ECE 101 Linear Systems

Problem Set 4 Solutions

Problem 1: CTFT and Differential Equation

4.34(a)

Since the system is stable using the frequency response we have

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{6 + (j\omega)^2 + +5j\omega} \tag{1}$$

Multiplying out the above equation we have

$$6Y(j\omega) + (j\omega)^2 Y(j\omega) + +5j\omega Y(j\omega) = j\omega X(j\omega) + 4X(j\omega), \tag{2}$$

and taking the inverse Fourier transform we have

$$\frac{d^2 y(t)}{dt} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t). \tag{3}$$

4.34(b)

Factorizing the impulse response we have

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}. \tag{4}$$

Expressing it in partial fractions we have

$$\frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)} = \frac{A}{(j\omega + 2)} + \frac{B}{(j\omega + 3)} \tag{5}$$

and multiplying it out we have

$$j\omega + 4 = A(j\omega + 3) + B(j\omega + 2). \tag{6}$$

Substituting $j\omega = -2$ we have $A = 2$ and substituting $j\omega = -3$ we have $B = -1$. Hence, we have

$$H(j\omega) = \frac{2}{(j\omega + 2)} + \frac{-1}{(j\omega + 3)}, \tag{7}$$

and taking inverse Fourier transform we have

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t). \tag{8}$$

4.34(c)

Taking the Fourier transform of the input we have

$$X(j\omega) = \frac{1}{j\omega+4} - \frac{1}{(j\omega+4)^2} = \frac{j\omega+3}{(j\omega+4)^2}. \tag{9}$$

The Fourier transform of the output is given by

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(j\omega+2)(j\omega+4)} \tag{10}$$

and expressing it in partial fractions we have

$$\frac{1}{(j\omega+2)(j\omega+4)} = \frac{A}{(j\omega+2)} + \frac{B}{(j\omega+4)}, \tag{11}$$

which implies

$$1 = A(j\omega+4) + B(j\omega+2). \tag{12}$$

Substituting $j\omega = -2$ we have $A = \frac{1}{2}$ and substituting $j\omega = -4$ we have $B = -\frac{1}{2}$. This implies

$$Y(j\omega) = \frac{1}{2(j\omega+2)} + \frac{-1}{2(j\omega+4)} \tag{13}$$

and taking inverse Fourier transform we have

$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t). \tag{14}$$

Problem 2: Sampling Theory

7.3(b)

Using the tables we obtain the Fourier transform as

$$\frac{\sin 4000\pi t}{\pi t} \longleftrightarrow \begin{cases} 1, & |\omega| < 4000\pi, \\ 0, & |\omega| > 4000\pi. \end{cases} \tag{15}$$

Since the signal is band limited to 4000π rad/sec and hence the Nyquist rate is 8000π rad/sec.

7.4(c)

Using the tables we obtain Fourier transform of $x^2(t)$ as

$$x^2(t) \longleftrightarrow \frac{1}{2\pi}X(j\omega) * X(j\omega) \tag{16}$$

Since the Nyquist rate of $x(t)$ is ω_0 implies that $X(j\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$. Hence the maximum value of ω such that $\frac{1}{2\pi}X(j\omega) * X(j\omega) = 0$ is $|\omega| > \omega_0$. Hence the Nyquist rate is $2\omega_0$.

Problem 3: Signal Modulation

8.8 (a)

Given $x(t)$ is real then $X(-j\omega) = X^*(j\omega)$ or $X(j\omega) = X^*(-j\omega)$. Let $z(t) := x(t)^\star h(t)$. Using the convolution property we have

$$Z(j\omega) = \mathcal{F}\{x(t)^\star h(t)\} = X(j\omega)H(j\omega), \tag{17}$$

$$= \begin{cases} jX(j\omega) & \text{if } -\omega_M < \omega < 0 \\ -jX(j\omega) & \text{if } 0 < \omega < \omega_M, \\ 0 & \text{if } \omega > \omega_M \text{ and } \omega < -\omega_M. \end{cases} \tag{18}$$

