Lecture 12

Fourier transform and properties

Preview of today's lecture

- Existence of the Fourier transform
  - Includes important connection to the FS

- Fourier transforms of some basic signals
  - Impact of filtering on the FS coefficients
  - Types of common filters: lowpass, highpass, bandpass, bandstop

- Fourier transform properties (time permitting)

- Midterm 2 is coming on Tuesday featuring the Fourier series

Summarizing the Fourier transform and its inverse

$\{x(t)\}$

Fourier transform (analysis)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier transform (synthesis)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

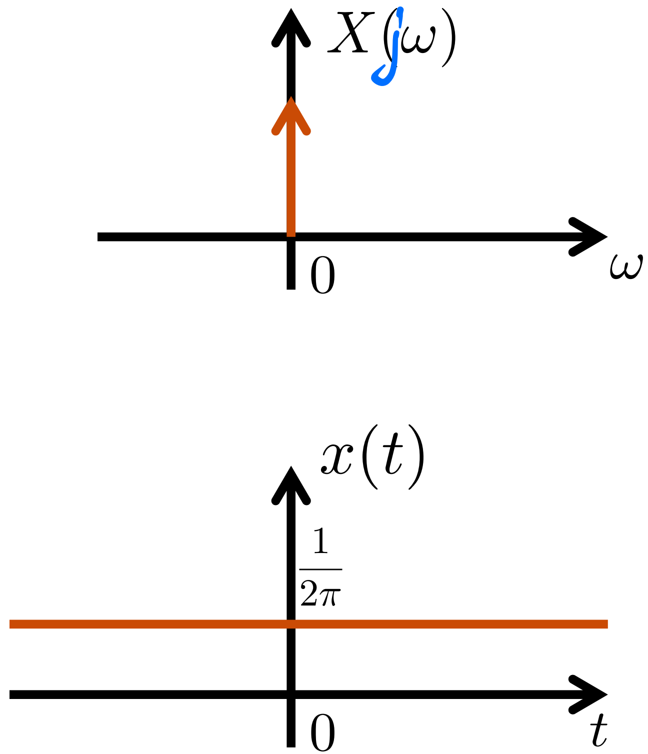
$$\boxed{x(t) \leftrightarrow X(j\omega)}$$

Example – Unit impulse in frequency

- Consider the signal $\boxed{X(j\omega) = \delta(\omega)}$

- Its FT is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega$$
$$= \frac{1}{2\pi}$$



Notice the duality here in each domain

Example – A few deltas in frequency

- Use the Fourier transform synthesis equation to determine the inverse Fourier transform of

$$X(\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$$

- Solution

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)\right) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\left(2\pi e^{j0t} + \pi e^{j4\pi t} + \pi e^{-j4\pi t}\right) \\ &= 1 + \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} \\ &= 1 + \cos(4\pi t) \end{aligned}$$

Sifting

Eulers

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 16 Fourier transform

Fourier

Existence of the Fourier transform

Key points

- Identify sufficient conditions for the "integrals to work"
- Define the Fourier transforms of CT and periodic CT signals

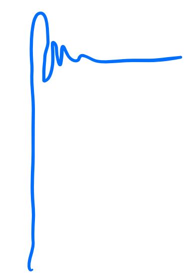
Sufficient condition: Finite energy

- Consider the class of signals that have finite energy $\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt} < \infty$

- For such signals the Fourier transform exists (is finite)

- Further there is zero-energy in the error in the sense that

$$\int_{-\infty}^{\infty} \left| x(t) - \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega}_{\hat{x}(t)} \right|^2 = 0$$



FT gives zero squared error for finite energy signals

Sufficient condition: satisfies Dirichlet

If true

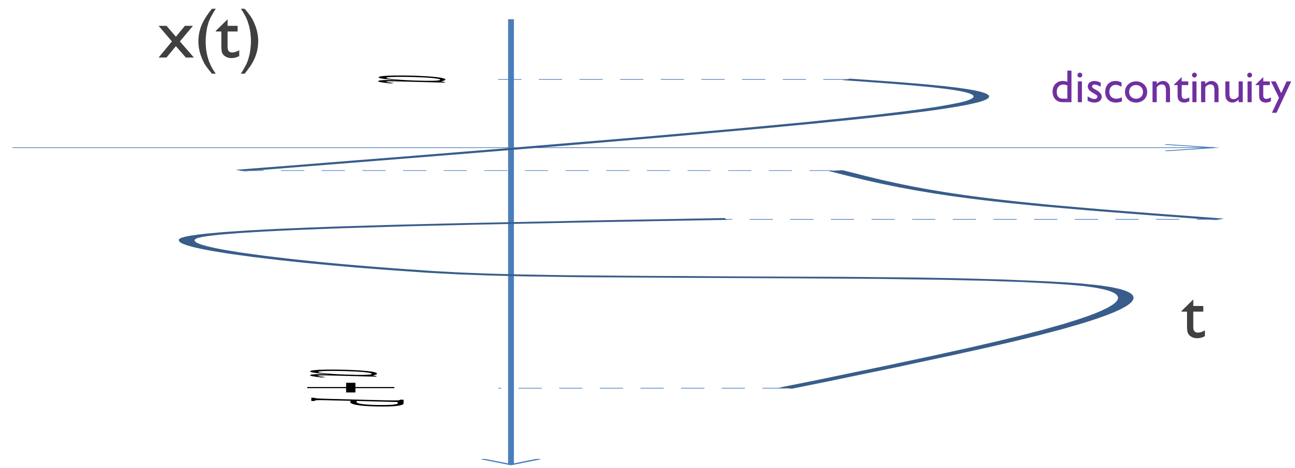
- An aperiodic signal $x(t)$ that satisfies the following conditions
  - (1) Absolute integrability $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
  - (2) Finite number of minima and maxima over a finite interval
  - (3) Finite number of discontinuities over a finite interval

Then

$x(t)$ and $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ are equal except at a discontinuity

