Lecture 17

Generalizing Bode plots

Preview of today's lecture

- LTI systems in the frequency domain
  - XXX
- Introduction to Bode plots
  - XXX

Frequency response of a low-pass RC filter

- The frequency response of this system is

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}} \qquad \omega_0 = \frac{1}{RC}$$

RC circuit

No relation to FS

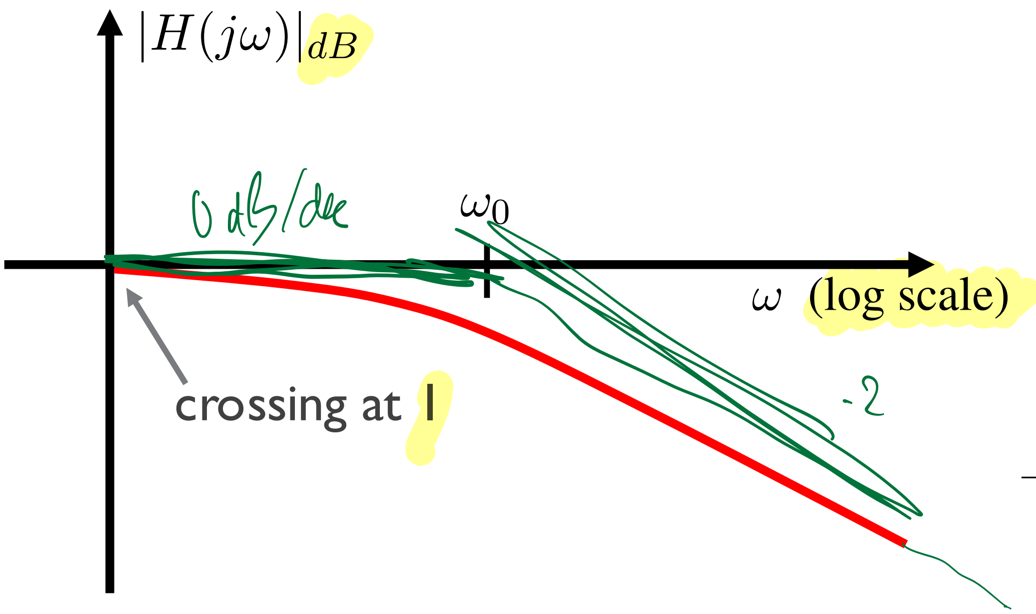
- The magnitude response in decibels is

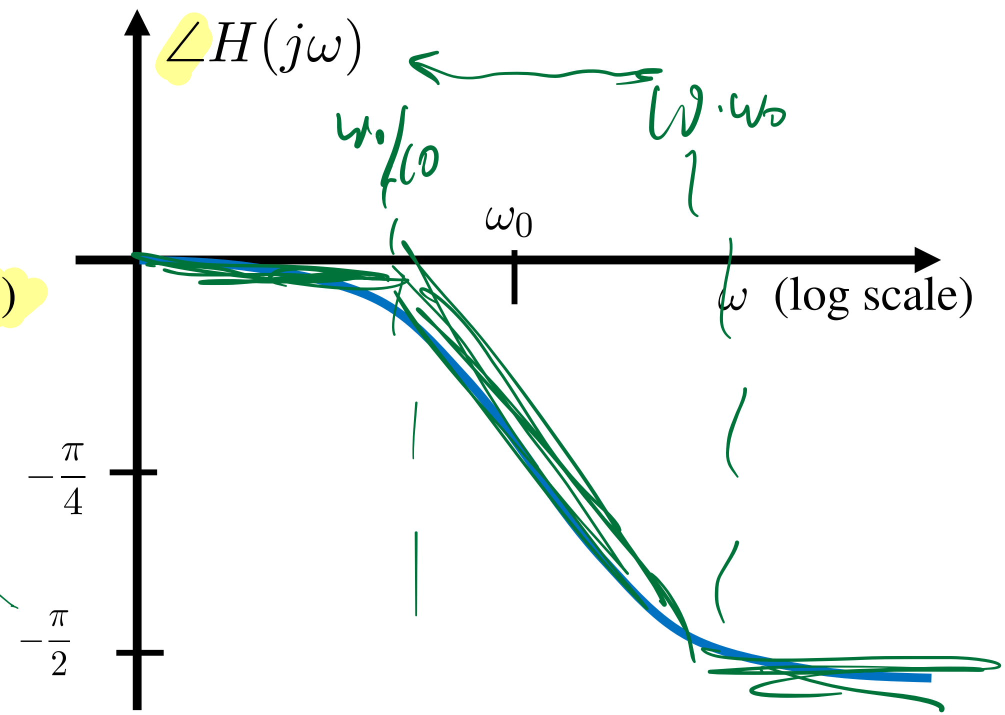
$$\begin{aligned} |H(j\omega)|_{\mathrm{dB}} &= 20\log_{10}|H(j\omega)| \\ &= -20\log_{10}\sqrt{1 + (\omega/\omega_0)^2} \end{aligned}$$

- The phase response in rad/s is

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

Plot of the frequency response of a low-pass RC filter

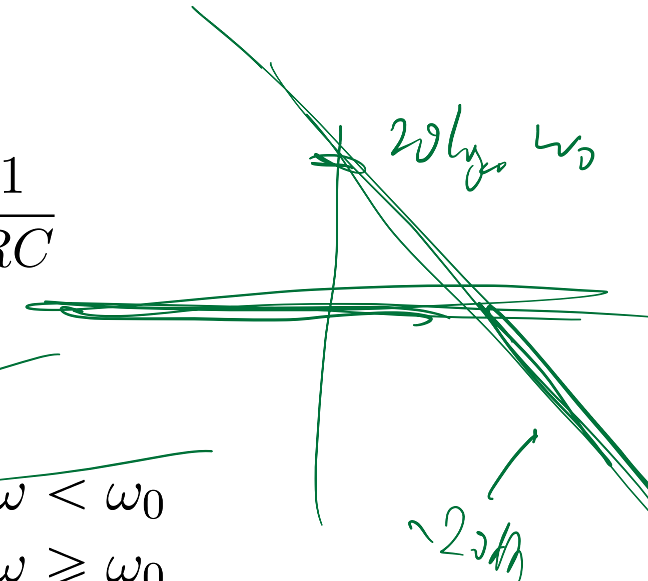




$\log_{10} 1 = 0$

1 2 3 4 5 6 .. 10 20 30 .. 100] log scale

Bode approximation of the frequency response



- The frequency response of this system is

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}} \qquad\qquad \omega_0 = \frac{1}{RC}$$

- The magnitude response is approximated as

2 lines

$$|H(j\omega)|_{dB} \approx \begin{cases} 0 \text{ dB} & \omega < \omega_0 \\ -20\log_{10}\omega + 20\log_{10}\omega_0 & \omega \geqslant \omega_0 \end{cases}$$

