ECE 45 – Circuits and Systems
Winter 2025
Homework #4

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Due: January 30 at 11:59 PM (via GradeScope)

1 Problem 1: Convolutions (50%)

Perform convolutions of the following functions, specifying the solution analytically and also sketching them by hand. Be aware that you may be able to compute some answers from other answers you already computed through the use of convolution properties. Make sure that each convolution clearly indicates the values for all values of t.

(a) $x(t) = e^{-2t}u(t), \quad h(t) = e^{-2t}u(t)$

The convolution integral is:

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Substituting $x(\tau) = e^{-2\tau}u(\tau)$ and $h(t-\tau) = e^{-2(t-\tau)}u(t-\tau)$:

$$(x * h)(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau.$$

Since $u(\tau)$ requires $\tau \geq 0$ and $u(t-\tau)$ requires $\tau \leq t$, the limits reduce to $0 \leq \tau \leq t$:

$$(x * h)(t) = \int_{0}^{t} e^{-2\tau}e^{-2(t-\tau)}d\tau.$$

Rewriting the exponentials:

$$e^{-2\tau}e^{-2(t-\tau)} = e^{-2t}e^{2\tau}.$$

Factoring out e^{-2t}:

$$(x * h)(t) = e^{-2t}\int_{0}^{t} e^{2\tau}d\tau.$$

Evaluating the integral:

$$\int e^{2\tau}d\tau = \frac{e^{2\tau}}{2} \quad \Rightarrow \quad \left[\frac{e^{2\tau}}{2}\right]_0^t = \frac{e^{2t}-1}{2}.$$

Multiplying by e^{-2t}:

$$(x * h)(t) = e^{-2t} \cdot \frac{e^{2t} - 1}{2} = \frac{1 - e^{-2t}}{2}, \quad t \geq 0.$$

Final Answer:

$$(x * h)(t) = \frac{1 - e^{-2t}}{2}, \quad t \geq 0.$$

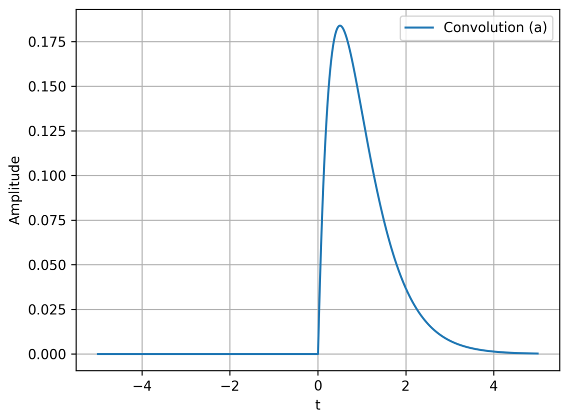

Figure 1: Convolution for part (a)

(b) $\quad x(t) = e^{-2t}u(t), \quad h(t) = e^{-2(t-1)}u(t-1)$

Substituting the given functions into the convolution integral:

$$(x * h)(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-2(t-\tau-1)}u(t-\tau-1)d\tau.$$

The unit step functions restrict the limits to $1 \leq \tau \leq t$, giving:

$$(x * h)(t) = \int_{0}^{t-1} e^{-2\tau}e^{-2(t-\tau-1)}d\tau.$$

Rewriting the exponentials:

$$e^{-2\tau}e^{-2(t-\tau-1)} = e^{-2(t-1)}e^{-2\tau}.$$

Factoring out $e^{-2(t-1)}$:

$$(x * h)(t) = e^{-2(t-1)} \int_{0}^{t-1} e^{-2\tau}d\tau.$$

Evaluating the integral:

$$\int e^{-2\tau}d\tau = \frac{e^{-2\tau}}{-2}\Big|_{0}^{t-1} = \frac{1 - e^{-2(t-1)}}{2}.$$

