UC San Diego

Lecture 1

Signals, systems, circuits and phasors

ECE 45

Preview of today's lecture

- Introduction
- Context for the course: Tying up loose ends
  - Phasors, complex sinusoids and Fourier
  - Circuits and linear time invariant systems
- Practical details
- Mathematical fundamentals

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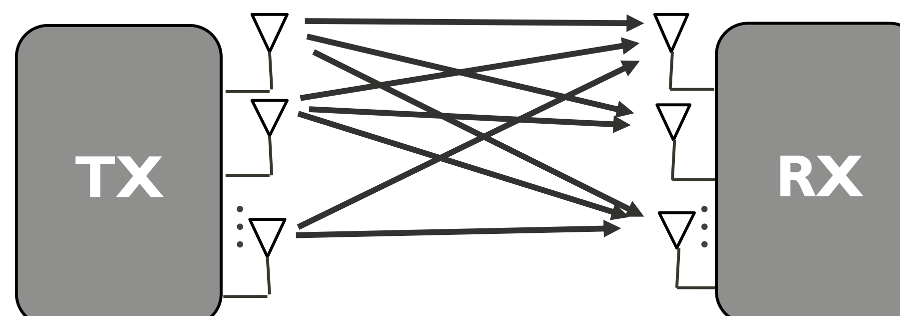
Quick facts

Graduated 52 Ph.D. students
> 800 papers, 65 US patents
99.5k citations, h-index 149
Fellow of IEEE, AAAS, NAI

3 textbooks, lab manual, monograph

MIMO communication



Context for the course

Learning objectives

- Recall key concepts from ECE 35
- Explain the holes that are plugged by ECE 45

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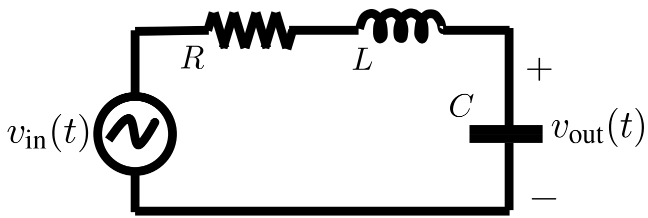
Circuits and linear systems



Analog design
ECE 35

$$x(t) \rightarrow \boxed{\text{system}} \rightarrow y(t)$$

Circuits and systems
ECE 45

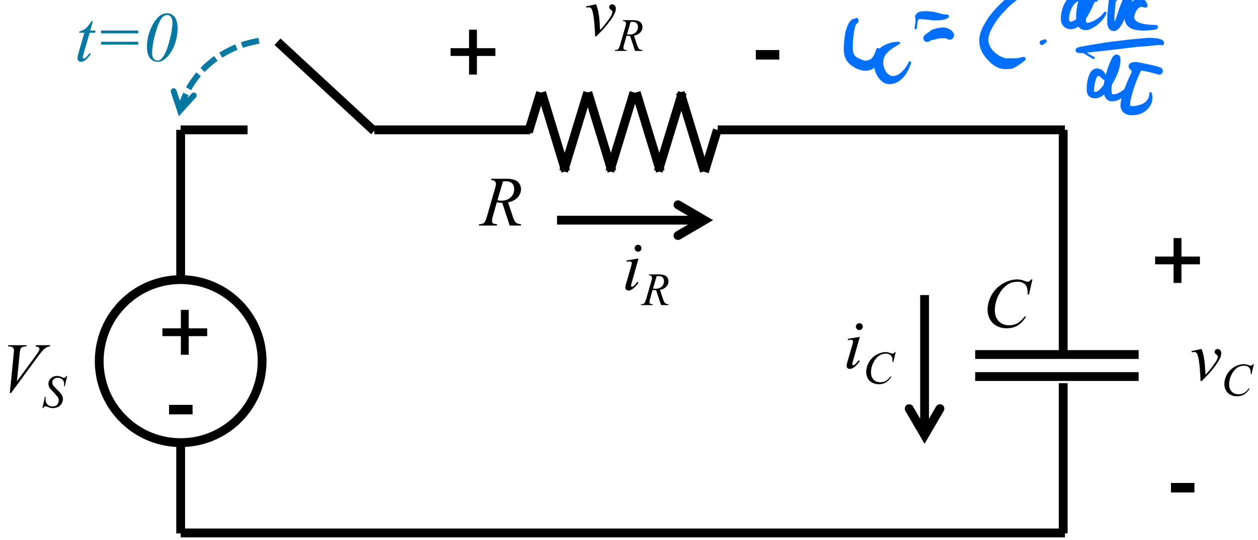
$$x[n] \rightarrow \boxed{\text{system}} \rightarrow y[n]$$

Linear systems
ECE 101

ECE 45 serves as an important transition from simple analog circuits to analog and digital signal processing

From C. Schurgers ECE 35 course materials

Solving for voltages and currents in a circuit



$i_c = C \cdot \frac{dv_c}{dt}$

$$i_C = i_R$$
$$V_S = v_R + v_C$$
$$i_R = \frac{v_R}{R}$$
$$i_C = C \cdot \frac{dv_C}{dt}$$

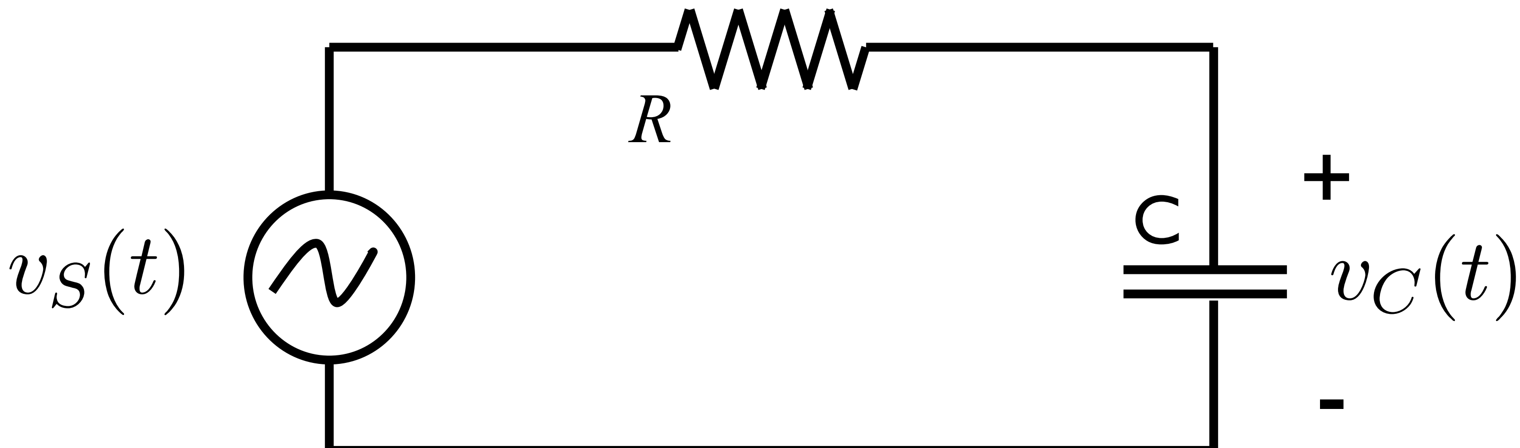
$v_C(0^-) = V_0$ Initial condition

$$C \cdot \frac{dv_C}{dt} = \frac{v_R}{R} = \frac{V_S - v_C}{R}$$
$$RC \frac{dv_C}{dt} + v_C = V_S$$

1st order differential equation

$$v_C(t) = (V_0 - V_S)e^{-\frac{t}{RC}} + V_S$$

The circuit as a system

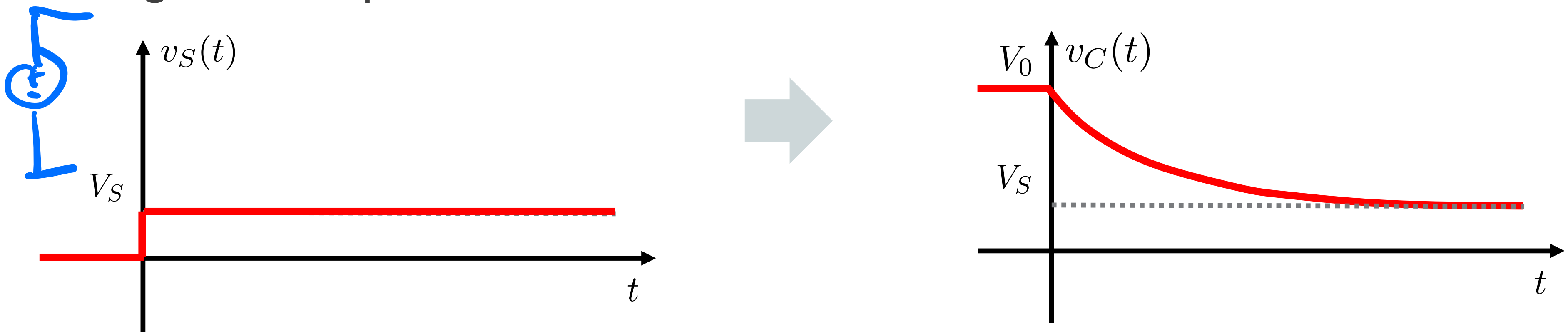


input
(voltage signal)

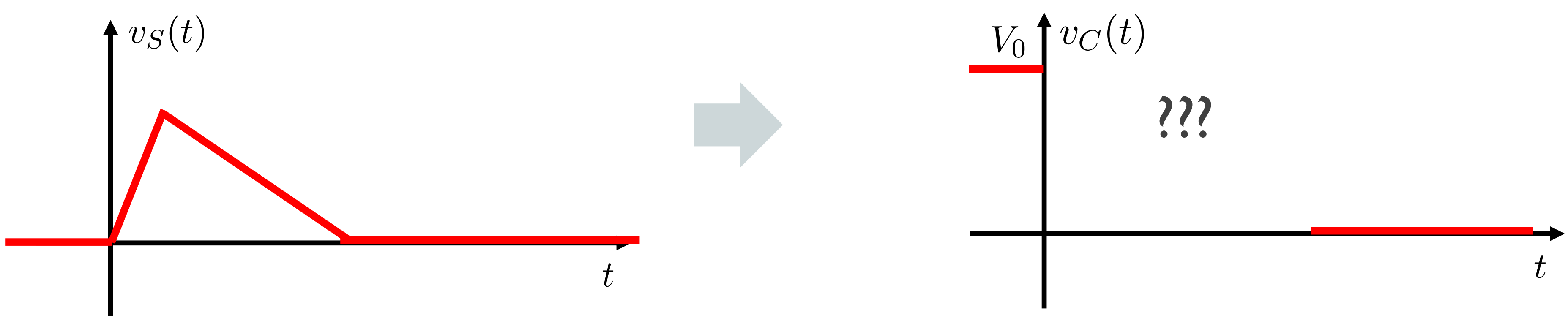
output
(voltage signal)

How to compute outputs for other inputs?

In the given example

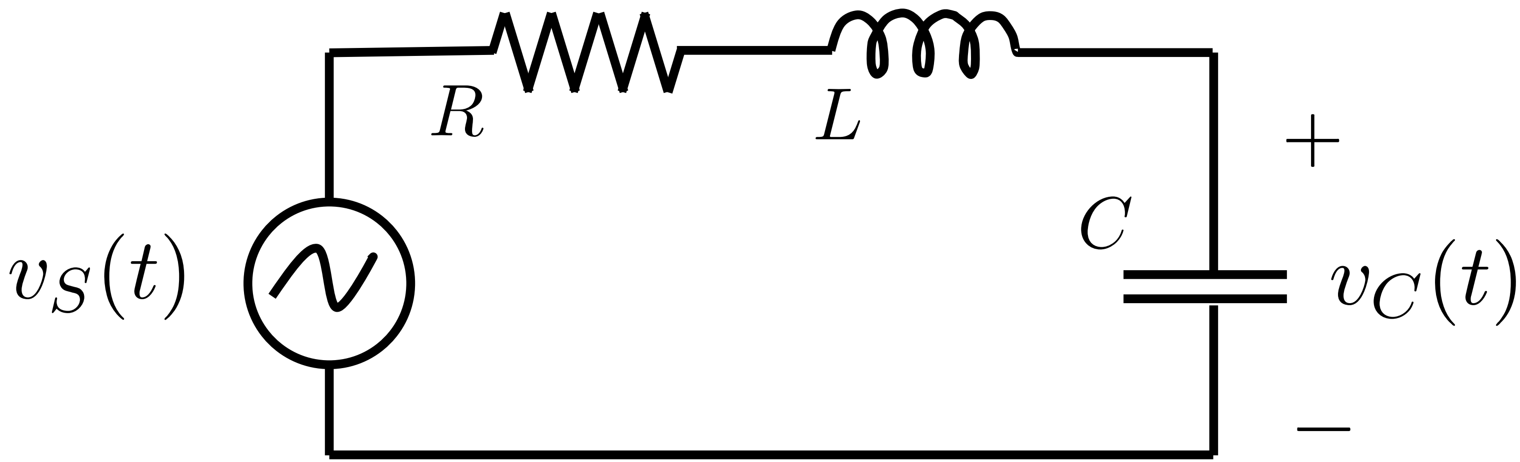


What about for a different input signal?



How to compute outputs for other circuits?

Example RLC circuit



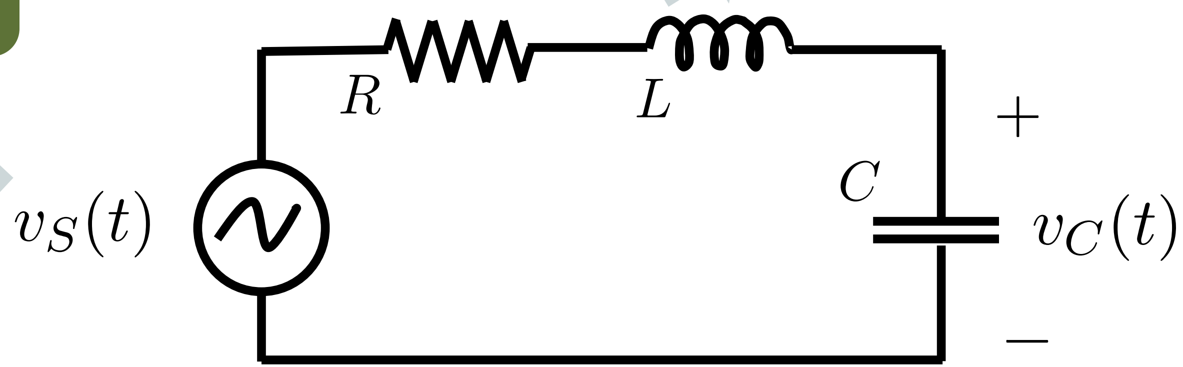
Use KVL to write the differential equation and solve ???

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

Approach in ECE 45

Recognize circuits as linear time invariant systems

Build upon phasor knowledge



Leverage important connections between sinusoids, linear time invariant systems and the Fourier transform

Complete solutions to differential equations, discrete-time signals, and the Laplace and Z transforms are deferred to ECE 101 ☹

Phasors and Fourier

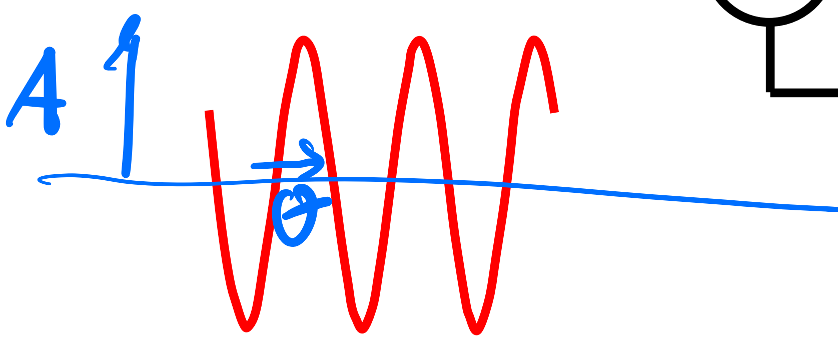
Learning objectives

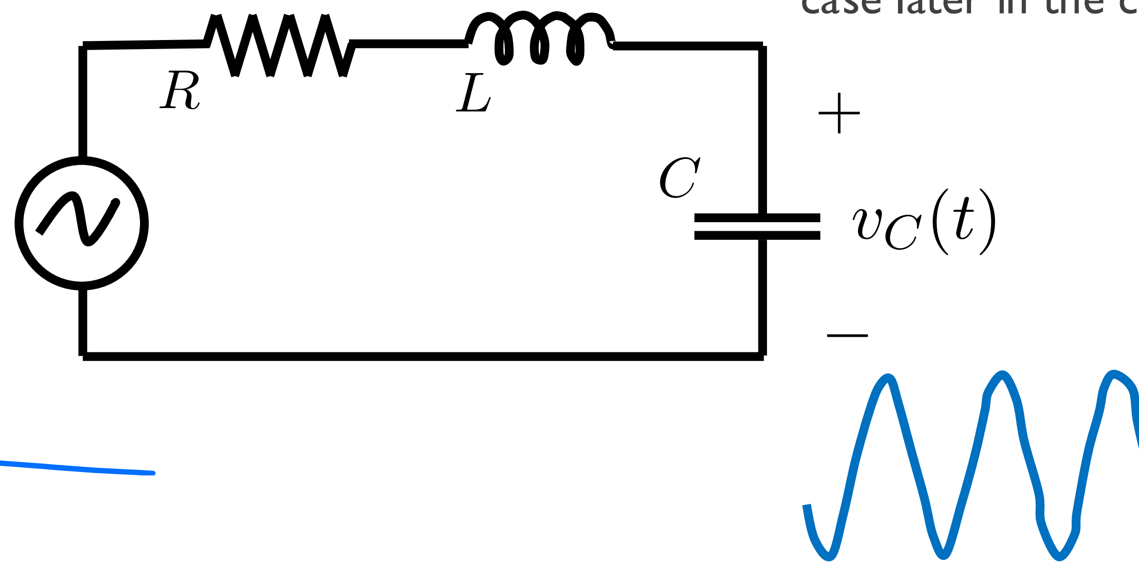
- Recall the use of phasors in AC circuit analysis
- Motivate the use of the Fourier transform in the context of phasors

Alternating current (AC) analysis of a circuit

$2\pi f$

$$v(t) = A\cos(\omega t + \theta)$$





* We prove why this is the case later in the course

Solve for desired output voltage or current assuming the input is a sinusoid

Works because all the measured voltages and currents are also sinusoids*

AC analysis of a circuit: Treat the input as a phasor

- Recall that a sinusoid with $A>0$ can be written as

$$A\cos(\omega t+\theta) = \mathrm{Re}\left[Ae^{j\theta}e^{j\omega t}\right]$$

$e^{j(\omega t+\theta)}$

$= e^{j\omega t}\, e^{j\theta}$

phasor

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$$v(t) = A\cos(\omega t+\theta) \longleftrightarrow \mathbf{V} = Ae^{j\theta}$$

time domain

real function

depends on time

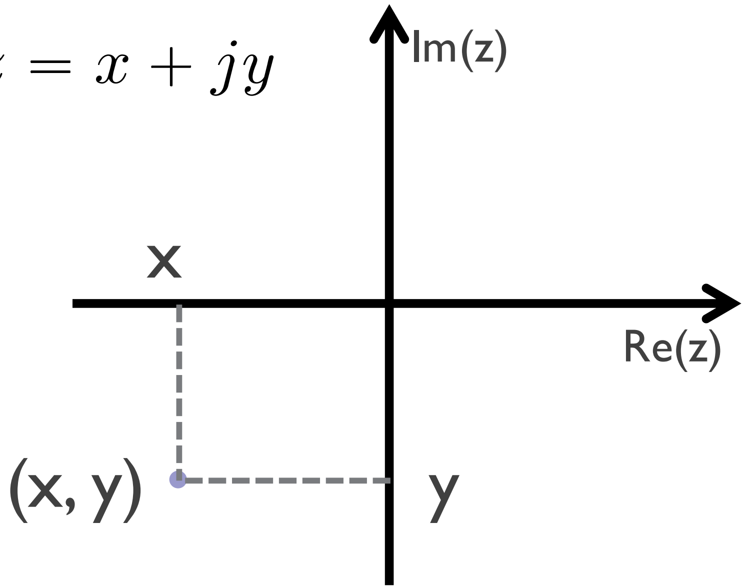
phasor domain

complex number

not a function of time

Complex numbers

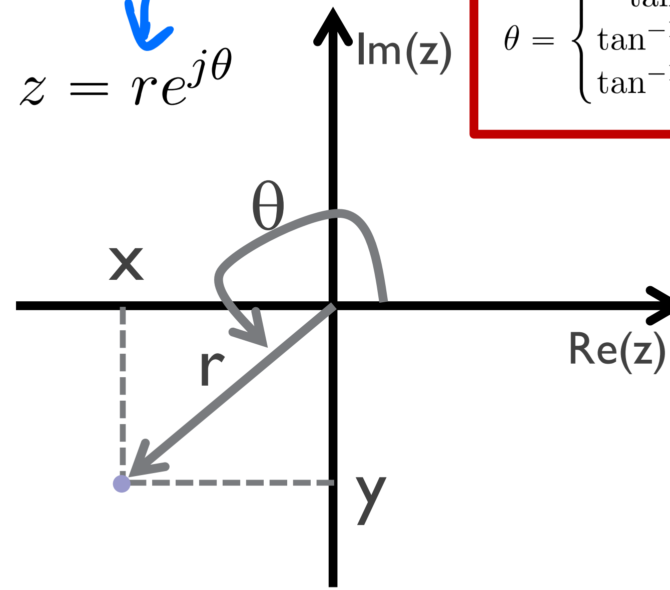
$$z = x + jy$$



Cartesian form

r>0

$$z = re^{j\theta}$$



Polar form

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & x < 0, y \geqslant 0 \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & x < 0, y < 0 \end{cases}$$

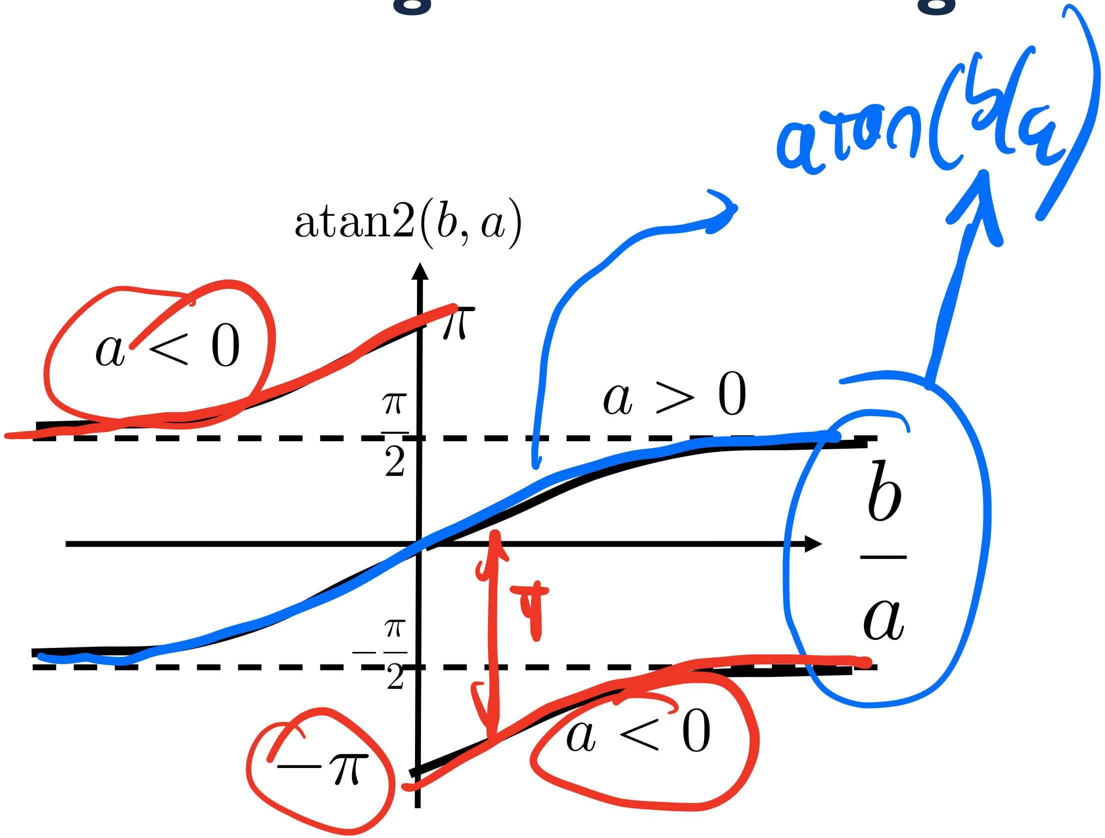
The quadrant matters!

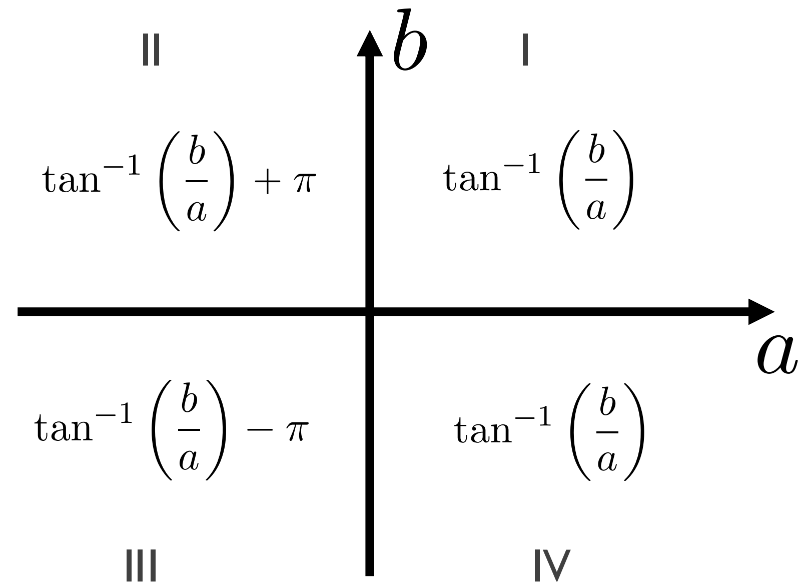
$\mathrm{atan2}(x, y)$

- Complex number is essentially a pair of independent real numbers z = (x, y)
  - Widely used in engineering and science
  - $j = \sqrt{-1}$ is the imaginary number (EE's use j because i is often current)

j

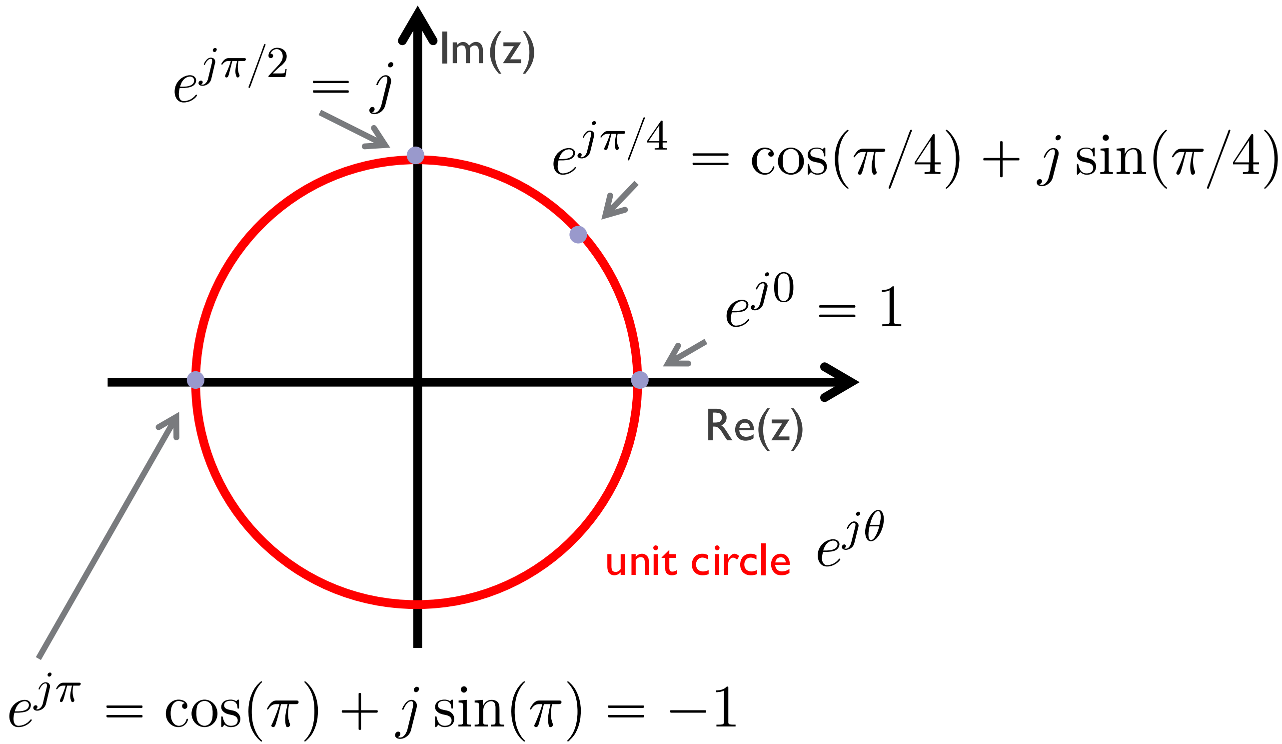
The 2-argument arctangent atan2(b,a)





Euler's formula

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$



Useful facts

$$e^{jx} = \cos x + j\sin x$$
$$e^{-jx} = \cos x - j\sin x$$
$$e^{jx} + e^{-jx} = 2\cos x$$
$$e^{jx} - e^{-jx} = 2j\sin x$$

$$\boxed{\begin{aligned}\cos(\theta) &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)\\ \sin(\theta) &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)\end{aligned}}$$

Working with complex numbers

$x(t) = a(t) + j b(t)$ $\quad$ $x^*(t) = a(t) - j b(t)$

consider these two complex numbers

$$z_1 = a + jb = r_1 e^{j\theta_1}$$

$$z_2 = c + jd = r_2 e^{j\theta_2}$$

addition

$$z_1 + z_2 = (a + jb) + (c + jd)$$
$$= (a + c) + j(b + d)$$

real $\quad$ imag

multiplication

$$z_1 z_2 = (a + jb)(c + jd)$$
$$= (ac - bd) + j(bc + ad)$$
$$= r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

conjugate

$$z_1^* = a - jb$$

$$z_1 + z_1^* = a - jb + a + jb$$
$$= 2a$$
$$= 2\mathrm{Re}(z_1)$$

$$z_1 z_1^* = r_1 r_1 e^{j(\theta_1 - \theta_1)}$$
$$= r_1^2$$
$$= x^2 + y^2$$

magnitude squared

division

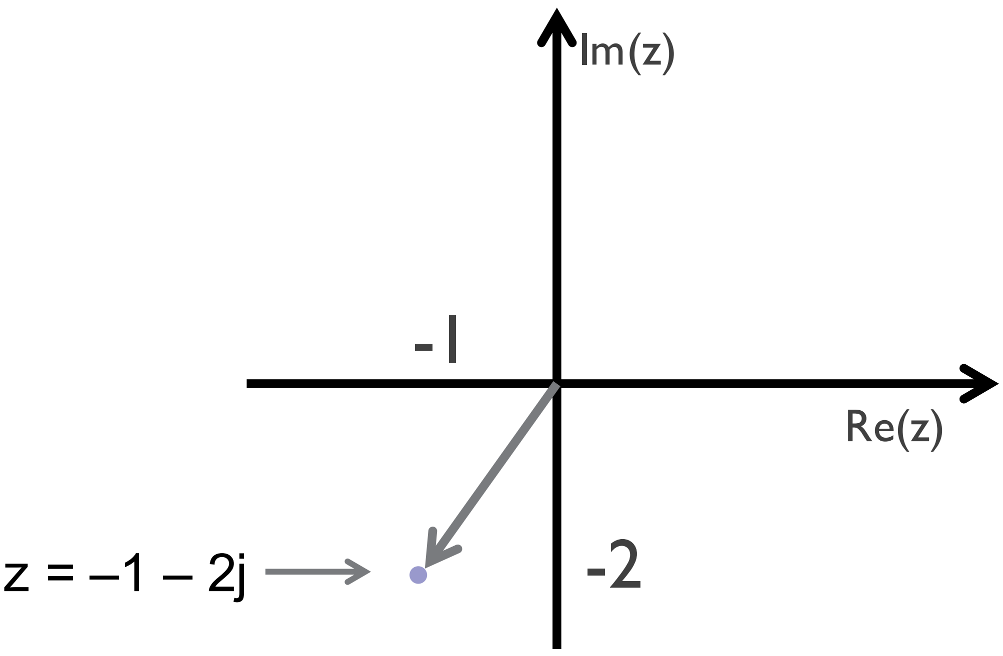
$$z_1 / z_2 = (r_1 / r_2) e^{j(\theta_1 - \theta_2)}$$

Example

- Consider z = –1 – 2j
  - Plot this complex number
  - Find its polar form

$$r = \sqrt{(-1)^2 + (-2)^2}$$
$$= \sqrt{5}$$

$$\theta = \tan^{-1}(-1/-2) - \pi$$
$$= -2.0344 \quad \text{radians}$$
$$= -0.6476\pi \quad \text{radians}$$



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Example

- Let $z_1 = 1 - j2$

  $z_2 = 2 + j3$

- Compute

$$z_1 + z_2$$

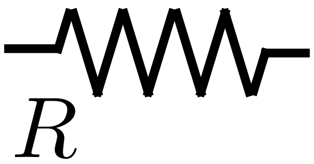
$$z_1 z_2$$

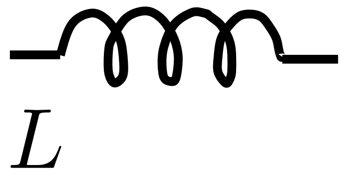
$$z_1 / z_2$$

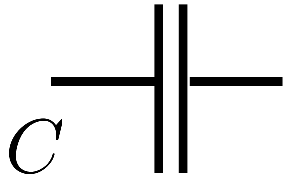
$$z_1 z_1^*$$

$$z_1 - z_1^*$$

AC analysis of a circuit: Convert to complex impedances







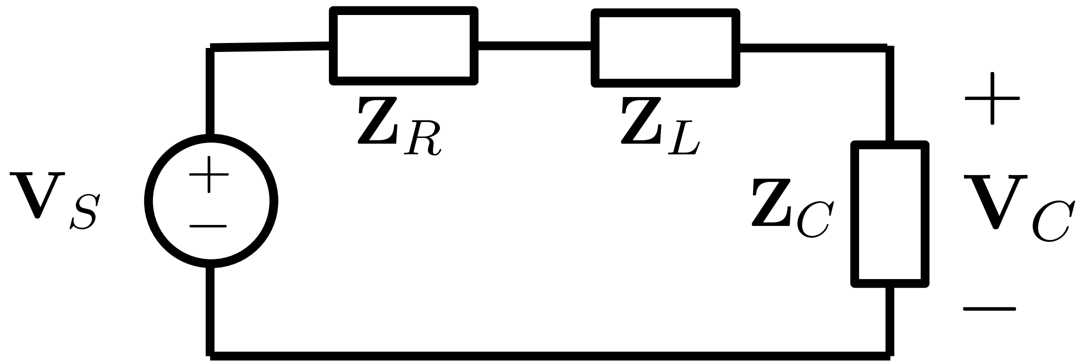
$$v(t) = Ri(t) \qquad v(t) = L\frac{di}{dt} \qquad i(t) = C\frac{dv}{dt}$$

$$\mathbf{V} = R\,\mathbf{I} \qquad \mathbf{V} = j\omega L\,\mathbf{I} \qquad \mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$$

$$\mathbf{Z}_R = R \qquad \mathbf{Z}_L = j\omega L \qquad \mathbf{Z}_C = \frac{1}{j\omega C}$$

freq

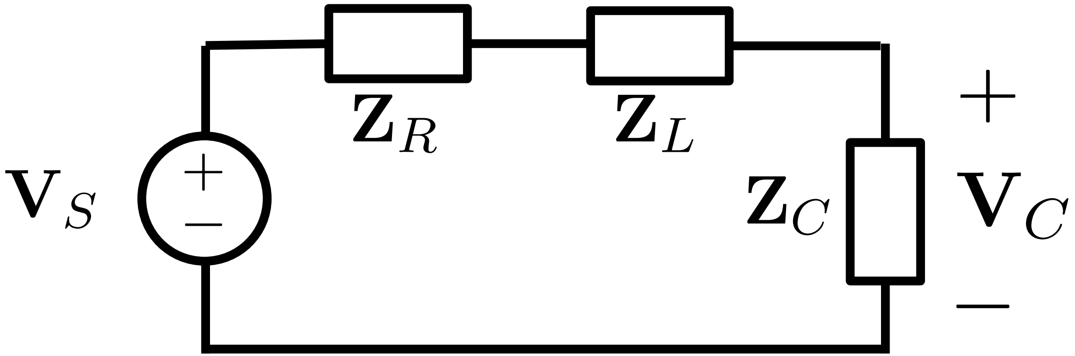
AC analysis of a circuit: Solve with impedances



$V_S = Ae^{j\theta}$

$$V_C = \frac{Z_C}{Z_R + Z_L + Z_C} V_S$$

AC analysis of a circuit: Solve with impedances



voltage divider

$Ae^{j\theta}$

$$\mathbf{V}_C = \frac{\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C}\mathbf{V}_S$$

$$= \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}\mathbf{V}_S$$

$$\cdot \frac{j\omega C}{j\omega C}$$

Frequency Response

$$= \boxed{\frac{1}{1 - \omega^2 LC + j\omega RC}}\mathbf{V}_S$$

AC analysis of a circuit: Convert to polar form

$$\mathbf{Z}_E = \frac{1}{1 - \omega^2 LC + jRC\omega}$$

$$\mathbf{V}_C = \mathbf{Z}_E \mathbf{V}_S$$

AC analysis of a circuit: Convert to polar form

$$\mathbf{Z}_E = \frac{1}{1 - \omega^2 LC + jRC\omega} \qquad \mathbf{V}_C = \mathbf{Z}_E \mathbf{V}_S$$

$$|\mathbf{Z}_E| = \sqrt{\frac{1}{(1 - \omega^2 LC)^2 + (RC\omega)^2}}$$

$$\frac{Ae^{j\theta_1}}{Be^{j\theta_2}} = \frac{A}{B} e^{j(\theta_1 - \theta_2)}$$

$$\phi_E = -\text{atan2}(1 - \omega^2 LC, \omega RC)$$

why?

AC analysis of a circuit: Convert back to the time domain

phasor

$$v_S(t) = A\cos(\omega t + \theta)$$

$$v_C(t) = \mathrm{Re}\left[|\mathbf{Z}_E|e^{j\phi_E}Ae^{j\theta}e^{j\omega t}\right]$$

AC analysis of a circuit: Convert back to the time domain

$$v_S(t) = A\cos(\omega t + \theta)$$

$$v_C(t) = \mathrm{Re}\left[|\mathbf{Z}_E|e^{j\phi_E}Ae^{j\theta}e^{j\omega}\right]$$

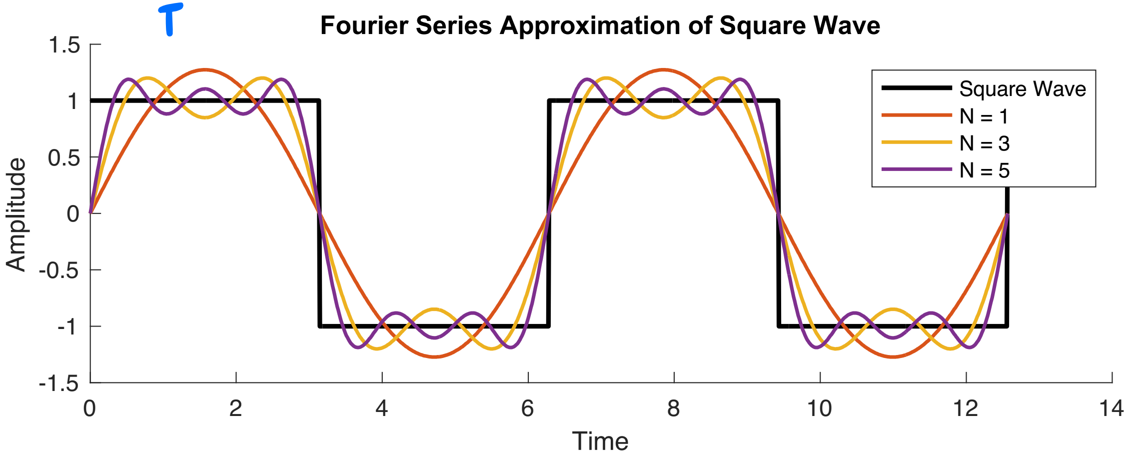
$$= A|\mathbf{Z}_E|\cos(\omega t + \theta + \phi_E)$$

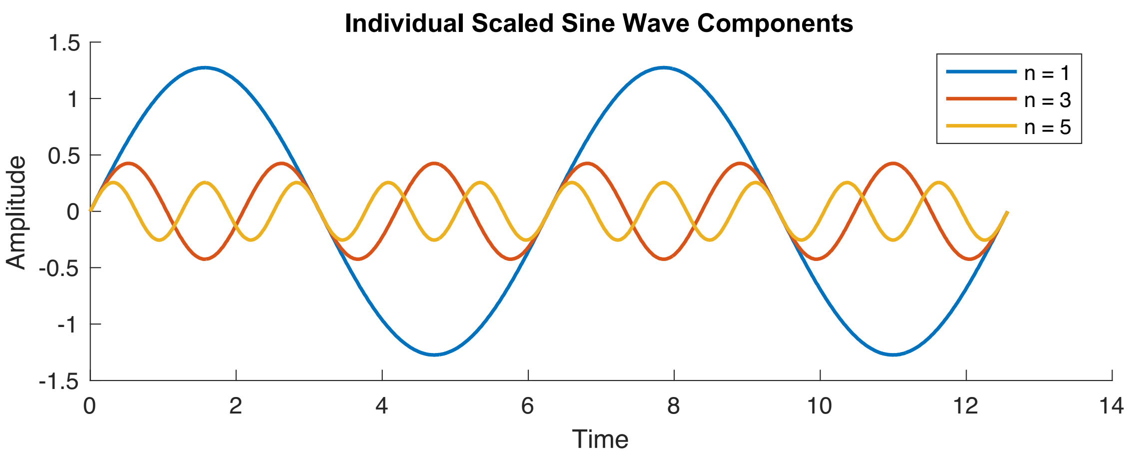
$$= A\sqrt{\frac{1}{(1-\omega^2LC)^2 + (\omega RC)^2}}\cos(\omega t + \theta + \mathrm{atan2}(1-\omega^2LC, \omega RC))$$

Amplitude and phase also depend on the frequency (but not time)

Why do we care about sinusoids? Part I

$2\pi \frac{t}{T}$

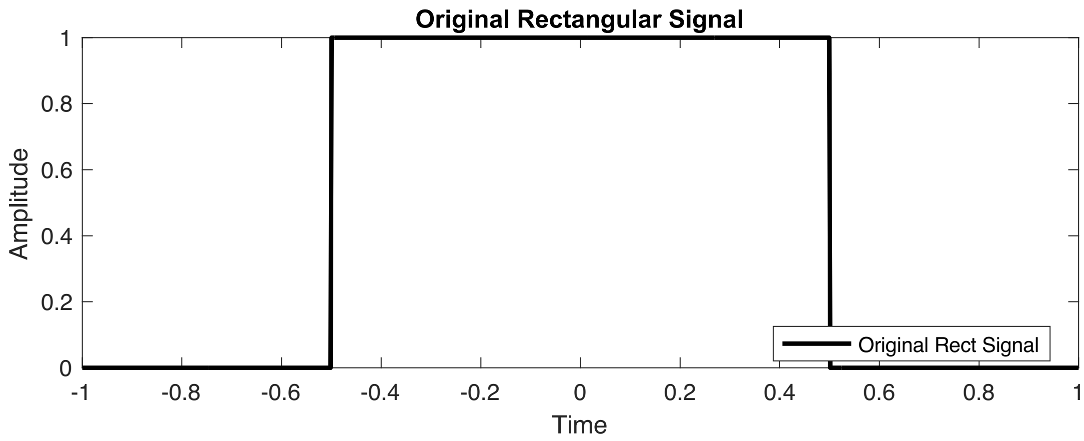


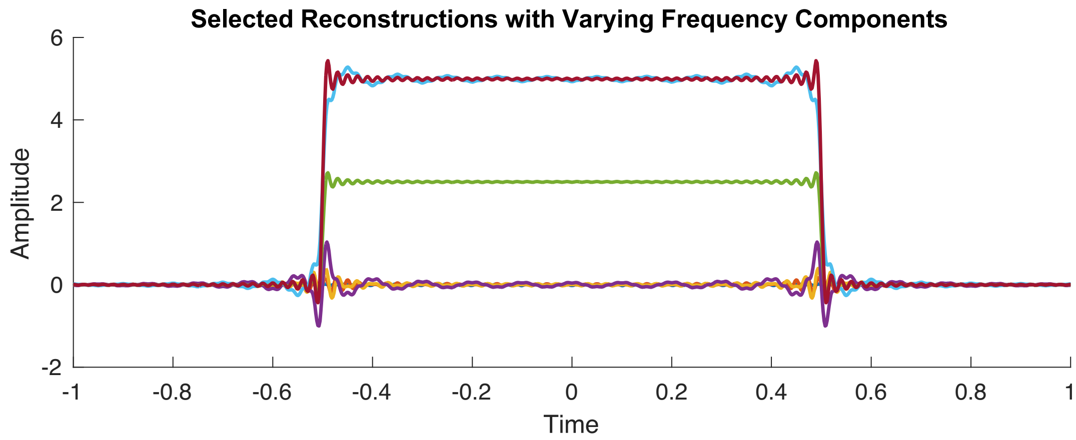


Fourier Series

It is possible to reconstruct periodic signals with sums of sinusoids!

Why do we care about sinusoids? Part II

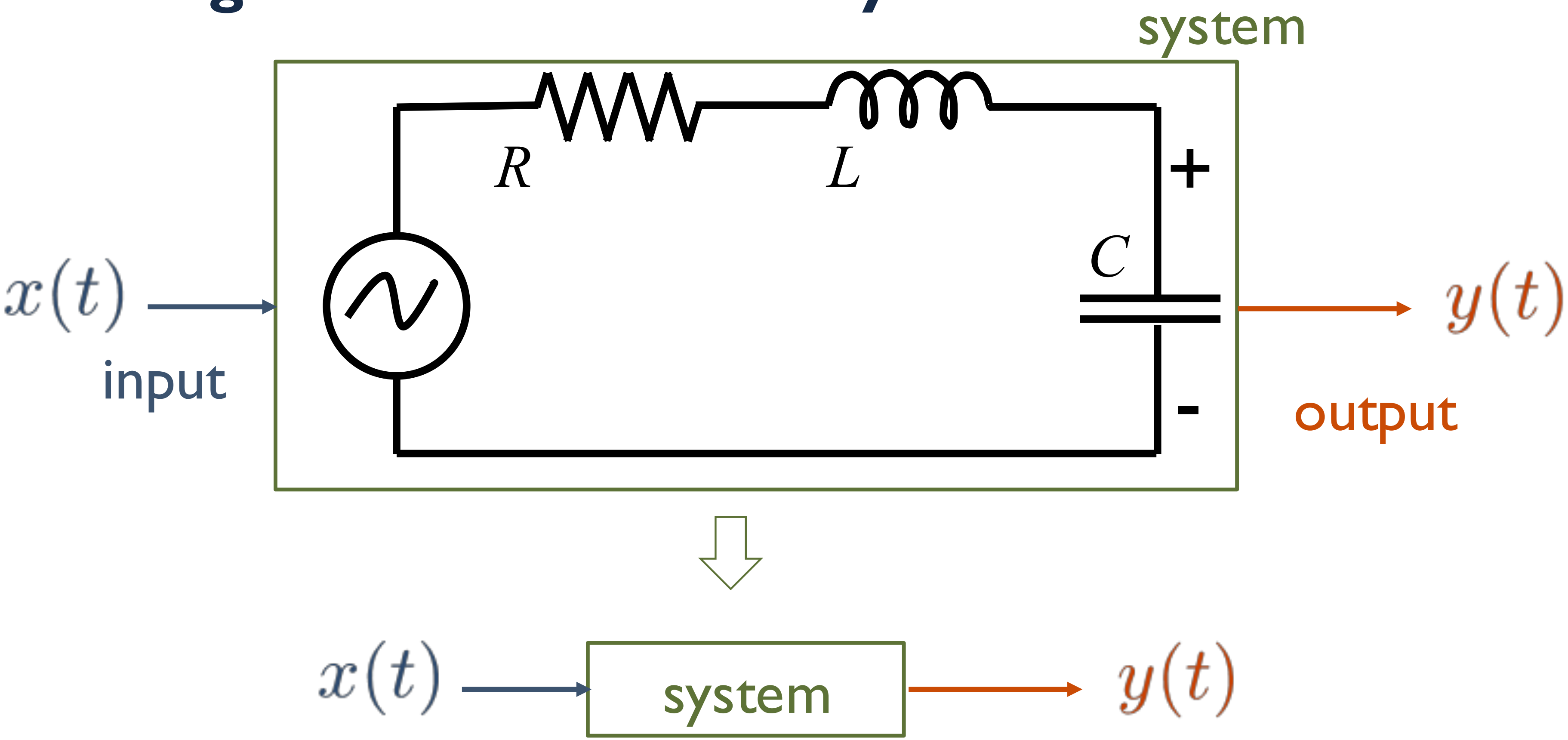




Fourier Transform

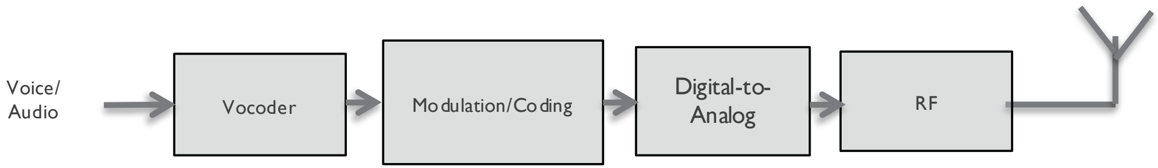
It is possible to reconstruct non-periodic signals using integrals of sinusoids!

