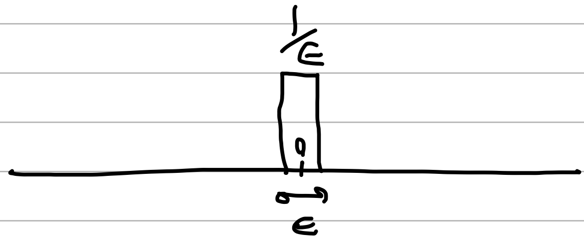
Discussion Session 2: Dirac Delta Function, linear, and time invariant systems.

The dirac delta function: limit of a sequence of functions that become increasing narrow and tall while maintaining an area of 1 under curve. Common approximation of the Dirac Delta function is rectangular pulse:



$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \operatorname{Rect}\left(\frac{t}{\epsilon}\right)$$

properties:

1) zero every where except at $t=0$: $\delta(t) = 0$, for all $t \neq 0$

2) Integral equal one: $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

3) Shifting Effect: $\int_{-\infty}^{\infty} f(t)\, \delta(t-t_0)\, dt = f(t_0)$

Importance:

The input $\delta(t)$ to a linear System yields the system's impulse response which characterizes the system completely.

Example:

a) $\int_{-\infty}^{\infty} (2t+3)\, \delta(t-2)\, dt =$

b) $\int_{-\infty}^{\infty} \cos(2\pi t) \left[\delta(t-1) + \delta(t+1) \right]$

c) $\int_{-\infty}^{\infty} \delta(at)\, dt$

$$\delta(a.x) = \frac{1}{|a|}\delta(x)$$

d) $\int_{-\infty}^{\infty} e^{-t^2}\, \delta(3t-6)\, dt$

Time - Invariant vs. Time - Variant Systems:

A system is time-invariant if its behavior does not change over time.

Mathematically $y(t) = \text{System}[x(t)]$ implies $y(t-t_0) = \text{System}[x(t-t_0)]$

Shifting the input $x(t)$ by t_0 results in the output $y(t)$ also being shifted by t_0.

A system is time-variant if its behavior depend on time, meaning a time shift in the input does not result in the same shift in the output.

How to check?

1) Shift the input signal $x(t)$ by t_0 : Replace $x(t)$ with $x(t-t_0)$
2) Pass the shifted input through the system and find new output $y'(t)$
3) Shift the original output $y(t)$ to get $y(t-t_0)$
4) compare $y'(t)$ and $y(t-t_0)$

Linear vs. Non Linear Systems:

A system is linear if it satisfies two properties.

1) Additivity: the response to sum of inputs equals the sum of the response to each input:

If $y_1(t) = System[x_1(t)]$ and $y_2(t) = System[x_2(t)]$
then $System[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$

2) Homogenity: The response to a scaled input is the scaled response:
$y(t) = System[x(t)]$, then $System[a \cdot x(t)] = a \cdot y(t)$

How to check?

1) Additivity: if $y_1(t) = System[x_1(t)]$ and $y_2(t) = System[x_2(t)]$, then $S[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$

2) Homogeneity (Scaling): If $y(t) = S[x(t)]$, then $S[a \cdot x(t)] = a \cdot y(t)$

Example:

a) $y(t) = x(2-t) + 1$

time-invariance:

b) $y(t) = t\sin(x(t))$

c) $y(t) = \int_{t}^{t+2} x(\tau)\, d\tau$

d) $y(t) = \int_{t-1}^{t+1} (t-\tau)\, x(\tau)\, d\tau$