Lecture 16

Introduction to bode plots

Preview of today's lecture

- LTI systems in the frequency domain
  - LTI systems act as filters on signals
  - Frequency response of a system characterizes the filter

- Introduction to Bode plots
  - Bode plots are a way to sketch the frequency response of a system
  - Plots are constructed with careful use of logarithms

- Bode plot of an RC filter
  - Bode plots are relevant for systems described by LCCDEs
  - Bode plots are built from the understanding of first order systems like those described by RC circuits

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 16 Fourier transform

Fourier

LTI systems in the frequency domain

Key points

- LTI systems act as filters on signals
- Frequency response of a system characterizes the filter

Linear and time invariant systems

- Fully characterized by their impulse response

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow h(t)$$

- Output computed from the impulse response for any input

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution

Response to a complex sinusoid

- Frequency response is used to characterize LTI systems

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

- The frequency response is computed from the impulse response

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

- This is the Fourier transform of the impulse response of the system

$$h(t) \overset{FT}{\longleftrightarrow} H(j\omega)$$

Specialization to real signals

- For any complex sinusoid

$$e^{j\omega_0 t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

- For the special case of real impulse sponses

$$\cos(\omega_0 t) \longrightarrow \boxed{H(j\omega)} \longrightarrow |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$$

magnitude and phase are important

Response to a periodic signal

- Periodic signals can be represented as a sum of harmonics of complex sinusoids with fundamental frequency

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- For a periodic signal the output of an LTI system looks like

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \longrightarrow \boxed{H(j\omega)} \longrightarrow \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt}$$

multiples of the fundamental frequency

Response to a periodic signal

- Aperiodic signals can be represented as a linear combination of complex sinusoids via the Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

- For a periodic signal, the output of an LTI system looks like

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) X(j\omega) e^{j\omega t} d\omega$$

Thinking about LTI systems as filters

- LTI systems in time domain: convolution with an impulse response

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- LTI systems in frequency domain: mult. with frequency response

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Can think of LTI systems in time or frequency domains

Introduction to Bode plots

Key points

- Bode plots are a way to sketch the frequency response of a system
- Plots are constructed with careful use of logarithms

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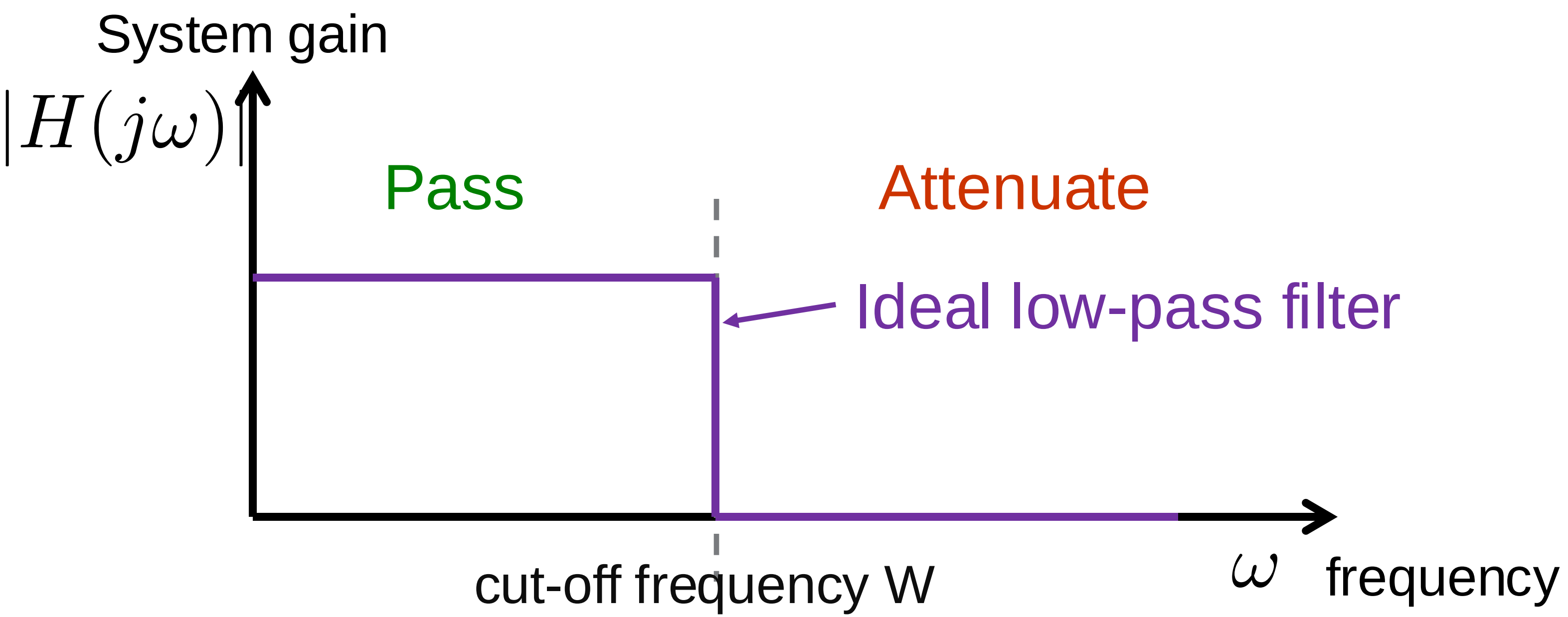
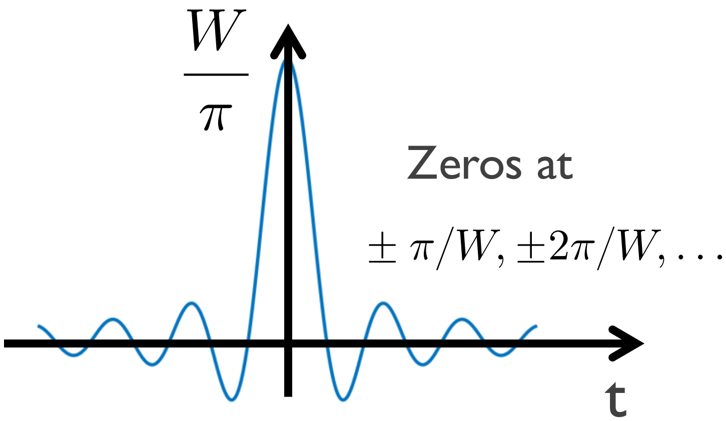
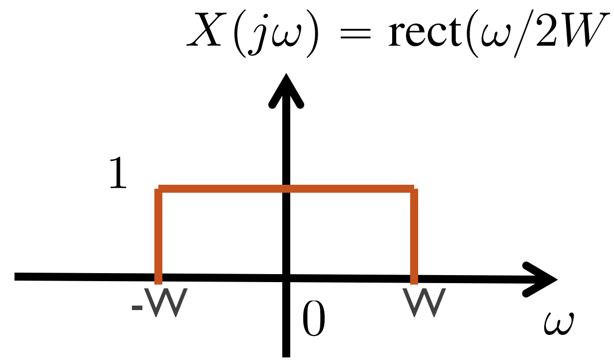
Recall the ideal low-pass filter



