ECE 101: Linear Systems Fundamentals

Spring 2025 - Lecture 4

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Today's topics

- Continuous-time unit impulse and unit step signals
- Continuous-time and discrete-time complex exponentials

Signals and Systems (2th Edition): sections 1.3 and 1.4.2

Continuous-time Unit Impulse and Unit Step Signals

Continuous-Time: Unit Step

- Continuous-time case: $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

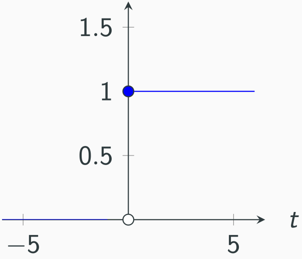
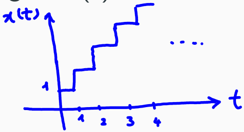


Figure 1: Plot of step signal, $u(t)$

- Example: Plot $x(t) = \sum_{k=0}^{\infty} u(t-k)$.



$x(t) = u(t) + u(t-1) + u(t-2) + u(t-3) + \cdots$

Continuous-Time: Unit Step

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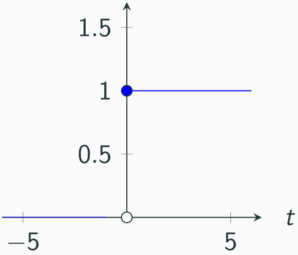
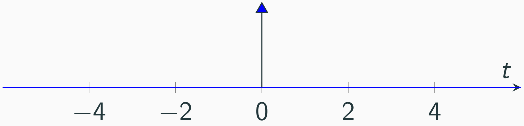


Figure 1: Plot of step signal, $u(t)$

- Example: Plot $x(t) = \sum_{k=0}^{\infty} u(t-k)$.

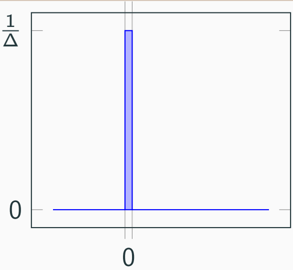
Continuous-time Unit Impulse: Mathematical formulation

- Define $\delta(t)$ to be the signal that satisfies the following properties:
  a. $\delta(t) = 0$ at any point $t \neq 0$
  b. $\delta(0) = \infty$ such that:
  $$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1.$$
  c. All the rules of calculus applies to it.
- a. and b. imply that $\int_{t_1}^{t_2} \delta(\tau) d\tau = 1$ for all $t_1 < 0 < t_2$.



Representing $\delta(t)$

Unit-impulse function



- Define $\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if } t \in [0, \Delta] \\ 0 & \text{else.} \end{cases}$
- For any signal $x(t)$ and small enough $\Delta > 0$:

$$\int_{-\infty}^{\infty} \delta_\Delta(\tau) d\tau = \int_0^\Delta \delta_\Delta(\tau) d\tau$$
$$= \frac{1}{\Delta} \int_0^\Delta d\tau = 1.$$

- We can **think** of $\delta(t)$ as $\lim_{\Delta \to 0} \delta_\Delta(t)$.

Properties of $\delta(t)$

Sampling Property

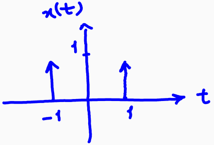
If $x(t)$ is continuous at τ, then:

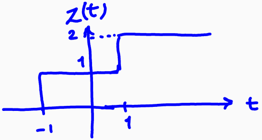
$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau).$$

- Example1: For $x(t) = (\delta(t-1) + \delta(t+1))\, t^2$, plot $z(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$$x(t) = t^2\,\delta(t-1) + t^2\,\delta(t+1) = \delta(t-1) + \delta(t+1)$$

$$z(t) = \int_{-\infty}^{t}\delta(\tau-1)\,d\tau + \int_{-\infty}^{t}\delta(\tau+1)\,d\tau = u(t-1) + u(t+1)$$





Properties of $\delta(t)$

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- Example 1: For $x(t) = \left(\delta(t-1) + \delta(t+1)\right) t^2$, plot $z(t) = \int_{-\infty}^{t} x(\tau) d\tau$

Properties of $\delta(t)$

Sampling Property

If $x(t)$ is continuous at τ, then:

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau).$$

Sifting Property

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = x(\tau).$$

Representation Property

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau,$$

Properties of $\delta(t)$

Representation Property

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau,$$

Proof.