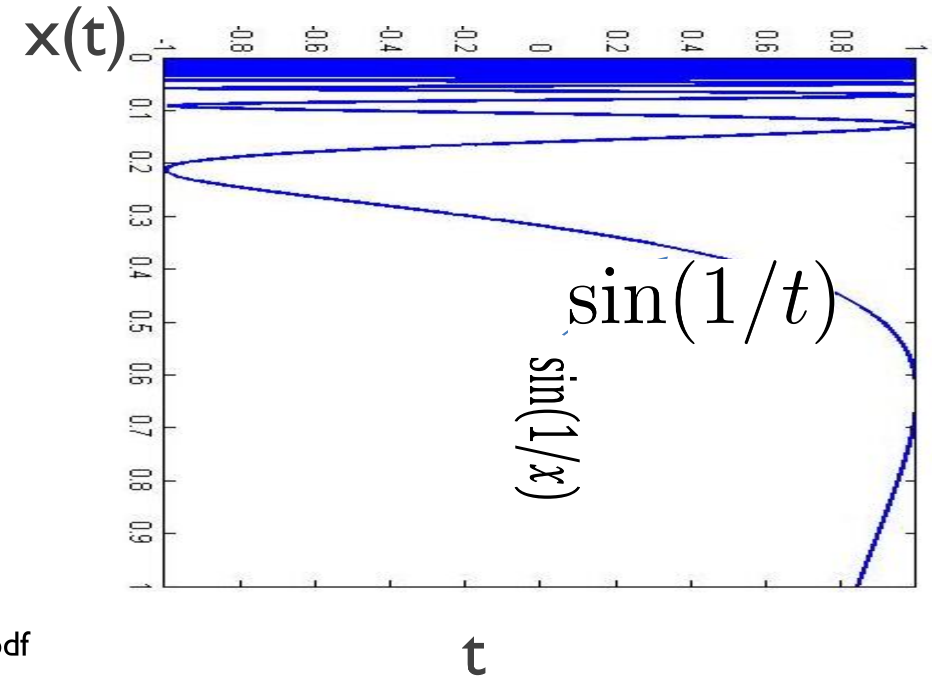
Example

Satisfied



number of maxima and minima tends to infinity as t approaches zero

Not satisfied



http://volkov.eng.ua.edu/ME501/2017-Fall-ME501-05-FourierAnalysis.pdf

What about periodic signals?

- Periodic signals do not satisfy either sufficient condition because

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |x(t)| dt = \infty$$

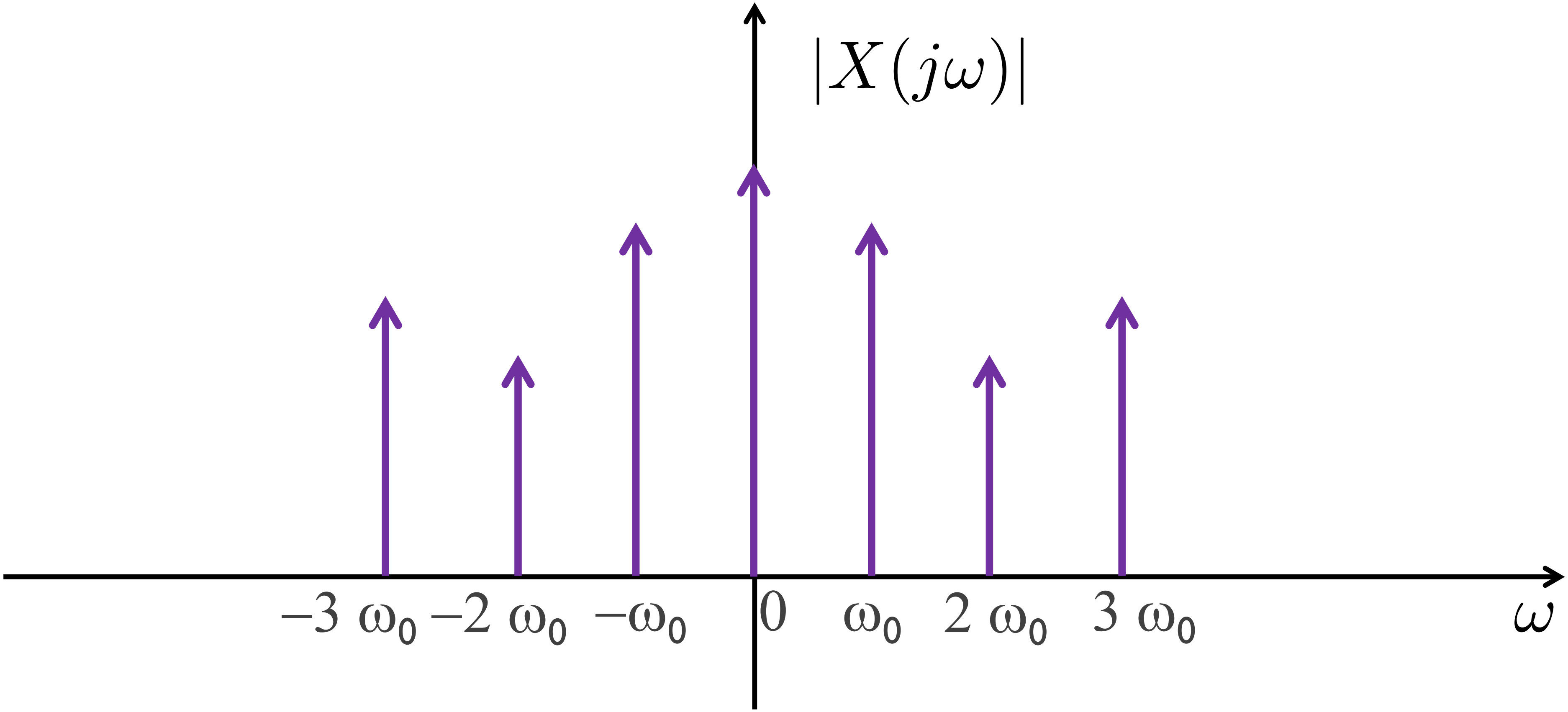
- If Dirac delta is acceptable $\delta(t)$ then can **define** the FT from FS
- Consider a periodic signal $x(t) \leftrightarrow \{a_k\}$ then the FT of $x(t)$ is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Fourier transform

FS coefficients

Periodic signals have a "line" spectrum at the harmonics of the fundamental frequency



Example – Shifted delta in frequency (again)

