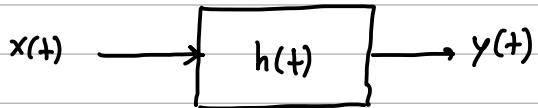


## Discussion Session 9:

Convolution property and Fourier Transform:



$h(t)$  = Impulse response of a system is its output when the input is a unit impulse function  $\delta(t)$ .

$H(j\omega)$  = The frequency response of a system describe how the system react to different frequency components.

$$H(j\omega) = F(h(t)) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

If we have an input signal  $x(t)$  and we want to find the output  $y(t)$  of a LTI system, we can compute it in two different ways.

1) Time-domain approach (Convolution)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

2) Frequency-domain approach (Multiplication)

$$\begin{aligned} X(j\omega) &= F(x(t)) \\ H(j\omega) &= F(h(t)) \end{aligned} \Rightarrow Y(j\omega) = X(j\omega) H(j\omega) \Rightarrow y(t) = F^{-1}(Y(j\omega))$$

Example:  $x(t) = e^{-t} u(t)$      $h(t) = u(t)$

Answer:

Time-Domain approach

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) u(t-\tau) d\tau = u(t) \int_0^t e^{-\tau} d\tau$$

$$= u(t) - e^{-t} \Big|_{\tau=0}^{t=t} = (1 - e^{-t}) u(t)$$

Frequency-domain approach:

$$X(j\omega) = F(e^{-t} u(t)) = \frac{1}{1+j\omega}$$

$$H(j\omega) = F(u(t)) = \frac{1}{j\omega} + \pi \delta(\omega)$$

$$Y(j\omega) = X(j\omega) H(j\omega) = \frac{1}{j\omega(1+j\omega)} + \frac{\pi}{1+j\omega} \delta(\omega) = \frac{1}{j\omega(1+j\omega)} + \pi \delta(\omega)$$

$$= \frac{A}{j\omega} + \frac{B}{1+j\omega} + \pi \delta(\omega)$$

$$A(1+j\omega) + B(j\omega) = A + (A+B)j\omega = 1 \rightarrow A=1, A+B=0 \rightarrow B=-1$$

$$Y(j\omega) = \frac{-1}{1+j\omega} + \frac{1}{j\omega} + \pi \delta(\omega)$$

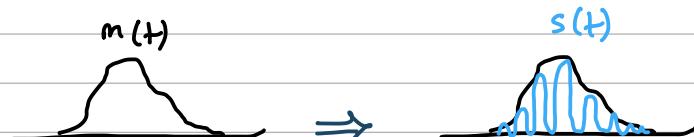
$$y(t) = F^{-1}\{Y(j\omega)\} = -e^{-t} u(t) + u(t) = (1 - e^{-t}) u(t)$$

Modulation:

Modulation is the process of modifying a carrier signal to transmit information.

AM modulation:  $s(t) = m(t) \cos(\omega_c t)$

message  $\xleftarrow{} \quad \overbrace{\quad}^{\text{carrier frequency}}$



It is called amplitude modulation (AM) as the message signal is embedded

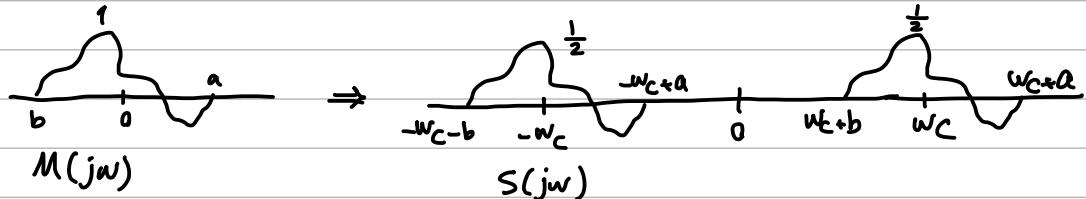
in the amplitude variation of the carrier signal.

To transmit the signal, we multiply the message with a high frequency cosine wave.

$$s(t) = m(t) \cos(\omega_c t) \longrightarrow \frac{1}{2} e^{j\omega_c t} + \frac{1}{2} e^{-j\omega_c t}$$

This corresponds to shifting the Fourier transform of signal to  $\pm \omega_c$ .

$$S(j\omega) = \frac{1}{2} M(j(\omega - \omega_c)) + \frac{1}{2} M(j(\omega + \omega_c))$$

