Chapter 1
Introduction

Signals and Systems

Brief review of fundamentals

Sinusoids

$$\boxed{x(t) = A\cos(\omega_0 t + \phi) \;=\; A\cos(2\pi f_0 t + \phi)}$$

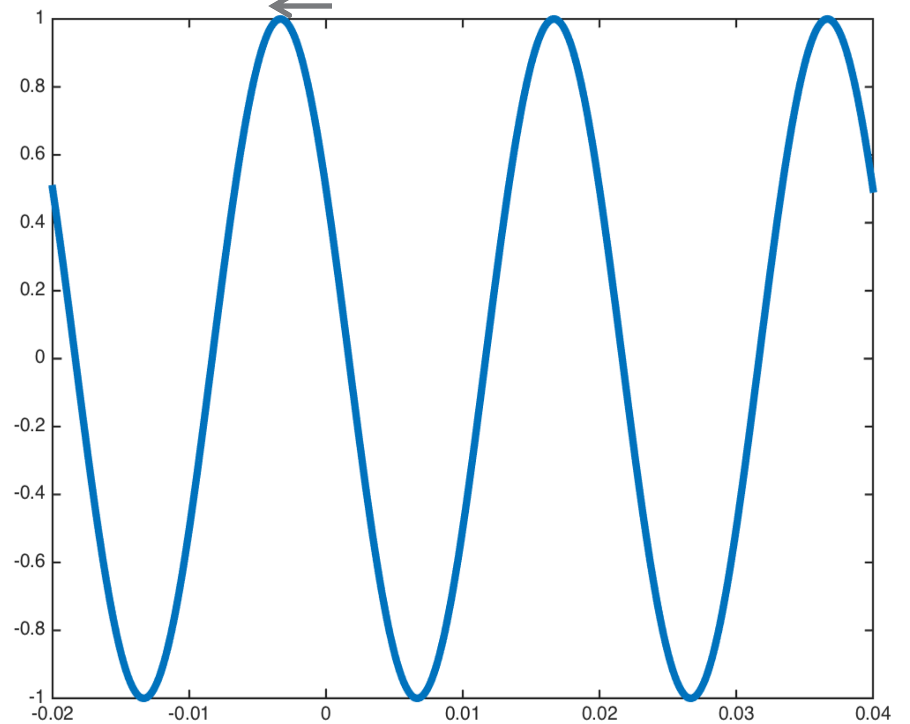
- A amplitude
- ϕ phase in radians
- ω_0 frequency (radians/sec)
- $f_0 = \dfrac{\omega_0}{2\pi}$ frequency (in Hertz)

- $T_0 = \dfrac{2\pi}{\omega_0} = \dfrac{1}{f_0}$ is the period

amplitude is 1

$\cos(2\pi 50t + \pi/4)$

$\pi/4$



frequency is 100π radians = 50 Hertz

period is T_0= 1/50 = 0.02 secs

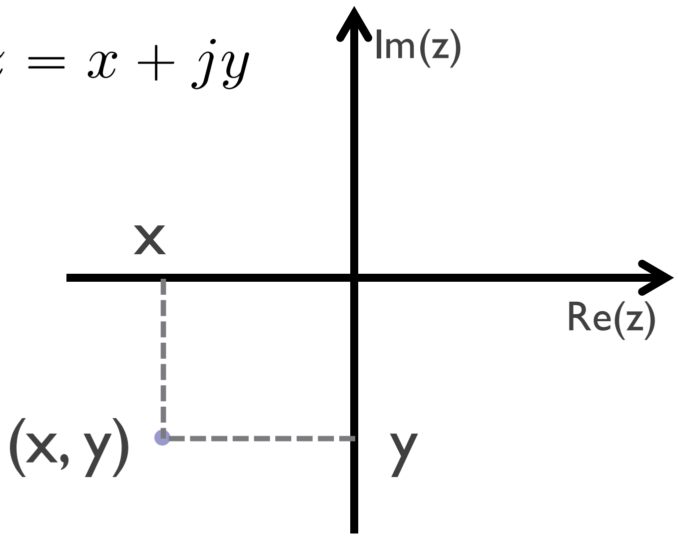
Complex numbers

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & x < 0, y \geqslant 0 \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & x < 0, y < 0 \end{cases}$$