- Using the sampling property:

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau).$$

- Integrating both sides:

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau \\ &= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = x(t).\end{aligned}$$

$\square$

Lecture 4 reading quiz

A continuous-time signal $x(t)$ is shown below. Assume

$$g(t) = (x(t) + x(-t))u(t)$$

Sketch and label $g(t)$, then find $g(0.5)$.

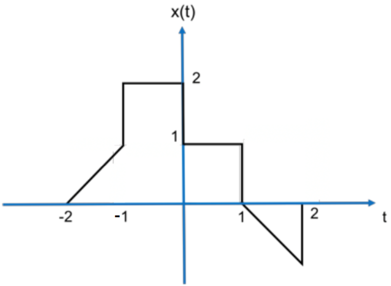
A. $g(0.5) = 3$

B. $g(0.5) = 2$

C. $g(0.5) = 0$

D. $g(0.5) = 1.5$

E. $g(0.5)$ cannot be determined



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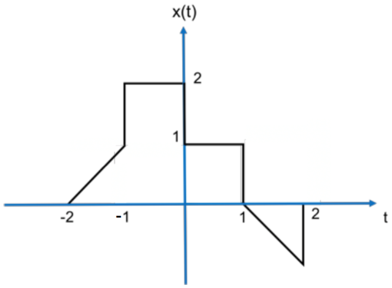


Exponential Signals

CT Exponential Signals

- Definition: A continuous-time (complex) exponential signal is a signal of the form:

$$x(t) = Ce^{at},$$

for $a, C \in \mathbb{C}$.

CT Real-valued Exponential Signals

Real-valued Exponential Signals:

- The case where $a = \alpha$ and $C = B$ are real numbers
- $\alpha > 0$ exponentially increasing
- $\alpha < 0$ exponentially decreasing

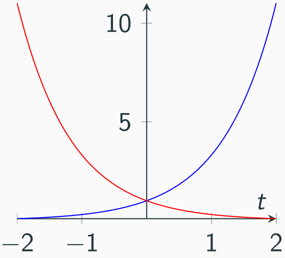
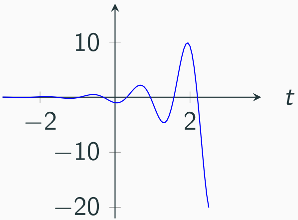


Figure 3: Plots of $x(t) = 1 \times\ e^{1.2t}$ and $x(t) = 1 \times\ e^{-1.2t}$

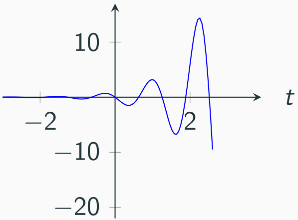
CT Exponential Signals: example

- The followings are the plots of real and imaginary parts of a complex exponential signal $x(t) = Ce^{at} = -1e^{(1.2+j5)t}$ where C is a real number equal to -1 and a is a complex number equal to $1.2 + j5$.

$\mathbb{R}\{x(t)\}$



$\mathbb{I}m\{x(t)\}$



CT Exponential Signals

- If $C = Ae^{j\theta}$ and $a = \alpha + j\omega_0$, then $x(t) = Ce^{at} = Ae^{\alpha t}e^{j(\omega_0 t + \theta)}$
- Decaying or growing exponent: real part of a
- Oscillatory behavior: imaginary part of a
- Phase: angle of C

CT Periodic Complex Exponential Signals

Periodic Complex Exponential:

- The case where $a = j\omega_0$
- In this case: $x(t) = Ae^{j(\omega_0 t+\theta)} = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta)$
- In this case, $x(t)$ is periodic with the fundamental period $\frac{2\pi}{|\omega_0|}$

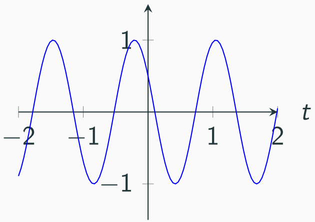


Figure 4: Plot of $\mathbb{R}e\{e^{j(5t+\frac{\pi}{3})}\}$

DT Exponential Signals

- Similar to CT: a discrete-time complex exponential signal is a signal of the form:
$$x[n] = Ce^{\beta n}.$$
- For technical reasons, it is more convenient to study:
$$x[n] = Ce^{\beta n} = C{\color{red}z}^n,$$
where $z = e^{\beta}$.