- The phase response is approximated as

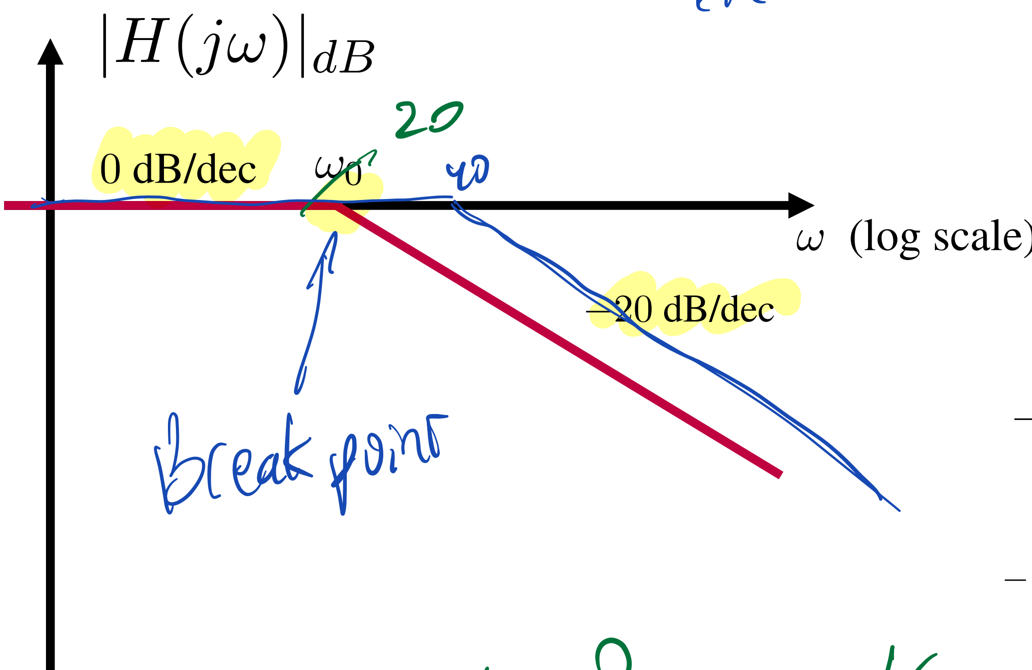
3 lines

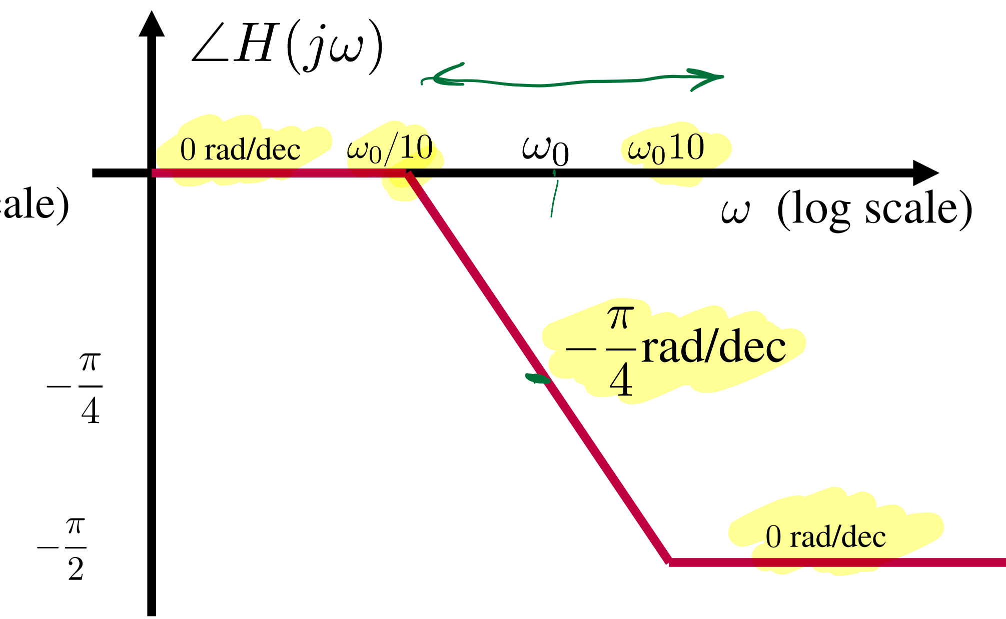
$$\angle H(j\omega) \approx \begin{cases} 0 & \omega < \omega_0/10 \\ -\frac{\pi}{4}\log_{10}\omega + \frac{\pi}{4}(\log_{10}\omega_0 - 1) & \frac{\omega_0}{10} < \omega < 10\omega_0 \\ -\frac{\pi}{2} & \omega > 10\omega_0 \end{cases}$$

Bode plot of the frequency response for the low-pass

$\omega_0 = 1/RC$

$1/RC = 1/40 \rightarrow \omega = 40$





Q: $RC = 1/20$? $\omega_0 = 1/(1/20) = 20$ rad/s

True responses approximated with a sequence of lines

Frequency response of a high-pass RC filter

$H_{LP}(j\omega) = \frac{1}{1 + j\omega/\omega_0}$

- The frequency response of this system is

$$H(j\omega) = 1 + j\frac{\omega}{\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

+'ve value

- The magnitude response in decibels is

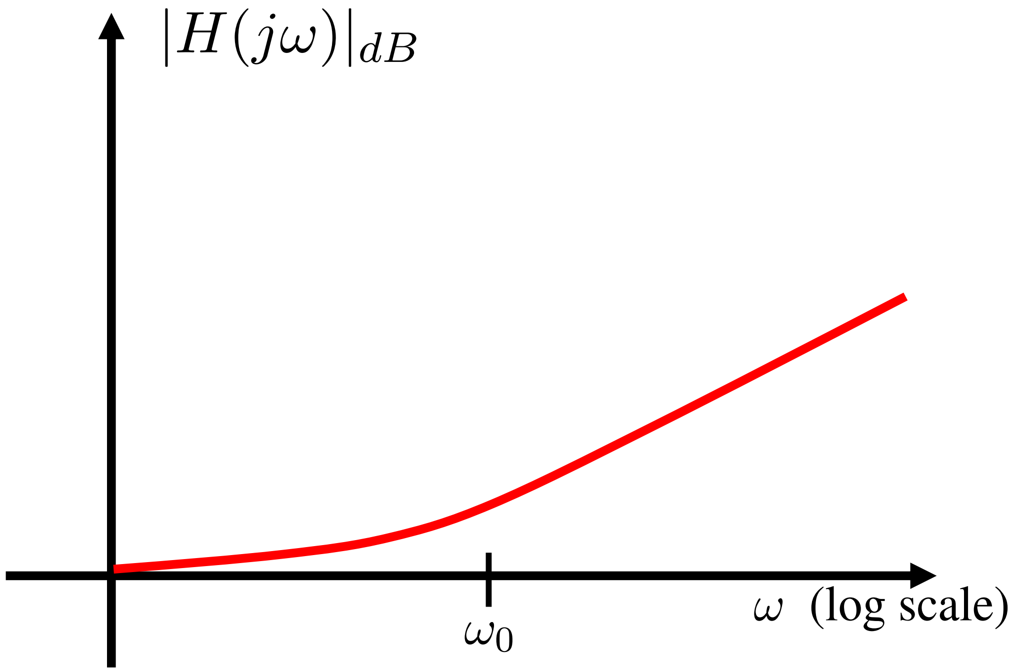
$$|H(j\omega)|_{\mathrm{dB}} = 20\log_{10}\sqrt{1 + (\omega/\omega_0)^2}$$

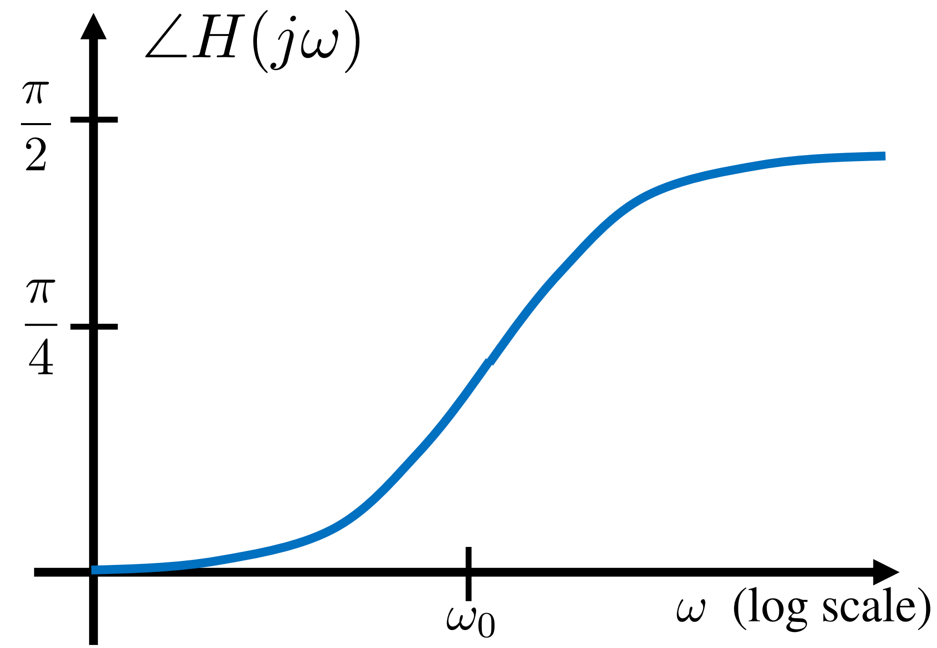
- The phase response in rad/s is

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

Note sign change versus the low-pass RC filter

Plot of the frequency response of a low-pass RC filter





Bode approximation of the frequency response

- The frequency response of this system is

$$H(j\omega) = 1 + j\frac{\omega}{\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

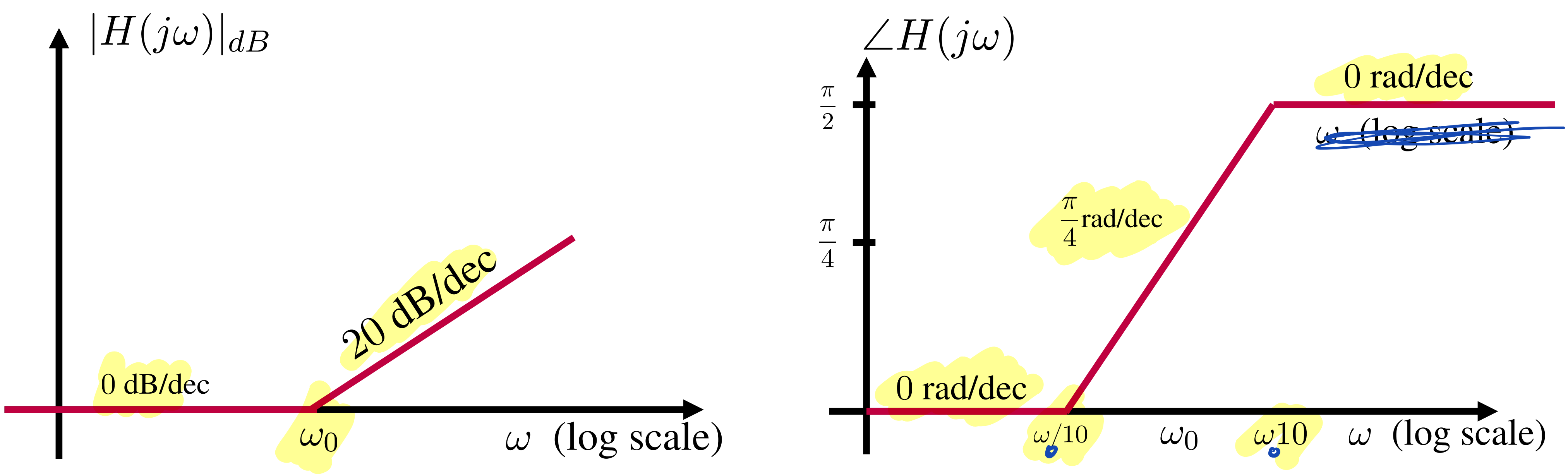
- The magnitude response is approximated as

$$|H(j\omega)|_{dB} \approx \begin{cases} 0 \text{ dB} & \omega < \omega_0 \\ 20\log_{10}\omega - 20\log_{10}\omega_0 & \omega \geqslant \omega_0 \end{cases}$$

- The phase response is approximated as

$$\angle H(j\omega) \approx \begin{cases} 0 & \omega < \omega_0/10 \\ \frac{\pi}{4}\log_{10}\omega - \frac{\pi}{4}(\log_{10}\omega_0 - 1) & \frac{\omega_0}{10} < \omega < 10\omega_0 \\ \frac{\pi}{2} & \omega > 10\omega_0 \end{cases}$$

Bode plot of the frequency response for the high-pass



True responses approximated with a sequence of lines

See how the sign difference manifests here vs the low-pass case

What is missing?

