**UNIVERSITY OF CALIFORNIA, SAN DIEGO**
**Electrical & Computer Engineering Department**
**ECE 101 - Fall 2023**
*Linear Systems Fundamentals*

**SOLUTIONS TO FINAL EXAM**
**You are allowed two 2-sided sheets of notes.**
**Tables 3.1, 3.2, 4.1, 4.2, 5.1, 5.2, 9.1, and 9.2 are attached.**
**No books, no other notes, no calculators.**

PRINT YOUR NAME Marc-Antoine Parseval des Chênes

Signature $\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|X(j\omega)\right|^2 d\omega$

Student ID Number $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$

| Problem | Weight | Score |
|---|---|---|
| 1 | 20 | 20 |
| 2 | 20 | 20 |
| 3 | 20 | 20 |
| 4 | 20 | 20 |
| 5 | 20 | 20 |
| 6 | 20 | 20 |
| Total | 120 | 120 |

**Please do not begin until told.**
**Show your work.**
**Use back of previous page and attached scratch sheets as needed.**
**Scan, upload, and tag your solutions on Gradescope at the end of the exam.**

**Good luck! You've got this!**

**Name/Student ID:** ______________________________

**Problem 1 [CTFT Properties]** (20 points, 5 points each part)

Let $x(t) = u(t+3) - u(t-1)$. Let $X(j\omega)$ denote its Fourier transform. In the following problems, refer to the attached tables, as necessary, to explain your solutions.

(a) Determine $X(j\omega)$.

(b) Determine $\int_{-\infty}^{\infty} X(j\omega) d\omega$.

(c) Let $y(t) = x(\frac{t}{2})$, with Fourier transform $Y(j\omega)$. Determine $Y(j0)$.

(d) Determine $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

**Write answers to parts (a) through (d) on the following pages.**

**Justify your answers.**

**Name/Student ID:** ______________________________

**Problem 1 [CTFT Properties] (cont.)**

Let $x(t) = u(t+3) - u(t-1)$. Let $X(j\omega)$ denote its Fourier transform.

(a) Determine $X(j\omega)$.

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \\
&= \int_{-3}^{1} e^{-j\omega t}dt = \frac{-1}{j\omega}e^{-j\omega t}\Big|_{-3}^{1} \\
&= \frac{-1}{j\omega}(e^{-j\omega} - e^{j3\omega}) \\
&= \frac{2}{2j\omega}e^{j\omega}(e^{j2\omega} - e^{-j2\omega}) = e^{j\omega}\frac{2\sin 2\omega}{\omega}
\end{aligned}$$

Alternatively, note that the signal $y(t) = x(t-1)$ is a rectangular pulse of amplitude 1 on the interval $[-2, 2]$.
From Table 4.2, we have $Y(j\omega) = \frac{2\sin(2\omega)}{\omega}$.
By the Time Shifting property in Table 4.1, we have $Y(j\omega) = e^{-j\omega}X(j\omega)$ implying

$$X(j\omega) = e^{j\omega}Y(j\omega) = e^{j\omega}\frac{2\sin(2\omega)}{\omega}.$$

Another acceptable solution:

$$u(t-t_0) \longleftrightarrow e^{-j\omega t_0}\left(\frac{1}{j\omega} + \pi\delta(\omega)\right)$$

$$\begin{aligned}
X(j\omega) &= e^{j\omega 3}\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) - e^{-j\omega}\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) \\
&= \frac{e^{j\omega 3} - e^{-j\omega}}{j\omega} + \pi(e^{j\omega 3} - e^{-j\omega})\delta(\omega) \\
&= \frac{e^{j\omega 3} - e^{-j\omega}}{j\omega}
\end{aligned}$$

since $(e^{j\omega 3} - e^{-j\omega})\delta(\omega) = \delta(\omega) - \delta(\omega) = 0$ by the sampling property of the unit impulse signal.

**Name/Student ID:** ______________________________

**Problem 1 [CTFT Properties] (cont.)**

Let $x(t) = u(t+3) - u(t-1)$. Let $X(j\omega)$ denote its Fourier transform.

(b) Determine $\int_{-\infty}^{\infty} X(j\omega) d\omega$.

The synthesis property states that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega.$$

Setting $t = 0$, this implies

$$\begin{aligned} x(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega 0} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) d\omega \end{aligned}$$

So,

$$2\pi x(0) = \int_{-\infty}^{\infty} X(j\omega) d\omega.$$

For this given signal, we have $x(0) = 1$, so we get

$$\int_{-\infty}^{\infty} X(j\omega) d\omega = 2\pi.$$

**Name/Student ID:** ______________________________

**Problem 1 [CTFT Properties] (cont.)**

Let $x(t) = u(t+3) - u(t-1)$. Let $X(j\omega)$ denote its Fourier transform.