DT Real-valued Exponential Signals

Real-valued exponential:

- The case where C and z are real.
- If $|z| > 1$ the signal is exponentially increasing.
- If $0 < |z| < 1$, the signal is exponentially decreasing.

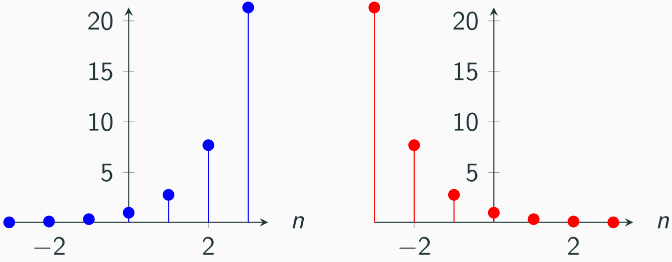


Figure 5: Plots of $x[n] = 1 \times \ (e^{1.02})^n$ and $x[n] = 1 \times \ (e^{-1.2})^n$

DT Periodic Exponential Signals

Periodic exponential:

- The case where $z = e^{j\Omega_0}$ is on the unit circle

- If $x[n] = Ce^{j\Omega_0 n}$ is periodic with period N, then:

$$\begin{aligned} x[n+N] &= x[n] \\ \Rightarrow Ce^{j\Omega_0(n+N)} &= Ce^{j\Omega_0 n}e^{j\Omega_0 N} = Ce^{j\Omega_0 n} \\ \Rightarrow e^{j\Omega_0 N} &= 1 \end{aligned}$$

- When is this true?

DT Periodic Exponential Signals

Periodic exponential:

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- When is this true?
- When $\Omega_0 = \frac{2\pi k}{N}$ for some integers k, N, and $N > 0$

DT Periodic Exponential Signals

- $x[n] = e^{j\Omega_0 n}$ is periodic if and only if Ω_0 is a rational multiple of 2π.

- The fundamental period is

$$N = \frac{2\pi m}{\Omega_0},$$

  where we assume that m and N are relatively prime, $\gcd(m, n) = 1$, i.e., $\frac{m}{N}$ is in reduced form.

Clicker question 1

- A continuous-time signal $x(t)$ is shown below.
  Determine the value of the integral

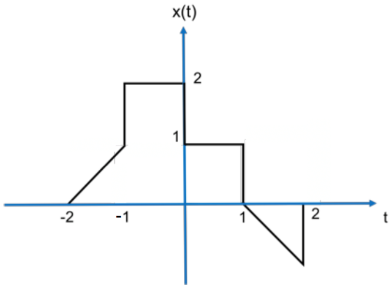
$$\int_{-\infty}^{\infty} x(t)\left(\delta(t+\frac{3}{2})-\delta(t-\frac{3}{2})\right)dt.$$

A. 0

B. 1

C. 2

D. 3



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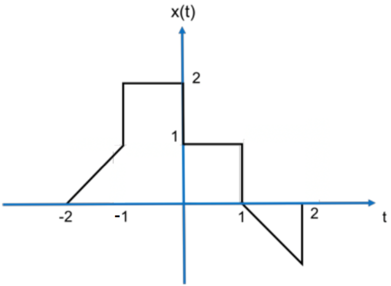


Clicker question 2

- Consider the continuous-time signal

$$x(t) = \delta(t+2) - \delta(t-2).$$

Determine the value of E_∞ for the signal

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau.$$

A. 0

B. 2

C. 4

D. 16

Clicker question 2

- Consider the continuous-time signal

$$x(t) = \delta(t+2) - \delta(t-2).$$

Determine the value of E_∞ for the signal

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau.$$

A. 0

B. 2

C. 4

D. 16

Class activity 1

- Consider a periodic signal

$$X(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. Sketch and label $x(t)$ and $\frac{dx(t)}{dt}$.

Class activity 1

- Consider a periodic signal

$$X(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. Sketch and label $x(t)$ and $\frac{dx(t)}{dt}$.

