UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2020
Linear Systems Fundamentals

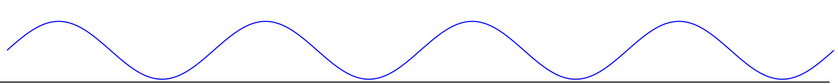
SOLUTIONS TO FINAL EXAM

Preface

- I recognize that final exams are stressful in the best of times, and this is an unusually hard time.
- Please remember that despite the stress, I am counting on you to uphold academic integrity while you complete your final exam.
- Posting or seeking exam questions or answers online, or by consulting unauthorized resources, is a gross violation of our principles of integrity and engineering ethics.
- To be fair to all students, any integrity violations discovered during the final exam will be reported to the Dean of Engineering and to the office of Academic Integrity.
- Please make sure you understand and follow the academic integrity guidelines for the exam. If you are not sure, ask me.
- An honest effort, no matter what the outcome, is something to be proud of, especially in these challenging times.
- I am very proud of the commitment and resilience you have displayed. You should feel proud, too.
- **Good luck on the exam!**

PRINT YOUR NAME J. B. Joseph Fourier

Student ID Number Transform Student ID Number 101

Signature 

Your signature confirms that you have completed this exam on your own and in accordance with the Academic Integrity Agreement.

Instructions

- Open Canvas website, no electronics allowed for problem solving.
- Tables from Chapters 3 ,4 5, and 9 of the textbook are attached to the exam for your convenience.
- Time allowed: 4 hours (including download and upload time).
- Write your solutions in the designated space in the exam.
- **Justify your answers.**
- If extra pages are needed to show all of your work, use the scratch pages at the end of the exam and submit them.
- **Upload your solutions by 7pm via Gradescope.**
- If you have not submitted the **Academic Integrity Agreement**, or are not sure if you did, **please do so now**.
- **If you do not submit the Academic Integrity Agreement form, you will receive an Incomplete grade for the quarter.**

Problem	Weight	Score
1	30 pts	30
2	30 pts	30
3	30 pts	30
4	30 pts	30
5	30 pts	30
6	30 pts	30
Total	180 pts	180

Good luck!

Name/Student ID: ______________________________

Problem 1 [DT Filtering] (30 points)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(a) Determine the frequency response $H(e^{j\omega})$ of the system.

(b) Determine and sketch precisely the magnitude $|H(e^{j\omega})|$ of the frequency response in the range $\omega \in [-\pi, \pi]$.

(c) Determine and sketch precisely the phase $\angle H(e^{j\omega})$ of the frequency response in the range $\omega \in [-\pi, \pi]$. (Add or subtract integer multiples of 2π to keep the phase in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.)

(d) Find a non-zero sinusoidal input signal $x[n]$ to the system that produces the output $y[n] = 2x[n]$.

Write your answers to parts (a), (b), (c), and (d) on the following 4 pages.

Name/Student ID: ______________________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(a) (8 points)

Determine the frequency response $H(e^{j\omega})$ of the system.

This can be solved either by applying the eigenfunction property of LTI systems or by applying the DTFT to both sides of the difference equation. We use the latter approach.

By the time-shifting property of the DTFT, we have

$$Y(e^{j\omega}) = X(e^{j\omega}) - e^{-j\omega 2} X(e^{j\omega}).$$

Thus

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 - e^{-j\omega 2}.$$

This can be rewritten as:

$$\begin{aligned} H(e^{j\omega}) &= e^{-j\omega}(e^{j\omega} - e^{-j\omega}) \\ &= 2je^{-j\omega}\sin(\omega). \end{aligned}$$

Name/Student ID: ______________________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

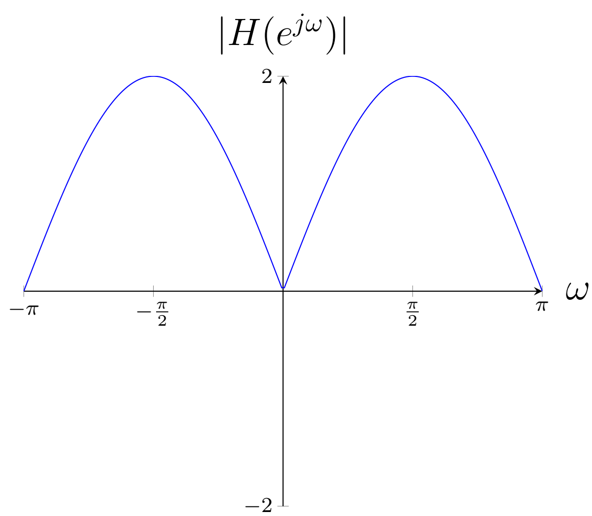
(b) (8 points)

Determine and sketch precisely the magnitude $|H(e^{j\omega})|$ of the frequency response in the range $\omega \in [-\pi, \pi]$.