- Consider the signal

$$\cancel{x(t) = \frac{1}{2\pi} e^{j\omega_0 t}}$$

- The signal is periodic with fundamental frequency ω_0

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \underbrace{\frac{1}{2\pi}}_{a_1} e^{j\omega_0 t}$$

k=1 in terms of FS

$$\frac{1}{2\pi} e^{j\omega_0 t} \leftrightarrow \delta(\omega - \omega_0)$$

more generally

$$\boxed{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

Fourier transform examples

Key points

- Use the Fourier transform synthesis and analysis equations
- Learn and use the transforms for common signals

Fourier transform of a causal exponential

- Consider the signal $\boxed{x(t) = e^{-at}u(t), \quad \mathrm{Re}\{a\} > 0}$

- Its FT is given by

$$X(j\omega) = \underbrace{\int_0^{\infty}}_{\text{b/c of } u(t)} \underbrace{e^{-at}}_{x(t)} e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

$$= \frac{1}{a+j\omega}$$

Fourier transform of a cosine

- Consider the signal $\boxed{x(t) = \cos \omega_0 t}$

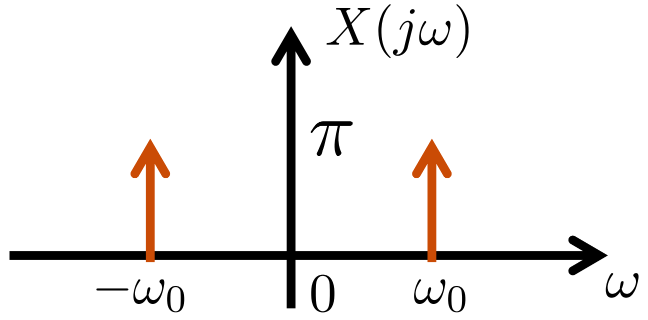
- This is a periodic signal, can find its FT using the FS

$$x(t) = \cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

- Leveraging the frequency impulse results

$$X(j\omega) = \frac{1}{2}\left(2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right)$$

$$= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier transform of a sine

- Consider the signal $\boxed{x(t) = \sin \omega_0 t}$

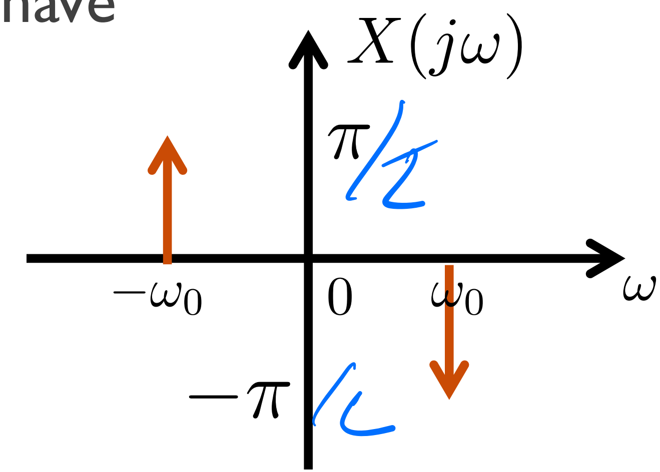
- To get its FT, we note that

$$x(t) = \sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

- Leveraging the frequency impulse results, we have

$$X(j\omega) = \frac{1}{2j}\left(2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right)$$

$$= \pi j\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)$$

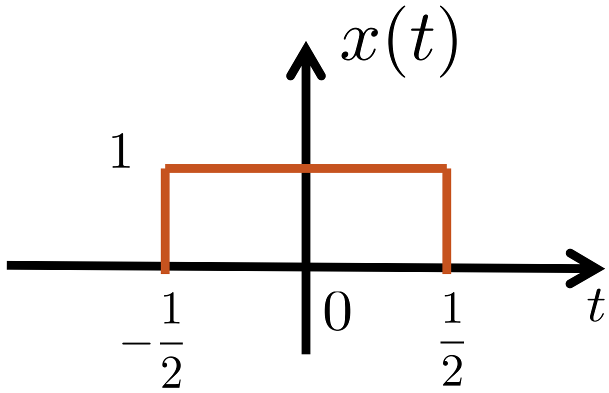


FT of a rectangle function is an important example

Fourier transform of a rectangle function

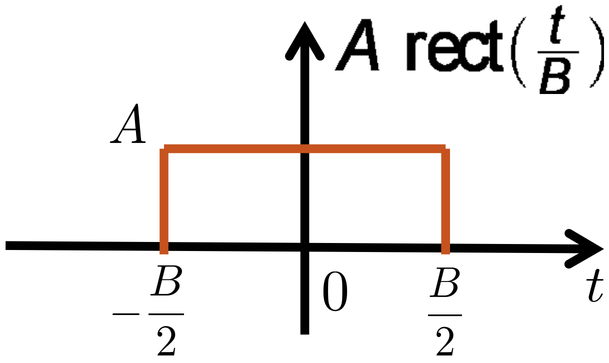
- Consider the signal

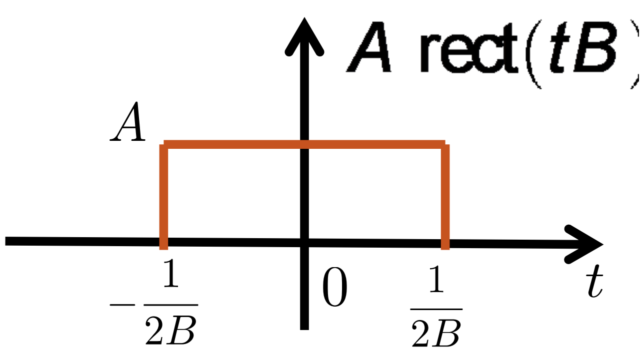
$$\boxed{x(t) = \mathrm{rect}(t) = \Pi(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leqslant \frac{1}{2} \end{cases}}$$



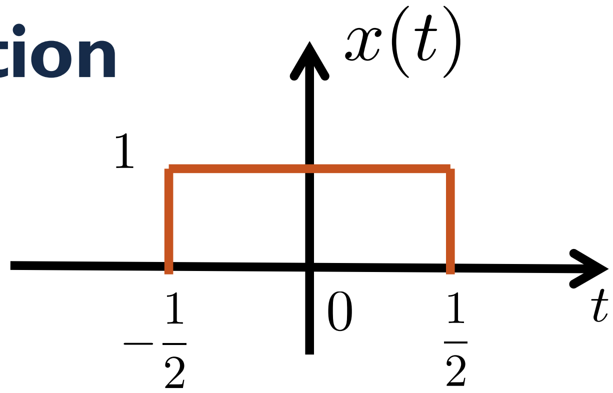
- Note that $$\int_{-\infty}^{\infty} |x(t)|^2 dt = 1$$

