Lecture 8

Eigenfunctions and differential equations

Preview of today's lecture

- Fourier series

- Orthogonality and computing the Fourier series

- Most important example: the rectangular pulse train

Eigenfunctions of a LTI system

$$x(t) = e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = H(s)e^{st}$$

Complex exponential with complex "frequency" s

Eigenvalue (a complex scalar)

eigenfunction

- CT complex exponentials are eigenfunctions of LTI systems
  - Attenuated and scaled according to $H(s)$ (system response)
  - The system response is Laplace transform of the impulse response
  - Will be studied in other courses like ECE 101

Special case of complex sinusoids

- Frequency response is used to characterize LTI systems

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

- The frequency response is computed from the impulse response

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

- This is the <u>Fourier transform</u> of the impulse response of the system

Frequency response for a LCCDE

- Consider a system where the input and output are related by a linear constant coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- The frequency response has a special form

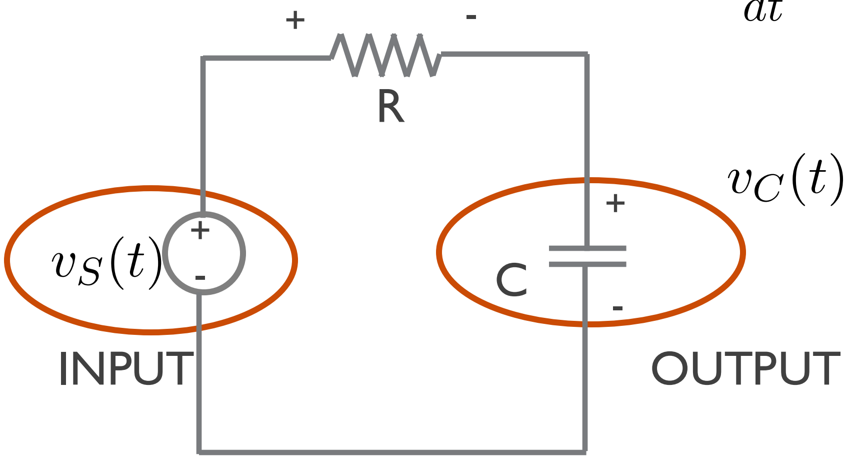
$$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

RLC circuits are modeled with differential equations

Example

- Recall this RC lowpass filter
  - Source voltage as the input
  - Capacitor voltage as the output

$$v_R(t) = iR = CR\frac{dv_C(t)}{dt}$$



Current through capacitor

$$i = C\frac{dv_C(t)}{dt}$$

Resulting differential equation

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

Frequency response of the circuit

- RLC circuits that are "at rest" are LTI (they have zero initial conditions, no stored charge, no memory)
- Consider the differential equation with input $v_S(t)$ and output $v_C(t)$

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

- The frequency response is

$$H(j\omega) = \frac{1}{RCj\omega + 1}$$

- If $v_S(t) = e^{j\omega t}$ then $v_C(t) = H(j\omega)e^{j\omega t}$

Response for the negative frequency

- Continuing the same example with input $v_S(t)$ and output $v_C(t)$

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t) \qquad H(j\omega) = \frac{1}{RCj\omega + 1}$$

- If we input $v_S(t) = e^{-j\omega t}$ then

$$v_C(t) = H(-j\omega)e^{-j\omega t}$$

Some simplifications

- Observe that

$$H(-j\omega) = \frac{1}{-RCj\omega + 1}$$

- But observe the

$$\begin{aligned} H^*(j\omega) &= \frac{1}{(RCj\omega + 1)^*} \\ &= \frac{1}{-RCj\omega + 1} \\ &= H(-j\omega) \end{aligned}$$

This is an example of conjugate symmetry

Inputting a cosine 1/2

- Now excite the circuit by a real cosine signal

$$\begin{aligned} v_S(t) &= A\cos(\omega t + \theta) \\ &= \frac{1}{2}Ae^{j\theta}e^{j\omega t} + \frac{1}{2}Ae^{-j\theta}e^{-j\omega t} \end{aligned}$$