From part (a),

$$\begin{aligned} |H(e^{j\omega})| &= |2je^{-j\omega}\sin(\omega)| \\ &= 2|\sin(\omega)|. \end{aligned}$$

A plot of $|H(e^{j\omega})|$ over one period $\omega \in [-\pi, \pi]$ is shown below.



Name/Student ID: ______________________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

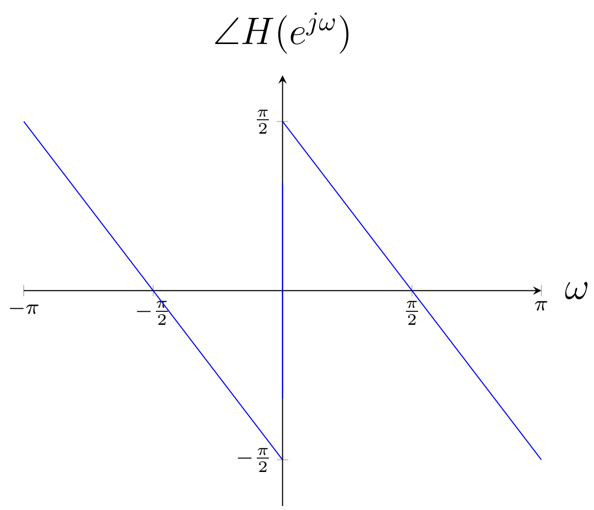
$$y[n] = x[n] - x[n-2].$$

(c) (8 points)

Determine and sketch precisely the phase $\angle H(e^{j\omega})$ of the frequency response in the range $\omega \in [-\pi, \pi]$. (Add or subtract integer multiples of 2π to keep the phase in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.)

From part (a), in the period $\omega \in [-\pi, \pi]$

$$\begin{aligned}
\angle H(e^{j\omega}) &= \angle 2je^{-j\omega}\sin(\omega) \\
&= \angle 2j + \angle e^{-j\omega} + \angle \sin(\omega) \\
&= \frac{\pi}{2} + (-\omega) + \pi \left\lfloor \frac{\omega}{\pi} \right\rfloor \\
&= \frac{\pi}{2} + (-\omega) + \begin{cases} 0, & 0 < \omega < \pi \\ \pi, & -\pi < \omega < 0 \end{cases}
\end{aligned}$$



Name/Student ID: ______________________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(d) (6 points)

Find a non-zero sinusoidal input signal $x[n]$ to the system that produces the output $y[n] = 2x[n]$.

The signal $x[n] = \cos(\frac{\pi}{2}n)$ satisfies

$$x[n] = \begin{cases} 1, & n = 0 \mod 4 \\ 0, & n = 1 \mod 4 \\ -1, & n = 2 \mod 4 \\ 0, & n = 3 \mod 4 \end{cases}$$

It is easy to verify that $x[n] - x[n-2] = 2x[n]$, so the system output when $x[n]$ is the input is $y[n] = 2x[n]$.

Name/Student ID: ______________________________

Problem 2 [DTFT] (30 points; 15 points each part)

(a) Find the DT Fourier transform (DTFT) $X(e^{j\omega})$ of the signal

$$x[n] = \left(\frac{1}{3}\right)^n u[n-2].$$

(b) Find the signal $x[n]$ whose DTFT in the period $[-\pi, \pi]$ is given by

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{3} \leq |\omega| \leq \frac{2\pi}{3} \\ 0, & 0 \leq |\omega| < \frac{\pi}{3} \text{ and } \frac{2\pi}{3} < |\omega| \leq \pi. \end{cases}$$

Express your answer in terms of sinusoidals signals, not exponential signals.

Write your answers to parts (a) and (b) on the following 2 pages.

Name/Student ID: ______________________________

Problem 2 (cont.)

(a) (15 points)

Find the DT Fourier transform (DTFT) $X(e^{j\omega})$ of the signal

$$x[n] = \left(\frac{1}{3}\right)^n u[n-2].$$

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\omega n} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{3}e^{-j\omega}\right)^n \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{3}e^{-j\omega}\right)^{m+2} \quad \text{(setting } m = n-2\text{)} \\
&= \left(\frac{1}{3}e^{-j\omega}\right)^2 \sum_{m=0}^{\infty} \left(\frac{1}{3}e^{-j\omega}\right)^m \\
&= \left(\frac{1}{9}\right) \frac{e^{-j2\omega}}{1-\frac{1}{3}e^{-j\omega}}
\end{aligned}$$

Name/Student ID: ____________________

Problem 2 (cont.)