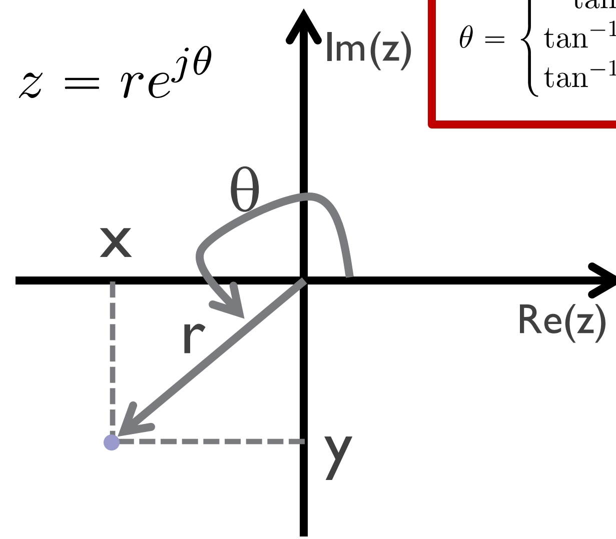
Note the quadrant matters!

$$z = x + jy$$



Cartesian form

$$z = re^{j\theta}$$

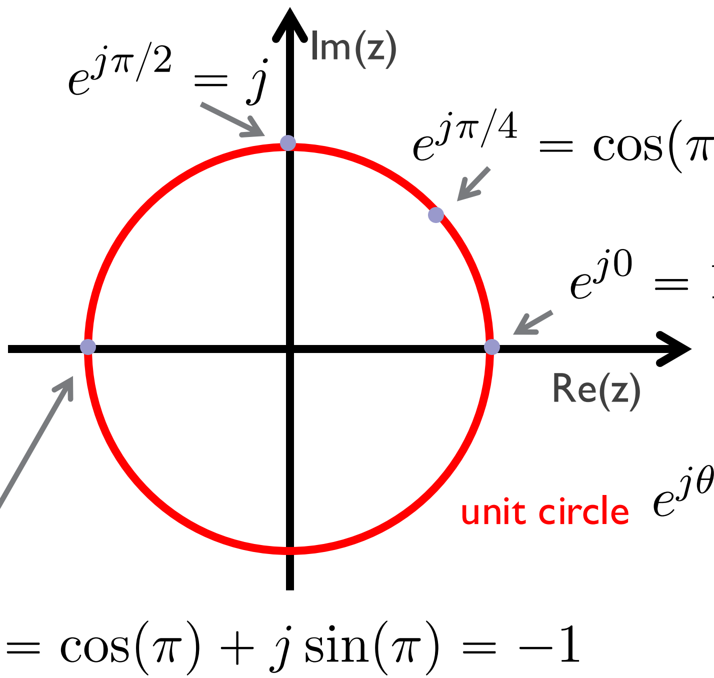


Polar form

- Complex number is essentially a pair of independent real numbers z = (x, y)
  - Widely used in engineering and science
  - $j = \sqrt{-1}$ is the imaginary number (EE's use j because i is historically used for current)

Euler's formula

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$



$$e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1$$

Useful facts

$$e^{jx} = \cos x + j\sin x$$
$$e^{-jx} = \cos x - j\sin x$$
$$e^{jx} + e^{-jx} = 2\cos x$$
$$e^{jx} - e^{-jx} = 2j\sin x$$

$$\boxed{\begin{aligned} \cos(\theta) &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin(\theta) &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned}}$$

Working with complex numbers

consider these two complex numbers

$$z_1 = a + jb = r_1 e^{j\theta_1}$$
$$z_2 = c + jd = r_2 e^{j\theta_2}$$

addition

$$\begin{aligned} z_1 + z_2 &= (a + jb) + (c + jd) \\ &= (a + c) + j(b + d) \end{aligned}$$

multiplication

$$\begin{aligned} z_1 z_2 &= (a + jb)(c + jd) \\ &= (ac - bd) + j(bc + ad) \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)} \end{aligned}$$

conjugate

$$\begin{aligned} z_1^* &= a - jb \\ z_1 + z_1^* &= a - jb + a + jb \\ &= 2a \\ &= 2\mathrm{Re}(z_1) \\ z_1 z_1^* &= r_1 r_1 e^{j(\theta_1 - \theta_1)} \\ &= r_1^2 \\ &= x^2 + y^2 \end{aligned}$$