- Because the system is LTI

$$v_C(t) = \frac{1}{2}Ae^{j\theta}H(j\omega)e^{j\omega t} + \frac{1}{2}H(-j\omega)Ae^{-j\theta}e^{-j\omega t}$$

- Because of conjugate symmetry

Inputting a cosine 2/2

- Because of conjugate symmetry

$$
\begin{aligned}
v_C(t) &= \frac{1}{2}Ae^{j\theta}H(j\omega)e^{j\omega t} + \frac{1}{2}H^*(j\omega)Ae^{-j\theta}e^{-j\omega t} \\
&= 2\mathrm{Re}\left[\frac{1}{2}Ae^{j\theta}H(j\omega)e^{j\omega t}\right]
\end{aligned}
$$

- Simplifying

$$
\begin{aligned}
v_C(t) &= A|H(j\omega)|\mathrm{Re}\left[e^{j\theta}e^{j\angle H(j\omega)}e^{j\omega t}\right] \\
&= A|H(j\omega)|\cos(\omega t + \theta + \angle H(j\omega))
\end{aligned}
$$

See the connection to phasors

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

- Consider a cosine input

$$v_S(t) = A\cos(\omega t + \theta)$$
$$= \mathrm{Re}\left[Ae^{j\theta}e^{j\omega t}\right]$$

- Solve the problem with phasors

$$\mathbf{V}_S = Ae^{j\theta}$$

KVL $\quad RCj\omega\mathbf{V}_C + \mathbf{V}_C = \mathbf{V}_S \;\Rightarrow\; \mathbf{V}_C = \frac{1}{RCj\omega + 1}\mathbf{V}_S$

$$= A\left|\frac{1}{RCj\omega + 1}\right| e^{j\theta + \angle 1/(RCj\omega+1)}$$

- Find the final time domain solution

$$v_C(t) = A\left|\frac{1}{RCj\omega + 1}\right|\cos(\omega t + \theta + \angle 1/(RCj\omega + 1))$$

What does it all mean?

- The phasor method is built upon the foundations of LTI systems
- When you use phasors, you are exploiting the property that complex sinusoids are eigenfunctions of LTI
- RLC circuits have real-valued components, which ensures the conjugate symmetry of the frequency response
- This makes it possible to connect input and output cosines

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

Continuous-time Fourier series

Learning objectives

- Explain the key idea of Fourier series representation of signals
- Specialize the Fourier series to real signals

Fourier series for C T periodic signals

- Consider the periodic signal $x(t)$ with period T: $x(t+T) = x(t)$
- The Fourier series representation of the periodic signal $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency

- The Fourier series coefficients of $x(t)$ are $\{a_k\}$ and a_0 is DC
- The k-th harmonic components of $x(t)$ are a_k and a_{-k}

Interpreting the Fourier series

- Can represent (most) periodic signals as

$$x(t) = \underbrace{a_0}_{\text{DC offset}} + \underbrace{a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}}_{1^{st} \text{ harmonic on fundamental term}} + \underbrace{a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t}}_{2^{nd} \text{ harmonic, at } 2\omega_0} + \cdots$$

- Checking periodicity

$$x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t+T)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \underbrace{e^{jk\omega_0 T}}_{=1}$$

$$= x(t)$$

$$\boxed{\begin{aligned} e^{jk\omega_0 T} &= e^{jk\frac{2\pi}{T}T} \\ &= e^{jk2\pi} \\ &= 1 \end{aligned}}$$

Example of Fourier series addition

- http://www.intmath.com/fourier-series/fourier-graph-applet.php

Special case of real signals

$$x^*(t) = x(t)$$

- Real signals have special symmetry in the Fourier series

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right)^*$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{jk\omega_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$$= \sum_{\ell=-\infty}^{\infty} a_{-\ell}^* e^{j\ell\omega_0 t}$$

$$\equiv \underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}_{x(t)}$$

$$\boxed{a_k = a_{-k}^*}$$ conjugate symmetry

Using the symmetry for real signals

- Suppose that x(t) is real

Decomposition is real, which is expected

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\
&= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right] \\
&= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right] \\
&= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{a_k e^{jk\omega_0 t}\}
\end{aligned}
$$

z z*

$$z + z^* = 2Re\{z\}$$

Writing the coefficients in polar form

- Let $a_k = A_k e^{j\theta_k}$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right)$$

