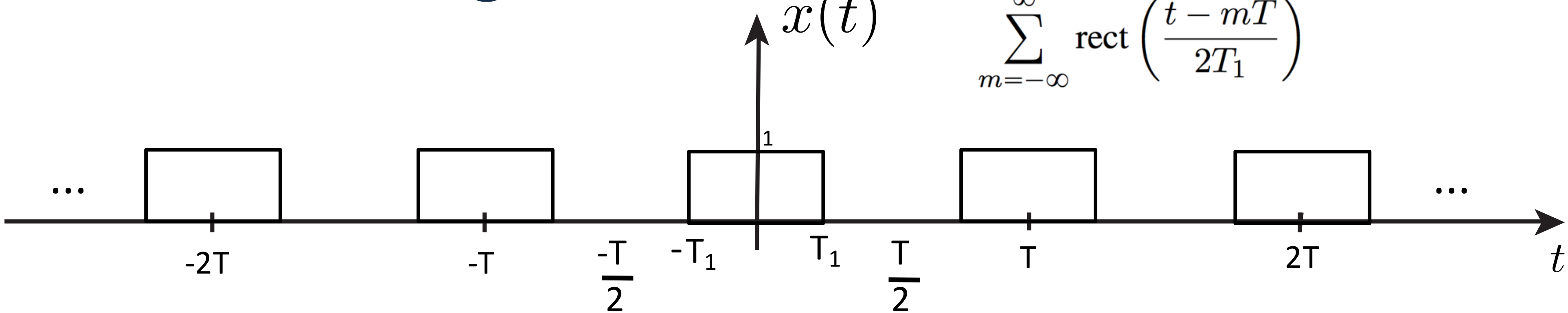
$x(t) \longleftrightarrow \{a_k\}$

	Time domain $x(t)$	Fourier coefficients a_k
Constant (periodic for any T)	c	$c\delta[k]$
Cosine	$\cos(\omega_0 t)$	$\frac{1}{2}\delta[k-1]+\frac{1}{2}\delta[k+1]$
Sine	$\sin(\omega_0 t)$	$\frac{1}{2j}\delta[k-1]-\frac{1}{2j}\delta[k+1]$
Impulse train	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}$

Pulse train in general

From O&W Example 3.5

$$\sum_{m=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-mT}{2T_1}\right)$$



From the book

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \qquad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$a_k = \frac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

with fundamental frequency

$$a_k = \frac{2T_1}{T}\mathrm{sinc}\left(\frac{k2T_1}{T}\right)$$

simplified

Output of an LTI system

$$\int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}$$

$$e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \overbrace{H(s)}e^{st}$$

$$\alpha e^{s_1 t} + \beta e^{s_2 t} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \alpha H(s_1)e^{s_1 t} + \beta H(s_2)e^{s_2 t}$$

New FS coefficients

$$\underbrace{\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}}_{x(t)} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} a_k \underline{H(j\omega_0 k)} e^{j\omega_0 kt}}_{y(t)}$$

Fourier series coefficients are modified by the frequency response of the system

Properties of Fourier series

Learning objectives

- Use Fourier series properties to simplify calculation & build intuition
- Analyze problems that include FS properties

Properties of the Fourier series

- The following notation is used to denote a signal and its FS coefficients

$$x(t) \stackrel{FS}{\longleftrightarrow} a_k$$

- Properties are used to figure out how transformations of the input signal lead to transformations of the FS coefficients, helps to avoid direct computation!

Fourier series properties

- Let $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) \overset{FS}{\longleftrightarrow} a_k \qquad\qquad y(t) \overset{FS}{\longleftrightarrow} b_k$$

	Time domain	FS domain
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shift	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$
Time reversal	$x(-t)$	a_{-k}

Fourier series properties (continued)

	Time domain	FS domain
Time scaling	$x(\alpha t)$	a_k $\quad T_{\text{new}} = \frac{T}{\alpha}$ period changes
Conjugate	$x^*(t)$	a^*_{-k}
Multiplication	$x(t)y(t)$	$\sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$
Derivative	$\frac{d}{dt}x(t)$	$a_k(jk\omega_0)$
Parseval's Theorem	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	