(c) Let $y(t) = x(\frac{t}{2})$, with Fourier transform $Y(j\omega)$. Determine $Y(j0)$.

From the analysis equation, we have

$$Y(j\omega) \;=\; \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt.$$

Evaluating at $t = 0$ gives

$$Y(j0) \;=\; \int_{-\infty}^{\infty} y(t)dt.$$

The signal $y(t)$ is a rectangular pulse of ampitude 1 on the interval $[-6.2]$, so

$$Y(j0) \;=\; \int_{-6}^{2} 1dt = 8.$$

Alternatively, by the Time and Frequency Scaling property in Table 4.1, we have

$$Y(j\omega) \;=\; 2X(j2\omega)$$

so

$$Y(j0) \;=\; 2X(j0).$$

Since $X(j0) = \int_{-\infty}^{\infty} x(t)dt = 4$, we conclude $Y(j0) = 8$.

**Name/Student ID:** ______________________________

**Problem 1 [CTFT Properties] (cont.)**

Let $x(t) = u(t+3) - u(t-1)$. Let $X(j\omega)$ denote its Fourier transform.

(d) Determine $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

By Parseval's Relation for Aperiodic Signals in Table 4.2,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

So,

$$\begin{aligned} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega &= 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= 2\pi \int_{-3}^{1} 1 dt \\ &= 8\pi. \end{aligned}$$

**Name/Student ID:** ______________________________

**Problem 2 [DTFT]** (20 points, 5 points each part)

This problem deals with the DT LTI system $S$ with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{-n} u[-n] + \left(\frac{1}{2}\right)^{n} u[n].$$

(a) Precisely sketch the signal $h[n]$ in the time interval $[-2, 2]$.

(b) Let $x[n] = u[n-1]$ be the input signal to the system $S$.
Determine the value of the corresponding output signal $y[n]$ at $n = 0$.

(c) Determine the frequency response $H(e^{j\omega})$ of the system $S$.
Express it in a form that does not use any exponential functions of $\omega$.

(d) Is the system $S$ invertible? If not, why not? If so, determine the frequency response $G(e^{j\omega})$ of its inverse system.

**Write answers to parts (a) through (d) on the following pages.**

**Justify your answers.**

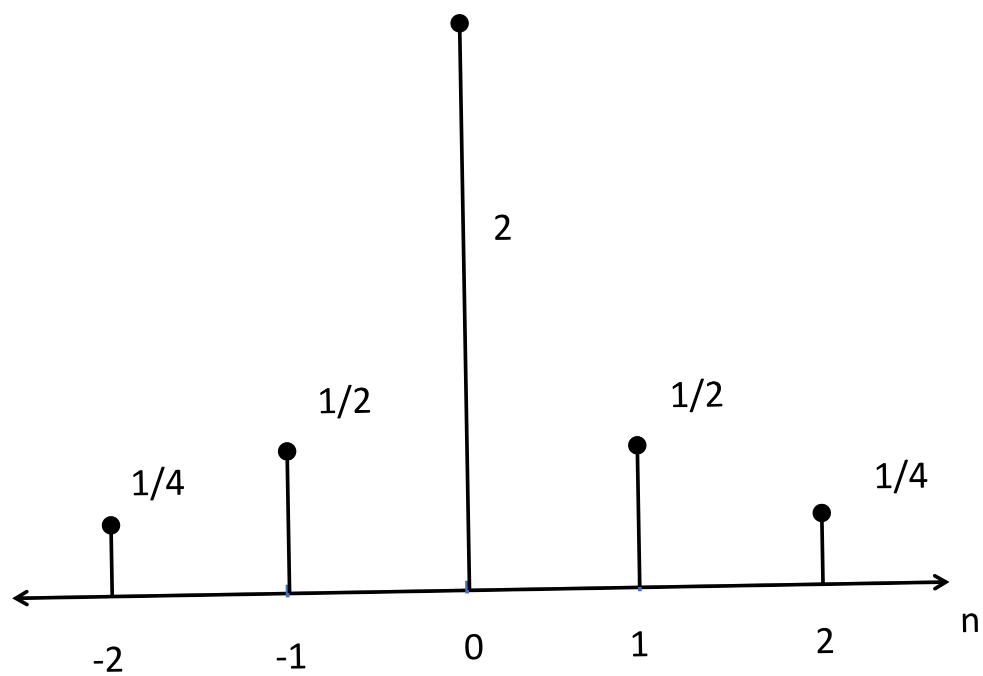
**Name/Student ID:** ______________________________

**Problem 2 [DTFT] (cont.)**

This problem deals with the DT LTI system $S$ with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{-n} u[-n] + \left(\frac{1}{2}\right)^{n} u[n].$$

(a) Precisely sketch the signal $h[n]$ in the time interval $[-2, 2]$.