Let $z_1(t) := z(t)\cos\omega_c t$ then using multiplication property we have

$$Z_1(j\omega) = \frac{1}{2\pi} Z(j\omega)^\star\, \pi\, (\delta(\omega - \omega_c) + \delta(\omega + \omega_c)) \tag{19}$$

$$= \frac{1}{2}\left(Z(j(\omega - \omega_c)) + Z(j(\omega + \omega_c))\right) \tag{20}$$

$$= \begin{cases} \frac{j}{2}X(j(\omega - \omega_c)) & \text{if } \omega_c - \omega_M < \omega < \omega_c, \\ -\frac{j}{2}X(j(\omega - \omega_c)) & \text{if } \omega_c < \omega < \omega_c + \omega_M, \\ \frac{j}{2}X(j(\omega + \omega_c)) & \text{if } -\omega_c - \omega_M < \omega < -\omega_c, \\ -\frac{j}{2}X(j(\omega + \omega_c)) & \text{if } -\omega_c < \omega < -\omega_c + \omega_M. \end{cases} \tag{21}$$

Now let $z_2(t) = x(t)\sin\omega_c t$ then using multiplication property we have

$$Z_2(j\omega) = \frac{1}{2\pi} X(j\omega)^\star \frac{\pi}{j}\, (\delta(\omega - \omega_c) - \delta(\omega + \omega_c)) \tag{22}$$

$$= -\frac{j}{2}X(j(\omega - \omega_c)) + \frac{j}{2}X(j(\omega + \omega_c)) \tag{23}$$

$$= \begin{cases} -\frac{j}{2}X(j(\omega - \omega_c)) & \text{if } \omega_c - \omega_M < \omega < \omega_c + \omega_M, \\ \frac{j}{2}X(j(\omega + \omega_c)) & \text{if } -\omega_c - \omega_M < \omega < -\omega_c + \omega_M. \end{cases} \tag{24}$$

Therefore, we have

$$Y(j\omega) = Z_1(j\omega) + Z_2(j\omega) = \begin{cases} -jX(j(\omega - \omega_c)) & \text{if } \omega_c < \omega < \omega_c + \omega_M, \\ jX(j(\omega + \omega_c)) & \text{if } -\omega_c - \omega_M < \omega < -\omega_c, \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

which implies

$$Y(-j\omega) = \begin{cases} -jX(j(-\omega - \omega_c)) & \text{if } \omega_c < -\omega < \omega_c + \omega_M, \\ jX(j(-\omega + \omega_c)) & \text{if } -\omega_c - \omega_M < -\omega < -\omega_c, \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

$$= \begin{cases} -jX(-j(\omega + \omega_c)) & \text{if } -\omega_c - \omega_M < \omega < -\omega_c, \\ jX(-j(\omega - \omega_c)) & \text{if } \omega_c < \omega < \omega_c + \omega_M \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Taking the conjugate and using the conjugate symmetry of $X(j\omega)$ we have

$$Y^*(-j\omega) = \begin{cases} jX(j(\omega + \omega_c)) & \text{if } -\omega_c - \omega_M < \omega < -\omega_c, \\ -jX(j(\omega - \omega_c)) & \text{if } \omega_c < \omega < \omega_c + \omega_M \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Therefore, we have shown that $Y(j\omega) = Y^*(-j\omega)$ which implies that $y(t)$ is real if $x(t)$ is real.

This problem can be solved graphically as well but the above method is a detailed solution for the underlying math.

8.8 (b)