- Write real signals as a sum of phase shifted cosines and DC term

http://en.wikipedia.org/wiki/Polar_bear

Writing the coefficients in Cartesian form

- Let $a_k = B_k + jC_k$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{(B_k + jC_k)(\cos k\omega_0 t + j\sin k\omega t)\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2(B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t))$$

- Write real signals as a sum of sine, cosine, and DC term

https://guinevereglasfurddotcom.files.wordpress.com/2014/11/rene-descartes_1214990c.jpg

Example 4

- A CT Periodic & real signal $x(t)$ has a fundamental period $T=8$. The non-zero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1} = 2$$

$$a_3 = a^*_{-3} = 4j$$

- Express $x(t)$ in exponential, polar and Cartesian forms

Example 4 - solution

$$
\begin{aligned}
x(t) &= a_1 e^{j\left(\frac{2\pi}{T}\right)t} + a_{-1} e^{-j\left(\frac{2\pi}{T}\right)t} + a_3 e^{j3\left(\frac{2\pi}{T}\right)t} + a_{-3} e^{-j3\left(\frac{2\pi}{T}\right)t} \\
&= 2e^{j\left(\frac{2\pi}{8}\right)t} + 2e^{-j\left(\frac{2\pi}{8}\right)t} + 4je^{j3\left(\frac{2\pi}{8}\right)t} - 4je^{-j3\left(\frac{2\pi}{8}\right)t} \\
&= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{3\pi}{4}t\right) \\
&= 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right)
\end{aligned}
$$

Exponential

Cartesian

Polar

Summary of Fourier series for CT periodic signals

General form of the Fourier series is

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{(jk\omega_0)t}}$$

where $\omega_0 = \dfrac{2\pi}{T}$ is the fundamental frequency

Special forms for when signal is real

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right)$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\left(B_k \cos\left(k\omega_0 t\right) - C_k \sin\left(k\omega_0 t\right)\right)$$

Fourier analysis and orthogonality

Learning objectives

- Connect Fourier series to the inner product of a vector
- Prove orthogonality of complex sinusoids

Fourier analysis and synthesis

time domain signal → **Fourier analysis** → Fourier coefficients (frequency domain)

figuring out what frequencies are used to build the time domain signal

Fourier coefficients (frequency domain) → **Fourier synthesis** → time domain signal

putting the frequencies together to build the time domain signal

Fourier theory involves decomposing signals into their Fourier coefficients and building signals from Fourier coefficients

Key Fourier equations

Finding the coefficients: Use the analysis equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}\mathrm{d}t = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}\mathrm{d}t$$

Integrate over the fundamental period T

Reconstructing the signal: Use the synthesis equations

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

coefficients

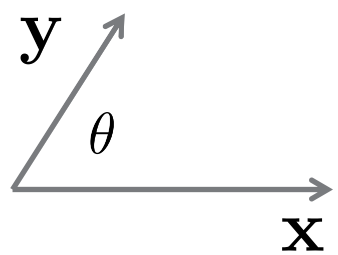
complex sinusoid

Signal *x(t)* that is periodic with fundamental period *T*

The analysis equation as an inner product

- Inner product of a pair of vectors

$$\mathbf{x}^*\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta)$$



- Inner product between a pair of periodic functions $x(t)$ and $y(t)$

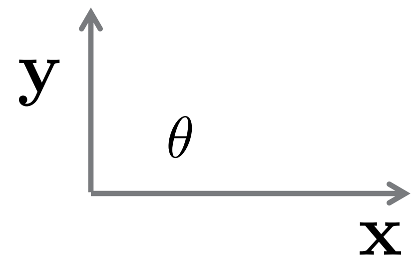
$$\langle x(t), y(t)\rangle = \int_0^T x(t)y^*(t)dt$$