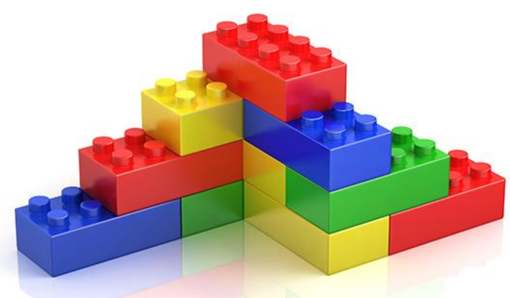
Thinking about circuits as a system

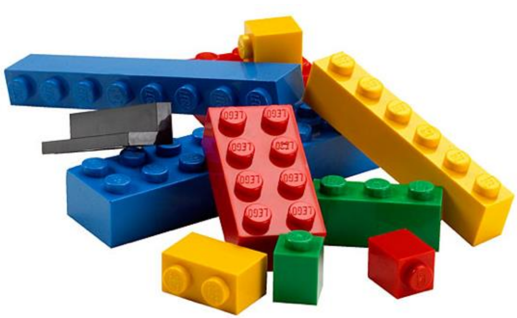


A mathematical description of how the input is transformed into the output

What is "systems thinking"?







A complicated object decomposed into functional blocks that interact

https://www.gwinnettpl.org/calendar/building-with-legos-ages-5-10-2/
https://www.tibco.com/blog/2015/05/05/lego-learning-the-building-blocks-of-data-visualization/
http://www.brickowl.com/catalog/lego-city-corner-set-7641

Why do we care about sinusoids? Part III

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

Special systems that satisfy the linear and time invariant (LTI) property like circuits at rest (zero initial conditions) are fully characterized by their frequency response!

$$e^{j\omega t} \longrightarrow \boxed{\text{system}} \longrightarrow \boxed{H(j\omega)}e^{j\omega t}$$

complex sinusoid

frequency response

Why do we care about sinusoids? Part IV

Fourier

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

Component of $x(t)$ represented by $e^{j\omega t}$

$$X(j\omega)e^{j\omega t} \longrightarrow \boxed{\text{system}} \longrightarrow \underbrace{X(j\omega)H(j\omega)}_{Y(j\omega)}e^{j\omega t}$$

Component of $y(t)$ represented by $e^{j\omega t}$

Sinusoidal outputs for LTI systems are easy to compute by using the Fourier Series or the Fourier transform

Connections to back to ECE 45

Lectures 2 - 4 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 5 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

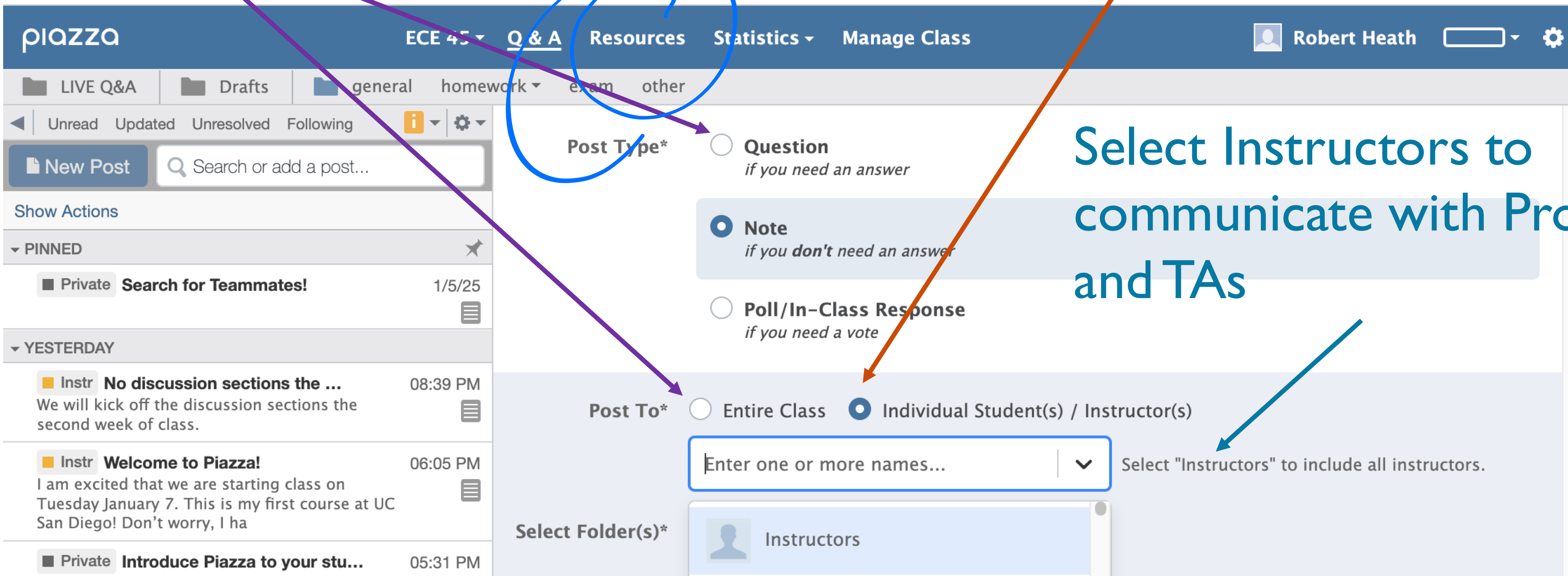
Fourier

Best way to send messages

Preferred for course questions

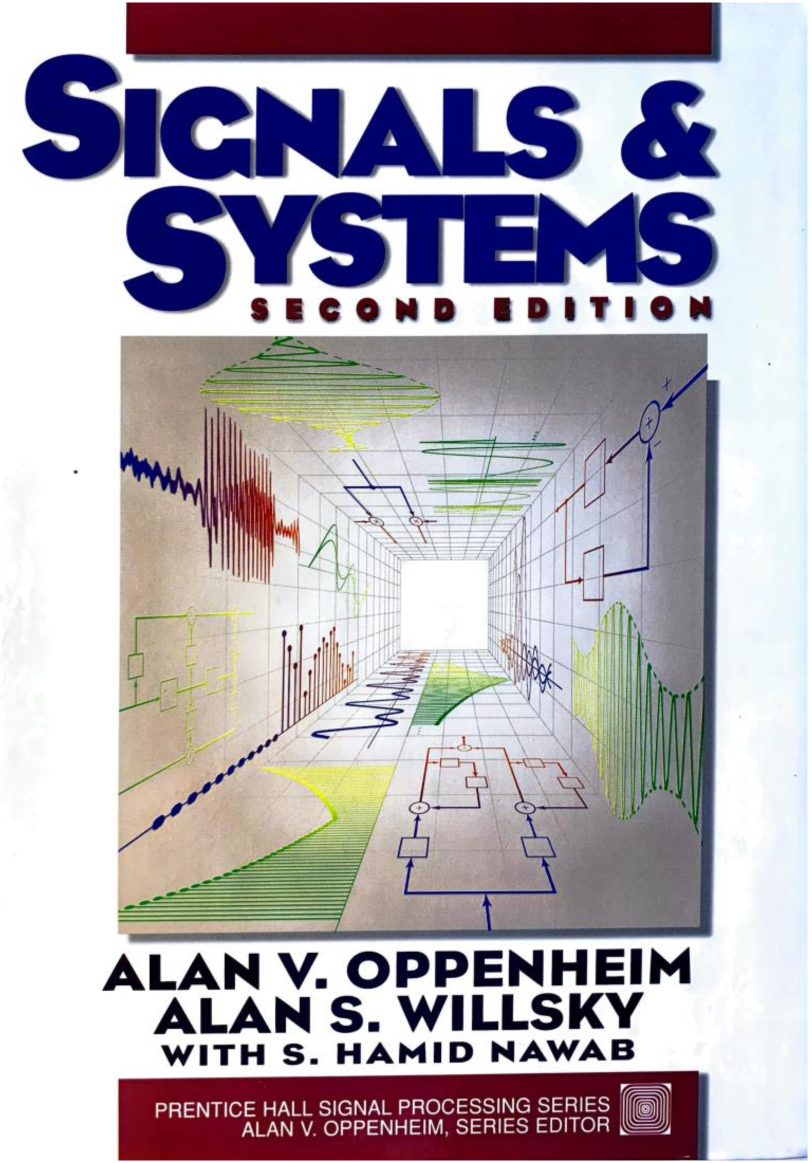
Used for specific communication, preferred over email

Select Instructors to communicate with Prof and TAs



Materials for the course

- Textbook
  - Signals & Systems, 2^{nd} edition
  - Classic book in ECE, worth buying
  - Many examples and problems
- Lecture notes
  - Developed based on the book
  - May have additional examples
- Supplemental materials
  - Material not covered in the book
  - Background material



How this class will work

- Lectures
  - Slides with annotations and handwritten examples
  - Advice: Take notes and be an active thinker
- Discussion sections
  - Practice problem solving
- Piazza will be used for course materials and discussions
  - Files will be uploaded in the resources folder
  - Use relevant tags when posting, e.g. hw1 for homework 1 discussions

	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Signal transformations, periodic, even and odd, rectangle, step	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, phasors	1.3		
4	1/16	Signals	Dirac delta, Kronecker delta, Sha function	1.4	HW3	HW2
5	1/21	LTI in time	Linear and time-invariant systems	2.1		
6	1/23	LTI in time	Convolution, convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Note: HW will normally be due at 11:59pm on Thursdays (not this week) piazza ECE 45 Q & A Resources Statistics Manage Class Robert Heath

Course Information Staff Resources

Edit Resource Sections

Chapter Notes

Manually sort using Sort on: - -

Chapter Notes	Actions
README_note_about_these_slides.pdf	Edit · Post a note · Update file · Delete
Chapter_1_ECE_45.pdf	Edit · Post a note · Update file · Delete
Chapter_2_ECE_45.pdf	Edit · Post a note · Update file · Delete
Chapter_3_ECE_45.pdf	Edit · Post a note · Update file · Delete
Chapter_4_ECE_45.pdf	Edit · Post a note · Update file · Delete
Chapter_7_ECE_45.pdf	Edit · Post a note · Update file · Delete

Add Links Add Files

Lecture Notes

Manually sort using Sort on: - -

How learning will be assessed

Homework	Problems related to material covered the previous week, graded on effort + corrections
Midterm exams	2 midterms, during class, each covering topics up to the midterm
Final exam	A comprehensive exam, held during final exam time

Gradescope

- Used for all homeworks and possibly for in-class assessments
  - Check out general instructions at http://www.gradescope.com
  - Cheat-sheet for scanning your homework is here
- Gradescope is linked to the course through Canvas
- You will upload your assignments at home and scan the areas relevant for the problems (required to receive credit)

Variable-length submissions

In most cases, your instructor will allow you to turn in a variable-length submission for written assignments. This means your answers may span across any number of pages and responses may not be located in the same place on each page. If your instructor will accept a variable-length submission, when you open or start the assignment, you'll have two options for submitting your work:

- **Individual images**, one or more per question
- **One PDF** that contains the whole submission; Gradescope will ask you to mark where each answer is

From https://help.gradescope.com/article/ccbpppziu9-student-submit-work

We use MATLAB, but be aware of Python

- Why Python?
  - It is free to you and your future employer
  - Fast prototyping language, similar to MATLAB, similar syntax
  - Used extensively in machine learning and data sciences
  - Many packages are available for scientific computing including manipulation of matrices (numpy) and plotting data (matlabplotlib), machine learning (scikit-learn)
  - Can be deployed on the cloud such as Amazon AWS
- Tons of free online tutorials: https://www.eecis.udel.edu/~boncelet/ipython.html
- MATLAB will be used but some examples in both languages

Tips for success

Seriously…

- Review your book and notes from ECE 35 and calculus book
  - You will need to perform integrals and derivatives
  - You will need trigonometric identities
  - You will need to master complex numbers and complex exponentials

- Don't fall behind!
  - Keep up with the material every week, read the book each week
  - Attend class and discussion sections
  - Get help: TA, office hours, your peers on Piazza, virtual study group

Attempt the homework problems on your own and get as far as you can using the book and course materials. This is critical to developing the problem-solving and self-teaching skills you need for this class and beyond.

Useful reference material

Trigonometric identities

Euler's theorem: $e^{\pm ju} = \cos u \pm j \sin u$

$\cos u = \frac{1}{2}(e^{ju} + e^{-ju})$

$\sin u = (e^{ju} - e^{-ju})/2j$

$\sin^2 u + \cos^2 u = 1$

$\cos^2 u - \sin^2 u = \cos 2u$

$2 \sin u \cos u = \sin 2u$

$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$

$\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$

$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$

$\sin u \cos v = \frac{1}{2}[\sin(u - v) + \sin(u + v)]$

From Signals and Systems: Continuous and Discrete by Ziemer, Tranter, and Fannin

Indefinite integrals

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$$

$$\int \sin^2(ax)\,dx = x/2 - \sin(2ax)/4a$$

$$\int \cos^2(ax)\,dx = x/2 + \sin(2ax)/4a$$

$$\int x\sin(ax)\,dx = [\sin(ax) - ax\cos(ax)]/a^2$$

$$\int x\cos(ax)\,dx = [\cos(ax) + ax\sin(ax)]/a^2$$

$$\int x^m \sin(x)\,dx = -x^m\cos(x) + m\int x^{m-1}\cos(x)\,dx$$

$$\int x^m \cos(x)\,dx = x^m\sin(x) - m\int x^{m-1}\sin(x)\,dx$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$$

$$\int \sin(ax)\cos(bx)\,dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right], \qquad a^2 \neq b^2$$

$$\int \cos(ax)\cos(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$$

$$\int e^{ax}\,dx = e^{ax}/a$$

$$\int x^m e^{ax}\,dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1}e^{ax}\,dx$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)]$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)]$$

From Signals and Systems: Continuous and Discrete by Ziemer, Tranter, and Fannin

Definite integrals

$$\int_0^\infty \frac{a\,dx}{a^2+x^2} = \pi/2, \qquad a > 0$$

$$\int_0^{\pi/2} \sin^n(x)\,dx = \int_0^{\pi/2} \cos^n(x)\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots(n)}\,\dfrac{\pi}{2}, & n \text{ even, } n \text{ an integer} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots(n)}, & n \text{ odd} \end{cases}$$

$$\int_0^\pi \sin^2(nx)\,dx = \int_0^\pi \cos^2(mx)\,dx = \pi/2, \qquad n \text{ an integer}$$

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = \int_0^\pi \cos(mx)\cos(nx)\,dx = 0, \quad m \neq n,\ m \text{ and } n \text{ integer}$$

$$\int_0^\pi \sin(mx)\cos(nx)\,dx = \begin{cases} 2m/(m^2-n^2), & m+n \text{ odd} \\ 0, & m+n \text{ even} \end{cases}$$

$$\int_0^\infty \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2}, \qquad a > 0$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty e^{-a^2x^2}\,dx = \sqrt{\pi}/2a, \qquad a > 0$$

$$\int_0^\infty x^n e^{-ax}\,dx = n!/a^{n+1}, \qquad n \text{ an integer and } a > 0$$

$$\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}, \qquad a > 0$$

$$\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2}, \qquad a > 0$$

$$\int_0^\infty e^{-a^2x^2}\cos(bx)\,dx = \frac{\sqrt{\pi}}{2a}\,e^{-b^2/4a^2}$$

From Signals and Systems: Continuous and Discrete by Ziemer, Tranter, and Fannin

Lecture 2

Rectangle, step functions, signal transformations, periodic, even and odd

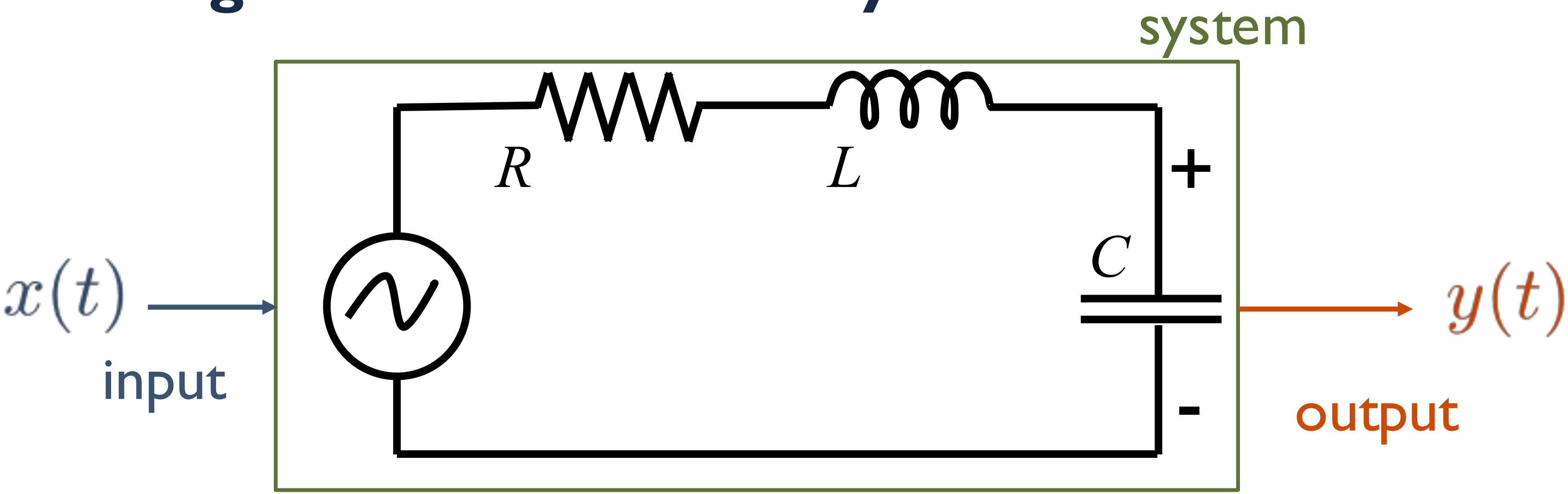
Preview of today's lecture

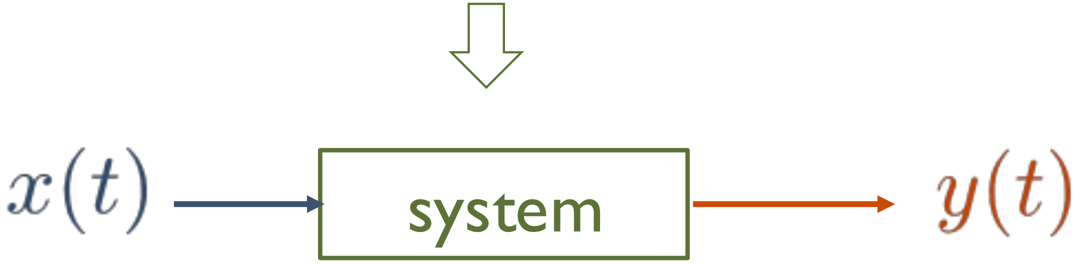
- Signals, unit–step and rectangle functions
  - Describe mathematically the unit step and rectangle functions

- Basic signal transformations
  - Apply different transformations on continuous-time signals
  - Create new signals from these transformations

- Signal characteristics: periodic, even, odd, and conjugate symmetric
  - Distinguish between periodic and aperiodic signals
  - Compute even and odd parts of an arbitrary signal

Thinking about circuits as a system

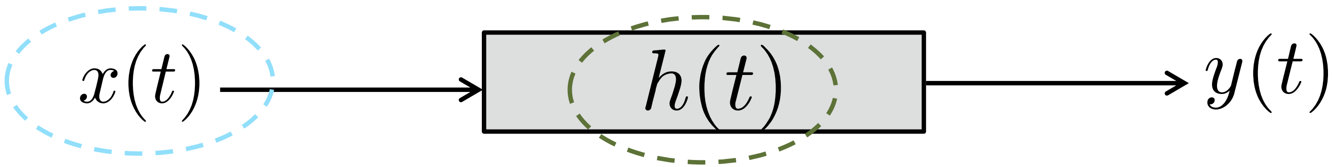




A mathematical description of how the input is transformed into the output

Connections back to ECE 45

Lectures 2 - 4 working with signals



Lectures 5 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

UC San Diego

	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, phasors	1.3		
4	1/16	Signals	Dirac delta, Kronecker delta, Sha function	1.4	HW3	HW2
5	1/21	LTI in time	Linear and time-invariant systems	2.1		
6	1/23	LTI in time	Convolution, convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Signals, unit–step and rectangle functions

Learning objectives

- Describe mathematically the unit step and rectangle functions

What is a "signal"?

ECE 45 focuses on CT signals

- Representation of a value/info. relative to an independent variable
  - Often a time variable but could be something else

- Continuous-time (CT) signals $x(t)$

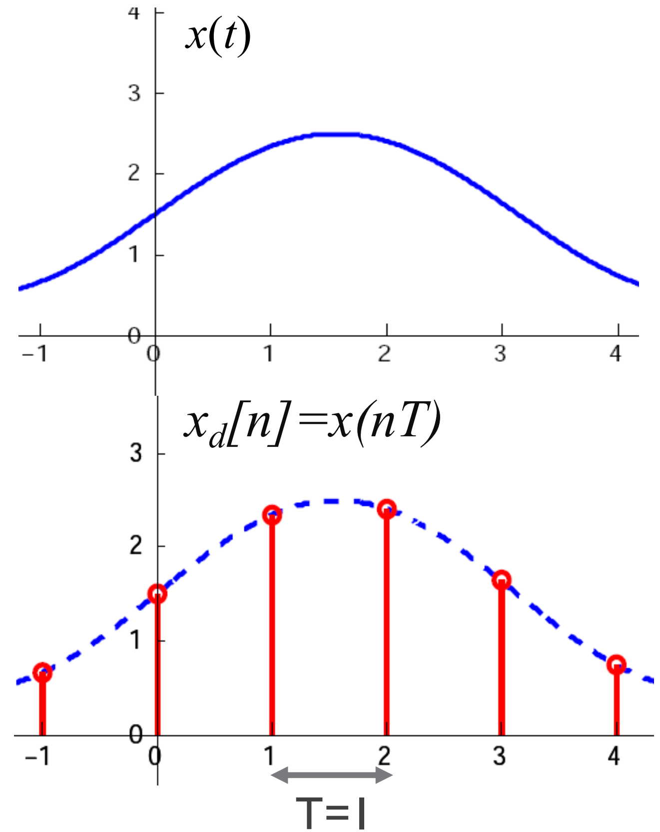
$t \in \mathbb{R}$

Bracket notation used to denote a discrete-time sequence

- Discrete-time (DT) signals $x[n]$
  - Often obtained by *sampling* CT signal
  - Taken at values nT

sampling period

$\cancel{x[n/2]}$



Complex signals

- Signals may take complex values (equivalently they have an amplitude and a phase when in polar form)

$$x(t) = \mathrm{Re}\{x(t)\} + j\mathrm{Im}\{x(t)\} \qquad x[n] = \mathrm{Re}\{x[n]\} + j\mathrm{Im}\{x[n]\}$$

- Complex signals are found in many practical problems
  - Most digital signal processors support complex operations
- Example: "in phase" (real) and "quadrature" (imag) EM signals

"rides" a cosine

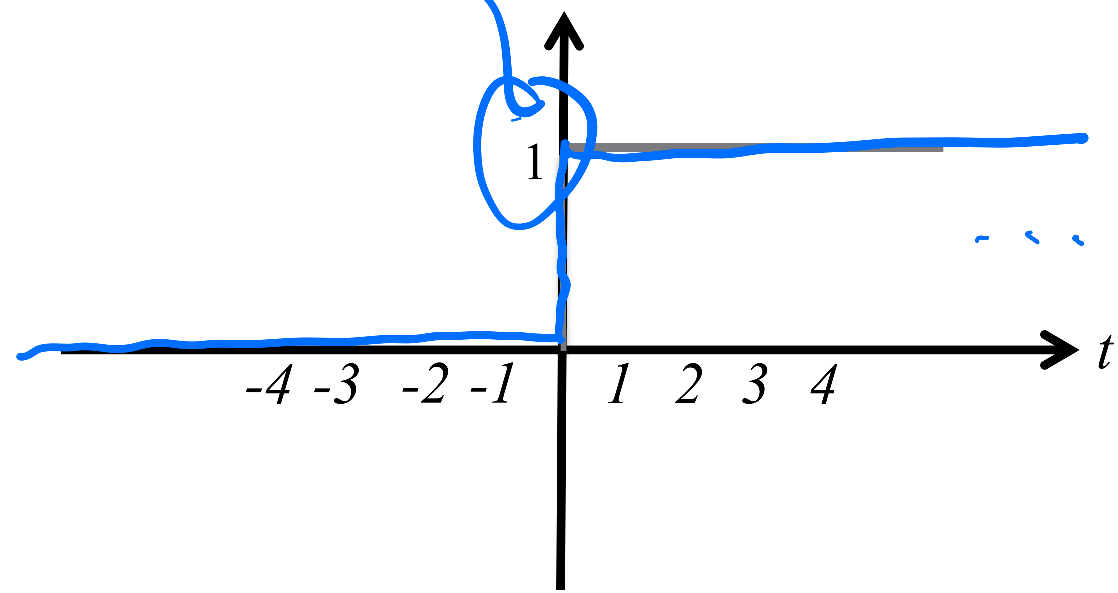
$$x(t) = \mathrm{Re}\{x(t)\} + j\mathrm{Im}\{x(t)\}$$

"rides" a sine

Many connections between complex signals, Maxwell's equations, phasors, etc.

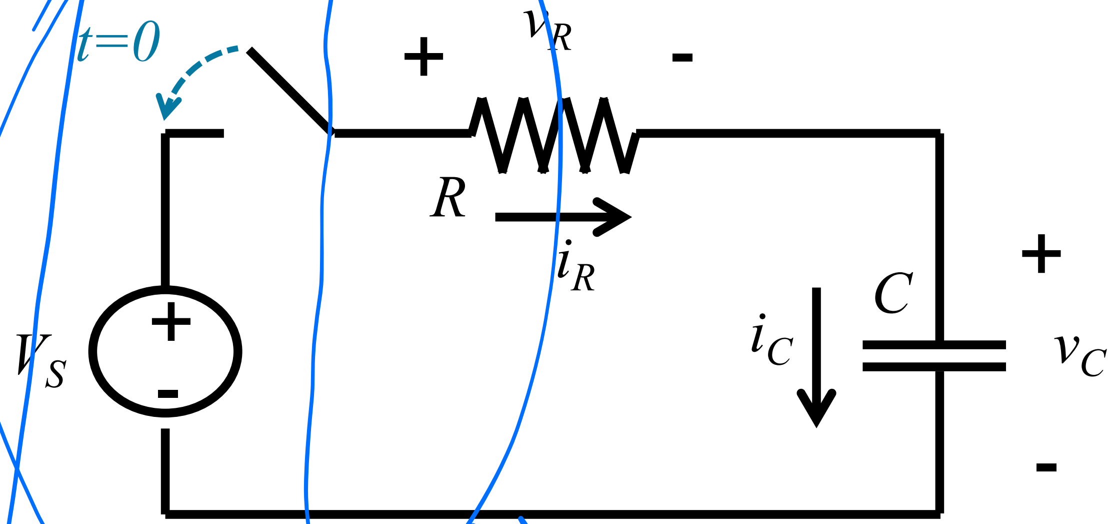
Unit step function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

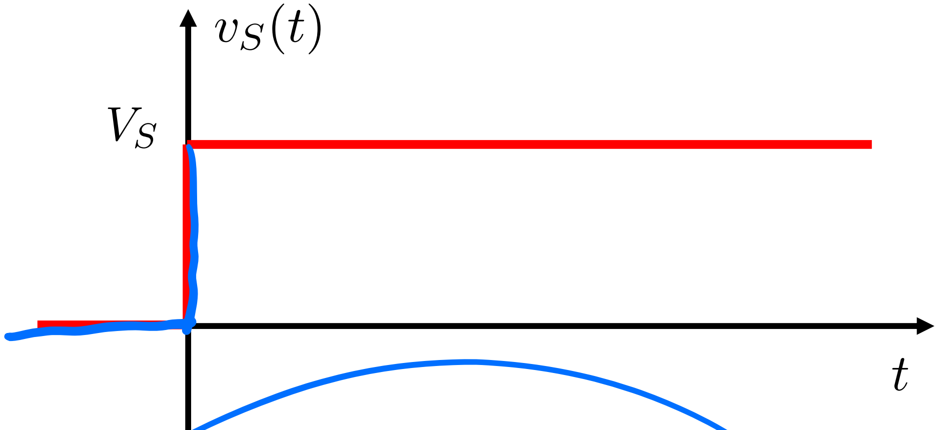


- At t=0, u(t) may be either 0, 1, or ½ depending on the book
  - The specific choice is only important in a mathematical analysis class

Where have you seen the unit step function?

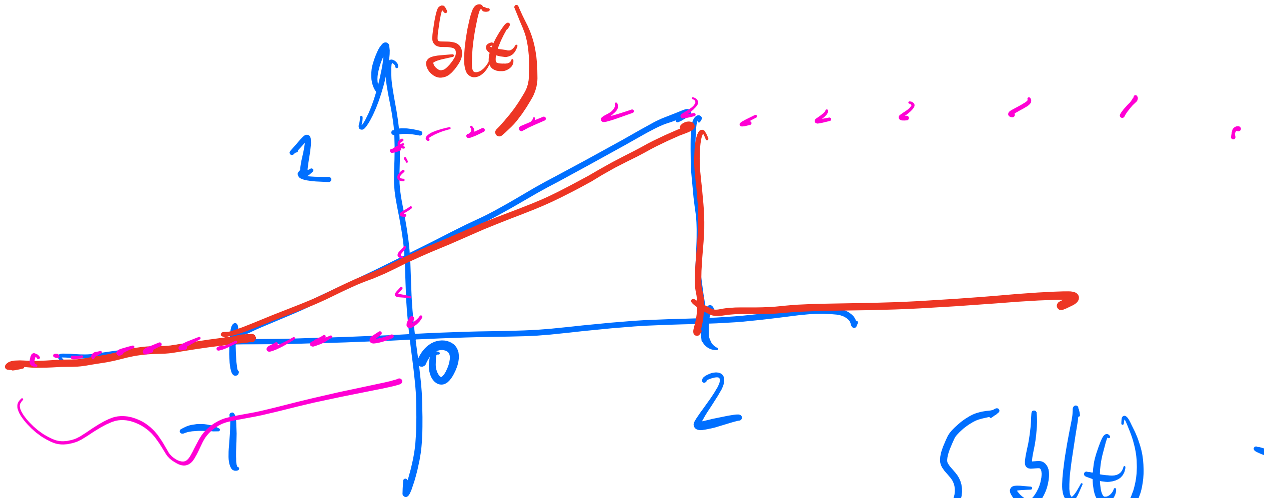


$v_S(t)$

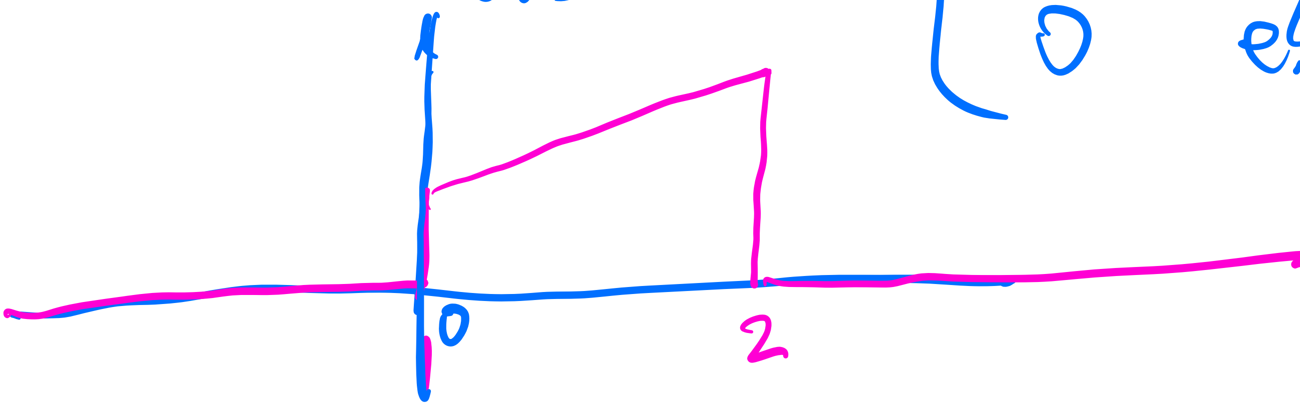


The voltage signal created from switching in a DC source is written using the unit step function

$$\boxed{v_S(t) = V_S\, u(t)}$$

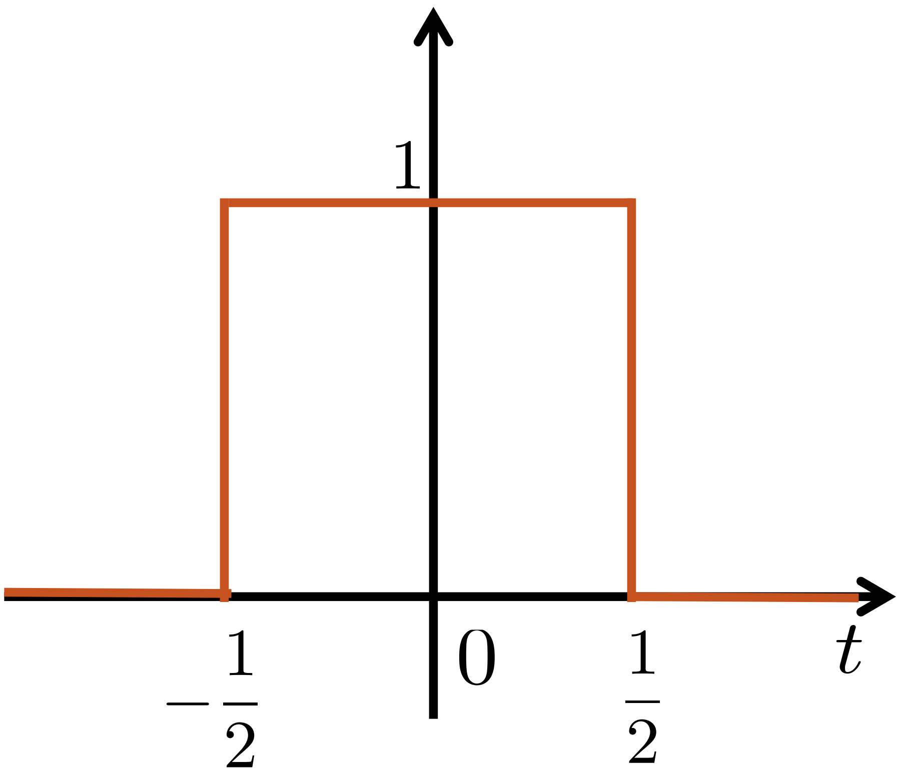


$$b(t)\,u(t) = \begin{cases} b(t) & t \geq 0 \\ 0 & \text{else} \end{cases}$$



Rectangle function

$$\mathrm{rect}(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leqslant \frac{1}{2} \end{cases}$$

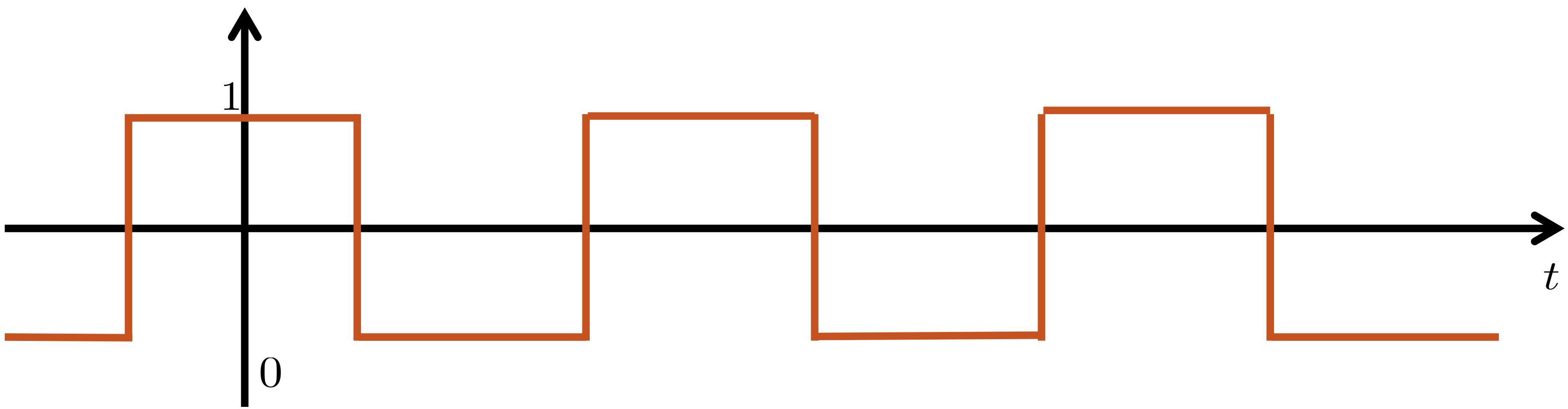


Also called the box function

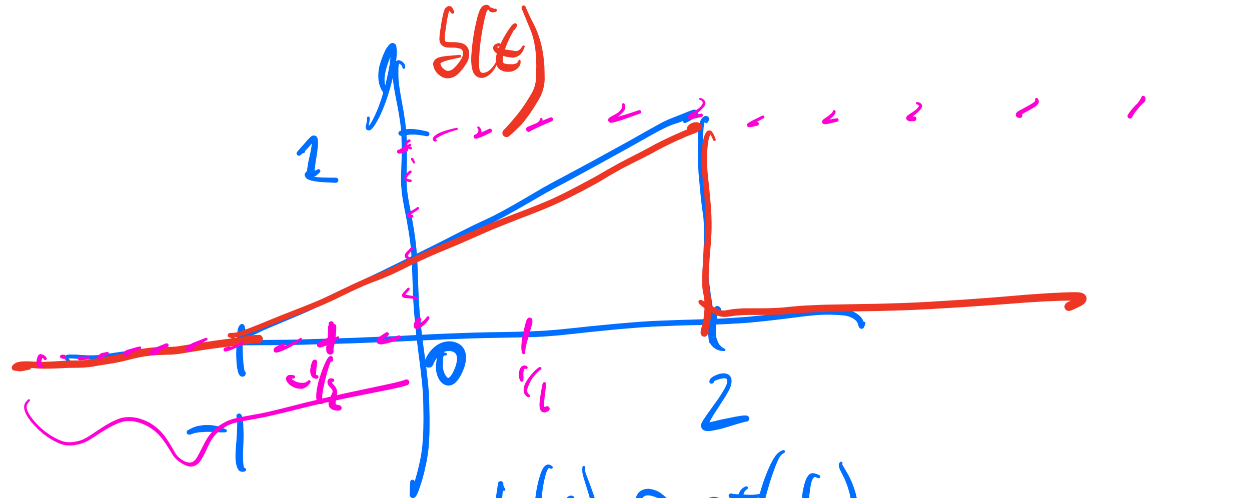
May have different notation in other books like $\Pi(t)$

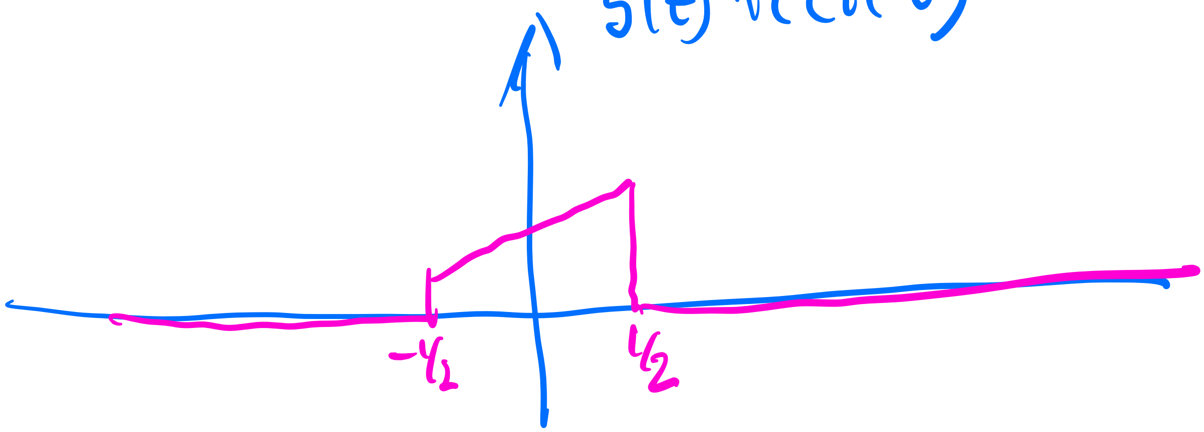
Where will you see the rectangle function?

- Part of many common convolution examples
- Used to build a square waves, which is analyzed via Fourier Series



- Shows up as part of "windowing," e.g. $x(t)\,\mathrm{rect}(t)$


f(t)
1
-1
-1/2
0
1/2
2


f(t) · rect(t)
-1/2
1/2

Why spend time on these basic signals?

- Unit step functions
  - An important building block signals for other parts of the course
  - Shows up often when signals ”start at zero”

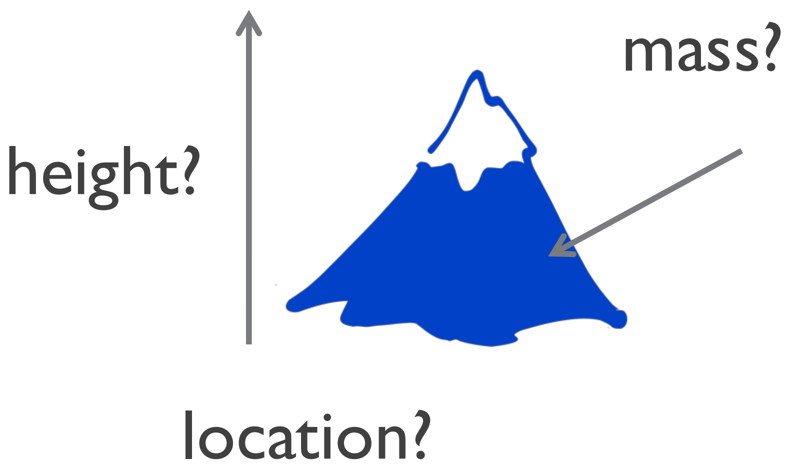
- Rectangle function
  - Another important building block signal
  - Used for many examples in convolution and Fourier

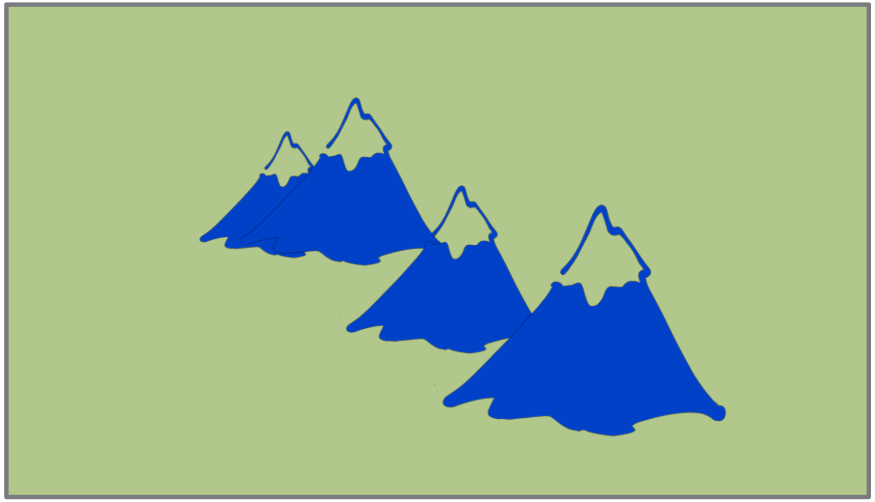
Basic signal transformations

Learning objectives

- Apply different transformations on continuous-time signals
- Create new signals from these transformations

Moving mountains (function is over space here)





- Use mountain description to build a map
  - Where are the mountains located?
  - What are their size?

Signal transformations are ways to describe and manipulate signals

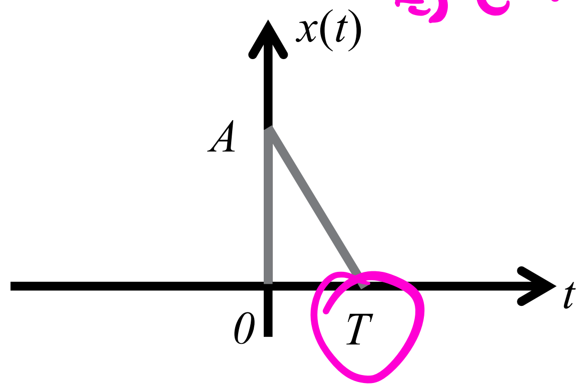
from clipartpanda.com

Example: time shift

$t_0 > 0$ in this example

$t-2=T$?

$\Rightarrow t=T+2$

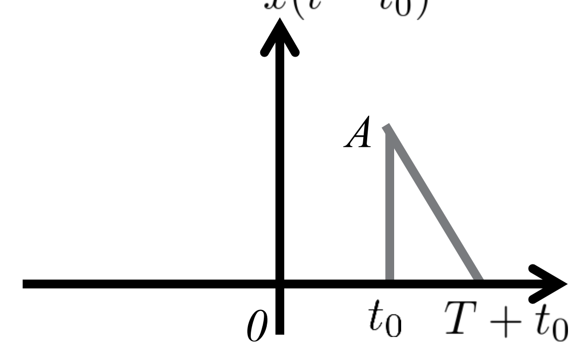


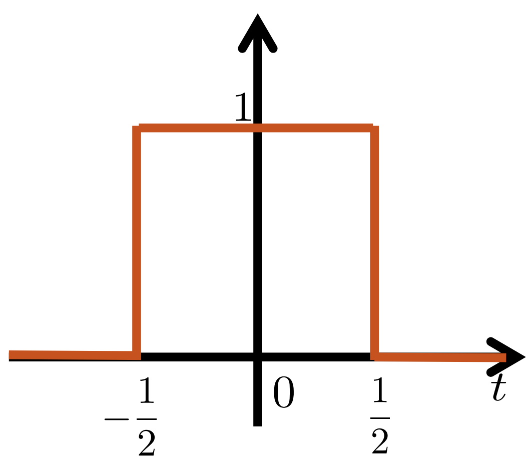
$$x(t-t_0) \xrightarrow{\text{right shift}}$$

$t_0=2$

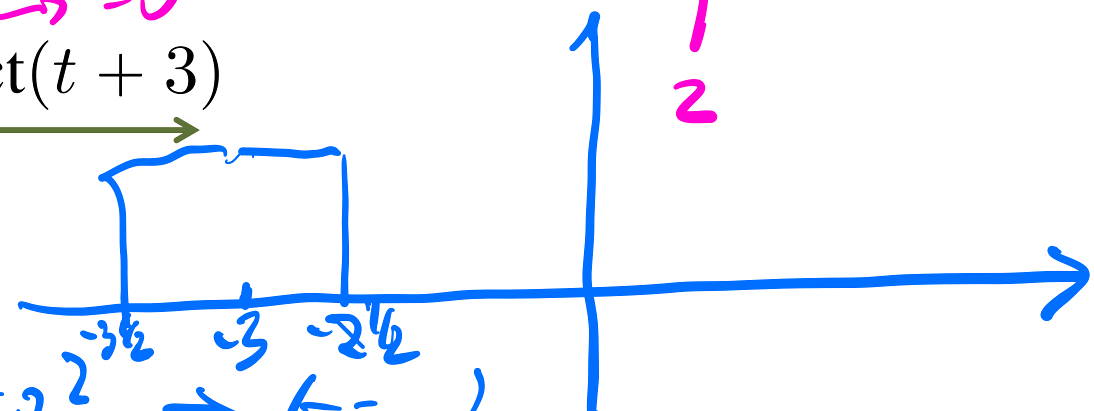
$x(t-2)$ $t=$?