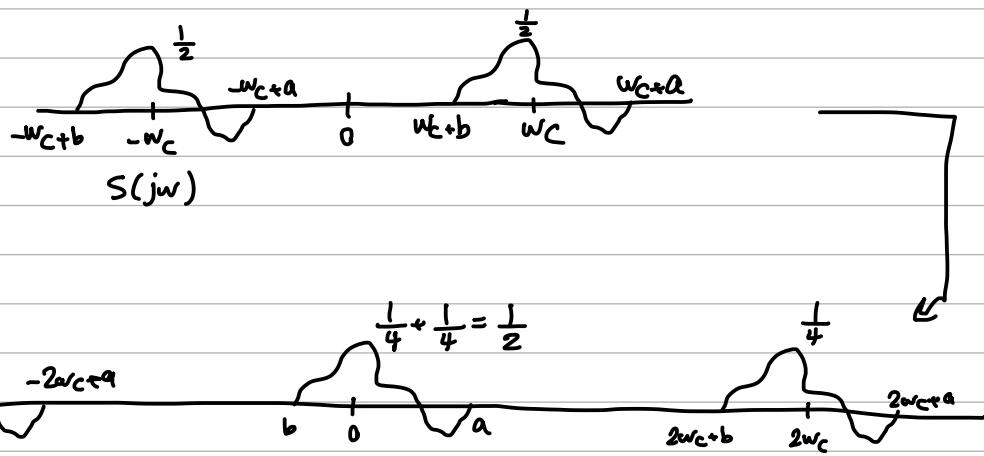


Demodulation:

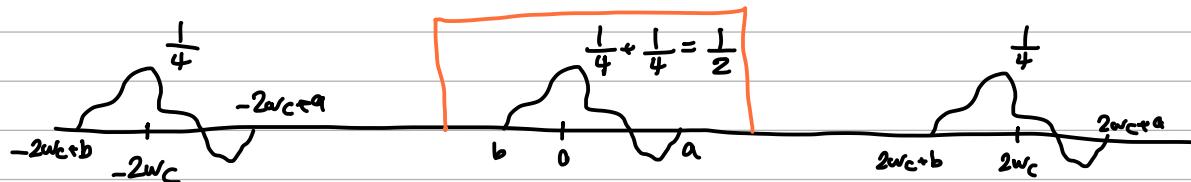
At receiver, the original signal is recovered by multiplying to  $\cos(\omega_c t)$

$$R(t) = S(t) \cos(\omega_c t) = m(t) \cos(\omega_c t) \cos(\omega_c t)$$

$$= m(t) \frac{1}{2} [1 + \cos(2\omega_c t)]$$

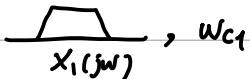


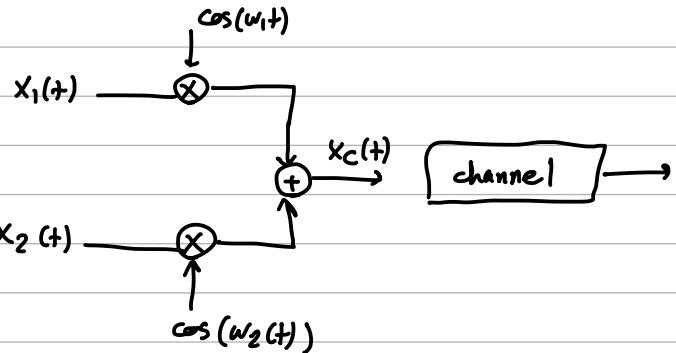
A low pass filter can recover the signal

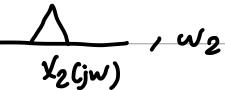


## Handling multiple signals.

In practical, multiple signals are transmitted simultaneously over different carrier frequency.

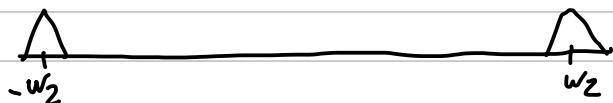
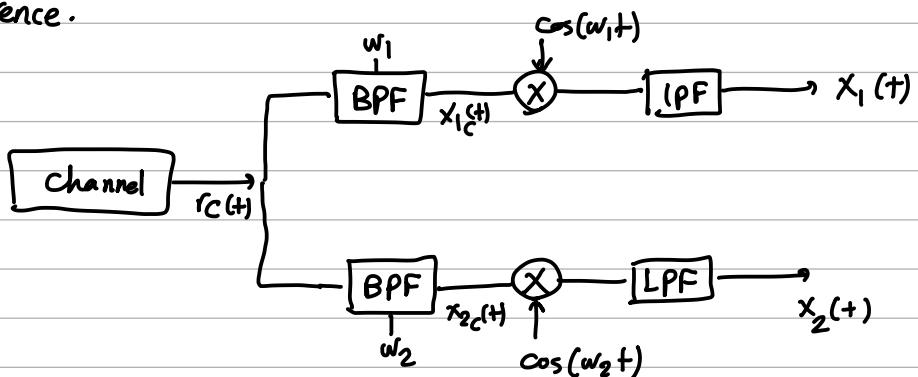
User 1:  ,  $w_{C1}$



User 2:  ,  $w_2$



At receiver, a band pass filter is used to separate desired signals, and avoid interference.