Fourier series and symmetry

	Time domain	FS domain
Conjugate symmetry	$x(t)$ real	$a_k = a^*_{-k}$
Real and even	$x(t)$ real and even	a_k real and even
Real and Odd	$x(t)$ real and odd	a_k imag. and odd

Property #1: Linearity

- If $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

$$z(t) = Ax(t) + By(t)$$

- Then $$z(t) \overset{FS}{\longleftrightarrow} Aa_k + Bb_k$$

FS of a sum of signals is the sum of their FS coefficients

Property #2: Time shifting

$\frac{2\pi}{\omega_0}$

- Let $x(t)$ have period $T = \cancel{\frac{2\pi}{\omega_0}}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$

- If $y(t) = x(t - t_0)$, $y(t)$ is periodic with the same period

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

- Then $b_k = a_k e^{-jk\omega_0 t_0}$ Note $|b_k| = |a_k|$ since $|e^{j\,k}| = 1$

Shift in time results in a phase shift in frequency

Example – Making use of the table

- Let $x(t)$ be a periodic signal with a fundamental period T, and FS coefficients a_k. Derive the FS coefficients of the following signal

$$x(t-t_0)+x(t+t_0)$$

$X(t) \longleftrightarrow \{a_k\}$

- Solution

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

#2

$$x(t-t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0}$$

#2

$$x(t+t_0) \overset{FS}{\longleftrightarrow} a_k e^{jk\omega_0 t_0}$$

$$x(t-t_0)+x(t+t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0} + a_k e^{jk\omega_0 t_0}$$

$a_k\left(e^{jk\omega_0 t_0} + e^{-jk\omega_0 t_0}\right)$

$$= 2\cos(k\omega_0 t_0)a_k$$

Property #3: Time reversal

$\frac{2\pi}{\omega_0}$

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$
- Then $y(t) = x(-t)$, $y(t)$ is periodic with the same period
- and

$$y(t) \overset{FS}{\longleftrightarrow} a_{-k}$$

Reverse in time results in reverse in frequency

Time reversal proof

- Suppose that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$-t$

- Then

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$$

- Changing variables

$$y(t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t} \quad \Rightarrow \quad y(t) \overset{FS}{\longleftrightarrow} a_{-k}$$

$m = -k$

Property #4: Time scaling

- Let $x(t)$ have period $T = \frac{2\square}{!_0}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$

- If $y(t) = x(\cancel{\alpha t})$, $\alpha > 0$
  $x(\alpha t)$

$\alpha < 1 \rightarrow$ stretching
$\alpha > 1 \rightarrow$ compression

- Then $y(t) = x(\cancel{\alpha t})$ is periodic with period $T_{\text{new}} = \frac{T_{\text{old}}}{\alpha}$
  $= x(\alpha t)$

$$x(\alpha t) \overset{FS}{\longleftrightarrow} a_k$$

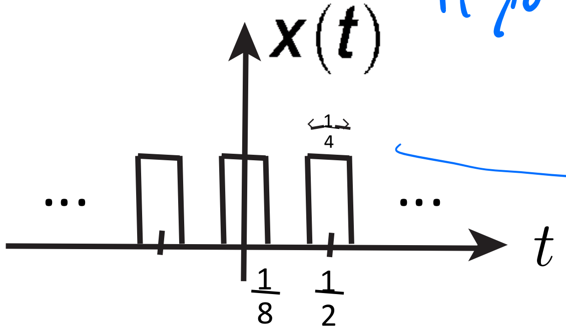
Scale in time does not change the FS coefficients

