Discussion Session 8

Fourier Transform : $$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dt$$

If $x(t)$ is peiodic signal, you can define the Fourier Transform from Fourier Series. $x(t) \longleftrightarrow \{a_k\} \Rightarrow X(jw) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(w - kw_o)$

Fourier Transfom

$\delta(t) \xleftrightarrow{F} 2\pi$ $\qquad$ $\frac{1}{2\pi} \xleftrightarrow{F} \delta(w)$

$\frac{e^{jw_o t}}{2\pi} \xleftrightarrow{F} \delta(w - w_o)$ $\qquad$ $\begin{cases} \cos(w_o t) \xleftrightarrow{F} \pi\, \delta(w - w_o) + \pi\, \delta(w + w_o) \\ \sin(w_o t) \xleftrightarrow{F} \pi j\, \delta(w - w_o) - \pi j\, \delta(w + w_o) \end{cases}$

$e^{-at} u(t) \longleftrightarrow \frac{1}{a + jw}$, $Re(a) > 0$ $\qquad$ $e^{-a|t|} \xleftrightarrow{F} \frac{2a}{a^2 + w^2}$, $Re(a) > 0$

$\begin{cases} rect(t) \longleftrightarrow sinc\left(\frac{w}{2\pi}\right) \\ rect\left(\frac{t}{2T_1}\right) \longleftrightarrow 2T_1\, sinc\left(\frac{wT_1}{\pi}\right) \end{cases}$ $\qquad$ $\delta(t) \xleftrightarrow{F} 1$, $\quad u(t) \xleftrightarrow{F} \frac{1}{jw} + \pi\, \delta(w)$

$\begin{cases} sinc(t) \longleftrightarrow rect\left(\frac{w}{2\pi}\right) \\ \frac{B}{2\pi}\, sinc\left(\frac{Bt}{2\pi}\right) \longleftrightarrow rect\left(\frac{w}{B}\right) \end{cases}$

Duality property of fourier transform:

If the fourier transform of $x(t)$ is $Y(jw)$, fourier transform of $Y(t)$ is $2\pi\, X(-w)$: $\quad F(x(t)) = Y(jw) \Rightarrow F(Y(t)) = 2\pi\, X(-w)$

Fourier transform and signal tranformation.

$x(t) \xleftrightarrow{F} X(jw) \qquad y(t) \xleftrightarrow{F} Y(jw)$

Linearity: $a\,x(t) + b\,y(t) \xleftrightarrow{F} a\,X(jw) + b\,Y(jw)$

Time Shift: $x(t-t_0) \xleftrightarrow{F} e^{-jwt_0}\,X(jw)$

Frequency Shift $x(t)\,e^{jw_0 t} \xleftrightarrow{F} X(j(w-w_0))$

Differentiation. $\frac{dx(t)}{dt} \xleftrightarrow{F} jw\,X(jw)$

especial case:
$x(-t) \longleftrightarrow X(-jw)$

Integration: $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{jw}X(jw) + \pi X(0)\,\delta(jw)$

Time scaling: $x(at) \xleftrightarrow{F} \frac{1}{|a|}\,X\left(\frac{jw}{a}\right)$

Frequency scaling: $\frac{1}{|b|}\,x\left(\frac{t}{b}\right) \xleftrightarrow{F} X(jbw)$

Time expansion lead to frequency compression and time compression lead to frequency expansion.

Convolution in time: $x(t) * y(t) \xleftrightarrow{F} X(jw)\,Y(jw)$

Multiplication in time: $x(t)\,y(t) \xleftrightarrow{F} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\,Y(j(w-\theta))\,d\theta$

$$= \frac{1}{2\pi}\left[X(jw) * Y(jw)\right]$$

Parseval's theorem $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(jw)|^2\,dw$

Example #1:

a) $F\{e^{-j4t}\ \text{rect}(2t-1)\}$

Answer: $\text{rect}(t) \xleftrightarrow{F} \text{sinc}\left(\frac{w}{2\pi}\right)$

$\text{rect}(t-1) \xleftrightarrow{F} e^{-jw}\ \text{sinc}\left(\frac{w}{2\pi}\right)$

$\text{rect}(2t-1) \xleftrightarrow{F} \frac{1}{2} e^{-j\frac{w}{2}}\ \text{sinc}\left(\frac{\frac{w}{2}}{2\pi}\right) = \frac{1}{2} e^{-jw/2}\ \text{sinc}\left(\frac{w}{4\pi}\right)$

$e^{-j4t}\ \text{rect}(2t-1) \xleftrightarrow{F} \frac{1}{2} e^{-j\frac{w+4}{2}}\ \text{sinc}\left(\frac{w+4}{4\pi}\right)$

b) $F^{-1}\{4\ \text{rect}(2w) - 2\}$

Answer: $\text{rect}\left(\frac{w}{2\pi}\right) \xleftrightarrow{F} \text{sinc}(t)$

$\text{rect}(2w) \xleftrightarrow{F} \frac{1}{4\pi}\ \text{sinc}\left(\frac{t}{4\pi}\right)$ $\left(\frac{1}{|b|} x\left(\frac{t}{b}\right) \xleftrightarrow{F} X(jbw)\right)$

$4\ \text{rect}(2w) \xleftrightarrow{F} \frac{1}{\pi}\ \text{sinc}\left(\frac{t}{4\pi}\right)$

$4\ \text{rect}(2w) - 2 \xleftrightarrow{F} \frac{1}{\pi}\ \text{sinc}\left(\frac{t}{4\pi}\right) - 2\ \delta(t)$

c) $F\left\{2 e^{-j2t}\ \text{sinc}\left(\frac{t-1}{2}\right)\right\}$

Answer: $\text{sinc}(t) \xleftrightarrow{F} \text{rect}\left(\frac{w}{2\pi}\right)$

$\text{sinc}(t-1) \xleftrightarrow{F} e^{-jw}\ \text{rect}\left(\frac{w}{2\pi}\right)$

$\text{sinc}\left(\frac{t-1}{2}\right) \xleftrightarrow{F} 2\ e^{-j(2w)}\ \text{rect}\left(\frac{2w}{2\pi}\right) = 2\ e^{-2jw}\ \text{rect}\left(\frac{w}{\pi}\right)$

$2\ e^{-j2t}\ \text{sinc}\left(\frac{t-1}{2}\right) \xleftrightarrow{F} 4\ e^{-2j(w+2)}\ \text{rect}\left(\frac{w+2}{\pi}\right)$

Example : Find $\int_{-\infty}^{\infty} |\text{sinc}(2000t)|^2\, dt$

Answer: parseval's theorem $\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(jw)|^2\, dw$

$x(t) = \text{sinc}(2000t) \xleftrightarrow{FT} \frac{1}{2000}\ \text{rect}\left(\frac{w}{2\pi \times 2000}\right) = \frac{1}{2000}\ \text{rect}\left(\frac{w}{4000\pi}\right)$

$$\int_{-\infty}^{\infty} |\text{sinc}(2000t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{2000} \text{rect}\left(\frac{w}{4000\pi} \right) \right)^2 dw$$

$$= \frac{1}{2\pi} \left(\frac{1}{2000} \right)^2 \int_{-2000\pi}^{2000\pi} 1 \, dw = \frac{1}{2\pi} \frac{1}{(2000)^2} \; 2 \times 2000\pi = \frac{1}{2000}$$