Systems passes low frequencies, attenuates high frequencies

Ideal low-pass filters



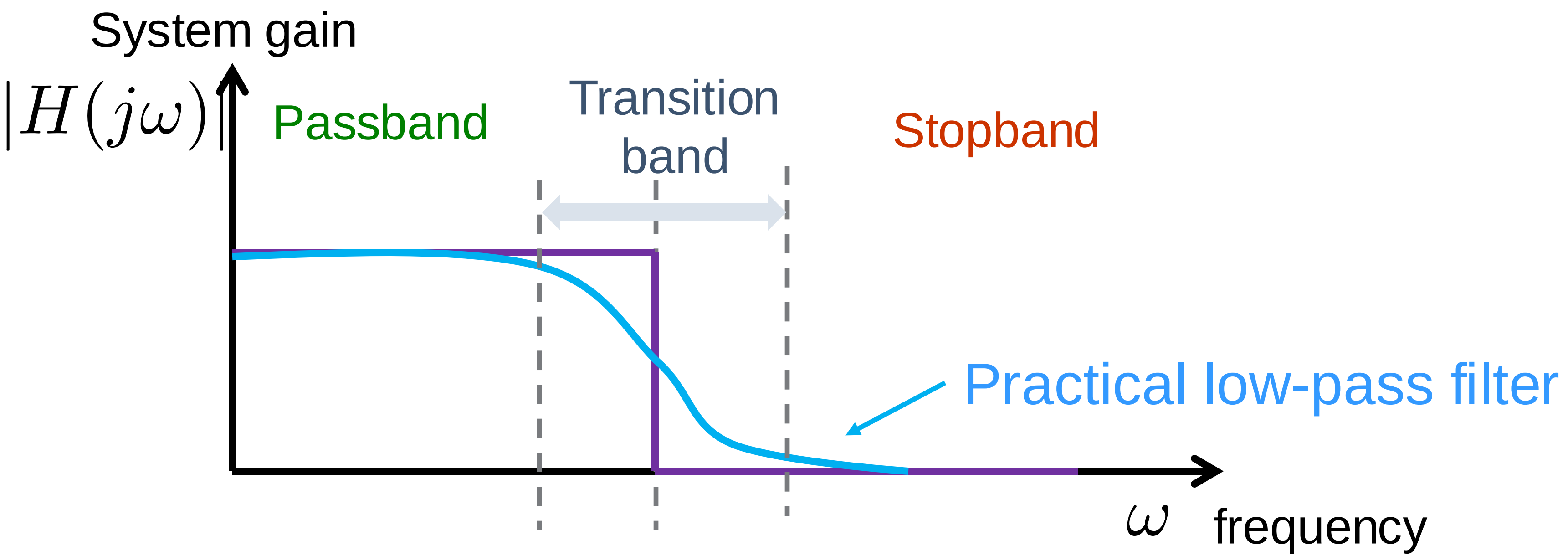


$$\frac{W}{\pi}\mathrm{sinc}\left(\frac{tW}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2W}\right)$$

Ideal low-pass filter is a sinc function

Practical low-pass filters

- Systems that pass low frequencies, attenuate high frequencies



Practical filters make some compromises

Filtering using circuits

- RLC circuits have systems that are described by linear constant coefficient differential equations

- Such systems have a frequency response of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \Rightarrow \quad \boxed{H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}}$$

Bode plots

- Bode plots are a way to sketch the frequency response

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

- Magnitude function

$$|H(j\omega)|$$

Used as an alternative or complement to computer-based simulation

- Phase function

$$\angle H(j\omega)$$

Works with LCCDE frequency responses

Digression on logs

$$c = \log_b a \;\leftrightarrow\; b^c = a$$

- Most common in signals and systems
  - Natural log $\ln = \log_e$ and log base 10 $\log_{10}$
- What is neat about logs?
  - Huge numbers → small positive numbers

$$\log_{10} \underbrace{10,345,034,896}_{\approx 10^{10}} = 10$$

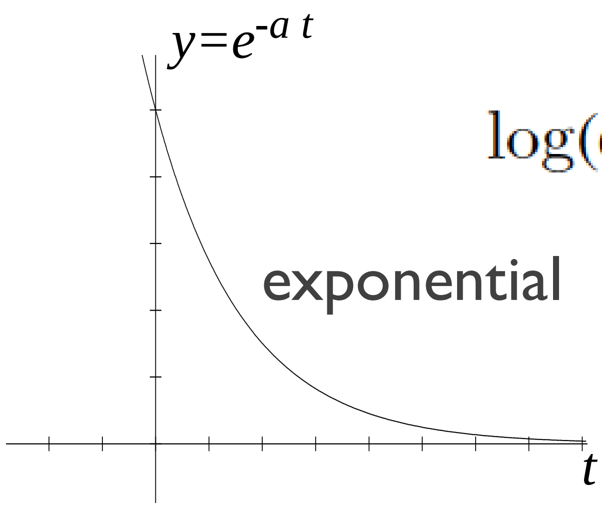
  - Tiny numbers → small negative numbers
  - Multiplication → addition
  - Division → subtraction
- Used to compute amplifier gain, antenna gain, losses, etc.

Napier

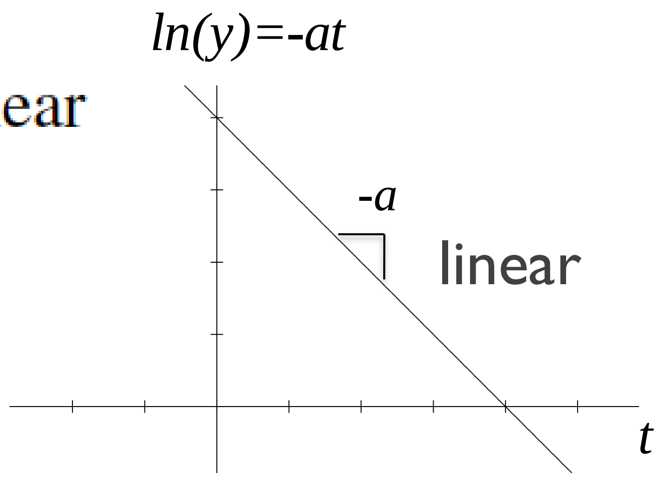
$$\log xy = \log x + \log y$$
$$\log x^y = y \log x$$
$$\log_b b = 1$$
$$\log_b 1 = 0$$
$$\log_b 0 = \text{undefined}$$