- Can create other related shapes





Fourier transform of a rectangle function



- Its FT is given by

$$X(j\omega) = \int_{-\infty}^{\infty} \mathrm{rect}(t) e^{-j\omega t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{j\omega} (e^{-\frac{j\omega}{2}} - e^{\frac{j\omega}{2}})$$

$$= \frac{2}{\omega} \cdot \frac{1}{2j} (e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}})$$

$$= \frac{2}{\omega} \sin \frac{\omega}{2}$$

$$= \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$$

$$= \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

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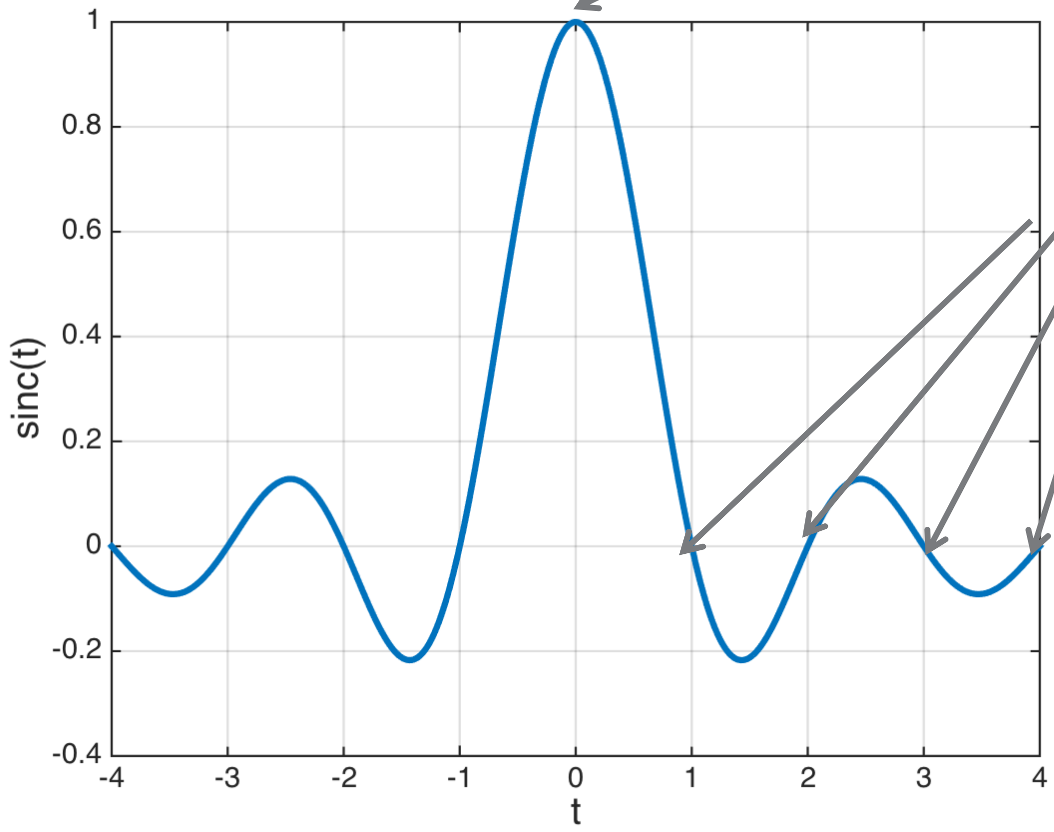
About the sinc function

Maximum value of 1



Zero crossings at +/-1, +/- 2, ….

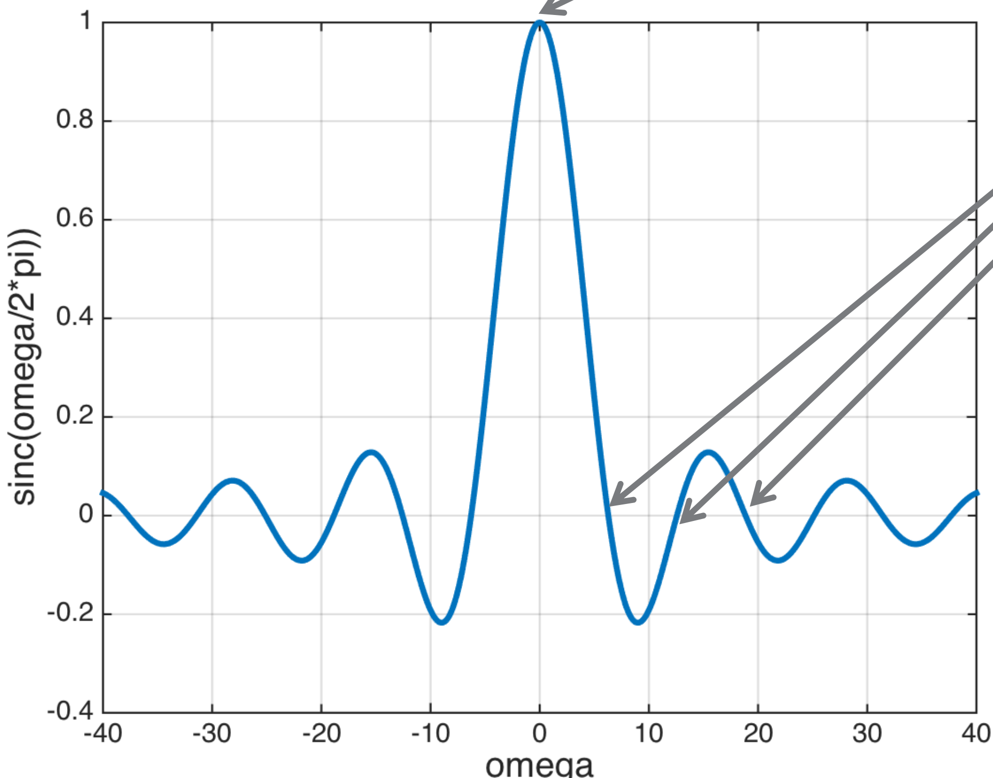
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Be aware, sometimes sinc is defined like this