## Bode plot:

It is a graphical representation of a system's frequency response, showing how system's gain (magnitude) and phase shift vary with frequency.

Constructing Bode plots by hand involves approximating these responses using straight line segments, which simplifies the analysis.

Steps to plot Bode Plot:

① Express the transfer function in standard form.

$$H(jw) = k \frac{(jw/z_1 + 1)(jw/z_2 + 1) \dots}{(jw/p_1 + 1)(jw/p_2 + 1) \dots}$$

Here,  $k$  is the gain,  $z_i$  are zeros, and  $p_i$  are the poles.

② Identify the corner (break) frequencies.

Poles and zeros of transfer function.

③ Magnitude plot:

$$\log |H(jw)| = \log |k| + \log \left| \frac{jw}{z_1} + 1 \right| + \log \left| \frac{jw}{z_2} + 1 \right| + \dots - \log \left| \frac{jw}{p_1} + 1 \right| - \log \left| \frac{jw}{p_2} + 1 \right| \dots$$

\* Initial gain: start by plotting the initial gain of  $20 \log |k|$  in decibels (dB)

\* Slope changes: at each zero frequency ( $w_z$ ), increase the slope by  $+20 \text{ dB/decade}$ . At each pole frequency, decrease the slope by  $-20 \text{ dB/decade}$ .

$$H(jw) = \frac{10(jw+1)}{\left(\frac{jw}{10}+1\right)\left(\frac{jw}{100}+1\right)} \Rightarrow \begin{aligned} \text{Initial gain: } &k = 10 \\ \text{zero: } &w_z = 1 \frac{\text{rad}}{\text{s}} \\ \text{poles: } &w_{p_1} = 10 \frac{\text{rad}}{\text{s}}, w_{p_2} = 100 \end{aligned}$$

Magnitude plot:

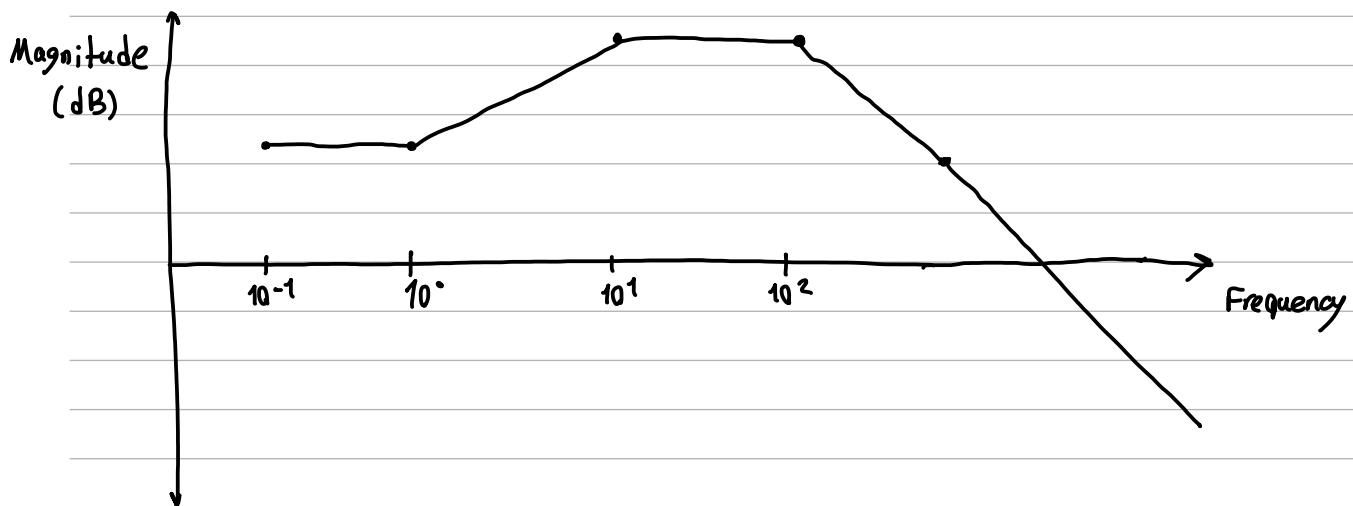
Initial gain:  $20 \log_{10}(10) = 20 \text{ dB}$

Low - Frequency Asymptote: Horizontal line at 20 dB for  $\omega < 1$ .