**Name/Student ID:** ____________________________________________

**Problem 2 [DTFT] (cont.)**

This problem deals with the DT LTI system $S$ with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{-n} u[-n] + \left(\frac{1}{2}\right)^{n} u[n].$$

(b) Let $x[n] = u[n-1]$ be the input signal to the system $S$.
Determine the value of the corresponding output signal $y[n]$ at $n = 0$.

By the Convolution Sum Formula:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Evaluating at $n = 0$, we get:

$$\begin{aligned}
y[0] &= \sum_{k=-\infty}^{\infty} x[k]h[-k] \\
&= \sum_{k=-\infty}^{\infty} u[k-1] \left( \left(\frac{1}{2}\right)^{k} u[k] + \left(\frac{1}{2}\right)^{-k} u[-k] \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} - 1 \\
&= \frac{1}{1-(1/2)} - 1 = 1.
\end{aligned}$$

**Name/Student ID:** ______________________________

**Problem 2 [DTFT] (cont.)**

This problem deals with the DT LTI system $S$ with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{-n} u[-n] + \left(\frac{1}{2}\right)^{n} u[n].$$

(c) Determine the frequency response $H(e^{j\omega})$ of the system $S$.
Express it in a form that does not use any exponential functions of $\omega$.

From the table of DTFT transform pairs, we have:

$$\left(\frac{1}{2}\right)^{n} u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

By the Time-Reversal property of the DTFT, we have:

$$\left(\frac{1}{2}\right)^{-n} u[-n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\frac{1}{2}e^{j\omega}}$$

So,

$$\begin{aligned}
H(e^{j\omega}) &= \frac{1}{1-\frac{1}{2}e^{j\omega}} + \frac{1}{1-\frac{1}{2}e^{-j\omega}} \\
&= \frac{1-\frac{1}{2}e^{-j\omega}+1-\frac{1}{2}e^{j\omega}}{\left(1-\frac{1}{2}e^{j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)} \\
&= \frac{2-\cos(\omega)}{1-\frac{1}{2}e^{-j\omega}-\frac{1}{2}e^{j\omega}+\frac{1}{4}} \\
&= \frac{2-\cos(\omega)}{\frac{5}{4}-\cos(\omega)}
\end{aligned}$$

Note that since $h[n]$ is real and even, its Fourier transform $H(e^{j\omega})$ is real and even.

**Name/Student ID:** ____________________

**Problem 2 [DTFT] (cont.)** This problem deals with the DT LTI system $S$ with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{-n} u[-n] + \left(\frac{1}{2}\right)^{n} u[n].$$

(d) Is the system $S$ invertible? If not, why not? If so, determine the frequency response $G(e^{j\omega})$ of its inverse system.

From part (c), we have that

$$H(e^{j\omega}) \;=\; \frac{2-\cos(\omega)}{\frac{5}{4}-\cos(\omega)}.$$

Let $G(e^{j\omega})$ be the formal inverse of $H(e^{j\omega})$:

$$G(e^{j\omega}) \;=\; \frac{\frac{5}{4}-\cos(\omega)}{2-\cos(\omega)}.$$

Noting that $|\cos(\omega)| \leq 1$, for all $\omega$, we see that $2-\cos(\omega) \neq 0$ and $\frac{5}{4}-\cos(\omega) \neq 0$, for all $\omega$. This means that $H(e^{j\omega})$ and $G(e^{j\omega})$ are well-defined and non-zero at all frequencies, so they each correspond to LTI systems that are invertible $^*$.

Since

$$H(e^{j\omega})G(e^{j\omega}) = 1$$

we conclude that $G(e^{j\omega})$ is the inverse system of the system $S$.

$^*$ These properties of $H(e^{j\omega})$ and $G(e^{j\omega})$ imply that for their respective LTI systems, there is no distinct pair of input signals $x_1(t)$ and $x_2(t)$ that produce the same output signal. For, if there were, then the input signal $x(t) = x_1(t) - x_2(t) \neq 0$ would produce the all-zero output signal $y(t) = 0$. By the Convolution Property of the DTFT in Table 5.2, we could conclude $X(e^{j\omega})H(e^{j\omega}) = Y(e^{j\omega}) = 0$. Since $H(e^{j\omega}) \neq 0$ for all $\omega$, this means $X(e^{j\omega}) = 0$ for all $\omega$, a contradiction.

**Name/Student ID:** ______________________________

**Problem 3 [Amplitude Modulation]** (20 points, 5 points each part)

Consider the signals $x_1(t) = \frac{\sin(10t)}{\pi t}$ and $x_2(t) = \frac{\sin(5t)}{\pi t}$.

(a) Determine the Fourier transforms $X_1(j\omega)$ and $X_2(j\omega)$. Sketch them precisely.