$$\mathrm{sinc}(t) = \frac{\sin(t)}{t}$$

Fourier transform of rectangle function

Maximum value of 1



Zero crossings at +/- 2π,+/- $4\pi\ldots$

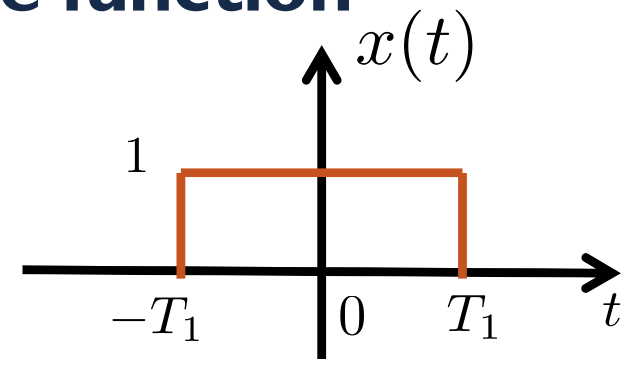
$$X(j\omega) = \operatorname{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\operatorname{sinc}(\omega/2\pi) = \frac{\sin(\pi\,\omega/2\pi)}{\pi\,\omega/2\pi}$$

$$= \frac{\sin(\omega/2)}{\omega/2}$$

Fourier transform of a scaled rectangle function

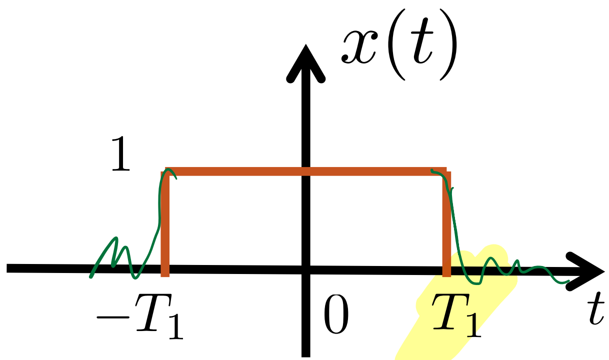


- Its FT is given by

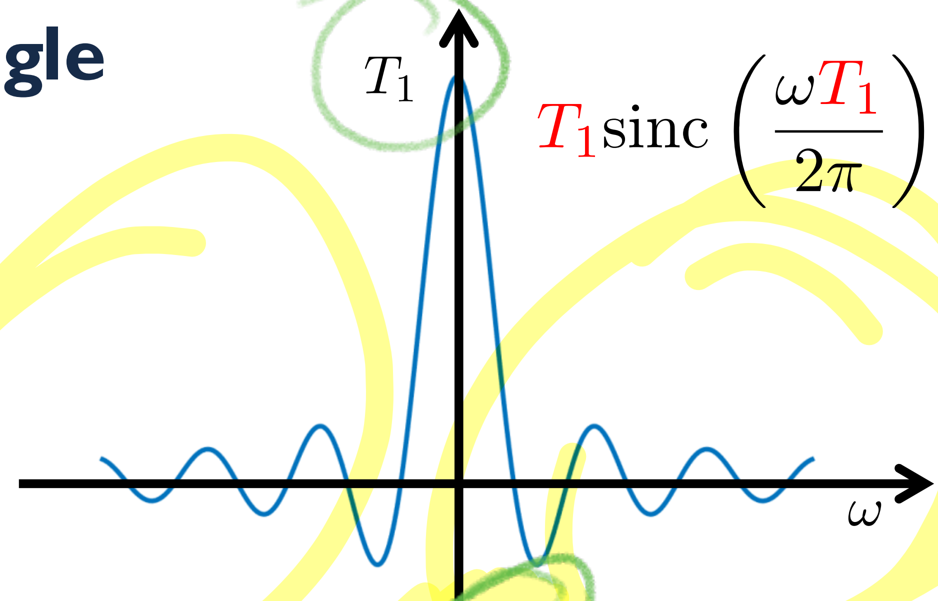
$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t/2T_1) e^{-j\omega t} dt \\
&= \int_{-T_1}^{T_1} e^{-j\omega t} dt \\
&= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} \\
&= -\frac{1}{j\omega}(e^{-j\omega T_1} - e^{j\omega T_1})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\omega} \cdot \frac{1}{2j}(e^{j\omega T_1} - e^{-j\omega T_1}) \\
&= \frac{2}{\omega} \sin \omega T_1 \\
&= T_1 \frac{\sin \frac{\omega T_1}{2}}{T_1 \frac{\omega}{2}} \\
&= T_1 \mathrm{sinc}\left(\frac{\omega T_1}{2\pi}\right)
\end{aligned}
$$

Fourier transform of rectangle



$\overset{F}{\longleftrightarrow}$



Zeros at $\pm\pi/T_1, \pm\pi/T_1, \ldots$

- Crossings at $\pm\pi/T_1, \pm\pi/T_1, \ldots$
- For $T_1 \to \infty$, FT is $\delta(!\ )$
- So
  - $T_1 \downarrow$, pulse narrow, sinc wide
  - $T_1 \uparrow$, sinc becomes narrow, pulse wide

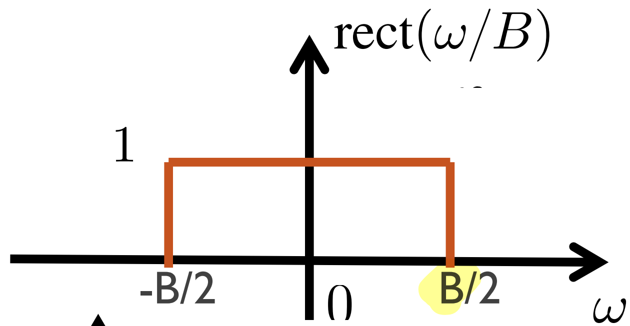
Narrow pulse in time is broad in frequency