$=0$





$$\mathrm{rect}(t+3) \longrightarrow$$



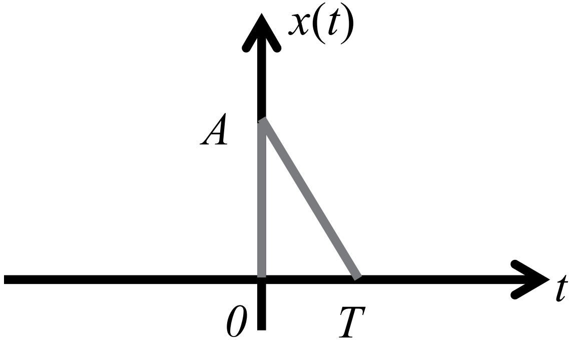
$t+3=0$? $\Rightarrow t=-3$

$x(t-(-3))$

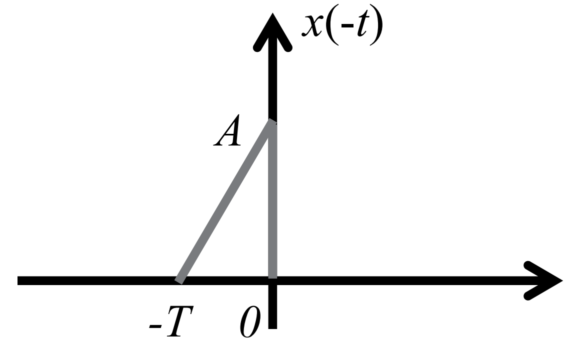
$t_0 = -3$?

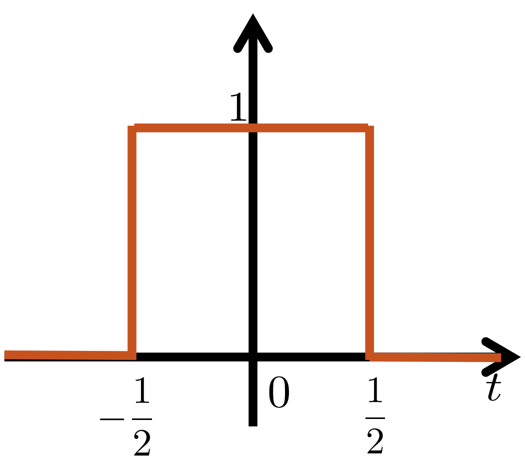
$x(t+3)$

Example: reflection or time reversal



$$\xrightarrow{x(-t)}$$





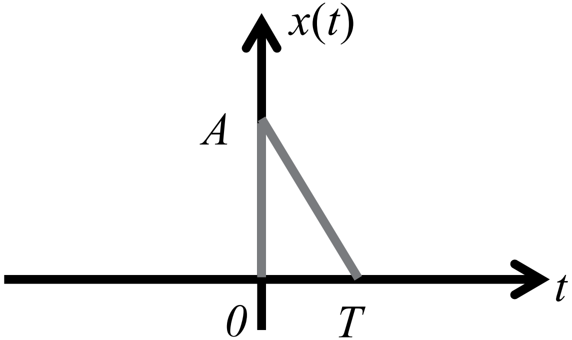
$$\xrightarrow{\mathrm{rect}(-t)}$$

$x(-(-t))$

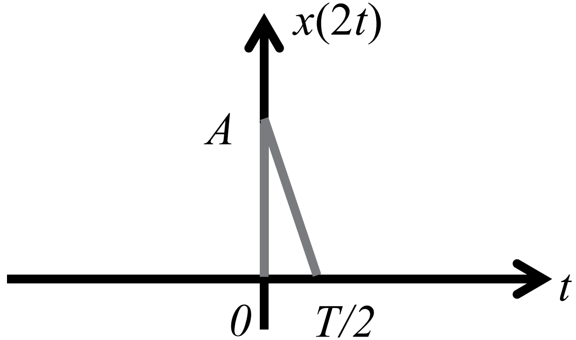
$= x(t)$

Same!!

Example: time compression / downsampling

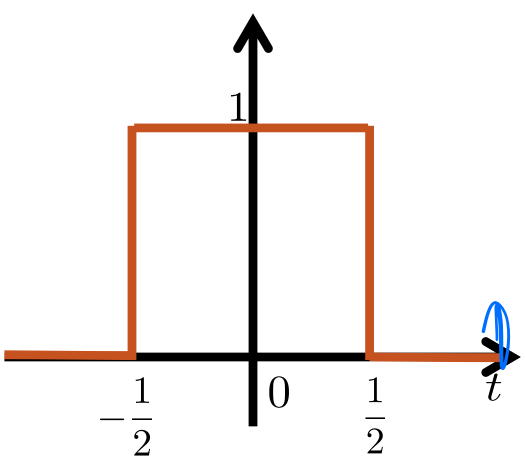


$\xrightarrow{x(2t)}$



$2t = T$?

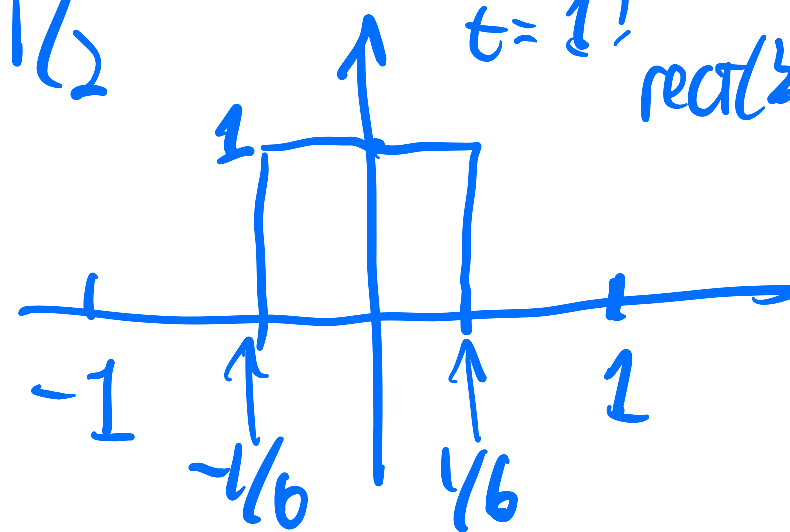
$t = T/2$



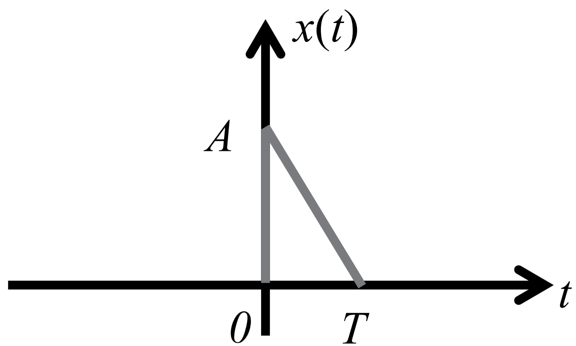
$\xrightarrow{\mathrm{rect}(3t)}$

$t = 1$?

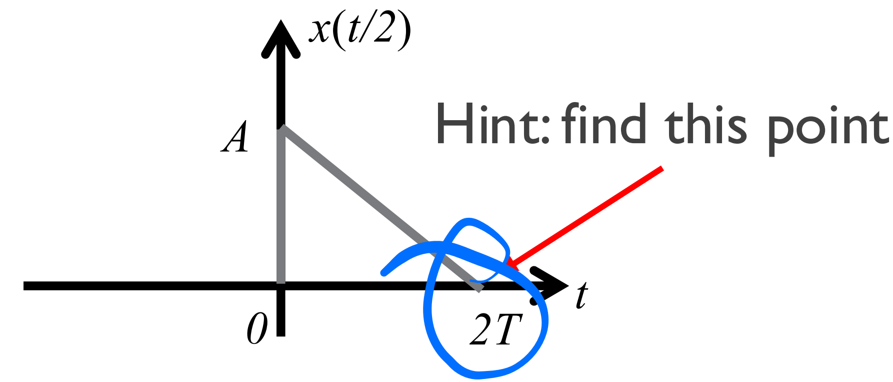
rect(3t)



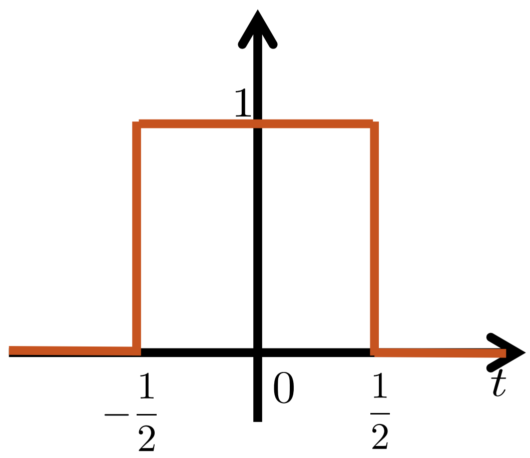
Example: time expansion (upsampling)



$$\xrightarrow{x(t/2)}$$

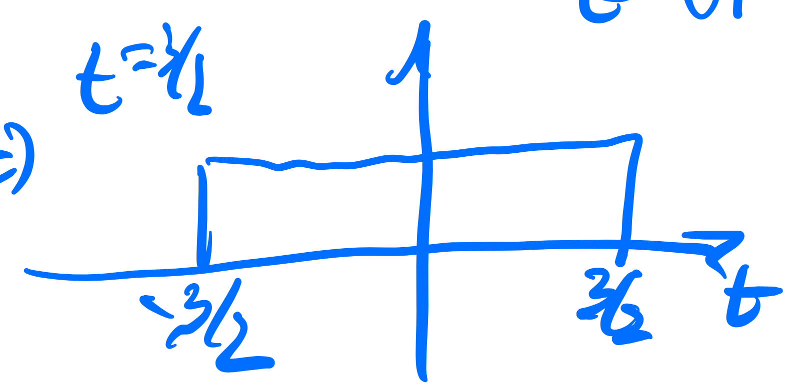


$t/2 = T \Rightarrow t = 2T$



$$\xrightarrow{\mathrm{rect}(t/3)}$$

$t/3 = 1/2 \Rightarrow t = 3/2$



Example

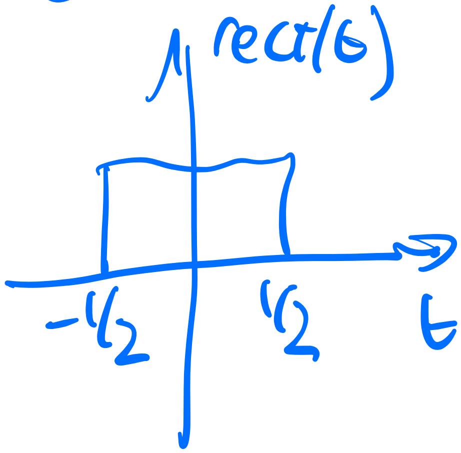
- Plot the signal $\mathrm{rect}(2t-3)$

① Let $a(t) = \mathrm{rect}(t-3)$

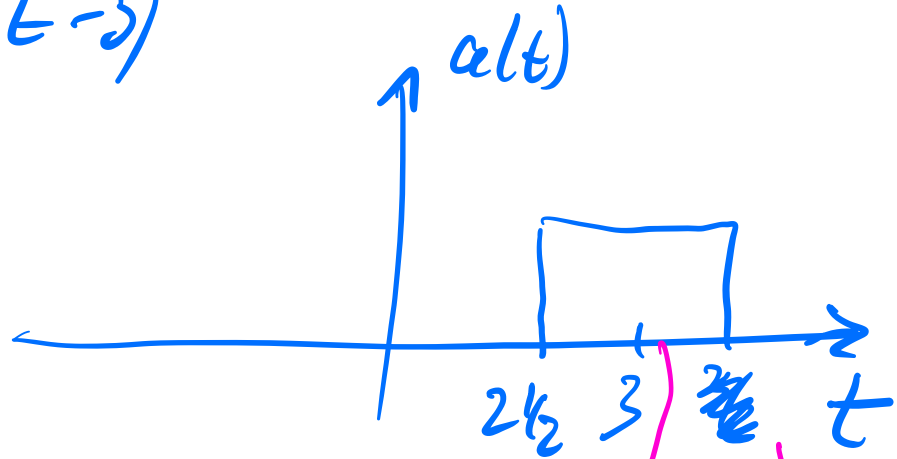
② $a(2t) = \mathrm{rect}(2t-3)$

- You may shift then scale, or scale then shift (carefully)

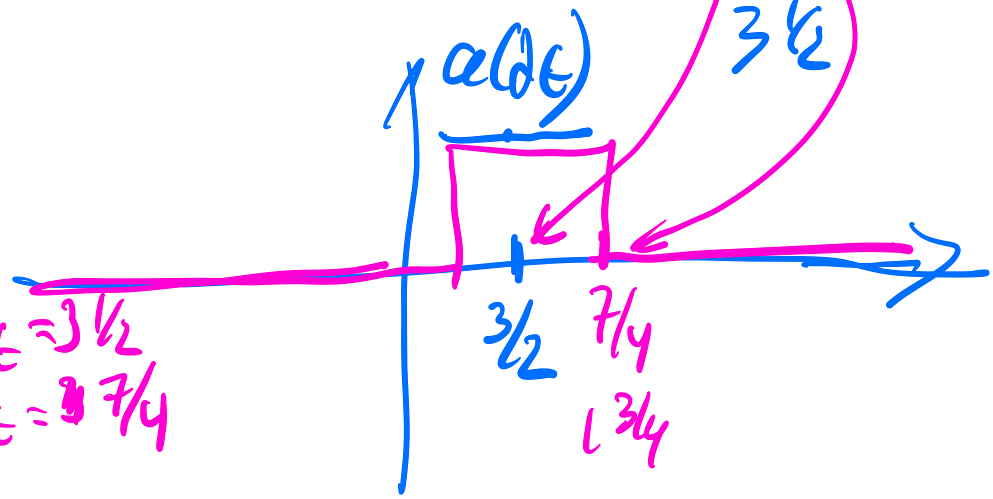
(1) $a(t) = \mathrm{rect}(t-3)$



$\Rightarrow$



$a(t) \longrightarrow$



$2t = 3$

$t = 3/2$

$2t = 3\,1/2$

$t = 7/4$

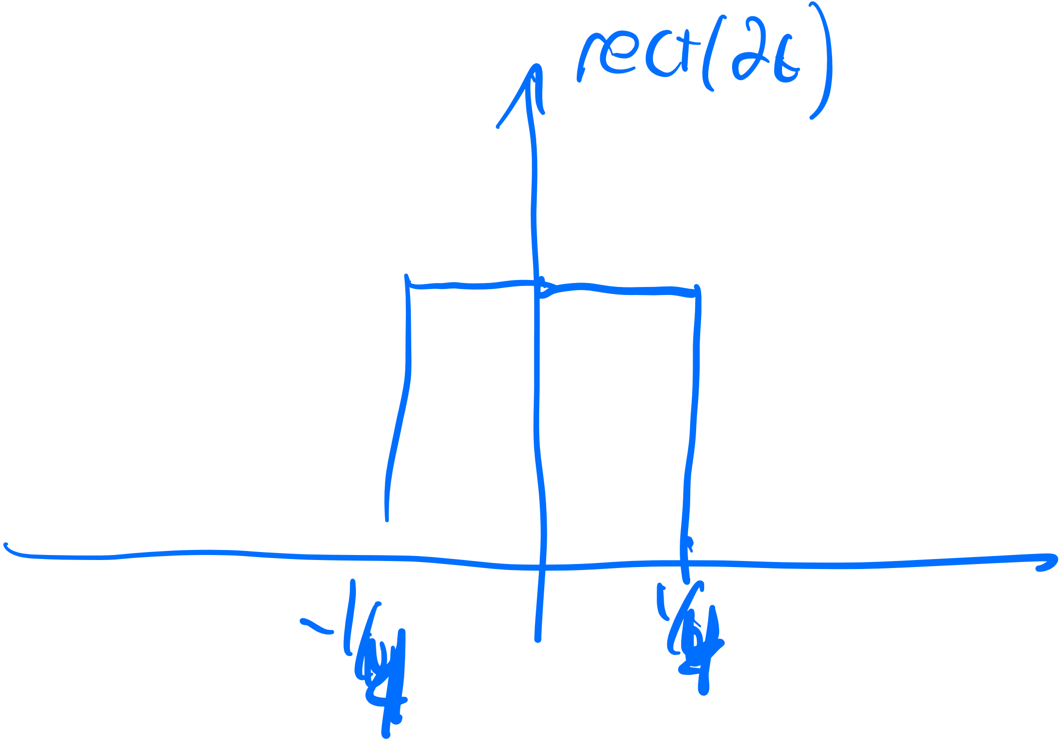
Example

- Plot the signal $\mathrm{rect}(2t-3) = \mathrm{rect}(2(t-3/2))$

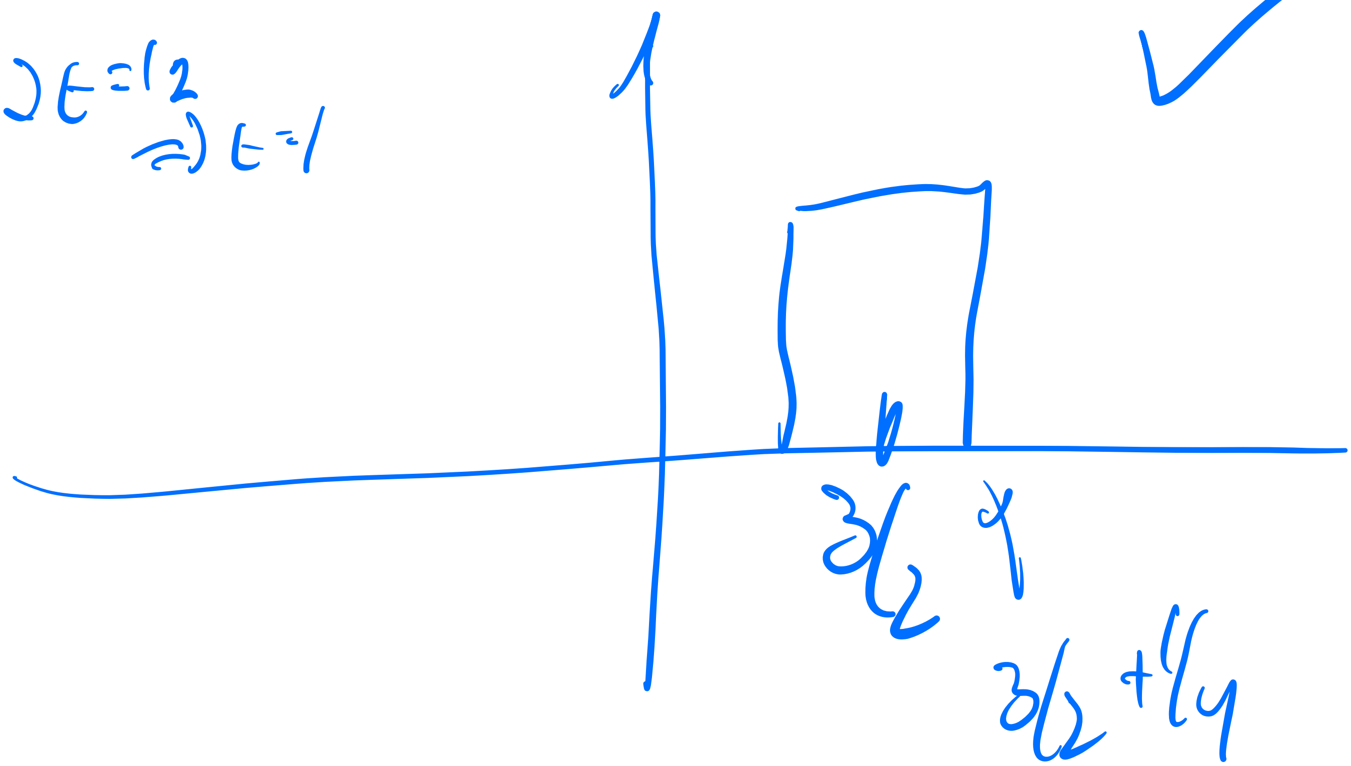
$$s(t) = \mathrm{rect}(2t)$$

$$\cancel{s(t)}\; c(t) = s(t-3/2) = \mathrm{rect}(2t-3)$$

- You may shift then scale, or scale then shift (carefully)



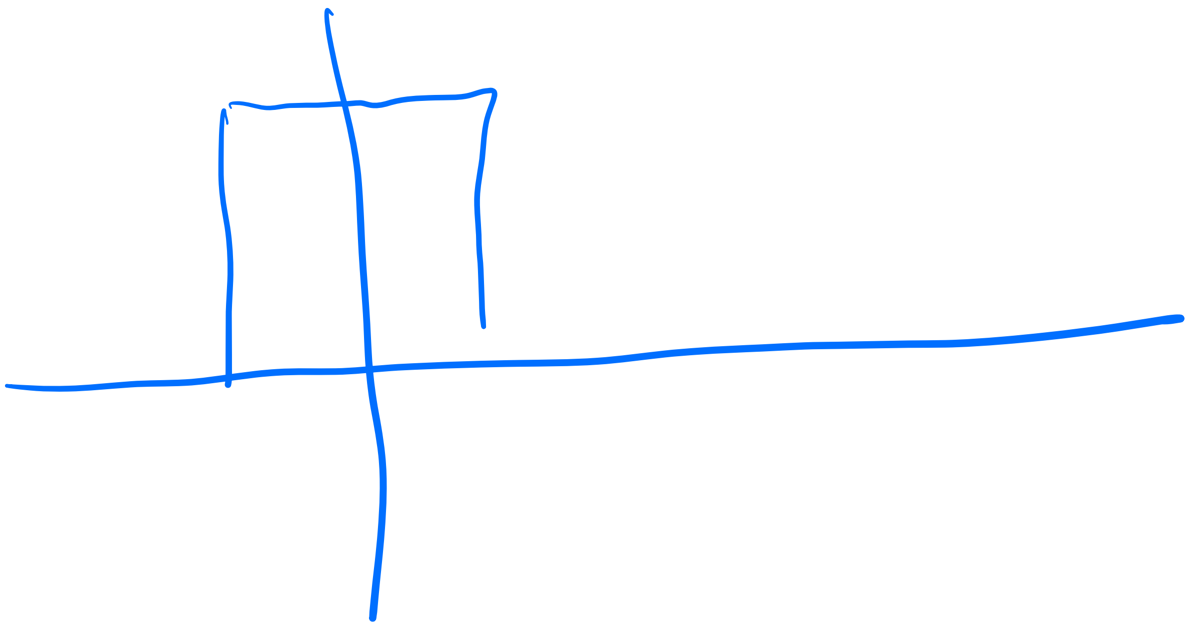
$2t = 1/2 \Rightarrow t = 1/4$



Signal transformation summary

- There are different ways to transform a signal
  - This section focused on transformations of the independent variable

- Transformations
  - Time shifting changes the starting point of a signal
  - Time scaling changes how fast the signal is "played"
  - Time reversal flips a signal
  - Compression / expansion change the "speed" of a signal



Signal characteristics: periodic, even, and odd

Learning objectives

- Distinguish between periodic and aperiodic signals
- Compute the period of a periodic signal
- Compute even and odd parts of an arbitrary signal

Periodic signals

$T > 0$

- Periodic signals satisfy for some finite non-zero T o

$$x(t) = x(t+T)$$

periodic with period T for all t

$\forall t$

- Period is the smallest non-zero solution
  - This is called the fundamental period
  - Normally period means fundamental period

$$x(t+2T) = x\big((t+T)+T\big) = x(t+T) = x(t)$$

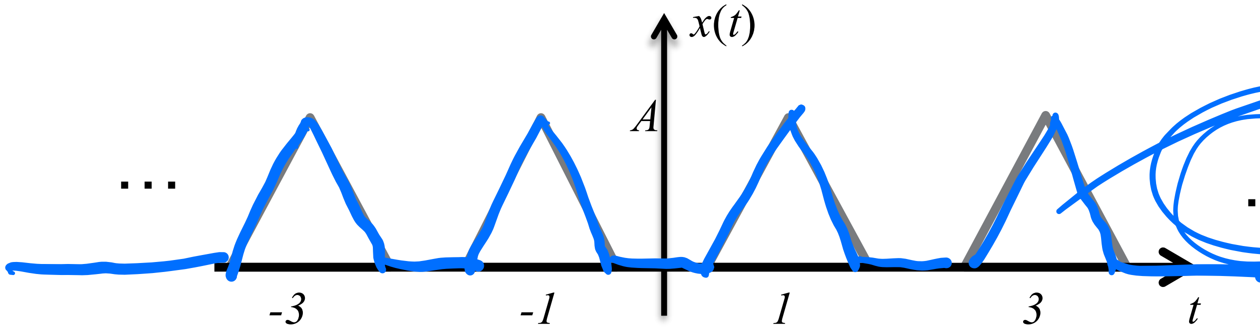
$$x(t+T) = x(t)$$

Periodic signals can be treated with special mathematical tools

Examples of periodic signals

Dots indicate that the signal continues

pulse train of triangles



period?

Establishing periodicity

- To prove a CT signal is periodic
  - Direct: find a $T>0$ such that $x(t)=x(t+T)$ for all t
  - Indirect: show that $x(t)$ is in a known class of periodic signals
- To prove a CT signal is aperiodic
  - Direct: Show that there is no $T>0$ such that $x(t)=x(t+T)$ for all t
- Remember the fundamental period is the smallest non-zero T

Example of direct method

- Determine whether or not the following signal is periodic? If it is periodic, determine its fundamental period.

$\omega = \frac{2\pi}{T} = 2$ $\quad T = \pi$

$$x(t) = \sin(2t) + \cos(4t + \pi/2)$$

$\sin\left(\frac{2\pi}{T} t\right) \quad = \frac{2\pi}{T}$

- Want to find T such that $x(t) = x(t+T)$

$$x(t+T) = \sin(2(t+T)) + \cos(4(t+T) + \pi/2)$$ shifted signal

$$\sin(2(t+T)) = \sin(2t) \text{ for } T = k\pi \text{ where } k \in \mathbb{Z}$$ multiple solutions

$$\cos(4(t+T) + \pi/2) = \cos(4t + \pi/2) \text{ for } T = k\pi/2 \text{ where } k \in \mathbb{Z}$$ multiple solutions

$$\boxed{T = \pi}$$ fundamental period

Example of indirect method

- Determine whether or not the following signal is periodic? If it is periodic, determine its fundamental period.

$$x(t) = [\cos(2t - \pi/3)]^2$$

- Recall that $\cos^2(x) = ½\,(1 + \cos 2x)$

$x(t) = [1 + \cos(4t - 2\pi/3)]/2$. Periodic, period $= 2\pi/(4) = \pi/2$.

- By showing this simplifies to a known periodic function, we can also conclude it is period and find the frequency

Even signals (real)

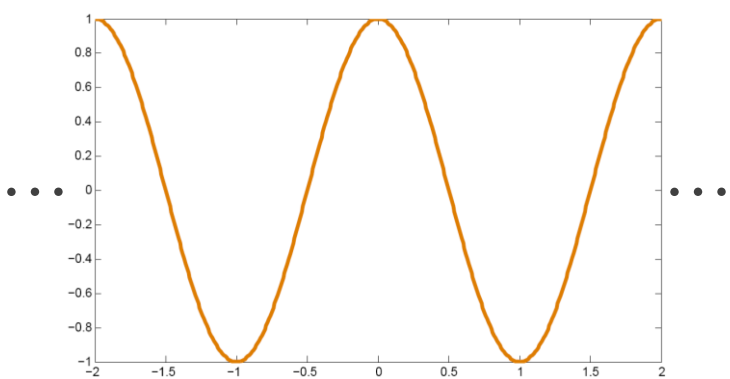
- Even signals satisfy

$$x(-t) = x(t)$$

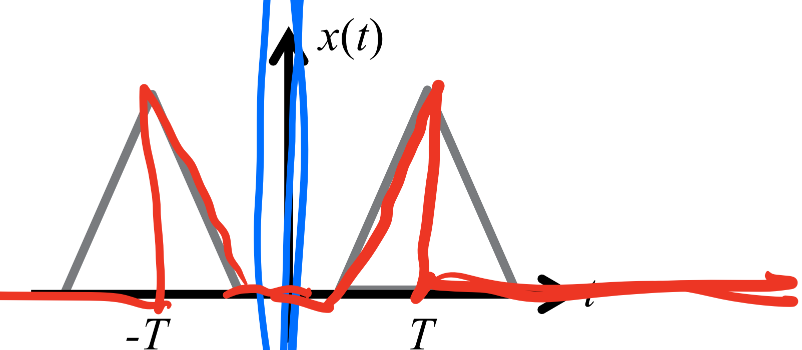
flipped

Same

- Examples (need not be periodic)



$\cos(2\pi f t)$



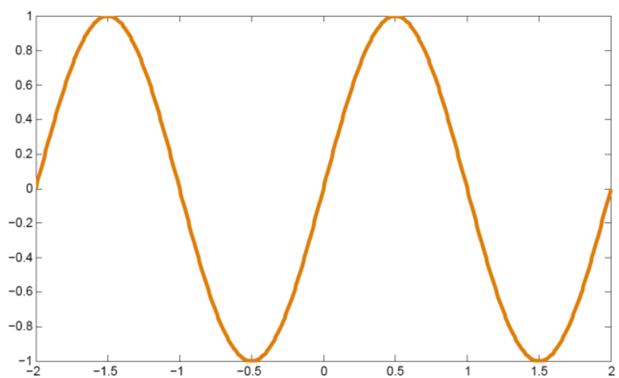
Odd signals (real)

- Odd signals satisfy $x(-t) = -x(t)$

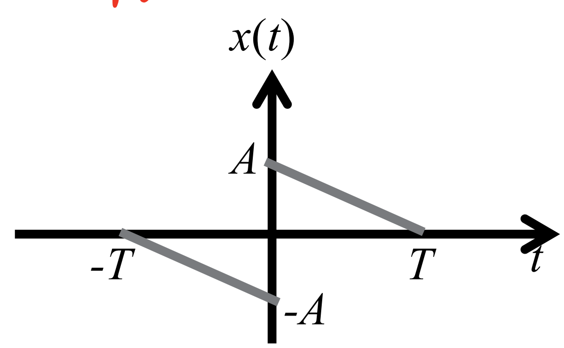
flipped

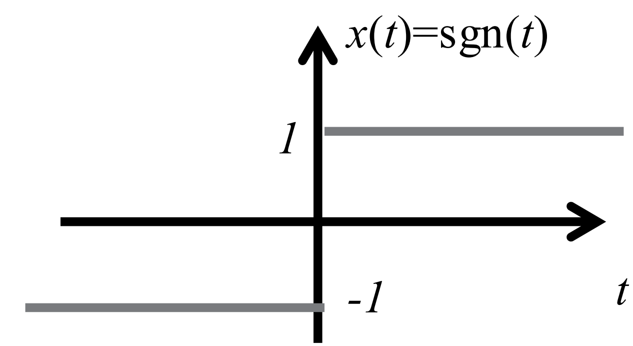
negated

- Examples



$\sin(2\pi f t)$





Generalization to complex signals

- Even becomes conjugate symmetric

$$x^*(-t) = x(t)$$

- Odd becomes conjugate antisymmetric

$$x^*(-t) = -x(t)$$

Interesting facts about even and odd functions

- Any real function can be written in terms of its even and odd parts

$$e(t) = \frac{1}{2}[f(t) + f(-t)]$$

$$o(t) = \frac{1}{2}[f(t) - f(-t)]$$

$$e(-t) = \frac{1}{2}[f(-t) + f(t)] = e(t)$$

$$o(-t) = \frac{1}{2}[f(-t) - f(t)] = -o(t)$$

$$\boxed{f(t) = e(t) + o(t)}$$

$e(-t) = \frac{1}{2}\{f(-t) + f(t)\}$

$= e(t)$

Interesting facts about even and odd functions

- Integration properties

$$\int_{-\infty}^{\infty} e(t)dt = 2\int_{0}^{\infty} e(t)dt$$

also true

$$\int_{-\infty}^{\infty} o(t)dt = 0 = \int_{-T_0}^{T_0} o(t)dt$$

- Multiplications of two odd or two even functions → even function

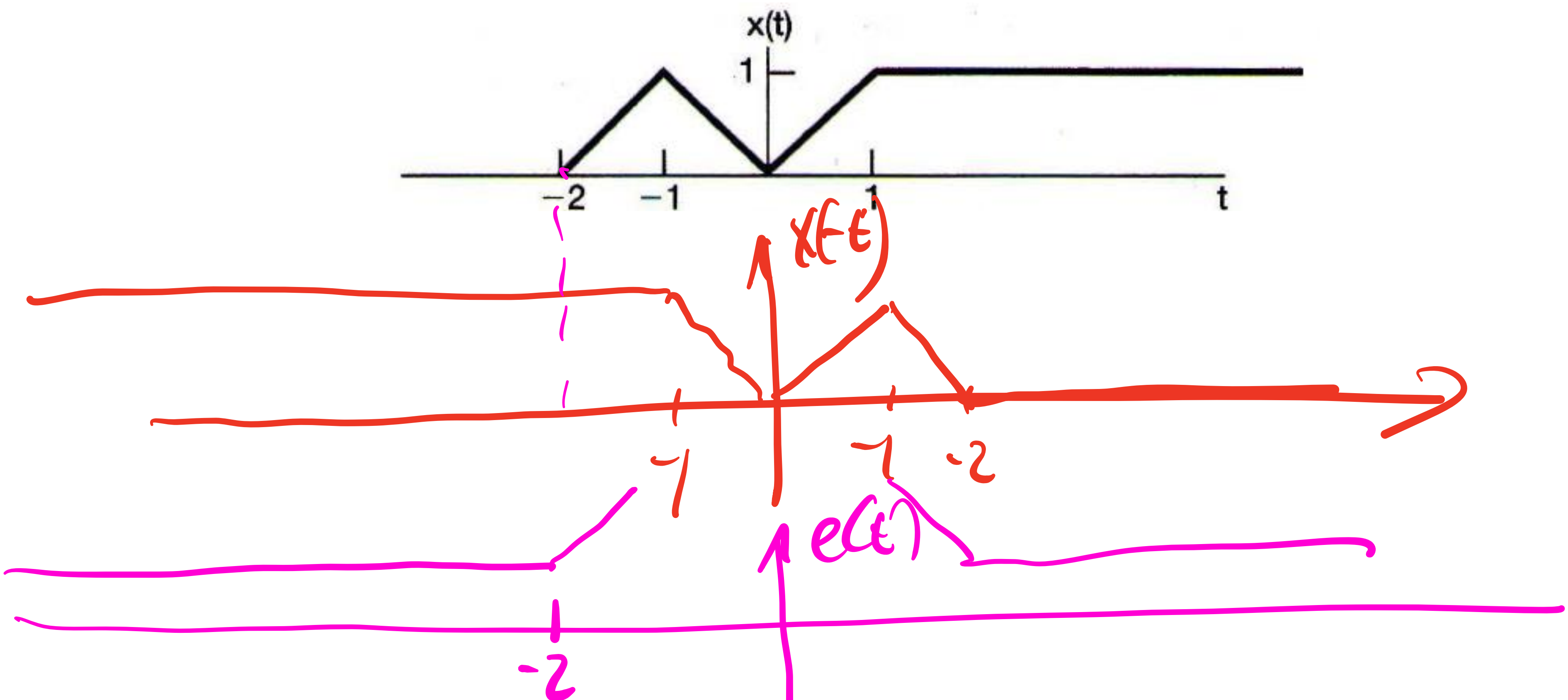
$$y(t) = o_1(t)o_2(t) = e(t)$$

$$y(t) = e_1(t)e_2(t) = \text{even function}$$

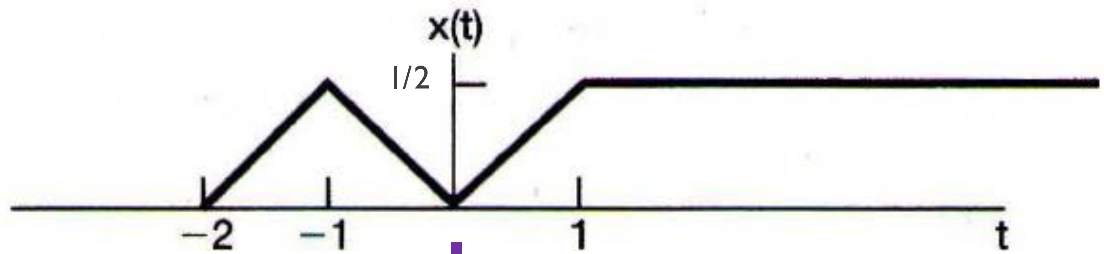
$$y(t) = o(t)e(t) = \text{odd function}$$

Example

- Determine and sketch the even and odd parts of the signal

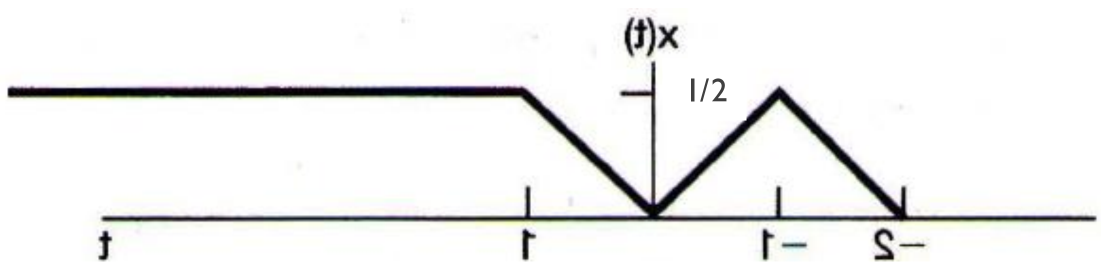


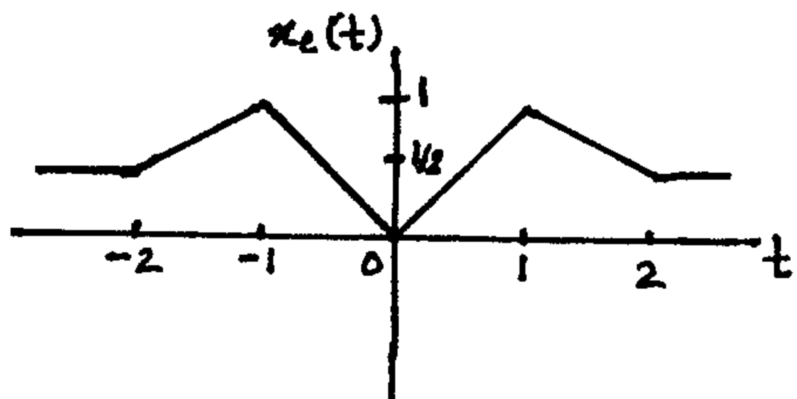
Sketching the even solution



plus

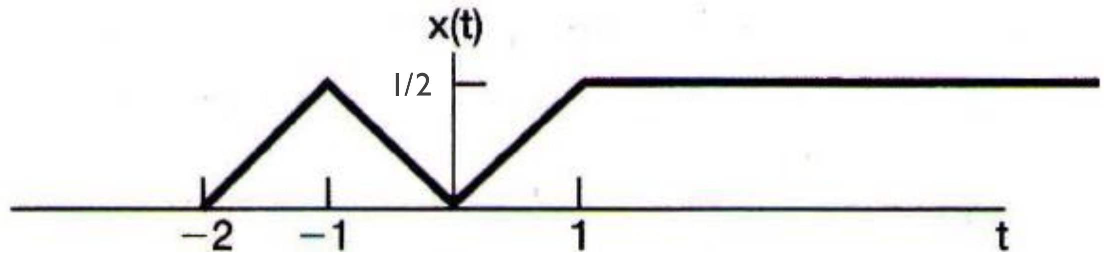
$$e(t) = \tfrac{1}{2}\,(x(t) + x(-t))$$



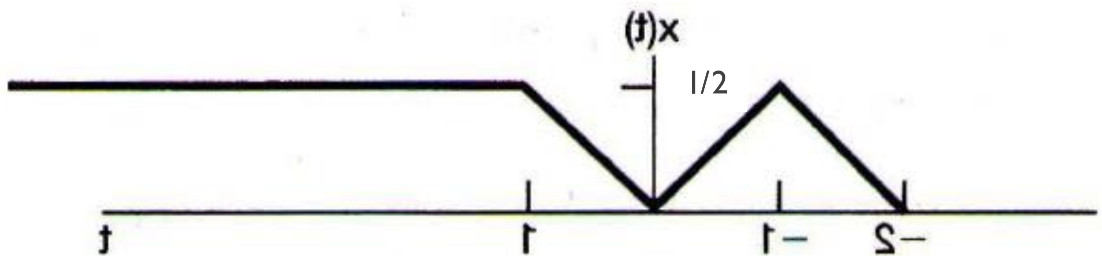


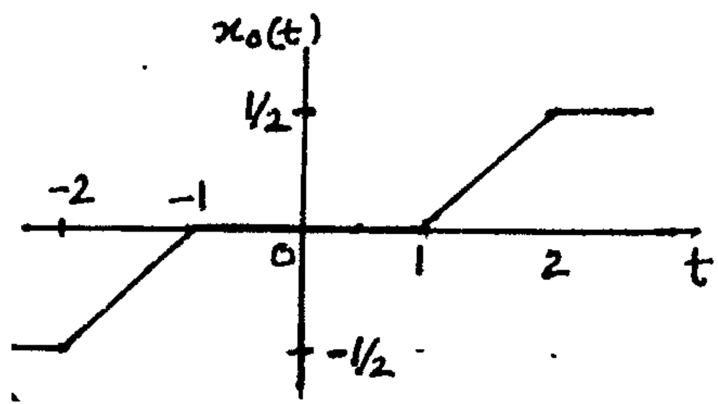
Sketching the odd solution

$o(t) = ½ (x(t) - x(-t))$



minus





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Similar properties for complex signals

$$E(t) = \frac{1}{2}[f(t) + f^*(-t)]$$ Conjugate symmetric

$$O(t) = \frac{1}{2}[f(t) - f^*(-t)]$$ Conjugate antisymmetric

$$E^*(-t) = \frac{1}{2}[f^*(-t) + f(t)] = E(t)$$

$$O^*(-t) = \frac{1}{2}[f^*(-t) - f(t)] = -O(t)$$

$$f(t) = E(t) + O(t)$$

Note: Conjugate symmetric & real is even (why?)