Multiplying by $e^{-2(t-1)}$:

$$(x * h)(t) = \frac{e^{-2(t-1)}(1 - e^{-2(t-1)})}{2}, \quad t \geq 1.$$

Final Answer:

$$(x * h)(t) = \frac{e^{-2(t-1)}(1 - e^{-2(t-1)})}{2}, \quad t \geq 1.$$

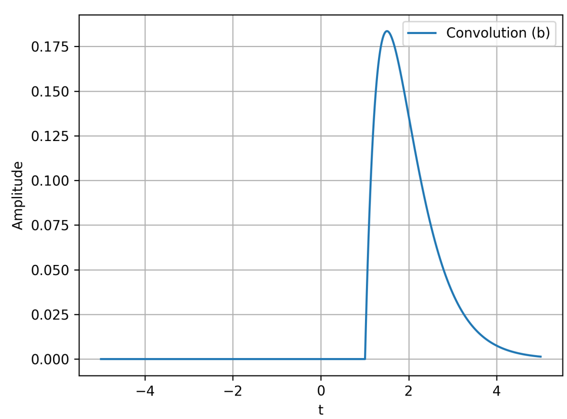

Figure 2: Convolution for part (b)

(c) $x(t) = \text{rect}(t - \frac{1}{2}), \quad h(t) = \text{rect}(2t)$

The convolution of two rectangular functions follows the property:

$$\text{rect}(t) * \text{rect}(t) = \text{tri}(t).$$

Thus,

$$(x * h)(t) = \frac{1}{2}\text{tri}(t - 1/2).$$

Final Answer:

$$(x * h)(t) = \frac{1}{2}\,\text{tri}(t - 1/2).$$

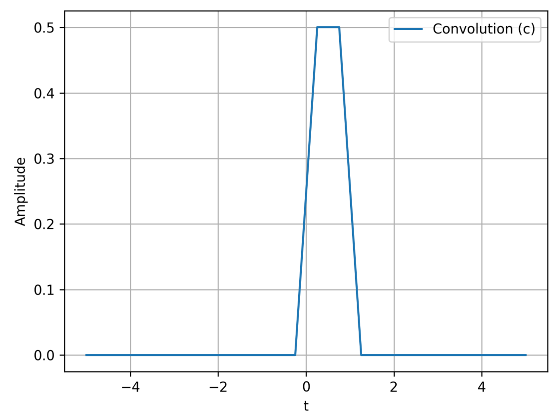

Figure 3: Convolution for part (c)

(d) $x(t) = e^{j\omega t}, \quad h(t) = e^{-\beta t}u(t)$

Taking the Fourier transform:

$$X(\omega) = 2\pi\delta(\omega - \omega_0), \quad H(\omega) = \frac{1}{\beta + j\omega}.$$

By the **convolution theorem**:

$$Y(\omega) = X(\omega)H(\omega) = 2\pi\delta(\omega - \omega_0) \cdot \frac{1}{\beta + j\omega}.$$

Since the delta function picks out ω_0:

$$(x * h)(t) = \frac{e^{j\omega t}}{\beta + j\omega}, \quad \text{for } \beta > 0.$$

Final Answer:

$$(x * h)(t) = \frac{e^{j\omega t}}{\beta + j\omega}, \quad \text{for } \beta > 0.$$

(e) $x(t) = e^{-2t}u(t), \quad h(t) = \text{rect}(t)$

The convolution integral:

$$(x * h)(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)\text{rect}(t-\tau)d\tau.$$

Since $\text{rect}(t-\tau)$ is 1 for $0 \leq t-\tau \leq 1$, the limits reduce to $0 \leq \tau \leq t+1$:

$$(x * h)(t) = \int_{0}^{t+1} e^{-2\tau}d\tau.$$

Evaluating the integral:

$$\int e^{-2\tau}d\tau = \frac{1-e^{-2(t+1)}}{2}.$$

Final Answer:

$$(x * h)(t) = \frac{1-e^{-2(t+1)}}{2}, \quad -1 \leq t \leq 1.$$

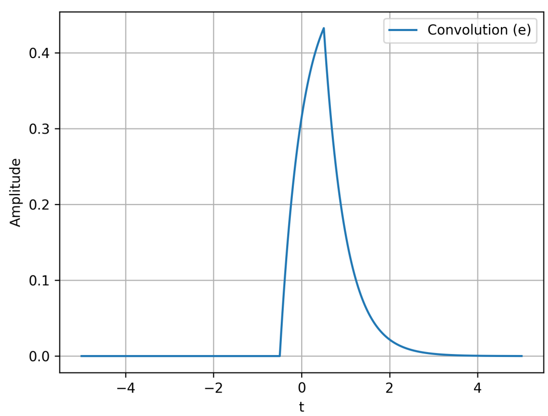

Figure 4: Convolution for part (e)