Visualizing time scaling

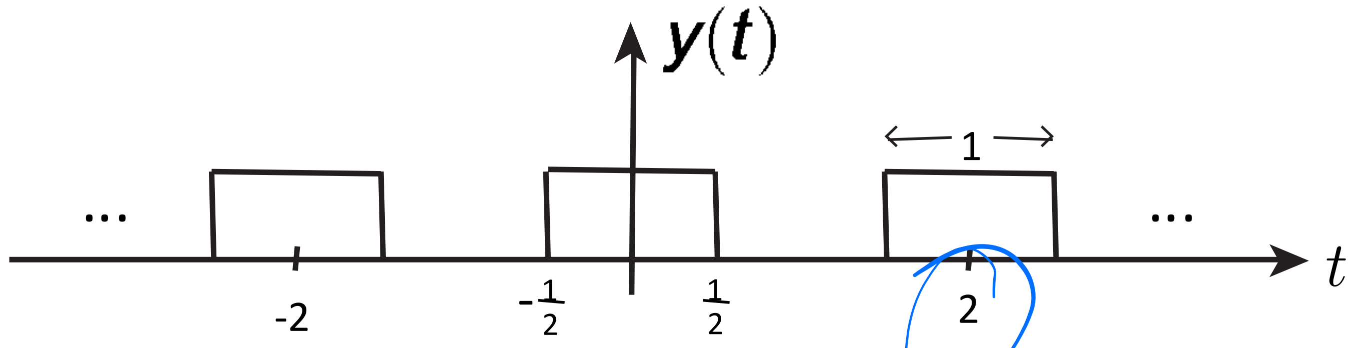
- Example $y(t) = x(\alpha t)$ $T = \frac{1}{2}$ $T_1 = 1/8$

$$\alpha = \frac{1}{4}$$

$y(t) = x(t/4)$



Stretched signal has same structure



$$a_k = \frac{2T_1}{T}\operatorname{sinc}\left(\frac{k2T_1}{T}\right)$$

$T = 1/2 \quad T_1 = 1/8$

$$\Rightarrow a_k = \frac{2 \cdot 1/8}{1/2} \cdot \operatorname{sinc}\left(\frac{k \cdot 2 \cdot 1/8}{1/2}\right)$$

$$= \frac{1}{2}\operatorname{sinc}\left(k/2\right)$$

$$T_{new} = T/\alpha$$

$$= 4 \cdot T$$

$$= 2$$

Time scaling proof

- Since $x(t) \stackrel{FS}{\longleftrightarrow} a_k$ it follows that $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

- Then

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

$$x(\alpha t) \stackrel{FS}{\longleftrightarrow} a_k$$

Reinforcing the time-scaling property

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$
- If $y(t) = x(\alpha t)$, $\alpha > 0$

$$\boxed{\begin{array}{l} \alpha < 1 \rightarrow \text{stretching} \\ \alpha > 1 \rightarrow \text{compression} \end{array}}$$

- Then $y(t) = x(\alpha t)$ is periodic with period $T_{\text{new}} = \dfrac{T_{\text{old}}}{\alpha}$

$$x(\alpha t) \overset{FS}{\longleftrightarrow} a_k$$

Scale in time does not change the FS coefficients

Property #5: Multiplication

- If $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$\frac{2\pi}{w_0}$$

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

Same

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

- Then for $z(t) = x(t)y(t)$

Product in time leads to a convolution sum in FS coefficients

$$z(t) = x(t)y(t) \overset{FS}{\longleftrightarrow} h_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

Property #6: Conjugation and symmetry

- If $x(t)$ is periodic with period $T = \cancel{\frac{2\pi}{\omega_0}}$ and $x(t) \xleftrightarrow{FS} a_k$

$\frac{2\pi}{w_0}$

- Then $x^*(t) \xleftrightarrow{FS} a^*_{-k}$

- Impliciations
  - If $x(t)$ is real, then the FS coefficients are conjugate symmetric

$$\boxed{a^*_{-k} = a_k}$$

  - If $x(t)$ is real and even, then the FS coefficients are real and even

$$a_k = a^*_k$$

  - If $x(t)$ is real and odd, then the FS coefficients are imaginary and odd

Property #7: Parseval's theorem

- Consider a periodic signals with FS representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- The power in the signal is

$$\boxed{\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$

Power is the same whether in the time or frequency domain

Proof of Parseval's theorem

for a complex number

$$|x|^2 = xx^*$$

$$\frac{1}{T}\int |x(t)|^2 dt = \frac{1}{T}\int_T \left| \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right|^2 dt$$

$$= \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k e^{jk\omega_0 t} a_\ell^* e^{-j\ell\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \frac{1}{T}\int_T a_k e^{jk\omega_0 t} a_\ell^* e^{-j\ell\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \frac{1}{T}\int_T e^{j(k-\ell)\omega_0 t} dt$$

Use orthogonality property

$\Rightarrow \delta[k-\ell]$

Proof of Parseval's theorem (cont.)