- So far only considered a simple RC filter
  - But general circuits can result in higher-order differential equations

Leverage Lecture 16

- Approach
  - Decompose the higher order frequency response into terms that look like simple high- and low-pass filters
  - Build the composite frequency response using the examples of the low and high pass responses
  - Main task will be to find break points and slopes

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Bode plots

Lectures 16, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 15 Fourier transform

Fourier

General Bode plots

Key points

- Generalize Bode plots for higher order filters building upon the foundations of the low-pass and high-pass filters
- Process the plots step-by-step, starting from the left-most breakpoint

Generalizing the frequency response

- A LTI system described by a LCCDE can in general be written in a factored form as

$$H(j\omega) = \frac{A\left(1 + \frac{j\omega}{z_1}\right)\left(1 + \frac{j\omega}{z_2}\right)\cdots}{(j\omega)^n\left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)\cdots}$$

maybe in the numerator or denominator with different integer values of n

Example

voltage @

output

- Find the factored form of the following LCCDE system

$$20\frac{d^2}{dt^2}y(t) + 100\frac{d}{dt}y(t) + 120y(t) = \frac{d^2}{dt^2}x(t) + \frac{d}{dt}x(t)$$

$x(t)$?

voltage @ input

- Hint: $20(z^2 + 5z + 6) = 20(z+2)(z+3)$

Second order

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$20(j\omega)^2 Y(j\omega) + 100(j\omega)Y(j\omega) + 120Y(j\omega)$$
$$= (j\omega)^2 X(j\omega) + (j\omega)X(j\omega)$$

Example (continued)

$$\left(20(j\omega)^2 + 100(j\omega) + 120\right) Y(j\omega) = \left((j\omega)^2 + (j\omega)\right) X(j\omega)$$

$$= j\omega(j\omega + 1) X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega(j\omega+1)}{20(j\omega)^2 + 100(j\omega) + 120}$$

Note: $20\left[(j\omega)^2 + 5(j\omega) + 6\right] = 20(j\omega+2)(j\omega+3)$

$$H(j\omega) = \frac{j\omega \cdot (j\omega + 1)}{20\,(j\omega + 2)\,(j\omega + 3)}$$

$$H(j\omega) = \frac{A\left(1 + \frac{j\omega}{z_1}\right)\left(1 + \frac{j\omega}{z_2}\right)\cdots}{(j\omega)^n\left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)\cdots}$$

$$= \frac{j\omega\,(j\omega + 1)}{20 \cdot 2\,(j\omega/2 + 1) \cdot 3\,(j\omega/3 + 1)}$$

$$= \frac{1}{120} \cdot \frac{j\omega\,(1 + j\omega)}{(1 + j\omega/2)\,(1 + j\omega/3)}$$

$$\boxed{\begin{array}{ll} n = -1 & z_1 = 1 \\ A = 1/120 & p_1 = 2 \\ & p_2 = 3 \end{array}}$$

Refactoring the frequency response

high-pass　　low pass

$$H(j\omega) = \frac{A\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots}{(j\omega)^n\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\cdots}$$

$$= A\frac{1}{(j\omega)^n}\overbrace{\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots}^{\text{high-pass}}\cdot\overbrace{\frac{1}{\left(1+\frac{j\omega}{p_1}\right)}\cdot\frac{1}{\left(1+\frac{j\omega}{p_2}\right)}}^{\text{low pass}}$$

$$= A(j\omega)^{-n}\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots\left(1+\frac{j\omega}{p_1}\right)^{-1}\left(1+\frac{j\omega}{p_2}\right)^{-1}\cdots$$

$$= A(j\omega)^{-n}\prod_{i=1}^{N}\left(1+\frac{j\omega}{\omega_i}\right)^{a_i}, \quad a_i = \pm 1$$

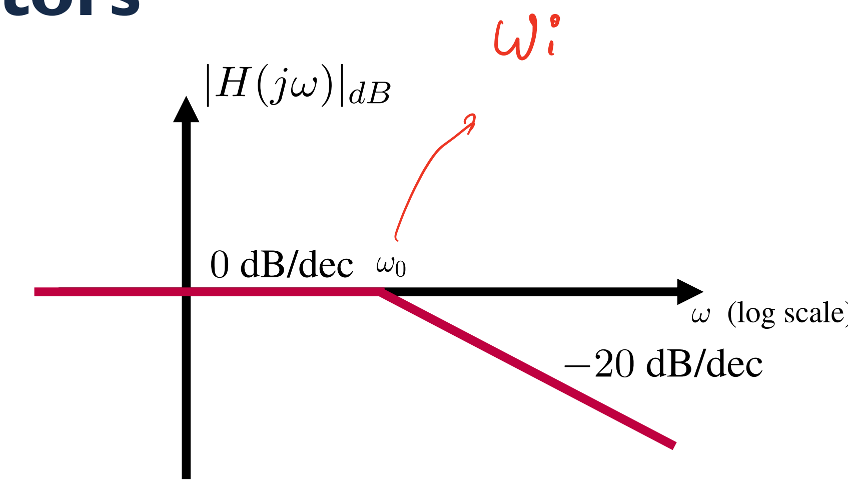
introduce to make notation compact

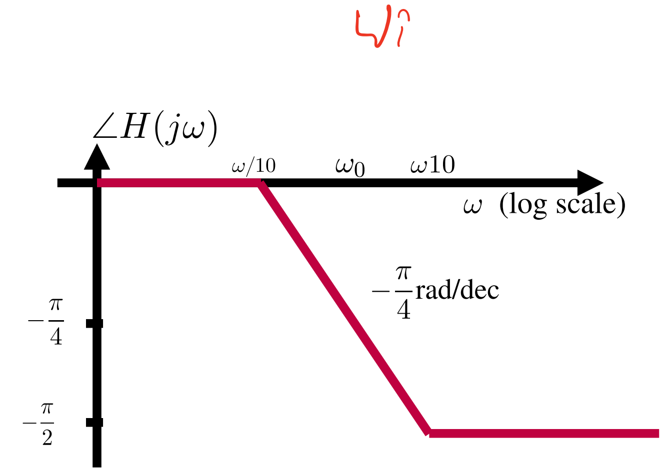
low-pass filter or high-pass filter depending on the factor

Key factors

$a_i = -1$

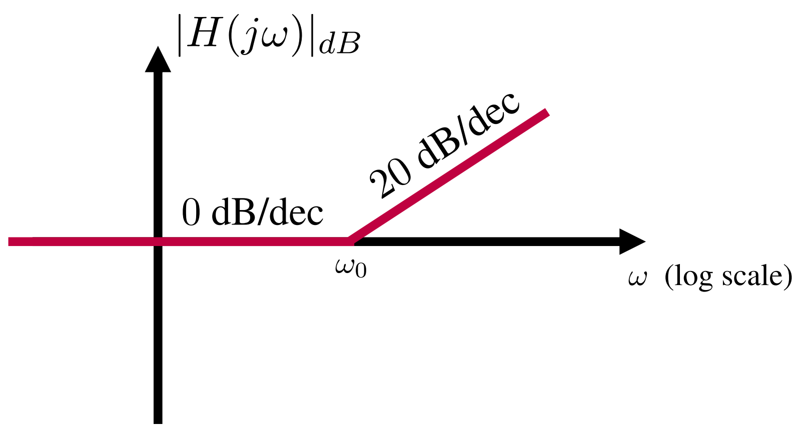
low-pass

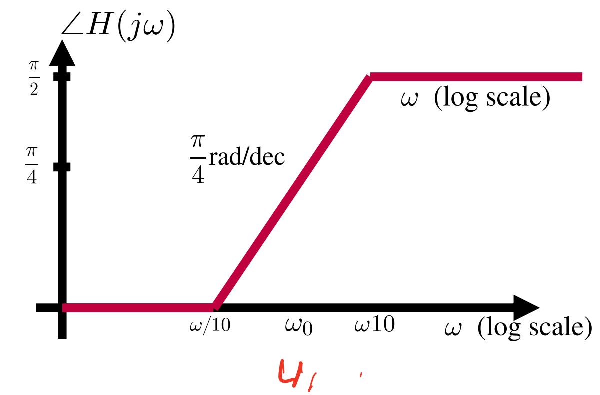




$a_i = 1$

high-pass





Magnitude response

$$H(j\omega) = \frac{A\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots}{(j\omega)^n\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\cdots}$$