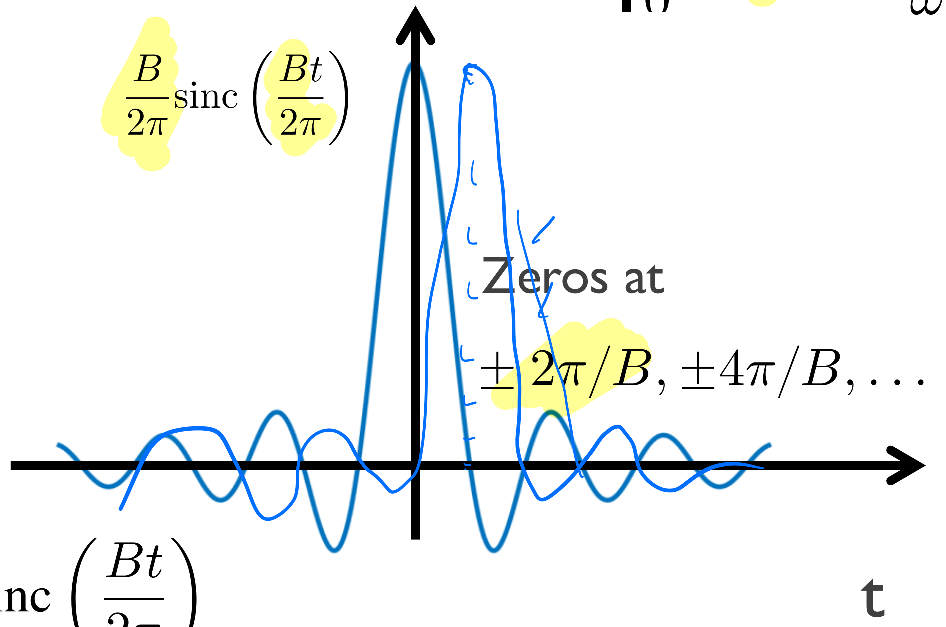
Inverse Fourier transform of the rectangle function

- Can compute the FT directly

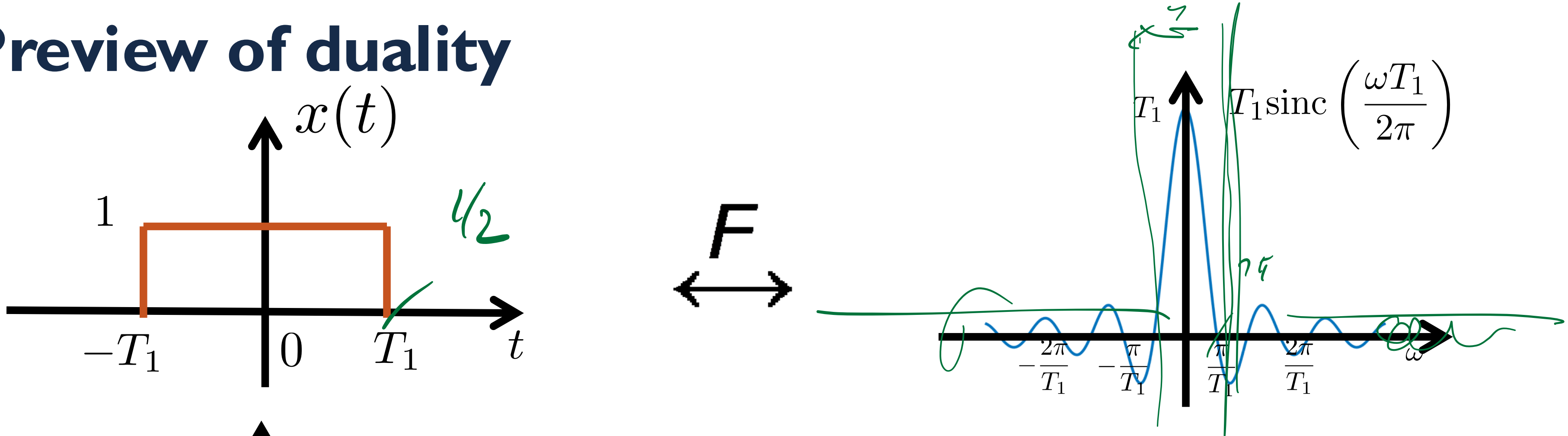
$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\frac{B}{2}}^{\frac{B}{2}} e^{j\omega t}d\omega \\
&= \frac{1}{j2\pi t}e^{j\omega t}\Big|_{-\frac{B}{2}}^{\frac{B}{2}} \\
&= \frac{1}{j2\pi t}(e^{\frac{jB}{2}t} - e^{-\frac{jB}{2}t}) \\
&= \frac{1}{\pi t}\sin(Bt/2) \\
&= \frac{B/2\pi}{\pi Bt/2\pi}\sin\left(\frac{Bt\pi}{2\pi}\right)
\end{aligned}
$$

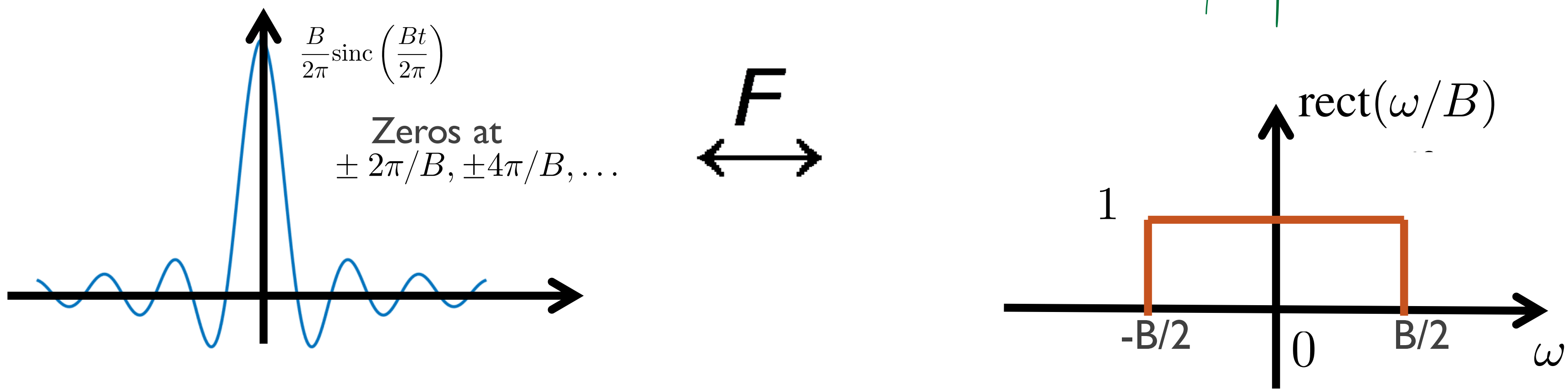
$$\Rightarrow \frac{B}{2\pi}\mathrm{sinc}\left(\frac{Bt}{2\pi}\right)$$