Consider the Fourier transform of $y(t)\sin(\omega_c t)$ given as follows

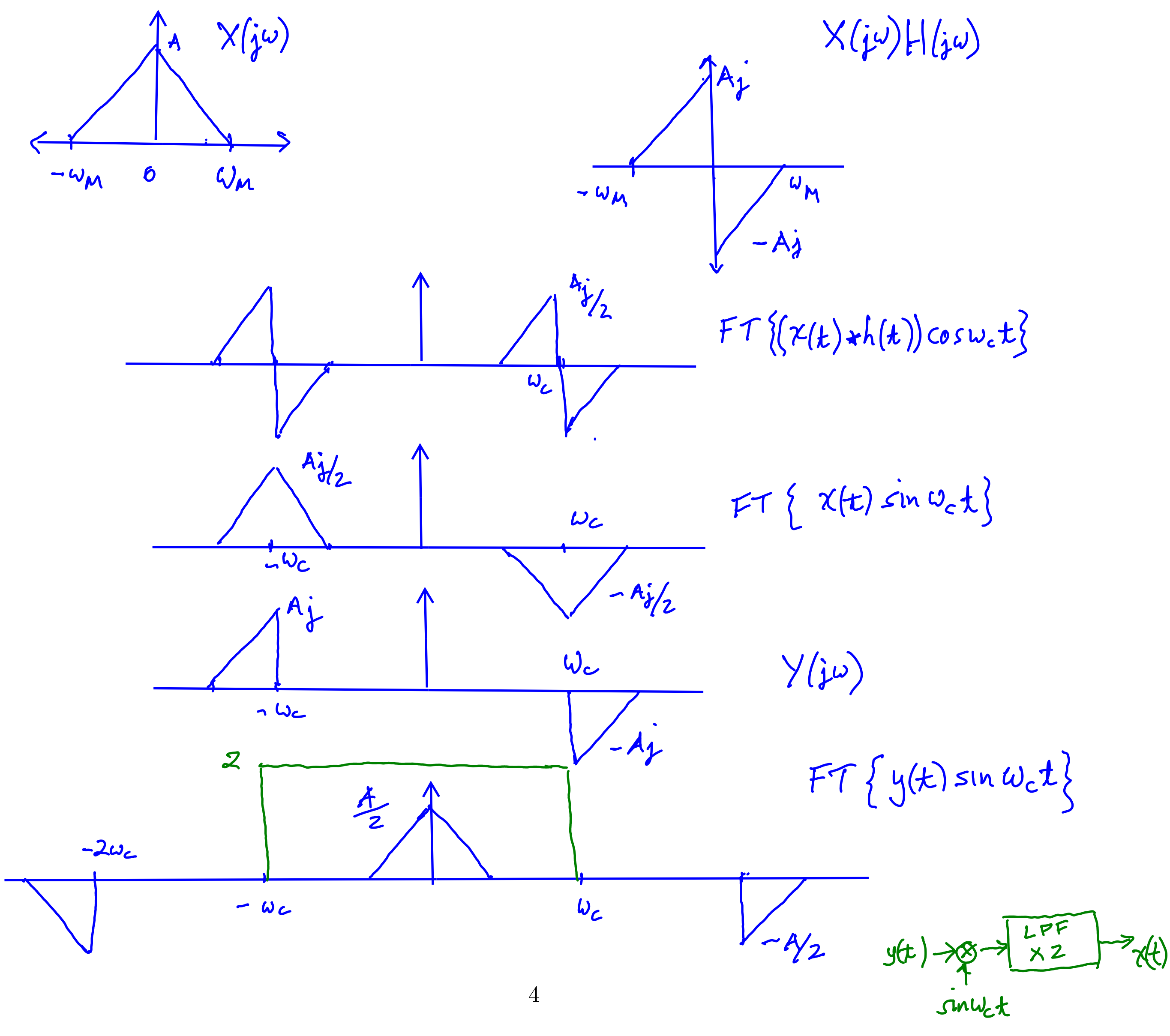
$$\mathcal{F}\{y(t)\sin(\omega_c t)\} = \frac{1}{2\pi}Y(j\omega) * \frac{\pi}{j}\left(\delta(\omega-\omega_c)-\delta(\omega+\omega_c)\right) \tag{29}$$

$$= \frac{j}{2}\left(-Y(j(\omega-\omega_c))+Y(j(\omega+\omega_c))\right) \tag{30}$$

$$= \begin{cases} -\frac{1}{2}X(j(\omega-2\omega_c)) & \text{if } 2\omega_c < \omega < 2\omega_c+\omega_M, \\ \frac{1}{2}X(j(\omega+2\omega_c)) & \text{if } -2\omega_c-\omega_M < \omega < -2\omega_c, \\ \frac{1}{2}X(j\omega) & \text{if } -\omega_M < \omega < \omega_M. \end{cases} \tag{31}$$

Hence, we can recover the signal $x(t)$ by passing $y(t)\sin(\omega_c t)$ through a low pass filter with a frequency response given below

$$H(j\omega) = \begin{cases} 2 & \text{if } -\omega_c < \omega < \omega_c, \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