$-20n$ dB/dec

overall gain

can be high or low-pass
in this case low-pass

$$|H(j\omega)|_{dB} = 20\log_{10}|A|$$

$$-\,20n\log_{10}\omega$$

high-pass terms

$$+\,20\log_{10}\left(1+\frac{j\omega}{z_1}\right) + 20\log_{10}\left(1+\frac{j\omega}{z_2}\right) + \ldots$$

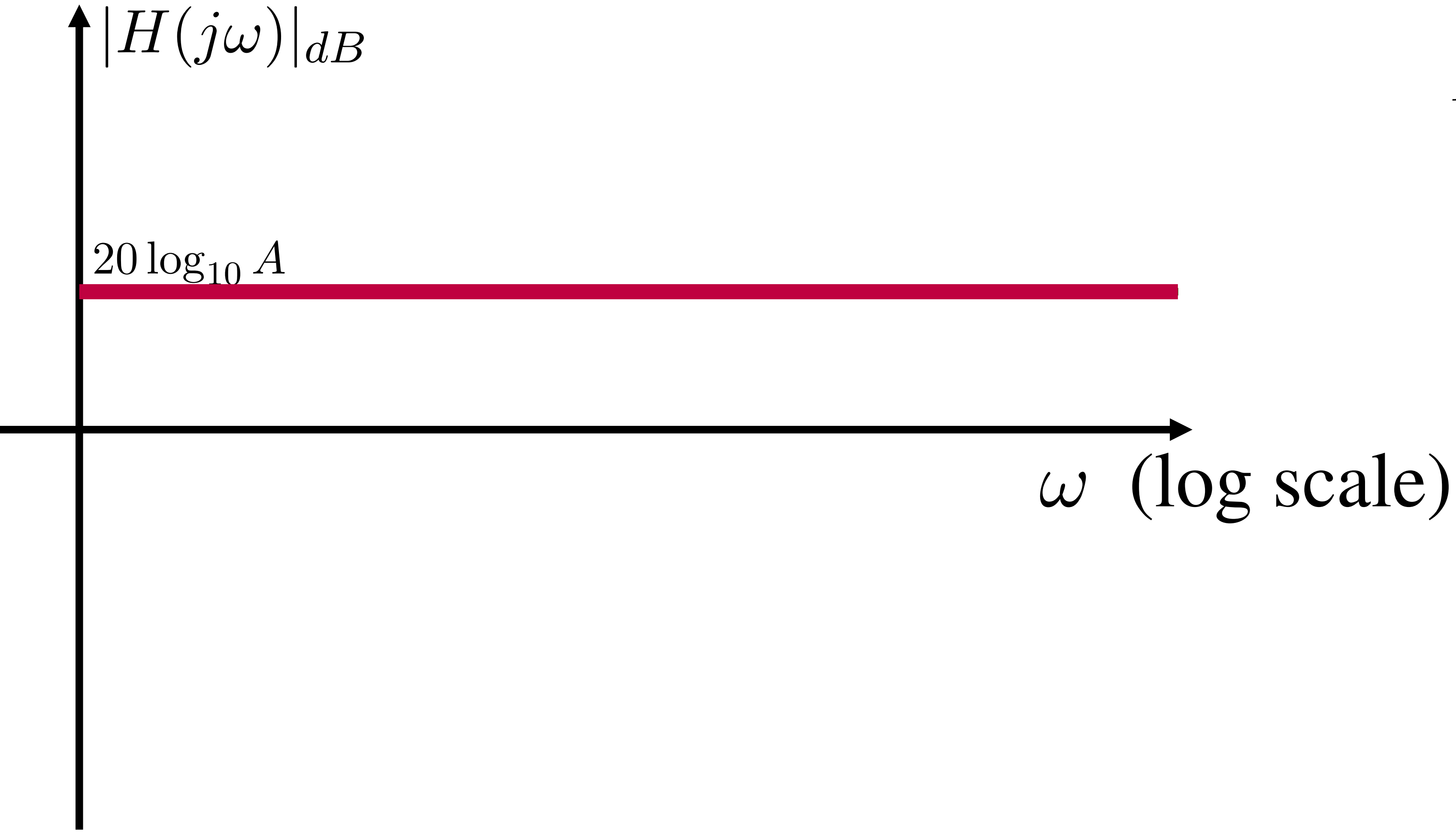
low-pass terms

$$-\,20\log_{10}\left(1+\frac{j\omega}{p_1}\right) - 20\log_{10}\left(1+\frac{j\omega}{p_2}\right) + \ldots$$

Sketching the magnitude response

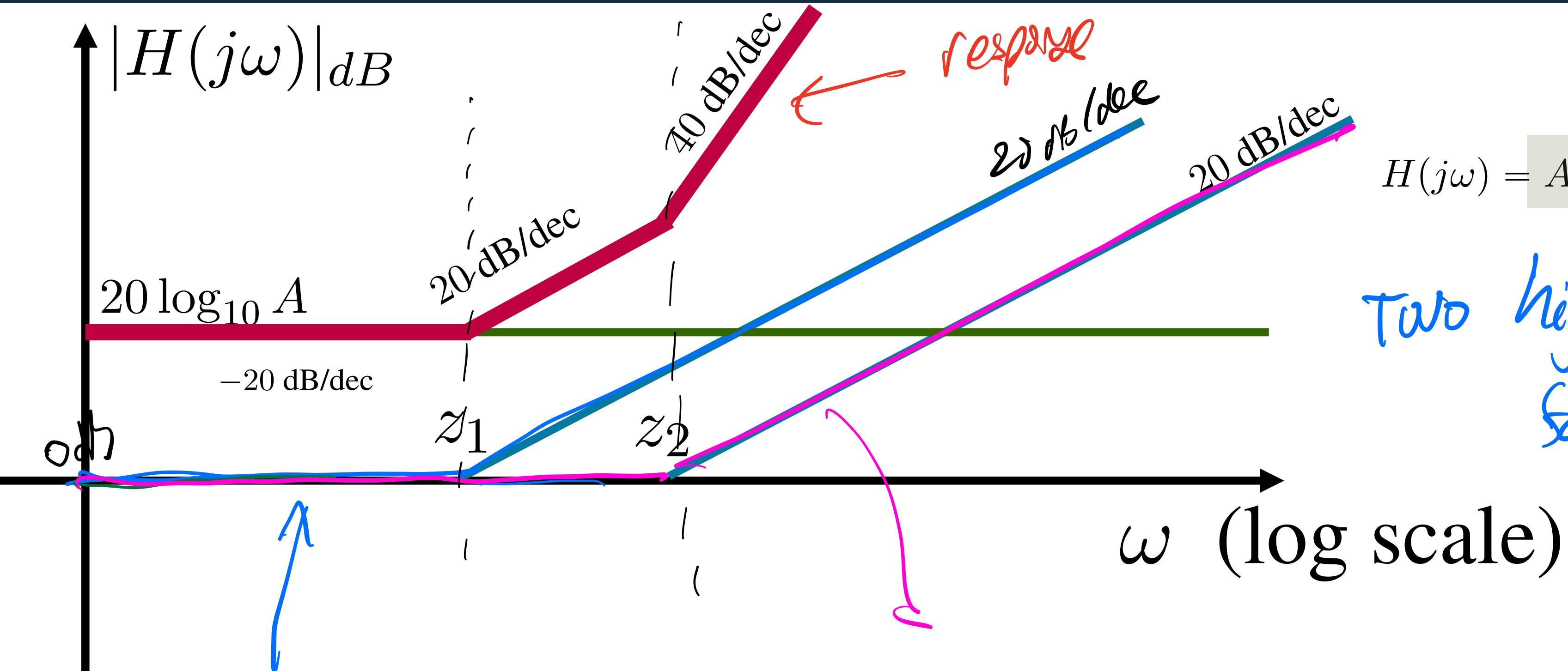
1. Identify all break-points
2. Plot the individual Bode plot for each term
3. Sum the result
4. Label all break points, slopes, and transition points

It is easiest to work from left to right, using the slopes of the lines to create the final plot