Preview of duality





It is possible to find one set of transforms from the other

Essential Fourier transforms

Key points

- Know these important FT pairs

Basic Fourier transform pairs 1/3

(more pairs in the book)

	Time domain *x(t)*	Frequency domain *X(j ω)*
Delta	$\delta(t)$	2π
Constant	$\dfrac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\dfrac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega - \omega_0)$
Causal exponential	$e^{-at}u(t)$ $\mathrm{Re}\{a\} > 0$	$\dfrac{1}{a + j\omega}$

(more pairs in the book)

Basic Fourier transform pairs 2/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Cosine	$\cos \omega_0 t$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
Sine	$\sin \omega_0 t$	$\pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

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(more pairs in the book)

Basic Fourier transform pairs 3/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Rectangle	$\mathrm{rect}(t)$	$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
Scaled rectangle	$\mathrm{rect}\left(\frac{t}{2T_1}\right)$	$2T_1\frac{\sin(\omega)}{\omega} = 2T_1\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$
Sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
Scaled sinc	$\frac{B}{2\pi}\mathrm{sinc}\left(\frac{Bt}{2\pi}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$

Fourier transform properties

Key points

- Use FT properties to simplify calculation & build intuition
- Analyze problems that include FT properties

Fourier transform properties I $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$ $y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)$

	Time domain	Fourier transform
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(j\omega)$

Fourier transform properties 2 $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

	Time domain	Fourier transform
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert}x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2d\omega$	

Fourier transform properties 3

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$$

	Time domain	Fourier transform
Convolution in time	$y(t) = h(t) * x(t)$	$Y(j\omega) = H(j\omega)X(j\omega)$
Multiplication in time	$y(t) = h(t)x(t)$	$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$

Linearity

- If $$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega), \quad y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$
- Then $$ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$$

Sums in time lead to sums in frequency

Linearity example

- Consider

$$\cos t \leftrightarrow \square[\delta(! \;-1) + \delta(! \;+1)]$$

$$\sin t \leftrightarrow \square j\,[\delta(! \;+1) - \delta(! \;-1)]$$

- By linearity

$$\cos t + j\,\sin t \leftrightarrow \square\delta(! \;-1) + \underbrace{\square\delta(! \;+1) - \square\delta(! \;+1)}_{0} + \square\delta(! \;-1)$$

$$= 2\square\delta(! \;-1)$$

$$= F\{e^{t}\}$$

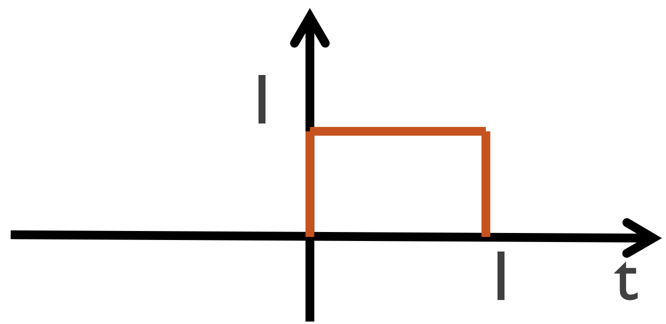
Time shifting

- If
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
- Then
$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$
- Time shifting does not change the magnitude in the freq. domain
$$|X(j\omega)e^{-j\omega t_0}| = |X(j\omega)||e^{-j\omega t_0}|$$
- Phase changes are linear with frequency $!$ and shift t_0
$$\angle(X(j\omega)e^{-j\omega t_0}) = \angle X(j\omega) - \omega t_0$$

Shift in time leads to linear phase shift in frequency

Time shifting example

- Find the Fourier transform of



- This signal is just a shifted rectangle function $\mathrm{rect}(t - 1/2)$

- Using the time shifting property

$$\mathrm{rect}(t - 1/2) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega/2}\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

Differentiation

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$\frac{dx}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

- Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\underbrace{\frac{dx}{dt}}_{\text{new func}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}(e^{j\omega t})d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega X(j\omega)}_{\text{new FT}} e^{j\omega t}d\omega$$

Differentiation example

- What is the FT of the system characterized by

$$\frac{dy}{dt} + ay(t) = x(t)$$

- Solution:
  - Take FT of both sides

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$
$$(j\omega + a)Y(j\omega) = X(j\omega)$$

  - Therefore $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{1}{j\omega + a}$

Integration

- If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

DC component

Time scaling

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Time **expansion** |a|<1 leads to frequency **compression**

Time **compression** |a|>1 leads to frequency **expansion**

Frequency scaling

- If
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

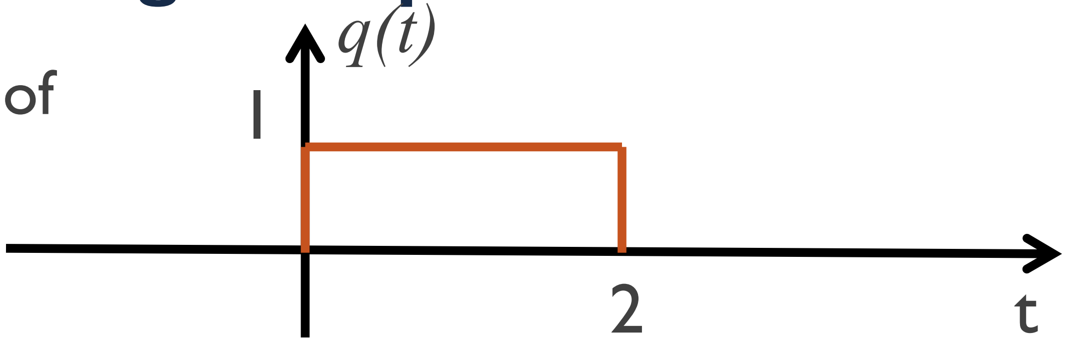
- Then
$$\frac{1}{|b|}x\left(\frac{t}{b}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(jb\omega)$$