Relationship between logarithms and exponentials

$$\log a^x = x \log a$$

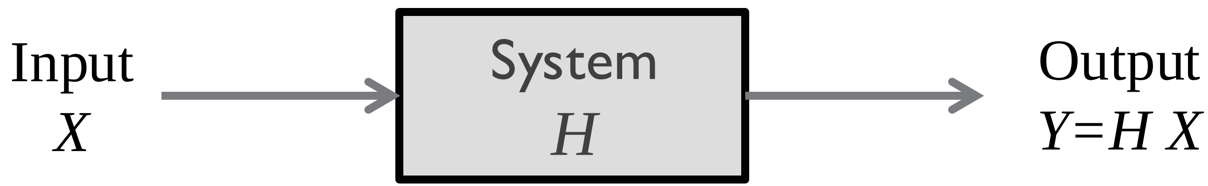


log(exponential) $\rightarrow$ linear



Use of logarithms to measure gain

- The Decibel (Deci = ten, bel = “Bell Labs”)



- General rule
  - Use $H\ [\mathrm{dB}] = 10\log_{10}|H|^2$ dealing with power
  - Use $H\ [\mathrm{dB}] = 20\log_{10}|H|$ dealing with amplitude

$$\mathrm{Gain}[\mathrm{dB}] = 10\log_{10}\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}$$

decibels are a unitless ratio of **powers**

Example dB calculation

- Determine the gain (in dB) of the an amplifier with linear gain

- H = 20

$$
\begin{aligned}
20\log_{10}20 &= 20\log_{10}2(10)\\
&= 2\cdot 10\log_{10}2 + 2\cdot 10\log_{10}10\\
&= 26\text{dB}
\end{aligned}
$$

- H = 0.1

$$20\log_{10}0.1 = 20(-1) = -20\text{dB}$$

- H = 50

$$
\begin{aligned}
20\log_{10}50 &= 20\log_{10}5(10)\\
&= 20\log_{10}5 + 20\log_{10}10\\
&= 14\text{dB} + 20\text{dB}\\
&= 34\text{dB}
\end{aligned}
$$

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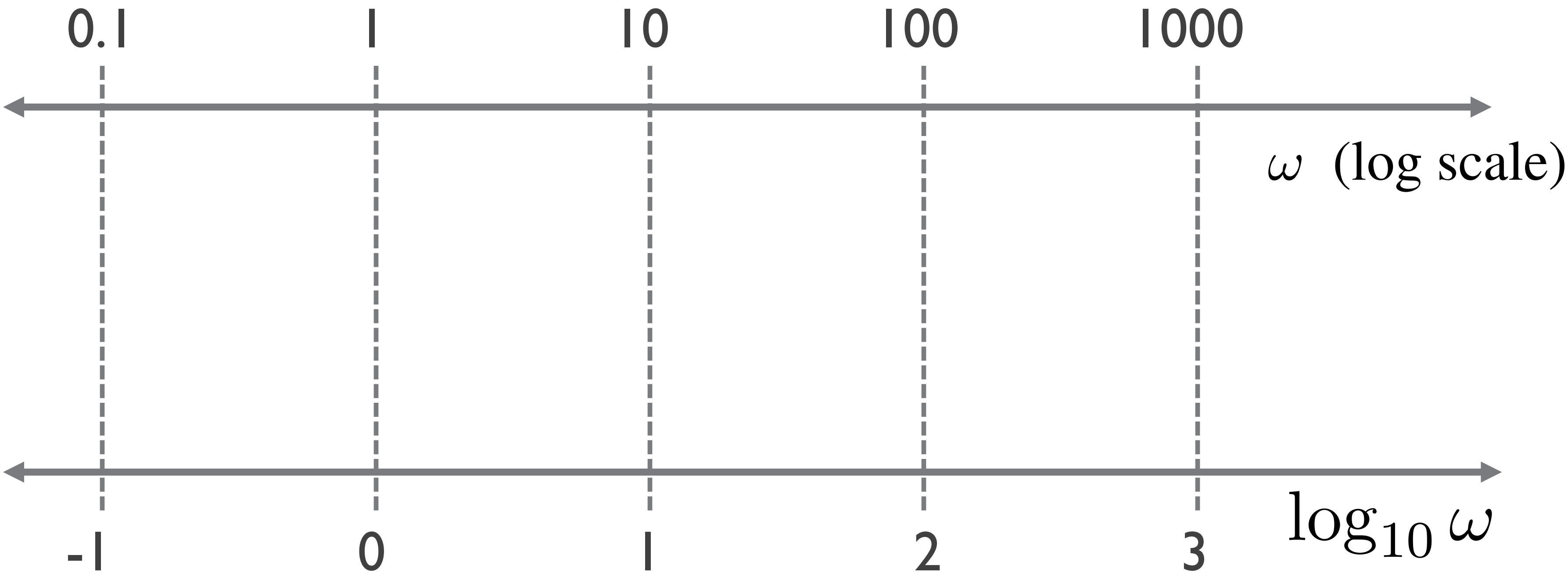
Aside about the log scale



Equivalent but the numbers are easier to read on the log scale

Back to Bode plots

- A Bode plot is a collection of two plots
- The x-axis is "semi-log x" (effectively plots $\log_{10}(\omega)$)
- The magnitude plot approximates

$$|H(j\omega)|_{\text{dB}} = 20\log_{10}|H(j\omega)|$$

- The phase plot approximates

$$\angle H(j\omega)$$

(recall this is implemented with arctan2)

Summary

- Bode plots are convenient ways to illustrate the frequency response of a filter in terms of the magnitude and the phase
- Magnitude is measured using the decibel system building upon base-ten logarithms with a 20 x (amplitude) or 10 x (power) factor
- X-axis is frequency in radians but plotted in a log scale (this is called semilogx in MATLAB)

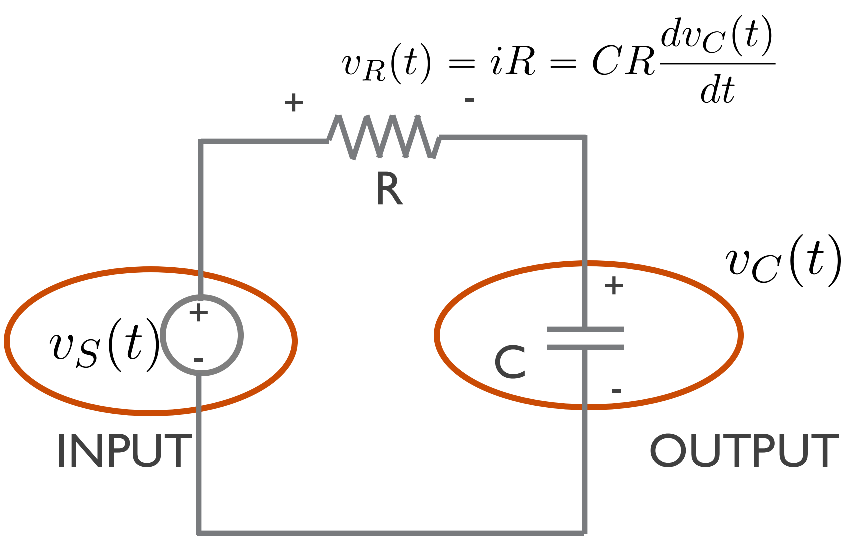
Bode plot of an RC filter

Key points

- Bode plots are relevant for systems described by LCCDEs
- Bode plots are built from the understanding of first order systems like those described by RC circuits