$H(j\omega) = \boxed{A}$

Note: when given actual numbers, also label the y-axis

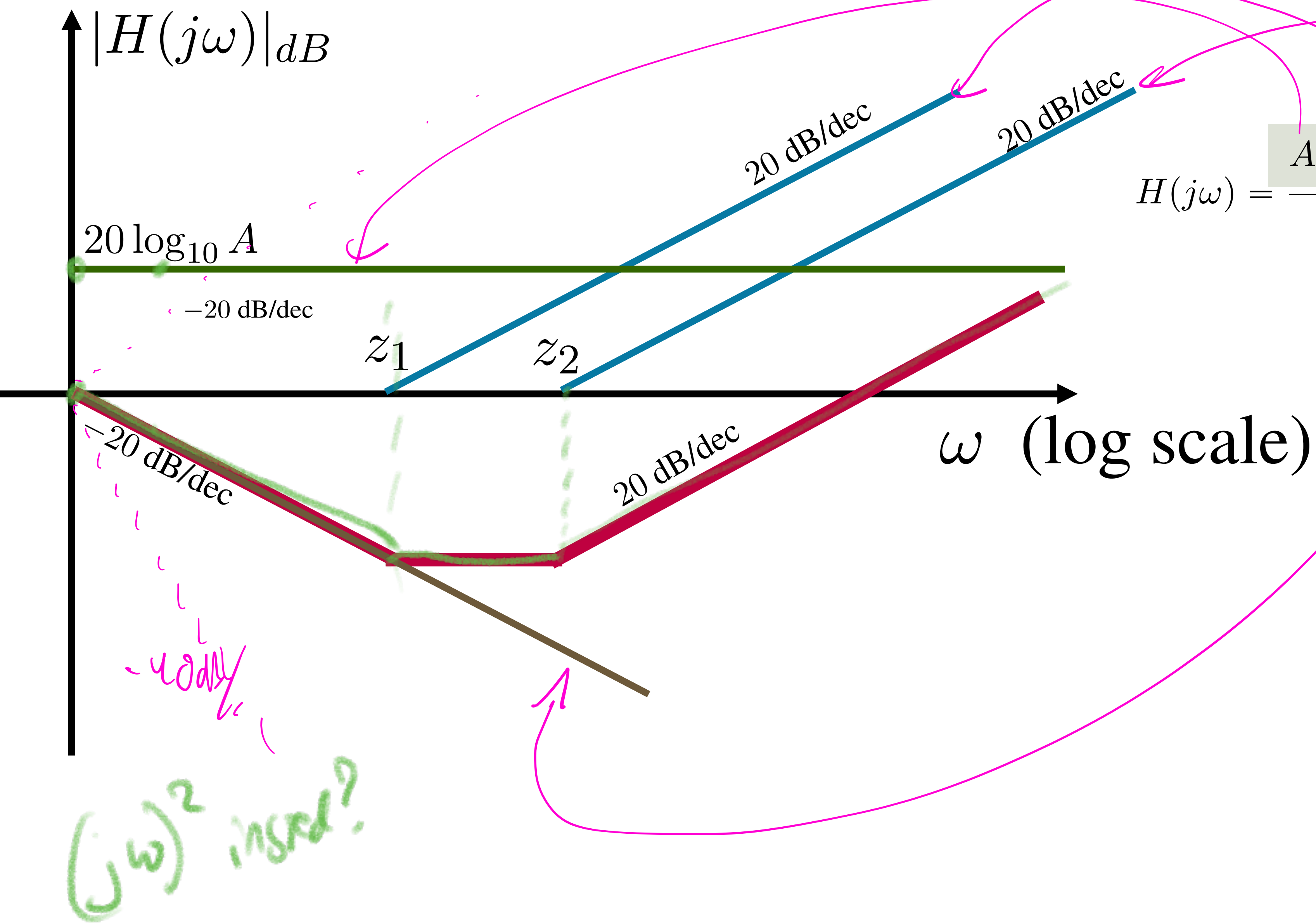


$$H(j\omega) = A\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)$$

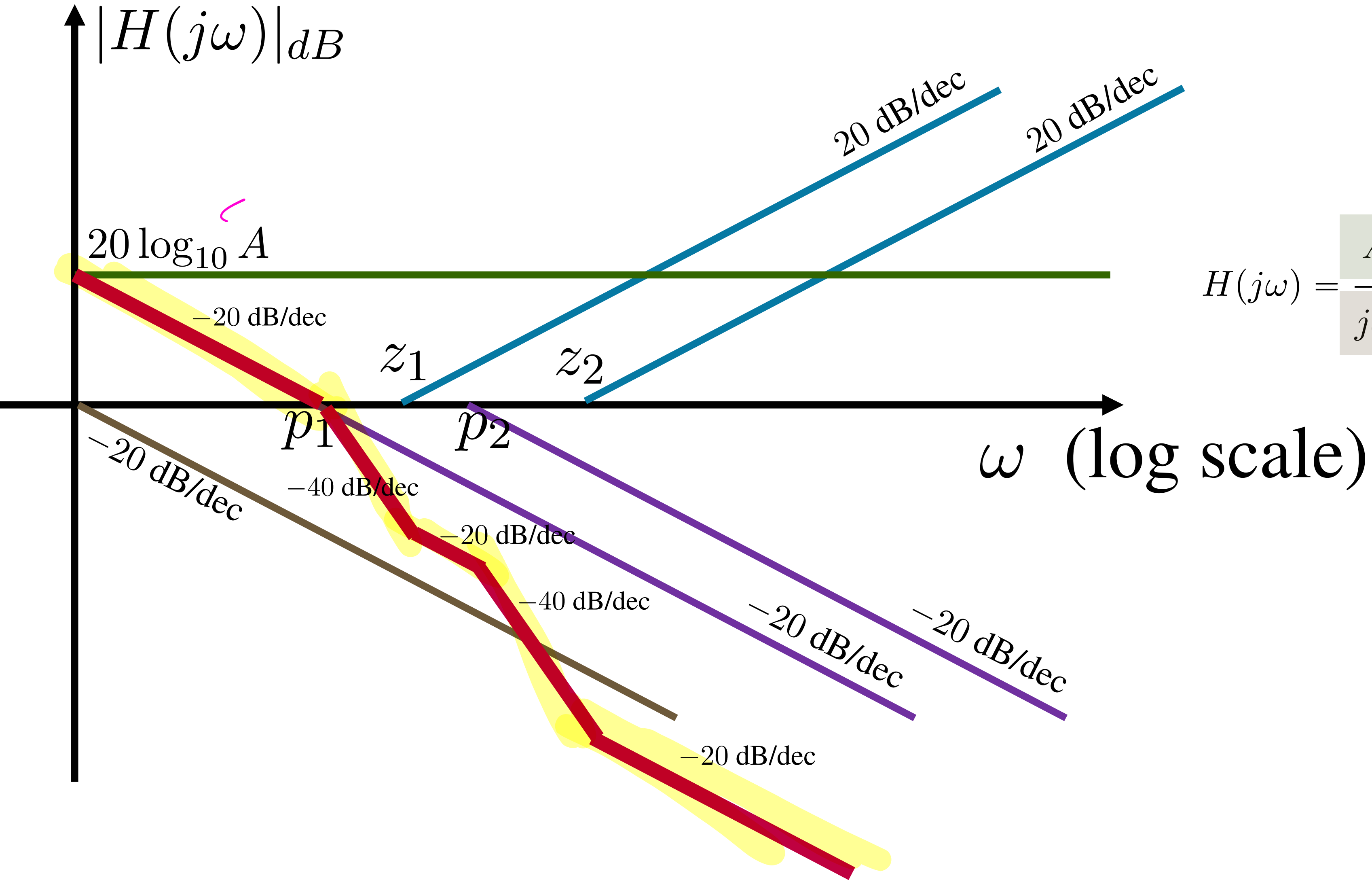
Two highpass sections

Corresponds to $(1+j\omega/z_1)$

Corresponds to $(1+j\omega/z_2)$



$$H(j\omega) = \frac{A\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)}{j\omega}$$



$$H(j\omega) = \frac{A\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)}{j\omega\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)}$$

Note: when given actual numbers, also label the y-axis

Phase response

zero

$$\angle H(j\omega) = \angle A$$

$$- \angle (j\omega)^n$$

constant offset can be (depending on value of n) $-\frac{\pi}{2}, -\pi, \frac{\pi}{2}, 0$

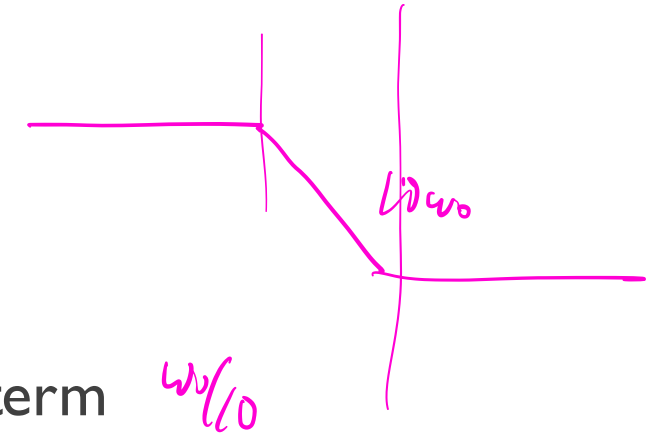
high-pass terms

$$+ \angle \left(1 + \frac{j\omega}{z_1}\right) + \angle \left(1 + \frac{j\omega}{z_1}\right) + \ldots$$

