ECE 45 – Circuits and Systems
Winter 2025

Homework #4

Due: January 30 at 11:59pm, submitted via GradeScope.

You can make multiple upload attempts to experiment with the system and the best way to upload. You must correctly mark the answers to the problems in GradeScope, e.g. problem 1, problem 2, problem 3, to get full credit. Note that you must tag your problems when uploading to GradeScope or they will not be graded and you will not receive credit. Any regrade requests must be placed through GradeScope within one week of the return of the homework.

Remember, discussion of homework questions is encouraged. Please be absolutely sure to submit your own independent homework solution.

1. (50 %) Perform convolutions of the following functions, specifying the solution analytically and also sketching them by hand. Be aware that you may be able to compute some answers from other answers you already computed through the use of convolution properties. Make sure that each convolution clearly indicates the values for all values of t.

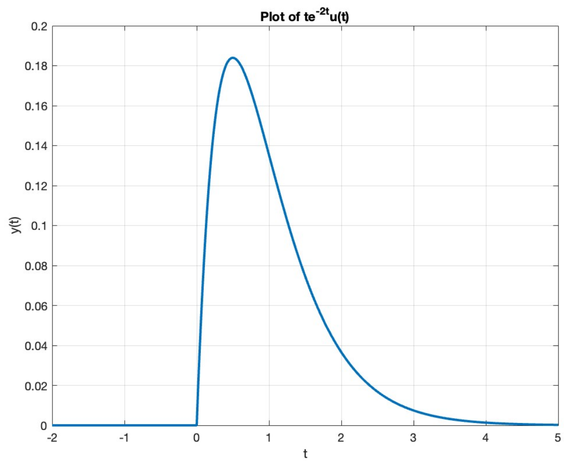
   (a) $x(t) = e^{-2t}u(t)$ and $h(t) = e^{-2t}u(t)$

   Solution:

$$\begin{aligned} x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)\, d\tau \end{aligned}$$

   Since $u(\tau) = 0$ for $\tau < 0$, the limits reduce to:

$$\begin{aligned} x(t) * h(t) &= u(t)\int_{0}^{t} e^{-2\tau}e^{-2(t-\tau)}\, d\tau \\ &= u(t)\int_{0}^{t} e^{-2t}\, d\tau \\ &= te^{-2t}u(t) \end{aligned}$$



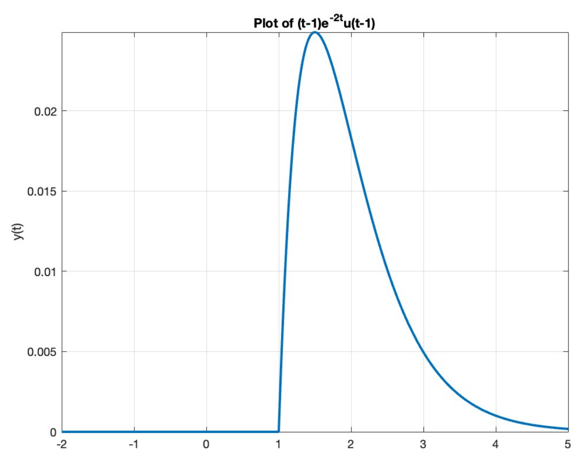
(b) $x(t) = e^{-2t}u(t)$ and $h(t) = e^{-2t}u(t-1)$

Solution:

$$\begin{aligned} x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau-1)\, d\tau \end{aligned}$$

Since $u(\tau) = 0$ for $\tau < 0$ and $u(t-\tau-1) = 0$ for $t-\tau-1 < 0$, it follows that the limits of integration are from 0 to $t-1$:

$$\begin{aligned} x(t) * h(t) &= u(t-1)\int_{0}^{t-1} e^{-2\tau}e^{-2(t-\tau)}\, d\tau \\ &= u(t-1)\int_{0}^{t-1} e^{-2t}\, d\tau \\ &= (t-1)e^{-2t}u(t-1). \end{aligned}$$



(c) $x(t) = \text{rect}(t - 1/2)$ and $h(t) = \text{rect}(2t)$

Solution:

$$\begin{aligned} x(t) * h(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \text{rect}(2\tau)\text{rect}(t-\tau-1/2)\,d\tau \end{aligned}$$