Recall the RC system

- Source voltage as the input
- Capacitor voltage as the output



Current through capacitor

$$i = C\frac{dv_C(t)}{dt}$$

Resulting differential equation

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

Finding the frequency response

- Think of this circuit as an LTI system

$$v_S(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow v_C(t)$$

- The frequency response of this system described by an LCCDE

$$\boxed{H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}} \quad \Rightarrow \quad H(j\omega) = \frac{1}{1 + jRC\omega}$$

Computing the magnitude

$$|H(j\omega)|_{\mathrm{dB}} = 20\log_{10}|H(j\omega)|$$

$$|H(j\omega)| = \frac{1}{|1+j\omega RC|}$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$= \frac{1}{\sqrt{1+(\omega/\omega_0)^2}}$$

where the cutoff frequency is

$$\omega_0 = \frac{1}{RC}$$

Now going to decibels

- Inputting into the log function

$$
\begin{aligned}
|H(j\omega)|_{dB} &= 20\log_{10}\frac{1}{\sqrt{1+(\omega/\omega_0)^2}} \\
&= 20\log_{10}1 - 20\log_{10}\sqrt{1+(\omega/\omega_0)^2} \\
&= -20\log_{10}\sqrt{1+(\omega/\omega_0)^2}
\end{aligned}
$$

- Summarizing the final expression

$$|H(j\omega)|_{dB} = -20\log_{10}\sqrt{1+(\omega/\omega_0)^2}$$

Asymptotic regimes for small frequencies

$$|H(j\omega)|_{dB} = -20\log_{10}\sqrt{1+(\omega/\omega_0)^2}$$

- For small frequencies

$$\omega \ll \omega_0 \Rightarrow \left(\frac{\omega}{\omega_0}\right)^2 \approx 0$$

- Incorporating this assumption

$$\begin{aligned}|H(j\omega)|_{dB} &= -20\log_{10}\sqrt{1+0}\\ &= -20\log_{10}1 = 0\end{aligned}$$

Asymptotic regimes for large frequencies

- For large frequencies

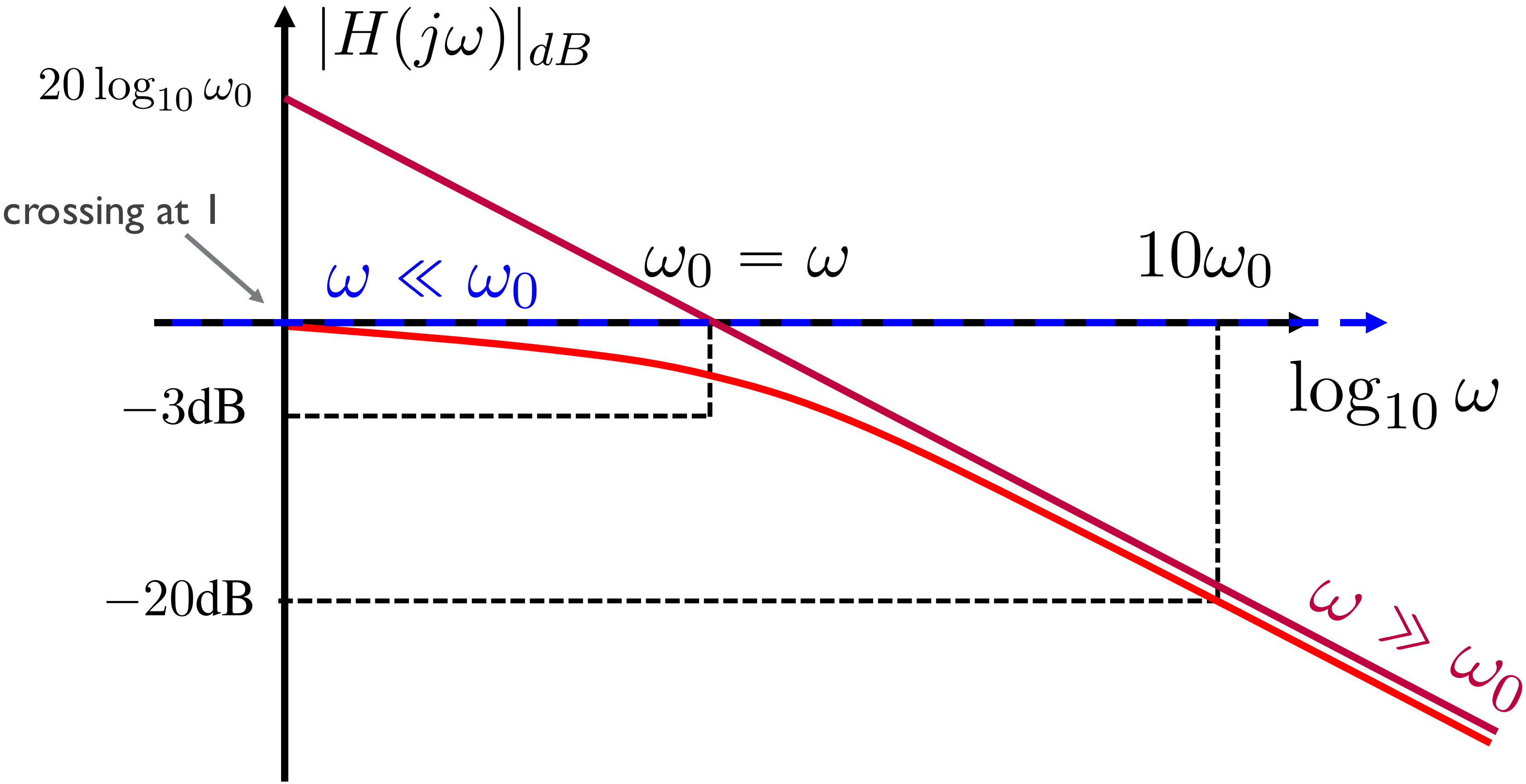
$$\omega \gg \omega_0 \Rightarrow 1 + \left(\frac{\omega}{\omega_0}\right)^2 \approx \left(\frac{\omega}{\omega_0}\right)^2$$