low-pass terms

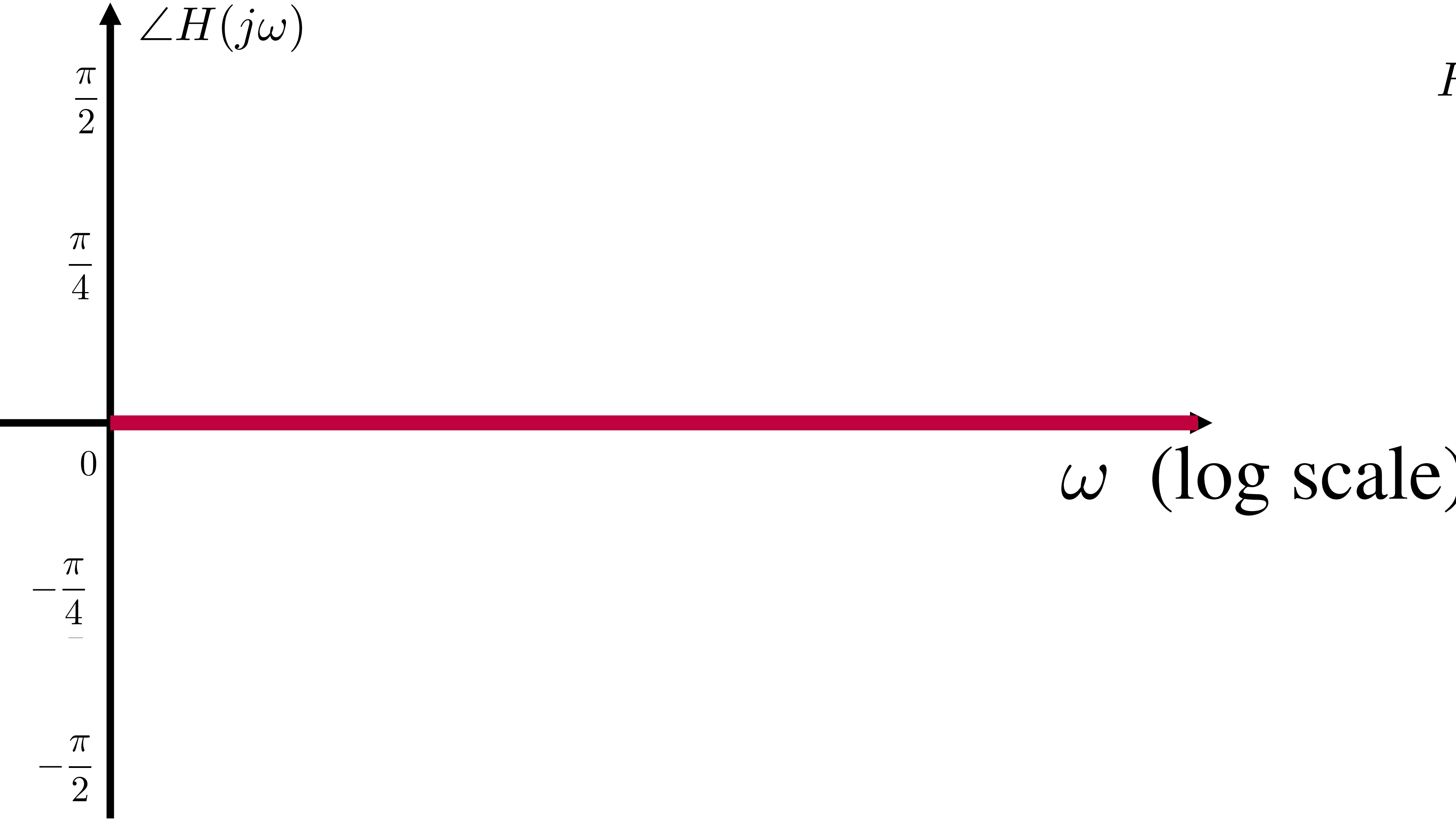
$$- \angle \left(1 + \frac{j\omega}{p_1}\right) - \angle \left(1 + \frac{j\omega}{p_2}\right) - \ldots$$

Sketching the phase response

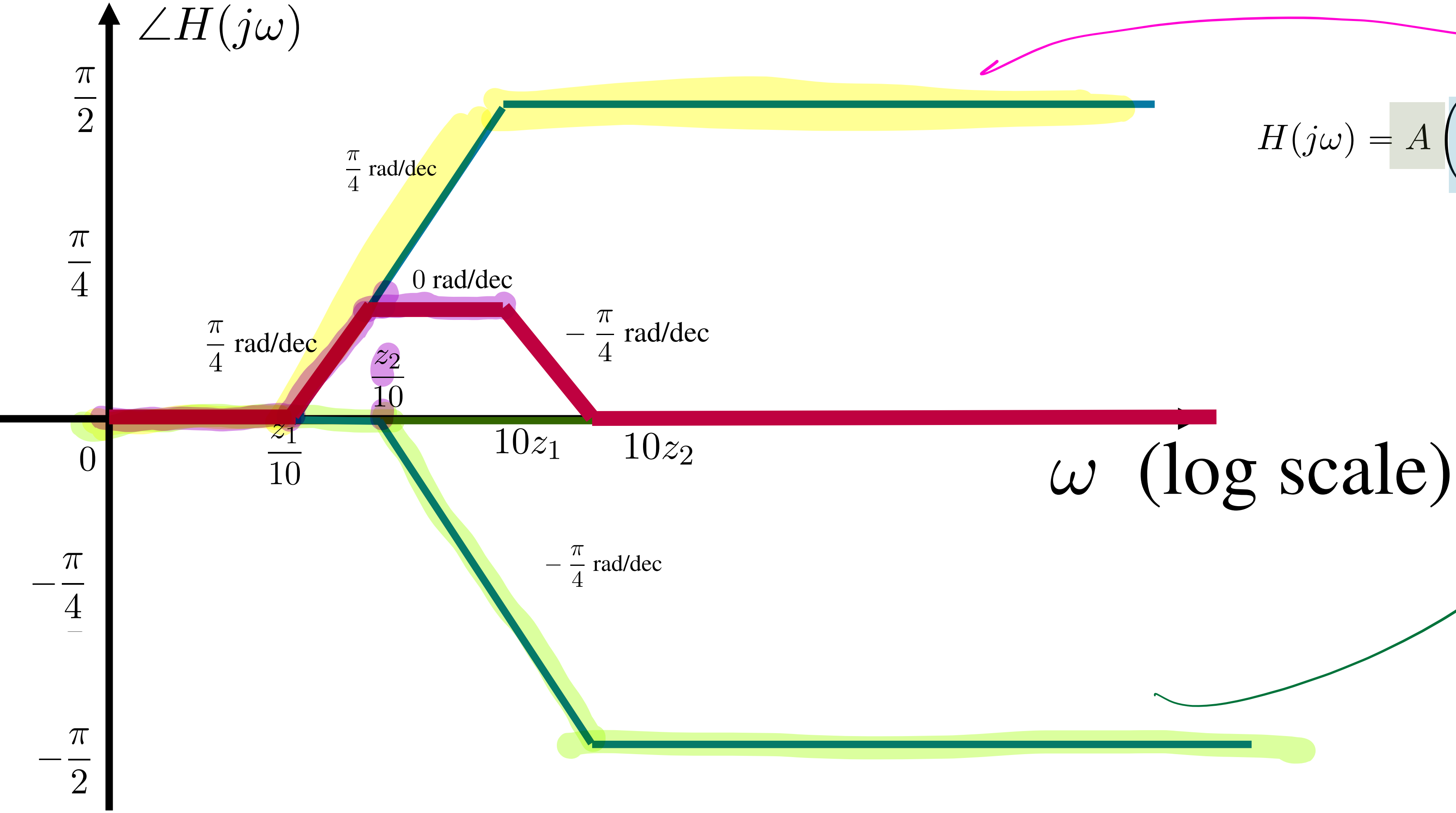


1. Identify all break-points
2. Plot the individual Bode plot for each term
3. Sum the result
4. Label all break points, slopes, and transition points