(b) (15 points)

Find the signal $x[n]$ whose DTFT in the period $[-\pi, \pi]$ is given by

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{3} \le |\omega| \le \frac{2\pi}{3} \\ 0, & 0 \le |\omega| < \frac{\pi}{3} \text{ and } \frac{2\pi}{3} < |\omega| \le \pi. \end{cases}$$

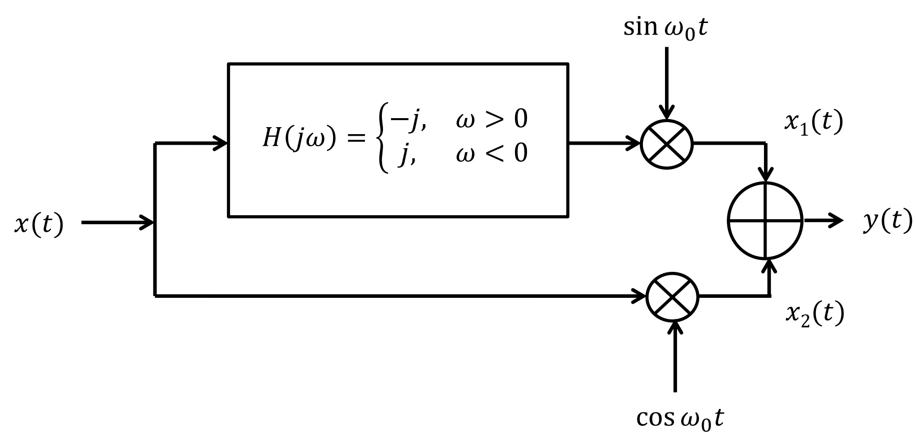
Express your answer in terms of sinusoidals signals, not exponential signals.

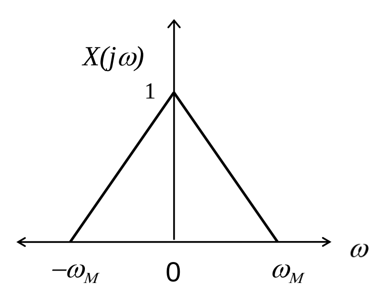
$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} e^{j\omega n}d\omega \\
&= \frac{e^{j\omega n}}{2\pi jn}\Big|_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} + \frac{e^{j\omega n}}{2\pi jn}\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\
&= \frac{e^{-j\frac{\pi}{3}n} - e^{-j\frac{2\pi}{3}n}}{2\pi jn} + \frac{e^{j\frac{2\pi}{3}n} - e^{j\frac{\pi}{3}n}}{2\pi jn} \\
&= \frac{e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n}}{2\pi jn} - \frac{e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}}{2\pi jn} \\
&= \frac{\sin(\frac{2\pi}{3}n)}{\pi n} - \frac{\sin(\frac{\pi}{3}n)}{\pi n}
\end{aligned}$$

Name/Student ID: ______________________________

Problem 3 [Amplitude Modulation] (30 points)

Consider the modulation system shown below. The input signal $x(t)$ has a Fourier transform $X(j\omega)$ that is zero for $|\omega| > \omega_M$, also shown below. Assume that $\omega_0 > \omega_M$





(a) Detemine and sketch precisely the Fourier transform $X_1(j\omega)$ of $x_1(t)$, the Fourier transform $X_2(j\omega)$ of $x_2(t)$, and the Fourier transform $Y(j\omega)$ of $y(t)$.

(b) Is there a demodulation scheme that recovers the signal $x(t)$ from $y(t)$? If so, describe one. If not, why not?

Write your answers to parts (a) and (b) on the following 2 pages.

Name/Student ID: ____________________

Problem 3 (cont.)

(a) (20 points)

Detemine and sketch the Fourier transform $X_1(j\omega)$ of $x_1(t)$, the Fourier transform $X_2(j\omega)$ of $x_2(t)$, and the Fourier transform $Y(j\omega)$ of $y(t)$.