- Incorporating this assumption

$$\begin{aligned}|H(j\omega)|_{dB} &= -20\log_{10}\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \\ &\approx -20\log_{10}\sqrt{\left(\frac{\omega}{\omega_0}\right)^2} \\ &= -20\log_{10}\frac{\omega}{\omega_0}\end{aligned}$$

$$\Rightarrow \quad |H(j\omega)|_{dB} = 20\log_{10}\omega_0 - 20\log_{10}\omega$$

Magnitude plot for a low-pass filter



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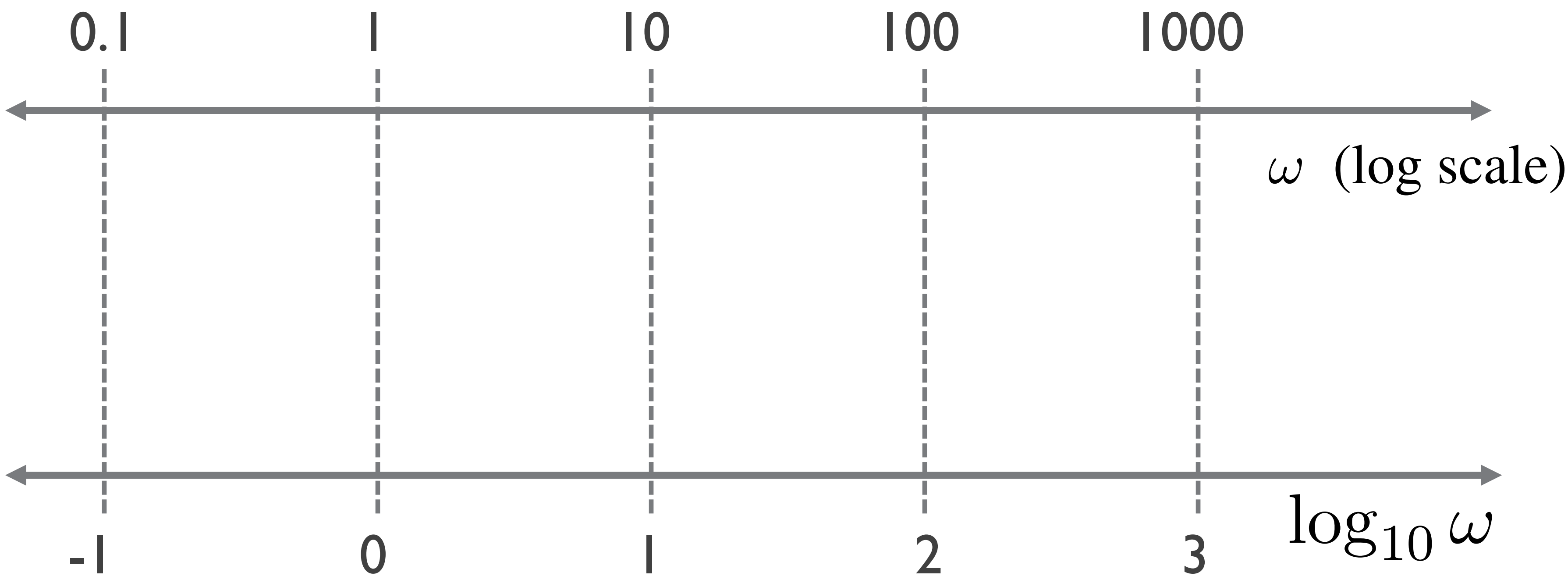
Aside about the log scale

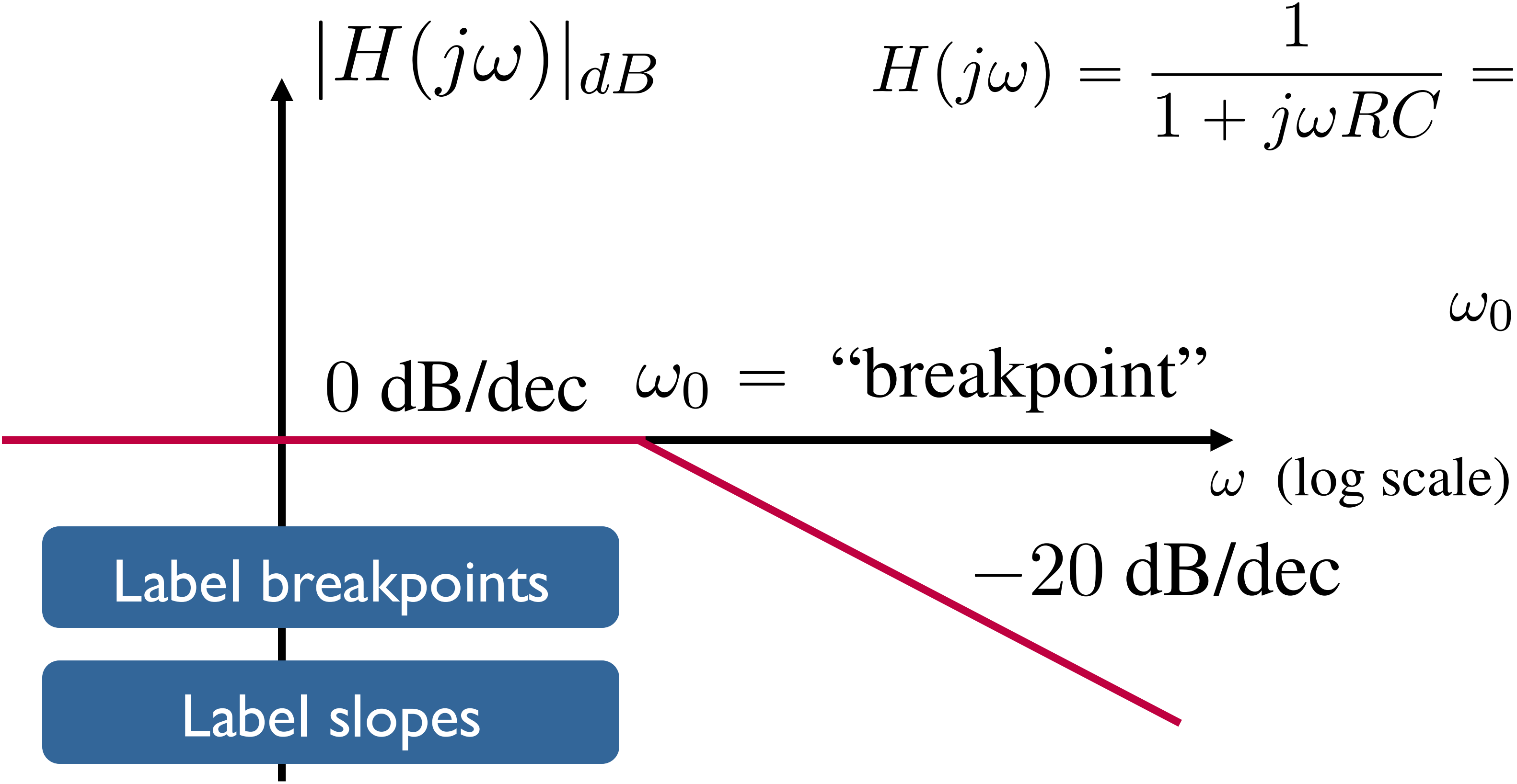


Equivalent but the numbers are easier to read on the log scale

Bode magnitude plot of the low-pass filter

$$H(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\frac{\omega}{\omega_0}}$$