Way to define magnitude and extent of overlap

Orthogonality

- A pair of vectors are orthogonal if

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta) = 0$$



- Two periodic functions *x(t)* and *y(t)* are orthogonal if

$$\langle x(t), y(t) \rangle = \int_0^T x(t) y^*(t) dt = 0$$

Orthogonality of complex sinusoids

- Consider the following periodic signals

$$x(t) = e^{jk\omega_0} \qquad y(t) = e^{jn\omega_0}$$

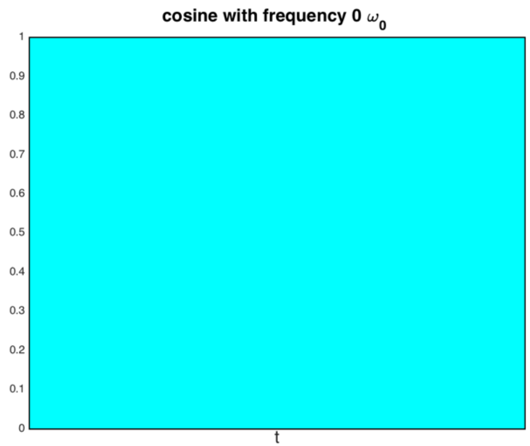
- Compute the inner product between these two signals

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos((k-n)\omega_0 t) + j\sin((k-n)\omega_0 t) dt$$

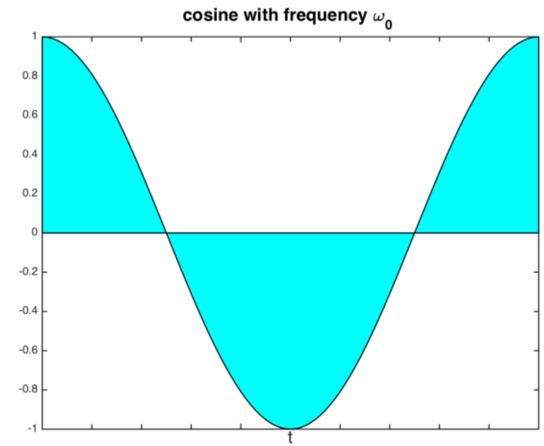
$$= \begin{cases} T & k = n \\ 0 & k \neq n \end{cases} \Rightarrow T\delta[k-n]$$

Complex sinusoids comprised of different harmonics of the fundamental frequency are orthogonal

Illustration of integrating the cosine

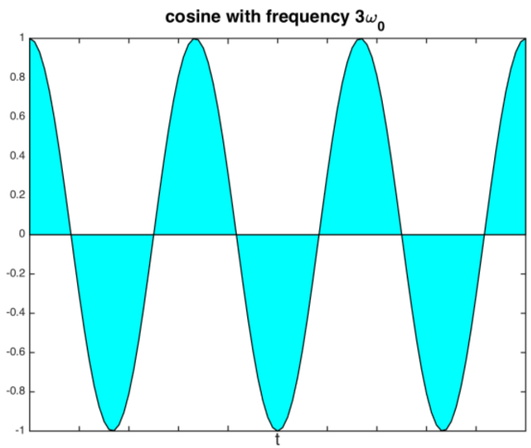


Area=T



Area=0

("negative area" below x-axis cancels "positive area")



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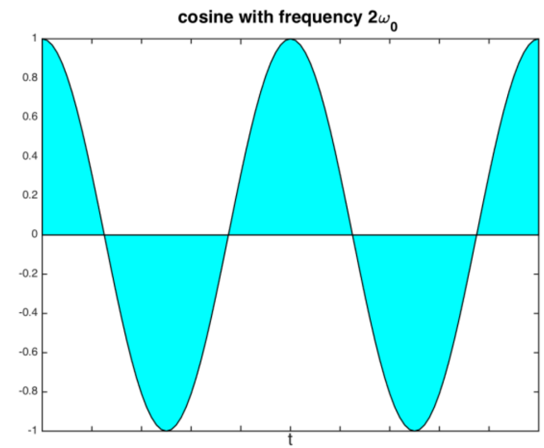


Area=0

Use orthogonality to check analysis & synthesis

- Inserting the synthesis into the analysis equations

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_0^T a_k e^{jk\omega_0 t}e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \frac{1}{T}\int_0^T e^{j(k-n)\omega_0 t}dt$$

$$\delta[k-n]$$

- Note: Assumes the order of integration & sum can be exchanged which is not always the case, relates to discussion in next lecture