Let $h(t)$ be the impulse response of the filter, and let $z(t) = x(t) * h(t)$. By the convolution property, we have

$$\begin{aligned} Z(j\omega) &= \mathcal{F}\{x(t) * h(t)\} \\ &= X(j\omega)H(j\omega) \\ &\begin{cases} -jX(j\omega), & 0 < \omega < \omega_M \\ jX(j\omega), & -\omega_M < \omega < 0 \\ 0, & |\omega| < \omega_M \end{cases} \end{aligned}$$

The Fourier transform of $x_1(t) = z(t)\sin(\omega_0 t)$ is

$$\begin{aligned} X_1(j\omega) &= \frac{1}{2\pi} Z(j\omega) * -j\pi(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)) \\ &= -\frac{j}{2}\left(Z(j(\omega-\omega_0)) - Z(j(\omega+\omega_0))\right) \\ &= \begin{cases} -\frac{1}{2}(X(j(\omega-\omega_0)), & \omega_0 < \omega < \omega_0 + \omega_M \\ \frac{1}{2}(X(j(\omega-\omega_0)), & \omega_0 - \omega_M < \omega < \omega_0 \\ \frac{1}{2}(X(j(\omega-\omega_0)), & -\omega_0 < \omega < -\omega_0 + \omega_M \\ -\frac{1}{2}(X(j(\omega-\omega_0)), & -\omega_0 - \omega_M < \omega < -\omega_0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The Fourier transform of $x_2(t) = x(t)\cos(\omega_0 t)$ is

$$\begin{aligned} X_2(j\omega) &= \frac{1}{2\pi} X(j\omega) * \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)) \\ &= \frac{1}{2}\left(X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right) \end{aligned}$$

So, the Fourier transform of the output $y(t) = x_1(t) + x_2(t)$ is

$$Y(j\omega) = \begin{cases} X(j(\omega-\omega_0), & \omega_0 - \omega_M < \omega < \omega_0 \\ X(j(\omega+\omega_0), & -\omega_0 < \omega < -\omega_0 + \omega_M \\ 0, & \text{otherwise} \end{cases}$$

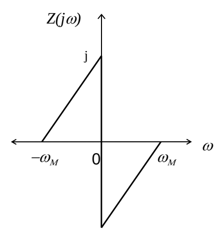
A sketch of these transforms is shown on the next page.

Name/Student ID: ______________________________

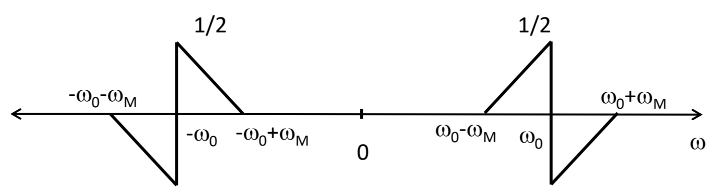
Problem 3 (cont.)

(a) (cont.)

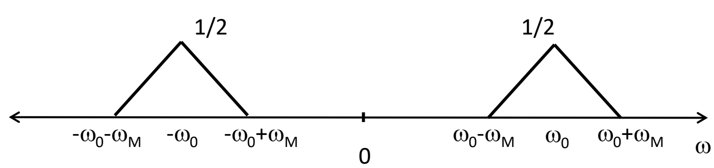
$Z(j\omega)$



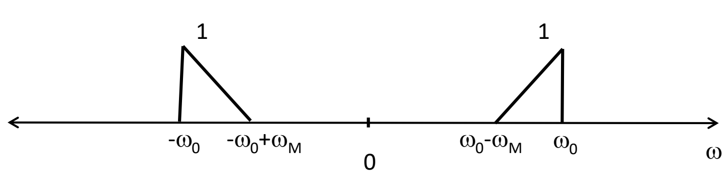
$X_1(j\omega)$



$X_2(j\omega)$



$Y(j\omega)$



Name/Student ID: ______________________________

Problem 3 (cont.)

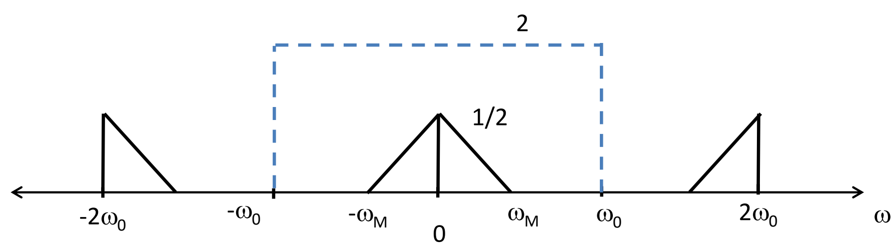
(b) (10 points)

Is there a demodulation scheme that recovers the signal $x(t)$ from $y(t)$? If so, describe one. If not, why not?