magnitude squared

division

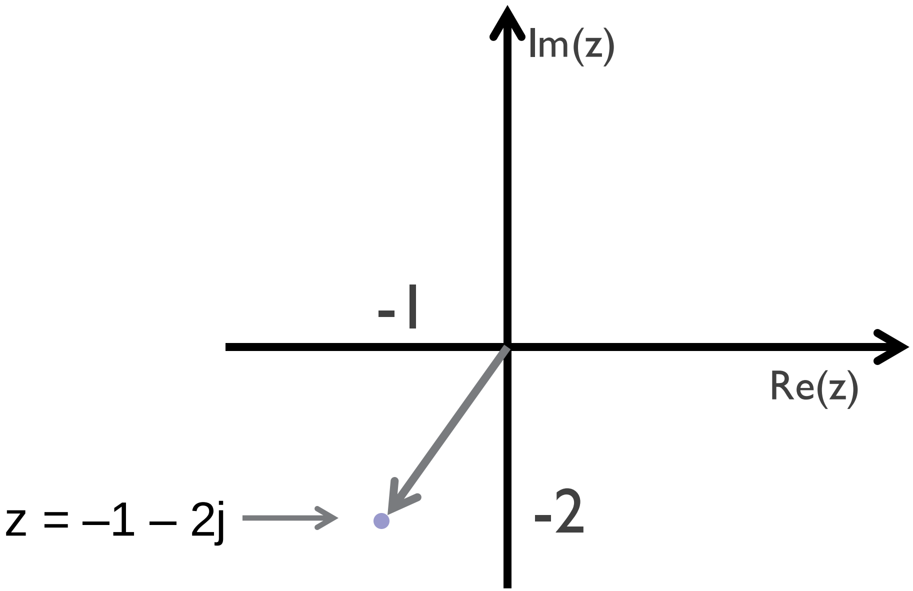
$$z_1/z_2 = (r_1/r_2)e^{j(\theta_1 - \theta_2)}$$

Example

- Consider z = –1 – 2j
  - Plot this complex number
  - Find its polar form

$$r = \sqrt{(-1)^2 + (-2)^2}$$
$$= \sqrt{5}$$

$$\theta = \tan^{-1}(-1/-2) - \pi$$
$$= -2.0344 \quad \text{radians}$$
$$= -0.6476\pi \quad \text{radians}$$



Example

- Let $$z_1 = 1 - j2$$ $$z_2 = 2 + j3$$
- Compute

$$z_1 + z_2$$
$$z_1 z_2$$
$$z_1 / z_2$$
$$z_1 z_1^*$$
$$z_1 - z_1^*$$

Logs

$$c = \log_b a \leftrightarrow b^c = a$$

- Most common in signals and systems
  - Natural log $\ln = \log_e$ and log base 10 $\log_{10}$

Napier

- What is neat about logs?
  - Huge numbers → small positive numbers

$$\log_{10} \underbrace{10,345,034,896}_{\approx 10^{10}} = 10$$

  - Tiny numbers → small negative numbers
  - Multiplication → addition
  - Division → subtraction
- Used to compute amplifier gain, antenna gain, losses, etc.

$$\log xy = \log x + \log y$$

$$\log x^y = y \log x$$

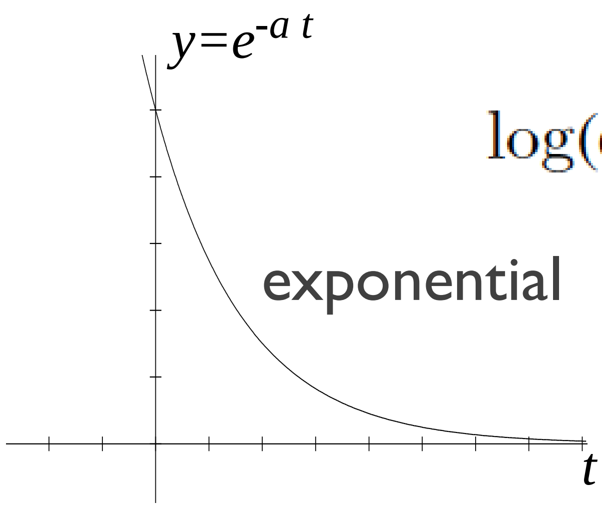
$$\log_b b = 1$$

$$\log_b 1 = 0$$

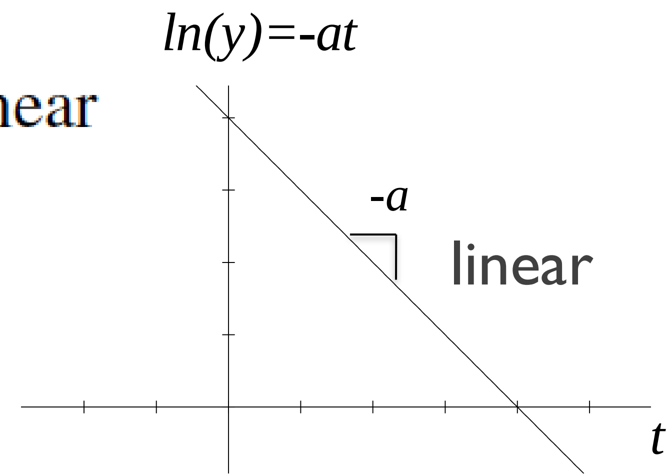
$$\log_b 0 = \text{undefined}$$

Relationship between logarithms and exponentials

$$\log a^x = x \log a$$



$$\log(\text{exponential}) \rightarrow \text{linear}$$



Application of logs in information theory/communications

noise signal

transmitted signal → + → received signal

Capacity of this communication channel is:

units are bits per second

$$C = B \log_2 \left(1 + \frac{P_{\text{signal}}}{P_{\text{noise}}}\right)$$

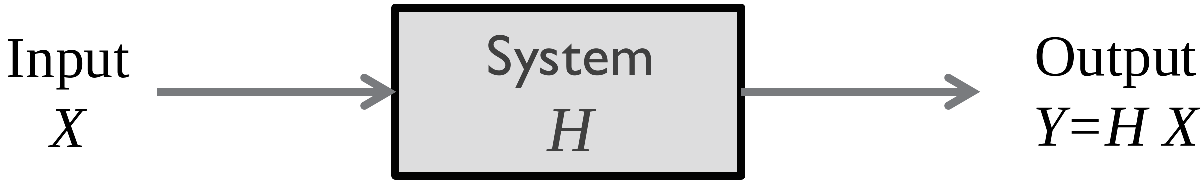
bandwidth of the communication channel

signal-to-noise ratio (SNR)

Claude Shannon (Bell Labs)

Decibel: describing system gain

- The Decibel (Deci = ten, bel = "Bell Labs")



  - If H > 1 → "gain"
  - If H < 1 → "attenuation" (also called gain)

$$H = \frac{Y}{X}$$

$$H(\mathrm{dB}) = 20\log_{10}\left|\frac{Y}{X}\right| = 10\log_{10}\left|\frac{Y}{X}\right|^2$$

$$10\log_{10} 10 = 10\ \mathrm{dB}$$
$$10\log_{10} 1 = 0\ \mathrm{dB}$$
$$10\log_{10} 2 = 3\ \mathrm{dB}$$
$$10\log_{10} 0.5 = -3\ \mathrm{dB}$$

decibels are a unitless ratio of **powers**

Example dB calculation

- Determine the gain (in dB) of the an amplifier with linear gain

- H = 20

$$\begin{aligned} 20\log_{10}20 &= 20\log_{10}2(10) \\ &= 2\cdot 10\log_{10}2 + 2\cdot 10\log_{10}10 \\ &= 26\text{dB} \end{aligned}$$

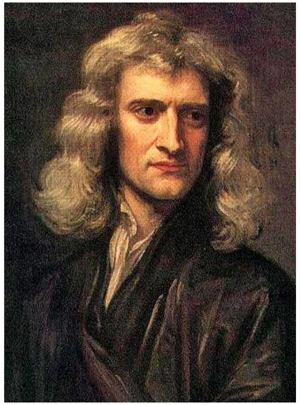
- H = 0.1

$$20\log_{10}0.1 = 20(-1) = -20\text{dB}$$

- H = 50

$$\begin{aligned} 20\log_{10}50 &= 20\log_{10}5(10) \\ &= 20\log_{10}5 + 20\log_{10}10 \\ &= 14\text{dB} + 20\text{dB} \\ &= 34\text{dB} \end{aligned}$$

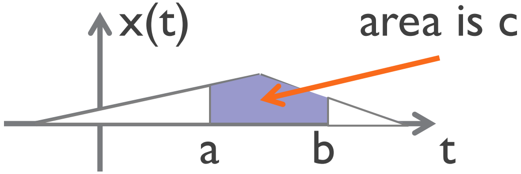
Integration



Newton

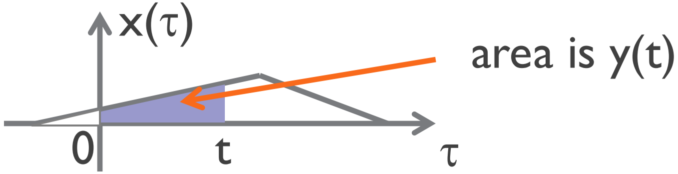
- Integrals are an essential part of calculus
  - Computes the area under a curve
  - “Opposite” of the derivative operation
- Most integrals in this course are definite integrals

$$c = \int_a^b x(t)dt$$



- Sometimes the limits will be a variable, not a fixed number

$$y(t) = \int_0^t x(\tau)d\tau$$



Integration example 1

$$\begin{aligned}\int_0^\infty e^{-2t}dt &= \frac{1}{-2}e^{-2t}\Big|_0^\infty \\ &= \frac{1}{-2}(0 - e^{-2\cdot 0}) \\ &= \frac{1}{2}\end{aligned}$$