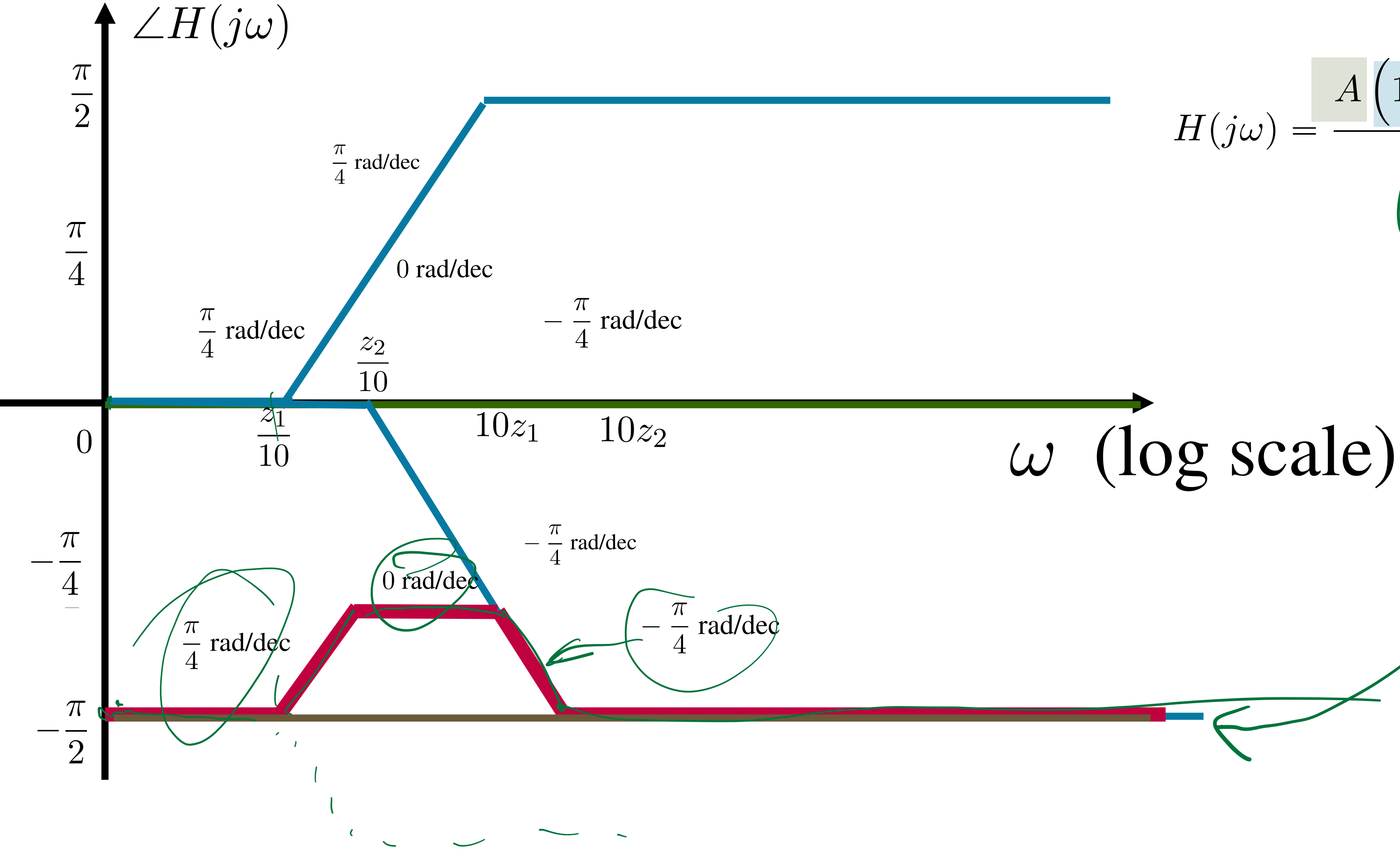
Be mindful that there are more break points



$$H(j\omega) = \boxed{A}$$

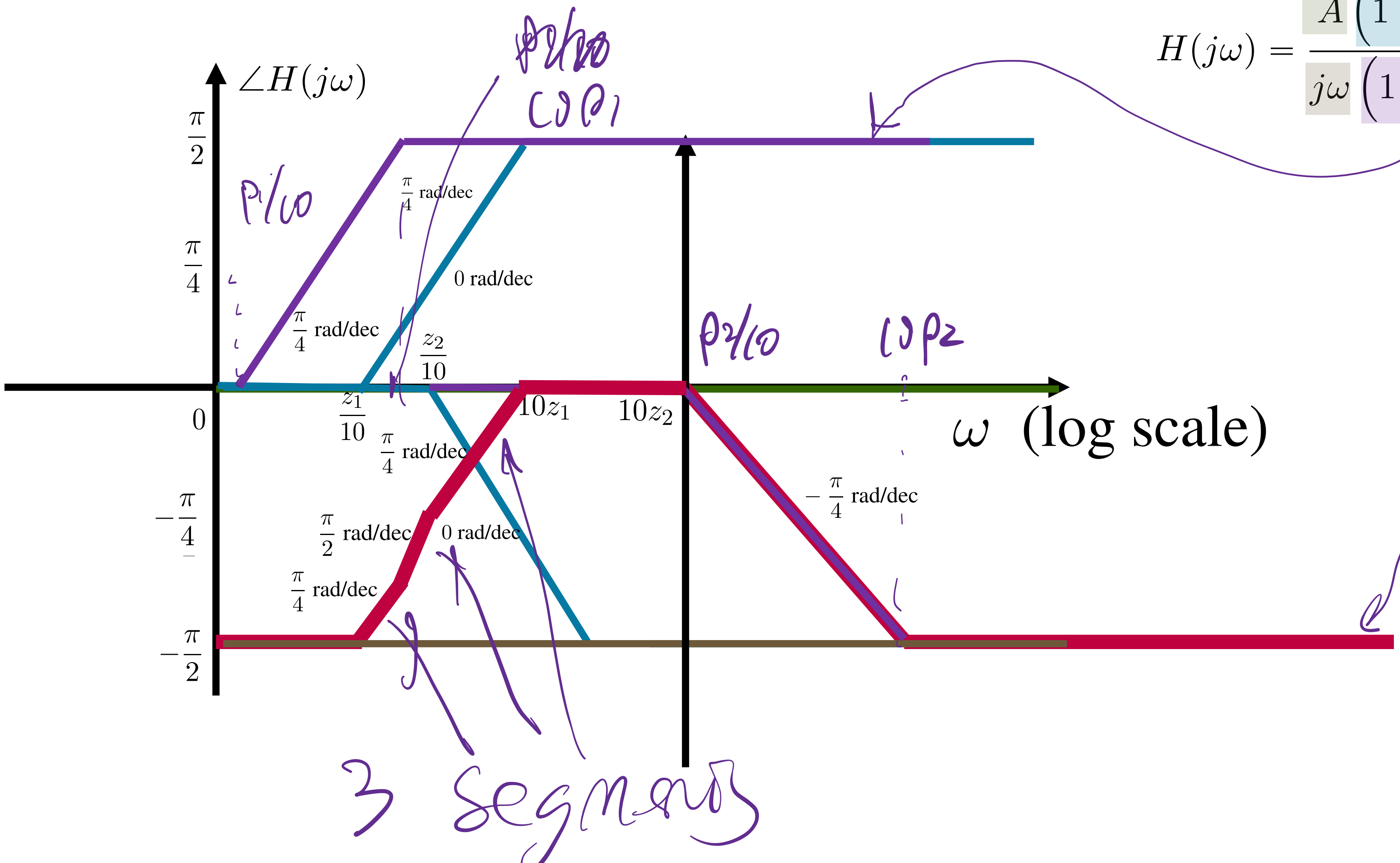


$$H(j\omega) = A\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)$$



$$H(j\omega) = \frac{A\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)}{j\omega}$$

$$H(j\omega) = \frac{A\left(1 + \frac{j\omega}{z_1}\right)\left(1 + \frac{j\omega}{z_2}\right)}{j\omega\left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)}$$