$$\omega_0 = \frac{1}{RC}$$



Bode phase of the low-pass filter

$$H(j\omega) = \frac{1}{1 + jRC\omega}$$

- Computing the phase

$$\begin{aligned}\angle H(j\omega) &= \text{atan2}(0, 1) - \text{atan2}(\omega RC, 1)\\ &= 0 - \tan^{-1}(\omega RC)\end{aligned}$$

- For large frequencies

$$\lim_{\omega\to\infty} - \tan^{-1}(\omega RC) = -\frac{\pi}{2}$$

- At the critical frequency

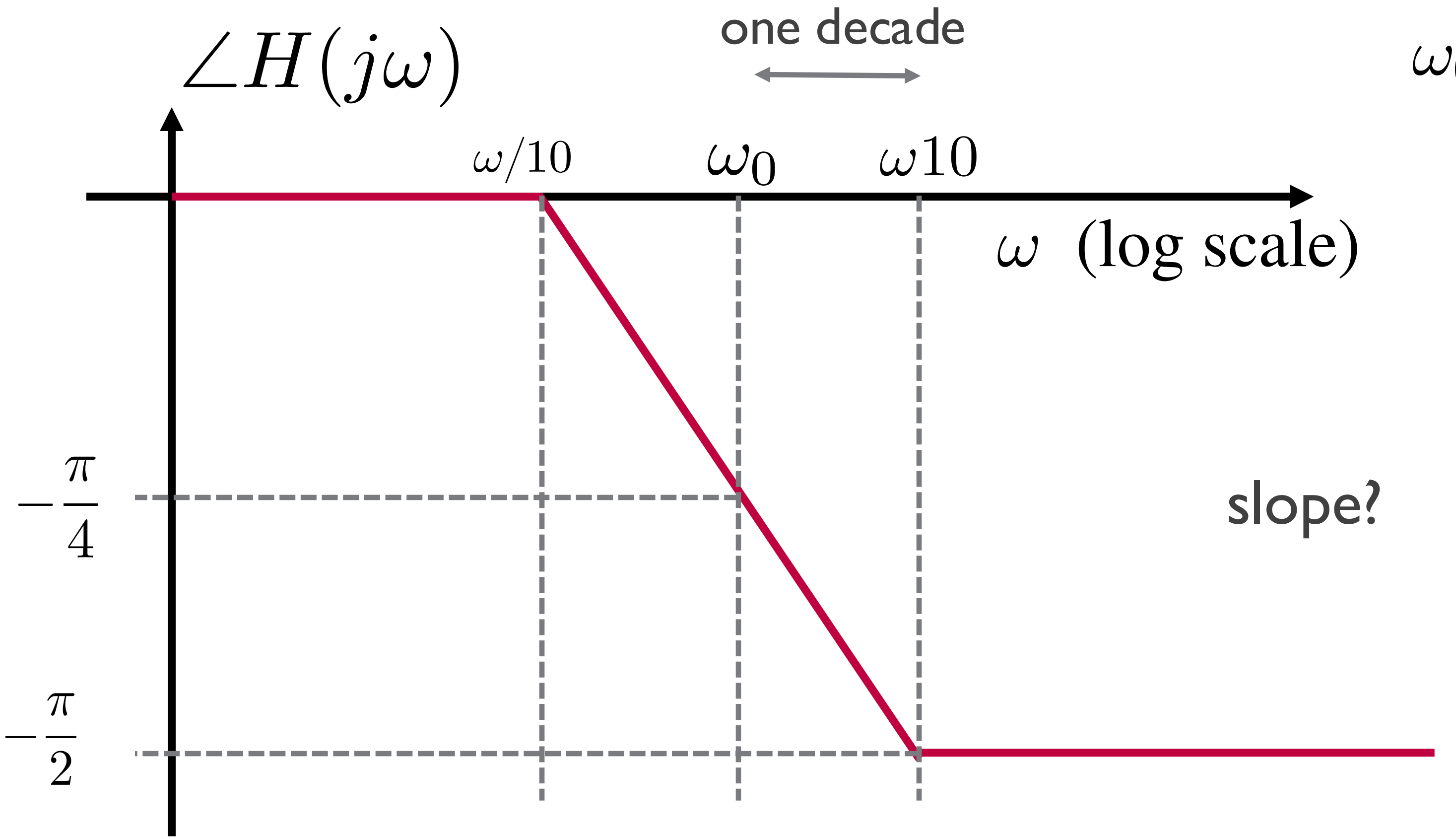
$$\omega = \omega_0 \Rightarrow -\tan^{-1}(1) = -\frac{\pi}{4}$$

- For small frequencies

$$\omega = 0 \Rightarrow -\tan^{-1}(0) = 0$$

Bode phase plot of the low-pass filter

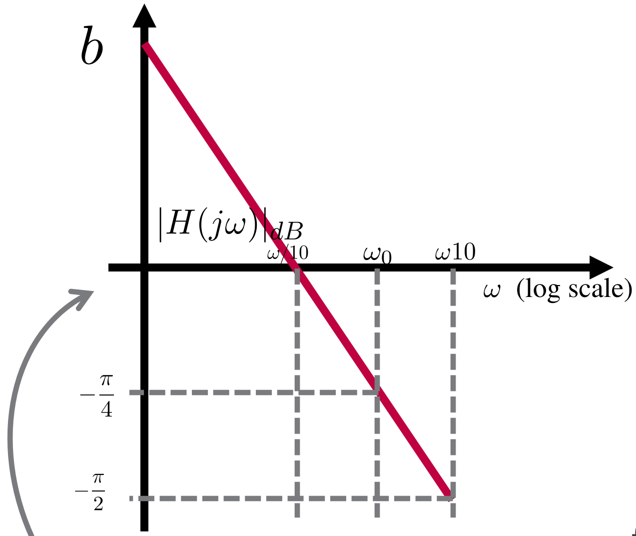
$$\omega_0 = \frac{1}{RC}$$



slope?