Integration example 2

Trig Identities

$$\int_0^3 \cos(\pi t)\sin(\pi 2t)dt = \frac{1}{2}\int_0^3 \sin((2\pi-\pi)t) + \sin((2\pi+\pi)t)dt$$

$$\text{since } \cos A \sin B = \frac{1}{2}\left[\sin(B-A) + \sin(B+A)\right]$$

$$= \frac{1}{2}\int_0^3 \sin(\pi t) + \sin(3\pi t)dt$$

$$= \frac{-1}{2\pi}\cos(\pi t)\Big|_0^3 + \frac{-1}{2\cdot 3\pi}\cos(3\pi t)\Big|_0^3$$

$$= \frac{1}{2\pi}(\cos(0) - \cos(3\pi)) + \frac{1}{3}(\cos(0) - \cos(3\pi))$$

$$= \frac{1}{2\pi}(1 + 1 + \frac{1}{3}(1+1)) = \frac{4}{3\pi}$$

Integration example 3

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

$$\int_0^\infty te^{-t}dt = t(-1)e^{-t}\Big|_0^\infty - \int_0^\infty (-1)e^{-t}dt$$

Integration by parts

$$= -\lim_{t\to\infty} te^{-t} - e^{-t}\Big|_0^\infty$$

$$= -\lim_{t\to\infty} \frac{t}{e^t} + 1$$

L'Hopital's rule

$$= 0 + 1$$

$$= 1$$

Geometric sum

- For discrete-time, will deal with sums instead of integrals

- Infinite geometric sum

$$\sum_{n=0}^{\infty} a^n = \begin{cases} \frac{1}{1-a} & |a| < 1 \\ \infty & \text{otherwise} \end{cases}$$

- Finite geometric sum

$$\sum_{n=0}^{N} a^n = \begin{cases} \frac{1-a^{N+1}}{1-a} & a \neq 1 \\ N+1 & a = 1 \end{cases}$$

What is a “signal” and what is a “system”?

Learning objectives

- Explain what is meant by the term “signal” and give examples
- Explain what is meant by the term “system” and give examples

What is a "signal"?

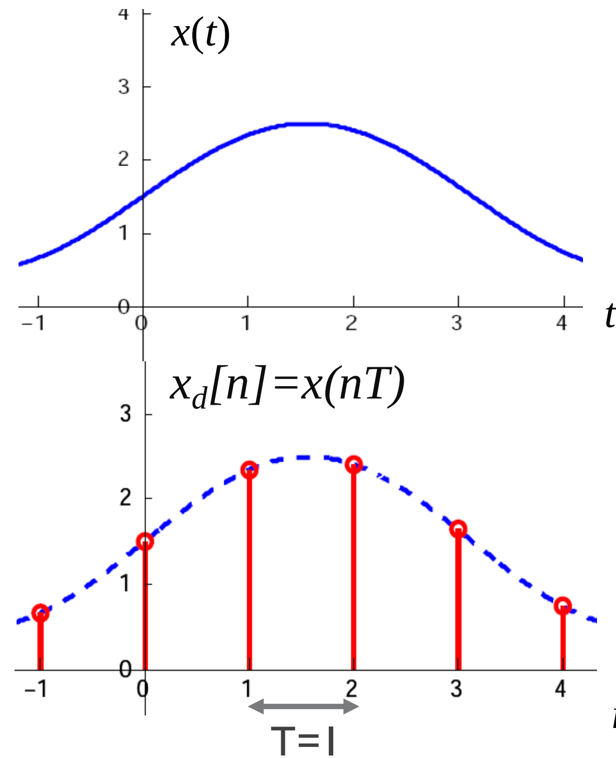
- Representation of a value/info. relative to an independent variable
  - Often a time variable but could be something else
- Continuous-time (CT) signals $x(t)$

Bracket notation used to denote a discrete-time sequence

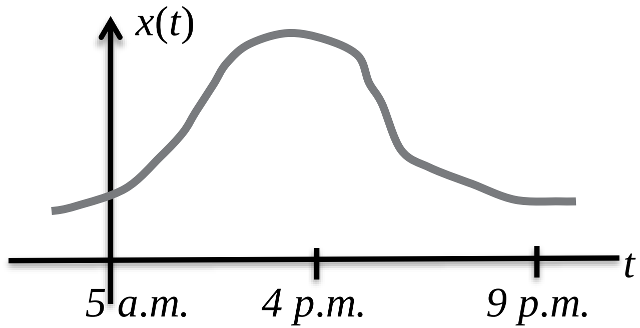
- Discrete-time (DT) signals $x[n]$
  - Often obtained by *sampling* CT signal
  - Taken at values nT

