ECE 101: Linear Systems Fundamentals

Summer Session II 2020 - Lecture 9

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Today's topics

- Definition of discrete-time Fourier series (DTFS)
- The synthesis and analysis equations for DTFS
- Examples of DTFS

Signals and Systems (2th Edition): sections 3.6 and 3.7

Fourier Series of Discrete-Time Signals

Fourier Series for Discrete-time Periodic Signals

Fourier Series of a Periodic DT Signal

For a periodic DT signal $x[n]$ with fundamental period $N \geq 1$ and fundamental frequency $\omega_0 = \frac{2\pi}{N}$, we have:

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \tag{1}$$

where

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}. \tag{2}$$

- Eq. (1) is called *the Fourier series* representation of $x[n]$.
- Each a_k is called a Fourier coefficient of $x[n]$.

Key Equations for DT FS

- For a **periodic** signal $x[n]$ with *fundamental period* N and **fundamental frequency** $\omega_0 = \frac{2\pi}{N}$
- **Synthesis Equation:**

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n},$$

- **Analysis Equation:**

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

- **Notation**

$$x[n] \xleftrightarrow{F.S.} a_k$$

- Important feature: $\{a_k\}$ is periodic with period N

$$a_{k+N} = a_k$$

a_k coefficients are periodic with period of N

- The Fourier series coefficients of DT signals are periodic with period N

$$a_{k+N} = a_k$$

Proof.

$$\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} \\
a_{k+N} &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-j(k+N)\omega_0 n} \\
&= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} e^{-jN\omega_0 n} \\
&= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} e^{-j2\pi n} \\
&= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = a_k
\end{aligned}$$

$\square$

DTFS Example

- Find the Fourier series coefficients of the DT signal

$$x[n] = cos(\frac{5\pi}{4}n) + jcos(\frac{\pi}{2}n) + 1$$

$$\cos\left(\frac{5\pi}{4}(n+N)\right) = \cos\left(\frac{5\pi}{4}n\right) \longrightarrow \cos\left(\frac{5\pi}{4}n + \frac{5\pi}{4}N\right) = \cos\left(\frac{5\pi}{4}n\right)$$

$$\frac{5\pi}{4}N = 2k\pi \longrightarrow N = \frac{8}{5}k \longrightarrow N_0 = 8$$

$$\cos\left(\frac{\pi}{2}n + \frac{\pi}{2}N\right) = \cos\left(\frac{\pi}{2}n\right) \longrightarrow \frac{\pi}{2}N = 2k\pi \longrightarrow N_0 = 4k \longrightarrow N_0 = 4$$

The period of $x[n]$ is LCM $(4, 8) = 8$, $N = 8$, $\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

$$x[n] = 1 + \frac{1}{2}e^{j\frac{5\pi}{4}n} + \frac{1}{2}e^{-j\frac{5\pi}{4}n} + \frac{j}{2}e^{j\frac{\pi}{2}n} + \frac{j}{2}e^{-j\frac{\pi}{2}n}$$

DTFS Example

- Find the Fourier series coefficients of the DT signal

$$x[n] = cos(\frac{5\pi}{4}n) + jcos(\frac{\pi}{2}n) + 1$$

$$x[n] = 1 + \frac{1}{2}e^{j\frac{5\pi}{4}n} + \frac{1}{2}e^{-j\frac{5\pi}{4}n} + \frac{j}{2}e^{j\pi/2\, n} + \frac{j}{2}e^{-j\pi/2\, n}$$

synthesis equation: $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=-5}^{2} a_k e^{jk\frac{\pi}{4}n}$

$$x[n] = a_{-5}e^{-j\frac{5\pi}{4}n} + a_{-4}e^{-j\pi n} + a_{-3}e^{-j\frac{3\pi}{4}n} + a_{-2}e^{-j\pi/2\, n} + a_{-1}e^{-j\pi/4\, n} + a_0 + a_1 e^{j\frac{\pi}{4}n} + a_2 e^{j\pi/2\, n}$$

because I selected the interval of k to be $[-5, 2]$, I need to find $a_{-5}, a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2$. a_k values are periodic with period of N.
In this example $N=8$, so we are looking for 8 distinct a_k values.

DTFS Example

- Find the Fourier series coefficients of the DT signal

$$x[n] = cos(\frac{5\pi}{4}n) + jcos(\frac{\pi}{2}n) + 1$$

By comparing $x[n]$ written in exponential format with its Fourier series representation:

$a_0 = 1$, $a_{-5} = \frac{1}{2}$, $a_{-2} = \frac{j}{2}$, $a_2 = \frac{j}{2}$, $a_{-3} = \frac{1}{2}$

- The only exponential term in $x[n]$ that is not accounted for is $\frac{1}{2} e^{j\frac{5\pi}{4}n}$

- One way to identify the order of this exponential is to compare it with $e^{jk\frac{\pi}{4}n}$ from the synthesis equation. $e^{j\frac{5\pi}{4}n} = e^{j\frac{k\pi}{4}n}$ for $k=5 \rightarrow a_5 = \frac{1}{2}$

- we are not looking for a_5. This is not in our chosen interval, but we know a_k coefficients are periodic $\Rightarrow a_5 = a_{5-8} = a_{-3} \Rightarrow a_{-3} = \frac{1}{2}$

DTFS Example

- Find the Fourier series coefficients of the DT signal

$$x[n] = cos(\frac{5\pi}{4}n) + jcos(\frac{\pi}{2}n) + 1$$

- another way of identifying the order of $e^{j\frac{5\pi}{4}n}$ is as following:

$e^{j\frac{5\pi}{4}n} = e^{j\frac{5\pi}{4}n} \times e^{-j2\pi n}$ $\longrightarrow$ because $e^{-j2\pi n} = \cos(-2\pi n) + j\sin(-2\pi n)$

$= 1$

$e^{j\frac{5\pi}{4}n} e^{-j2\pi n} = e^{j(\frac{5\pi}{4}n - 2\pi n)}$

$= e^{-j\frac{3\pi}{4}n}$ $\Rightarrow \frac{1}{2}e^{j\frac{5\pi}{4}n} = \frac{1}{2}e^{-j\frac{3\pi}{4}n}$

If you look at the expanded version of the synthesis equation in the selected range, $e^{-j\frac{3\pi}{4}n}$ is present in the range and $a_{-3} = \frac{1}{2}$

DTFS Example

- Find the Fourier series coefficients of the DT signal

$$x[n] = cos(\frac{5\pi}{4}n) + jcos(\frac{\pi}{2}n) + 1$$

a_k values for $[-5, 2]$ are:

$\boxed{a_{-5} = \frac{1}{2}}$, $\boxed{a_{-4} = 0}$, $\boxed{a_{-3} = \frac{1}{2}}$, $\boxed{a_{-2} = \frac{j}{2}}$, $\boxed{a_{-1} = 0}$, $\boxed{a_{0} = 1}$, $\boxed{a_{1} = 0}$, $\boxed{a_{2} = \frac{j}{2}}$

Properties of discrete-time Fourier series

Properties of DTFS (please read Section 3.7)

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [\text{x[n] real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [\text{x[n] real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Parseval's relation for DTFS

- Parseval's relation

Parseval's relation

For a periodic signal $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$, we have:

$$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

- Note that $|a_k|^2$ is the average energy of the k'th harmonic component
- Parseval relation: Energy of a signal is equal to sum of the energy of its harmonic components!