Signal characteristics summary

- Periodic signals
  - Special type of signals that repeat
  - Need to determine if a signal is periodic and its period

- Even, odd, conjugate symmetric, conjugate antisymmetric
  - Signals with symmetry about the y axis
  - Fact used to simplify computations and derive intuition

- These special signal structures will be used in Fourier analysis

Lecture 3

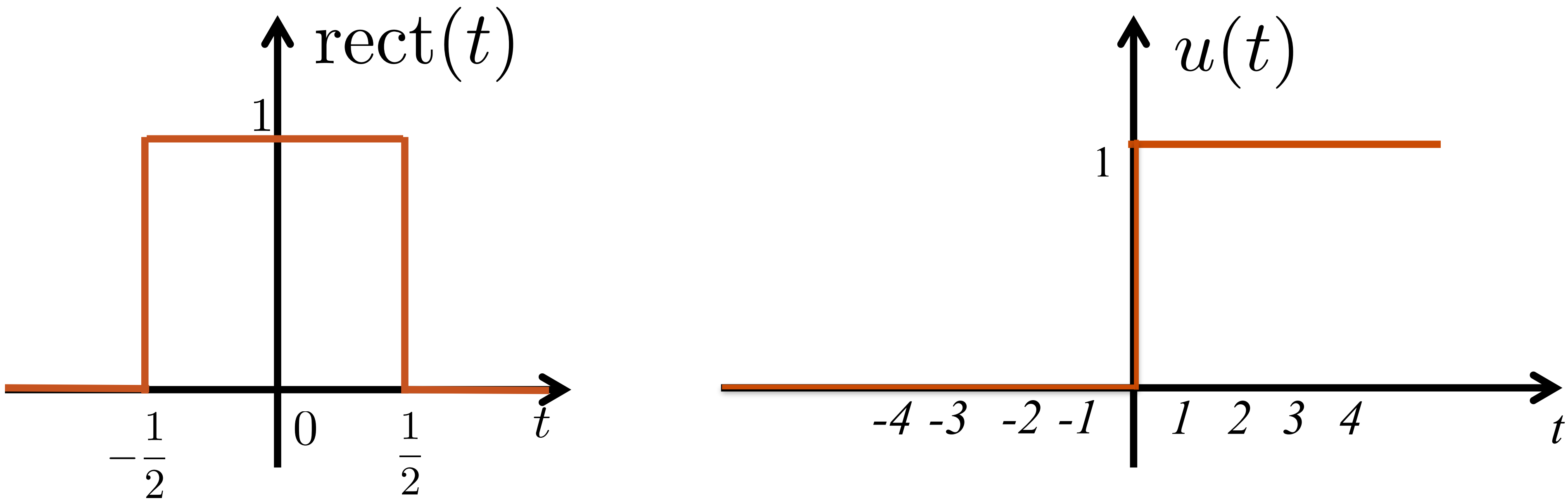
Exponential, sinusoids, complex exponentials, and the delta function

Preview of today's lecture

- CT sinusoids and exponentials
  - Determine the key parameters of a complex sinusoid and exponential
  - Sketch a complex exponential based on its form

- Unit-impulse function also known as the Dirac delta function
  - Explain the properties of delta unit impulse function
  - Exploit the sifting property to simplify expressions with deltas
  - Exploit the integration property to simplify expressions with deltas

From Lecture #2: two common functions



Examples: Shifting, inverting and scaling

- Let

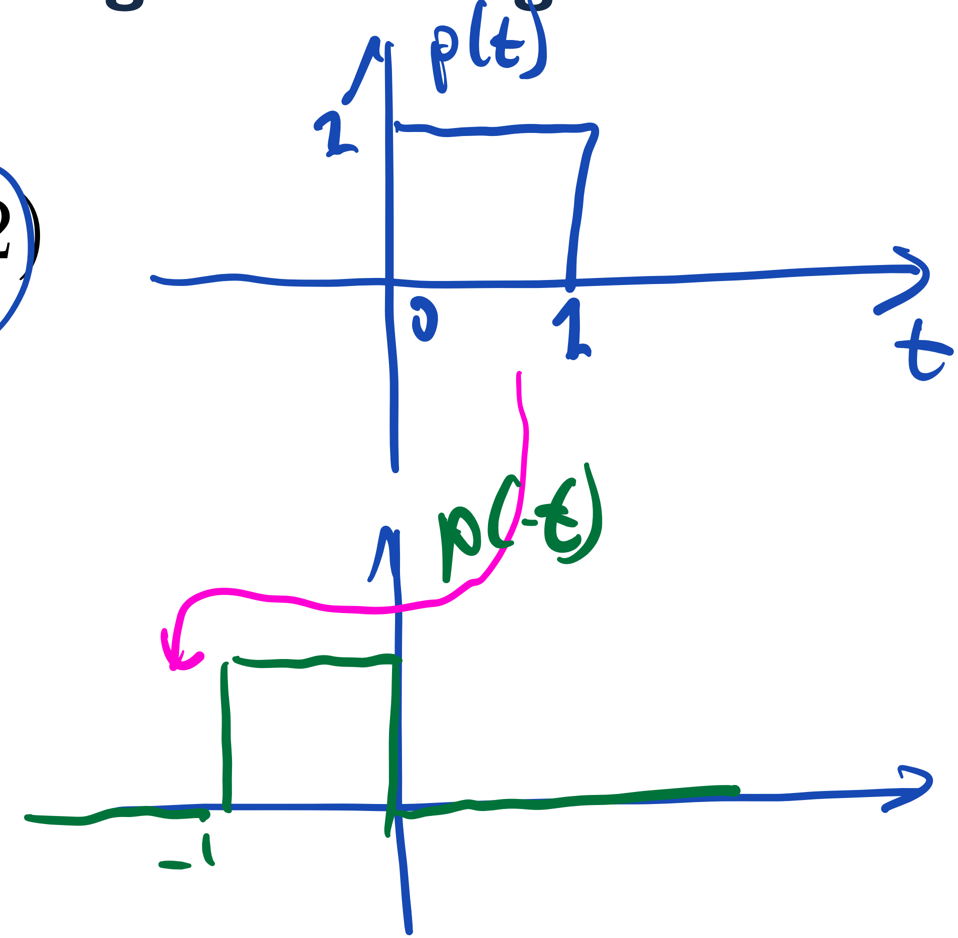
$$p(t) = \mathrm{rect}(t - 1/2)$$

- Find and plot

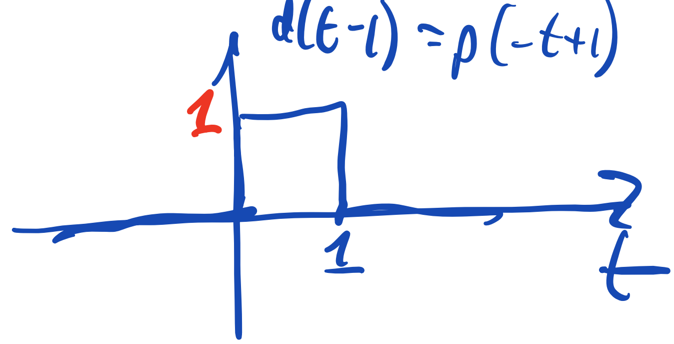
$$p(-t) = d(t)$$

$$p(-t+1) = b(t)$$

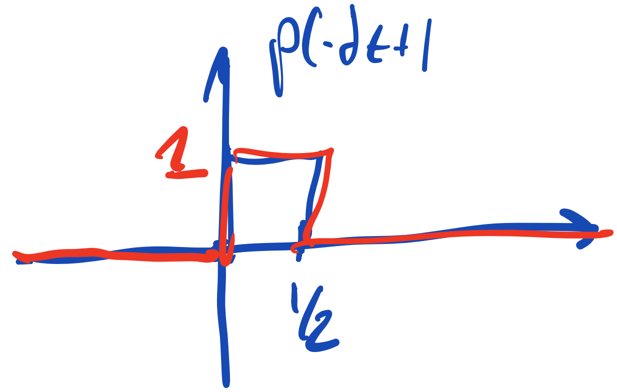
$$p(-2t+1)$$



$$p(-t+1) = p(-(t-1))$$
$$= d(t-1)$$



$$p(-2t+1) = d(2t)$$
$$= p(-2 \cdot t + 1)$$



$d(t)$ $t=1$ $2t=1$

$t = 1/2$

Connections back to ECE 45

Lectures 2 - 4 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 5 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, phasors	1.3		
4	1/16	Signals	Dirac delta, Kronecker delta, Sha function	1.4	HW3	HW2
5	1/21	LTI in time	Linear and time-invariant systems	2.1		
6	1/23	LTI in time	Convolution, convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

CT sinusoids and exponentials

Learning objectives

- Determine the key parameters of a complex sinusoid and exponential
- Sketch a complex exponential based on its form

CT real sinusoid

$\mathbb{R}$

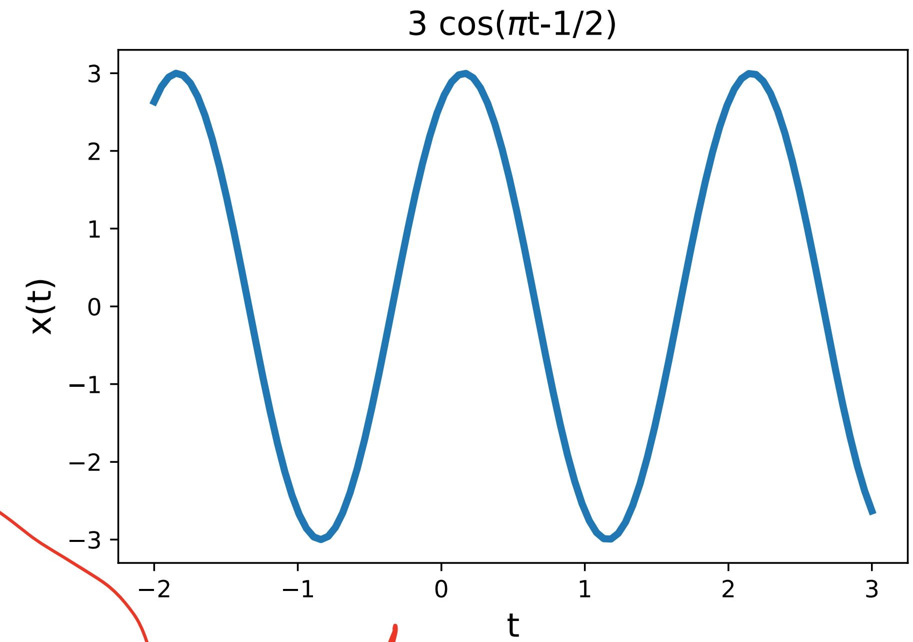
amplitude > 0 phase

$$x(t) = C\cos(\omega_0 t + \theta)$$

frequency in radians/s

Periodic with period $T = \dfrac{2\pi}{\omega_0}$

$\cos(\omega_0 t + \pi/3)$



$C = 3$

$\omega_0 = \pi$

$\theta = -1/2$

$\pi/3$

$$\cos\Big(\omega_0\big(\underbrace{t+\frac{2\pi}{\omega_0}}_{T}\big)\Big)$$

$$=\cos\left(\omega_0 t+\frac{\cancel{\omega_0}\cdot 2\pi}{\cancel{\omega_0}}\right)$$

$$=\cos(\omega_0 t+2\pi)=\cos(\omega_0 t)$$

Complex numbers

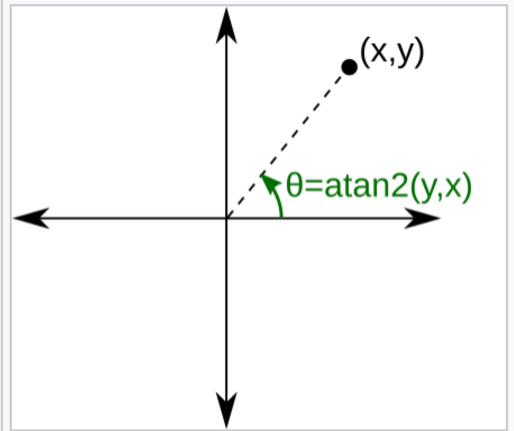
$$C = x + jy \qquad \text{Cartesian}$$

$$= |C|\, e^{j\theta} \qquad \text{Polar}$$

Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\boxed{C = \underbrace{|C|\cos(\theta)}_{\mathrm{Re}\,\{C\}} + j\,\underbrace{|C|\sin(\theta)}_{\mathrm{Im}\,\{C\}}}$$



atan2(y, x) returns the angle θ between the ray to the point (x, y) and the positive *x*-axis, confined to $(-\pi, \pi]$.

https://en.wikipedia.org/wiki/Atan2

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

CT complex sinusoid

$C \in \mathbb{R}$, $C > 0$

complex frequency in radians/s

$$x(t) = Ce^{j\omega_0 t} \qquad \text{period } T = \frac{2\pi}{\omega_0}$$

$$= \overbrace{|C|e^{j\theta}}\, e^{j\omega_0 t}$$

$$= \boxed{|C|e^{j(\omega_0 t + \theta)}}$$

amplitude phase

$$= |C|\cos(\omega_0 t + \theta) + j|C|\sin(\omega_0 t + \theta)$$

real and imaginary portions related through Eulers

Note: if the amplitude C is negative, then we could simply compensate for the negative by shifting the phase by pi

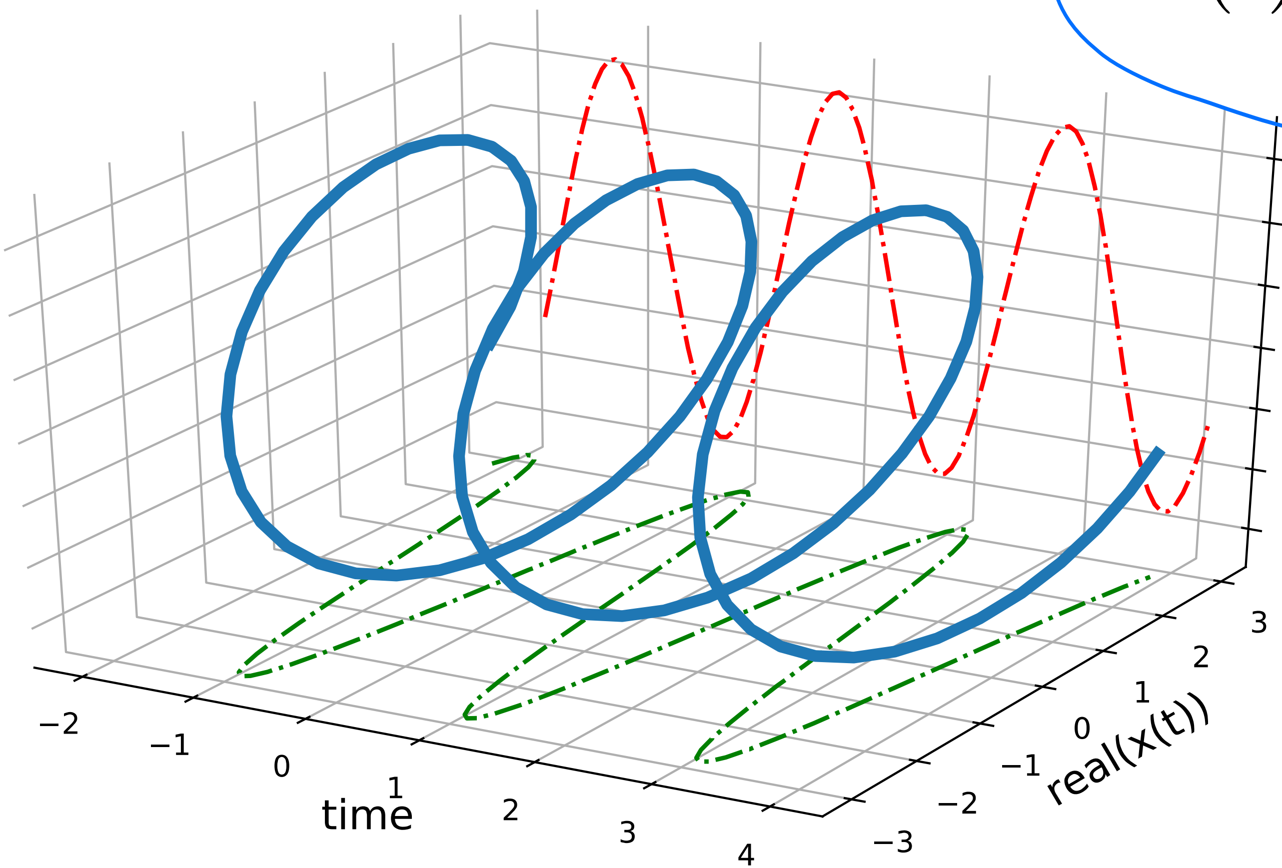
Visualizing a complex sinusoid

$$x(t) = 3e^{j(\pi t - \frac{1}{2})}$$

$|x(t)|$

$|3e^{j(\pi t - 1/2)}|$

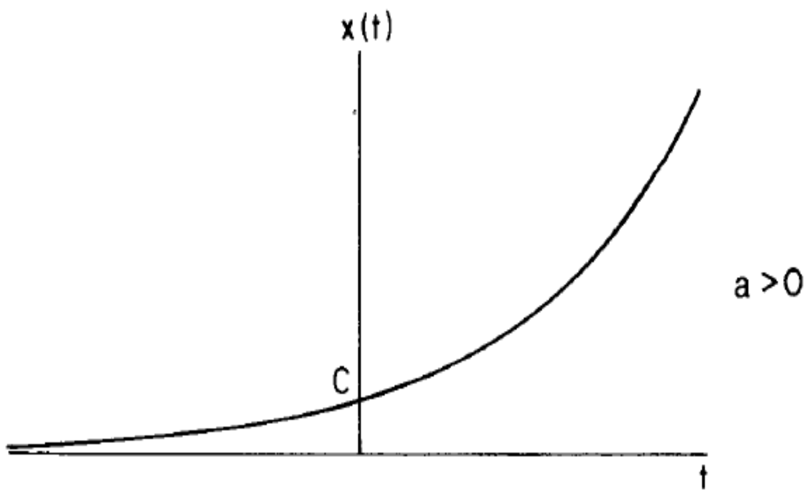
$= 3$



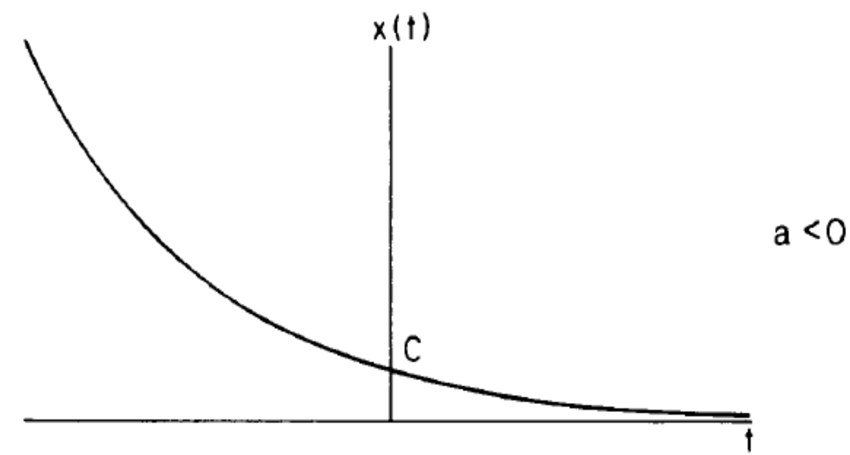
CT real exponential signal

$$x(t) = Ce^{at}$$

real real



Increasing exponential
(a > 0, C > 0)



Decaying exponential
(a < 0, C > 0)

CT complex exponential: general case

$$x(t) = Ce^{at}$$

complex complex

polar of C

car of a

$C = c_\sigma + jc_\omega$; cartesian

$= |C|e^{j\theta}$, polar

$a = r + j\omega_0$, cartesian

$$x(t) = Ce^{at} = |C|e^{j\theta}\overbrace{e^{(r+j\omega_0)t}}$$

$e^{rt}e^{j\omega_0 t}$

$$\boxed{x(t) = |C|e^{rt}e^{j(\omega_0 t+\theta)}}$$

General case includes real exponential, real sinusoid, and complex sinusoid as special cases

Visualizing CT complex exponentials

$$x(t) = \overbrace{|C|e^{rt}}^{\text{real}} e^{j(\omega_0 t+\theta)}$$

Increasing or decaying exponential "envelope"

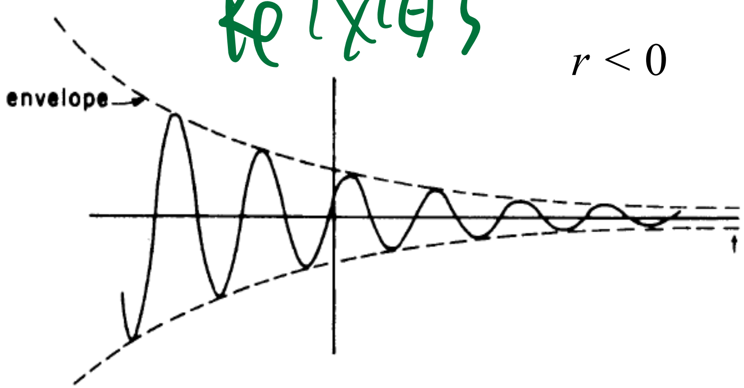
Complex sinusoid

$$Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t + \theta)$$
$$Im\{x(t)\} = |C|e^{rt}\sin(\omega_0 t + \theta)$$

$$e^{rt} \text{ is the "damping" term} \longrightarrow \begin{cases} r > 0 \to \text{ blows up} \\ r < 0 \to \text{ decays to zero} \end{cases}$$

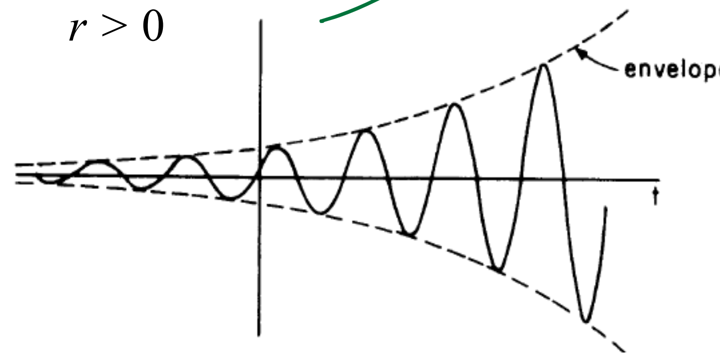
Re{x(t)}

$r < 0$



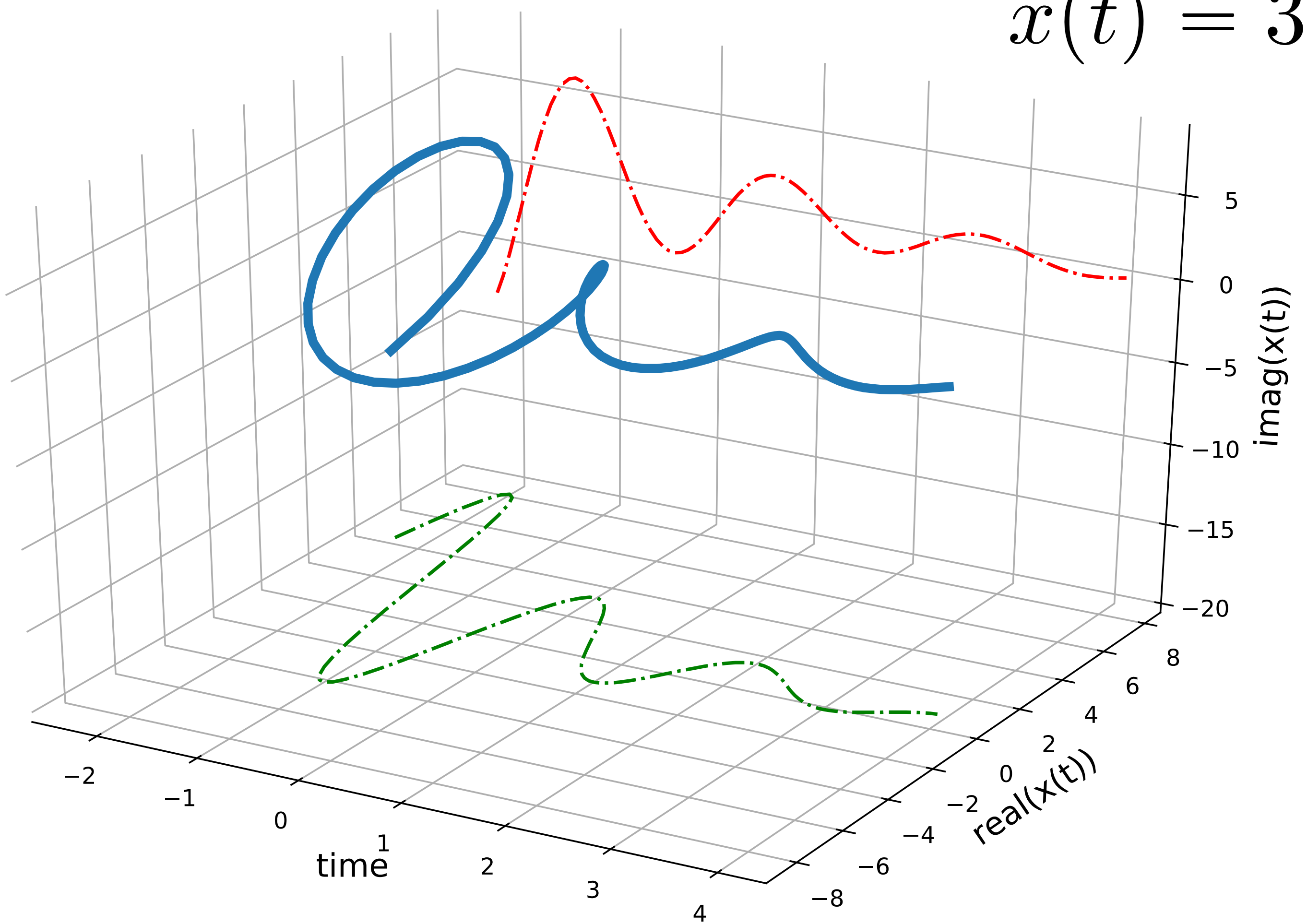
Re{x(t)}

$r > 0$



Visualizing CT complex exponentials in 3D

$$x(t) = 3e^{-t/2}e^{j(\pi t - 1/2)}$$



Example: Sketching a complex exponential

- Consider a complex exponential with the following values

$$x(t) = Ce^{at}$$

$$C = 3 - j, \quad a = 1 + 10j$$

- The problem:
  - Express x(t) in terms of its envelope and complex sinusoidal parts
  - Express the real and imaginary parts of x(t)
  - Express the magnitude of x(t)
  - Plot all of these from time t = 0 to t = 3, showing the envelope

Solution I

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

- Convert $C = 3 - j$ to polar form

$$\begin{aligned} C &= \sqrt{3^1 + 1} e^{j \operatorname{atan2}(-1,3)} \\ &= \sqrt{10} e^{-j0.3218} \end{aligned}$$

- Substitute to get the general form (recall $a = 1 + 10j$)

$$\begin{aligned} x(t) &= Ce^{at} \\ &= \sqrt{10} e^{-j0.3218} e^{t(1+10j)} \\ &= \sqrt{10} e^t e^{j(10t - 0.3218)} \end{aligned}$$

See more about the two argument arc tangent here https://en.wikipedia.org/wiki/Atan2

Solution 2

- Apply Euler's to get real and imaginary parts:

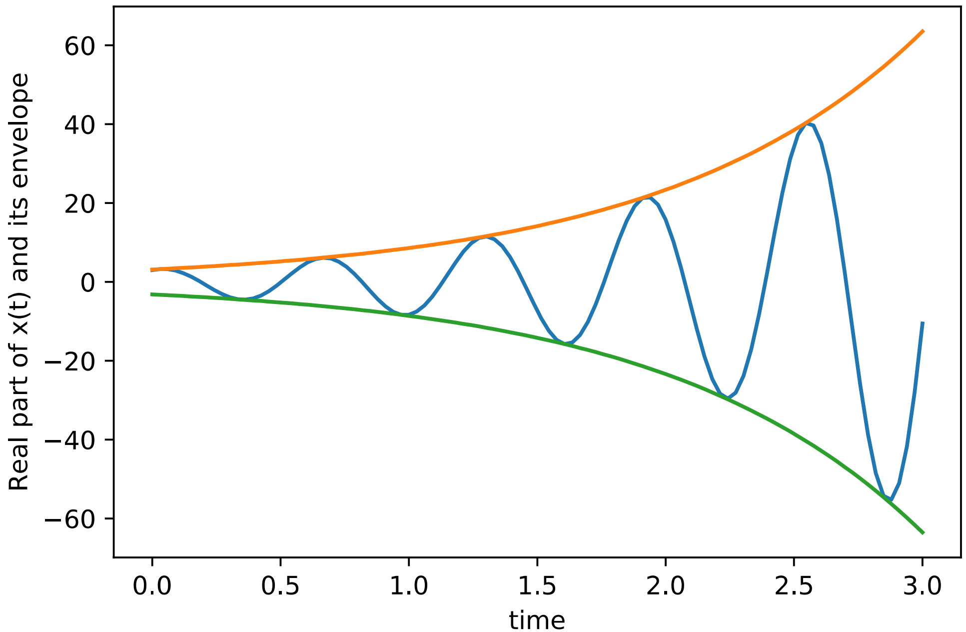
$$Re\{x(t)\} = \sqrt{10}e^{t}\cos(10t - 0.3218)$$
$$Im\{x(t)\} = \sqrt{10}e^{t}\sin(10t - 0.3218)$$

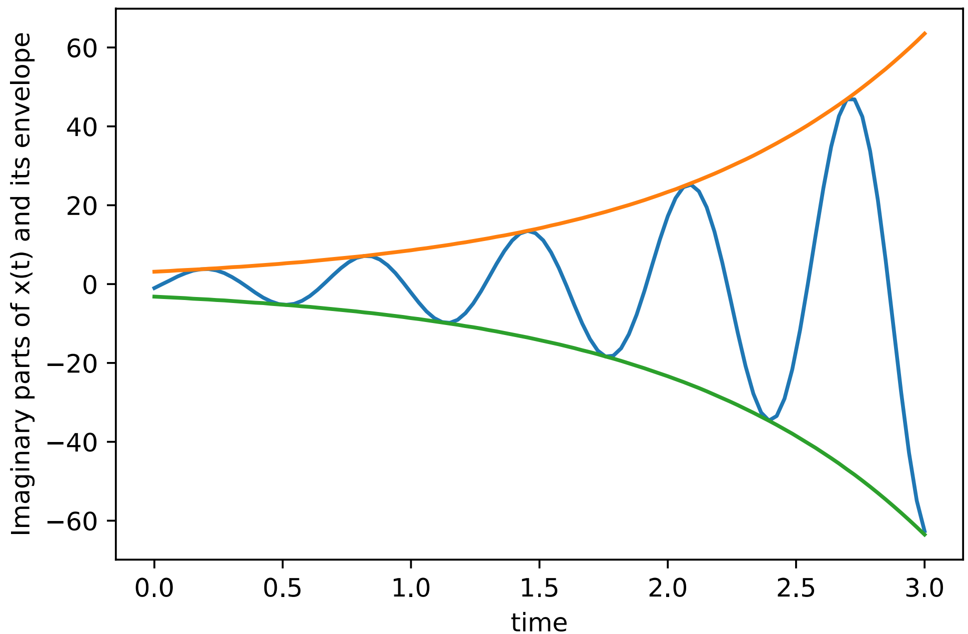
- Use the general form to find the magnitude

$$|x(t)| = \sqrt{10}e^{t}$$

- Sketch the signal behavior

Real and imaginary parts are just π/2 shifts (magnitude is just the positive envelope)





$$Re\{x(t)\} = \sqrt{10}e^{t}\cos(10t - 0.3218)$$
$$Im\{x(t)\} = \sqrt{10}e^{t}\sin(10t - 0.3218)$$

MATLAB to create these plots

```
C = 3-j;
a = 1+10j;
t = 0:.01:3; % equivalent to linspace (0,3,101)
x = C*exp(a*t);
env = abs(C)*exp(real(a) *t);

figure(1);
plot(t,real(x),t,env,t,-env);
xlabel('time');
ylabel('Real part of x(t) and its envelope');

figure(2);
plot(t,imag(x),t,env,t,-env);
xlabel('time');
ylabel('Imaginary parts of x(t) and its envelope');
```

CT exponential and sinusoidal signals in summary

- Complex exponentials and sinusoids
  - Important building blocks for future lectures
  - Can be understood through Euler's identity
  - Related to real exponentials and real sinusoids
  - Will become close friends by the end of the course ☺

- You should be able to
  - Identify complex exponentials and complex sinusoids
  - Determine the period of a complex sinusoid
  - Plot a complex exponential sketch and in MATLAB

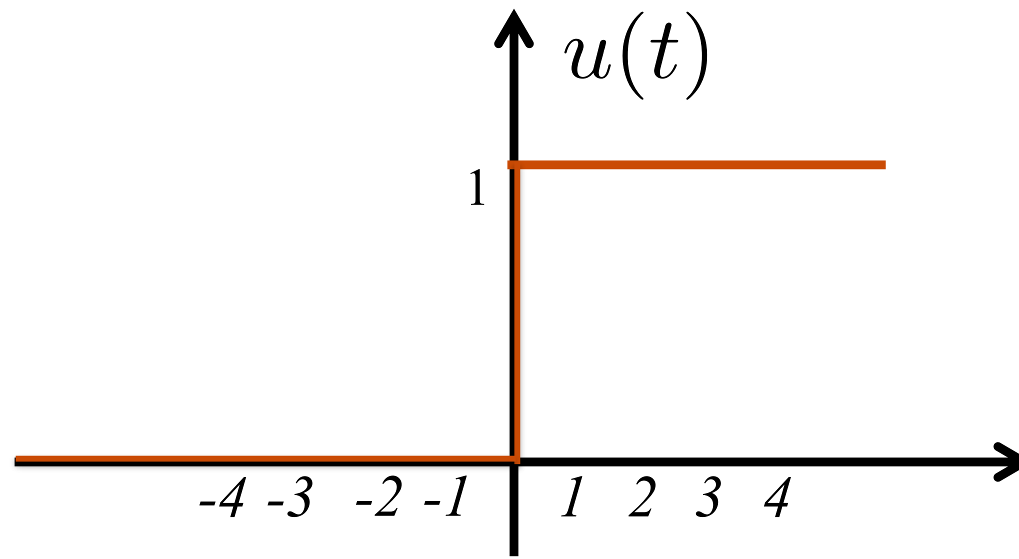
Unit-impulse function
also known as the Dirac delta function

Learning objectives

- Explain the properties of delta unit impulse function
- Exploit the sifting property to simplify expressions with deltas
- Exploit the integration property to simplify expressions with deltas

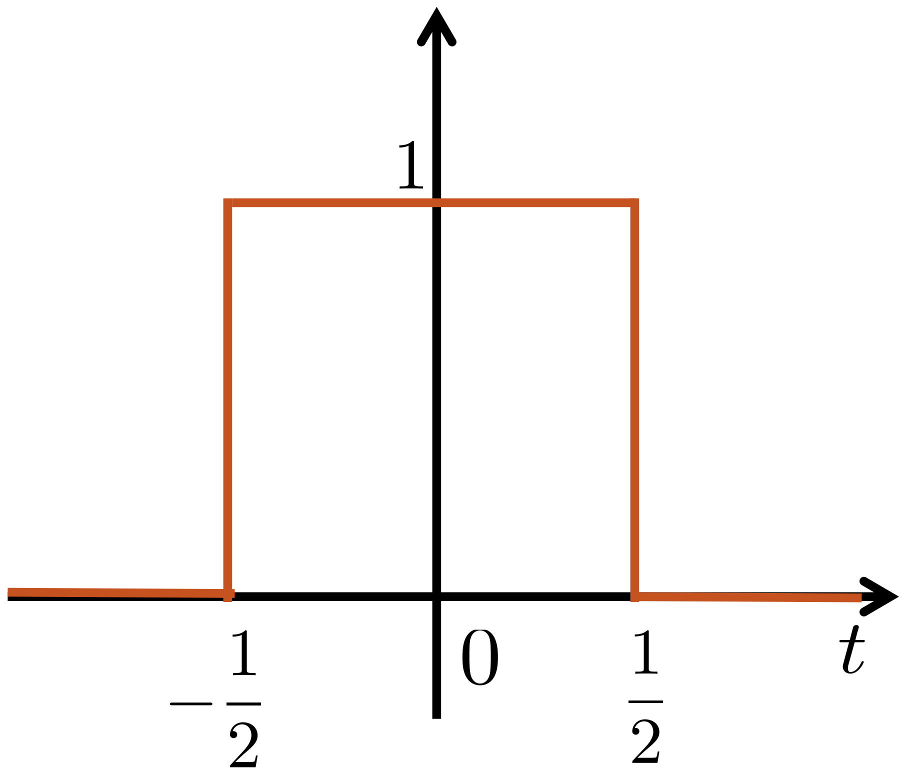
Unit step function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

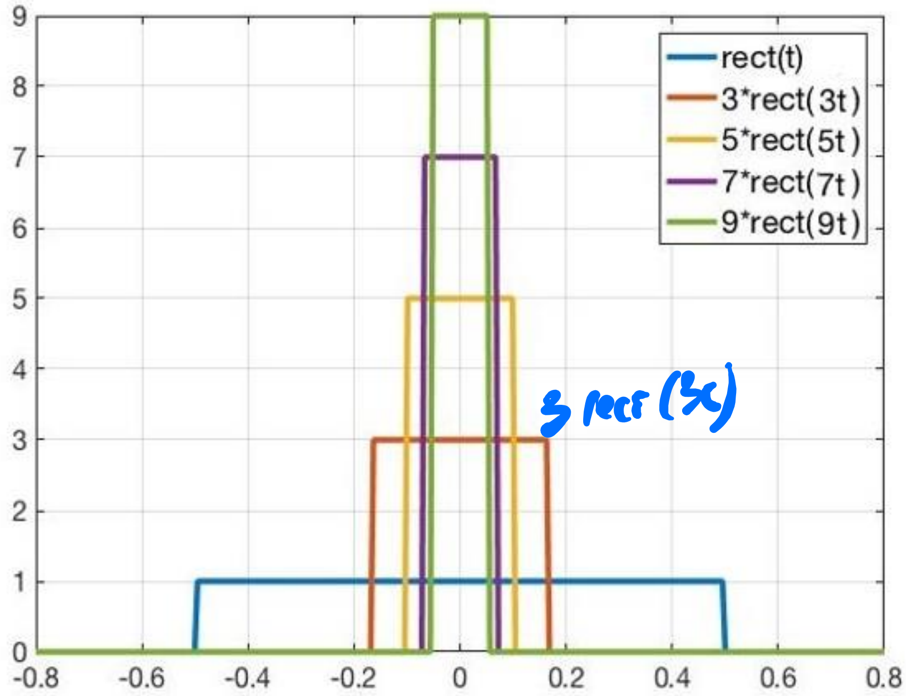


Rectangle function

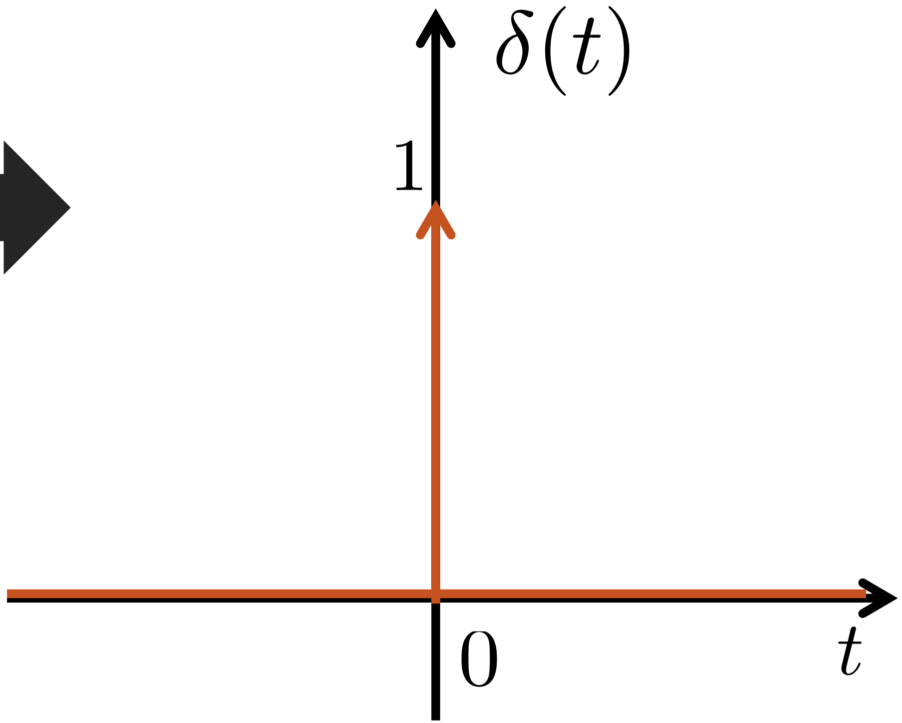
$$\mathrm{rect}(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leqslant \frac{1}{2} \end{cases}$$



Consider the following sequence

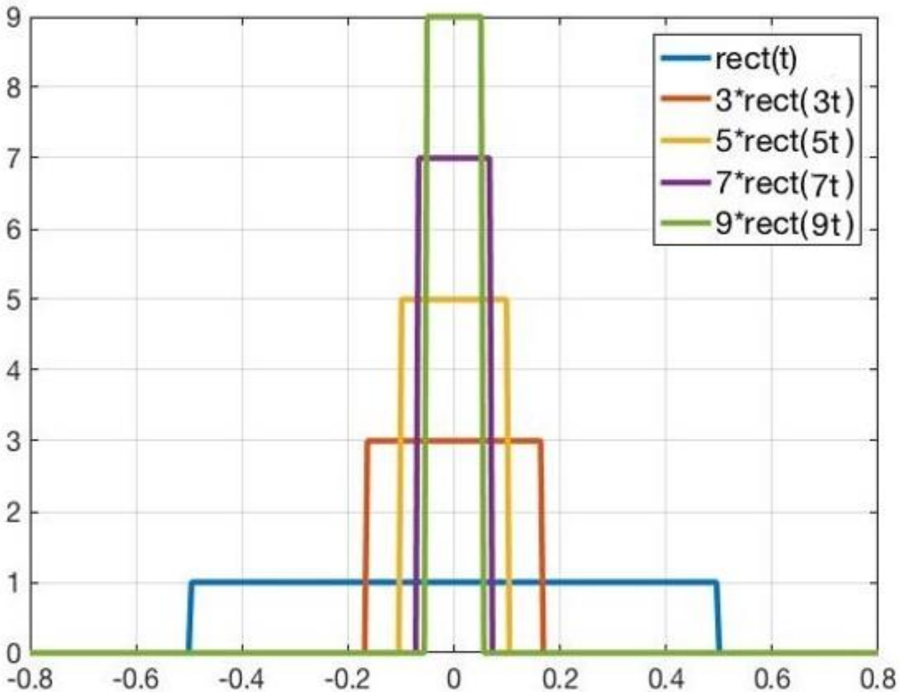


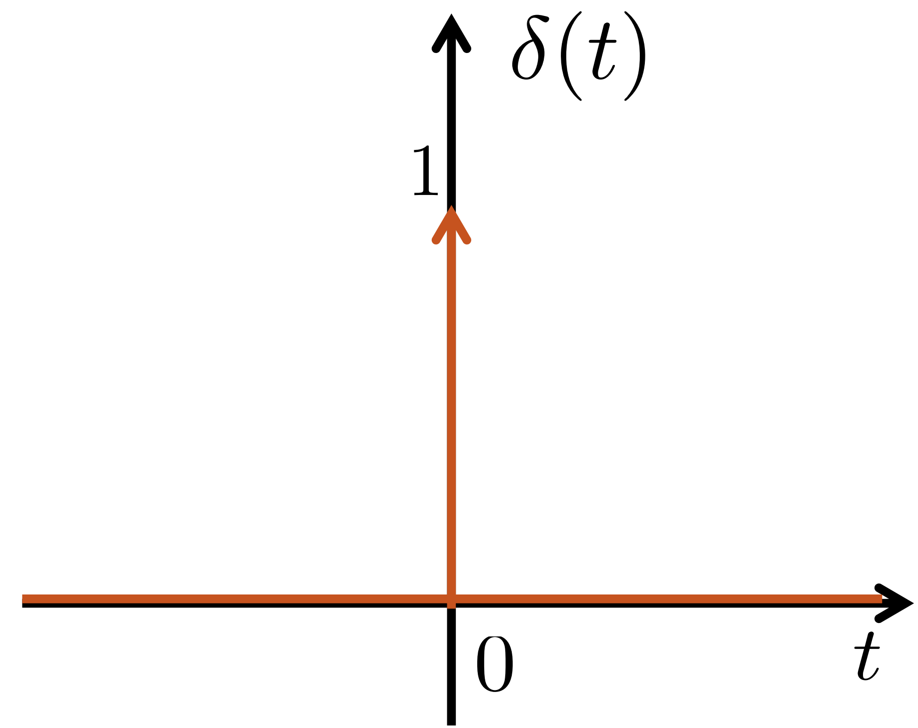
Dirac delta or unit-impulse function



Delta function is a generalized function that requires some care

Unit area property



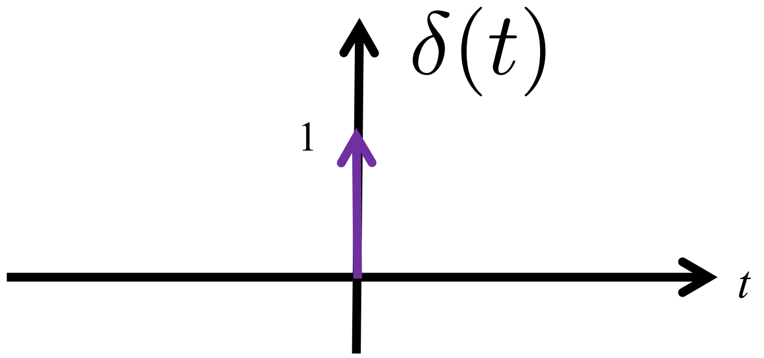


Bt

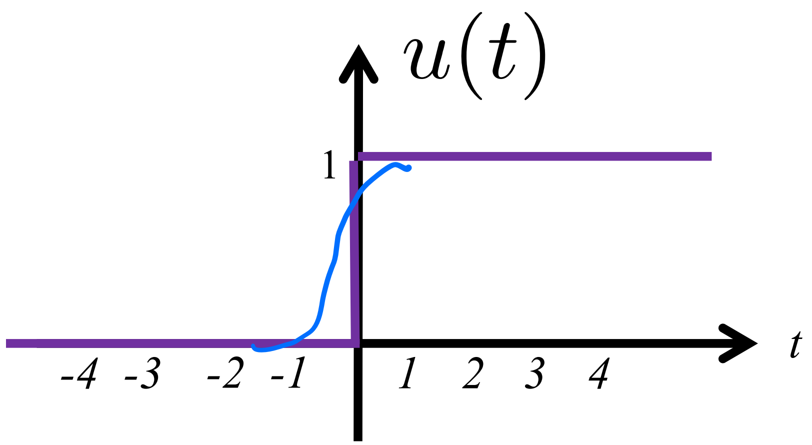
$$\int_{-\infty}^{\infty} B \,\mathrm{rect}\left(\frac{t}{B}\right) dt = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

Connection to the unit step function

$$\frac{du(t)}{dt} = \delta(t)$$



$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$



Sifting property at zero

any function

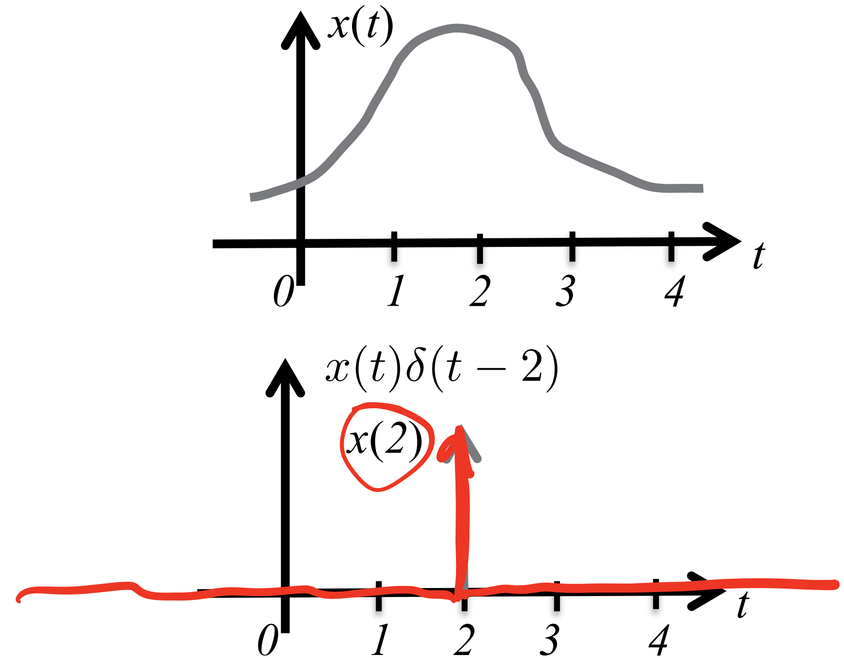
$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Avoid common error: be sure to leave in delta function!

From https://www.epicurious.com/ingredients/do-you-really-need-to-sift-flour-article

Example



Sifting property example

- Consider the following signal

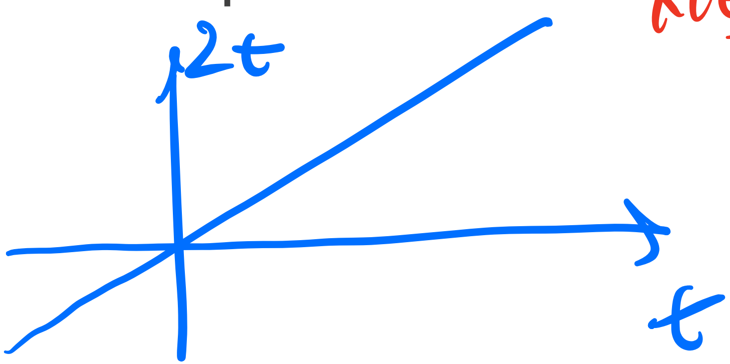
$$x(t) = 2t$$

- Find a simplified expression for

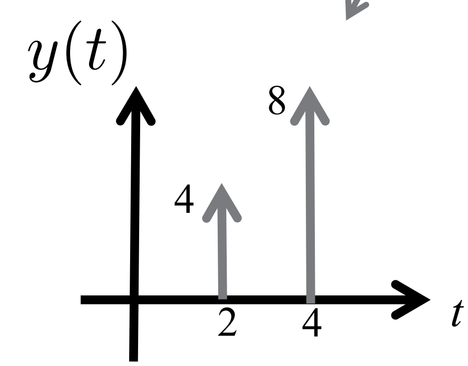
$$y(t) = x(t)(\delta(t-2) + \delta(t-4))$$

and plot the result

$$x(t)\delta(t-2)$$
$$+ x(t)\delta(t-4)$$
$$= x(2)\delta(t-2) + x(4)\delta(t-4)$$



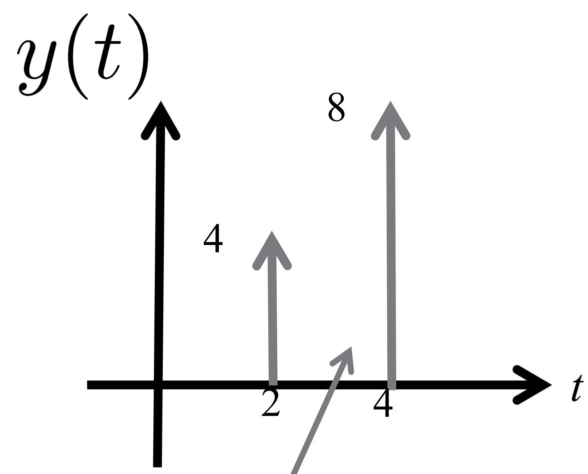
We draw deltas like this



Sifting property example

$$x(t) = 2t$$

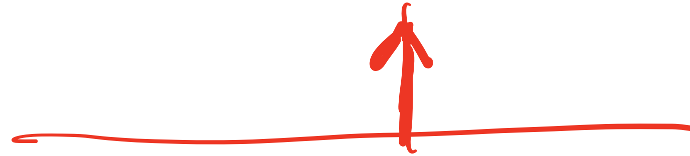
$$\begin{aligned} y(t) &= x(t)(\delta(t-2) + \delta(t-4)) \\ &= 2t(\delta(t-2) + \delta(t-4)) \\ &= 2t\delta(t-2) + 2t\delta(t-4) \\ &= 4\delta(t-2) + 8\delta(t-4) \end{aligned}$$



We label deltas like this

$$\int_{-\infty}^{\infty} y(t)\,dt = \int_{-\infty}^{\infty} 4\delta(t-2)\,dt + \int_{-\infty}^{\infty} 8\delta(t-4)\,dt = 12$$

Integration property



$$\int_{-\infty}^{\infty} x(\tau)\delta(\tau)d\tau = x(0)$$

$$\int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau = x(t)$$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

There are no delta functions in the final expression, they were integrated out

Integration property examples

- Simplify the following expressions

$$\int_{-\infty}^{\infty} \cos\left(\frac{\pi t^2}{2}\right)\delta(t+2)dt$$

$$= \int_{-\infty}^{\infty} \cos\left(\frac{\pi(-2)^2}{2}\right)\delta(t+2)\,dt$$

$$= \cos(2\pi)\cdot\underbrace{\int_{-\infty}^{\infty}\delta(t+2)\,dt}_{1}$$

$$= 1$$

$$\int_{-\infty}^{\infty} \delta(t-2)\delta(t+2)dt = 0$$

$g(t)$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,d\tau$$

$$x(t)\int_{-\infty}^{\infty}\delta(t-\tau)\,d\tau = x(t)$$

$$\int_{-\infty}^{\infty} x(\eta)\,\delta(t-\eta)\,d\eta = g(t)$$

Integration property examples

- Simplify the following expressions

$\int_{-\infty}^{\infty} \cos\left(\frac{\pi t^2}{2}\right) \delta(t+2)dt$	$\int_{-\infty}^{\infty} \cos\left(\pi \frac{(-2)^2}{2}\right) \delta(t+2)dt = \cos(2\pi) = 1$
$\int_{-\infty}^{\infty} \delta(t-2)\delta(t+2)dt$	0
$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$	$x(t)$

Time scaling the delta function (1/3)

- What is $\int_{-\infty}^{\infty} \delta(a\tau)d\tau$?