Yes, $x(t)$ can be recovered by multiplying $y(t)$ by $\cos(\omega_0 t)$ to produce $y_1(t) = y(t)\cos(\omega_0 t)$ and passing $y_1(t)$ through a low-pass reconstruction filter with gain 2 and cutoff frequency ω_0 and frequency response

$$H_r(\omega) = \begin{cases} 2, & |\omega| < \omega_0 \\ 0, & \text{otherwise} \end{cases}$$

The frequency domain perspective is shown below.



Name/Student ID: ______________________________

Problem 4 [Sampling] (30 points; 10 points each part)

Consider the sinusoidal signal $x(t) = \sin(100\pi t)$. Apply continuous-time impulse train sampling to this signal, with sampling frequency ω_s.

(a) Assume $\omega_s = 80\pi$, producing the sampled signal $x_1(t)$. Determine $X_1(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_1(t)$?

(b) Assume $\omega_s = 120\pi$, producing the sampled signal $x_2(t)$. Determine $X_2(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_2(t)$?

(c) Suppose you applied a reconstruction filter with frequency response

$$H(j\omega) = \begin{cases} \frac{1}{40}, & |\omega| < 80\pi \\ 0, & |\omega| > 80\pi \end{cases}$$

to the signals $x_1(t)$ and $x_2(t)$ of parts (a) and (b). Determine the corresponding outputs $z_1(t)$ and $z_2(t)$.

Write your answers to parts (a), (b), and (c) on the following 3 pages.

Name/Student ID: ______________________________

Problem 4 (cont.)

Consider the sinusoidal signal $x(t) = \sin(100\pi t)$. Apply continuous-time impulse train sampling to this signal, with sampling frequency ω_s.

(a) (10 points)

Assume $\omega_s = 80\pi$, producing the sampled signal $x_1(t)$. Determine $X_1(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_1(t)$?

The sampling period is $T = \frac{2\pi}{\omega_s} = \frac{2\pi}{80\pi} = \frac{1}{40}$.

$$X_1(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$
$$= 40\sum_{k=-\infty}^{\infty} X(j(\omega - k80\pi))$$

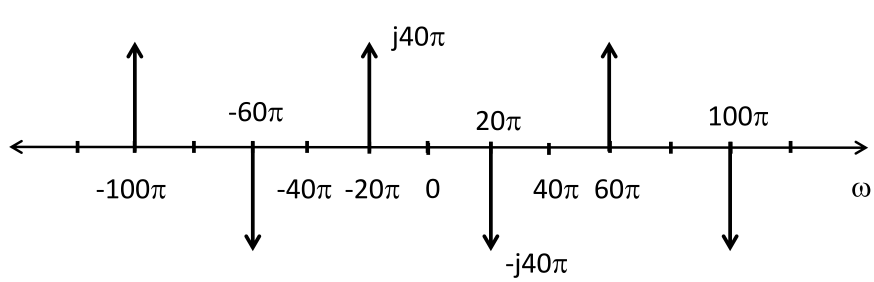
Here

$$X(j\omega) = -j\pi(\delta(\omega - 100\pi) - \delta(\omega + 100\pi))$$
$$= -j\pi\delta(\omega - 100\pi) + j\pi\delta(\omega + 100\pi)$$

So,

$$X_1(j\omega) = \sum_{k=-\infty}^{\infty} -j40\pi\delta(\omega - 100\pi - k80\pi) + j40\pi\delta(\omega + 100\pi - k80\pi)$$

$X_1(j\ \omega)$ (sampling frequency ω_s=80π)



The output $y(t) = \sin(20\pi t)$ can be generated by passing $x_1(t)$ through a low-pass reconstruction filter with gain $\frac{1}{40}$ and cut-off frequency $\frac{\omega_s}{2} = 40\pi$

Name/Student ID: ____________________________

Problem 4 (cont.)

Consider the sinusoidal signal $x(t) = \sin(100\pi t)$. Apply continuous-time impulse train sampling to this signal, with sampling frequency ω_s.

(b) (10 points)

Assume $\omega_s = 120\pi$, producing the sampled signal $x_2(t)$. Determine $X_2(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_2(t)$?