(f) $x(t) = e^{-2t}u(t), \quad h(t) = e^{-2t}u(t) + \text{rect}(t)$

By linearity:

$$(x * h)(t) = (x * e^{-2t}u(t)) + (x * \text{rect}(t)).$$

Using results from parts (a) and (e):

$$(x * h)(t) = \frac{1 - e^{-2t}}{2} + \frac{1 - e^{-2(t+1)}}{2}.$$

Final Answer:

$$(x * h)(t) = \frac{1 - e^{-2t}}{2} + \frac{1 - e^{-2(t+1)}}{2}, \quad -1 \leq t \leq 1.$$

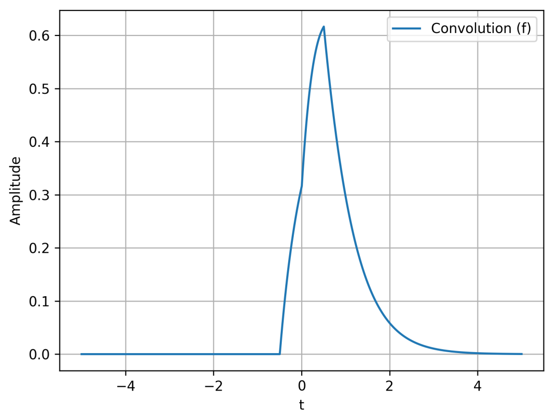

Figure 5: Convolution for part (f)

(g) $x(t) = \frac{1}{t^2+1}u(-t), \quad h(t) = u(t-1)$

The convolution integral:

$$(x * h)(t) = \int_{-\infty}^{1} \frac{1}{\tau^2+1} d\tau.$$

Using the standard integral:

$$\int \frac{1}{t^2+1} dt = \tan^{-1}(t).$$

Evaluating from $-\infty$ to 1:

$$\left[\tan^{-1}(\tau)\right]_{-\infty}^{1} = \tan^{-1}(1) - \tan^{-1}(-\infty).$$

Since $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(-\infty) = -\frac{\pi}{2}$:

$$(x * h)(t) = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}, \quad t \geq 1.$$

Final Answer:

$$(x * h)(t) = \frac{3\pi}{4}, \quad t \geq 1.$$

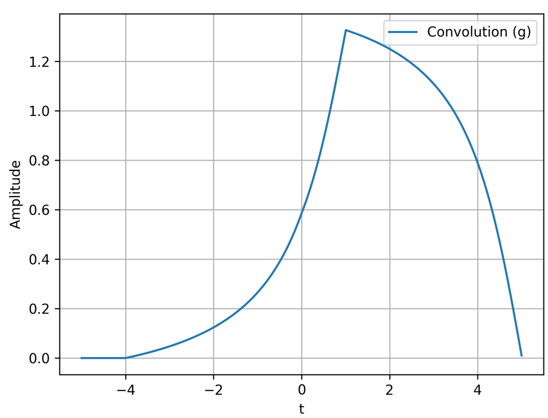

Figure 6: Convolution for part (g)

2 Problem 2: Additional Convolutions (20%)

Consider the function $x(t) = \text{rect}(t - \frac{1}{2})$. With $*$ denoting convolution, compute the following convolutions and sketch them by hand.

(a) $(x * x)(t)$

The convolution integral:

$$(x * x)(t) = \int_{-\infty}^{\infty} \text{rect}(\tau - \frac{1}{2})\text{rect}(t - \tau - \frac{1}{2})d\tau.$$

Since the convolution of two rectangular functions is a triangular function:

$$(x * x)(t) = \text{tri}(t - 1).$$

Final Answer:

$$(x * x)(t) = \text{tri}(t - 1).$$

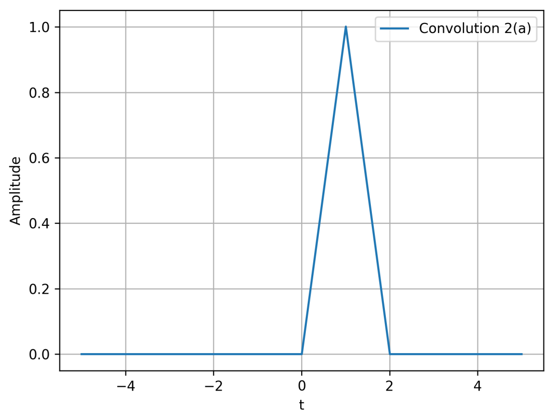

Figure 7: Convolution for part (a)