- Suppose that $a > 0$

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(a\tau)d\tau &= \int_{-\infty}^{\infty} \frac{1}{a}\delta(t)dt \\ &= \frac{1}{a}\end{aligned}$$

real number

$$\int_{-\infty}^{\infty} \delta(a\tau)\, d\tau \qquad a>0$$

$$t = a\tau \qquad dt = a\, d\tau$$

$$d\tau = \frac{1}{a}\, dt$$

$$\int_{-\infty}^{\infty} \frac{1}{a}\, \delta(t)\, d\tau = \frac{1}{a}$$

$a < 0$

$$\int_{-\infty}^{\infty} \delta(at)\, dt$$

$t = a\tau \qquad dt = a\, d\tau$

$$\int_{+\infty}^{-\infty} \frac{1}{a}\, \delta(t)\, dt$$

$$= -\int_{-\infty}^{\infty} \frac{1}{a}\, \delta(t)\, dt = \frac{1}{-a}$$

Time scaling the delta function (2/3)

- Suppose that $a < 0$

$$\int_{-\infty}^{\infty} \delta(a\tau)d\tau = \int_{-\infty}^{\infty} \frac{1}{-a}\delta(t)dt$$

$$= \frac{1}{-a}$$

Time scaling the delta function (3/3)

- Conclude that for any real value a

$$\int_{-\infty}^{\infty} \delta(a\tau)d\tau = \frac{1}{|a|}$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\int_{-\infty}^{\infty} \cos\left((\pi t)^2/2\right) \delta(2 \cdot t)\, dt$$

$$\frac{1}{2}\delta(t)$$

$$\int_{-\infty}^{\infty} \frac{1}{2} \cdot \cos\left((\pi t)^2/2\right) \delta(t)\, dt$$

$$\frac{1}{2} \cdot \cos\left((\pi \cdot 0)^2/2\right) = \frac{1}{2}$$

Cautionary notes on the delta function

- The unit-impulse function, also called the Direct delta function, is really a *generalized function*
  - It does not really behave like a normal function
  - It is either zero or undefined
- We should technically only be using $\delta(t)$ under the integral sign
  - It is well defined in the integral sign
  - Some Professors will complain if not in the integral sign (but not me)
  - Take real analysis in the math department for further enlightenment
- Despite these quirks, <u>the delta function is extremely useful</u> for modeling and understanding signals and systems

"All models are wrong, but some are useful" – George Box (statistician)

Summary of the delta function

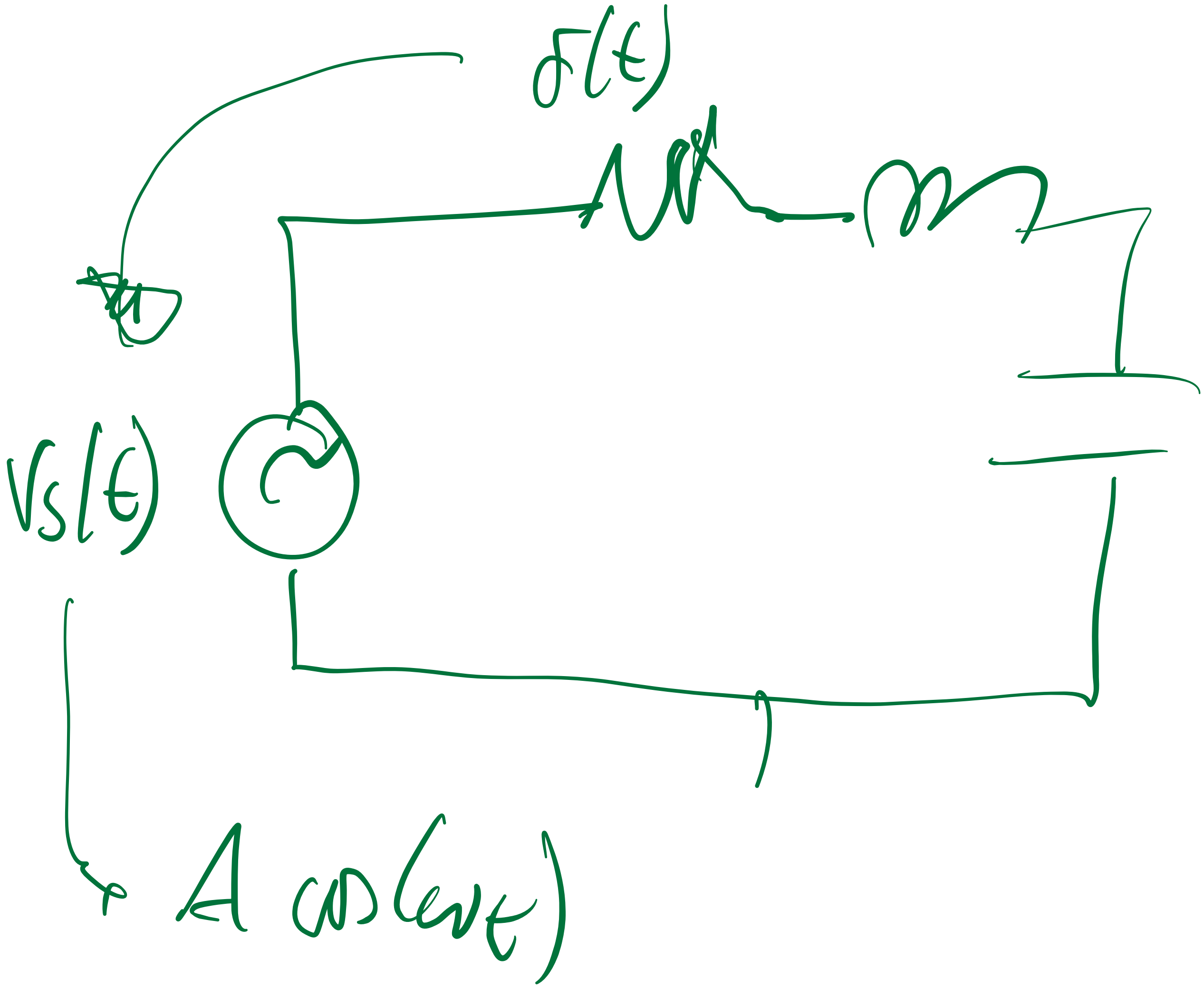
- Unit-impulse or Dirac delta in important basic signal
  - Generalized function of time

- Sifting with deltas pulls out the signal value but leaves the delta

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

- Integrating with deltas eliminates the delta and gives a value

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

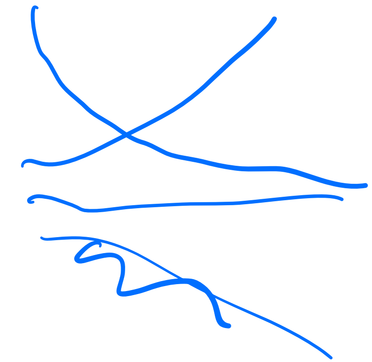

$\delta(t)$
$V_s(t)$
$A\cos(\omega t)$

Lecture 4

~~Exponential, sinusoids, complex exponentials, and the delta function~~

Systems, Time invariance, linearity

Complex exponential signals



- General form of complex exponential

$$x(t) = Ce^{at}$$

- Simplifying with $C = |C|e^{j\theta}$ and $a = r + j\omega_0$

$$x(t) = |C|e^{j\theta}e^{rt+j\omega_0 t}$$
$$= |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

not periodic

- Complex sinusoid is the special case $r = 0$

$$x(t) = |C|e^{j(\omega_0 t+\theta)}$$

$\omega_0 = \frac{2\pi}{T}$

Example

- Suppose $C = 0.25\, e^{j0.2\pi}$

$$a = \underbrace{0.5}_{r} + \underbrace{j2\pi 1000}_{\omega_0}$$

$$|x(t)| = 0.25\, e^{t/2}$$

$$\angle x(t) = 0 + 0 + \underbrace{2\pi 1000 t + 0.2\pi}_{\text{unwrapped phase}}$$

- Determine

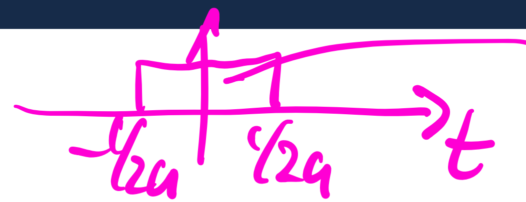
$$x(t) = 0.25\, e^{t/2}\, e^{j(2\pi 1000 t + 0.2\pi)}$$

$$0.25\, e^{t/2} \{\cos(2\pi 1000 t + 0.2\pi) + j \sin(2\pi 1000 t + 0.2\pi)\}$$

$$\mathrm{Re}\{x(t)\} = 0.25\, e^{t/2} \cos(2\pi 1000 t + 0.2\pi)$$

Delta function

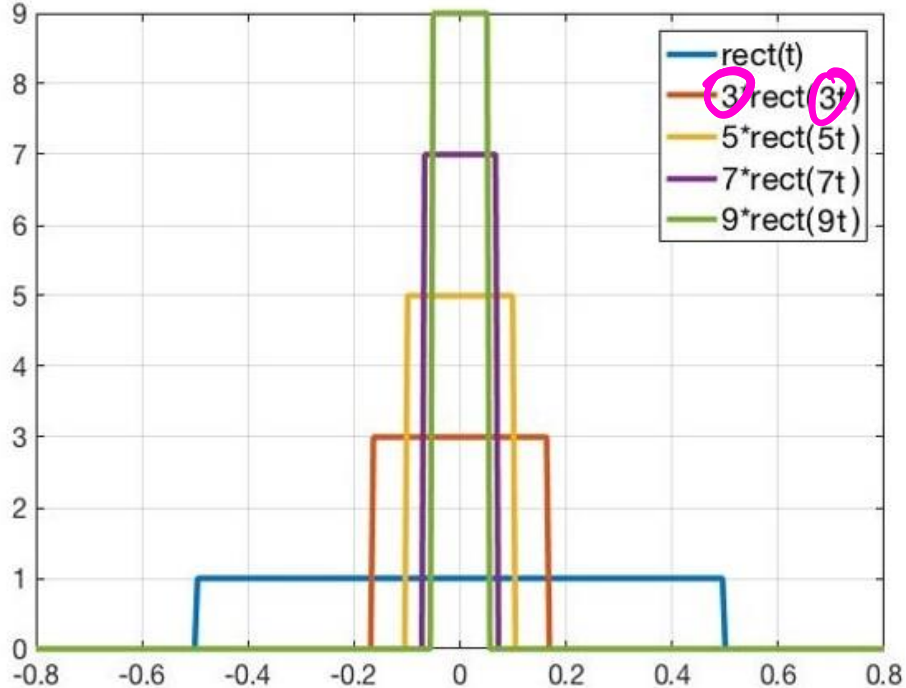
rect(at)

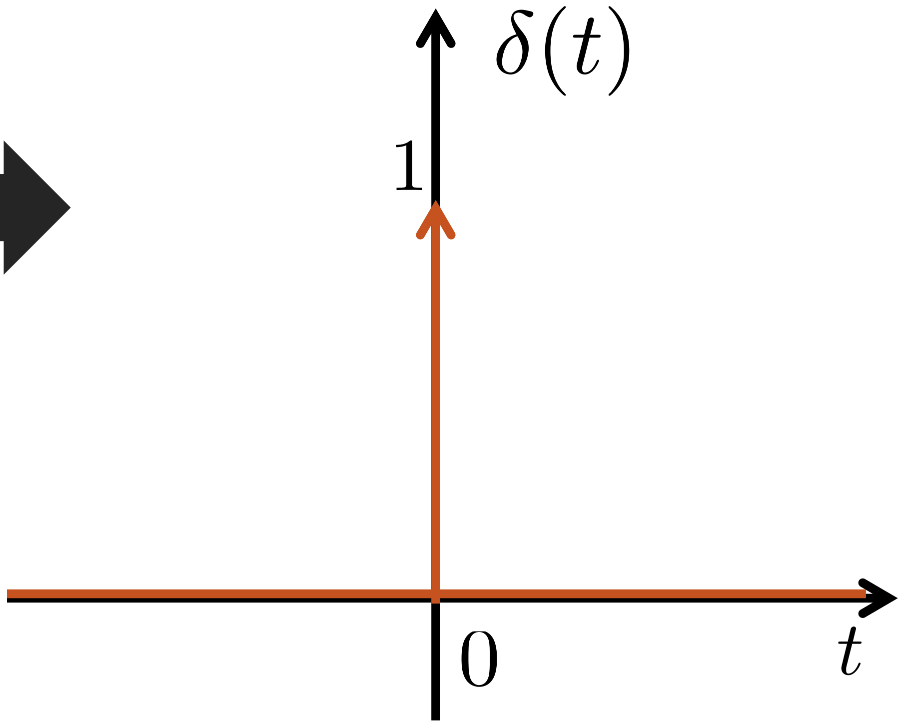


1/a

at=1/2

Dirac delta or unit-impulse function





Delta function as the limit of a sequence of every narrowing unit energy rectangles

Summary of the delta function

- Sifting with deltas pulls out the signal value but leaves the delta

$$\underbrace{x(t)}\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

1

- Integrating with deltas eliminates the delta and gives a value

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

$$\int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty} \cancel{\delta(t-t_0)\,dt}$$

- Other properties

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\boxed{\delta(at) = \frac{1}{|a|}\delta(t)}$$

Examples

- Simplify the following

1 $$(3r^2 + 2r + 1)\delta(r - 1) = (3 \cdot 1^2 + 2 \cdot 1 + 1)\delta(r-1)$$
$$= \boxed{6\delta(r-1)}$$

2 $$\int_{-\infty}^{\infty} (3r^2 + 2r + 1)\delta(r - 1)dr = 6$$

3 $$3r^2\delta(2r) = 3r^2 \cdot \frac{1}{2} \cdot \delta(r) = \frac{3r^2}{2} \cdot \delta(r-0)$$
$$= 0^2 \cdot \delta(r) = 0$$

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

UC San Diego

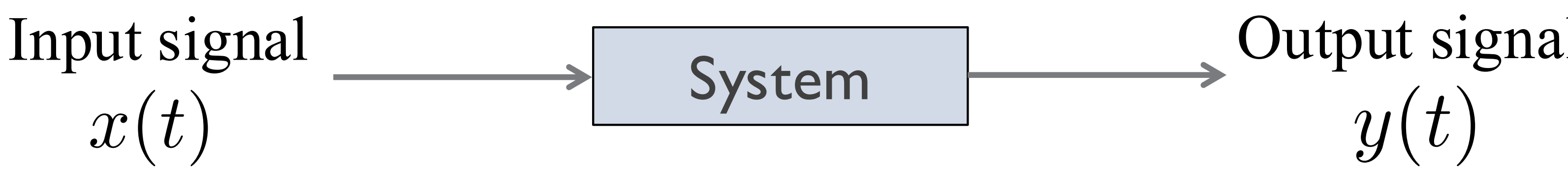
	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, Dirac Delta	1.3 - 1.4		
4	1/16	LTI in time	Systems, linearity, time invariance	1.6.5, 1.6.6	HW3	HW2
5	1/21	LTI in time	Impulse response and convolution	2.2		
6	1/23	LTI in time	Convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Introduction to Systems

Learning objectives

- Describe the output of the systems in terms of their inputs
- Give examples of continuous-time systems

Continuous-time systems



Amplifier (or all-pass amplifier)

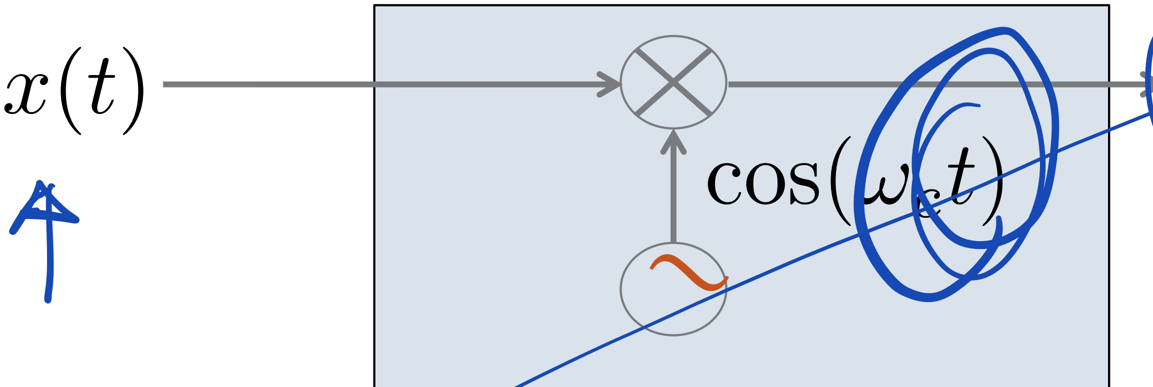
$\cos(24\pi t)$ $\quad\quad$ $A\cos(24\pi t)$

$$x(t) \longrightarrow \boxed{A} \longrightarrow y(t) = Ax(t)$$

$A > 0$ "good"

- A is the gain $\quad A > 0$
- Passes all input frequencies equally (makes more sense w/ Fourier)

Amplitude modulation



$$y(t) = x(t)\cos(\omega_c t)$$

Information "rides" the carrier

$\omega_c = 2\pi \cdot 800{,}000$

Amplitude Modulation transmitter (AM radio)

- At the receiver:

$$(x(t)\cos(\omega_c t))\cos(\omega_c t) = \cos^2(\omega_c t)x(t)$$

$$= \frac{1}{2}(1 + \cancel{\cos(2\omega_c t)})x(t)$$

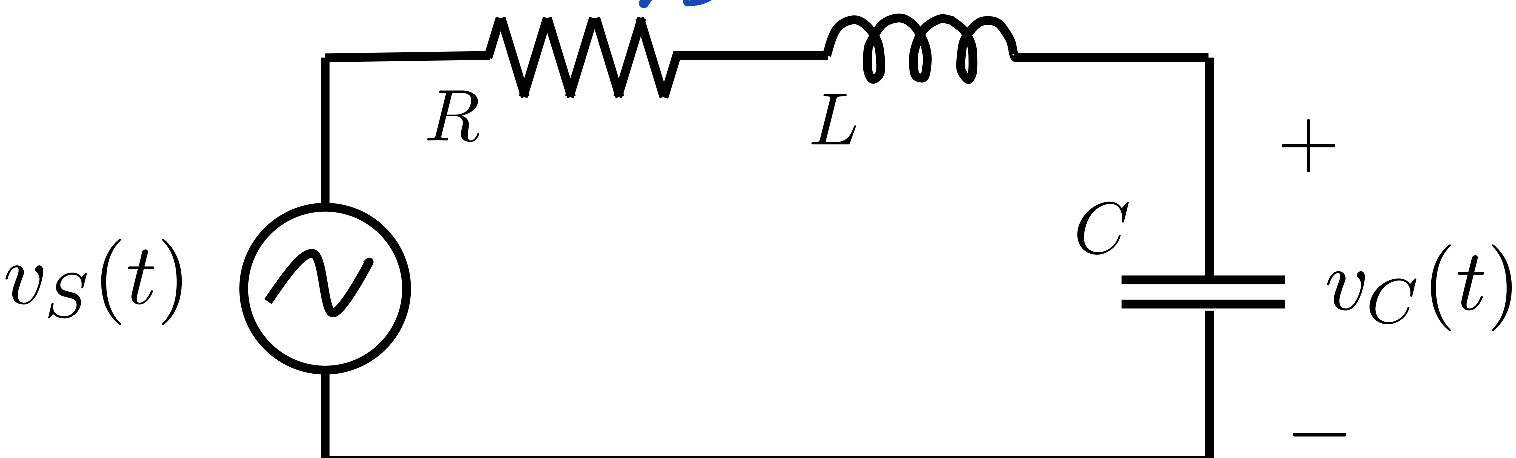
$$\longrightarrow x(t)$$

Filter out high frequencies

Note: The filter operation only works for certain bandlimited signals x(t), not true in general (we will cover concepts of filtering and bandlimited in subsequent lectures)

RLC circuit

Example RLC circuit



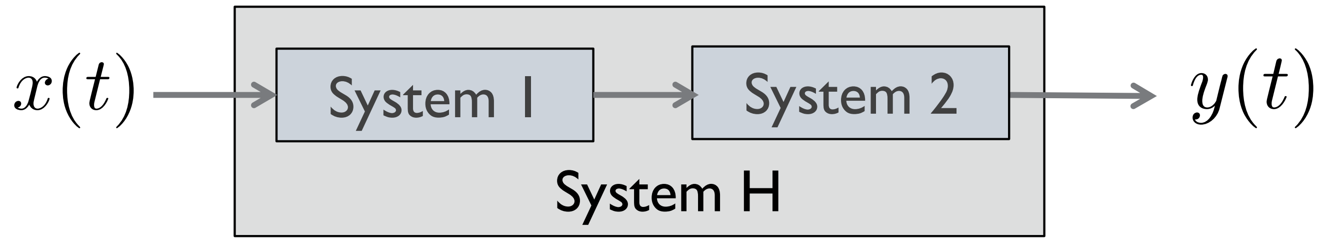
output

in

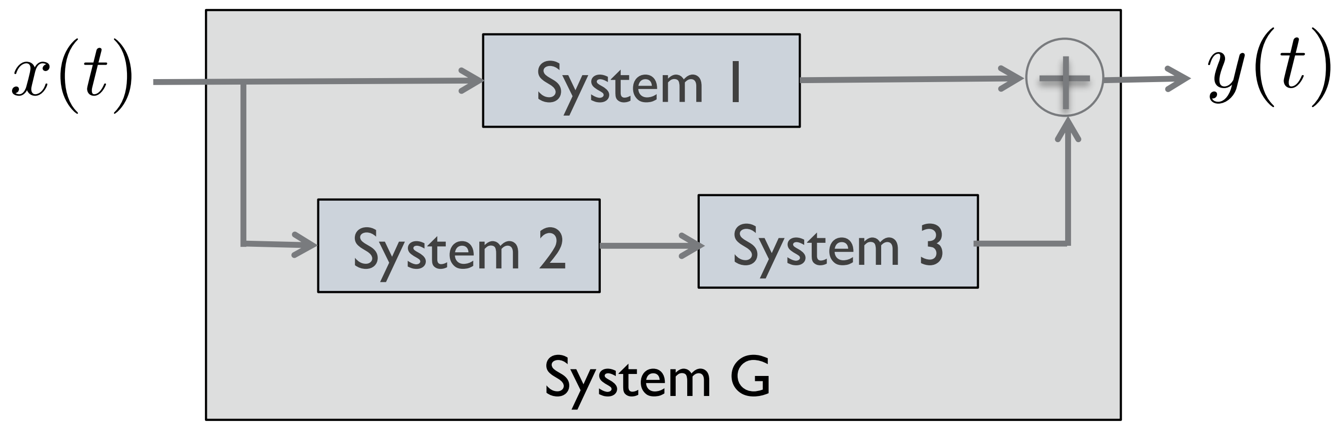
$$LC\frac{d^2 v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

Inter-connected systems – "systems-of-systems"

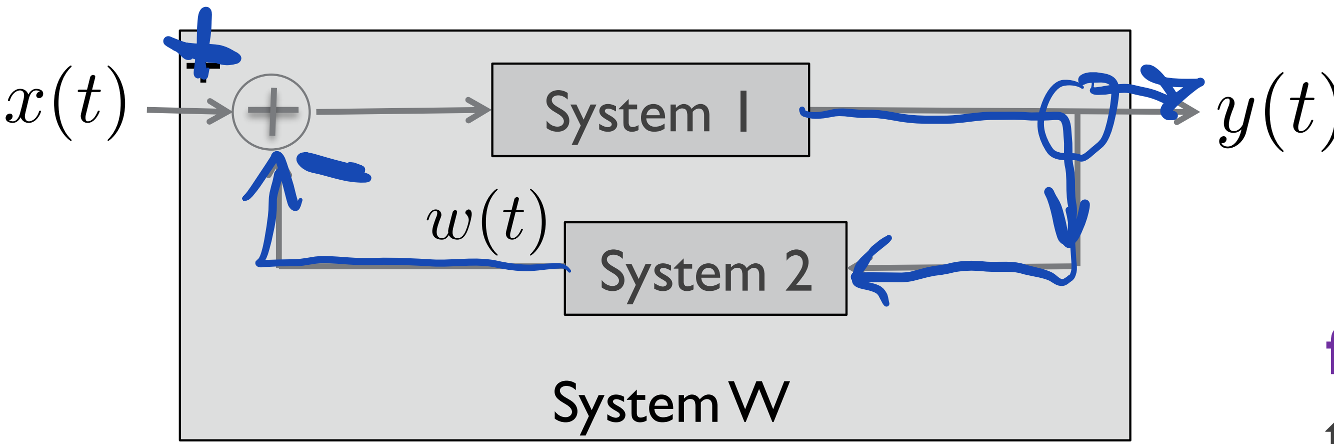
Serial



Parallel



Interconnected systems with feedback



feedback is when the output goes back into the input

Feedback is used in control systems

System introduction in summary

- A system is a functional unit that relates an input signal to an output signal
- Systems can be described in many ways including mathematically or via a block diagram
- Continuous-time systems have continuous inputs and outputs

Time invariance

Learning objectives

- Determine if a system is time invariant or time varying

Time invariant (TI)

A system is time invariant if behaves in the same way regardless of the current time

Formally: consider the system

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If

$$x(t - t_o) \longrightarrow \boxed{\text{System}} \longrightarrow y(t - t_o)$$

For all t_0 then the system is time invariant otherwise it is time varying.

How to check if the system is time-invariant?

- Method #1 (direct approach)
  - Consider the system

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

  - Apply the shifted input $x_2(t) = x_1(t - t_0)$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

  - Does the $\boxed{y_2(t) = y_1(t - t_0)?}$ If yes, the system is TI

$\forall t_0$

How to check if the system is time-invariant?

- Method #2 (counter example)

  If we suspect the system is time variant, find an example where time invariance fails "a counterexample"

  - This method is often quicker

- Note:
  - If you can not find a counter example, then you have to use Method 1
  - The counter example is just a simple way to **disprove** TI

- Hint: Usually, but not always, if the output includes any function of time other than $x(t)$, it is time-varying

Time invariance example I

System described by $y(t) = 3tx(t-3)$

$$y_1(t) = 3tx_1(t-3)$$ Output for generic input $x_1(t)$

$$y_2(t) = 3tx_2(t-3)$$ Output for generic input $x_2(t)$

Now, let: $x_2(t) = x_1(t-t_0)$

$$y_2(t) = 3tx_1(t-t_0-3)$$

$$y_1(t-t_0) = 3(t-t_0)x_1(t-t_0-3)$$

As $y_2(t)$ does not match $y_1(t-t_0) \longrightarrow$ **time-variant system**

Time invariance example 2

System described by $y(t) = x(t)x(t-1)$

$$y_1(t) = x_1(t)x_1(t-1)$$

$$y_2(t) = x_2(t)x_2(t-1)$$

$y_1(t-t_0)$
$= x_1(t-t_0)$
$\times x_1(t-t_0-1)$

Let $x_2(t) = x_1(t-t_0)$

$$y_2(t) \overset{?}{=} x_1(t-t_0)x_1(t-t_0-1)$$

$$= y_1(t-t_0)$$

The system is **time invariant (TI)**

Time invariance example 3

System with AM modulation $y(t) = x(t)\cos(\omega_c t)$

Counterexample, let $x_1(t) = \delta(t), x_2(t) = \delta(t - \frac{\pi}{2\omega_c})$

$$y_1(t) = \delta(t)$$

$$y_2(t) = \delta\left(t - \frac{\pi}{2\omega_c}\right)\cos(\omega_c t) = 0$$

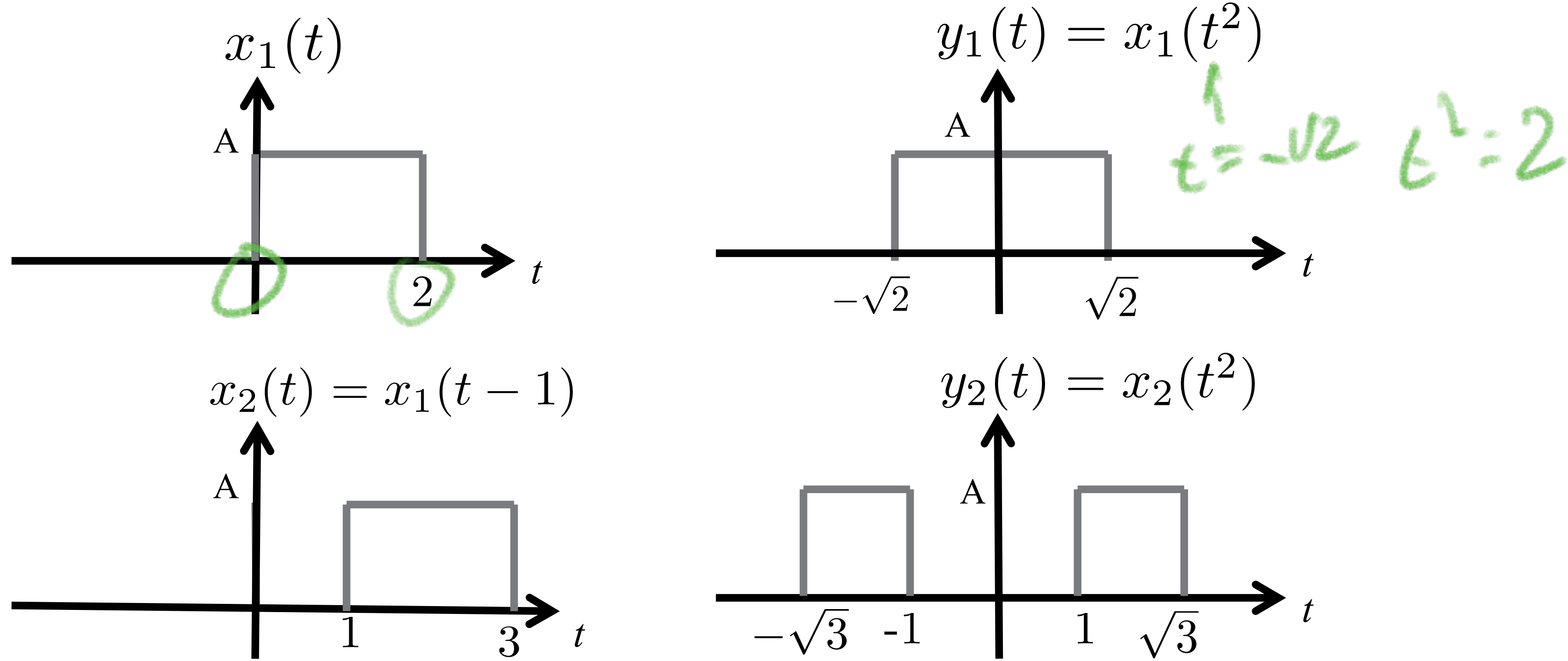
$$y_2(t) \neq y_1\left(t - \frac{\pi}{2\omega_c}\right)$$

$$= \delta\left(t - \frac{\pi}{2\omega_c}\right)$$

Time-variant

Time invariance example 4

$$y(t) = x(t^2)$$



Time-variant

(hint didn't work, Method 1 also tricky to see)

Time invariance summary

- A system is time invariant if behaves in the same way regardless of the current time
- Time invariant systems are much easier to design and analyze compared to time varying systems
- The main way to check time invariance is to shift the input and see if the output is always shifted
- A counter example is sufficient to show a system is not time invariant

Linearity

Learning objectives

- Determine if a system is linear or nonlinear
- Understand sub-properties of superposition and scaling

Scaling the input

If a system obeys the scaling property, then scaled inputs lead to scaled outputs

Consider the following system:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If the scaling property is satisfied then for any scalar value A

$$Ax(t) \longrightarrow \boxed{\text{System}} \longrightarrow Ay(t)$$

Superposition property

If a system obeys the superposition property, then system acts in the same way onto each system

If for two different inputs:

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

then superpositon holds if for input $x(t) = x_1(t) + x_2(t)$ then:

$$x_1(t) + x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t) + y_2(t)$$

Linear systems

A system is linear if scaling is preserved and superposition holds

Consider two different inputs and outputs

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

If the following holds for any scalar A and B then the system is linear

$$Ax_1(t) + Bx_2(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow Ay_1(t) + By_2(t)$$

How to check if the system linear?

- Direction approach
  - Check that scaling holds $Ax(t) \Rightarrow Ay(t)$
  - Check the superposition holds $x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)$
- Slightly faster direct approach
  - Check that $Ax_1(t) + Bx_2(t) \Rightarrow Ay_1(t) + By_2(t)$
- Find a counter example
  - One that may be useful (from the scaling property)

$$0 \longrightarrow \boxed{\text{System}} \longrightarrow 0$$

  - If a system generates a non-zero output to a signal that is zero for all time then it is non-linear

Linearity example I

$$y(t) = x(t)x(t-1)$$

Let us check if the scaling property holds $x_1(t) = Ax(t)$

$$\begin{aligned} y_1(t) &= x_1(t)Ax_1(t-1) \\ &= Ax(t)Ax(t-1) \\ &= A^2x(t)x(t-1) \\ &\neq Ay(t) \end{aligned}$$

Scaling fails → Nonlinear

Linearity example I – alternative solution

Consider inputs

$$y(t) = x(t)x(t-1)$$

$$x_1(t) \rightarrow y_1(t) = x_1(t)x_1(t-1)$$

$$x_2(t) \rightarrow y_2(t) = x_2(t)x_2(t-1)$$

$$x_3(t) = Ax_1(t) + Bx_2(t)$$

Note that

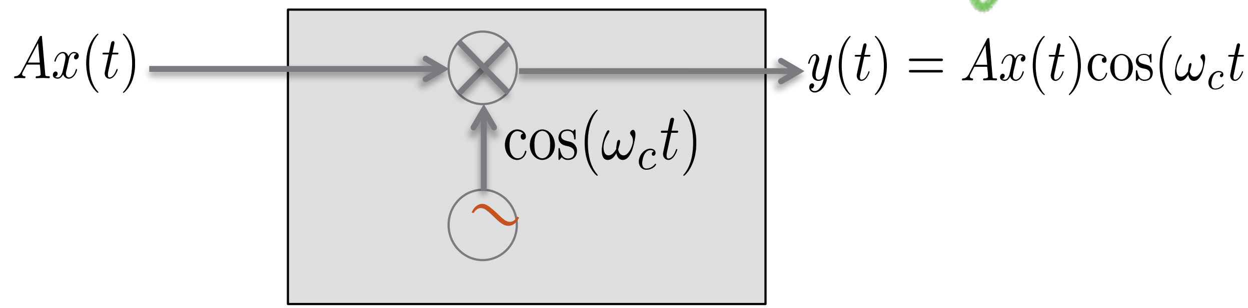
Nonlinear

$$\begin{aligned} y_3(t) &= x_3(t)x_3(t-1) \\ &= (Ax_1(t) + Bx_2(t))\,(Ax_1(t-1) + Bx_2(t-1)) \\ &= A^2x_1(t) + B^2x_2(t) + ABx_1(t)x_2(t-1) + ABx_1(t-1)x_2(t) \\ &\neq Ay_1(t) + By_t(t) = Ax_1(t)x_1(t-1) + Bx_2(t)x_2(t-1) \end{aligned}$$

Linearity example 2

System with AM modulation

Scaling is preserved



What about superposition?

$$y_1(t) = x_1(t)\cos\omega_c t$$

$$y_2(t) = x_2(t)\cos\omega_c t$$

$$x(t) = x_1(t) + x_2(t) \longrightarrow y(t) = (x_1(t) + x_2(t))\cos\omega_c t$$

$$= x_1(t)\cos\omega_c t + x_2(t)\cos\omega_c t$$

$$= y_1(t) + y_2(t)$$

Linear system

Linearity example 3

- Consider the affine system

$$y[n] = 2x[n] + 1$$

- Suppose that

$$x_1[n] \rightarrow y_1[n]$$

$$x_2[n] \rightarrow y_2[n] \text{ where } x_2[n] = Ax_1[n]$$

- Now observe that

$$\begin{aligned} y_2[n] &= 2x_2[n] + 1 \\ &= 2Ax_1[n] + 1 \\ &\neq Ay_1[n] \end{aligned}$$

Scaling does not hold therefore is a **nonlinear system**

Linearity summary

- A system linear if *scaled inputs* lead to *scaled outputs* and the *sum of inputs* leads to a *sum of outputs* if the inputs were applied separately
- Linear systems are easier to design and analyze
- Many systems in practice are nonlinear but are designed to be as linear as possible, or are only used with inputs where they behave in a linear fashion
- Need to check both the scaling and superposition properties to prove that a system is linear

Reference example with details

- Is the following system linear? Time-invariant?

$$y(t) = t^2 x(t-1)$$

Reference example – checking time invariance

Consider the output to input $x_1(t)$

$$y_1(t) = t^2 x_1(t-1)$$

Define a new input

$$x_2(t) = x_1(t - t_0)$$

Compute the output

$$\begin{aligned} y_2(t) &= t^2 x_2(t-1) \\ &= t^2 x_1(t - 1 - t_0) \end{aligned}$$

Not time invariant!

Compare with a shifted version of the first output

$$y_1(t - t_0) = (t - t_0)^2 x_1(t - 1 - t_0) \neq y_2(t)$$

Reference example – checking linearity

Consider the inputs and outputs

$$x_1(t) \to y_1(t) = t^2 x_1(t-1)$$

$$x_2(t) \to y_2(t) = t^2 x_2(t-1)$$

Define a new input

$$x_3(t) = ax_1(t) + bx_2(t)$$

Compute the output

Linear!

$$\begin{aligned} y_3(t) &= t^2 x_3(t-1) \\ &= t^2(ax_1(t-1) + bx_2(t-1)) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Lecture 5

Impulse response and convolution

Preview of today's lecture

- Impulse response of an LTI system
  - Define the impulse response of a continuous-time system
  - Connect the impulse response to the convolution integral

- Convolution
  - Determine the output of an LTI system using the convolution
  - Compute the continuous-time convolution between two signals

Time invariant (TI)

A system is time invariant if behaves in the same way regardless of the current time

Formally: consider the system

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If

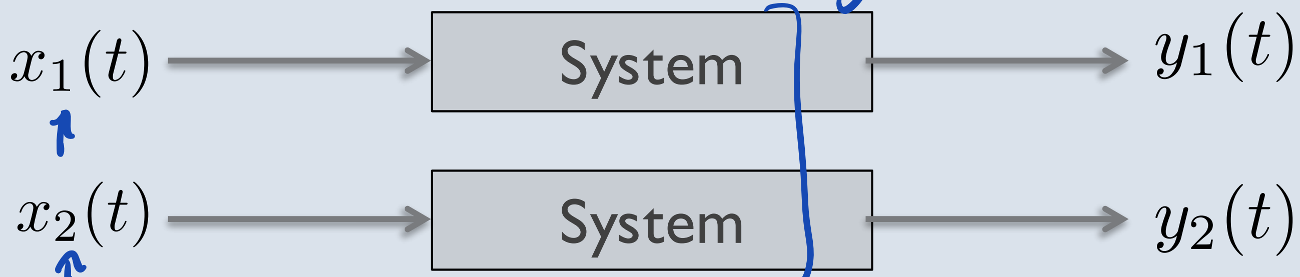
$$x(t - t_o) \longrightarrow \boxed{\text{System}} \longrightarrow y(t - t_o)$$

For all t_0 then the system is time invariant otherwise it is time varying.

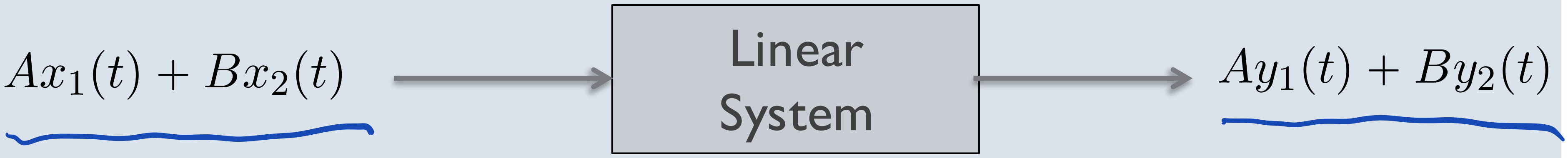
Linear systems

A system is linear if scaling is preserved and superposition holds

Consider two different inputs and outputs same

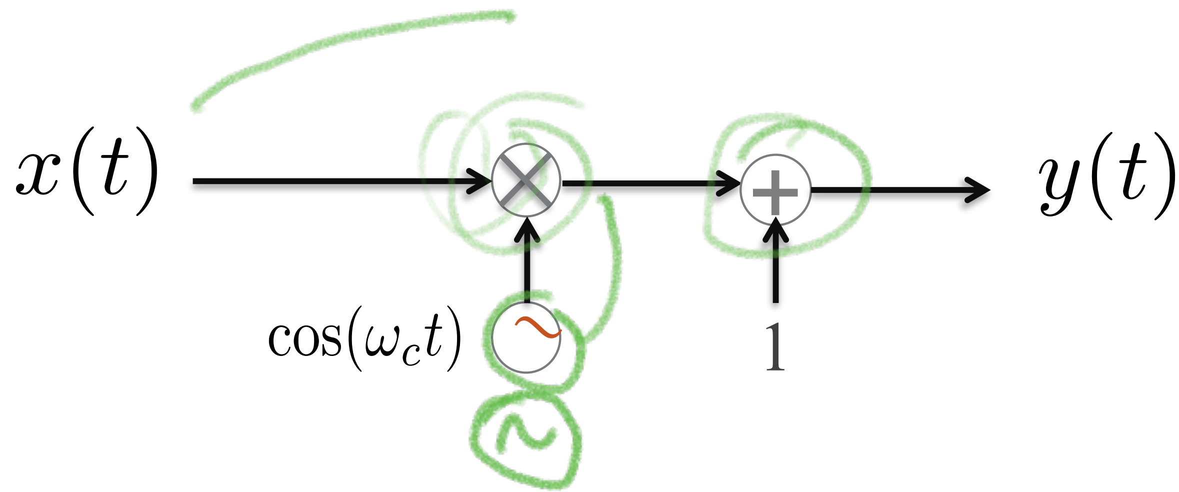


If the following holds for any scalar A and B then the system is linear



Example

- Determine if the following systems are time-invariant and/or linear
- System 1



- System 2

$$y(t) = \int_0^1 x(t-q)dq$$

Linearity

$$y(t) = \left(1 + \cos(\omega_c t)\right) x(t)$$

Check scaling

$$A x(t) \rightarrow A y(t)$$

$$x_1(t) = A x(t) \rightarrow y_1(t) = \left(1 + \cos(\omega_c t)\right) x_1(t)$$

$$= \left(1 + \cos(\omega_c t)\right) A x(t)$$

$$= \left(1 + A\cos(\omega_c t)\right) x(t)$$

$$A y(t) = \left(A + A\cos(\omega_c t)\right) x(t)$$

$$\neq \left(1 + A\cos(\omega_c t)\right) x(t)$$

Scaling does not hold

Counterexample

Let $x(t) = 0 \quad \forall t$

Then by scaling property

$y(t) = 0 \quad \forall t$ But

in this case

$$y(t) = 1 + \cos(\omega_c t)\, x(t)$$

$$= 1 + \cos(\omega_c t) \cdot 0$$

$$= \underline{1} \neq 0$$

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

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	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, Dirac Delta	1.3 - 1.4		
4	1/16	LTI in time	Systems, linearity, time invariance	1.6.5, 1.6.6	HW3	HW2
5	1/21	LTI in time	Impulse response and convolution	2.2		
6	1/23	LTI in time	Convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Impulse response of an LTI system

Learning objectives

- Define the impulse response of a continuous-time system
- Connect the impulse response to the convolution integral

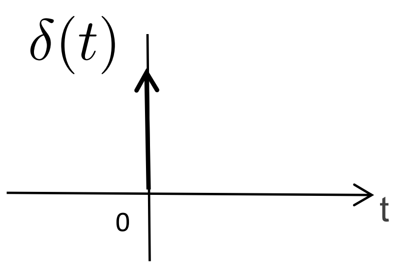
System impulse response

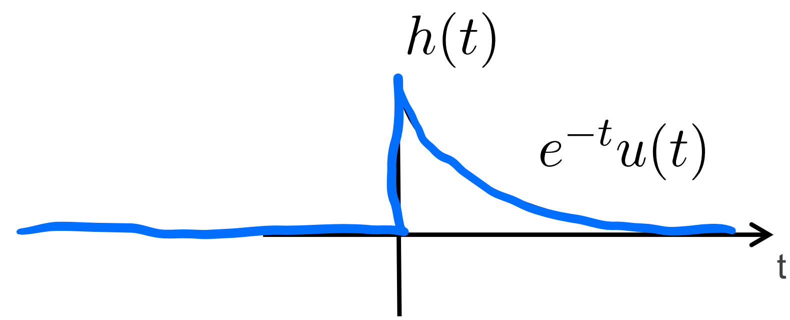
$$x(t) = \delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t)$$

- Consider the input signal $\delta(t)$
- The output corresponding to this input is the **impulse response**
  - The resulting sequence is usually called $h(t)$
- All systems have an impulse response, but:
  - The impulse response is special <u>only</u> for LTI systems
  - Focus on LTI systems throughout this course

Example (typical "first-order" differential system)

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t) = e^{-t}u(t)$$





delta function input

Exponential function out

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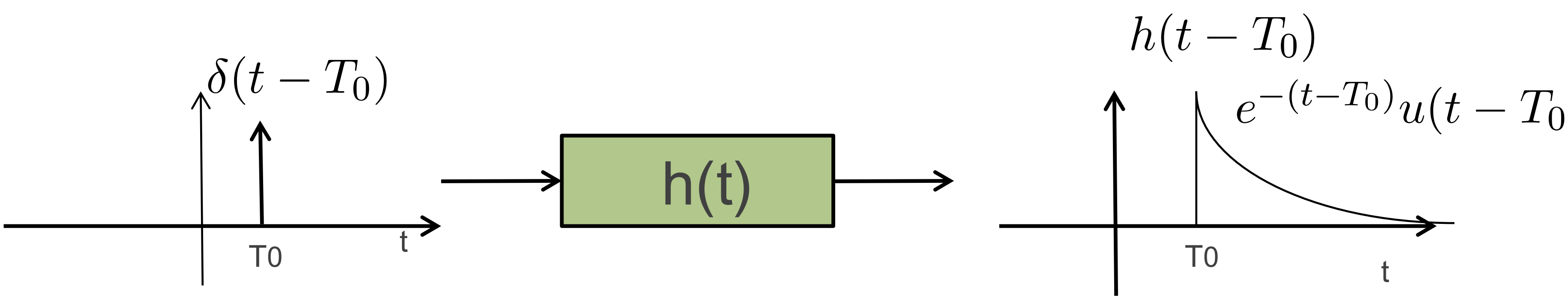
Time invariance

- Because of time invariance

$$\delta(t-\underbrace{T_0}_{\text{fixed}}) \longrightarrow \boxed{h(t)} \longrightarrow h(t-T_0)$$

When written like this, implies an LTI system with impulse response h(t).