$$\frac{1}{T}\int |x(t)|^2 dt = \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \delta[k-\ell]$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

Orthogonality is key to the proof

Parseval's theorem – Example

- Consider the signal $x(t) = \cos(\omega_0 t)$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

- The FS coefficients: $a_0 = 0, \quad a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0 \text{ else}$
  - Find the power using Parseval's theorem

$$\frac{1}{T}\int_T |\cos(\omega_0 t)|^2 dt = \sum |a_k|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

  - Find the power directly in the time domain

$$\cos^2 \omega_0 t = \frac{1}{2}(1 + \cos 2\omega_0 t) \qquad \frac{1}{T}\int_0^T \frac{1}{2} dt + \frac{1}{T}\int_T \cos 2\omega_0 t\, dt = \frac{1}{2}$$

Property #8: Derivative

- Consider a periodic signal $x(t)$ with $T = \frac{2\pi}{\omega_0}$ and

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

- Then

$$\frac{dx(t)}{dt} \leftrightarrow a_k(jk\omega_0)$$

Each FS coefficient scaled as a function of the frequency

Proof of the derivative property

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\frac{d}{dt}x(t) = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k (jk\omega_0) e^{jk\omega_0 t}$$

Example - Using Parseval's theorem

- Consider the signal $x(t) = \cos(\omega_0 t)$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

- The FS coefficients: $a_0 = 0, \quad a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0 \text{ else}$
  - Find the power using Parseval's theorem

$$\frac{1}{T}\int_T |\cos(\omega_0 t)|^2 dt = \sum |a_k|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

  - Find the power directly in the time domain

$$\cos^2 \omega_0 t = \frac{1}{2}(1 + \cos 2\omega_0 t) \qquad \frac{1}{T}\int_0^T \frac{1}{2} dt + \frac{1}{T}\int_T \cos 2\omega_0 t \, dt = \frac{1}{2}$$

Summary of Fourier series properties

- Fourier series properties relate transformations of signals in the time domain and transformations of Fourier series coefficients
- Understanding the properties is valuable for developing intuition on how signals behave in the time and frequency domains
- Exploiting the properties has the practical advantage of avoiding tedious Fourier Series or inverse Fourier Series calculations
- While you can refer to the table for solving homework and exam problems, you must internalize the properties in your brain to use in the real world

Application of Fourier series properties

Learning objectives

- Use the Fourier series properties to infer information about signals

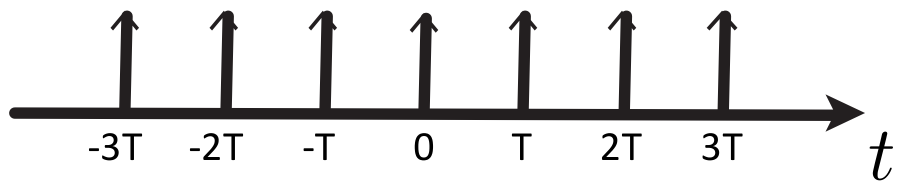
Several important examples are provided here for reference, they may be may be covered in the discussion section

Application Example I

This impulse train signal shows up later in the course in sampling. This kind of signal can be used to build waveforms that are good for radar for example.

- Consider the impulse train signal

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \text{period } T$$

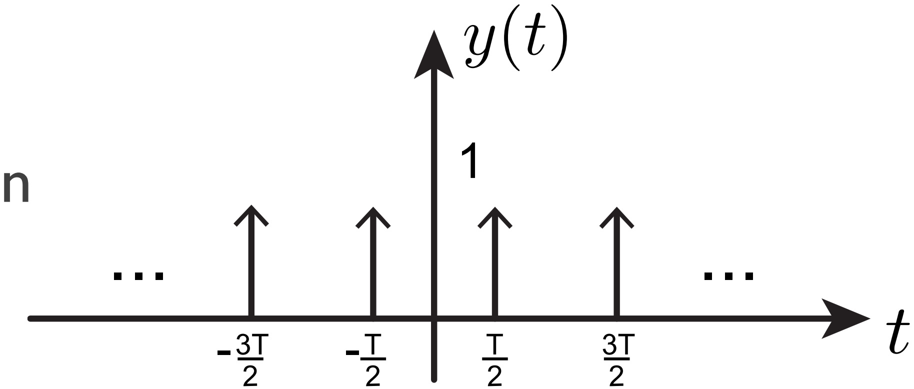


- Calculate the FS coefficients

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\frac{2\pi}{T}t}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{0}dt \\ &= \frac{1}{T} \quad \forall k \end{aligned}$$

