Discussion week 3:

Convolution: $$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

For discrete-time signals:

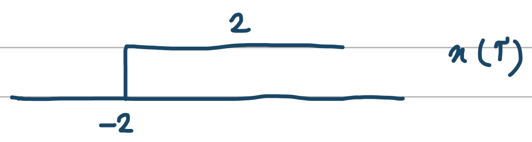
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

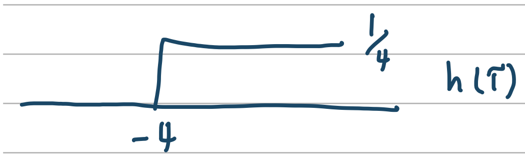
1. Flip $h(\tau)$ to get $h(-\tau)$
2. Shift $h(-\tau)$ by t to get $h(t-\tau)$
3. Multiply $x(\tau)$ and $h(t-\tau)$
4. Integrate over τ for all time.

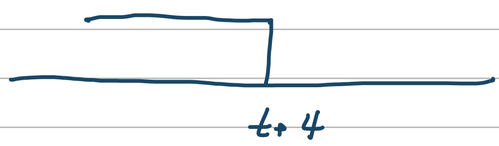
Example #1: $x(t) = 2\,u(t+2)$, $h(t) = \frac{1}{4}\int_{-\infty}^{t+5} \delta(\lambda-1)$, what is $y(t)$?

$$h(t) = \frac{1}{4}\int_{-\infty}^{t+5} \delta(\lambda-1)\, d\lambda = \frac{1}{4}\left[u[t+5-1] - u(-\infty)\right] = \frac{1}{4}u(t+4)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$



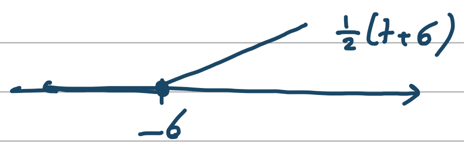




Case 1: $-\infty < t < -6$: no overlap, so $y(t) = 0$

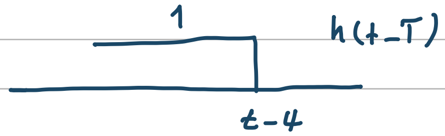
Case 2: $t > -6$: $y(t) = \int_{-2}^{t+4} 2 \times \frac{1}{4}\, d\tau = \frac{1}{2}\left[t+4-(-2)\right] = \frac{1}{2}(t+6)$

$$y(t) = \begin{cases} 0 & t < -6 \\ \frac{1}{2}(t+6) & t > -6 \end{cases}$$

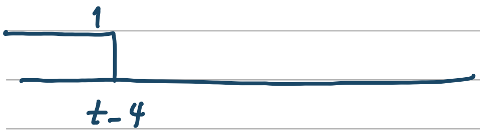


example # 2 : $x(t) = 4\left[u(t-2) - u(t-7)\right]$, $h(t) = u(t-4)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

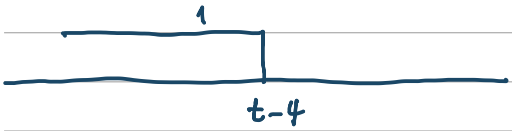


case 1:



$\Rightarrow$ $-\infty < t < 6$: No overlap, $y(t) = 0$

Case 2:



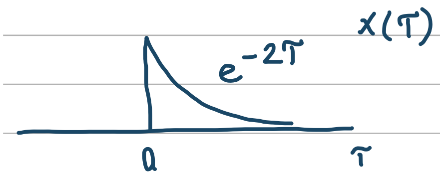
$\Rightarrow$ $6 < t < 11$: $y(t) = \int_{2}^{t-4} 4\, d\tau = 4(t-6)$

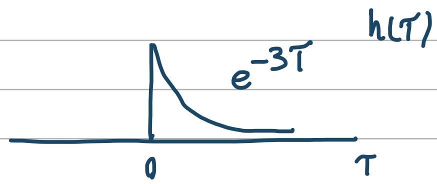
Case 3:

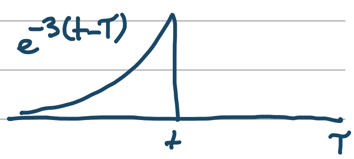
$\Rightarrow$ $11 < t < \infty$: $y(t) = \int_{2}^{7} 4\, d\tau = 4 \times 5 = 20$

$$y(t) = \begin{cases} 0 & t < 6 \\ 4(t-6) & 6 < t < 11 \\ 20 & t > 11 \end{cases}$$