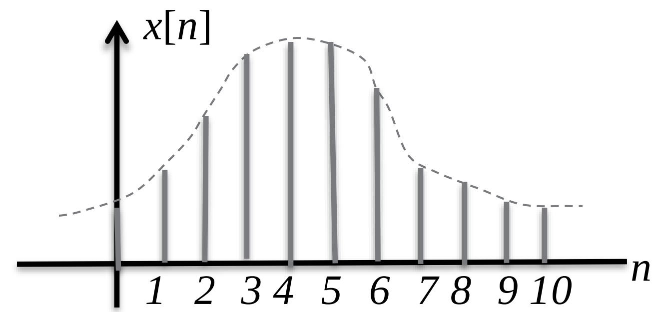
sampling period

$\cancel{x[n/2]}$



Signal example: Temperature





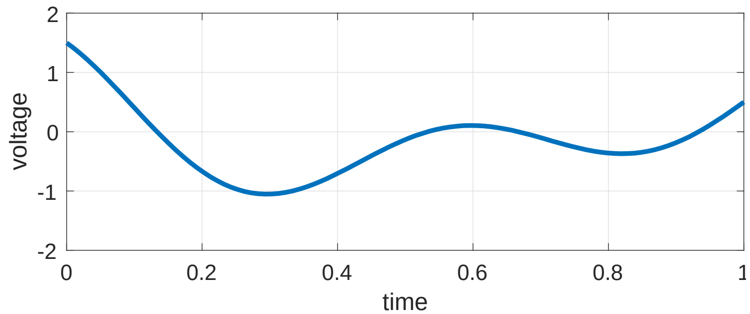
- Computing the average temperature

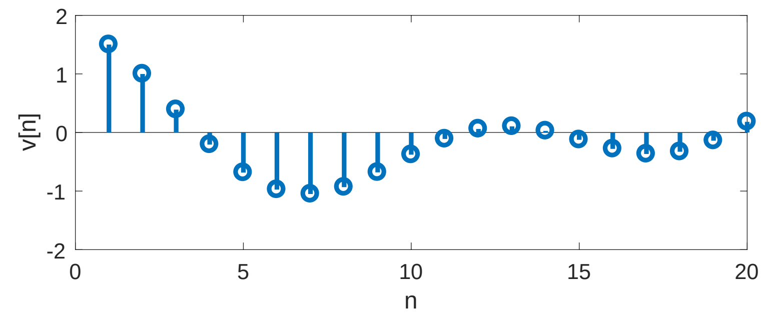
CT: $\bar{x} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t)dt$

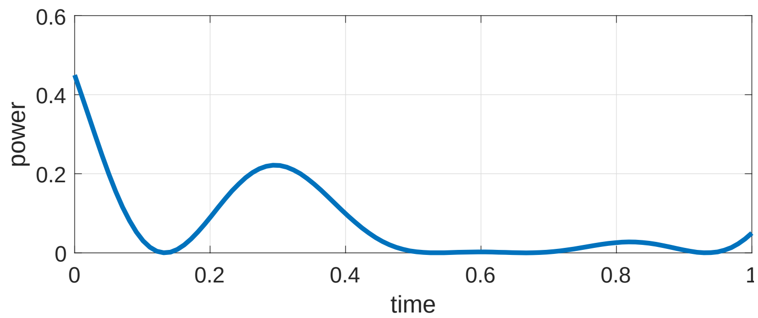
DT: $\bar{x} = \frac{1}{24} \sum_{n=0}^{23} x[n]$

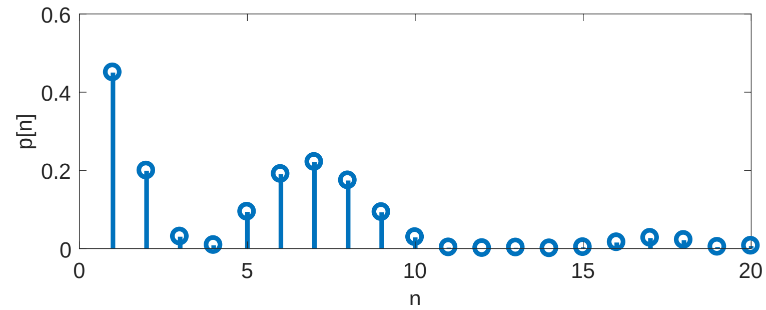
integral vs sum

Signal example: Voltage received by a cell phone antenna









$$P(t) = v^2(t)/R$$

$$v[n] = v(nT)$$
$$P[n] = p(nT)$$

(units not same)