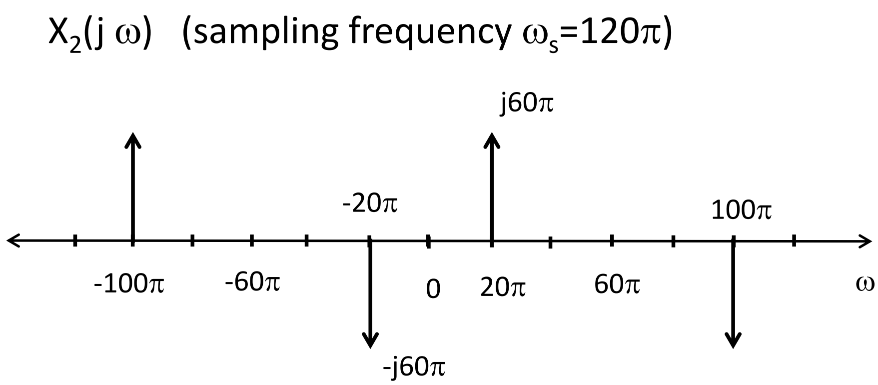
The sampling period is $T = \frac{2\pi}{\omega_s} = \frac{2\pi}{120\pi} = \frac{1}{60}$.

$$\begin{aligned} X_2(j\omega) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \\ &= 60 \sum_{k=-\infty}^{\infty} X(j(\omega - k120\pi)) \end{aligned}$$

Here
$$\begin{aligned} X(j\omega) &= -j\pi(\delta(\omega - 100\pi) - \delta(\omega + 100\pi)) \\ &= -j\pi\delta(\omega - 100\pi) + j\pi\delta(\omega + 100\pi) \end{aligned}$$

So,

$$X_2(j\omega) = \sum_{k=-\infty}^{\infty} -j60\pi\delta(\omega - 100\pi - k120\pi) + j60\pi\delta(\omega + 100\pi - k120\pi)$$



The output $y(t) = \sin(20\pi t)$ can be generated by passing $x_2(t)$ through a low-pass reconstruction filter with gain $-\frac{1}{60}$ and cut-off frequency $\frac{\omega_s}{2} = 60\pi$

Name/Student ID: ______________________________

Problem 4 (cont.)

(c) (10 points)

Suppose you applied a reconstruction filter with frequency response

$$H(j\omega) = \begin{cases} \frac{1}{40}, & |\omega| < 80\pi \\ 0, & |\omega| > 80\pi \end{cases}$$

to the signals $x_1(t)$ and $x_2(t)$ of parts (a) and (b). Determine the corresponding outputs $z_1(t)$ and $z_2(t)$.

The output $z_1(t) = \sin(20\pi t) - \sin(60\pi t)$.

The output $z_2(t) = -\frac{3}{2}\sin(20\pi t)$.

Name/Student ID: ______________________________

Problem 5 [Laplace Transform and ROC] (30 points)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(a) Sketch precisely the pole-zero plot associated with $H(s)$. Indicate the order of all poles and zeros.

(b) Identify all possible regions of convergence (ROC) associated with $H(s)$ and indicate for each ROC whether the corresponding system is causal and/or stable.

(c) For each possible ROC in part (b), indicate whether the corresponing impulse response $h(t)$ is left-sided, right-sided, two-sided, or finite-duration, and whether or not it has a Fourier transform $H(j\omega)$.

(d) Determine a linear constant-coefficient differential equation that characterizes the LTI systems with system function $H(s)$.

Write your answers to parts (a), (b), (c), and (d) on the following 4 pages.

Name/Student ID: ______________________________

Problem 5 (cont.)

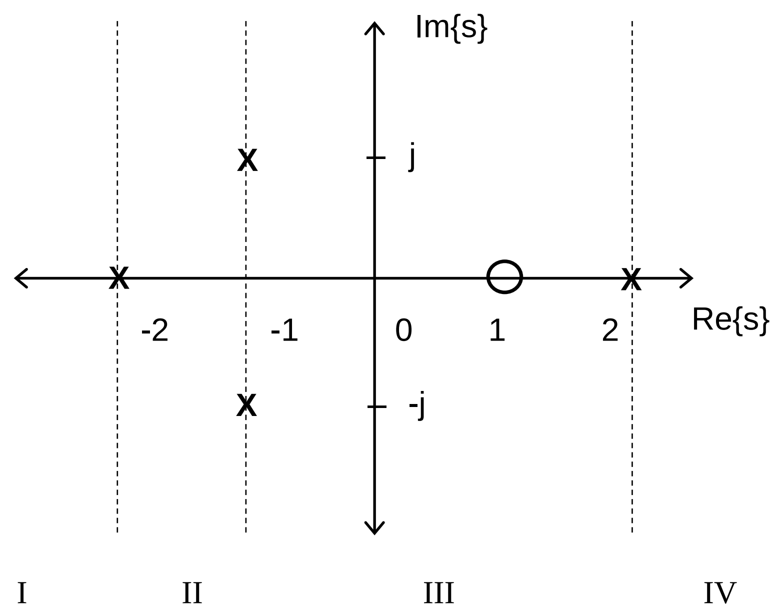
Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(a) (8 points)

Sketch precisely the pole-zero plot associated with $H(s)$. Indicate the order of all poles and zeros.