Summary of finding the Fourier series coefficients

- The Fourier series coefficients are computed from

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

- The signal is reconstructed from its coefficients using

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

- The Fourier coefficients give insight into "how much" of the frequency $k\omega_0$ is contained in the signal

Fourier coefficients of a rectangular pulse train

Learning objectives

- Find the Fourier series coefficients of a classic example
- Use the results of this derivation in future lectures

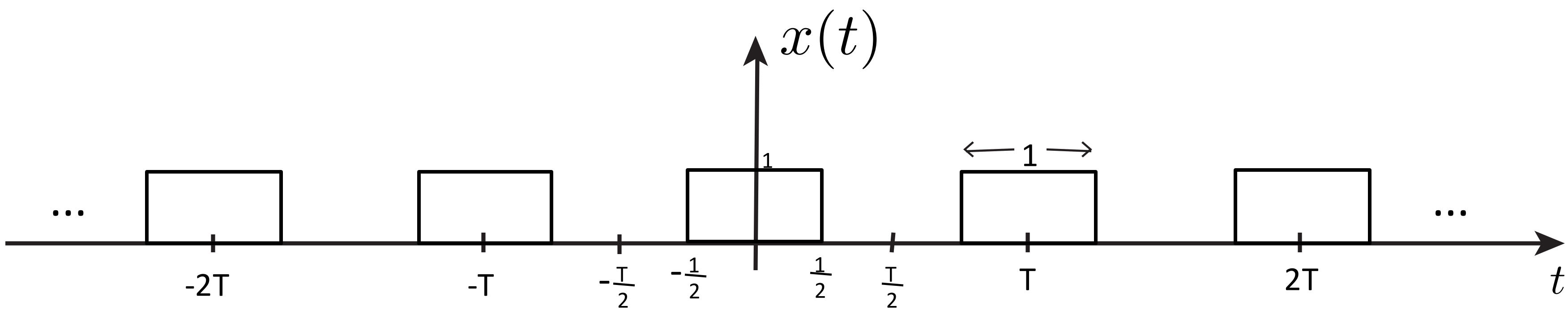
Pulse train I

This is an important reference example. It may not be covered in class. We will use the general result though in other example problems as this is an interesting and relevant signal used in circuits.

- Find the Fourier series coefficients of the unit pulse train

$$x(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \frac{1}{2} < |t| < \frac{T}{2} \end{cases}$$

and is repeated every T



Pulse train 2

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt, \quad \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} 1\cdot e^{-jk\omega_0 t}dt$$

$$= \frac{-1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{jk\omega_0 T}\left(e^{\frac{-jk\omega_0}{2}} - e^{\frac{jk\omega_0}{2}}\right)$$

$$= \frac{2}{k\omega_0 T}\frac{1}{2j}\left(e^{\frac{jk\omega_0}{2}} - e^{-\frac{jk\omega_0}{2}}\right)$$

Pulse train 3

$$a_k = \frac{1}{T}\frac{\sin\left(\frac{k\omega_0}{2}\right)}{\frac{k\omega_0}{2}}$$

$$= \frac{1}{T}\frac{\sin\left(\pi\frac{k\omega_0}{2\pi}\right)}{\pi\frac{k\omega_0}{2\pi}}$$