(b) Let signal $y(t) = y_1(t) + y_2(t)$, where $y_1(t) = x_1(t)\cos(20t)$ and $y_2(t) = x_2(t)\cos(40t)$. Express the Fourier transform $Y(j\omega)$ of $y(t)$ in terms of $X_1(j\omega)$ and $X_2(j\omega)$. Sketch $Y(j\omega)$ precisely.

(c) The signal $y(t)$ is demodulated using multiplication by $\cos(20t)$ to form the signal $z(t) = y(t)\cos(20t)$. Express the Fourier transform $Z(j\omega)$ in terms of $Y_1(j\omega)$ and $Y_2(j\omega)$. Then express $Z(j\omega)$ in terms of $X_1(j\omega)$ and $X_2(j\omega)$. Sketch precisely the portion of $Z(j\omega)$ in the frequency range $[-50, 50]$.

(d) (i) Describe precisely a scheme to recover $x_1(t)$ from $z(t)$.

(ii) Describe a scheme to recover $x_2(t)$ from the signal $y(t)$.

**Write answers to parts (a) through (d) on the following pages.**

**Justify your answers.**

**Name/Student ID:** ______________________________

**Problem 3 [Amplitude Modulation] (cont.)**

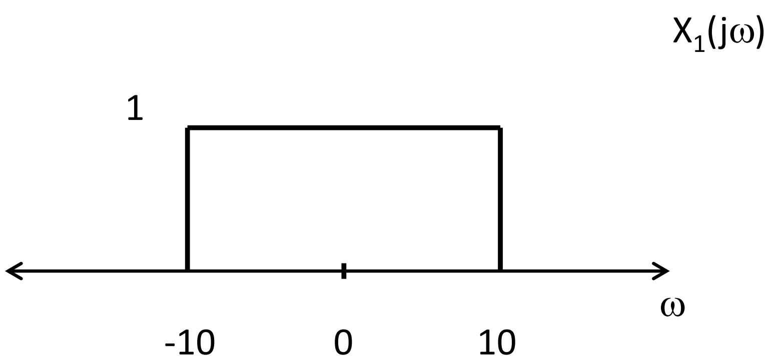
Consider the signals $x_1(t) = \frac{\sin(10t)}{\pi t}$ and $x_2(t) = \frac{\sin(5t)}{\pi t}$.

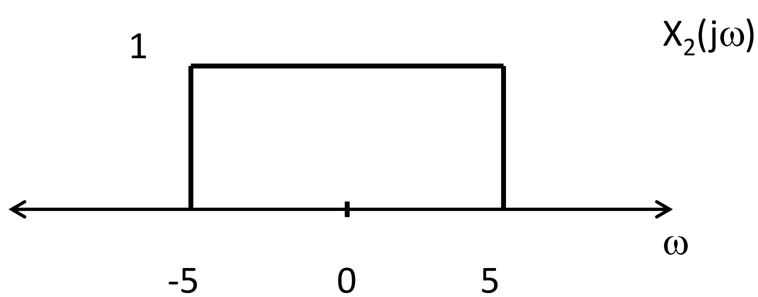
(a) Determine the Fourier transforms $X_1(j\omega)$ and $X_2(j\omega)$. Sketch them precisely.

From Table 4.2:

$$X_1(j\omega) = \begin{cases} 1, & |\omega| < 10 \\ 0, & |\omega| > 10 \end{cases}$$

$$X_2(j\omega) = \begin{cases} 1, & |\omega| < 5 \\ 0, & |\omega| > 5 \end{cases}$$



**Name/Student ID:** ______________________________

**Problem 3 [Amplitude Modulation] (cont.)**

(b) Let signal $y(t) = y_1(t) + y_2(t)$, where $y_1(t) = x_1(t)\cos(20t)$ and $y_2(t) = x_2(t)\cos(40t)$. Express the Fourier transform $Y(j\omega)$ of $y(t)$ in terms of $X_1(j\omega)$ and $X_2(j\omega)$. Sketch $Y(j\omega)$ precisely.

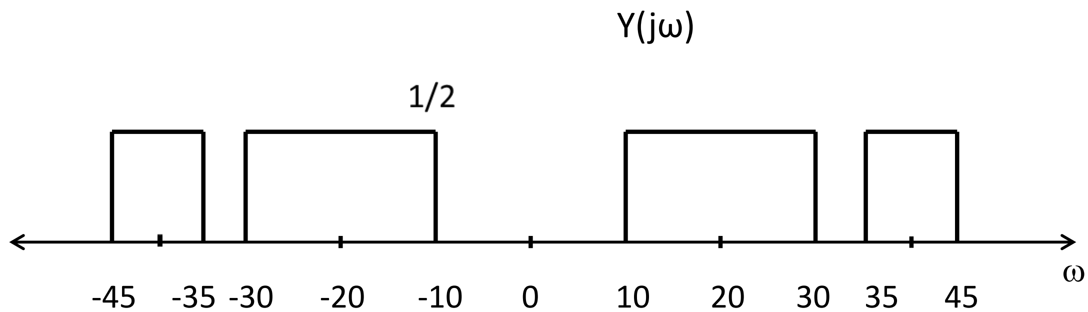
Let $Y_1(j\omega)$ be the Fourier transform of $y_1(t)$ and $Y_21(j\omega)$ be the Fourier transform of $y_2(t)$.