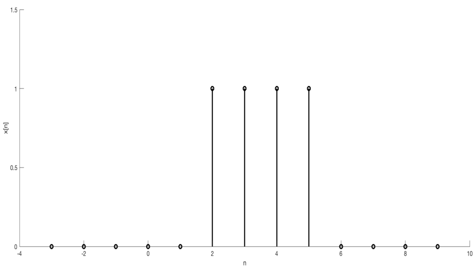


Problem 4: DTFT Analysis

5.21. Compute the fourier transform of each of the following signals:

5.21.(a). $x[n] = u[n-2] - u[n-6]$

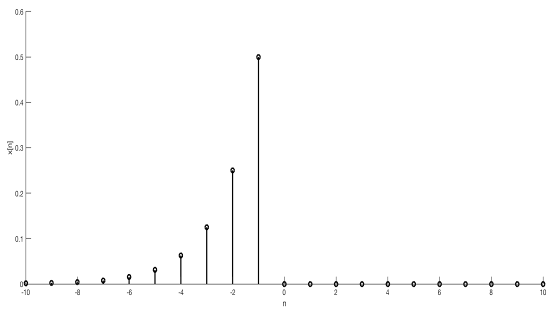
$x[n] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$



$$X(e^{jw}) = e^{-2jw} + e^{-3jw} + e^{-4jw} + e^{-5jw} = e^{-7j\omega/2}\, sin(2\omega)/sin(\omega/2)$$

5.21.(b). $x[n] = (\frac{1}{2})^{-n}u[-n-1]$

Using the analysis equation we have:



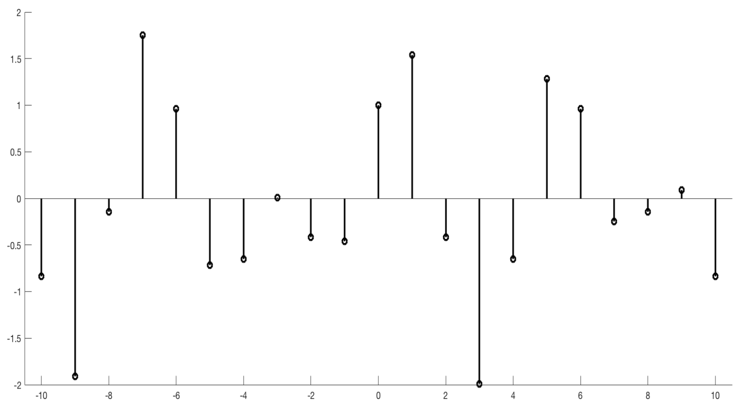
$X(e^{jw}) = \sum_{-\infty}^{\infty} x[n]e^{-jwn} = \sum_{-\infty}^{\infty} (\frac{1}{2})^{-n}u[-n-1]e^{-jwn} = \sum_{-\infty}^{-1} (\frac{1}{2})^{-n}e^{-jwn}$

$X(e^{jw}) = \sum_{-\infty}^{-1} (\frac{1}{2}e^{jw})^{-n} = \sum_{1}^{\infty} (\frac{1}{2}e^{jw})^{n} = \frac{\frac{1}{2}e^{jw}}{1-\frac{1}{2}e^{jw}}$

5.21.(g). $x[n] = sin(\frac{\pi n}{2}) + cos(n)$

Using Euler's formula we have:

$x[n] = \frac{1}{2j}(e^{j\pi n/2} - e^{-j\pi n/2}) + \frac{1}{2}(e^{jn} + e^{-jn})$



Now using table 5.2. we can calculate $X(e^{jw})$:

$X(e^{jw}) = \frac{2\pi}{2j}(\delta(w - \pi/2) - (w + \pi/2)) + \frac{2\pi}{2}(\delta(w - 1) + (w + 1)) \quad for \quad -\pi < w < \pi$

$X(e^{jw}) = \frac{\pi}{j}(\delta(w-\pi/2)-(w+\pi/2))+\pi(\delta(w-1)+(w+1)) \quad for \; -\pi < w < \pi$

Problem 5: DTFT Synthesis

5.22. The following are the Fourier transform of discrete time signals. Determine the signal corresponding to each transform.

5.22.(a).

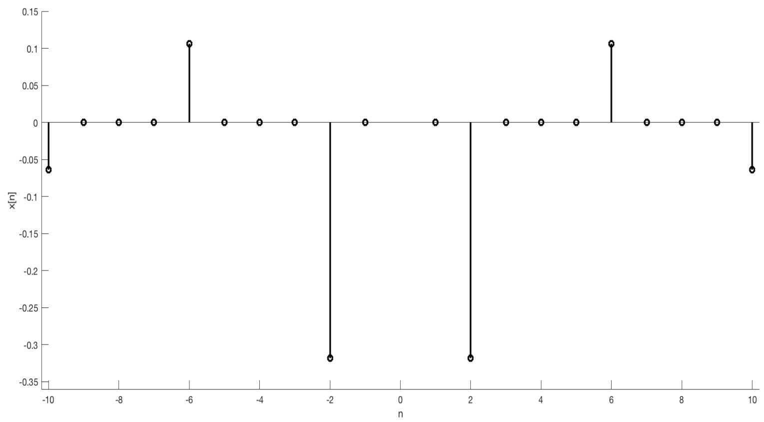
$$X(e^{jw}) = \begin{cases} 1 & \pi/4 \leq |w| \leq 3\pi/4 \\ 0 & 3\pi/4 \leq |w| \leq \pi, \quad 0 \leq |w| < \pi/4 \end{cases}$$