Application Example I

- Now consider the impulse pulse train



- Calculate the FS coefficients
- Because $y(t) = x\left(t - \frac{T}{2}\right)$

$$
\begin{aligned}
b_k &= a_k e^{-jk\omega_0 t_0} \\
&= a_k e^{-jk\frac{2\pi}{T}\frac{T}{2}} \\
&= a_k e^{-jk\pi} \\
&= \frac{1}{T}\cos k\pi \\
&= \frac{(-1)^k}{T}
\end{aligned}
$$

Application Example 2

- Let $x(t)$ be a periodic signal with a fundamental period T, and FS coefficients a_k. Derive the FS coefficients of the following signal

$$\frac{d^2x(t)}{dt^2}$$

Application Example 2

- From the definition of the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}$$

- Differentiating

$$\frac{d^2 x(t)}{dt^2} = \frac{d^2}{dt^2} \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}$$

$$= \sum_{k=-\infty}^{\infty} \frac{d^2}{dt^2} a_k e^{j(2\pi/T)kt}$$

$$= \sum_{k=-\infty}^{\infty} \boxed{-k^2 \frac{4\pi^2}{T^2} a_k} e^{j(2\pi/T)kt}$$

Application Example 3

(from O&W 3.23 (a))

- Consider the FS coefficients of a CT signal that is periodic with period 4. Determine the signal $x(t)$

$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \dfrac{\sin(k\pi/4)}{k\pi}, & \text{otherwise} \end{cases}$$

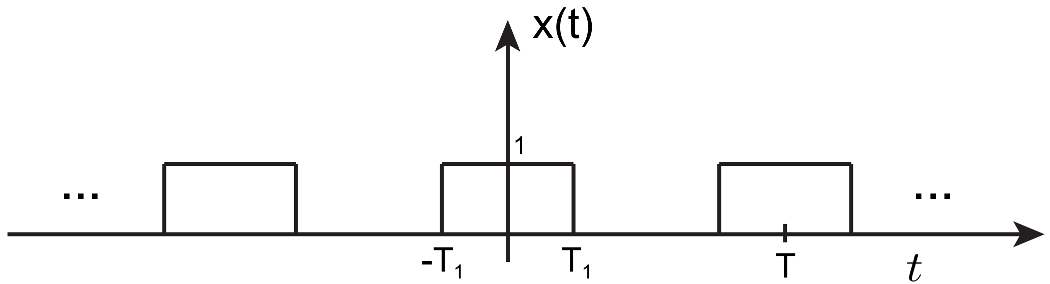
- Approach
  - Start with a known FS
  - Make transformations to reach the required signal

This is a detailed example that involves working backwards from the FS coefficients to find the underlying signal. It requires look at the expression and thinking about the properties differently than other problems as we are doing some detective work here.

Application Example 3 (continued)

- Use the known FS and the FS properties to recover signals from their FS coefficients

- Consider this function from an earlier lecture

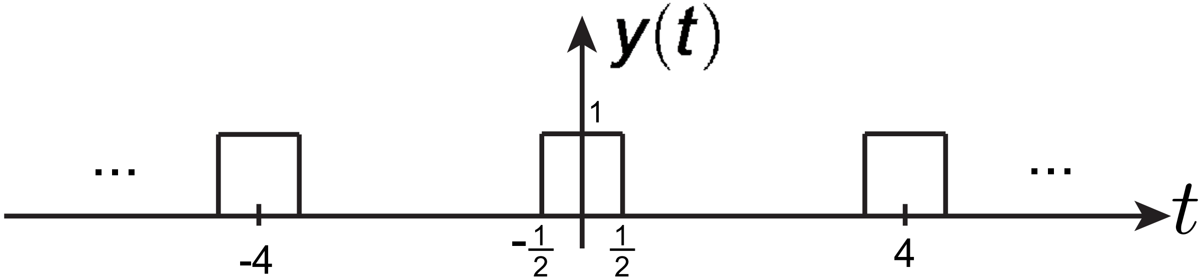


$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi}$$

Application Example 3 (continued)