Then $Y(j\omega) = Y_1(j\omega) + Y_2(j\omega)$, where

$$Y_1(j\omega) = \frac{1}{2}X_1(j(\omega - 20)) + \frac{1}{2}X_1(j(\omega + 20))$$

and

$$Y_2(j\omega) = \frac{1}{2}X_2(j(\omega - 40)) + \frac{1}{2}X_2(j(\omega + 40))$$



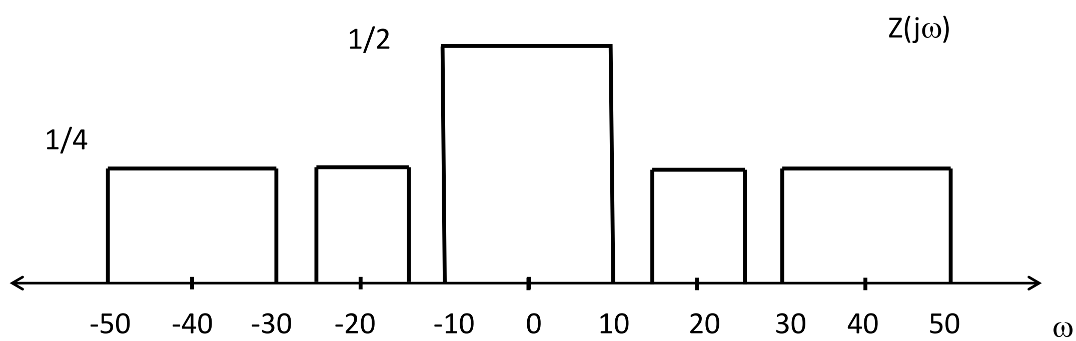
**Name/Student ID:** ______________________________

**Problem 3 [Amplitude Modulation] (cont.)**

(c) The signal $y(t)$ is demodulated using multiplication by $\cos(20t)$ to form the signal $z(t) = y(t)\cos(20t)$. Express the Fourier transform $Z(j\omega)$ in terms of $Y_1(j\omega)$ and $Y_2(j\omega)$. Then express $Z(j\omega)$ in terms of $X_1(j\omega)$ and $X_2(j\omega)$. Sketch precisely the portion of $Z(j\omega)$ in the frequency range $[-50, 50]$.

$$Z(j\omega) = \frac{1}{2}Y_1(j(\omega-20)) + \frac{1}{2}Y_1(j(\omega+20)) + \frac{1}{2}Y_2(j(\omega-20)) + \frac{1}{2}Y_2(j(\omega+20))$$

$$\begin{aligned} Z(j\omega) = &\frac{1}{4}X_1(j(\omega-40)) + \frac{1}{2}X_1(j\omega) + \frac{1}{4}X_1(j(\omega+40)) \\ &+ \frac{1}{4}X_2(j(\omega-60)) + \frac{1}{4}X_2(j(\omega-20)) \\ &+ \frac{1}{4}X_2(j(\omega+20)) + \frac{1}{4}X_2(j(\omega+60)) \end{aligned}$$



**Name/Student ID:** ______________________________

**Problem 3 [Amplitude Modulation] (cont.)**

(d) (i) Describe precisely a scheme to recover $x_1(t)$ from $z(t)$.

Apply a low-pass filter with cut-off frequency $10 < \omega_{co} < 15$ and gain 2.

Note: A cutoff frequency exactly equal to 10 or 15 is acceptable here, i.e., $|\omega_{co}| \leq 10$ or $|\omega_{co}| \leq 15$, but not the best answer.

(ii) Describe a scheme to recover $x_2(t)$ from the signal $y(t)$.

First demodulate by multiplying $y(t)$ by $\cos(40t)$. Then apply a low-pass filter with cut-off frequency $5 < \omega_{co} < 10$ and gain 2.

Note: A cutoff frequency exactly equal to 5 or 10 is acceptable here, i.e., $|\omega_{co}| \leq 5$ or $|\omega_{co}| \leq 10$, but not the best answer.

**Name/Student ID:** ____________________

**Problem 4 [Sampling Theory]** (20 points, 5 points each part)

For each of the following signals, determine if a sampling period of $T = \frac{1}{4}$ seconds is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples. Justify your answers by explicit reference to the Fourier transforms of the signals.