(b) $(x * x(-t))(t)$

The convolution integral:

$$(x * x(-t))(t) = \int_{-\infty}^{\infty} \text{rect}(\tau - \frac{1}{2})\text{rect}(-t + \tau - \frac{1}{2})d\tau.$$

Since $x(-t)$ is a flipped version of $x(t)$, the result remains the same:

$$(x * x(-t))(t) = \text{tri}(t - 1).$$

Final Answer:

$$(x * x(-t))(t) = \text{tri}(t - 1).$$

Note: This is known as the **autocorrelation** of a signal, which is used in communications. Explain why it is interesting.

Autocorrelation is interesting because it measures how well a signal matches a time-shifted version of itself. It helps analyze periodicity, detect repeating patterns, and is commonly used in radar, speech recognition, and error detection systems.

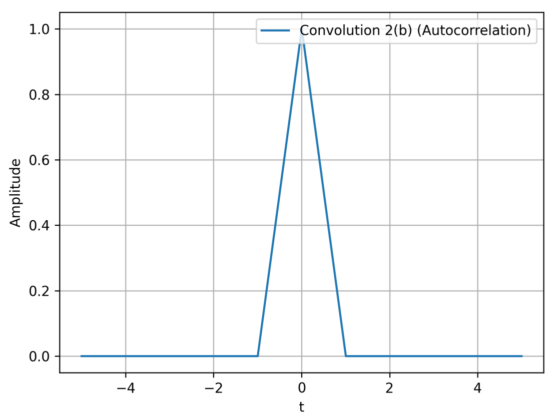

Figure 8: Convolution for part (b)

(c) $(x * x(t-1))(t)$

Since shifting in the convolution domain results in shifting in the time domain:

$$(x * x(t-1))(t) = \text{tri}(t-2).$$

Final Answer:

$$(x * x(t-1))(t) = \text{tri}(t-2).$$

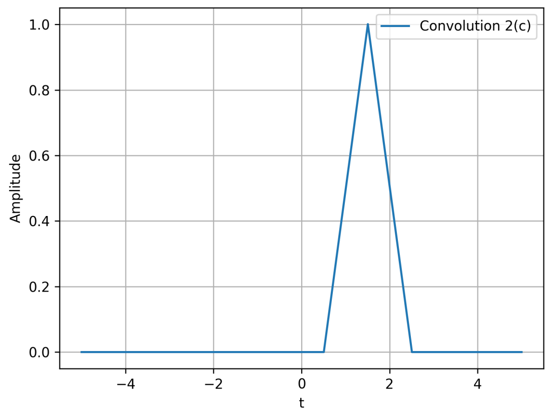

Figure 9: Convolution for part (c)

(d) $(x * \delta(t-2))(t)$

Since convolving with a delta function shifts the function:

$$(x * \delta(t-2))(t) = x(t-2).$$

Final Answer:

$$(x * \delta(t-2))(t) = \mathrm{rect}(t - \frac{5}{2}).$$

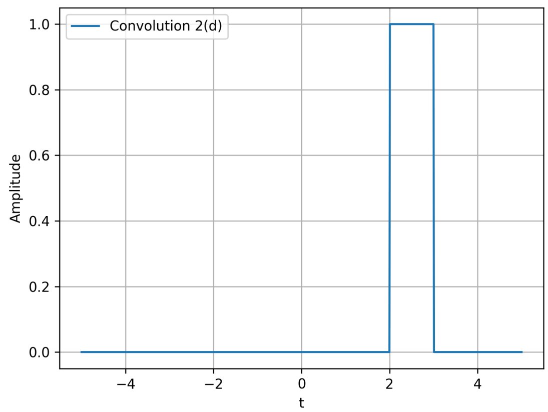

Figure 10: Convolution for part (d)