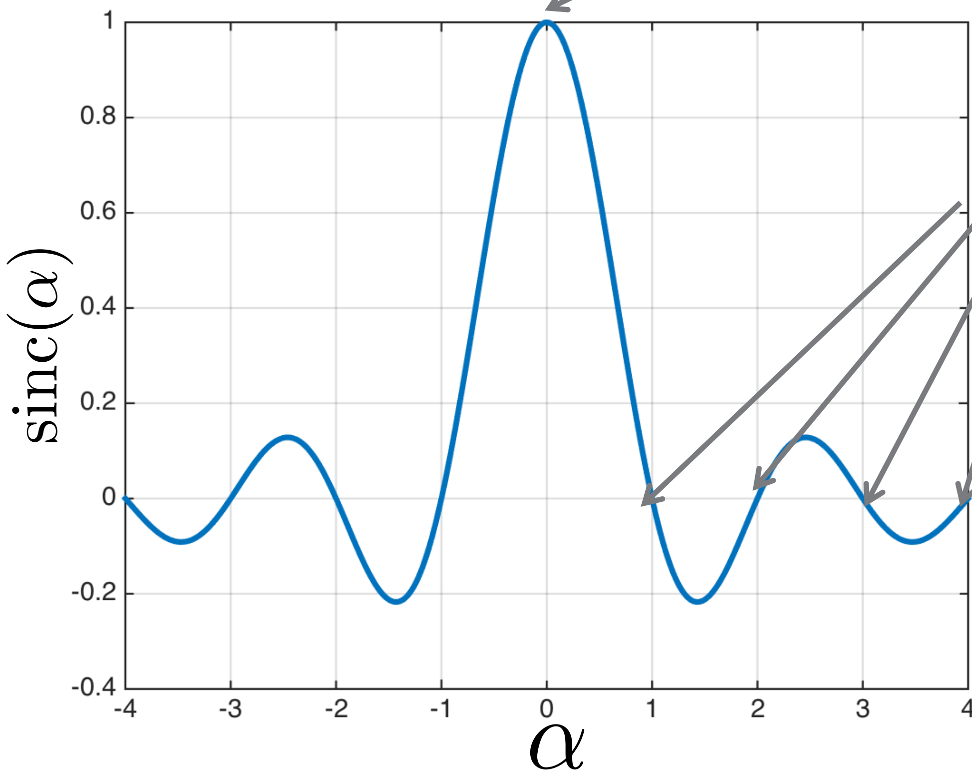
$$= \frac{1}{T}\operatorname{sinc}\left(\frac{k\omega_0}{2\pi}\right)$$

We define the sinc as

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Pulse train 4

Maximum value of 1, i.e. $\mathrm{sinc}(0) = 1$



Zero crossings at +/-1, +/- 2, ….

$$\mathrm{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

Be aware, sometimes sinc is defined like this

$$\mathrm{sinc}(\alpha) = \frac{\sin(\alpha)}{\alpha}$$

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Pulse train 5

- What about k = 0?

$$a_0 = \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{x(t)}_{1}\, dt = \frac{1}{T}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{T}$$