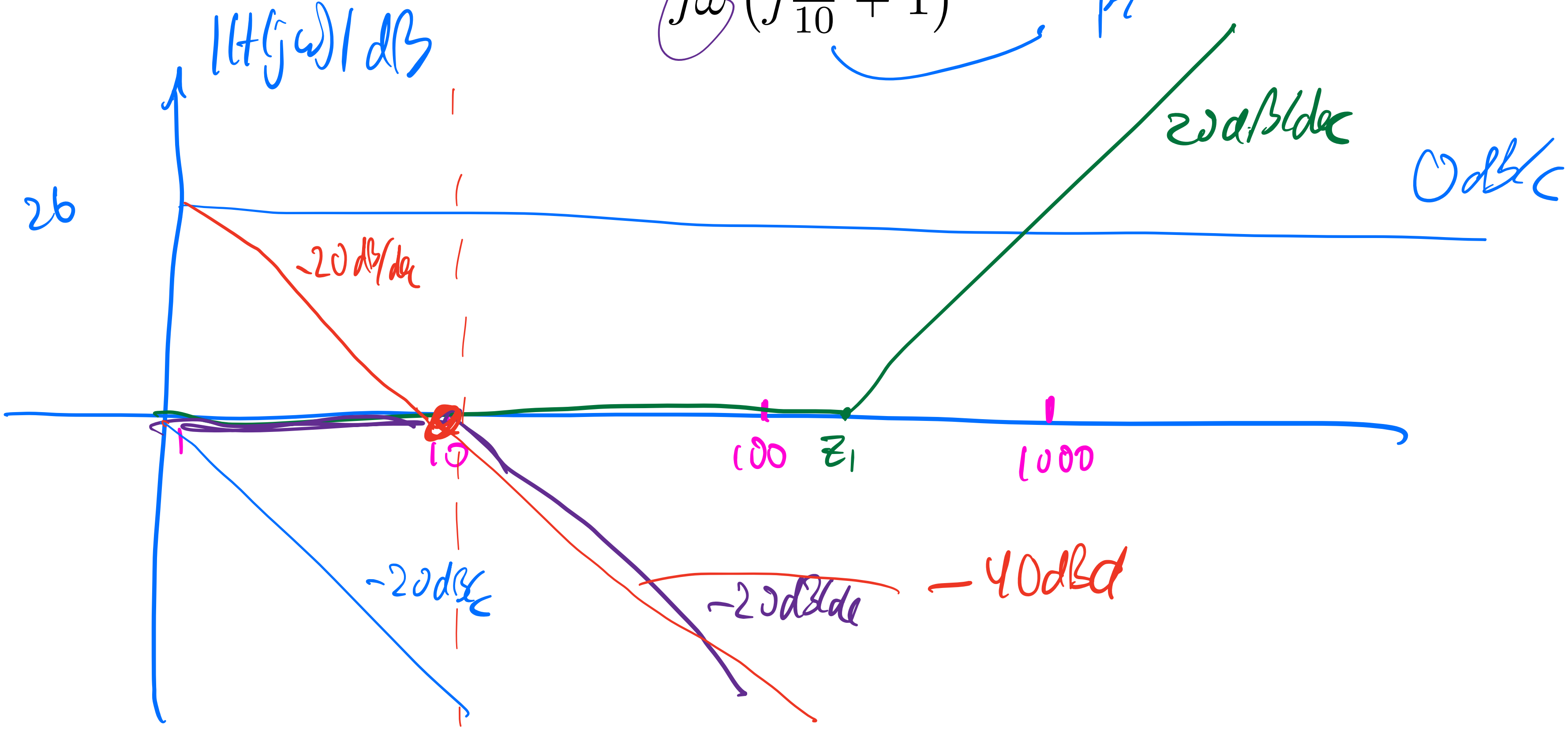
Example

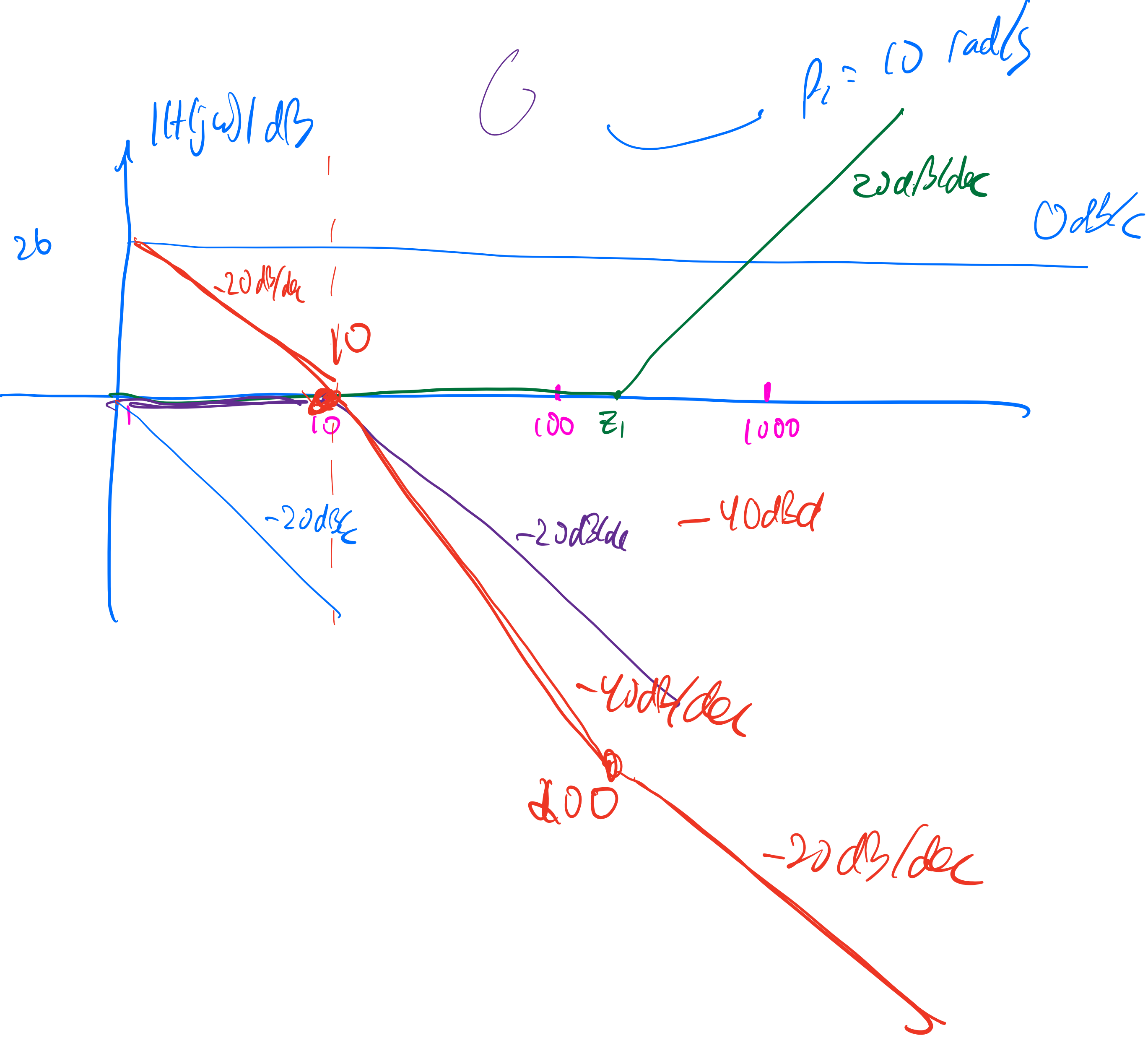
$20 \log_{10} 20$

$z_1 = 200$ rad/s

$$H(j\omega) = \frac{20\left(j\frac{\omega}{200} + 1\right)}{j\omega\left(j\frac{\omega}{10} + 1\right)}$$

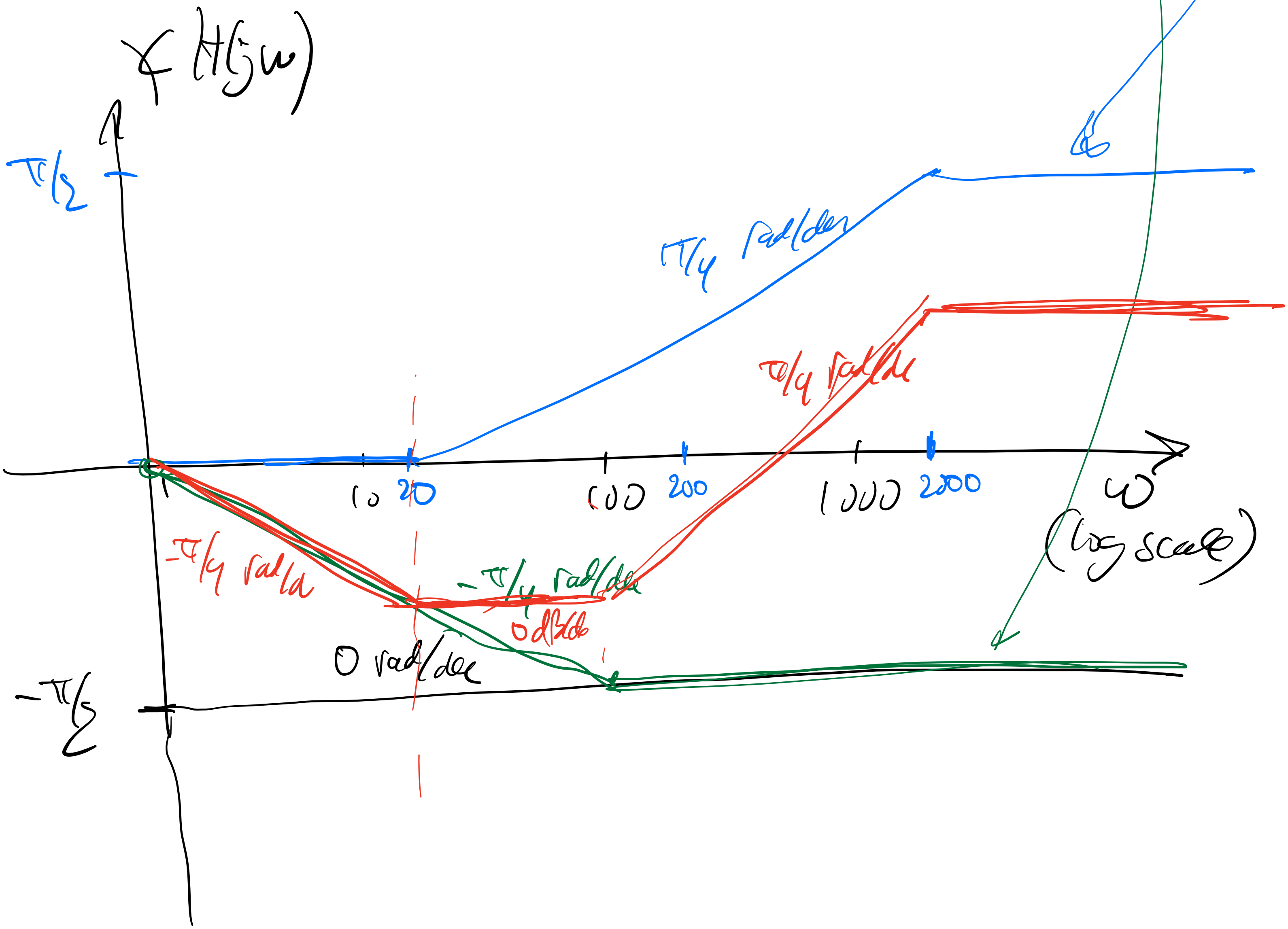
$p_1 = 10$ rad/s




$p_1 = 10$ rad/s
$|H(j\omega)|$ dB
20
-20dB/dec
10
20dB/dec
0dB/c
1
10
100
z_1
1000
-20dB/dec
-20dB/dec
-40dB/d
-40dB/dec
200
-20dB/dec

$$H(j\omega) = \frac{20\left(j\frac{\omega}{200}+1\right)}{j\omega\left(j\frac{\omega}{10}+1\right)}$$

3



Example (continued)

Example (continued)

Example (continued)