- Example



Shifts in the input shift the output

Linearity

- Because of the homogenous property

$$\alpha\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow \alpha h(t)$$

Scaling the input scales the output

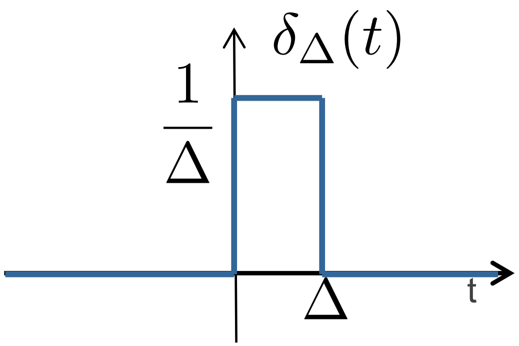
- Because of the additive property

$$\alpha\delta(t) + \beta\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow \overbrace{\alpha h(t)}^{\alpha} + \overbrace{\beta h(t - T_0)}^{\beta}$$

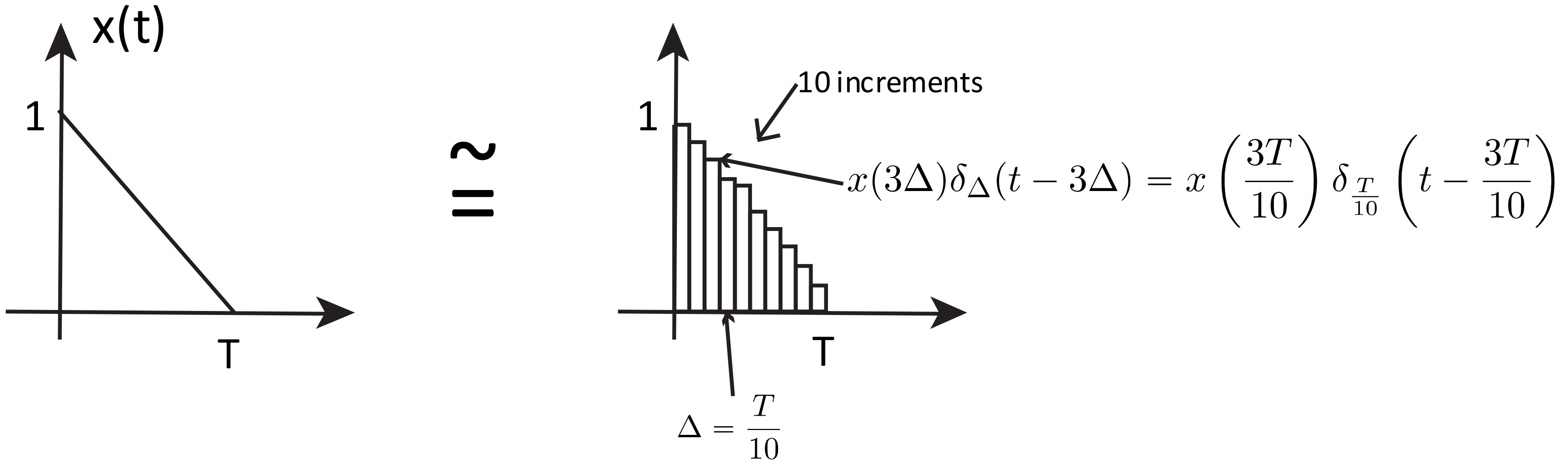
A sum of inputs leads to a sum of outputs

Stair step approximation of an input signal

- Consider the rectangle function



- Suppose that we approximate a signal using this function

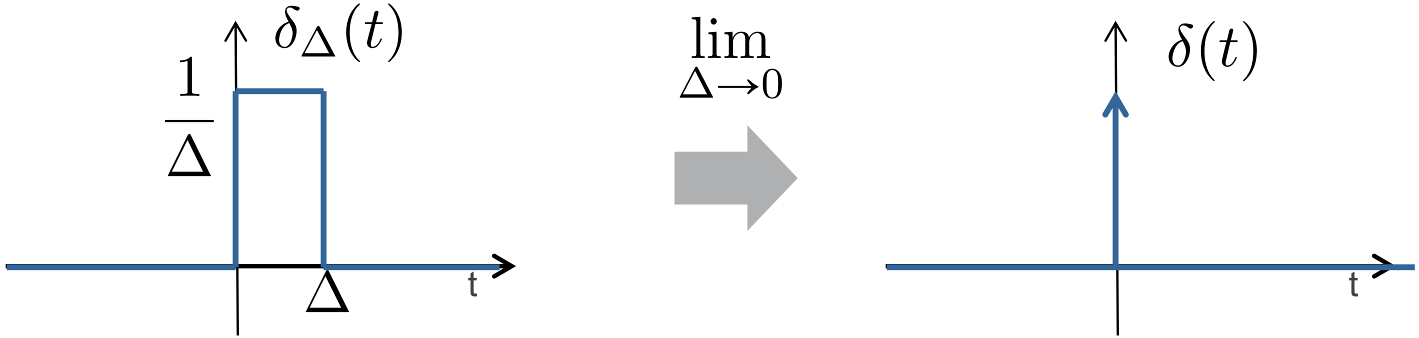


Stair step approximation of an input signal

- Write the stair case approximation of a function as

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

- Recall from the derivation of the delta function



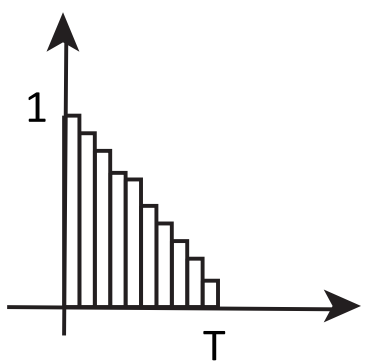
Stair step approximation of an input signal

- Now taking the limit

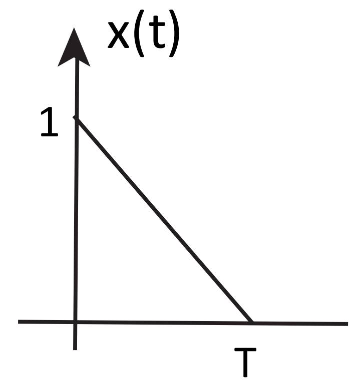
$$\lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$= x(t)$$

Easy to see via sifting property that this must be true



$$\lim_{\Delta \to 0}$$



Any signal can be written as an integral of itself with shifted deltas

Back to the LTI system

- What if we put in

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{h(t)} \longrightarrow x(\tau)h(t-\tau)$$

LTI

- How about

$$x(\tau_1)\delta(t-\tau_1) + x(\tau_2)\delta(t-\tau_2) \longrightarrow \boxed{h(t)} \longrightarrow$$

$$x(\tau_1)h(t-\tau_1) + x(\tau_2)h(t-\tau_2)$$

Uncovering the convolution

- Now putting x(t) in the integral format

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{h(t)} \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

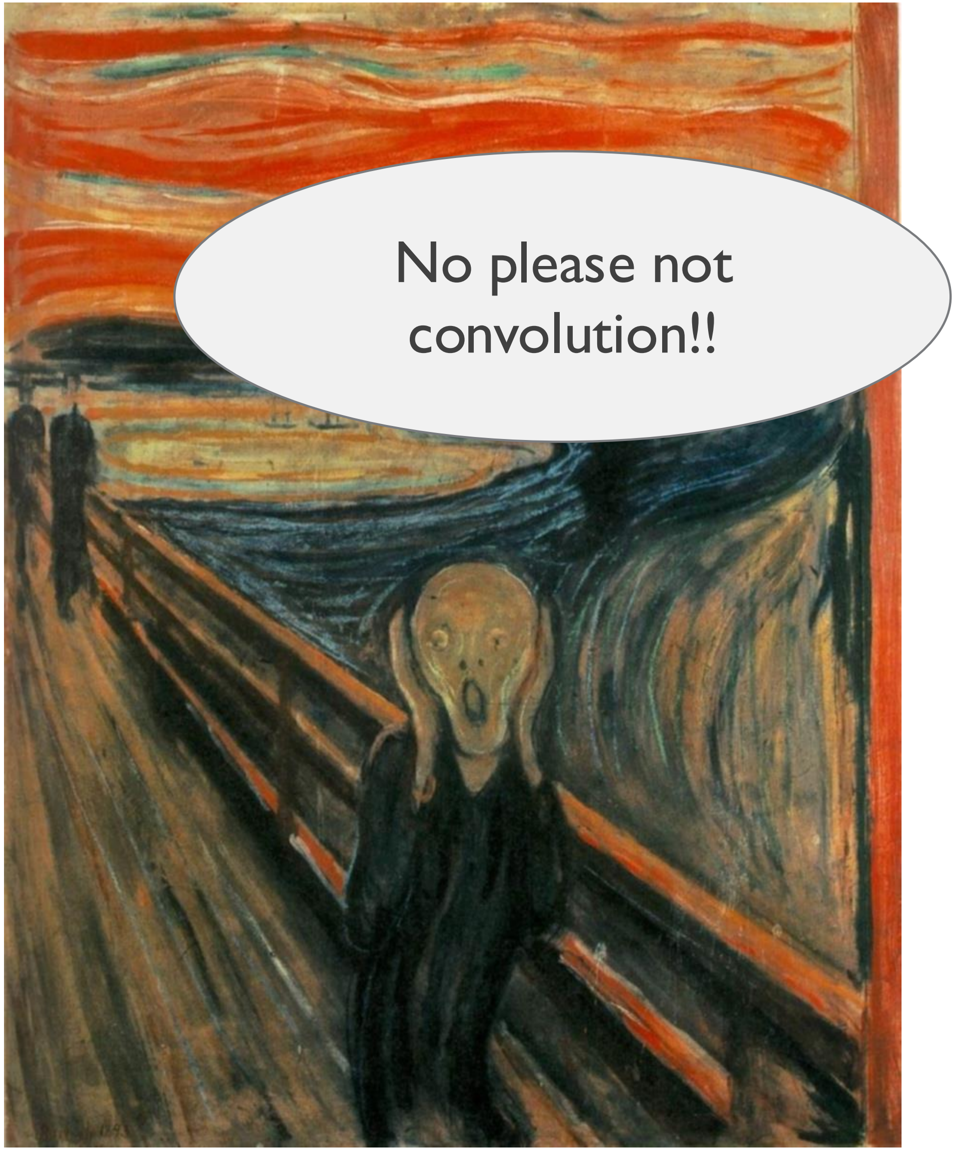
- Thus, the input and output of an LTI system are related via the **convolution integral:**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution

Learning objectives

- Determine the output of an LTI system using the convolution
- Compute the continuous-time convolution between two signals



https://commons.wikimedia.org/wiki/File:The_Scream.jpg with additions by Prof. Heath

Uncovering the convolution

- Consider an LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- The output can be computed from the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Basic convolution properties

- Commutative

Shorthand notation

$$y(t) = x(t) * h(t) = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau$$

Choose option that makes it easy!

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

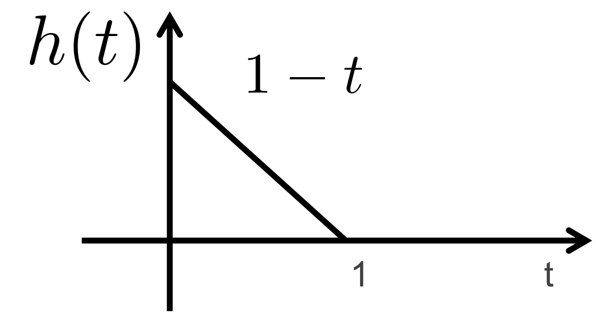
Use properties to simplify convolutions (more next lecture)

CT convolution example #1

This is a typical example using two finite length signals. Rectangles and triangle functions are common in examples / HW as they give results that are easy to integrate. It is important here to understand the different **intervals** in the convolutions.

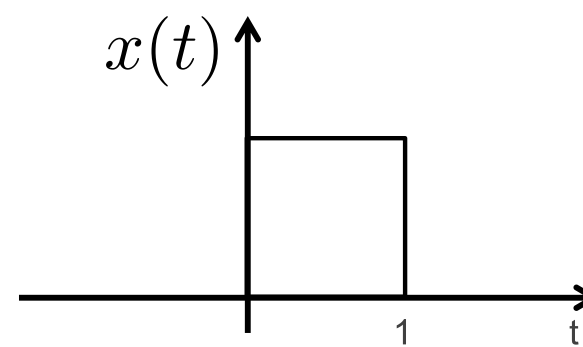
- Find the output of a system with impulse response

$$h(t) = (1-t)[u(t) - u(t-1)]$$



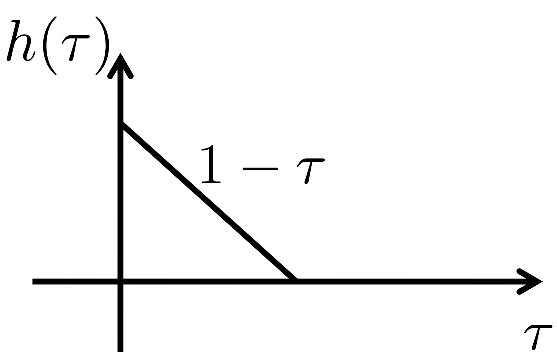
for the input

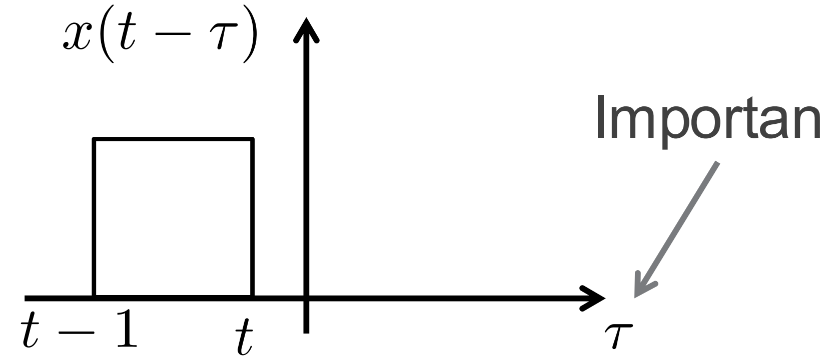
$$x(t) = u(t) - u(t-1)$$



CT convolution example #1: Graphical solution

- Plot one signal versus τ
- Flip the second signal and shift it by *t*
  - Here, plot it to the left of h(τ)
  - So plotted *t* has a negative value, usually



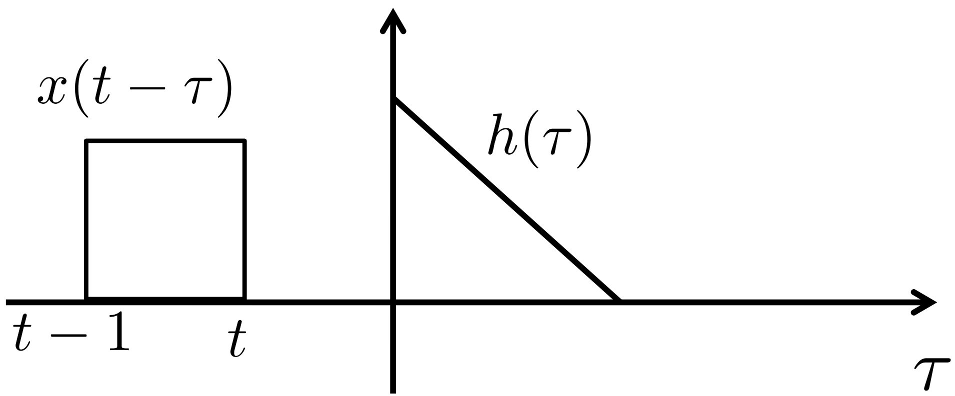


- There are 4 intervals (why?):

$t < 0$ $\quad 0 \leqslant t \leqslant 1$ $\quad 1 \leqslant t \leqslant 2$ $\quad 2 < t$

CT convolution example #1: First interval

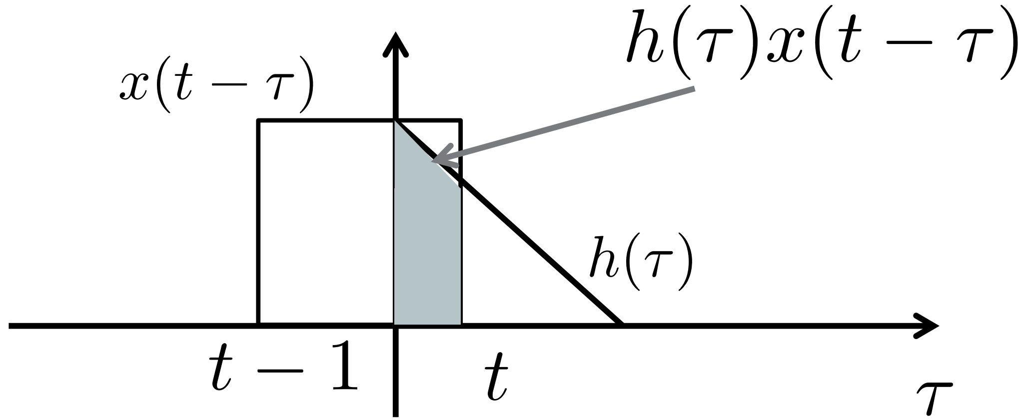
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



- First interval: there is no overlap!

$t < 0$ $\qquad h(\tau)x(t-\tau) = 0 \qquad \Rightarrow \qquad y(t) = 0$

CT convolution example #1: Second interval



- Second interval $0 \leqslant t \leqslant 1$

$$y(t) = \int_0^t (1-\tau)d\tau = \tau - \frac{\tau^2}{2}\Big]_0^t$$

$$= t - \frac{t^2}{2}$$

CT convolution example #1: Third interval

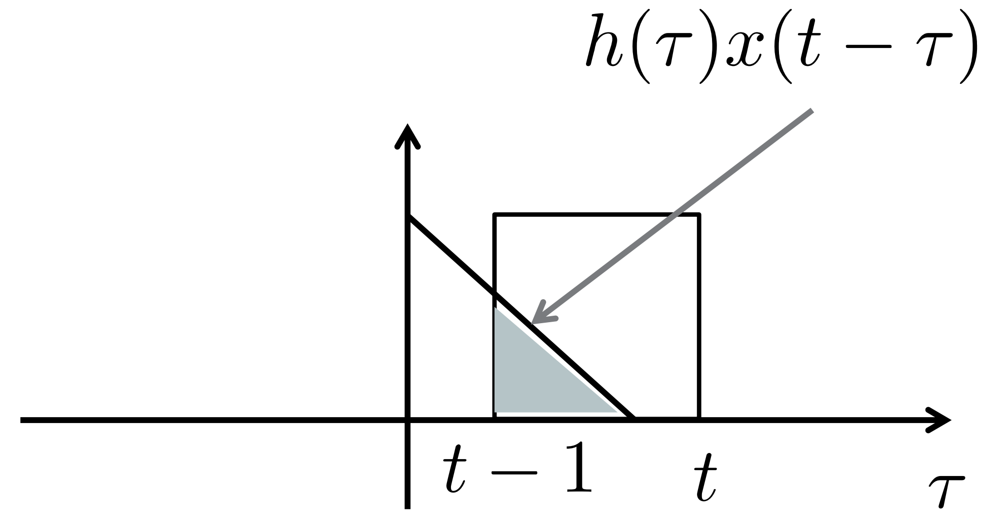
- Third interval $1 \leqslant t \leqslant 2$

$$y(t) = \int_{t-1}^{1} (1-\tau)d\tau$$

$$= \tau - \frac{\tau^2}{2}\Big]_{t-1}^{1}$$

$$= 1 - \frac{1}{2} - \left(t - 1 - \frac{(t-1)^2}{2}\right)$$

$$= \frac{t^2}{2} - 2t + 2$$

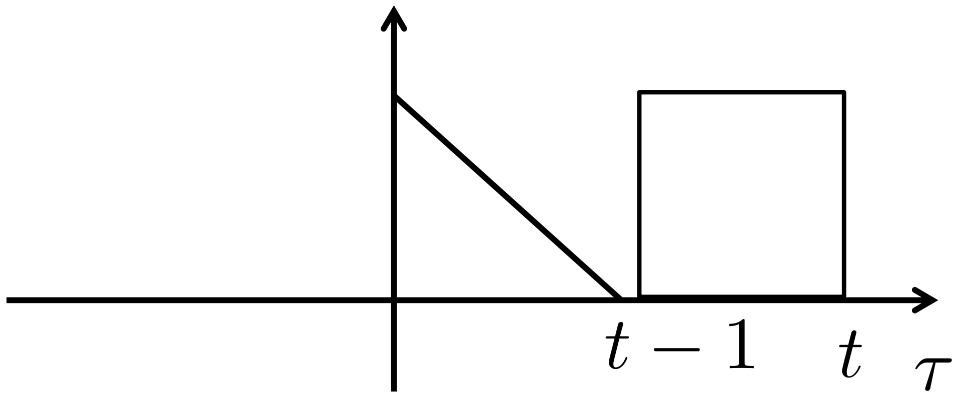


Integrating the same function, but with different integration limits

CT convolution example #1: Fourth and final interval

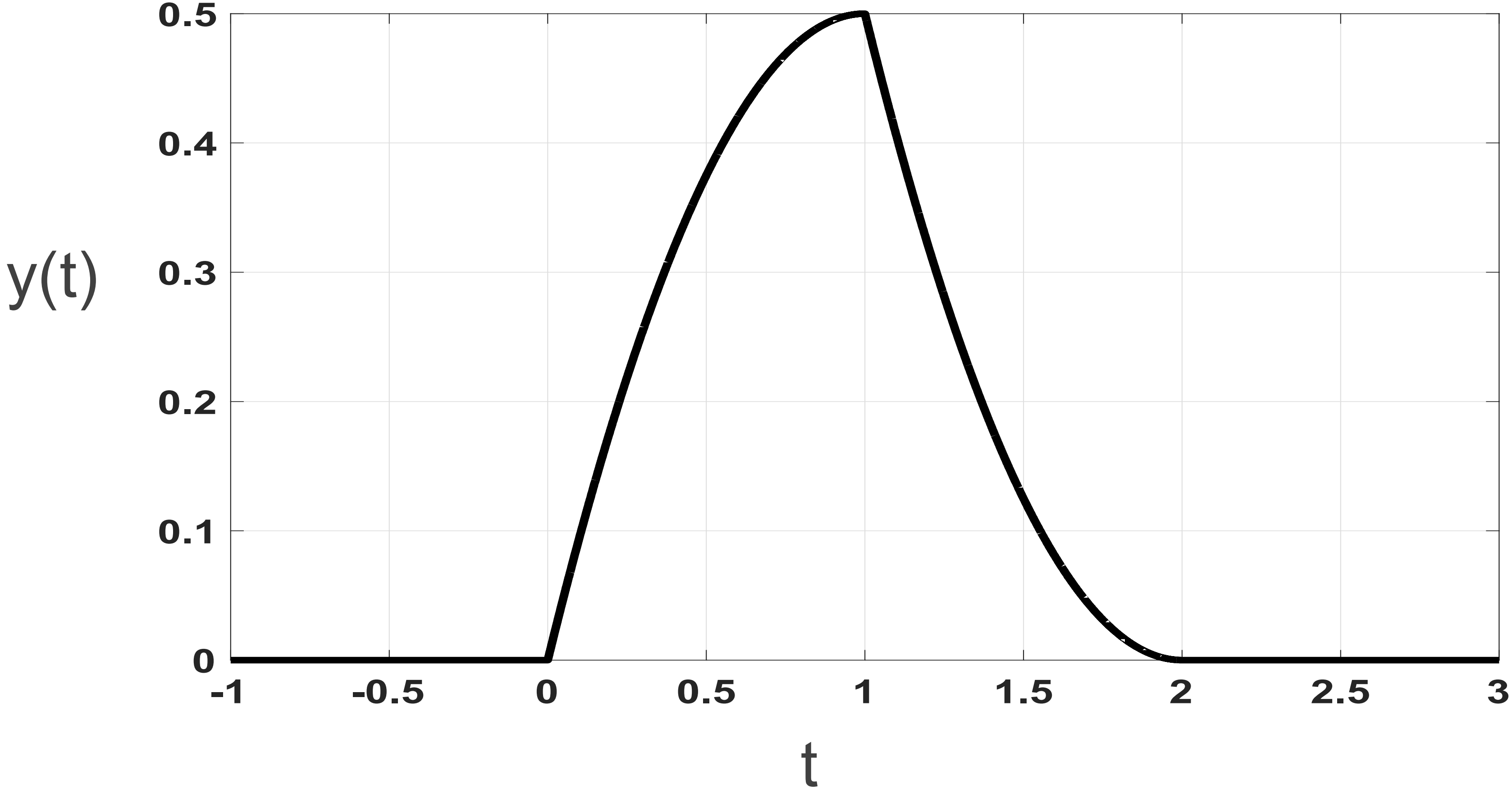
- Fourth interval $2 < t$

$$y(t) = 0$$

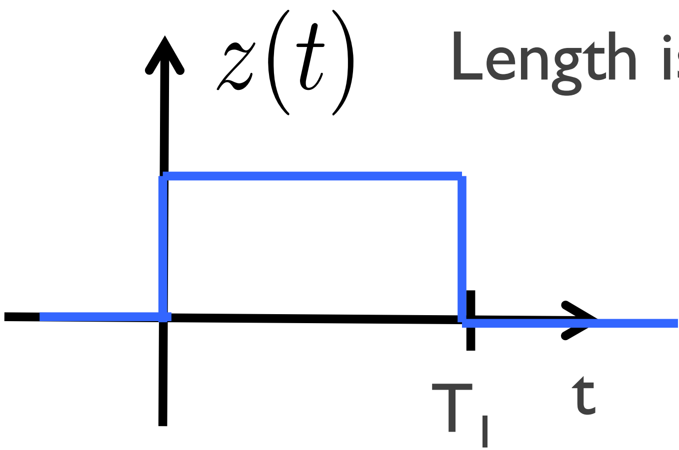


- Sanity checks
  - Check at $t=1$, the output should be the same for intervals 2 & 3 in order for it to be continuous
  - Same for t = 0 and t = 2 (should be zero there)
  - Duration of output should be $T_1 + T_2 = 1+1 = 2$

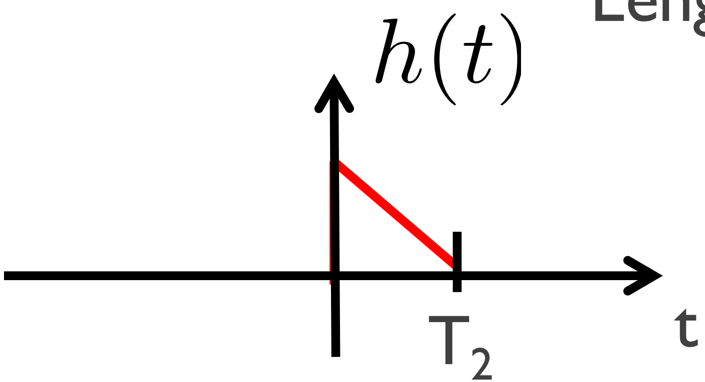
CT convolution example #1: Plot of y(t)



Length / duration of a convolution in continuous time

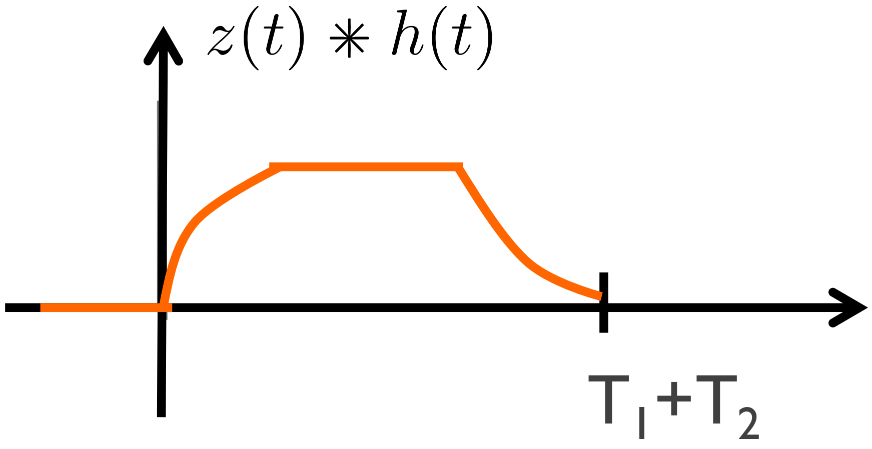


Length is T_1

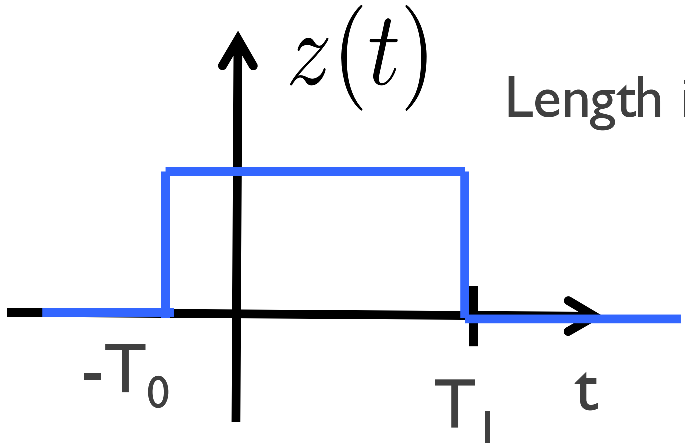


Length is T_2

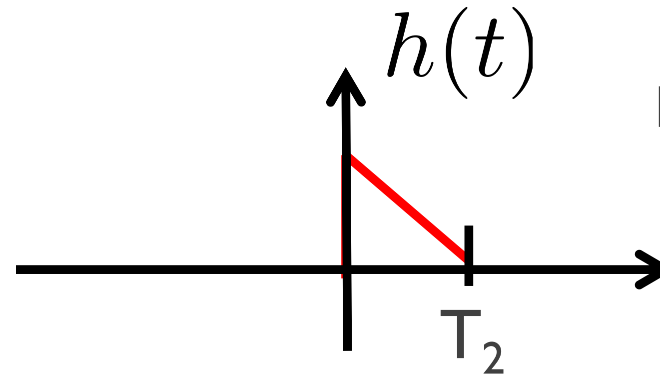
Length is T_1+T_2



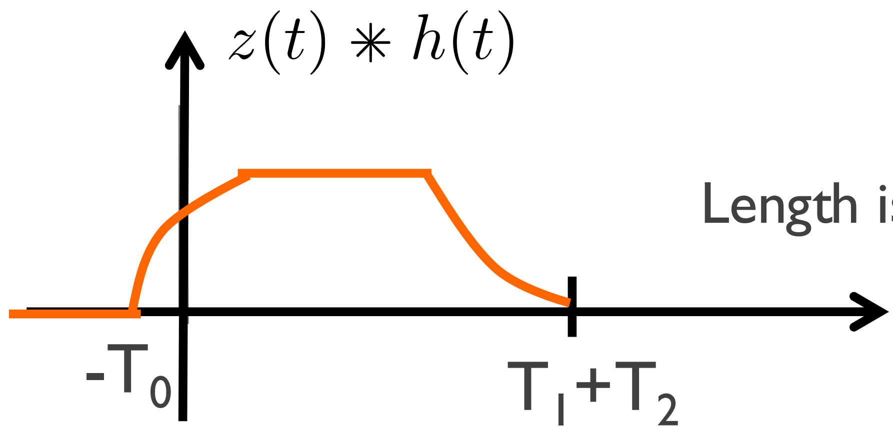
Length / duration of a convolution in continuous time



Length is $T_{0+}T_1$



Length is T_2



Length is $T_0 + T_1+T_2$

Animation example

http://www.cse.yorku.ca/~asif/spc/ConvolutionIntegral_Final3.swf

CT convolution example #2

This is an example with a simple input that consists of a few delta functions. The convolution is easy to compute using the LTI property of convolution and the definition of impulse response.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Determine and sketch the convolution of the following input

$$x(t) = \delta(t) + 2\delta(t-1)$$

and system with impulse response

$$h(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1, \\ 2-t, & 1 < t \leqslant 2, \\ 0, & \text{elsewere} \end{cases}$$

CT convolution example #2: Solution approach

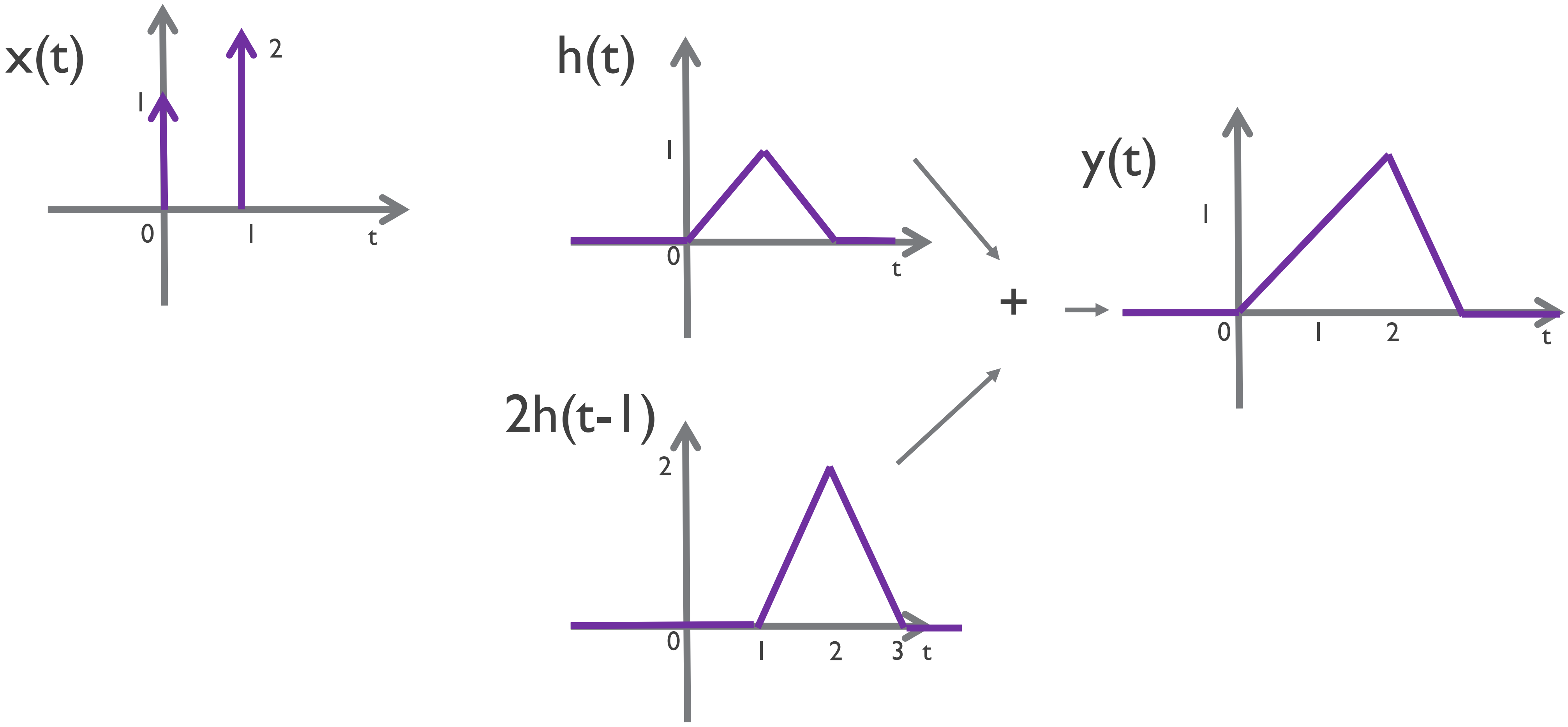
- Recall the property

$$\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t - T_0)$$

- Therefore

$$x(t) = \delta(t) + 2\delta(t-1) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) + 2h(t-1)$$

CT convolution example #2: Solution sketch

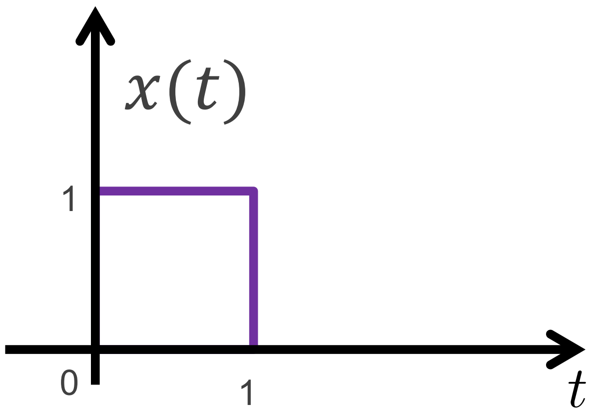


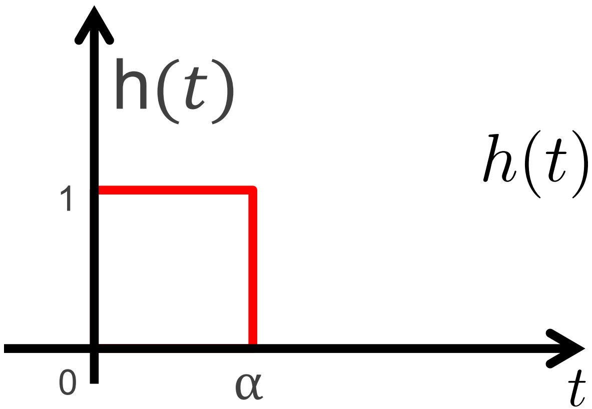
CT convolution example #3

This is an important example involving the convolution of two rectangles of different widths. The solution will be different depending on the widths of the rectangles. Notice what happens in each interval.

- Determine and sketch $y(t) = x(t) * h(t)$, where

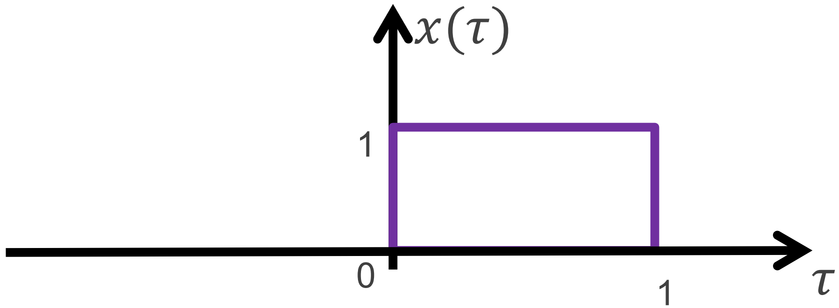
$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

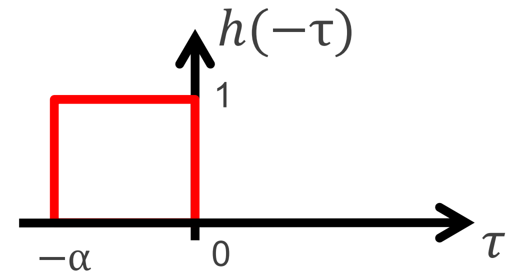




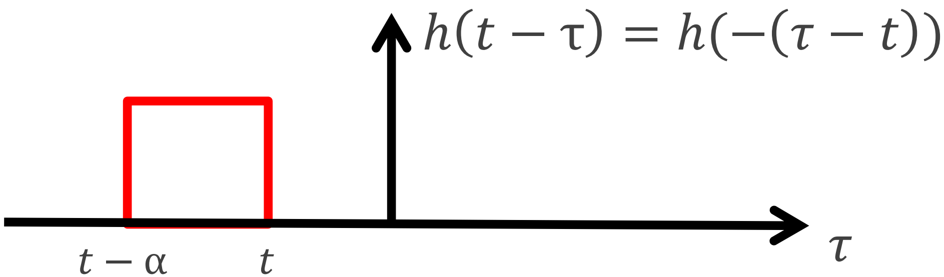
$$h(t) = x(t/\alpha) \qquad 0 < \alpha \leq 1$$

CT convolution example #3: Solution



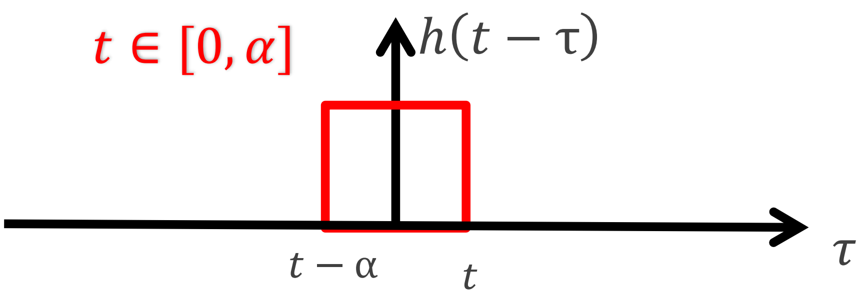


$t < 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0 \quad t < 0$$

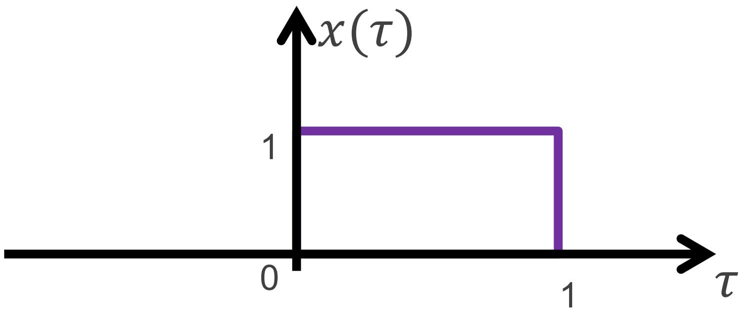
$t \in [0, \alpha]$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} 1 d\tau \quad t \in [0, \alpha]$$

$$= t \quad t \in [0, \alpha]$$

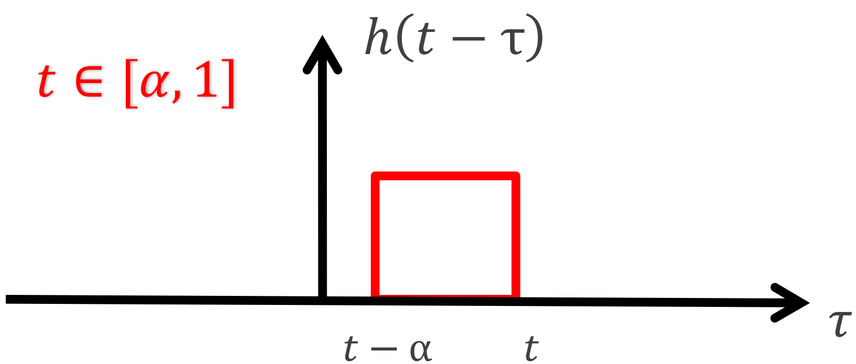
CT convolution example #3: Solution



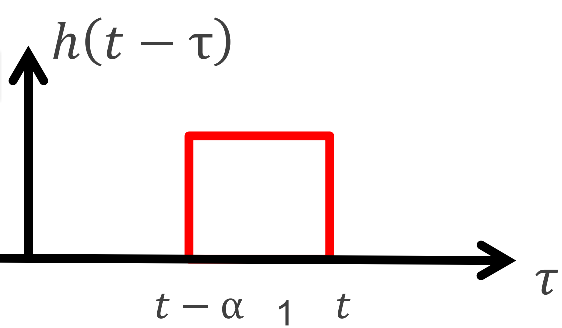
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{t} 1 d\tau \quad t \in [\alpha, 1]$$

$$= \alpha \quad t \in [\alpha, 1]$$

$t \in [\alpha, 1]$



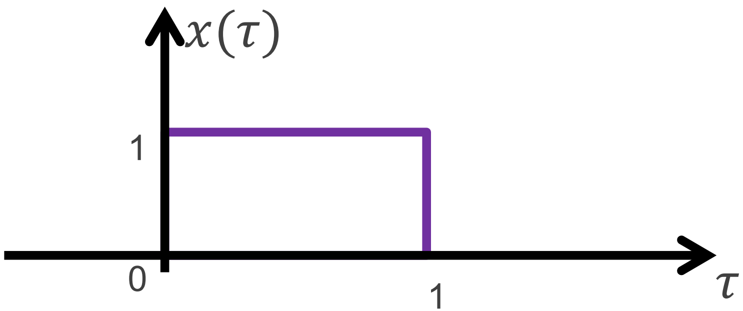
$t \in [1, 1+\alpha]$



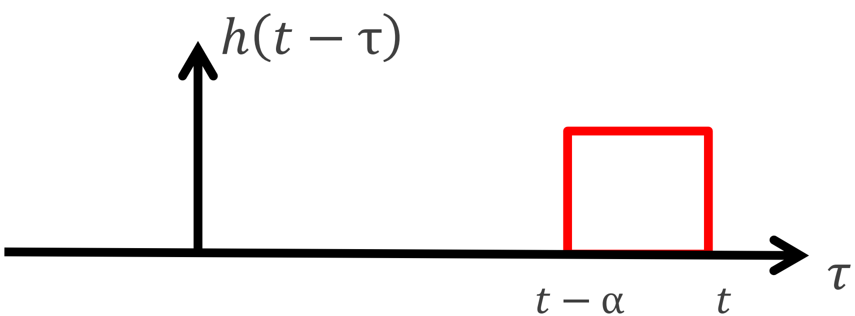
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{1} 1 d\tau \quad t \in [1, 1+\alpha]$$

$$= 1 - (t-\alpha) \quad t \in [1, 1+\alpha]$$

CT convolution example #3: Solution



$t \in [1+\alpha, \infty]$

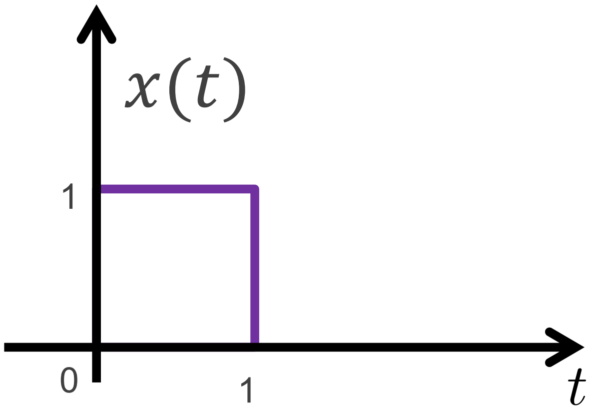


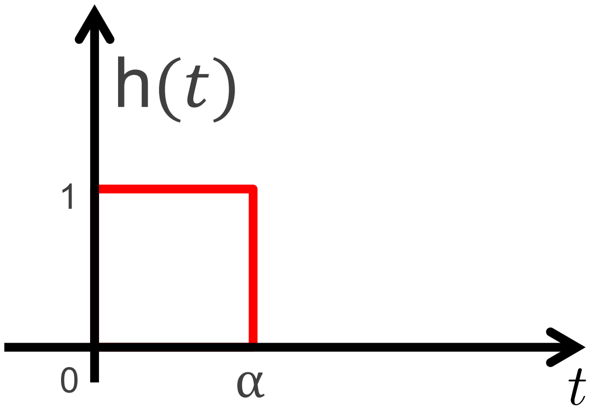
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{1}^{t} 0 d\tau \quad t \in [\alpha+1, \infty]$$

$$= 0 \quad t \in [\alpha+1, \infty]$$

CT convolution example #3: Solution (summary)

$$y(t) = x(t) * h(t)$$





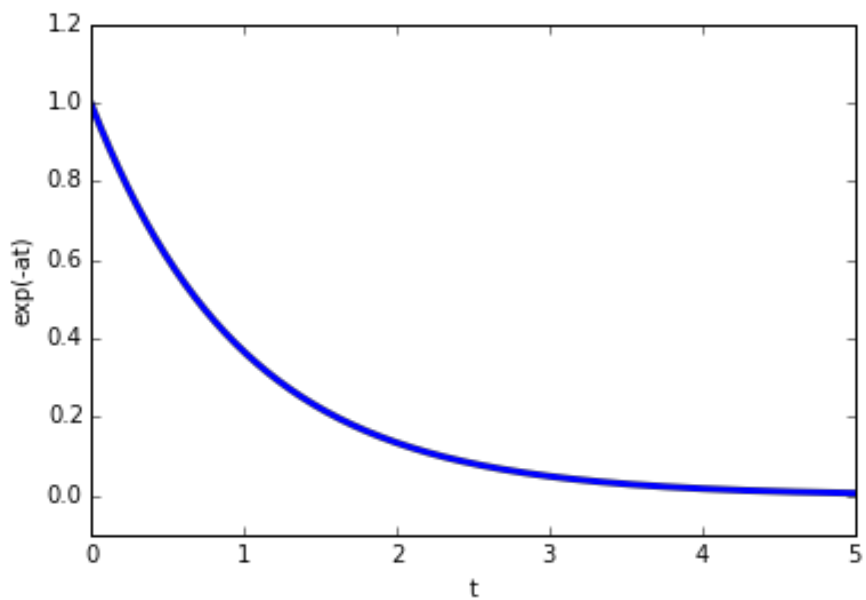
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} t & < & 0 & 0 \\ t & \in & [0,\alpha] & t \\ t & \in & [\alpha,1] & \alpha \\ t & \in & [1,1+\alpha] & 1-(t-\alpha) \\ t & > & 1+\alpha & 0 \end{cases}$$

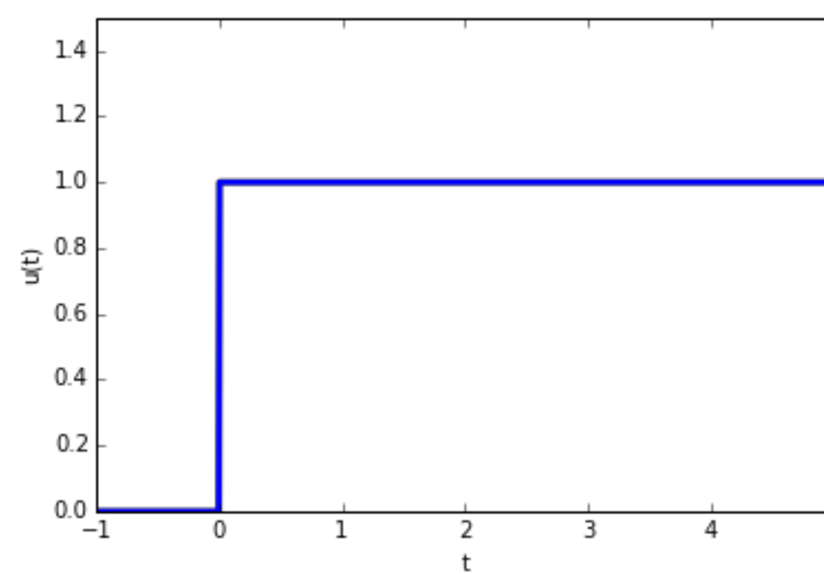
CT convolution example #4

This is an important example of convolution to causal signals together. It will be solved by using the graphical approach to help set up the integrals.