Since

$$\text{rect}(2\tau) = 0 \quad \text{for } \tau \in (-\infty, -1/4] \cup [1/4, \infty]$$

and

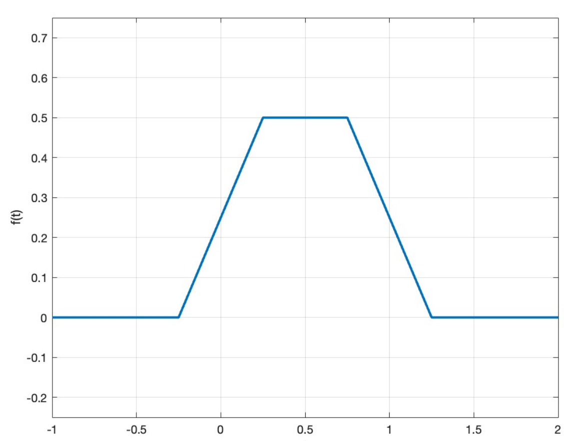
$$\text{rect}(t-\tau-1/2) = 0 \quad \text{for } |t-\tau-1/2| > 0.5,$$

the integral evaluates as follows:

$$x(t) * h(t) = \begin{cases} \int_{-1/4}^{t} 1 \times 1\,d\tau, & -1/4 < t < 1/4 \\ \int_{-1/4}^{1/4} 1 \times 1\,d\tau, & 1/4 \leqslant t < 3/4 \\ \int_{t-1}^{1/4} 1 \times 1\,d\tau, & 3/4 \leqslant t < 5/4 \\ 0, & \text{otherwise} \end{cases}$$

Evaluating the integrals:

$$x(t) * h(t) = \begin{cases} t + 1/4, & -1/4 < t < 1/4 \\ 1/2, & 1/4 \leqslant t < 3/4 \\ 5/4 - t, & 3/4 \leqslant t < 5/4 \\ 0, & \text{otherwise} \end{cases}$$



(d) $a(t) = e^{j\omega t}$ and $b(t) = e^{-\beta t}u(t)$ (be careful about possible special cases for different values of β and ω) [only analytically, no plot]

Solution:

$$\begin{aligned}
a(t) * b(t) &= \int_{-\infty}^{\infty} b(\tau)a(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} e^{j\omega\tau}e^{-\beta(t-\tau)}u(t-\tau)\,d\tau \\
&= \int_{-\infty}^{t} e^{j\omega\tau-\beta t+\beta\tau}\,d\tau \\
&= e^{-\beta t}\int_{-\infty}^{t} e^{(j\omega+\beta)\tau}\,d\tau \\
&= e^{-\beta t}\left[\frac{e^{(j\omega+\beta)\tau}}{j\omega+\beta}\right]_{\tau=-\infty}^{\tau=t} \\
&= e^{-\beta t}\left[\frac{e^{(j\omega+\beta)t}}{j\omega+\beta} - 0\right] \\
&= \frac{e^{j\omega t}}{j\omega+\beta}
\end{aligned}$$

For the exponential term to decay to 0 as $\tau \to -\infty$, the real part of $j\omega + \beta$ must be positive. This means that $\mathrm{Re}(\beta) > 0$ for convergence.

(e) $x(t) = e^{-2t}u(t)$ and $\mathrm{rect}(t)$

Solution:

$$\begin{aligned}
x(t) * h(t) &= \int_{-\infty}^{\infty} \mathrm{rect}(\tau)x(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} u(\tau+1/2)e^{-2(t-\tau)}u(t-\tau)\,d\tau - \int_{-\infty}^{\infty} u(\tau-1/2)e^{-2(t-\tau)}u(t-\tau)\,d\tau
\end{aligned}$$

Now, evaluating the two integrals:

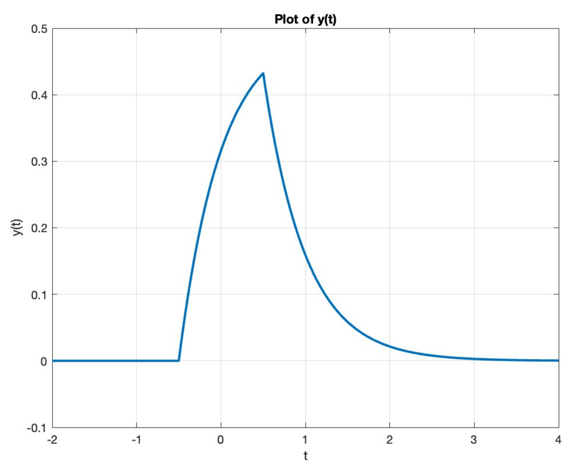
$$\begin{aligned}
\int_{-\infty}^{\infty} u(\tau+1/2)e^{-2(t-\tau)}u(t-\tau)\,d\tau &= u(t+1/2)\int_{-1/2}^{t} e^{-2(t-\tau)}\,d\tau \\
&= u(t+1/2)e^{-2t}\int_{-1/2}^{t} e^{2\tau}\,d\tau \\
&= u(t+1/2)e^{-2t}\frac{e^{2\tau}}{2}\Big|_{-1/2}^{t} \\
&= \frac{1}{2}e^{-2t}\left(e^{2t} - e^{-1}\right)u(t+1/2)
\end{aligned}$$