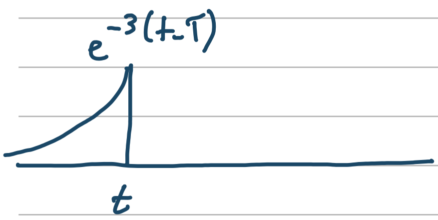
Example #3: $x(t) = e^{-2t}\, u(t)$, $h(t) = e^{-3t}\, u(t)$





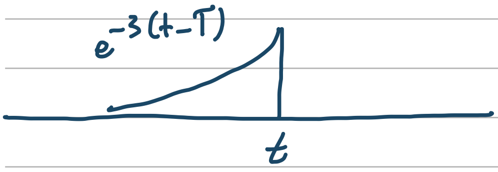


case 1:



$t < 0$: No overlap : $y(t) = 0$

Case 2:



$t>0$, $y(t) = \int_0^t e^{-2\tau} e^{-3(t-\tau)} \, d\tau$

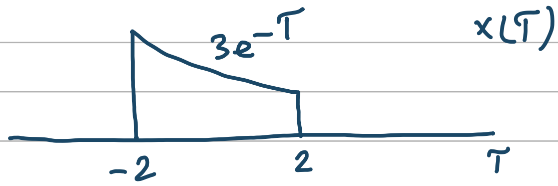
$= \int_0^t e^{-3t} e^{\tau} \, d\tau = e^{-3t}(e^t - 1) = e^{-2t} - e^{-3t}$

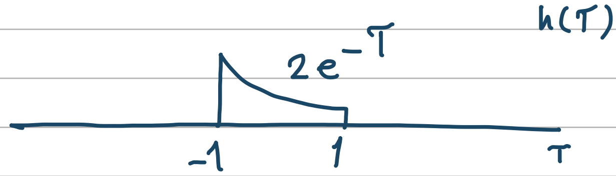
$$y(t) = \begin{cases} 0 & t<0 \\ e^{-2t} - e^{-3t} & t>0 \end{cases} \Rightarrow y(t) = \left[e^{-2t} - e^{-3t}\right] u(t)$$

Example #4: $x(t) = 3e^{-t} \, \text{rect}\left(\frac{t}{4}\right)$ $\quad h(t) = \int_{t-1}^{t+1} e^{-(t-\lambda)} \delta(\lambda) \, d\lambda$

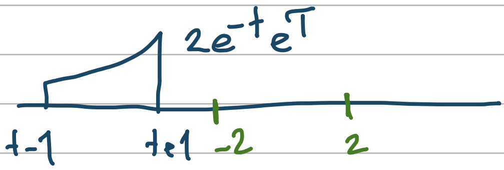
$h(t) = \int_{t-1}^{t+1} e^{-(t-\lambda)} \delta(\lambda) \, d\lambda = \int_{t-1}^{t+1} e^{-t} \delta(\lambda) \, d\lambda = e^{-t}\left[u(t+1) - u(t-1)\right]$

$y(t) = \int_{-\infty}^{\infty} x(\tau) \, h(t-\tau) \, d\tau$





Case #1:

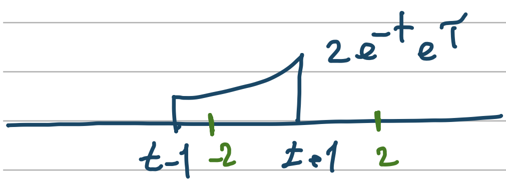


$-\infty < t < -3$:

No overlap: $y(t) = 0$

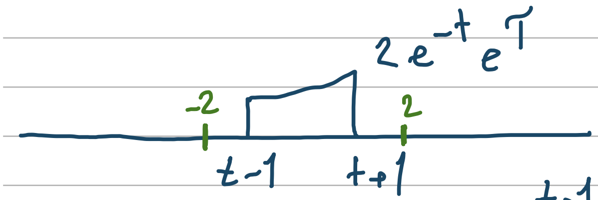
Case #2:

$-3 < t < -1$: $y(t) = \int_{-2}^{t+1} 3e^{-\tau} \, 2e^{-t} e^{\tau} \, d\tau$



$= \int_{-2}^{t+1} 6e^{-t} \, d\tau = 6e^{-t}(t+3)$

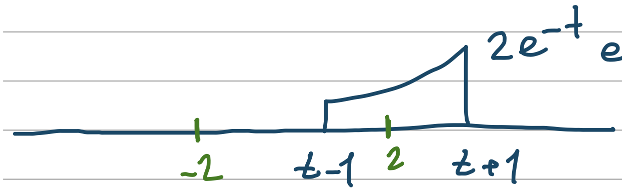
Case 3:



$$-1<t<1: \quad y(t)=\int_{t-1}^{t+1} 3e^{-T}\, 2e^{-t}e^{T}\, dT$$

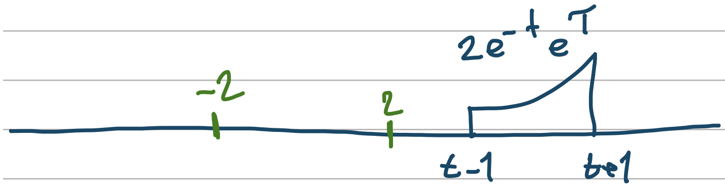
$$=\int_{t-1}^{t+1} 6e^{-t}\, dT = 6e^{-t}\,(t+1-(t-1)) = 12\, e^{-t}$$