$s^2+2s+2 = (s-(-1+j))(s-(-1-j))$ and $s^2-4 = (s-2)(s+2)$.



The poles at $s = 2, -2, -1+j, -1-j$ all have order 1.

The zero at $s = 1$ has order 1.

Name/Student ID: ______________________________

Problem 5 (cont.)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(b) (8 points)

Identify all possible regions of convergence (ROC) associated with $H(s)$ and indicate for each ROC whether the corresponding system is causal and/or stable.

The ROCs and corresponding system properties are:

I	$\mathcal{R}e(s) < -2$	not causal	not stable
II	$-2 < \mathcal{R}e(s) < -1$	not causal	not stable
III	$-1 < \mathcal{R}e(s) < 2$	not causal	stable
IV	$2 < \mathcal{R}e(s)$	causal	not stable

Name/Student ID: ______________________________

Problem 5 (cont.)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(c) (8 points)

For each possible ROC in part (b), indicate whether the corresponing impulse response $h(t)$ is left-sided, right-sided, two-sided, or finite-duration, and whether or not it has a Fourier transform $H(j\omega)$.

The ROCs and corresponding impulse response properties are:

I	$\mathcal{R}e(s) < -2$	left-sided	no Fourier transform
II	$-2 < \mathcal{R}e(s) < -1$	two-sided	no Fourier transform
III	$-1 < \mathcal{R}e(s) < 2$	two-sided	yes Fourier transform
IV	$2 < \mathcal{R}e(s)$	right-sided	no Fourier transform

Name/Student ID: ____________________

Problem 5 (cont.)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(d) (6 points)

Determine a linear constant-coefficient differential equation that characterizes the LTI systems with system function $H(s)$.

The differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

has system function

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.$$

Here, $H(s)$ can be written as

$$H(s) = \frac{s-1}{s^4+2s^3-2s^2-8s-8}$$

so the differential equation corresponding to the given $H(s)$ is

$$\frac{d^4y(t)}{dt^4} + 2\frac{d^3y(t)}{dt^3} - 2\frac{d^2y(t)}{dt^2} - 8\frac{dy(t)}{dt} - 8y(t) = \frac{dx(t)}{dt} - x(t).$$

Name/Student ID: ______________________________

Problem 6 [Laplace Transform and LTI Systems] (30 points)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(a) Determine and sketch precisely the magnitude $|H(j\omega)|$ of the frequency response $H(j\omega)$. How would you describe the filter characteristics of the system - lowpass, highpass, bandpass, allpass?

(b) Determine (but do not sketch) the phase $\angle H(j\omega)$ of the frequency response $H(j\omega)$.

(c) Determine the impulse response $h(t)$ of the system.

Hint: Write $(s-1)^2$ in the form $(s+1)^2 + f(s)$.

(d) Suppose the input to the system is $x(t) = te^t u(t)$. Determine the corresponding output $y(t)$.

Write your answers to parts (a), (b), (c), and (d) on the following 4 pages.

Name/Student ID: ______________________________

Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

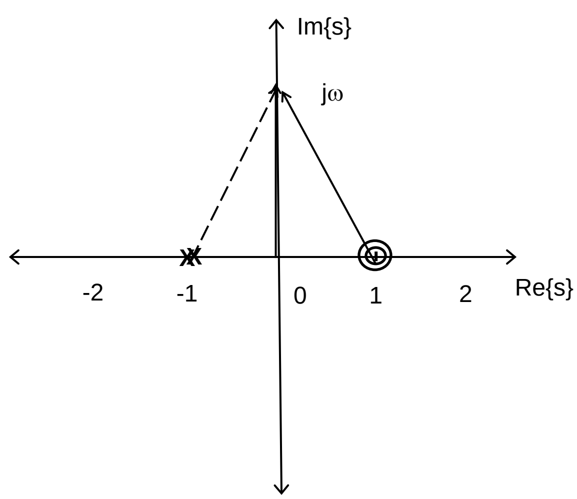
(a) (8 points) Determine and sketch precisely the magnitude $|H(j\omega)|$ of the frequency response $H(j\omega)$. How would you describe the filter characteristics of the system - lowpass, highpass, bandpass, allpass?