(a) $x(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \end{cases}$

(b) $x(t) = \cos(\frac{3}{2}\pi t)e^{-j\frac{3}{2}\pi t}$

(c) $x(t) = \frac{\sin(3\pi t)}{\pi t} * \frac{\sin(6\pi t)}{\pi t}$

(d) $x(t) = e^{-3\pi t}u(t)$

**Write answers to parts (a) through (d) on the following pages.**

**Justify your answers.**

**Name/Student ID:** ______________________________

**Problem 4 [Sampling Theory]**

For each of the following signals, determine if a sampling period of $T = \frac{1}{4}$ seconds is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples. Justify your answers by explicit reference to the Fourier transforms of the signals.

The corresponding sampling frequency is $\omega_2 = \frac{2\pi}{T} = 8\pi$

(a) $x(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \end{cases}$

From Table 4.2, we find

$$X(j\omega) = \frac{2\sin(\omega/2)}{\omega}.$$

The signal is not band-limited, so aliasing occurs.

The signal cannot be reconstructed.

**Name/Student ID:** ______________________________

**Problem 4 [Sampling Theory] (cont.)**

For each of the following signals, determine if a sampling period of $T = \frac{1}{4}$ seconds is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples. Justify your answers by explicit reference to the Fourier transforms of the signals.

(b) $x(t) = \cos(\frac{3}{2}\pi t)e^{-j\frac{3}{2}\pi t}$

Let $x_1(t) = \cos(\frac{3}{2}\pi t)$ with Fourier transform from Table 4.2 given by $X_1(j\omega) = \pi\delta(\omega - \frac{3}{2}\pi) + \pi\delta(\omega + \frac{3}{2}\pi)$. This has maximum frequency $\frac{3}{2}\pi$.

Let $x_2(t) = e^{-j\frac{3}{2}\pi t}$ with Fourier transform from Table 4.2 given by $2\pi\delta(\omega + \frac{3}{2}\pi)$.

Applying the Multiplication Property of CTFT in Table 4.2,

$$\begin{aligned} X(j\omega) &= \frac{1}{2\pi}X_1(j\omega) * X_2(j\omega) \\ &= \pi\delta(\omega) + \pi\delta(\omega + 3\pi) \end{aligned}$$

It follows that the maximum frequency of $X(j\omega)$ is $W = 3\pi$.

Since $\omega_s > 2W$, no aliasing occurs.

The signal can be reconstructed.

**Name/Student ID:** ____________________

**Problem 4 [Sampling Theory] (cont.)**

For each of the following signals, determine if a sampling period of $T = \frac{1}{4}$ seconds is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples. Justify your answers by explicit reference to the Fourier transforms of the signals.

(c) $x(t) = \frac{\sin(3\pi t)}{\pi t} * \frac{\sin(6\pi t)}{\pi t}$

Let $x_1(t) = \frac{\sin(3\pi t)}{\pi t}$ and $x_2(t) = \frac{\sin(6\pi t)}{\pi t}$.

From Table 4.2, we find its Fourier transform $X_1(j\omega)$ and $X_2(j\omega)$:

$$\begin{aligned} X_1(j\omega) &= \begin{cases} 1, & |\omega| < 3\pi \\ 0, & |\omega| > 3\pi \end{cases} \\ X_2(j\omega) &= \begin{cases} 1, & |\omega| < 6\pi \\ 0, & |\omega| > 6\pi \end{cases} \end{aligned}$$

By the Convolution Property in Table 4.1,

$$\begin{aligned} X(j\omega) &= X_1(j\omega)X_2(j\omega) \\ &= X_1(j\omega). \end{aligned}$$

The signal has maximum frequency $W = 3\pi$.

Since $\omega_s > 2W$, no aliasing occurs.

The signal can be reconstructed.

**Name/Student ID:** ______________________________

**Problem 4 [Sampling Theory] (cont.)**

For each of the following signals, determine if a sampling period of $T = \frac{1}{4}$ seconds is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples. Justify your answers by explicit reference to the Fourier transforms of the signals.

(d) $x(t) = e^{-3\pi t}u(t)$

From Table 4.2, we find

$$X(j\omega) = \frac{1}{3\pi + j\omega} = \frac{3\pi - j\omega}{9\pi^2 + \omega^2}.$$

$|X(j\omega) \neq 0$ for all $\omega$, so the signal is not band-limited.

Therefore aliasing occurs.

The signal cannot be reconstructed.

**Name/Student ID:** ________________________________________

**Problem 5 [Laplace Transform and LTI Systems]** (20 points)

The system function of a stable LTI system $S$ is

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

(a) Determine the poles and zeros and sketch precisely the pole-zero plot.

(b) Let $H(j\omega)$ be the frequency response of the system $S$. Determine the magnitude $|H(j\omega)|$ at $\omega = 0, 1$, and in the limit as $\omega \to \pm\infty$.