Case 4:



$$1<t<3: \quad y(t)=\int_{t-1}^{2} (3e^{-T})\,(2e^{-t}e^{T})\, dT$$

$$=\int_{t-1}^{2} 6e^{-t}\, dT = 6e^{-t}\,(2-(t-1)) = 6e^{-t}(3-t)$$



$t>3$, no overlap

$y(t)=0$

$$y(t)=\begin{cases} 6e^{-t}\,[t+3] & -3<t<-1 \\ 12\, e^{-t} & -1<t<1 \\ 6e^{-t}\,[-t+3] & 1<t<3 \\ 0 & \text{otherwise} \end{cases}$$

Example #5: $x(t)=\text{rect}\left(\frac{t}{2}\right)$, $h(t)=\text{rect}(t-1)$

Solve analytically.

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(T)\, h(t-T)\, dT$$

$$=\int_{-\infty}^{\infty} \text{rect}\left(\frac{T}{2}\right)\, \text{rect}(t-T-1)\, dT$$

$\text{rect}\left(\frac{\tau}{2}\right)$ is none zero for $\tau \in [-1,1]$ and $\text{rect}(t-\tau-1)$ is none zero for $\tau \in \left[t-\frac{3}{2}, t-\frac{1}{2}\right]$

if $t-\frac{1}{2} < -1$ or $t-\frac{3}{2} > 1$, we have no overlap.

so for $t < \frac{-1}{2}$ and $t > \frac{5}{2}$, $y(t) = 0$

if $t-\frac{3}{2} < -1 < t-\frac{1}{2}$, $y(t) = \int_{-1}^{t-\frac{1}{2}} d\tau = t-\frac{1}{2}-(-1) =$ $t+\frac{1}{2}$, so for $\frac{-1}{2} < t < \frac{1}{2}$, $y(t) = t+\frac{1}{2}$

if $-1 < t-\frac{1}{2} < 1$, $y(t) = \int_{t-1/2}^{1} 1\, d\tau = 1-\left(t-\frac{1}{2}\right) = \frac{3}{2}-t$

$-1 < t-\frac{3}{2} < t-\frac{1}{2} < 1$, $y(t) = \int_{t-\frac{3}{2}}^{t-\frac{1}{2}} 1\times 1\, d\tau =$

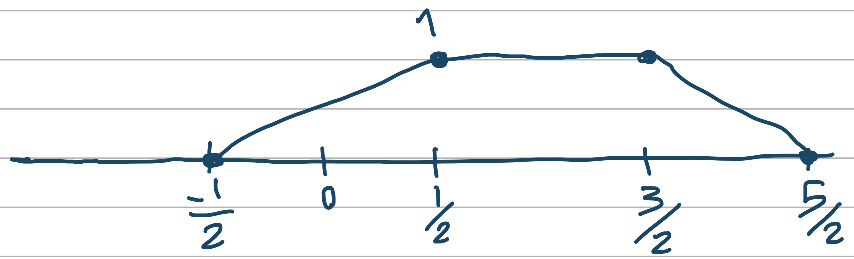
$\left(t-\frac{1}{2}\right) - \left(t-\frac{3}{2}\right) = 1$

so for $t > \frac{1}{2}$, $t < \frac{3}{2}$, $y(t) = 1$

if $t-\frac{3}{2} < 1 < t-\frac{1}{2}$, $y(t) = \int_{t-\frac{3}{2}}^{1} 1\times 1\, d\tau = 1-\left(t-\frac{3}{2}\right) =$

$= \frac{5}{2} - t$

so for $t > \frac{3}{2}$, $t < \frac{5}{2}$, $y(t) = t-\frac{3}{2}$



Some properties of convolution

1. $x(t) * h(t) = h(t) * x(t)$
2. $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$
3. $y(t) = x(t) * h(t) \Rightarrow y'(t) = x(t-t_0) * h(t) = y(t-t_0)$
4. $x(t) * \delta(t) = x(t)$, $x(t) * \delta(t-t_0) = x(t-t_0)$