Frequency expansion |b|<1 leads to time compression

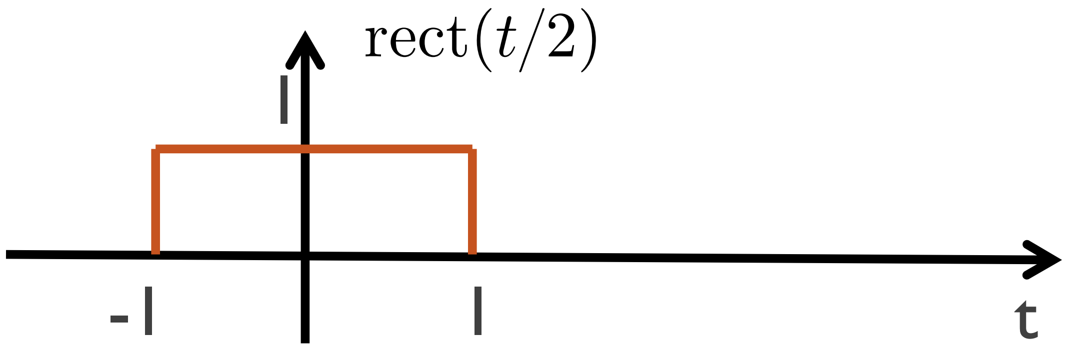
Frequency compression |b|>1 leads to time expansion

Shift and scaling example

- Find the FT of



- To start, notice that



- Shifting gives our function $q(t) = \mathrm{rect}\left(\frac{t-1}{2}\right) = \mathrm{rect}\left(\frac{t}{2} - \frac{1}{2}\right)$

Shift and scaling example (continued)

- From the scaling property

$$\mathrm{rect}\left(\frac{t}{2}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\mathrm{sinc}\left(\frac{2\omega}{2\pi}\right) = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

- From the shift property

$$\mathrm{rect}\left(\frac{t-1}{2}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega}2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

Another scaling example

- What is the inverse Fourier transform of $\text{sinc}(\omega)$?
- We know that
  - From the rect-sinc Fourier pair $\quad \text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\frac{\omega}{2\pi}\right)$
  - From the scaling law $\quad x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$
- Using the scaling property $\quad \text{rect}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\text{sinc}\left(\omega\right)$
- Therefore using linearity $\quad \frac{1}{2\pi}\text{rect}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\omega\right)$

Inversion

- Find the FT of $x(-t)$

- This is just a special case of time and frequency scaling

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- With $a = -1$

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$$

Example using scaling and time shift

- Determine the Fourier transform of $\mathrm{sinc}(1-2t)$

$$
\begin{aligned}
x(t) &= \mathrm{sinc}(1-2t) \\
&= y(2t) \\
y(t) &= \mathrm{sinc}(1-t) \\
&= \mathrm{sinc}(-(t-1)) \\
&= z(t-1) \\
z(t) &= \mathrm{sinc}(-t) \\
&= \mathrm{sinc}(t)
\end{aligned}
$$

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}(\omega/2\pi)$$

$$\mathrm{sinc}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\mathrm{rect}(-\omega) = 2\pi\mathrm{rect}(\omega)$$

$$\mathrm{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}(\omega/(2\pi))$$

$$
\begin{aligned}
Z(j\omega) &= \mathrm{rect}(\omega/(2\pi)) \\
Y(j\omega) &= e^{-j\omega} Z(j\omega) \\
X(j\omega) &= \frac{1}{2} Y\left(j\frac{\omega}{2}\right) \\
&= \frac{1}{2} e^{-j\omega/2} \mathrm{rect}(\omega/(4\pi))
\end{aligned}
$$

Frequency shifting

- If $$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
- Then $$x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

this is called modulation

- Corollary $$x(t)\cos\omega_0 t \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$$

Modulate in time leads to shift in frequency

Example combining shift and scaling

- Determine the inverse Fourier transform of

$$X(j\omega) = \frac{2\sin\left(3(\omega - 2\pi)\right)}{(\omega - 2\pi)}$$

Example (continued)

- Given

$$X(j\omega) = \frac{2\sin\left(3(\omega - 2\pi)\right)}{(\omega - 2\pi)}$$

- Use the following fact

$$\mathrm{rect}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$$

- For convenience let $R(j\omega) = \dfrac{\sin(\omega/2)}{\omega/2}$

- Rewrite as another shifted function

$$X(j\omega) = Y(j(\omega - 2\pi))$$

$$Y(j\omega) = \frac{2\sin(3\omega)}{\omega}$$

Example (continued)

- Rewrite again as

$$Y(j\omega) = 3\frac{2\sin(6\omega/2)}{6\omega/2}$$
$$= 6R(j6\omega)$$

- Using the scaling property

$$Y(j\omega) = 6R(j6\omega) \Rightarrow y(t) = \mathrm{rect}(t/6)$$

- Using the shift property

$$X(j\omega) = Y(j(\omega - 2\pi)) \Rightarrow x(t) = e^{j2\pi t}\,\mathrm{rect}(t/6)$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Energy of the signal in the time domain

Energy of the signal in the frequency domain

- This is a result of conservation of energy
- Scaling factor is because of radians

Example usign Parseval's theorem

- If the signal $x(t)$ has the FT below $X(j\omega)$

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2 \qquad X(j\omega) = \begin{cases} \frac{j}{2\pi}, & -2 \leqslant \omega < 0 \\ -\frac{j}{2\pi}, & 0 \leqslant \omega \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$$

- Calculate

$$A = \int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

Example with Parseval's theorem (cont.)

$$\int_{-\infty}^{\infty} t^2 \left(\frac{\sin(t)}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi^3}$$

Duality in the Fourier transform

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$X(jt) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

If you know one Fourier pair then you know the other Fourier pair

Applications of duality

- Reproving frequency shift
  - Consider $$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$
  - Then $$x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))$$

- Impulse in time and frequency
  - Consider $$\delta(t) \overset{F}{\longleftrightarrow} 1$$
  - Then $$1 \overset{F}{\longleftrightarrow} 2\pi\delta(\omega)$$

Duality example

- Consider

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

- Then

$$\mathrm{sinc}\left(\frac{t}{2\square}\right) \overset{F}{\longleftrightarrow} 2\square \cdot \mathrm{rect}(-!\ )$$

$$= 2\square \cdot \mathrm{rect}(!\ )$$

Since rect is an even function.