(e) $(x * \delta(t-2) * x(t+1))(t)$

By associativity of convolution:

$$(x * \delta(t-2) * x(t+1))(t) = (x * x)(t-2).$$

Using the result from (a):

$$(x * x)(t-2) = \mathrm{tri}(t-3).$$

Final Answer:

$$(x * \delta(t-2) * x(t+1))(t) = \mathrm{tri}(t-3).$$

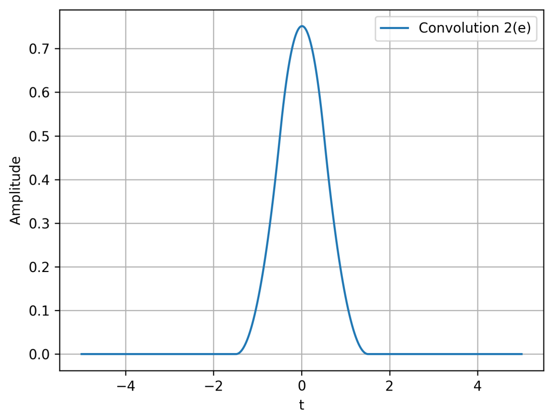

Figure 11: Convolution for part (e)

(f) $[(x + x(t-2)) * x](t)$

By linearity:

$$(x * x)(t) + (x(t-2) * x)(t).$$

Using the result from (a):

$$\text{tri}(t-1) + \text{tri}(t-3).$$

Final Answer:

$$(x + x(t-2)) * x = \text{tri}(t-1) + \text{tri}(t-3).$$

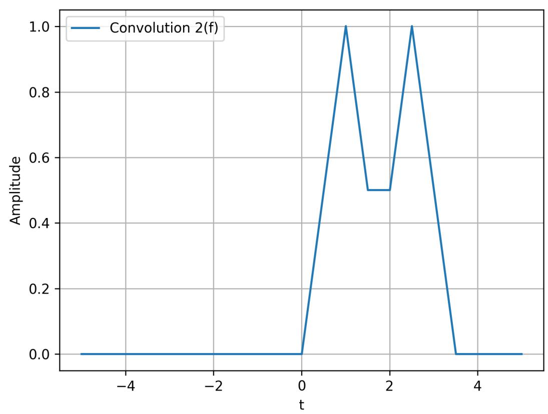

Figure 12: Convolution for part (f)

3 Problem 3: Associativity of Convolution (10%)

Prove the equality:

$$[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)]$$

by showing that both sides of the equation equal:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)d\tau d\sigma$$

Proof: By the definition of convolution, we first compute the left-hand side:

$$[y * g](t) = \int_{-\infty}^{\infty} y(\lambda)g(t-\lambda)d\lambda,$$

where

$$y(t) = [x * h](t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Substituting $y(t)$:

$$([x * h] * g)(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)h(\lambda-\tau)d\tau\right] g(t-\lambda)d\lambda.$$

Rearrange the order of integration:

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(\lambda-\tau)g(t-\lambda)d\lambda\right] d\tau.$$

Recognizing the inner integral as a convolution:

$$= \int_{-\infty}^{\infty} x(\tau)[h * g](t-\tau)d\tau.$$

Thus, we obtain:

$$[x * h] * g = x * [h * g].$$

Since both sides are equal, convolution is associative.

4 Problem 4: Corrections (20%)

Based on the solutions, correct your previous week's homework using a colored pen (or annotation) so it's obvious what you've corrected. If you got a problem exactly right, just use a red check mark to indicate it.

HW Corrections attached on the next page.

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4) (20%) HW Corrections

1)

a) $(1 + v + v^2)\delta(t)$

$(1 + v + v^2)\delta(t) = (1 + 0 + 0^2)\delta(t) = \delta(t)$ **CORRECT**

b) $\int_{-\infty}^{\infty} \delta(t - 1)x(1 - t)dt$

$\int_{-\infty}^{\infty} \delta(t - a)f(t)dt = f(a)$

$\int_{-\infty}^{\infty} \delta(t - 1)x(1 - t)dt = x(1 - 1) = x(0)$ **CORRECT**

c) $\int_{-\infty}^{\infty} \delta(t - 4)\frac{sin(\pi t^2)}{\pi t^2}dt = \frac{sin(\pi 4^2)}{\pi 4^2} = \frac{sin(16\pi)}{16\pi} = 0$ **CORRECT**

d) $\sum_{n=0}^{\infty} (t + 1)^n\delta(t)$

$(t + 1)^n = (0 + 1)^n = 1$ for all n, when substituting $t = 0$.