- Solution:
  - Consider the signal $y(t) \leftrightarrow b_k$ with FS coefficients $$b_k = \frac{\sin\frac{k\pi}{4}}{k\pi}$$
  - As T=4

$$b_k = \frac{\sin\frac{k\pi}{4}}{k\pi} = \frac{\sin\left(k\cdot\frac{2\pi}{T}\cdot T_1\right)}{k\pi}$$
$$= \frac{\sin\left(k\frac{\pi}{2}\cdot T_1\right)}{k\pi} \rightarrow T_1 = \frac{1}{2}$$



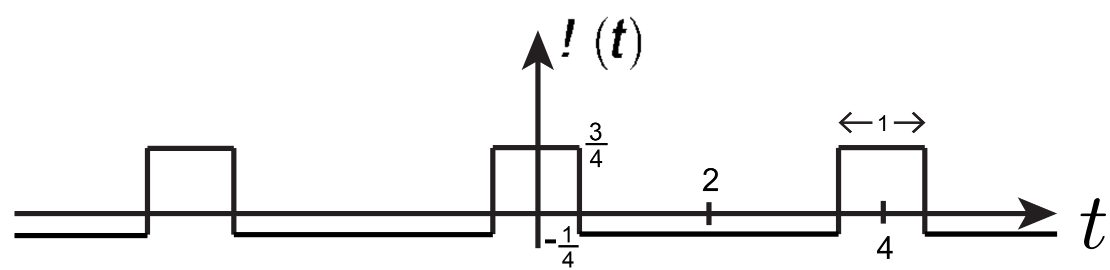
Application Example 3 (continued)

- The dc component of the signal $y(t)$ is

$$b_0 = \frac{1}{T}\int_T y(t)dt$$
$$= \frac{1}{4}\cdot 1 = \frac{1}{4}$$

- But the DC component of $y(t)$ is 0, so subtract it
  - Define the signal $w(t) \leftrightarrow c_k$ as $\omega(t) = y(t) - \frac{1}{4}$
  - Then $c_0 = 0$

$$c_k = \frac{\sin\frac{\pi k}{4}}{\pi k}$$



Application Example 3 (continued)

- Now, what is remaining is to add j^k

  - We know that $j^k = (e^{j\frac{\square}{2}})^k = e^{j\frac{\square}{2}k}$

  - So, now consider $x(t) = w(t - t_0)$

  - Using the FS properties

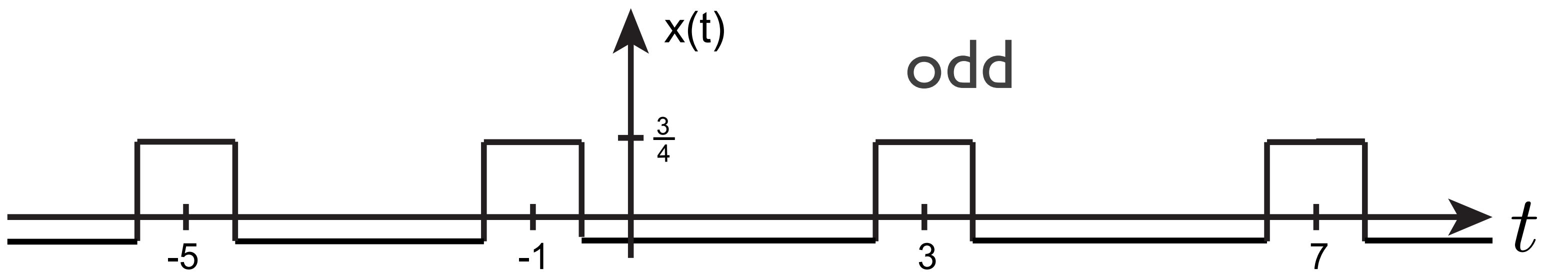
If $w(t) \leftrightarrow c_k$ then $w(t - t_0) \leftrightarrow c_k \cdot e^{j\frac{\square}{2}k} = a_k$

$$e^{j\frac{\pi}{2}k} = e^{-jk\omega_0 t_0}, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$= e^{-jk\frac{\pi}{2}t_0} \longrightarrow t_0 = -1$$