- Determine and sketch $y(t) = x(t) * h(t)$, where

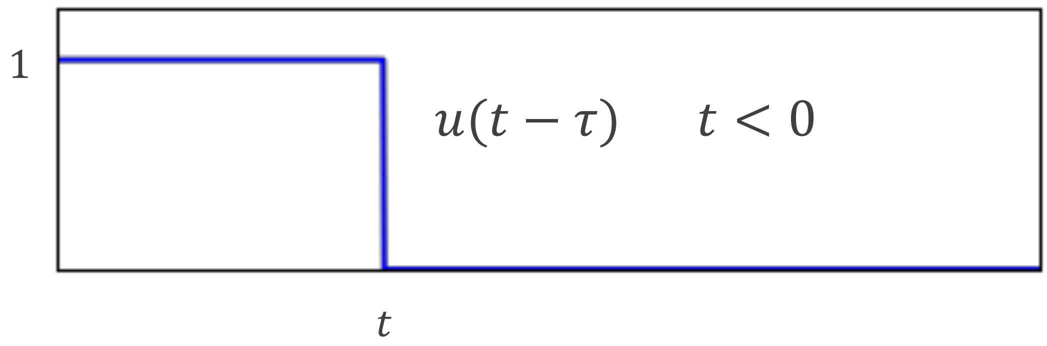
$$x(t) = e^{-at}u(t), \qquad a > 0 \qquad\qquad h(t) = u(t)$$

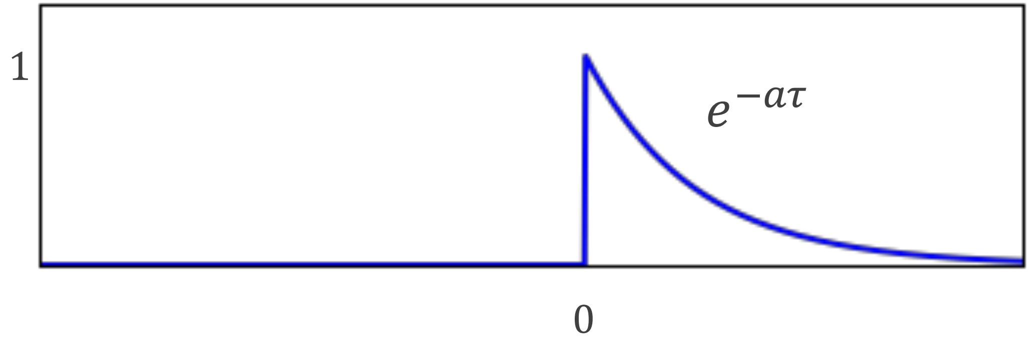




CT convolution example #4: Solution

$$y(t) = x(t) * h(t)$$

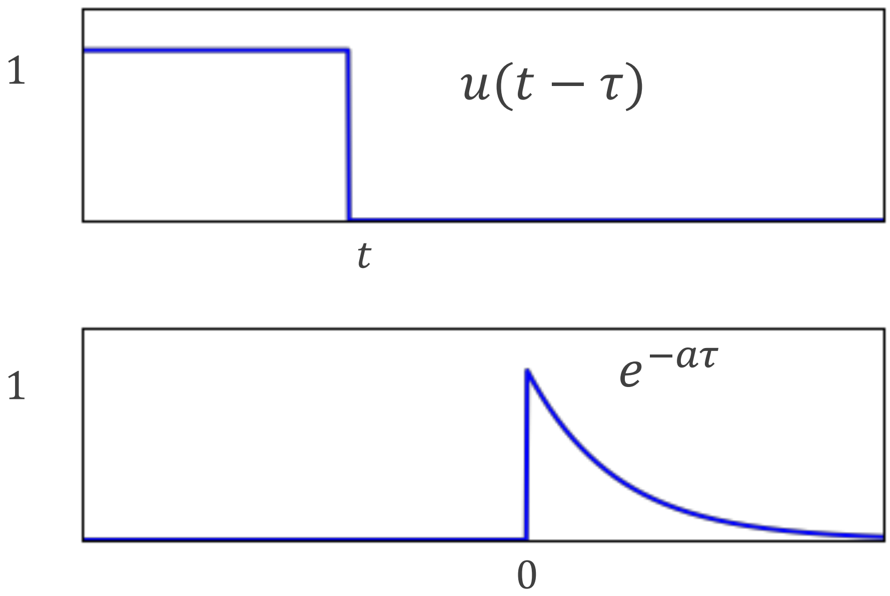




$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-a\tau}u(t-\tau)d\tau \end{aligned}$$

CT convolution example #4: Solution

For $t < 0$

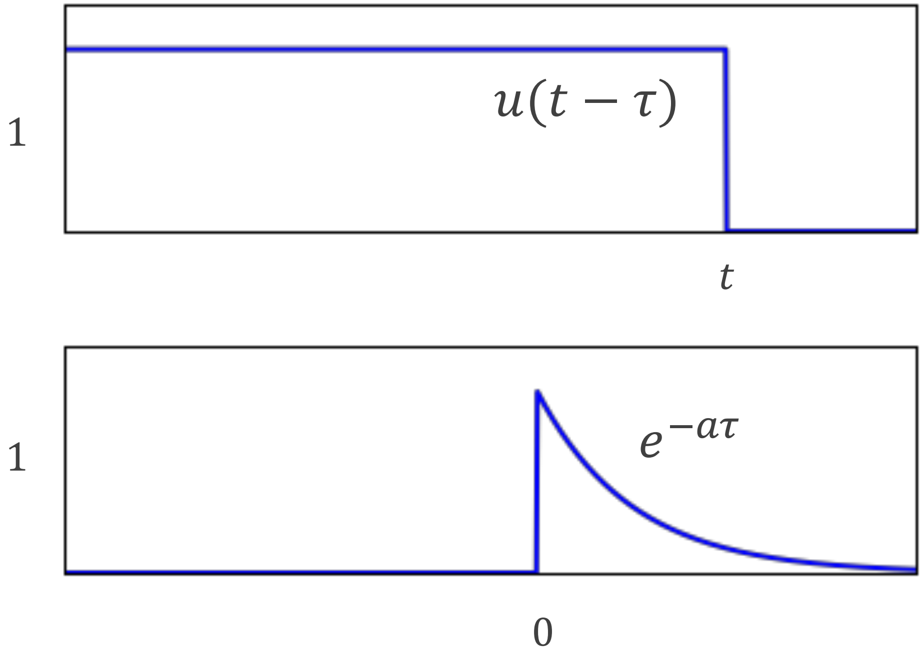


$$x(t) * h(t) = \int_{-\infty}^{\infty} 0 d\tau$$

$$= 0$$

CT convolution example #4: Solution

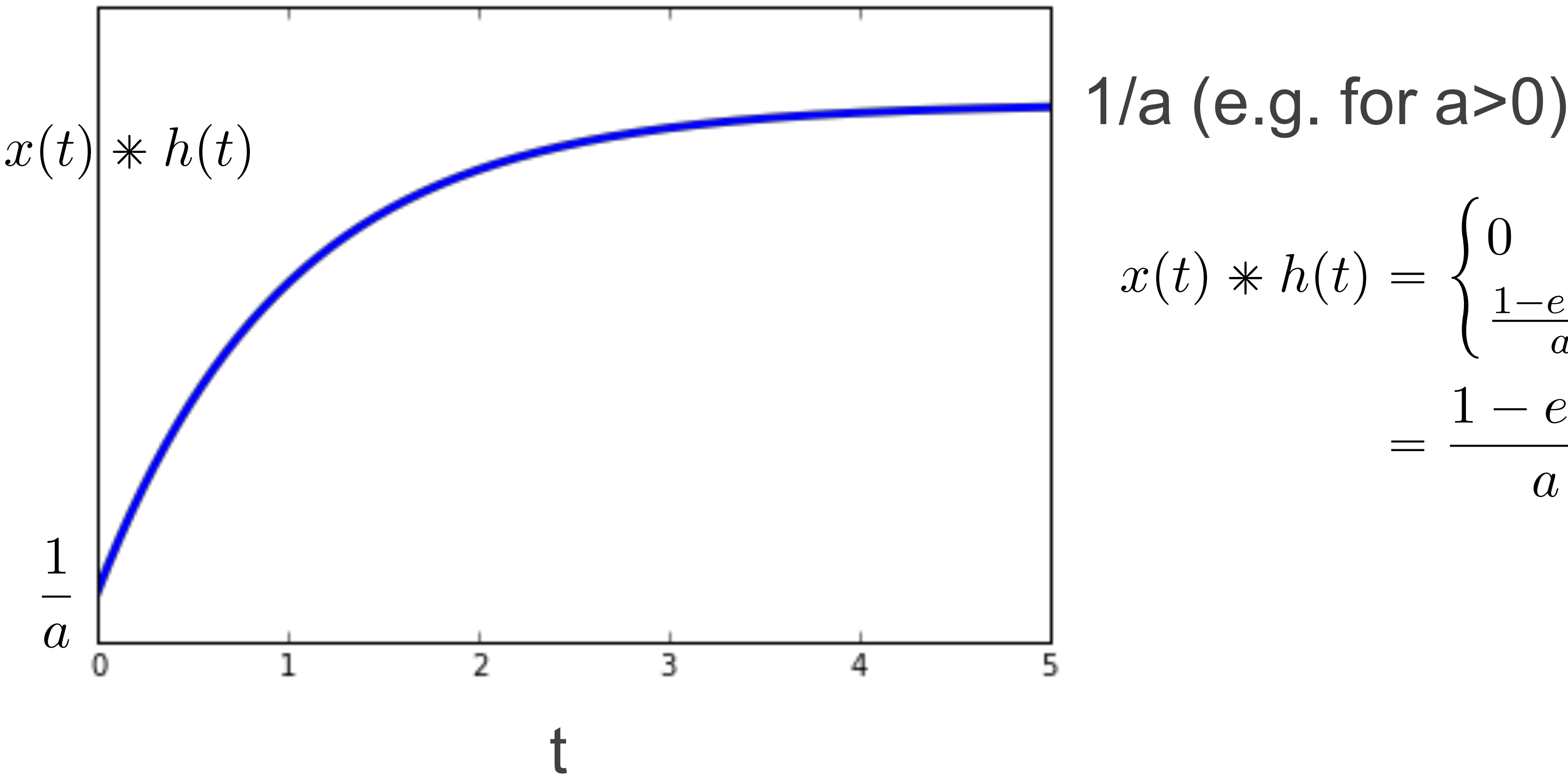
For $t \geq 0$



$$x(t) * h(t) = \int_0^t e^{-a\tau} u(t-\tau) d\tau$$

$$= \frac{1 - e^{-at}}{a}$$

CT convolution example #4: Solution



$$x(t) * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1-e^{-at}}{a} & t \geqslant 0 \end{cases}$$

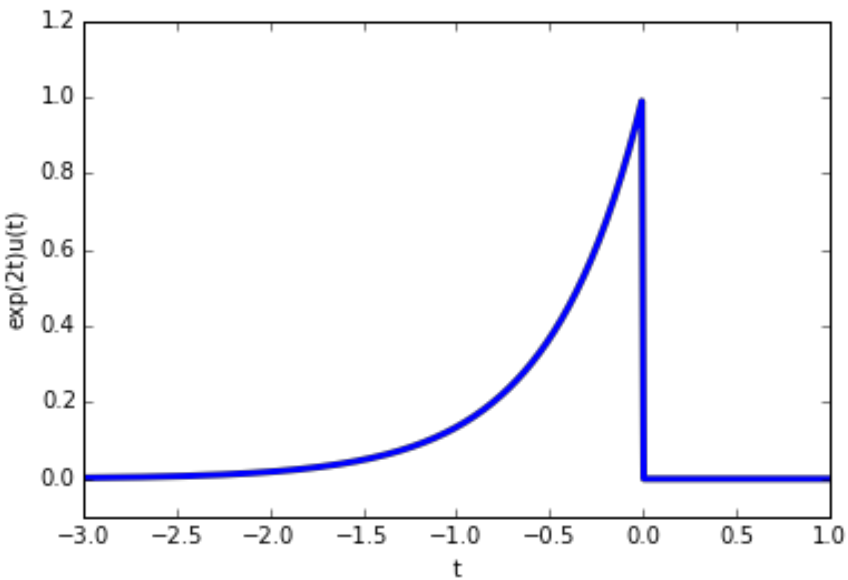
$$= \frac{1 - e^{-at}}{a} u(t)$$

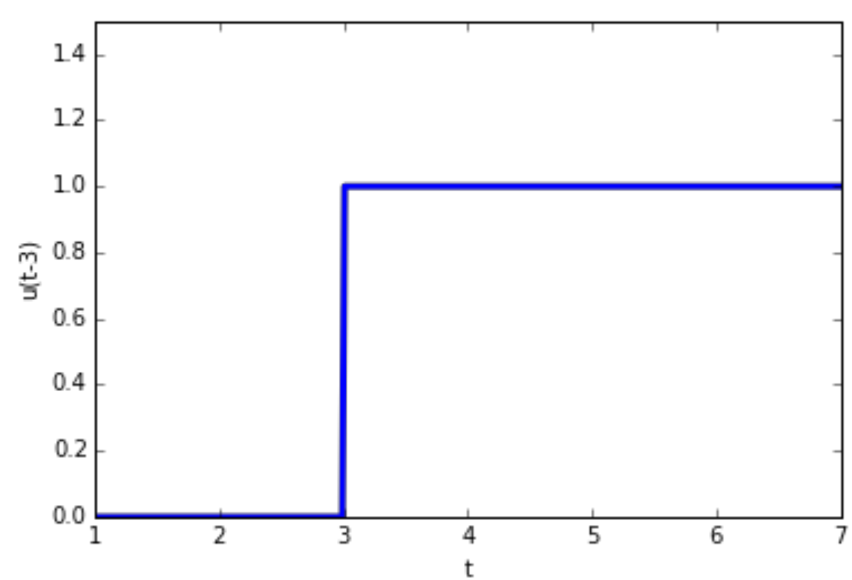
CT convolution example #5

This is an example where an anti-causal and a causal signal are convolved together, unlike the previous examples. Notice the difference intervals in this case.

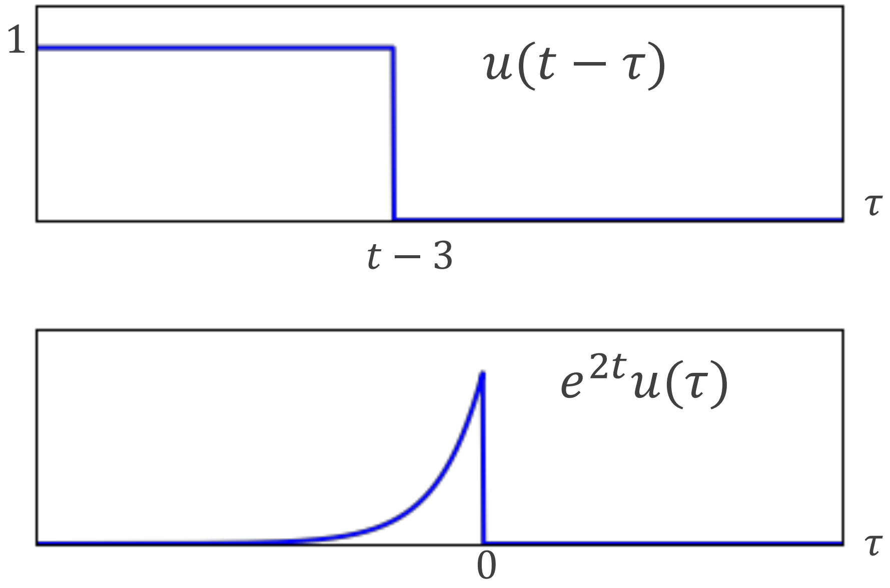
- Determine and sketch $y(t) = x(t) * h(t)$, where

$$x(t) = e^{2t}u(-t) \qquad h(t) = u(t-3)$$





CT convolution example #5: Solution



For t-3 < 0

$$x(t) * h(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau$$

$$= \frac{e^{2(t-3)}}{2}$$

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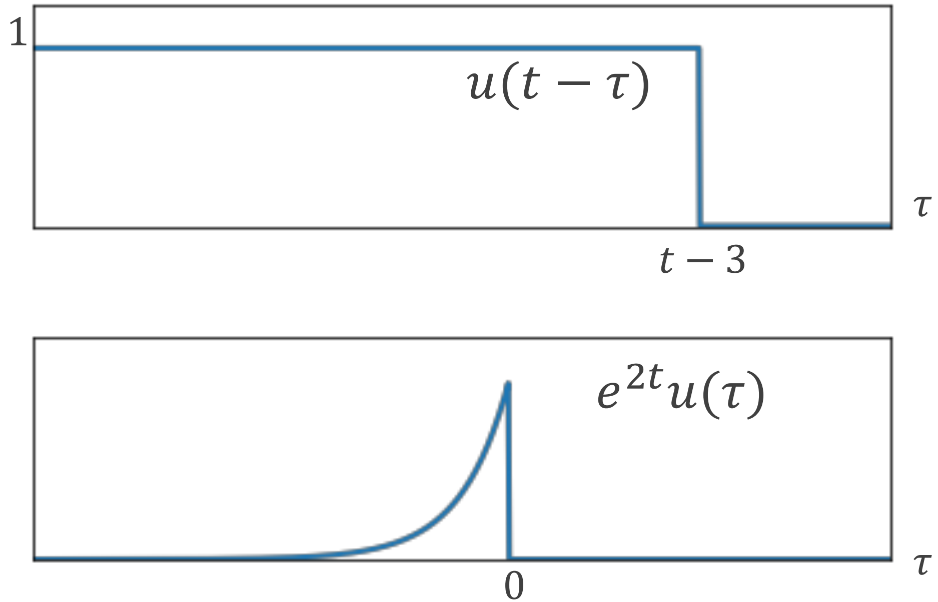
CT convolution example #5: Solution



For t-3 > 0

$$x(t) * h(t) = \int_{-\infty}^{0} e^{2\tau} d\tau$$

$$= \frac{1}{2}$$

CT convolution example #5: Solution

$$
\begin{aligned}
x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \begin{cases} \frac{e^{2(t-3)}}{2} & t < 3 \\ \frac{1}{2} & t \geqslant 3 \end{cases}
\end{aligned}
$$

Basic convolution properties

- Commutative

$$y(t) = x(t) * h(t) \qquad = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) \qquad = \int h(\tau)x(t-\tau)d\tau$$

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

Use properties to simplify convolutions

Words of wisdom on convolution

- Convolution is a fact of life
  - All real world LTI systems generate outputs in the time domain given by the convolution sum or integral (yes RLC circuits are doing convolution for you)
- You need to practice convolutions
  - There are many examples in the book and videos online
- In future lectures, we will cover tools and ideas that allow us to avoid computing convolutions (most of the time)
  - But you still need to learn how to do it in the time domain (don’t wait)

Lecture 6

More convolution

Preview of today's lecture

- Convolution
  - Determine the output of an LTI system using the convolution
  - Compute the continuous-time convolution between two signals

- Convolution properties
  - Summarize key properties of convolution
  - Leverage properties of convolution to simplify its calculations

Review

- Output of an LTI system is completely characterized by the impulse response of the system

$$\delta(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow h(t)$$

- Input and output of an LTI system are related through convolution

output

input

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

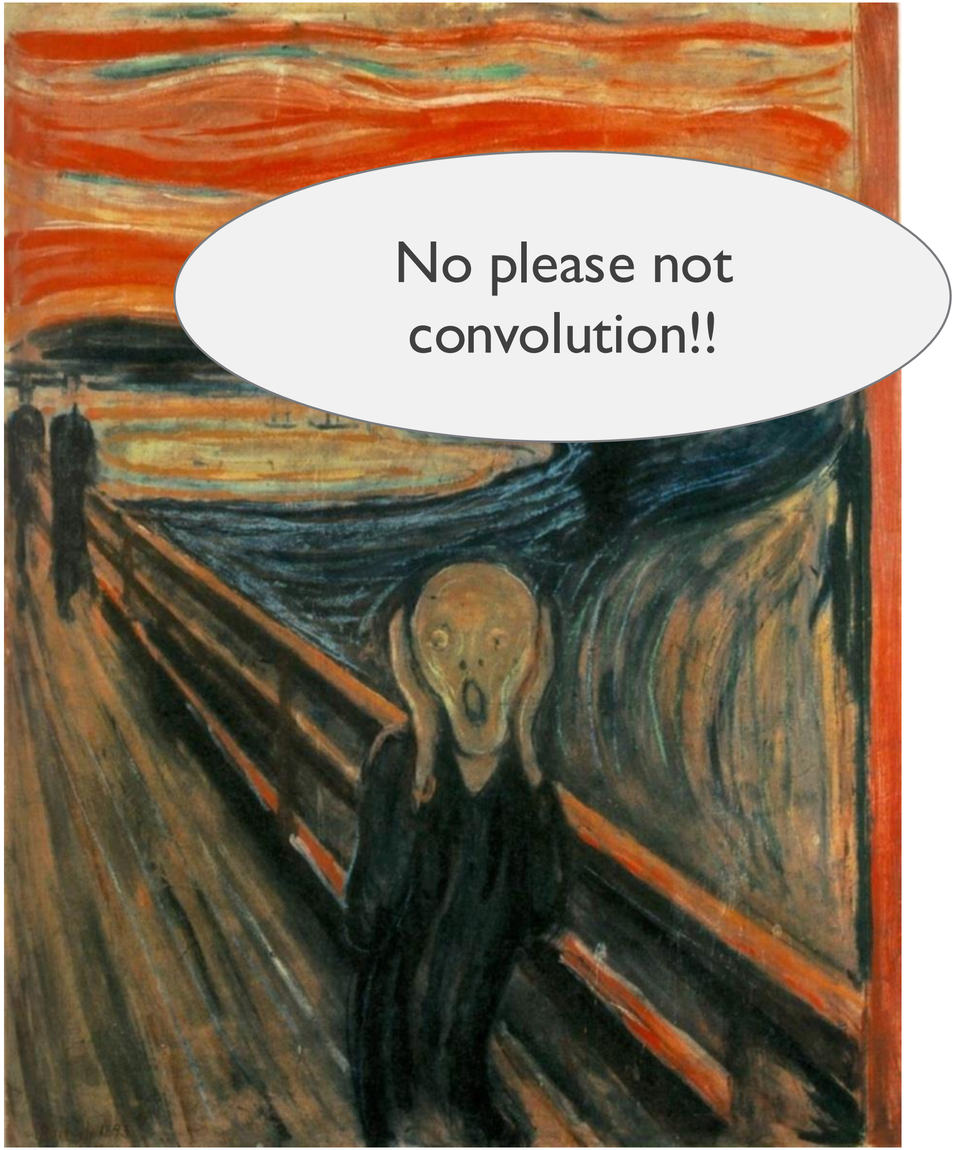
Fourier

	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, Dirac Delta	1.3 - 1.4		
4	1/16	LTI in time	Systems, linearity, time invariance	1.6.5, 1.6.6	HW3	HW2
5	1/21	LTI in time	Impulse response and convolution	2.2		
6	1/23	LTI in time	Convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Convolution

Learning objectives

- Determine the output of an LTI system using the convolution
- Compute the continuous-time convolution between two signals



https://commons.wikimedia.org/wiki/File:The_Scream.jpg with additions by Prof. Heath

Uncovering the convolution

- Consider an LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- The output can be computed from the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Basic convolution properties

- Commutative

$$y(t) = x(t) * h(t)$$

$$= h(t) * x(t)$$

Shorthand notation

$$= \int x(\tau)h(t-\tau)d\tau$$

$$= \int h(\tau)x(t-\tau)d\tau$$

Choose option that makes it easy!

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

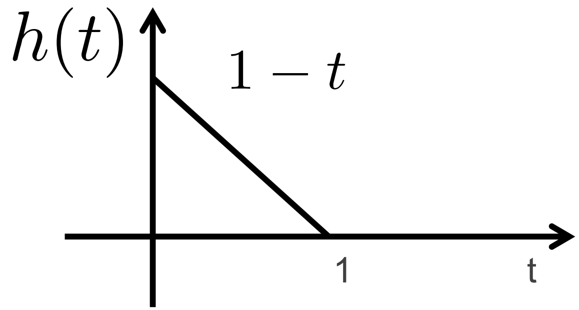
Use properties to simplify convolutions (more next lecture)

CT convolution example #1

This is a typical example using two finite length signals. Rectangles and triangle functions are common in examples / HW as they give results that are easy to integrate. It is important here to understand the different **intervals** in the convolutions.

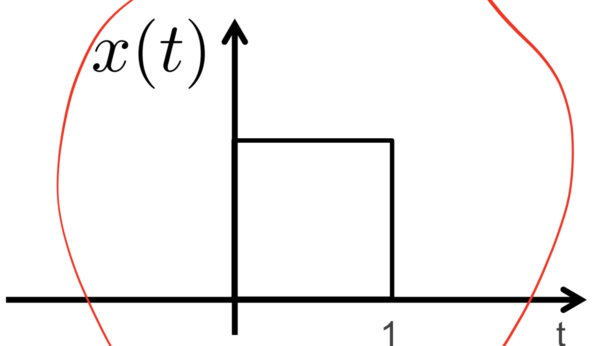
- Find the output of a system with impulse response

$$h(t) = (1-t)[u(t) - u(t-1)]$$



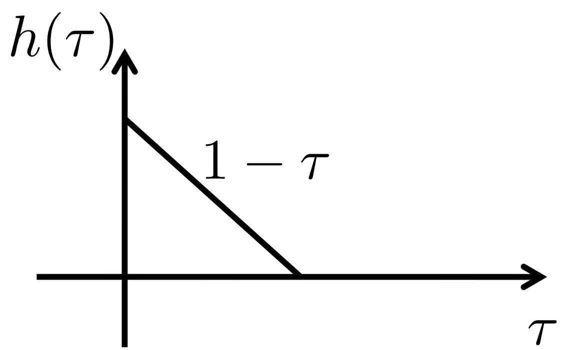
for the input

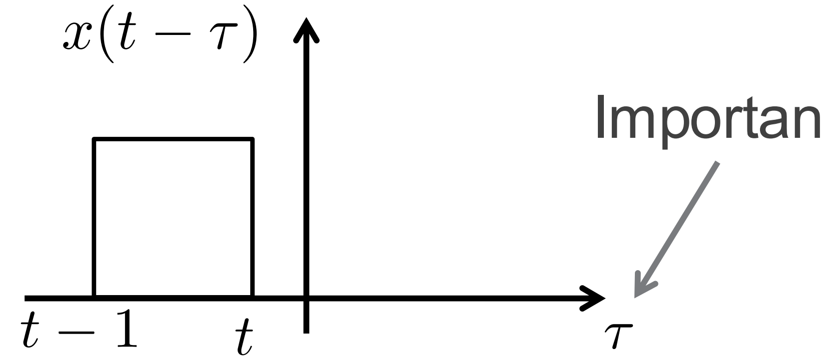
$$x(t) = u(t) - u(t-1)$$



CT convolution example #1: Graphical solution

- Plot one signal versus τ
- Flip the second signal and shift it by *t*
  - Here, plot it to the left of h(τ)
  - So plotted *t* has a negative value, usually





- There are 4 intervals (why?):

$$t < 0 \qquad 0 \leqslant t \leqslant 1 \qquad 1 \leqslant t \leqslant 2 \qquad 2 < t$$

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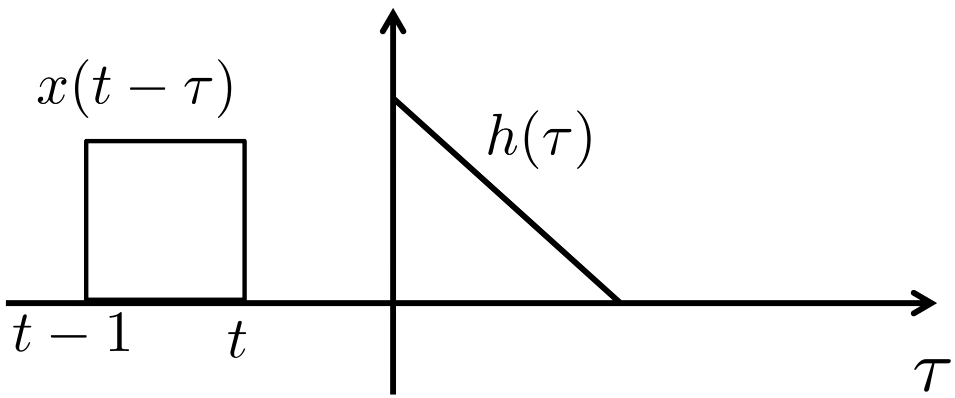
CT convolution example #1: First interval

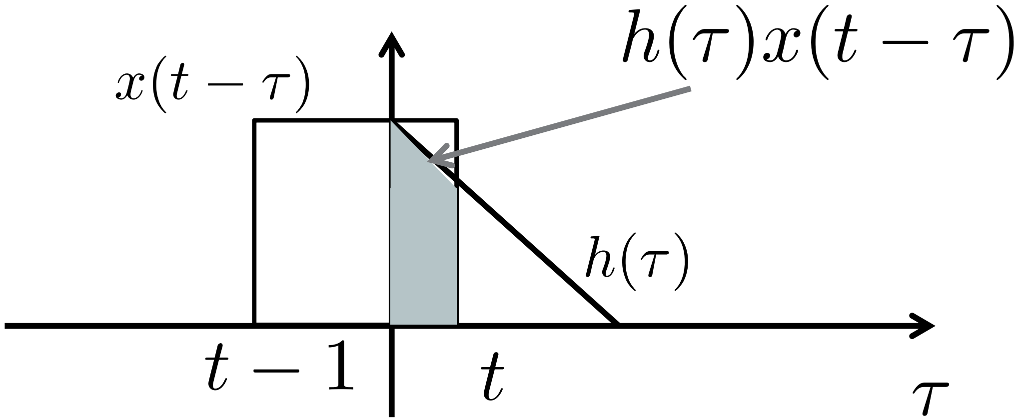
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



- First interval: there is no overlap!

$t < 0$ $\quad h(\tau)x(t-\tau) = 0 \quad \Rightarrow \quad y(t) = 0$

CT convolution example #1: Second interval



- Second interval $0 \leqslant t \leqslant 1$

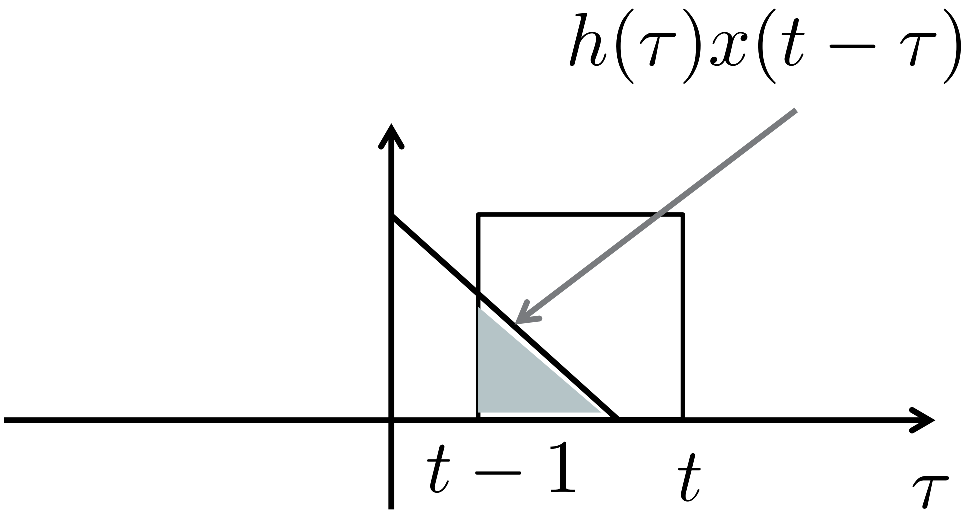
$$y(t) = \int_0^t (1-\tau)d\tau = \tau - \frac{\tau^2}{2}\Big]_0^t$$

$$= t - \frac{t^2}{2}$$

CT convolution example #1: Third interval

- Third interval $1 \leqslant t \leqslant 2$

$$y(t) = \int_{t-1}^{1} (1-\tau)d\tau$$
$$= \tau - \frac{\tau^2}{2}\Big]_{t-1}^{1}$$
$$= 1 - \frac{1}{2} - \left(t - 1 - \frac{(t-1)^2}{2}\right)$$
$$= \frac{t^2}{2} - 2t + 2$$

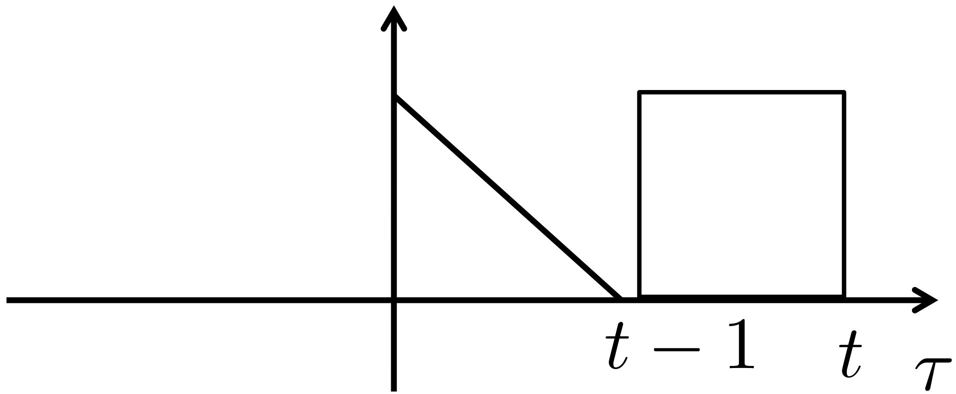


Integrating the same function, but with different integration limits

CT convolution example #1: Fourth and final interval

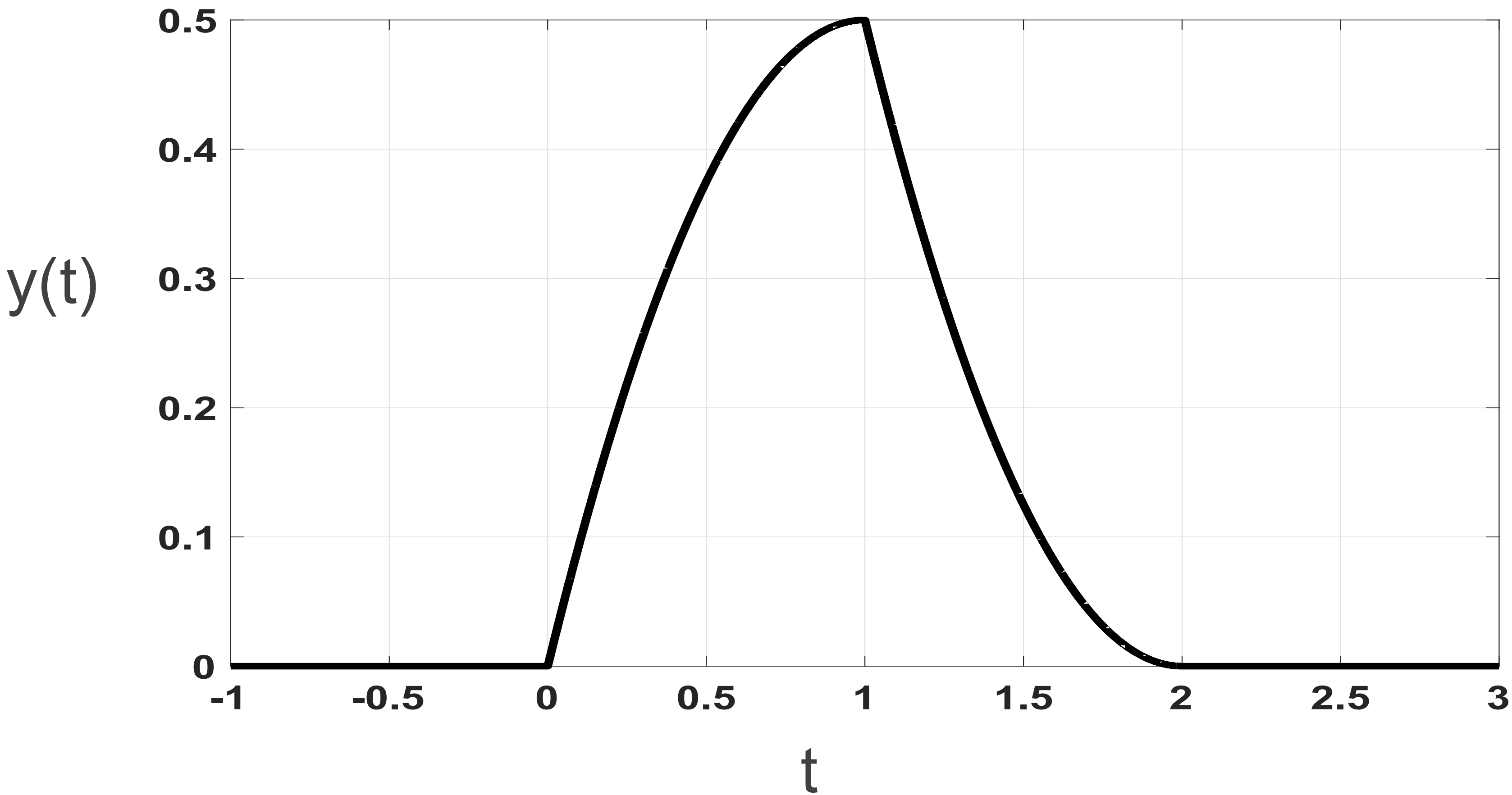
- Fourth interval $2 < t$

$$y(t) = 0$$

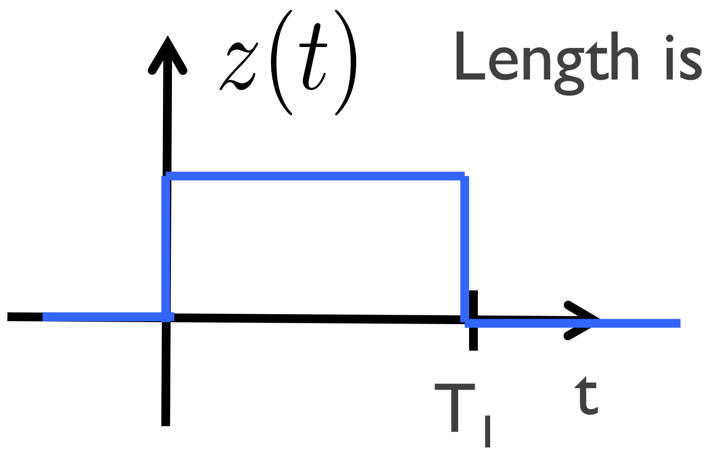


- Sanity checks
  - Check at $t=1$, the output should be the same for intervals 2 & 3 in order for it to be continuous
  - Same for t = 0 and t = 2 (should be zero there)
  - Duration of output should be $T_1 + T_2 = 1+1 = 2$

CT convolution example #1: Plot of y(t)

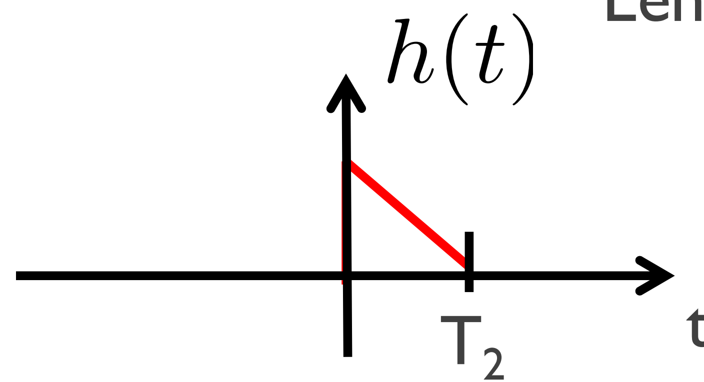


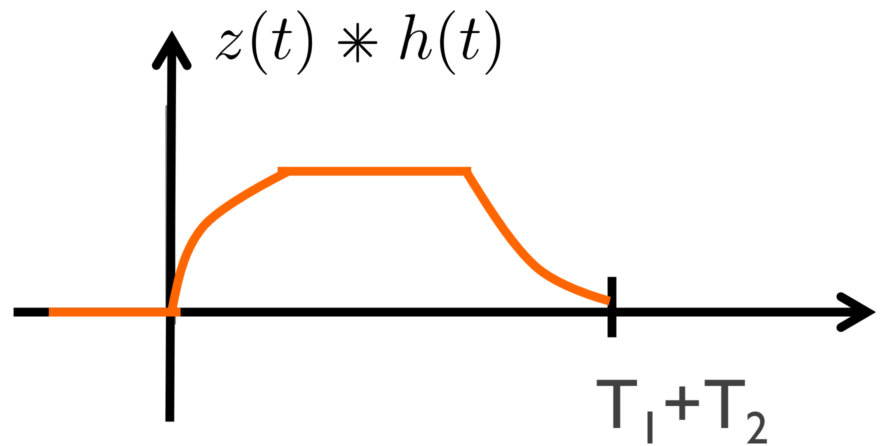
Length / duration of a convolution in continuous time



Length is T_1

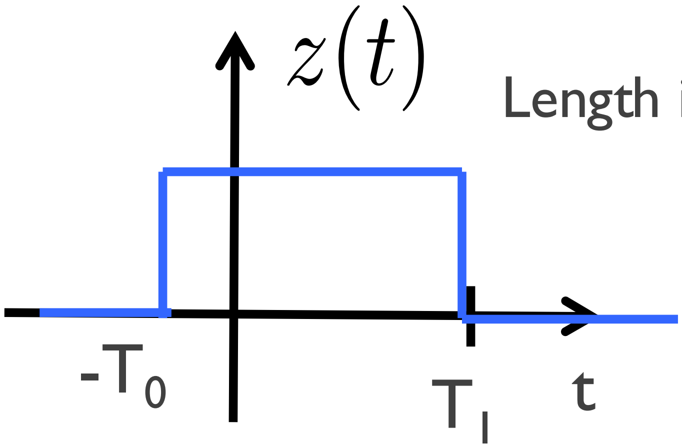
Length is T_2



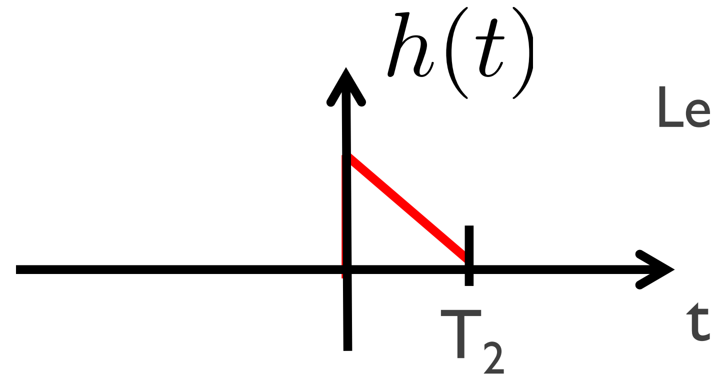


Length is T_1+T_2

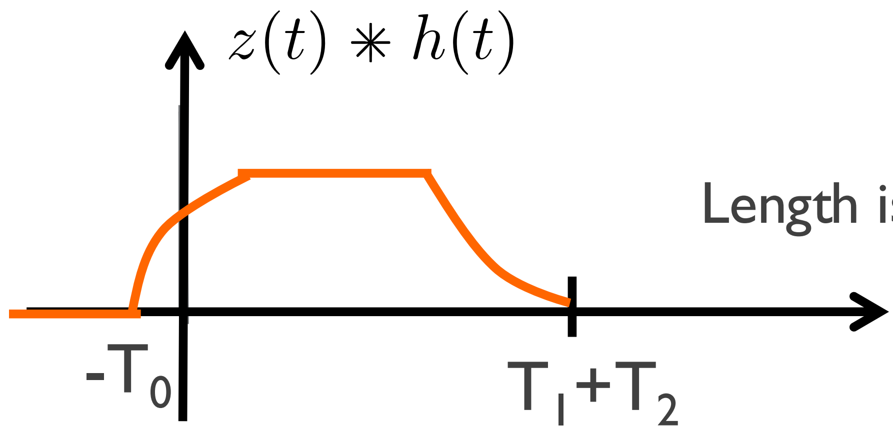
Length / duration of a convolution in continuous time



Length is T_0+T_1



Length is T_2



Length is $T_0 + T_1 + T_2$

Animation example

http://www.cse.yorku.ca/~asif/spc/ConvolutionIntegral_Final3.swf

CT convolution example #2

This is an example with a simple input that consists of a few delta functions. The convolution is easy to compute using the LTI property of convolution and the definition of impulse response.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Determine and sketch the convolution of the following input

$$x(t) = \delta(t) + 2\delta(t-1)$$

and system with impulse response

$$h(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1, \\ 2-t, & 1 < t \leqslant 2, \\ 0, & \text{elsewere} \end{cases}$$

CT convolution example #2: Solution approach

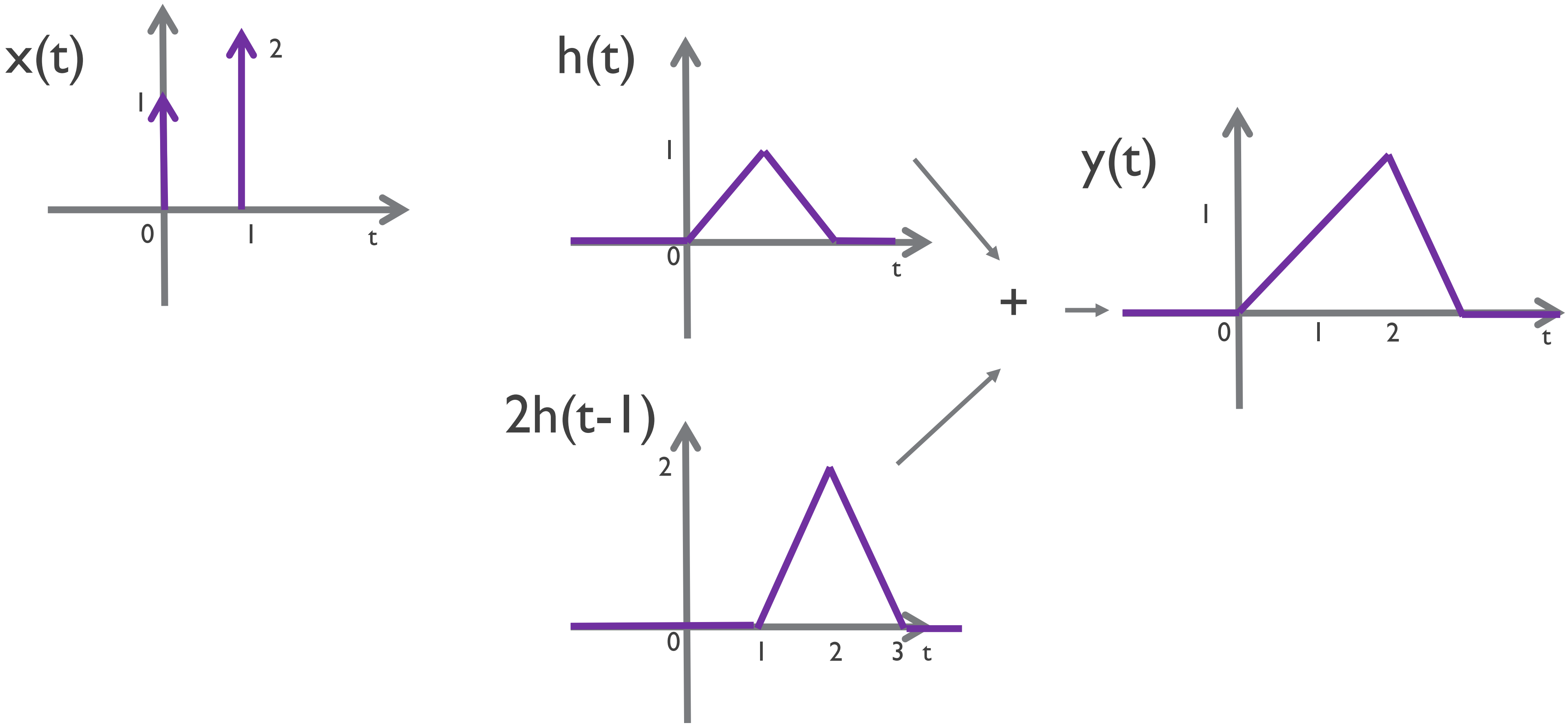
- Recall the property

$$\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t - T_0)$$

- Therefore

$$x(t) = \delta(t) + 2\delta(t-1) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) + 2h(t-1)$$

CT convolution example #2: Solution sketch

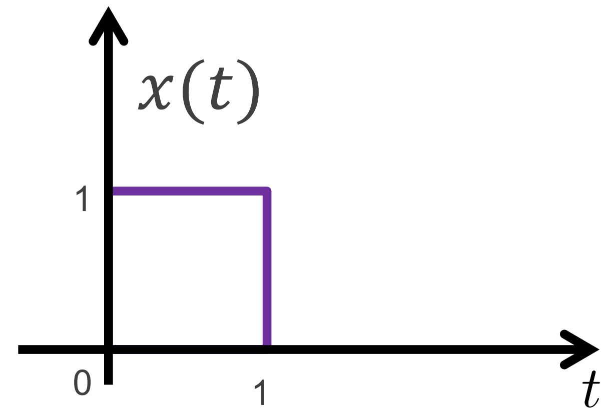


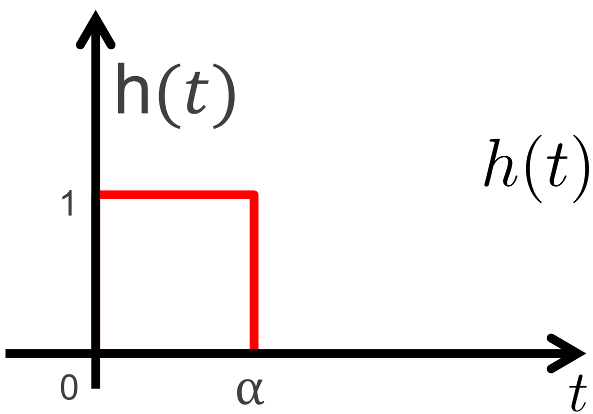
CT convolution example #3

This is an important example involving the convolution of two rectangles of different widths. The solution will be different depending on the widths of the rectangles. Notice what happens in each interval.