Complex signals

- Signals generally take complex values (equivalently they have an amplitude and a phase)

$$x(t) = \mathrm{Re}\{x(t)\} + j\mathrm{Im}\{x(t)\} \qquad\qquad x[n] = \mathrm{Re}\{x[n]\} + j\mathrm{Im}\{x[n]\}$$

- Complex signals are found in many practical problems
  - Most digital signal processors support complex operations
- Example: "in phase" (real) and "quadrature" (imag) EM signals

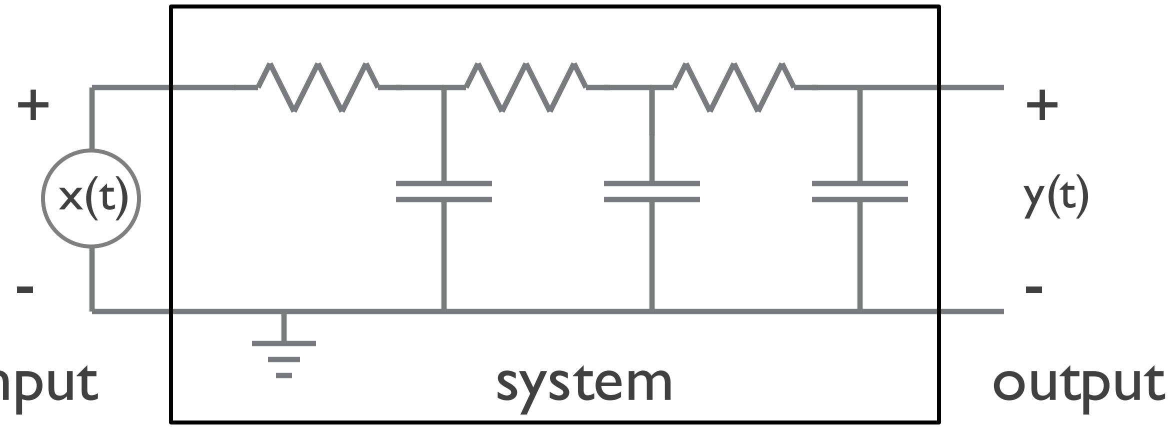
$$x(t) = \overbrace{\mathrm{Re}\{x(t)\}}^{\text{"rides" a cosine}} + j\underbrace{\mathrm{Im}\{x(t)\}}_{\text{"rides" a sine}}$$

Many connections between complex signals, Maxwell's equations, phasors, etc.

What is a "system"?