Likewise:

$$
\begin{aligned}
\int_{-\infty}^{\infty} u(\tau-1/2)e^{-2(t-\tau)}u(t-\tau)\,d\tau &= u(t-1/2)\int_{1/2}^{t} e^{-2(t-\tau)}\,d\tau \\
&= u(t-1/2)e^{-2t}\int_{1/2}^{t} e^{2\tau}\,d\tau \\
&= u(t-1/2)e^{-2t}\frac{e^{2\tau}}{2}\bigg|_{1/2}^{t} \\
&= \frac{1}{2}e^{-2t}\left(e^{2t}-e^{1}\right)u(t-1/2)
\end{aligned}
$$

Therefore, the convolution is:

$$
y(t) = \frac{1}{2}e^{-2t}\left(e^{2t}-e^{-1}\right)u(t+1/2) - \frac{1}{2}e^{-2t}\left(e^{2t}-e^{1}\right)u(t-1/2)
$$



(f) $x(t) = e^{-2t}u(t)$ and $h(t) = e^{-2t}u(t) + \mathrm{rect}(t)$

Solution:

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= e^{-2t}u(t) * \left(e^{-2t}u(t) + \mathrm{rect}(t)\right) \\
&= e^{-2t}u(t) * e^{-2t}u(t) + e^{-2t}u(t) * \mathrm{rect}(t)
\end{aligned}
$$

From problem 1(a),

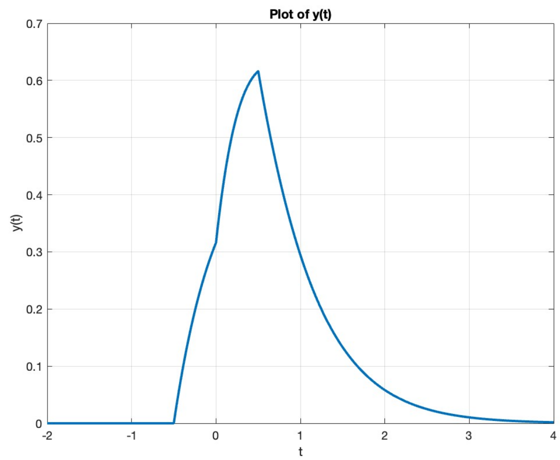
$$
e^{-2t}u(t) * e^{-2t}u(t) = te^{-2t}u(t)
$$

and from problem 1(e),

$$
e^{-2t}u(t) * \mathrm{rect}(t) = \frac{1}{2}e^{-2t}\left(e^{2t}-e^{-1}\right)u(t+1/2) - \frac{1}{2}e^{-2t}\left(e^{2t}-e^{1}\right)u(t-1/2)
$$

Thus, using the linearity property of the convolution, we get:

$$
\begin{aligned}
y(t) = te^{-2t}u(t) + \frac{1}{2}e^{-2t}\left(e^{2t} - e^{-1}\right)u(t+1/2) \\
- \frac{1}{2}e^{-2t}\left(e^{2t} - e^{1}\right)u(t-1/2)
\end{aligned}
$$



(g) $x(t) = \frac{1}{t^2+1}u(-t)$, $h(t) = u(t-1)$

Solution:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(\tau - t)\, d\tau \\
&= \int_{-\infty}^{\infty} \frac{1}{\tau^2+1} u(-\tau)u(t-\tau-1)\, d\tau
\end{aligned}
$$

For $t < 1$:

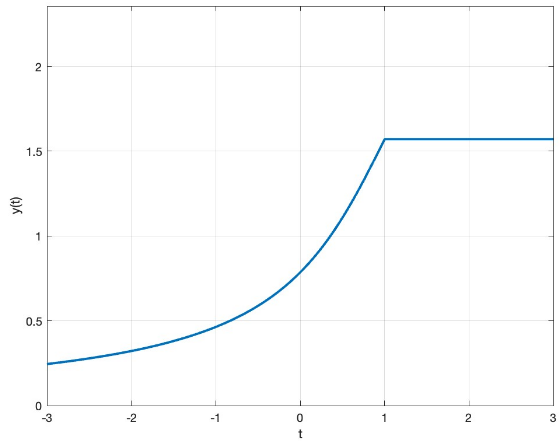
$$
\begin{aligned}
&= \int_{-\infty}^{t-1} \frac{1}{\tau^2+1}\, d\tau \\
&= \tan^{-1}(\tau)\Big|_{-\infty}^{t-1} \\
&= \tan^{-1}(t-1) + \frac{\pi}{2}
\end{aligned}
$$

For $t \geqslant 1$:

$$
\begin{aligned}
&= \int_{-\infty}^{0} \frac{1}{\tau^2+1}\, d\tau \\
&= \tan^{-1}(\tau)\Big|_{-\infty}^{0} \\
&= \frac{\pi}{2}
\end{aligned}
$$

Thus, we have:

$$
y(t) = \begin{cases} \tan^{-1}(t-1) + \frac{\pi}{2}, & \text{for } t < 1 \\ \frac{\pi}{2}, & \text{for } t \geqslant 1 \end{cases}
$$



2. (20 %) Consider the functions $x(t) = \mathrm{rect}(t-1/2)$. With $\ast$ denoting the shorthand for convolution, compute the following convolutions and sketch them by hand.

   (a) $x(t) \ast x(t)$

   Solution:

   From $x(t) = \mathrm{rect}(t-1/2)$, we get $x(t) = \begin{cases} 1, & \text{if } 0 \leqslant t \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$. Let

$$\begin{aligned} y(t) &= x(t) \ast x(t) \\ &= \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau \end{aligned}$$

   For $0 \leqslant t \leqslant 1$:

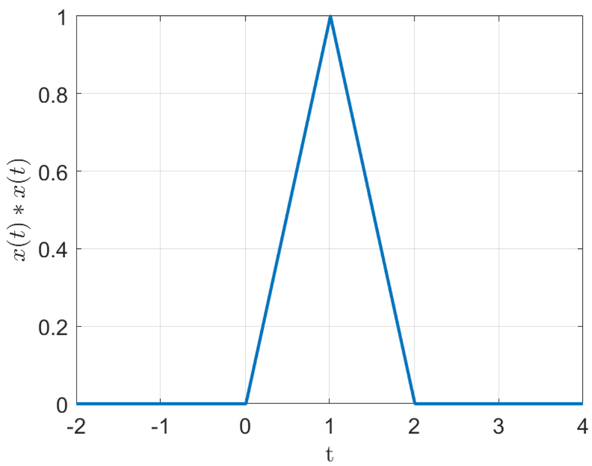
$$\begin{aligned} y(t) &= \int_0^t 1 d\tau \\ &= \tau\Big|_0^t \\ &= t \end{aligned}$$

   For $1 < t \leqslant 2$:

$$\begin{aligned} y(t) &= \int_{t-1}^1 1 d\tau \\ &= \tau\Big|_{t-1}^1 \\ &= 2-t \end{aligned}$$

   Therefore,

$$y(t) = \begin{cases} t, & \text{if } 0 \leqslant t \leqslant 1 \\ 2-t, & \text{if } 1 < t \leqslant 2 \\ 0, & \text{otherwise} \end{cases}.$$



(b) $x(t) * x(-t)$. This is also known as the autocorrelation of a signal. It is used in communications. Also explain why it might be interesting.

Solution:

$$\begin{aligned} y(t) &= x(t) * x(-t) \\ &= \int_{-\infty}^{\infty} x(\tau)x(\tau - t)\, d\tau \end{aligned}$$

For $-1 \leqslant t \leqslant 0$:

$$\begin{aligned} y(t) &= \int_{0}^{t+1} 1 d\tau \\ &= \tau\big|_{0}^{t+1} \\ &= t + 1 \end{aligned}$$