Class activity 2

- Show that DT periodic exponentials are periodic in frequency in addition to being periodic in time.

$$x[n] = e^{j\Omega_0 n} = e^{j(\Omega_0 + 2k\pi)n}$$

where $\Omega_0 \in \mathbb{R}$.

DT periodic exponentials are periodic in frequency

Periodicity in frequency:

- Suppose $x[n] = e^{j\Omega_0 n}$, where $\Omega_0 \in \mathbb{R}$.

$$e^{j(\Omega_0 + 2\pi)n} = e^{j\Omega_0 n} e^{j2\pi n}.$$

  Since $n \in \mathbb{Z}$, $e^{j2\pi n} = 1$, and

$$e^{j(\Omega_0 + 2\pi)n} = e^{j\Omega_0 n}.$$

  More generally, for any $k \in \mathbb{Z}$, we have

$$x_k[n] = e^{j(\Omega_0 + 2\pi k)n} = e^{j\Omega_0 n} = x[n].$$

DT periodic exponentials are periodic in frequency

- DT periodic exponentials are periodic in frequency in addition to being periodic in time.

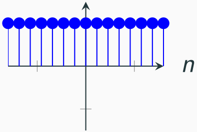
$$x[n] = e^{j\Omega_0 n} = e^{j(\Omega_0 + 2k\pi)n}$$

where $\Omega_0 \in \mathbb{R}$.

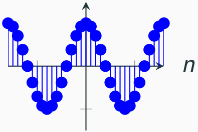
- The periodicity in frequency applies to the periodic complex exponential signals, so we have a different notion of low and high frequencies in the discrete-time setting.

DT periodic exponentials are periodic in frequency

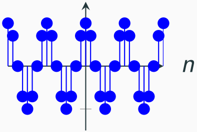
$x[n] = cos(0n)$



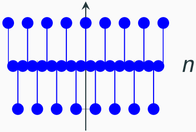
$x[n] = cos(\pi n/8)$



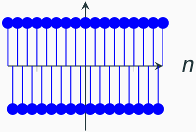
$x[n] = cos(\pi n/4)$



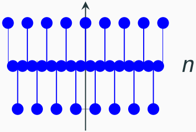
$x[n] = cos(\pi n/2)$



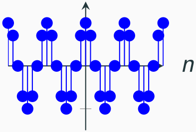
$x[n] = cos(\pi n)$



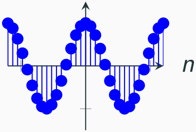
$x[n] = cos(3\pi n/2)$



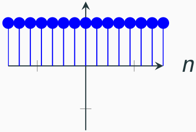
$x[n] = cos(7\pi n/4)$



$x[n] = cos(15\pi n/8)$



$x[n] = cos(2\pi n)$



Differences between CT and DT complex exponentials

1. CT complex exponential $e^{j\omega_0 t}$ is periodic for any ω_0 value, but its DT counterpart $e^{j\Omega_0 n}$ is only periodic if $\Omega_0 = (m2\pi)/N$, or Ω_0 is a rational multiple of 2π.

$$x[n] = e^{j\Omega_0 n}$$
$$x[n+N] = x[n]$$
$$\Rightarrow e^{j\Omega_0 N} = 1 = e^{jm2\pi}, m \in \mathbb{Z}$$

2. For $x(t) = e^{j\omega_0 t}$ as you increase ω_0, the rate of oscillations increases, however, $x[n] = e^{j\Omega_0 n}$ is periodic in frequency with period of 2π.

$$e^{j(\Omega_0+2\pi)n} = e^{j\Omega_0 n}e^{j2\pi n} = e^{j\Omega_0 n}$$

Class activity 3

- Determine if $x(t)$ is periodic or not. If it is periodic, find its fundamental frequency.

$$x(t) = \mathcal{E}v\{sin(3\pi t)u(t)\}$$

Class activity 4

- Determine if $x(t)$ is periodic or not. If it is periodic, find its fundamental frequency.

$$x(t) = \mathcal{O}dd\{sin(3\pi t)u(t)\}$$

MATLAB Class Demo

- Lecture 4 Demo
  - Visualizing CT complex exponential signals
  - DT versus CT sinusoidal signals
- You can download the MATLAB demo from the Canvas module.