At  $\omega = 1 \text{ rad/s}$  (zero): Increase the slope by  $20 \text{ dB/decade}$ .

At  $\omega = 10 \text{ rad/s}$  (pole): Decrease slope by  $-20 \text{ dB/decade}$

At  $\omega = 100 \text{ rad/s}$  (pole), Decrease slope by  $-20 \text{ dB/decade}$



④ Construct the phase plot.

\* Initial phase: start with the phase angle contributed by the gain  $k$ . If  $k$  is positive, the initial phase is  $0^\circ$ , if negative, it is  $\pi$ .

\* Phase contribution of poles and zeros.

zero at  $w_z$ . Contribute  $+\frac{\pi}{4}$  per decade, starting from one decade before  $w_z$  and leveling off at  $\frac{\pi}{2}$  one decade after.

pole at  $w_p$ : Contribute  $-\frac{\pi}{4}$  per decade, starting from one decade  $w_p$  and leveling off at  $-\frac{\pi}{2}$  one decade after.

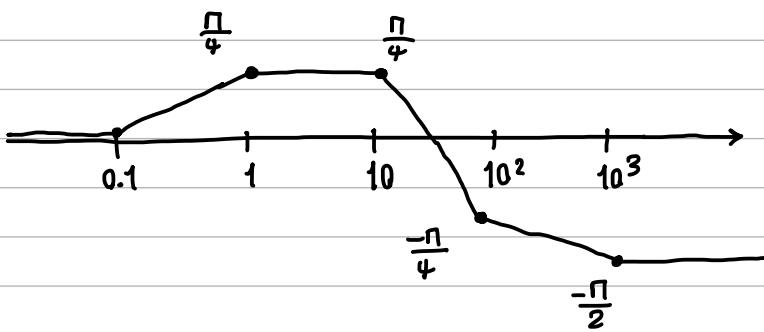
Example:  $H(jw) = \frac{10(jw+1)}{(jw/10+1)(jw/100+1)}$

Initial phase:  $0^\circ$  ( $k$  is positive)

$w_z = 1 \rightarrow$  From  $w = \frac{1}{10}$  to  $10$  rad: phase increase  $\frac{\pi}{4}$  per decade

$w_p = 10 \rightarrow$  From  $w = \frac{10}{10} = 1$  to  $10 \times 10 = 100$ , phase decrease  $\frac{\pi}{4}$  per decade.

$w_{p_2} = 100 \rightarrow$  From  $w = \frac{100}{10} = 10$  to  $100 \times 10 = 1000$ , phase decreases  $\frac{\pi}{4}$  per decade.



## Notes:

1) Handling  $(\frac{j\omega}{2} + 1)^n$  or  $(\frac{j\omega}{p} + 1)^n$

\* If zero or pole is repeated n times, it amplifies the effect.

Magnitude: instead of  $20 \text{ dB/decade}$ , change slope by  $20n \text{ dB/decade}$ .

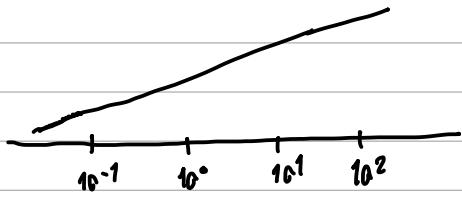
Phase: Each zero contributes  $+90^\circ$  per factor, and each pole contribute  $-90^\circ$  per factor.

$$H(j\omega) = \frac{1}{(j\omega + 1)^2} \quad \left\{ \begin{array}{l} \text{Magnitude: start at } 0 \text{ dB, drops by } -40 \text{ dB/decade after } \omega=1 \\ \text{phase: start at } 0^\circ, \text{ then drops to } -\frac{\pi}{2} \text{ at } \omega=1 \text{ and settles at } -\pi. \end{array} \right.$$

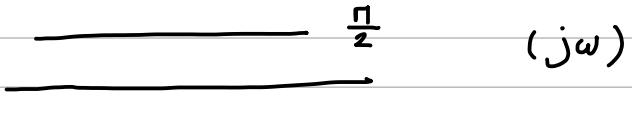
2) Handling  $j\omega$  and  $\frac{1}{j\omega}$ .

$j\omega$ : contributes  $+20 \text{ dB/decade}$  in definil and a constant  $+\frac{\pi}{2}$  phase shift.

$\frac{1}{j\omega}$ : contributes  $-20 \text{ dB/decade}$  and a constant  $-\frac{\pi}{2}$  phase shift.



Magnitude



Phase