- As an side, for the sinc function

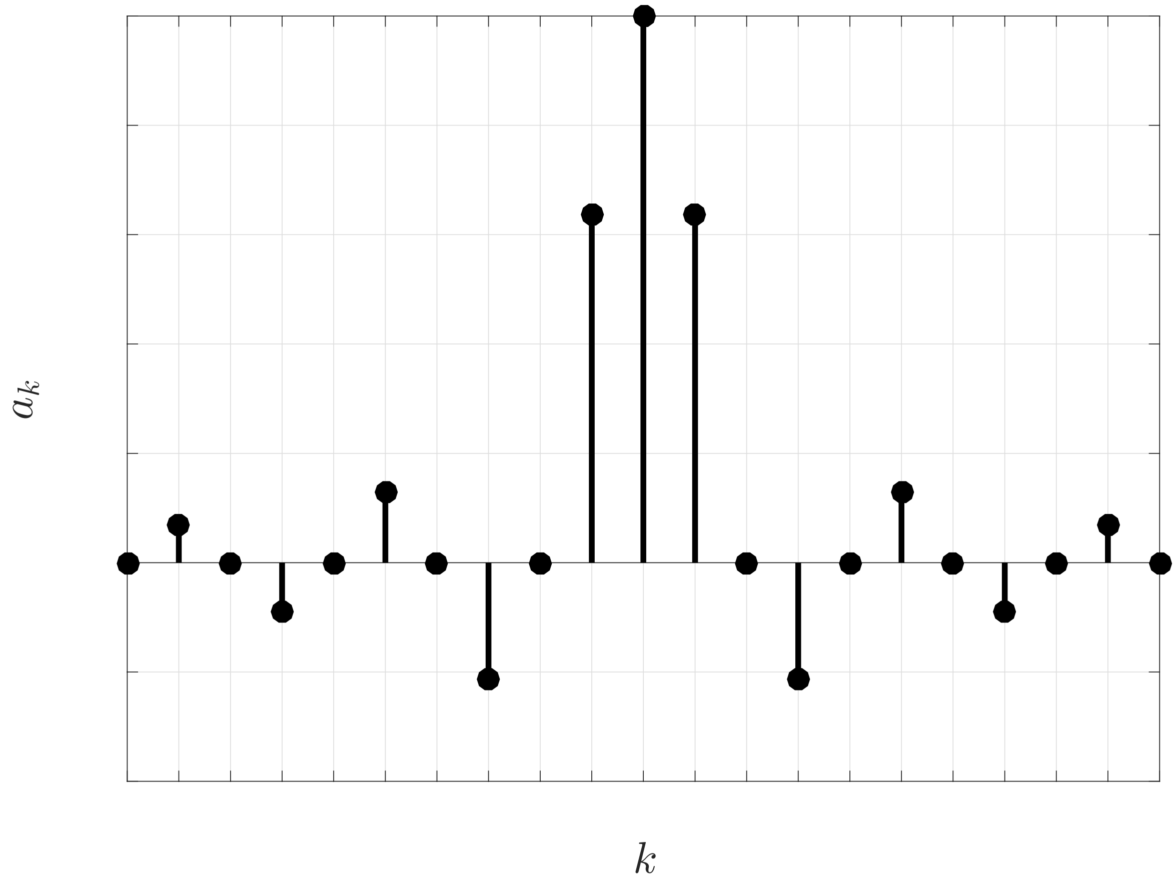
$$\lim_{t\to\infty} \frac{\sin(\pi t)}{\pi t} = \lim_{t\to\infty} \frac{\pi\cos(\pi t)}{\pi} = 1$$

- Therefore the following holds for all values of k

$$a_k = \frac{1}{T}\operatorname{sinc}\left(\frac{k\omega_0}{2\pi}\right) = \frac{\omega_0}{2\pi}\operatorname{sinc}\left(\frac{k\omega_0}{2\pi}\right)$$

Pulse train 6

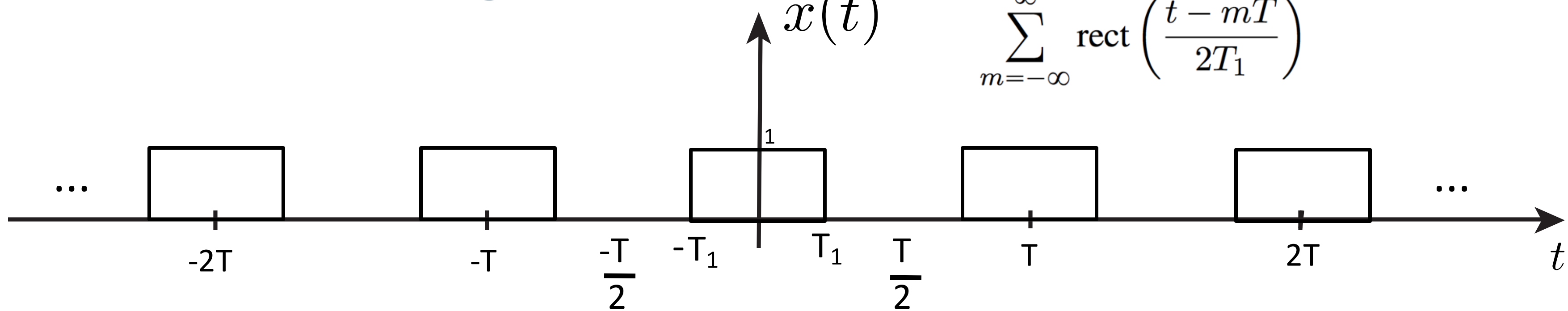
Example with T=2



Pulse train in general

From O&W Example 3.5

$$\sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{t-mT}{2T_1}\right)$$



From the book

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \qquad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

$$a_k = \frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

with fundamental frequency

$$a_k = \frac{2T_1}{T} \text{sinc}\left(\frac{k2T_1}{T}\right)$$

simplified