Using the synthesis equation we have:

$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw = \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4} e^{jwn}dw + \frac{1}{2\pi}\int_{-3\pi/4}^{-\pi/4} e^{jwn}dw$

$= \frac{1}{2\pi}\frac{e^{jwn}}{jn}]_{\pi/4}^{3\pi/4} + \frac{1}{2\pi}\frac{e^{jwn}}{jn}]_{-3\pi/4}^{-\pi/4} = \frac{1}{2\pi}\frac{e^{jn3\pi/4} - e^{jn\pi/4} + e^{-jn\pi/4} - e^{-jn3\pi/4}}{jn}$

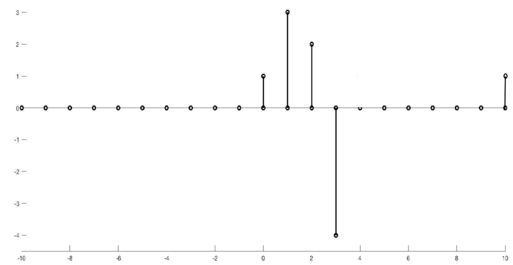
$= \frac{1}{2\pi jn}(e^{jn3\pi/4} - e^{-jn3\pi/4} + e^{-jn\pi/4} - e^{jn\pi/4}) = \frac{1}{\pi n}(sin(3\pi n/4) - sin(\pi n/4))$



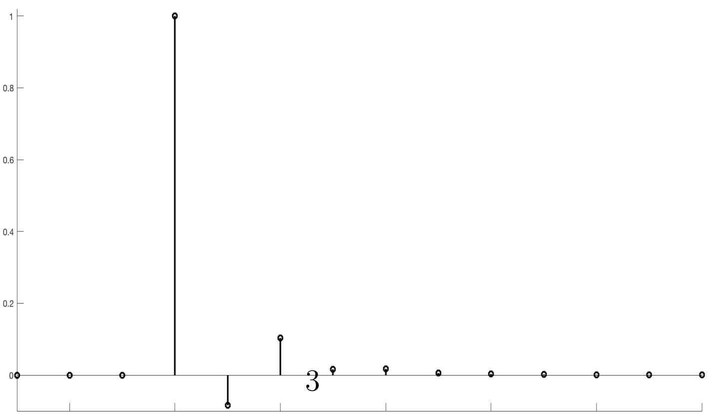
5.22.(b).
$X(e^{jw}) = 1 + 3e^{-jw} + 2e^{-j2w} - 4e^{-j3w} + e^{-j10w}$

According the table, we know that $\delta[n - n_0] \leftrightarrow e^{-jwn_0}$. Therefore:

$x[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2] - 4\delta[n-3] + \delta[n-10]$



5.22.(g).
$X(e^{jw}) = \frac{1-\frac{1}{3}e^{-jw}}{1-\frac{1}{4}e^{-jw}-\frac{1}{8}e^{-2jw}}$
In order to be able to use the table, we must first convert the fraction to one of the forms included in the table. First order fractions are in the table. Therefore:



$$X(e^{jw}) = \frac{1-\frac{1}{3}e^{-jw}}{(1-\frac{1}{2}e^{-jw})(1+\frac{1}{4}e^{-jw})} = \frac{A}{(1-\frac{1}{2}e^{-jw})} + \frac{B}{(1+\frac{1}{4}e^{-jw})}$$

$$\rightarrow A = 2/9, B = 7/9$$

$$X(e^{jw}) = \frac{2/9}{(1-\frac{1}{2}e^{-jw})} + \frac{2/9}{(1+\frac{1}{4}e^{-jw})}$$

$$x[n] = \frac{2}{9}(\frac{1}{2})^n u[n] + \frac{7}{9}(\frac{-1}{4})^n u[n]$$

Please note that in this plot the values of x[n] are very small as n increases. On the other hand, the values of x for $n < 0$ are actually 0.