Application Example 3 (concluded)

- Hence, $x(t) = w(t - t_0) = w(t + 1)$



$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \, \dfrac{\sin(k\pi/4)}{k\pi}, & \text{otherwise} \end{cases}$$

Application Example 4

(similar to O&W 3.23 (d))

- Consider the FS coefficients of a CT signal that is periodic with period 4. Determine the signal $x(t)$

$$a_k = \begin{cases} 1, & k \text{ odd} \\ 2, & k \text{ even} \end{cases}$$

- Solution: Use the fact that

$$\sum_{k=-\infty}^{\infty} \delta(t-k) \overset{FS}{\longleftrightarrow} a_k = 1 \ \forall k$$

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Application Example 4 (continued)

Step 1

- Consider a train of deltas with period 4

$$\sum_{k=-\infty}^{\infty} \delta(t-4k) = \sum_{k=-\infty}^{\infty} \delta(4(t/4-k))$$

- Using the scaling property of the delta function $\delta(t/a) = |a|\delta(t)$

$$\sum_{k=-\infty}^{\infty} \delta(t-4k) = \frac{1}{4}\sum_{k=-\infty}^{\infty} \delta(t/4-k)$$

- Now from the time scaling property and linearity property

$$\frac{1}{4}\sum_{k=-\infty}^{\infty} \delta(t/4-k) \stackrel{FS}{\longleftrightarrow} b_k = \frac{1}{4} \ \forall k$$

Application Example 4 (continued)

- As a result we can conclude from

$$\sum_{k=-\infty}^{\infty} \delta(t-k) \overset{FS}{\longleftrightarrow} a_k = 1 \ \forall k$$

- That

$$\sum_{k=-\infty}^{\infty} \delta(t-4k) \overset{FS}{\longleftrightarrow} b_k = \frac{1}{4} \ \forall k$$

- And

$$4 \sum_{k=-\infty}^{\infty} \delta(t-4k) \overset{FS}{\longleftrightarrow} b_k = 1 \ \forall k$$

Application Example 4 (continued)

Step 2

- Consider a signal with

$$c_k = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

- Inserting into the synthesis equation

$$x_2(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{4}kt}$$

$$= \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{4}2kt}$$

$$= \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{2}kt}$$

But this is just the synthesis of a signal with period 2 and FS coefficients {1}

$$\Rightarrow \quad 2\sum_{k=-\infty}^{\infty} \delta(t-2k)$$

Application Example 4 (concluded)

Step 3

- Write signal with period 4 and FS coefficients

$$a_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$$

- As the sum of signals with FS coefficients

$$b_k = 1 \qquad \text{and} \qquad c_k = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

- Time domain signal is then

$$\boxed{x(t) = 4 \sum_{k=-\infty}^{\infty} \delta(t - 4k) + 2 \sum_{k=-\infty}^{\infty} \delta(t - 2k)}$$

Application Example 5

(from O&W 3.23 (c))

- In the following, we specify the FS coefficients of a CT signal that is periodic with period 4. Determine the signal $x(t)$

$$a_k = \begin{cases} jk & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$$

- Solution:

$$\begin{aligned} x(t) &= a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t} \\ &= j e^{j\omega_0 t} - j e^{-j\omega_0 t} + 2j e^{j2\omega_0 t} - 2j e^{-j2\omega_0 t} \\ &= -2\sin(\omega_0 t) - 4\sin(2\omega_0 t) \end{aligned}$$

Application Example 6

(from O&W 3.26)

- Let $x(t)$ be a periodic signal whose FS coefficients are

$$a_k = \begin{cases} 2 & k = 0 \\ j(1/2)^{|k|} & \text{otherwise} \end{cases}$$

- Is $x(t)$ real?
  - Real signals must satisfy $x(t) = x^*(t)$ or $a_k = a^*_{-k}$ not satisfied here
- Is $x(t)$ even?
  - Even signals satisfy $x(t) = x(-t)$ or $a_k = a_{-k}$ yes is satisfied
- Is $\frac{dx(t)}{dt}$ even?
  - The FS coefficients of $\frac{dx(t)}{dt}$ are $(j\omega_0 k)a_k$ for which $(j\omega_0 k)a_k \neq -(j\omega_0 k)a_{-k}$