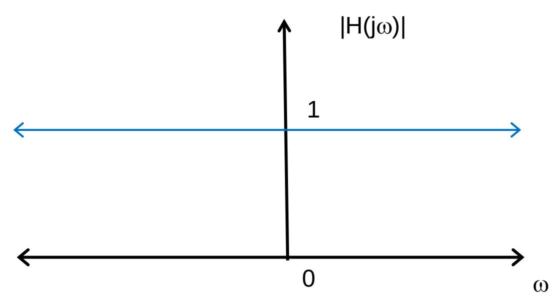
The system has a second-order zero at $s = 1$ and a second-order pole at $s = -1$.

$$|H(j\omega)| = \frac{|j\omega - 1|^2}{|j\omega + 1|^2}.$$

Since $|j\omega - 1| = |j\omega + 1| = \sqrt{1 + \omega^2}$, we have

$$|H(j\omega)| = 1, \ \forall \omega.$$





The system has the characteristics of an **allpass** filter.

Name/Student ID: ______________________________

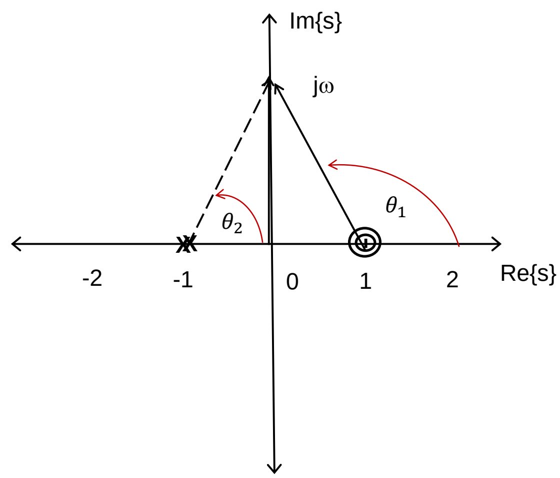
Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(b) (8 points)

Determine (but do not sketch) the phase $\angle H(j\omega)$ of the frequency response $H(j\omega)$. How would you describe the filter characteristics of the system - lowpass, highpass, bandpass, allpass?



$$\begin{aligned}\angle H(j\omega) &= 2\angle(j\omega - 1) - 2\angle(j\omega + 1)\\ &= 2\theta_1 - 2\theta_2\\ &= 2(\pi - \theta_2) - 2\theta_2\\ &= 2\pi - 4\theta_2\end{aligned}$$

Now, $\theta_2 = \tan^{-1}(\omega/1) = \tan^{-1}(\omega)$, so

$$\angle H(j\omega) = 2\pi - 4\tan^{-1}(\omega).$$

The system has the characteristics of an **allpass** filter.

Name/Student ID: ______________________________

Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(c) (8 points)

Determine the impulse response $h(t)$ of the system.

Hint: Write $(s-1)^2$ in the form $(s+1)^2 + f(s)$.

Following the hint, we note that $(s-1)^2 = (s+1)^2 - 4s$, so

$$H(s) = \frac{(s+1)^2 - 4s}{(s+1)^2} = 1 - \frac{4s}{(s+1)^2}.$$

Rewrite the (negative of the) second term using partial fraction expansion:

$$\frac{4s}{(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1}$$

Multiplying both sides by $(s+1)^2$ and evaluating at $s = -1$ yields

$$4s\Big|_{s=-1} = A = -4$$

Using this value of A and evaluating both sides at $s = 0$ yields $B = 4$. So,

$$H(s) = 1 + \frac{4}{(s+1)^2} - \frac{4}{s+1}.$$

Using the table of Laplace transform pairs, and taking into account the ROC, we get

$$h(t) = \delta(t) + 4te^{-t}u(t) - 4e^{-t}u(t).$$

Name/Student ID: ____

Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(d) (6 points)

Suppose the input to the system is $x(t) = te^t u(t)$. Determine the correponding output $y(t)$.

From the table of Laplace transform pairs

$$x(t) = te^t u(t) \leftrightarrow X(s) = \frac{1}{(s-1)^2}, \ \mathcal{R}e(s) > 1.$$

Now,

$$Y(s) = X(s)H(s) = \frac{1}{(s+1)^2}$$

with

$$ROC_Y \supseteq ROC_X \cap ROC_H = \{s \mid \mathcal{R}e(s) > 1\}.$$

Thus ROC_Y enlarges to $\mathcal{R}e(s) > -1$ and, referring to the table again, we find

$$y(t) = te^{-t} u(t).$$