Finding the slope

$$y = mx + b$$



zero point

$$\log_{10} 1 = 0$$

finding the slope

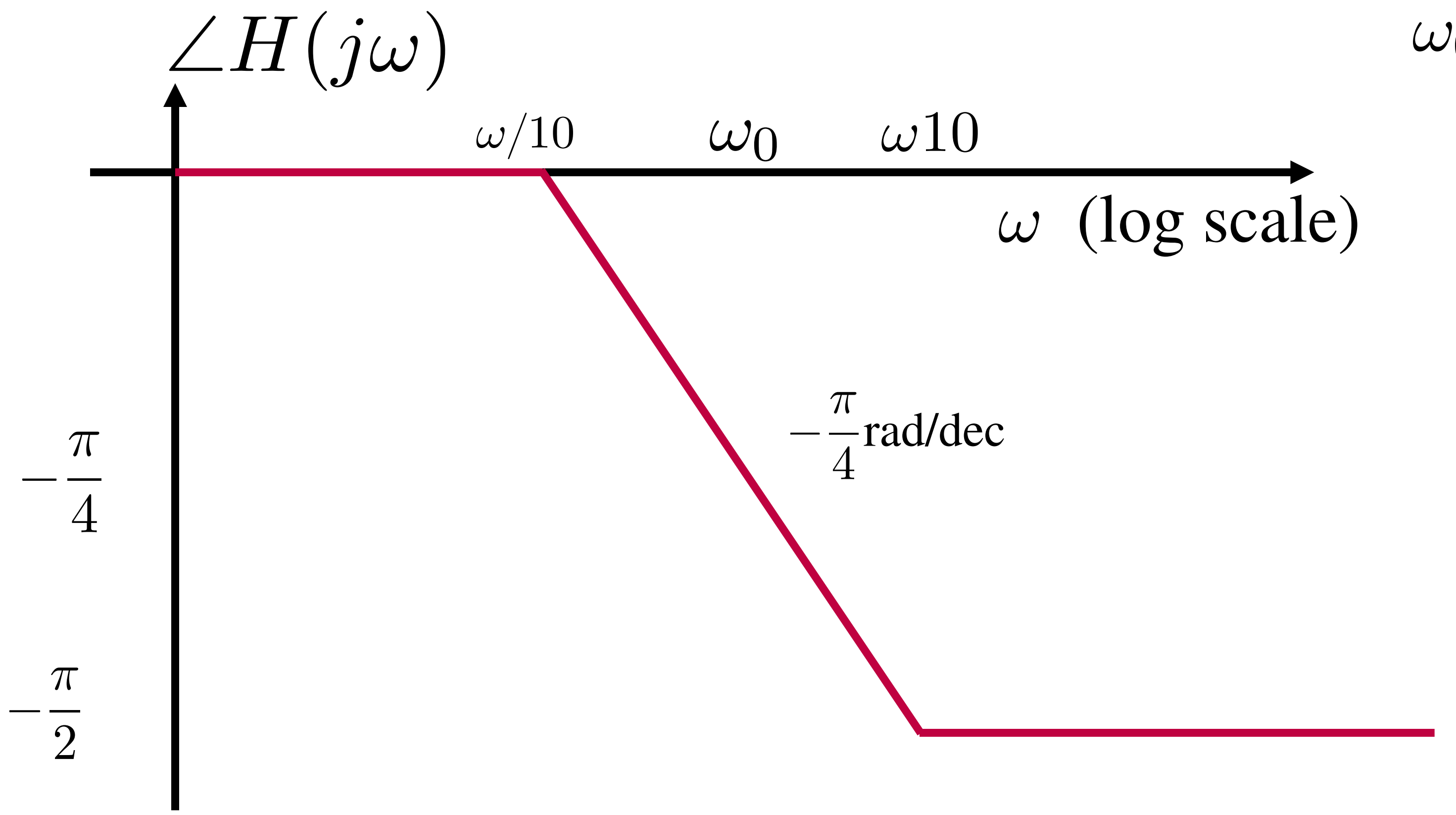
$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{\frac{\pi}{4}}{\log_{10}\omega_0 - \log_{10} 10\omega_0} \\ &= \frac{\frac{\pi}{4}}{\log_{10}\omega_0 - \log_{10}\omega_0 - \log_{10} 10} \\ &= -\frac{\pi}{4} \end{aligned}$$

finding the intercept

$$\begin{aligned} 0 &= m \log_{10} \frac{\omega_0}{10} + b \\ b &= -m \log_{10} \frac{\omega_0}{10} \\ &= \frac{\pi}{4} \left(\log_{10}\omega_0 - 1\right) \end{aligned}$$

Bode phase plot of the low-pass filter

$$\omega_0 = \frac{1}{RC}$$



Summary of the Bode equations for a low-pass filter

- Magnitude

$$|H(j\omega)|_{dB} \approx \begin{cases} 0 \text{ dB} & \omega < \omega_0 \\ -20\log_{10}\omega + 20\log_{10}\omega_0 & \omega \geqslant \omega_0 \end{cases}$$