- Determine and sketch $y(t) = x(t) * h(t)$, where

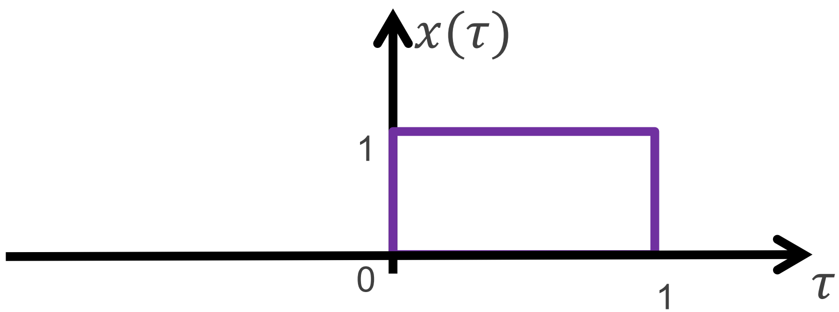
$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

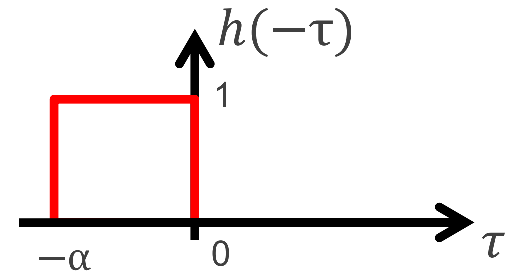




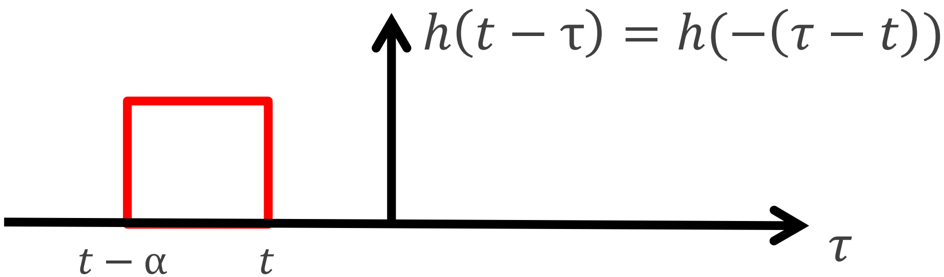
$$h(t) = x(t/\alpha) \qquad 0 < \alpha \leq 1$$

CT convolution example #3: Solution



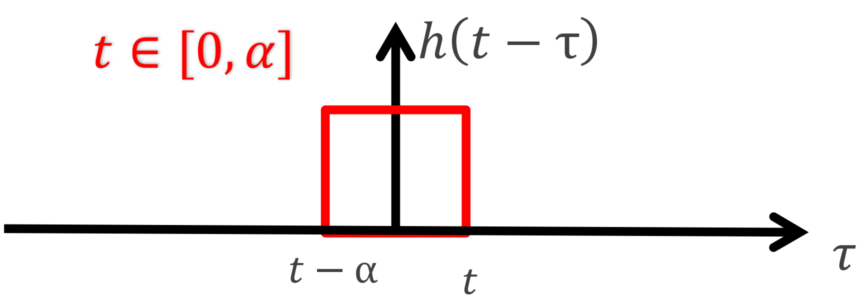


$t < 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0 \quad t < 0$$

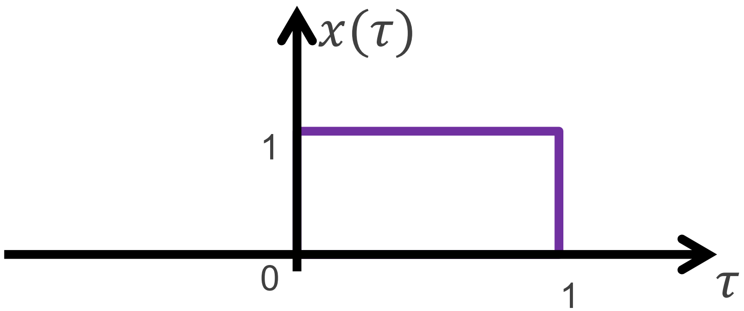
$t \in [0, \alpha]$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} 1 d\tau \quad t \in [0, \alpha]$$

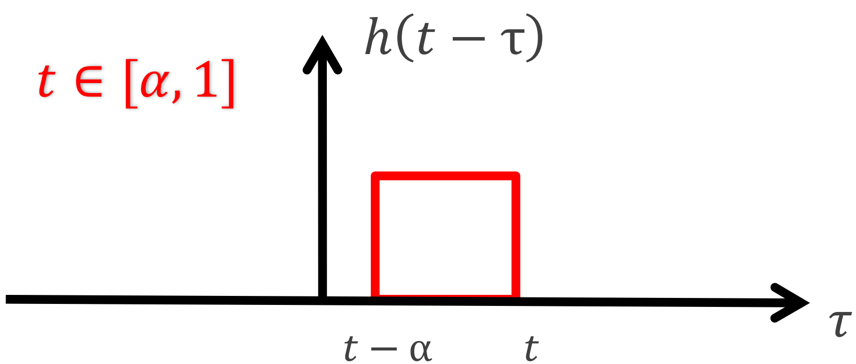
$$= t \quad t \in [0, \alpha]$$

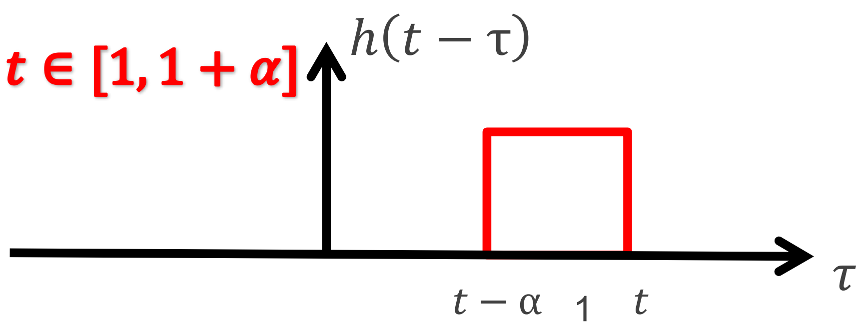
CT convolution example #3: Solution



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{t} 1 d\tau \quad t \in [\alpha, 1]$$

$$= \alpha \quad t \in [\alpha, 1]$$

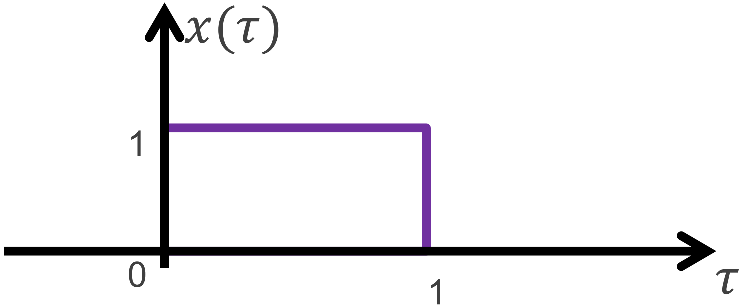




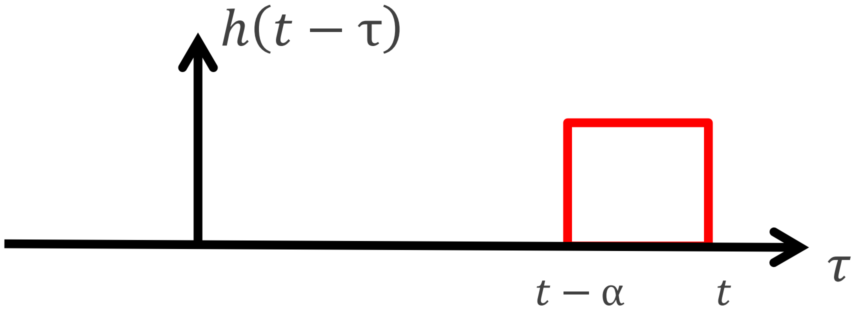
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{1} 1 d\tau \quad t \in [1, 1+\alpha]$$

$$= 1-(t-\alpha) \quad t \in [1, 1+\alpha]$$

CT convolution example #3: Solution



$t \in [1+\alpha, \infty]$

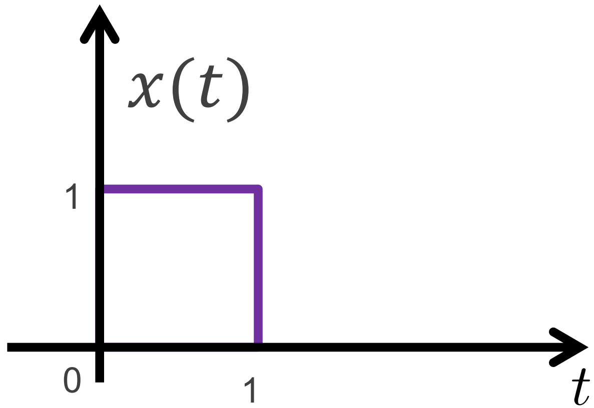


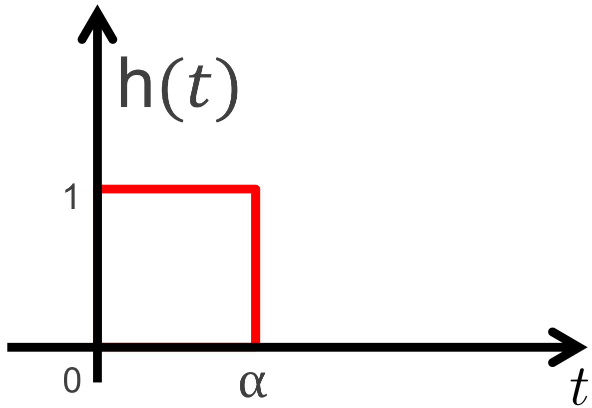
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{1}^{t} 0 d\tau \quad t \in [\alpha+1, \infty]$$

$$= 0 \quad t \in [\alpha+1, \infty]$$

CT convolution example #3: Solution (summary)

$$y(t) = x(t) * h(t)$$





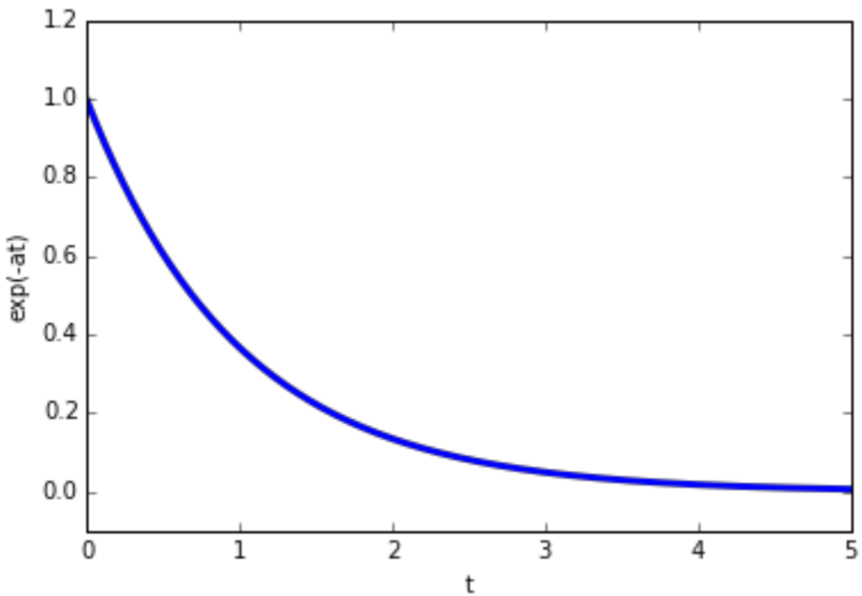
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} t & < & 0 & 0 \\ t & \in & [0,\alpha] & t \\ t & \in & [\alpha,1] & \alpha \\ t & \in & [1,1+\alpha] & 1-(t-\alpha) \\ t & > & 1+\alpha & 0 \end{cases}$$

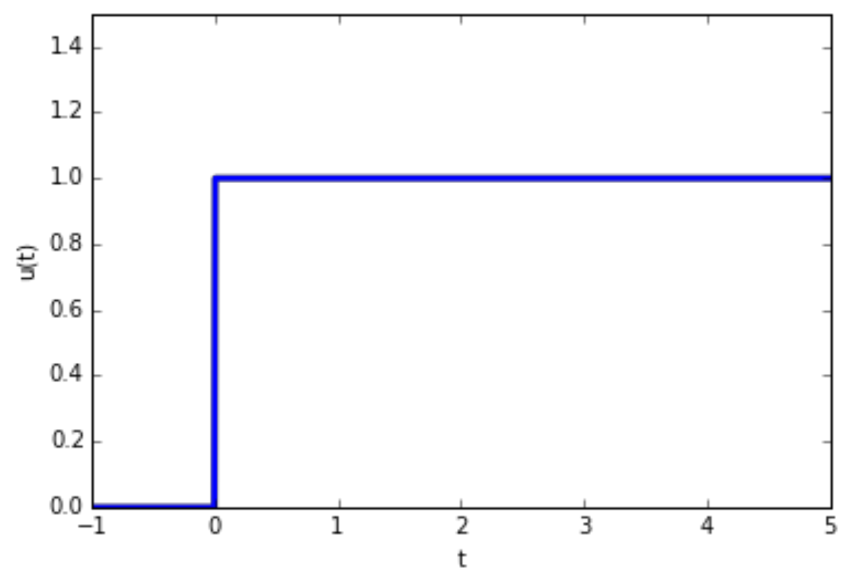
CT convolution example #4

This is an important example of convolution to causal signals together. It will be solved by using the graphical approach to help set up the integrals.

- Determine and sketch $y(t) = x(t) * h(t)$, where

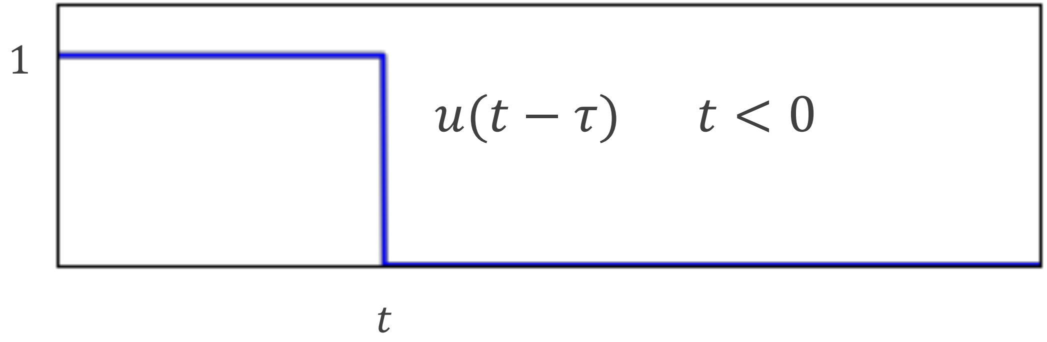
$$x(t) = e^{-at}u(t), \qquad a > 0 \qquad\qquad h(t) = u(t)$$

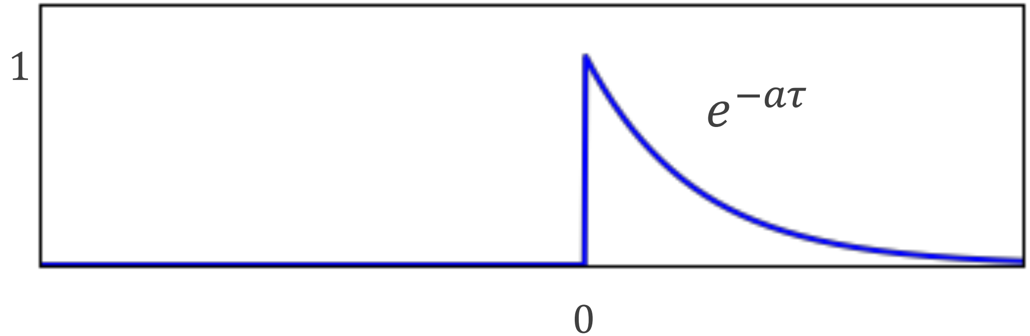




CT convolution example #4: Solution

$$y(t) = x(t) * h(t)$$

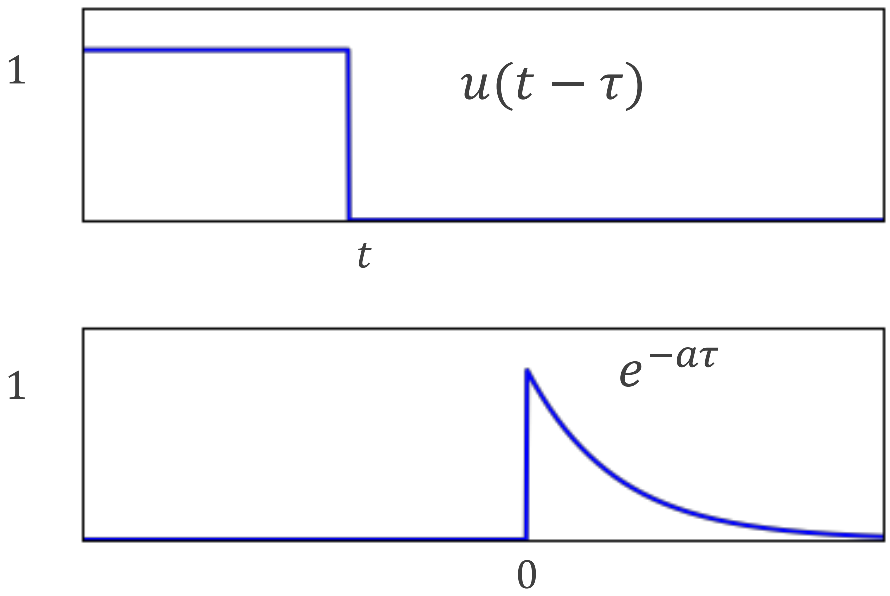




$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-a\tau}u(t-\tau)d\tau \end{aligned}$$

CT convolution example #4: Solution

For $t < 0$

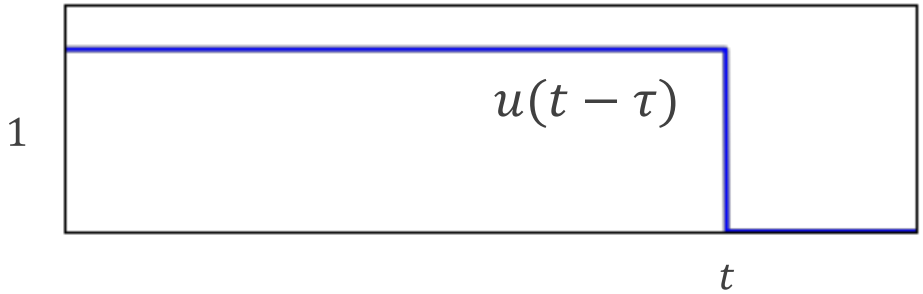


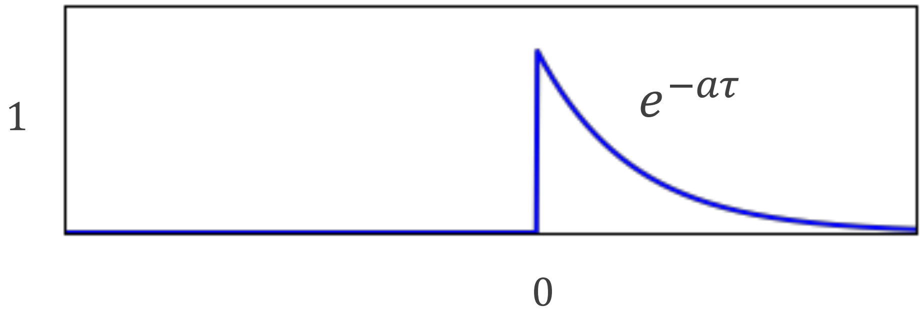
$$x(t) * h(t) = \int_{-\infty}^{\infty} 0 d\tau$$

$$= 0$$

CT convolution example #4: Solution

For $t \geq 0$

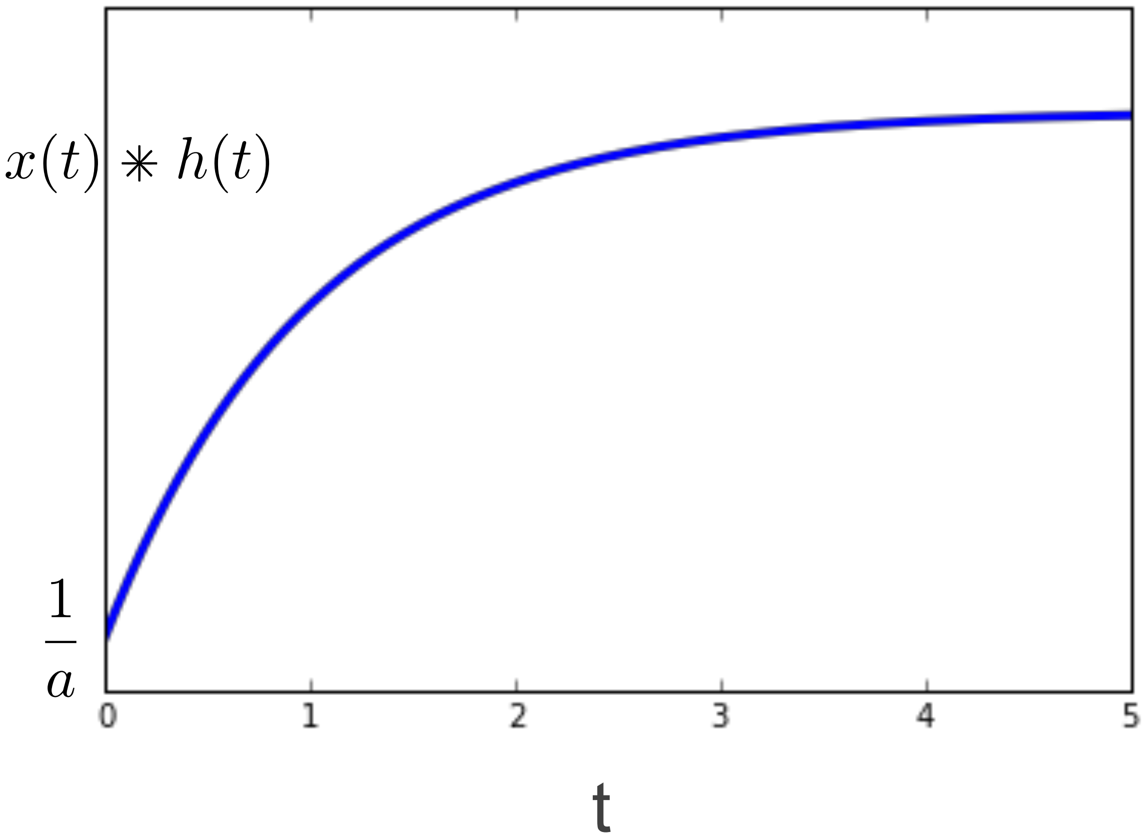




$$x(t) * h(t) = \int_0^t e^{-a\tau} u(t-\tau) d\tau$$

$$= \frac{1 - e^{-at}}{a}$$

CT convolution example #4: Solution



1/a (e.g. for a>0)

$$x(t) * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1-e^{-at}}{a} & t \geqslant 0 \end{cases}$$

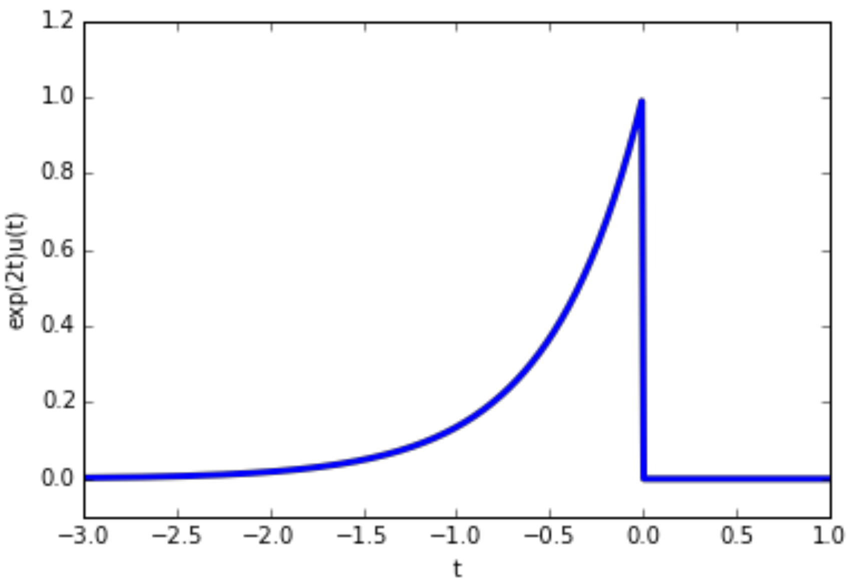
$$= \frac{1-e^{-at}}{a} u(t)$$

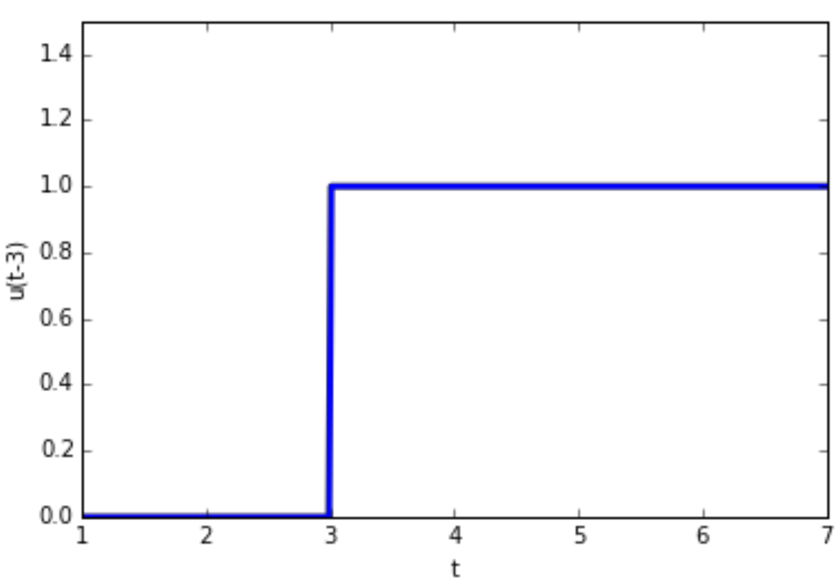
CT convolution example #5

This is an example where an anti-causal and a causal signal are convolved together, unlike the previous examples. Notice the difference intervals in this case.

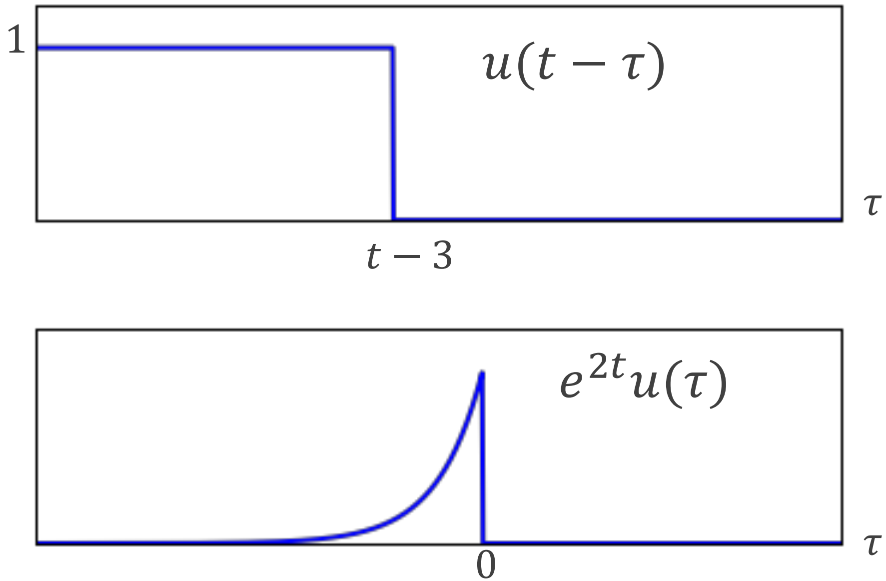
- Determine and sketch $y(t) = x(t) * h(t)$, where

$$x(t) = e^{2t}u(-t) \qquad h(t) = u(t-3)$$





CT convolution example #5: Solution

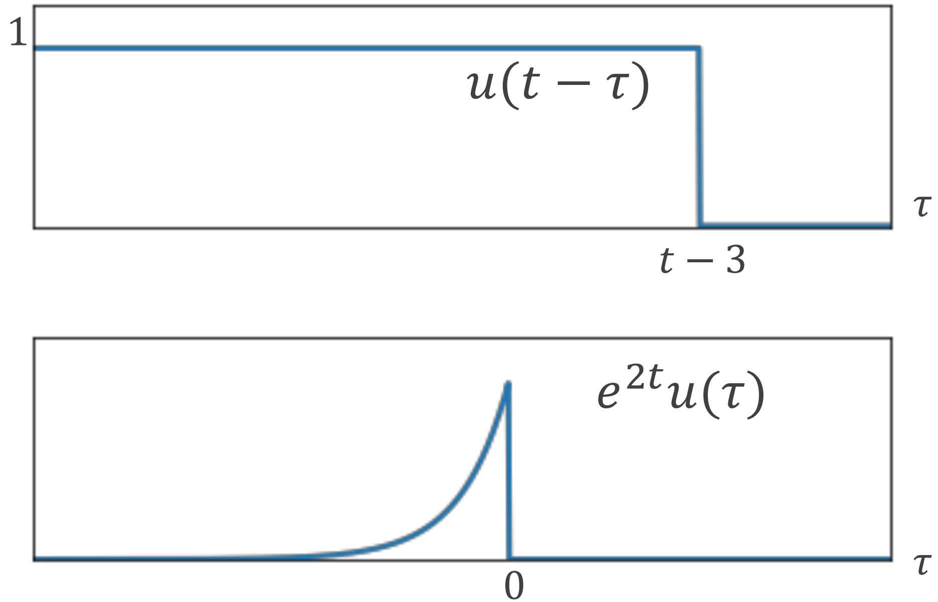


For t-3 < 0

$$x(t) * h(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau$$

$$= \frac{e^{2(t-3)}}{2}$$

CT convolution example #5: Solution



For t-3 > 0

$$x(t) * h(t) = \int_{-\infty}^{0} e^{2\tau} d\tau$$

$$= \frac{1}{2}$$

CT convolution example #5: Solution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \begin{cases} \frac{e^{2(t-3)}}{2} & t < 3 \\ \frac{1}{2} & t \geqslant 3 \end{cases}$$

Basic convolution properties

- Commutative

$$y(t) = x(t) * h(t) \qquad = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) \qquad = \int h(\tau)x(t-\tau)d\tau$$

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

Use properties to simplify convolutions

Words of wisdom on convolution

- Convolution is a fact of life
  - All real world LTI systems generate outputs in the time domain given by the convolution sum or integral (yes RLC circuits are doing convolution for you)
- You need to practice convolutions
  - There are many examples in the book and videos online
- In future lectures, we will cover tools and ideas that allow us to avoid computing convolutions (most of the time)
  - But you still need to learn how to do it in the time domain (don't wait)

How does research in ECE (especially signal processing) get done at a university?

Or do professors really write all of those papers they publish?

Composition of authors on a paper

2020 IEEE 21st International Workshop on Signal Processing Advances in Wireless Communications (SPAWC)

Deep Learning-based Carrier Frequency Offset Estimation with One-Bit ADCs

Graduate research assistant (GRA)

Industry collaborator (recent PhD)

Ryan M. Dreifuerst, Robert W. Heath Jr.
Department of Electrical and Computer Engineering
The University of Texas at Austin
Email: {ryandry1st, rheath}@utexas.edu

Mandar N. Kulkarni, Jianzhong Charlie Zhang
Standards and Mobility Innovation Lab
Samsung Research America
Email: {mandar.kulkarni, jianzhong.z}@samsung.com

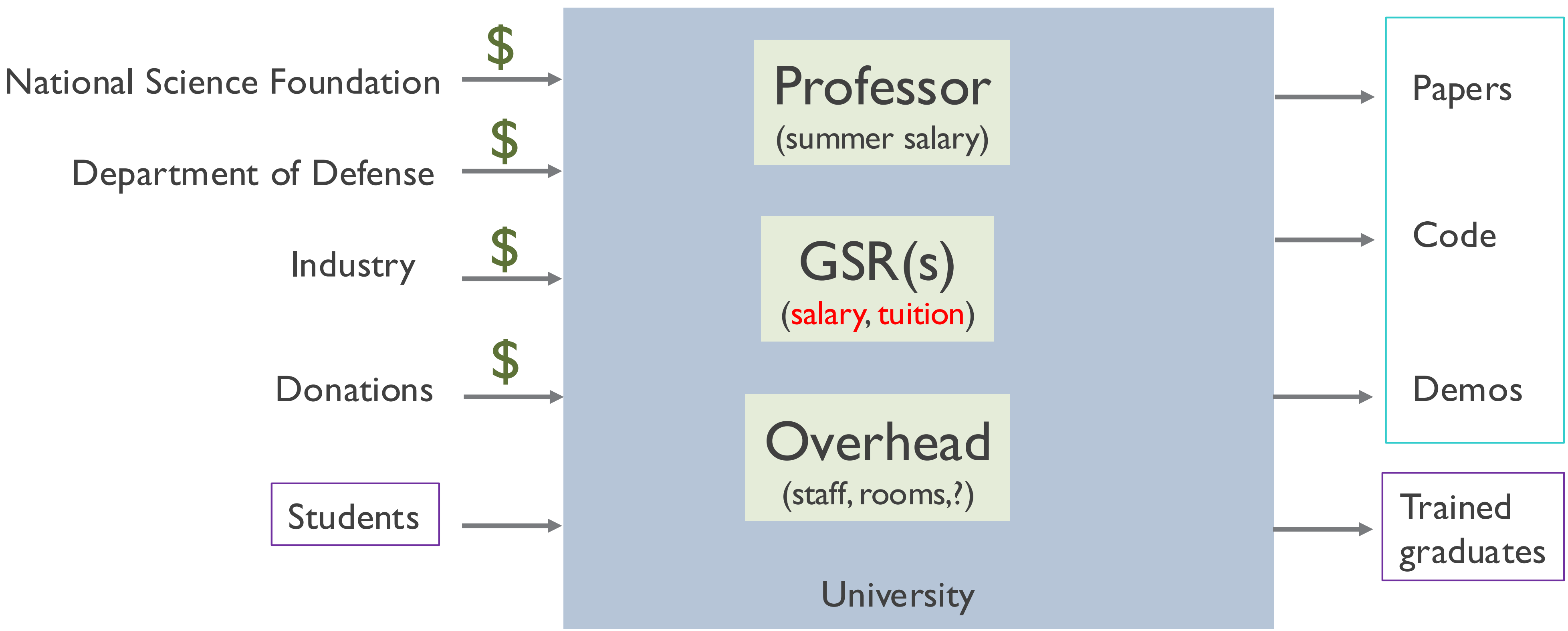
Professor

Industry collaborator (lab director)

Percent effort

75% 15% 8% 2%

Summary of system



Note that GSRs (PhD bound) are paid to do the research and attend school

How the work gets done 1/2

- Professors
  - Identify technical directions
  - Raise funding to support research
  - Provide input, on publications and presentations
  - General mentoring
- GRAs
  - Understand the state-of-the-art
  - Identify problems that can be solved
  - Develop solutions, simulate results, build prototype, etc
  - Create conference paper, journal paper, dissertation, etc

Type of work that may be done by a USR

How the work gets done 2/2

- Industry collaborators
  - Give insight into the state-of-the-art
  - Suggest important problems to solve
  - Fund the research (sometimes)

- Academic collaborators (usually Professor or Professor + GRA)
  - Contribute to the development of the research, e.g. bringing a complementary set of expertise or domain knowledge
  - Collaborate on the research results

Collaboration is increasingly important to identify and solve important problems

Convolution properties

Learning objectives

- Summarize key properties of convolution
- Exploit the properties to simplify convolution

Basic convolution properties

- Commutative

$$y(t) = x(t) * h(t)$$

Shorthand notation

$$= \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t)$$

$$= \int h(\tau)x(t-\tau)d\tau$$

Choose option that makes it easy!

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

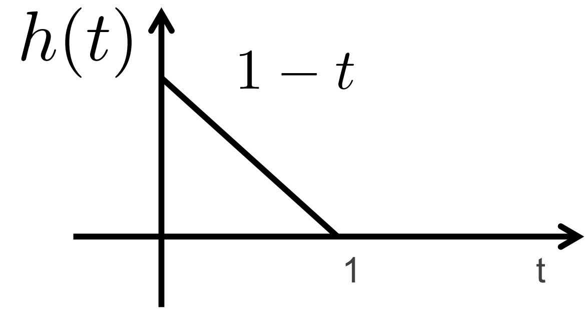
$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

Use properties to simplify convolutions

Recall this convolution

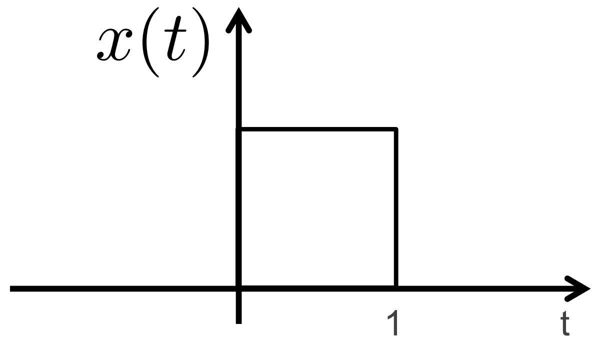
- Find the output of a system with impulse response

$$h(t) = (1-t)[u(t) - u(t-1)]$$



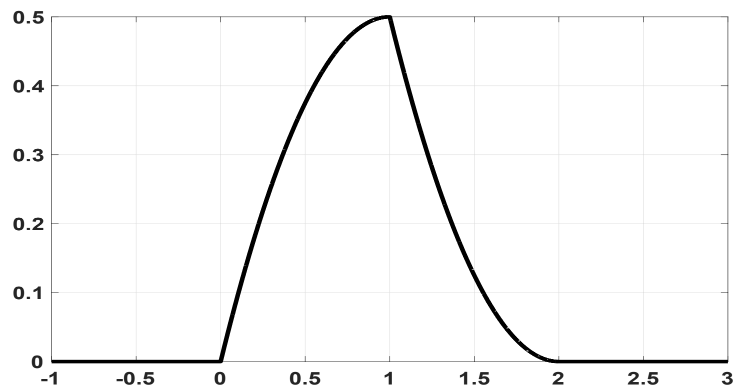
for the input

$$x(t) = u(t) - u(t-1)$$



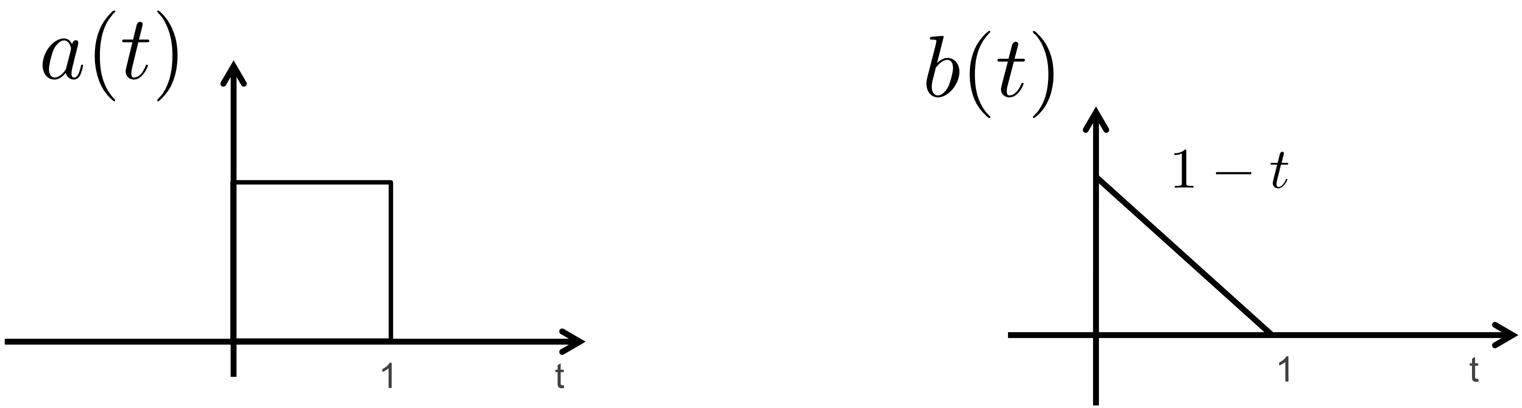
Solution to the convolution

$$y(t) = \begin{cases} 0 & t < 0 \\ t - \frac{t^2}{2} & 0 \leqslant t \leqslant 1 \\ \frac{t^2}{2} - 2t + 2 & 1 \leqslant t \leqslant 2 \\ 0 & 2 < t \end{cases}$$



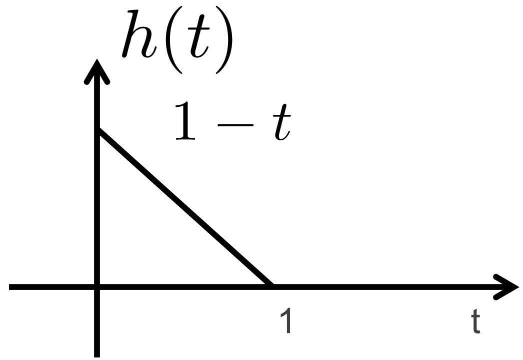
For later use, let us refer to this convolution as $f(t)$

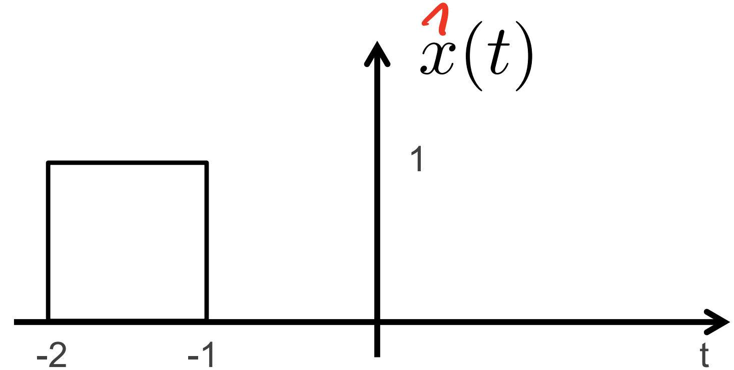
Use the properties to solve the following integral



$$c(\gamma) = \int_{-\infty}^{\infty} b(q)a(\gamma - q)dq$$

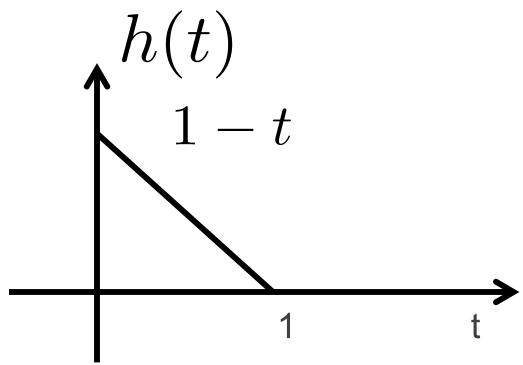
Use the properties to convolve the following

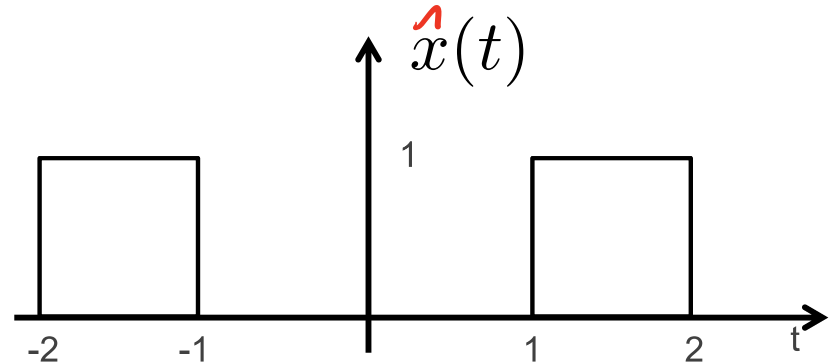




$$\hat{x}(t) = \delta(t-2) * x(t)$$

Use the properties to convolve the following





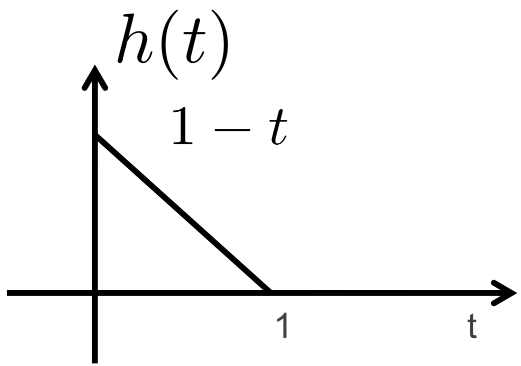
$x(t+2)$ $\qquad$ $x(t-1)$

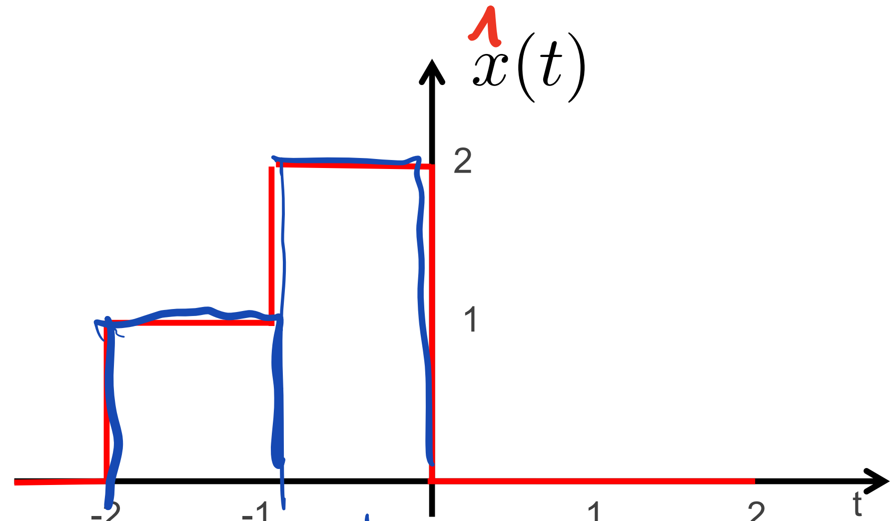
$$h(t) * \big(x(t+2) + x(t-1) \big)$$

$$h(t) * x(t+2) + h(t) * x(t-1)$$

$$= s(t+2) + s(t-1)$$

Use the properties to convolve the following





$x(t+2)$

$2x(t+1)$

$$h(t) * \left(x(t+2) + 2x(t+1) \right) = f(t+2) + 2f(t+1)$$

Summary about properties

- Convolution satisfies commutative, associative and distributed properties
  - Properties follow primarily from the integral structure
  - Use these properties to simplify the calculation of a convolution