(c) Sketch $|H(j\omega)|$ approximately. Is the system low-pass, high-pass, band-pass, or all-pass in nature?

**Write answers to parts (a) through (c) on the following pages. Justify your answers.**

**Name/Student ID:** ______________________________

**Problem 5 [[Laplace Transform and LTI Systems] (cont.)**

The system function of a stable LTI system $S$ is

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

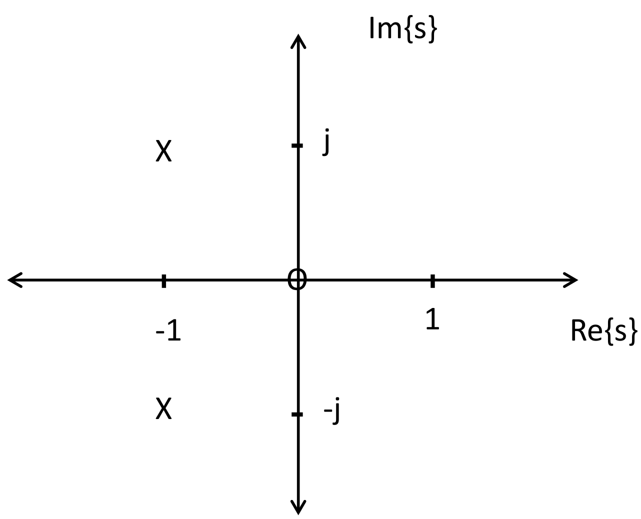
(a) (8 points)
Determine the poles and zeros and sketch precisely the pole-zero plot.

Factor the denominator (using the quadratic formula) to get

$s + 2s + 2 = (s - (-1+j))\,(s - (-1-j))$

First-order zero at $s = 0$.

First-order poles at $s = -1 \pm j$.



**Name/Student ID:** ____________________

**Problem 5 [[Laplace Transform and LTI Systems] (cont.)**

The system function of a stable LTI system $S$ is

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

(b) (8 points)
Let $H(j\omega)$ be the frequency response of the system $S$. Determine the magnitude $|H(j\omega)|$ at $\omega = 0, 1$, and in the limit as $\omega \to \pm\infty$.

$$\begin{aligned}
|H(j0)| &= 0 \\
|H(j1)| &= \frac{|j - 0|}{|j - (-1{+}j)| \cdot |j - (-1{-}j)|} = \frac{1}{\sqrt{5}} \\
\lim_{\omega\to\infty} |H(j\omega)| &= \lim_{\omega\to\infty} \frac{|j\omega|}{|j\omega - (-1{+}j)| \cdot |j\omega - (-1{-}j)|} \\
&\approx \lim_{L\to\infty} \frac{L}{L \cdot L} \\
&= 0 \\
\lim_{\omega\to-\infty} |H(j\omega)| &= \lim_{\omega\to-\infty} \frac{|j\omega|}{|j\omega - (-1{+}j)| \cdot |j\omega - (-1{-}j)|} \\
&\approx \lim_{L\to\infty} \frac{L}{L \cdot L} \\
&= 0
\end{aligned}$$

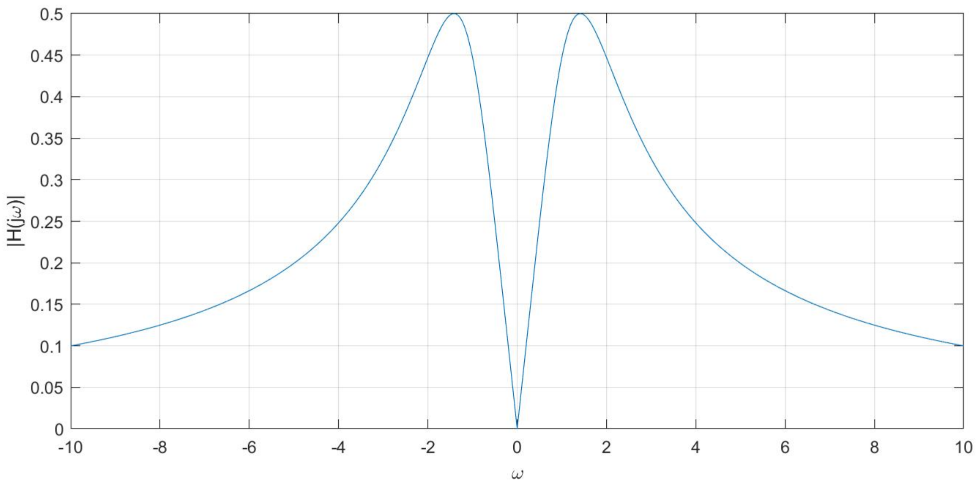
**Name/Student ID:** ______________________________

**Problem 5 [Laplace Transform and LTI Systems] (cont.)**

The system function of a stable LTI system $S$ is

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

(c) (4 points)
Sketch $|H(j\omega)|$ approximately. Is the system low-pass, high-pass, band-pass, or all-pass in nature?