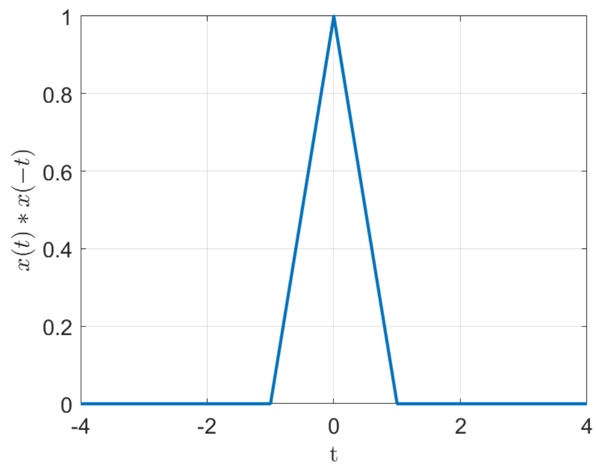
For $0 < t \leqslant 1$:

$$\begin{aligned} y(t) &= \int_{1}^{t} 1 d\tau \\ &= \tau\big|_{1}^{t} \\ &= 1 - t \end{aligned}$$

Therefore,

$$y(t) = \begin{cases} 1 + t, & \text{if } -1 \leqslant t \leqslant 0 \\ 1 - t, & \text{if } 0 < t \leqslant 1 \\ 0, & \text{otherwise} \end{cases} .$$

The autocorrelation measures the similarity between a signal and its time-shifted version. It is widely used in applications like radar where the receiver needs to find a pulse that was transmitted. Looking at the plot, you can see that the peak occurs where the two pulse line up exactly.



(c) $x(t) \ast x(t-1)$

Solution: Write $x(t-1)$ as

$$x(t-1) = x(t) \ast \delta(t-1)$$

We have

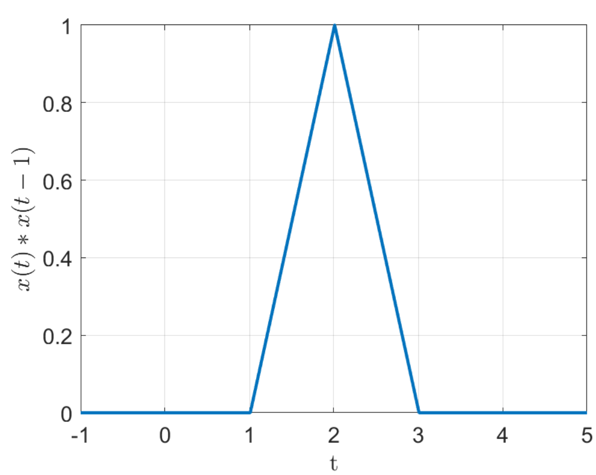
$$\begin{aligned} y(t) &= x(t) \ast x(t-1) \\ &= (x(t) \ast x(t)) \ast \delta(t-1) \end{aligned}$$

Let $r(t) = x(t) \ast x(t)$ as acquired in 2(a),

$$y(t) = r(t-1)$$

Namely,

$$y(t) = \begin{cases} t-1, & \text{if } 1 \leqslant t \leqslant 2 \\ 3-t, & \text{if } 2 < t \leqslant 3. \\ 0, & \text{otherwise} \end{cases}$$



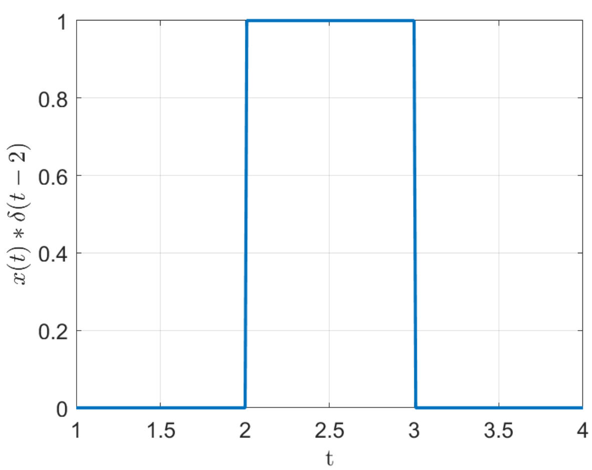
(d) $x(t) * \delta(t-2)$

Solution: Convolution with $\delta(t-2)$ shifts the signal 2 units to the right:

$$\begin{aligned} y(t) &= x(t) * \delta(t-2) \\ &= x(t-2) \end{aligned}$$

Therefore,

$$y(t) = \begin{cases} 1, & \text{if } 2 \leqslant t \leqslant 3 \\ 0, & \text{otherwise} \end{cases}.$$



(e) $x(t) * \delta(t-2) * x(t+1)$

Solution:

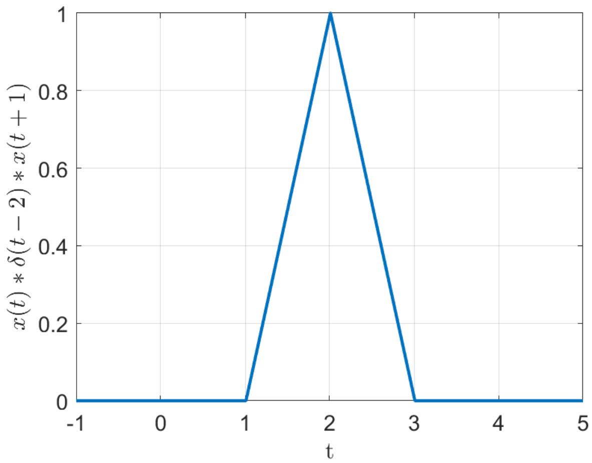
$$\begin{aligned} y(t) &= x(t) * \delta(t-2) * x(t+1) \\ &= x(t) * x(t-2+1) \\ &= x(t) * x(t) * \delta(t-1) \end{aligned}$$

Let $r(t) = x(t) * x(t)$ as acquired in 2(a),

$$y(t) = r(t-1)$$

Namely,

$$y(t) = \begin{cases} t-1, & \text{if } 1 \leqslant t \leqslant 2 \\ 3-t, & \text{if } 2 < t \leqslant 3. \\ 0, & \text{otherwise} \end{cases}$$



(f) $(x(t) + x(t-2)) * x(t)$

Solution:

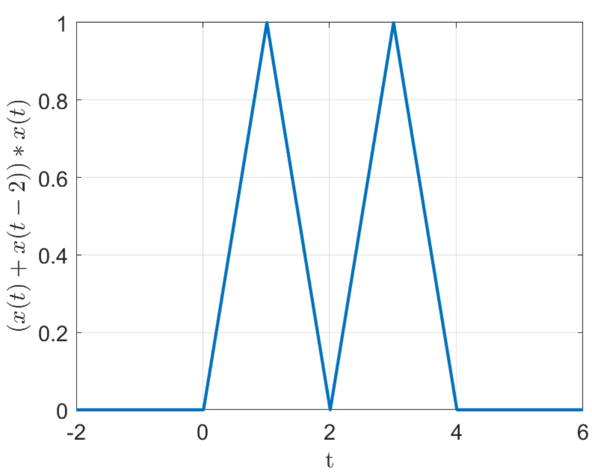
$$\begin{aligned} y(t) &= (x(t) + x(t-2)) * x(t) \\ &= x(t) * x(t) + x(t-2) * x(t) \\ &= x(t) * x(t) + \delta(t-2) * (x(t) * x(t)) \end{aligned}$$

Let $r(t) = x(t) * x(t)$ as acquired in 2(a),

$$y(t) = r(t) + r(t-2)$$

Therefore,

$$y(t) = \begin{cases} t, & \text{if } 0 \leqslant t \leqslant 1 \\ 2-t, & \text{if } 1 < t \leqslant 2 \\ t-2, & \text{if } 2 < t \leqslant 3 \\ 4-t, & \text{if } 3 < t \leqslant 4 \\ 0, & \text{otherwise} \end{cases}$$



3. (10 %) Associativity property of convolution. Prove the equality

$$[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)].$$

by showing that both sides of the above equation equal

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)d\tau d\sigma$$

Solution:

$$\begin{aligned}
[x(t) * h(t)] * g(t) &= \left[\int_{-\infty}^{\infty} x(\tau_1)h(t-\tau_1)\mathrm{d}\tau_1\right] * g(t) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau_1)h(t-\tau_2-\tau_1)g(\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\
x(t) * [h(t) * g(t)] &= x(t) * \left[\int_{-\infty}^{\infty} h(\tau_1)g(t-\tau_1)\mathrm{d}\tau_1\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau_2)h(\tau_1)g(t-\tau_2-\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau_1)h(\tau_2)g(t-\tau_2-\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\
&\overset{(*)}{=} -\int_{\infty}^{-\infty}\int_{-\infty}^{\infty} x(\tau_1)h(t-a-\tau_1)g(a)\mathrm{d}\tau_1\mathrm{d}a = [x(t) * h(t)] * g(t),
\end{aligned}$$

where $(*)$ follows from employing a change of variables $a = t - \tau_2 - \tau_1$, hence $\mathrm{d}a = -\mathrm{d}\tau_2$.

4. (20 %) Based on the solutions, correct your previous week's homework using a colored pen (or annotation) so it's obvious what you've corrected. If you got a problem exactly right, just use a red check mark to indicate as such.