- Phase

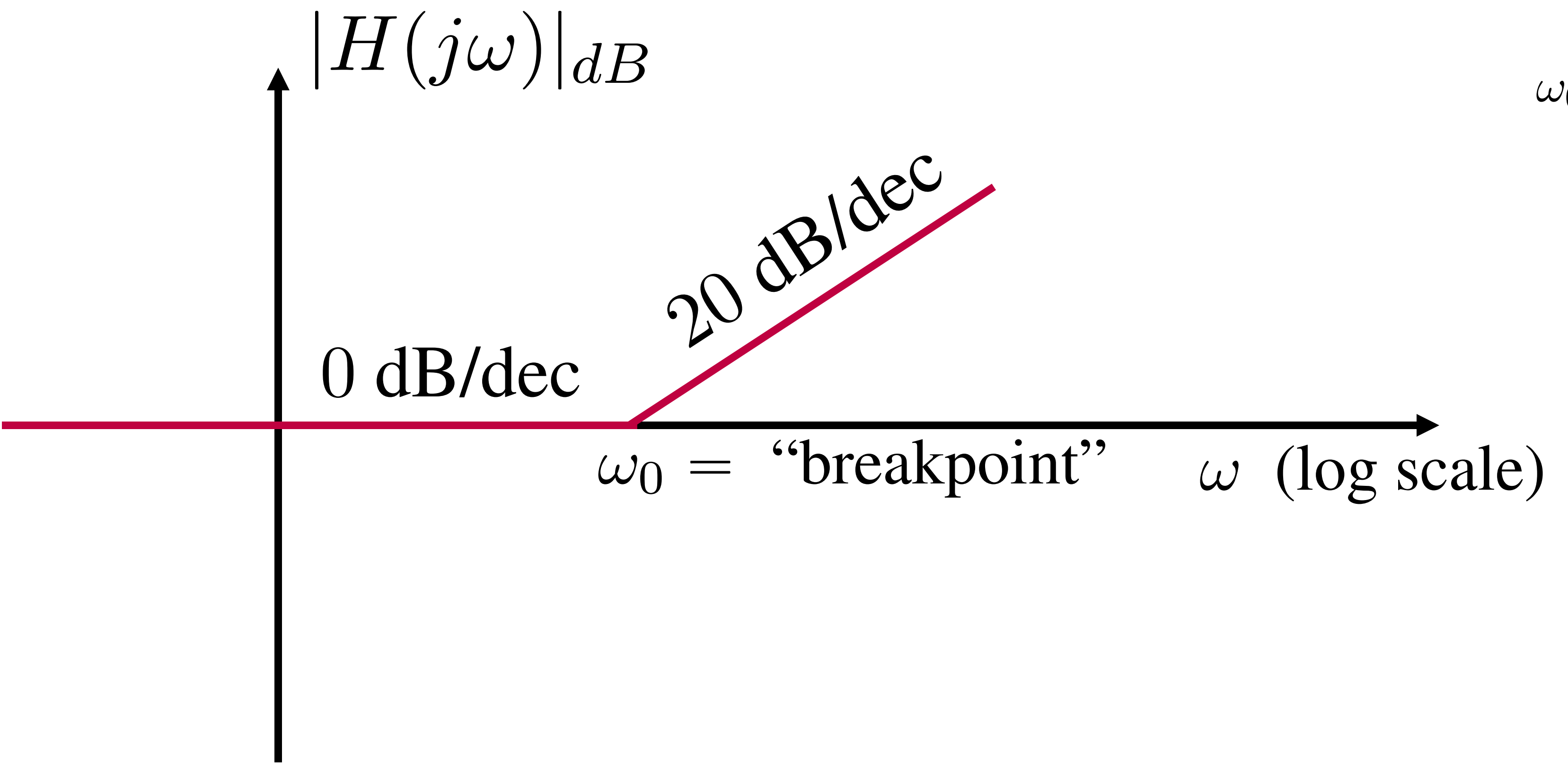
$$\angle H(j\omega) \approx \begin{cases} 0 & \omega < \omega_0/10 \\ -\frac{\pi}{4}\log_{10}\omega + \frac{\pi}{4}(\log_{10}\omega_0 - 1) & \frac{\omega_0}{10} < \omega < 10\omega_0 \\ -\frac{\pi}{2} & \omega > 10\omega_0 \end{cases}$$

What about a high-pass filter?

$$H(j\omega) = 1 + jRC\omega$$

$$|H(j\omega)|_{\mathrm{dB}} = 20\log_{10}\sqrt{1 + (\omega/\omega_0)^2}$$

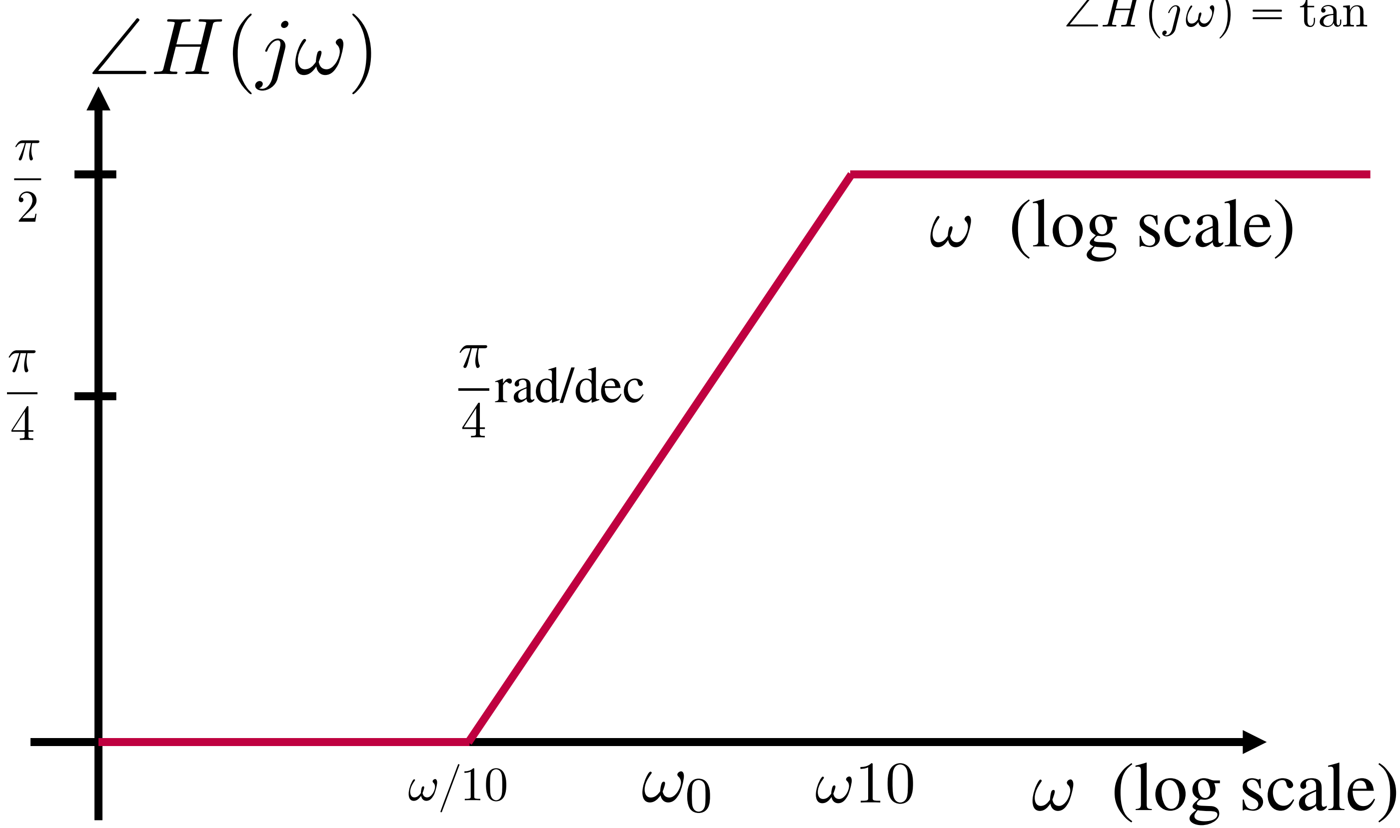
$$\omega_0 = \frac{1}{RC}$$



What about a high-pass filter?

$$H(j\omega) = 1 + jRC\omega$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$



Summary

- The Bode plot of a simple RC filter is approximated using three lines

- Magnitude plot
  - Plot frequency on the log scale vs magnitude in dB
  - Label the breakpoint and slope

- Phase plot
  - Plot the frequency on the log scale vs phase in radians
  - Label the endpoints of the transition region, critical frequency and slope

- Low-pass and high-pass filters have the “opposite” response