$\sum_{n=0}^{\infty} (t + 1)^n\delta(t) = \sum_{n=0}^{\infty} \delta(t)$ **CORRECT (assigned homework had limit to infinity, not 10)**

e) $f(t) = \int_{-\infty}^{t} \delta(\tau - 3)d\tau$

$\int_{-\infty}^{t} \delta(\tau - 3)d\tau = \{0\ if\ t < 3,\ 1\ if\ t \geq 3\}$

$f(t) = u(t - 3)$ **CORRECT**

f) $sin(2\pi t)\delta(\frac{1}{2} - 2t)$

$\frac{1}{2} - 2t = 0$ $\quad t = \frac{1}{4}$

$\delta(at - b) = \frac{1}{|a|}\delta(t - \frac{b}{a})$ $\quad a = -2, b = \frac{1}{2}$

$sin(2\pi t) = sin(2\pi * \frac{1}{4}) = sin(\frac{\pi}{2}) = 1$

$\frac{1}{2}\delta(t - \frac{1}{4})$ **CORRECT**

g) $\int_{-\infty}^{\infty} (\frac{du(t)}{dt} - rect(t))dt$

$\frac{du(t)}{dt} = \delta(t)$

$\int_{-\infty}^{\infty} rect(t)dt = 1$

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$$\int_{-\infty}^{\infty} \left(\frac{du(t)}{dt} - rect(t)\right)dt = 1 - 1 = 0$$ **CORRECT**

2) Determine Time Invariance/Linearity:
   a) $y(t) = 2x(t - 3)$
      i) When we shift the input, the output also shifts:
      $x(t) \rightarrow x(t - t_0)$ $\quad y(t) \rightarrow 2x((t - t_0) - 3) = 2x(t - t_0 - 3)$
      If we shift the output $y(t) \rightarrow y(t - t_0)$, we get the same answer, so the system **is time-invariant.** **CORRECT**
      ii) The system satisfies additivity $y(t) = x(t - 3) + x(t - 3)$ and homogeneity (scaling both by constant k), therefore, **is linear.** **CORRECT**
   b) $y(t) = \int_{-\infty}^{t} x(\gamma)d\gamma$
      i) When we shift the input, the output also shifts:
      $x(t) \rightarrow x(t - t_0)$ $\quad y(t) = \int_{-\infty}^{t} x(\gamma - t_0)d\gamma$
      When we shift the output, however, we get $\int_{-\infty}^{t-t_0} x(\gamma)d\gamma$.
      The limits of integration are different and therefore do not match, so the system **is not time-invariant.** **INCORRECT, the equations are equivalent.**
      ii) The system satisfies additivity $\int_{-\infty}^{t} [x_1(\gamma) + x_2(\gamma)]d\gamma$ and homogeneity (scaling both by constant k), therefore, **is linear.** **CORRECT**
   c) $y(t) = Re\{x(t)\}$
      i) $Re\{x(t - t_0)\}$ matches $y(t) = Re\{x(t - t_0)\}$, therefore the system **is time-invariant.** **CORRECT**
      ii) The system satisfies additivity $Re\{x_1(t) + x_2(t)\}$ and homogeneity (scaling both by constant k), therefore, **is linear.** **INCORRECT, not linear when scaling factor is complex.**
   d) $y(t) = x(t - 2) + x(2 - t)$
      i) Both shifted match at $y(t) = x((t - t_0) - 2) + x(2 - (t - t_0))$, system **is time-invariant.** **CORRECT**
      ii) Additivity is already included in the $x(t - 2) + x(2 - t)$ and homogeneity holds, system **is linear.** **CORRECT**
   e) $y(t) = log_2(1 + |x(t)|^2)$
      i) Both match at $y(t) = log_2(1 + |x(t - t_0)|^2)$, system **is time-invariant.**
      CORRECT

ii) Logarithmic and magnitude-squared operations are not linear, therefore the system **is not linear.** **CORRECT**

f) $y(t) = cos(x(t))$

i) Both shifts match at $y(t) = cos(x(t - t_0))$. System **is time-invariant.** **CORRECT**

ii) Cosine operation is not additive (same wave) and it is not homogeneous as scaling x(t) does not scare the system linearly. Therefore this system **is not linear.** **CORRECT**

g) $y(t) = \{0\ if\ x(t) < 1, \int_0^1 x(t - \tau)d\tau\ if\ x(t) \geq 0\}$

i) The thresholds mean that the behavior is dependent on input, therefore the system **is not time-invariant.** **INCORRECT, the system is time-invariant.**

ii) The piecewise definition and integration means this system **is not linear.** **CORRECT**