University of California San Diego

ECE 101 - Linear System Fundamentals
Midterm, Fall 2024

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Please read the exam rules. NO NOT BEGIN UNTIL TOLD!

- The exams must be done in a blue book. Remember to write your name on your bluebook.
- On the first page of your blue book, please write out and sign the following pledge: "*I pledge my honor that I have not violated the honor code or the rules specified by the instructor during this examination.*"
- This midterm exam is closed-book, closed notes, and closed external links. No MATLAB is required. No calculators are permitted.
- Except the one page of cheat sheet, you are not allowed to use any other materials during the exam.
- You cannot communicate with anyone during the exam. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing.*
- Please write your solutions as clear as possible. You should also provide sufficient details for your solutions. We will give your partial credits even if the final answer is incorrect but the details make sense. Please be rigorous and to the point in your answers.
- Some problems might be harder than others, and there is no particular order. Good luck!

Problem 1: CT & DT Signals [36 points]

(a) We have discussed two important classes of signals: 1) impulse signals and 2) complex exponentials. Using explanations and relevant equations, describe why these two classes of signals are fundamental in this course. Please be rigorous and to the point in your answer (You DO NOT have long explanations). [6 points]

(b) We claim that $x(t) * y(t-4) = x(t-4) * y(t)$ for all CT signals $x(t), y(t)$, where $*$ denotes the CT convolution. Please indicate whether this claim is **True** or **False**, and justify your answer. [6 points]

(c) Let $y[n] = u[n-3] * x[n]$, where $x[n]$ is any discrete-time signal, $u[n]$ denotes the unit step signal, and $*$ denotes the DT convolution. We claim that

$$y[0] = \sum_{k=-\infty}^{3} x[k].$$

Please indicate whether this claim is **True** or **False**, and justify your answer. [6 points]

(d) Consider a DT signal

$$x[n] = e^{j\frac{\pi}{6}n} + \sum_{k=-\infty}^{\infty} (-1)^k \delta[n-5k] \tag{1}$$

What is the fundamental frequency of ω_0? [6 pts]

(e) Consider the same signal $x[n]$ in Eq. (1) above. What is the odd part of the signal $x[n]$, i.e., $f[n] = Odd\{x[n]\}$? [6 pts]

(f) Let $x[n] = u[n+2] - 2u[n+1] + u[n-2]$, where $u[n]$ is the unit step signal. Please represent $x[n]$ mathematically as a linear combination of time-shifted unit impulse signals, and sketch $x[n]$ precisely. [6 pts]

Problem 2: DT LTI systems [36 pts]

Consider a DT LTI system defined by the difference equation

$$y[n] = x[n-1] - x[n+2].$$

1) Determine its impulse response $h[n]$ and sketch $h[n]$ precisely. [6 points]

2) Determine its step response $s[n]$ (i.e., the system has a unit step input $x[n] = u[n]$). Sketch $s[n]$ precisely. [6 points]

3) Determine its system function $H(z)$. [6 points]

4) Does there exist a complex number $z \neq 0$ such that $H(z) = 0$. If not, why not? If so give an example of such a number z. [6 points]

5) Please indicate whether or not this LTI system satisfies the specified properties and Justify your answer [12 points, 4 points each]

   (a) This system is invertible. True or False.

   (b) This system is stable. True or False.

   (c) This system is causal. True or False

Problem 3: DTFS and DF Filter (28 pts)

1) Let $x[n]$ be a discrete-time periodic signal with fundamental period $N = 4$, and Fourier series coefficient a_k. Let $y[n] = x[n+1]$. The Fourier series coefficients b_k of $y[n]$ are $b_k = (-1)^k$ for all k. [10 points, 5 points each]

   (a) Determine the Fourier series coefficients a_k of the signal $x[n]$.

   (b) Represent the signal $x[n]$ mathematically, and sketch $x[n]$ precisely.

2) Find the fundamental period N and the discrete-time Fourier series coefficients of the signal

$$x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$$

Use the DTFS coefficients to decide if the signal is even or odd. [8 points]

3) Let $x[n]$ be a discrete-time signal with fundamental period $N = 4$ and discrete-time Fourier series coefficients

$$a_0 = 2, \quad a_1 = 1, \quad a_2 = -2, \quad a_3 = 1.$$

Consider the discrete-time LTI system whose frequency response in the interval $[-\pi, \pi]$ is given by

$$H(e^{j\omega}) = \begin{cases} 1, & \text{if} \quad |\omega| \le \frac{2\pi}{3} \\ 0, & \text{otherwise} \end{cases}$$

Suppose the signal $x[n]$ is the input to this system. Determine and sketch precisely the output $y[n]$ in the interval $[0, 1, 2, 3]$. [10 points]

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\left\{\begin{array}{ll} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$