The system is band-pass in nature.

**Name/Student ID:** ____________________

**Problem 6 [Laplace Transform and Differential Equations]** (20 points)

Consider an LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}.$$

(a) Determine the system function $H(s)$ and the possible regions of convergence (ROC). For each region of convergence, determine whether the corresponding system is causal or stable. [Hint: Your answer to Problem 5(a) will be helpful here.]

(b) Determine the impulse response $h(t)$ when the system is causal. Express your answer in the form $Ce^{-\alpha t}\cos(\omega_0 t)u(t) + De^{-\alpha t}\sin(\omega_0 t)u(t)$, where $C, D, \alpha, \omega_0$ are constants. Use partial fraction expansion, and verify your answer using Table 9.2.

(c) Carefully sketch the canonical (direct-form) block diagram representing the causal LTI system in part (b).

**Write answers to parts (a) through (c) on the following pages.**

**Justify your answers.**

**Name/Student ID:** ________________________________

**Problem 6 [Laplace Transform and Differential Equations] (cont.)**

Consider an LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} = 2y(t) = \frac{dx(t)}{dt}.$$

(a) (8 points)

Determine the system function $H(s)$ and the possible regions of convergence (ROC). For each region of convergence, determine whether the corresponding system is causal or stable. [Hint: Your answer to Problem 5(a) will be helpful here.]

The differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

corresponds to the system function

$$H(S) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.$$

So, in this case,

$$H(s) = \frac{s}{s^2 + 2s + 2}.$$

This is the same system function as in Problem 5. From the pole-zero plot, we see that there are two possible regions of convergence:

**ROC I:** $\mathcal{R}e\{s\} > -1$.

(i) The system is causal because the ROC is a right half-plane.

(ii) The system is stable because the ROC contains the $j\omega$-axis.

**ROC II:** $\mathcal{R}e\{s\} < -1$.

(i) The system is not causal because the ROC is a left half-plane.

The system is not stable because the ROC does not contains the $j\omega$-axis.

**Name/Student ID:** ______________________________

**Problem 6 [Laplace Transform and Differential Equations] (cont.)**

Let $S$ be the causal LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} = 2y(t) = \frac{dx(t)}{dt}.$$

(b) (8 points)

Determine the impulse response $h(t)$ when the system is causal. Express your answer in the form $Ce^{-\alpha t}\cos(\omega_0 t)u(t) + De^{-\alpha t}\sin(\omega_0 t)u(t)$, where $C, D, \alpha, \omega_0$ are constants. Use partial fraction expansion, and verify your answer using Table 9.2.

Let $a = -1 + j$ and $a^* = -1 - j$.

Expand $H(s)$ using Partial Fraction Expansion:

$$\begin{aligned} H(s) &= \frac{A}{s-a} + \frac{B}{s-a^*} \\ &= \frac{(1+j)/2}{s-a} + \frac{(1-j)/2}{s-a^*} \end{aligned}$$

where we have used

$$\begin{aligned} A &= (s-a)H(s)\Big|_{s=a} = \frac{s}{s-a^*}\Big|_{s=a} = \frac{-1+j}{2j} = \frac{1+j}{2} \\ B &= (s-a^*)H(s)\Big|_{s=a^*} = \frac{s}{s-a}\Big|_{s=a^*} = \frac{-1-j}{-2j} = \frac{1-j}{2}. \end{aligned}$$

For the causal region of convergence, $\mathcal{R}e\{s\} > -1$, using standard Laplace transform pairs in Table 9.2, we have

$$\begin{aligned} h(t) &= \frac{1+j}{2}e^{at}u(t) + \frac{1-j}{2}e^{a^*t}u(t) \\ &= \frac{1}{2}(e^{at} + e^{a^*t})u(t) + \frac{j}{2}(e^{at} - e^{a^*t})u(t) \\ &= \frac{1}{2}e^{-t}(e^{jt} + e^{-jt})u(t) + \frac{j}{2}e^{-t}(e^{jt} - e^{-jt})u(t) \\ &= e^{-t}\cos(t)u(t) - e^{-t}\sin(t)u(t). \end{aligned}$$

So, $C = 1, D = -1, \alpha = 1, \omega_0 = 1$.

Using lines 13 and 14 of Table 9.2 with the constants above, we verify: $H(s) = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} = \frac{s}{s^2+2s+2}$ with ROC $\mathcal{R}e\{s\} > -1$.

**Name/Student ID:** ______________________________

**Problem 6 [Laplace Transform and Differential Equations] (cont.)**

Consider an LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} = 2y(t) = \frac{dx(t)}{dt}.$$

(c) (4 points)

Carefully sketch the canonical (direct-form) block diagram representing the causal LTI system in part (b).