input $x(t) \longrightarrow$ system $\longrightarrow y(t)$ output

mathematical description of how the input is transformed into the output



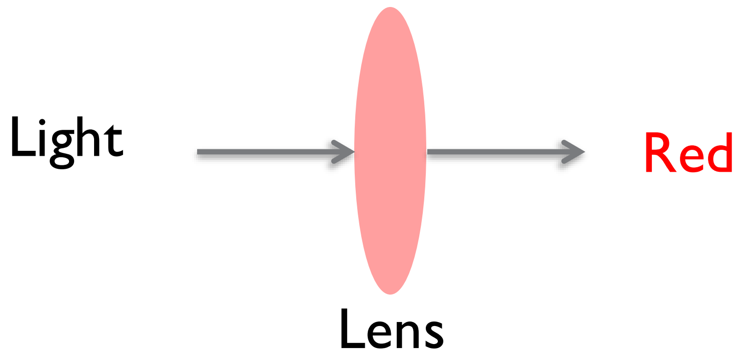
convolution

transfer function

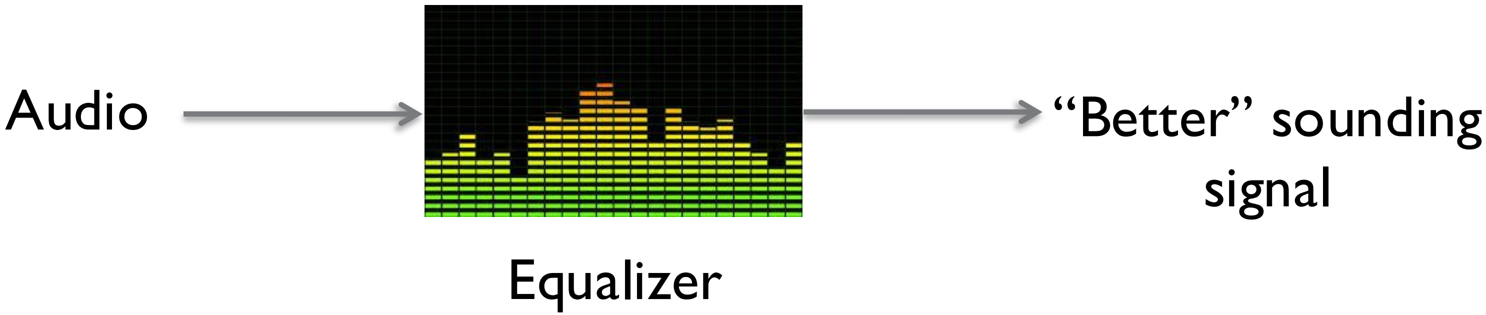
frequency response

System example: filters

- Optical filter

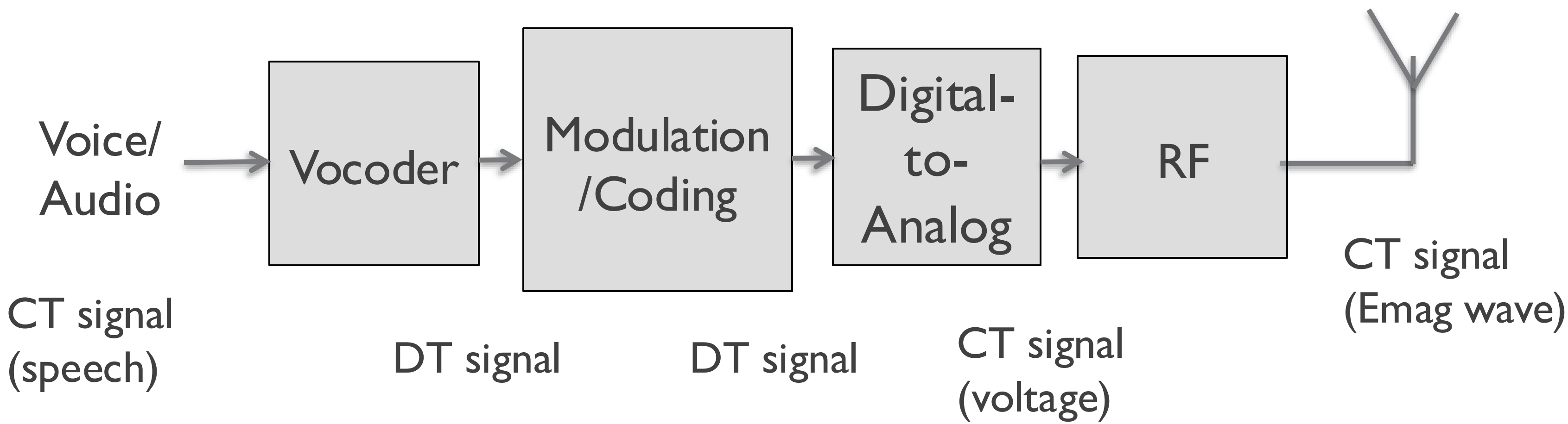


- Digital filter

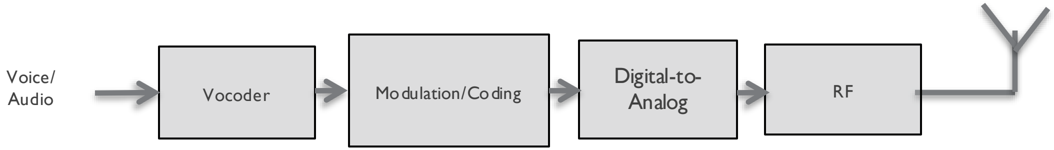


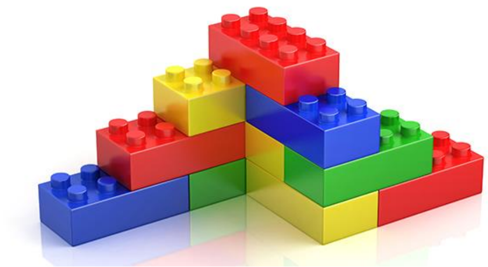
Systems can be complicated with many sub-systems

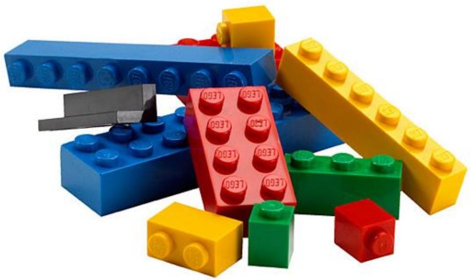
- Cell phone



What is "systems thinking"?







A complicated object decomposed into functional blocks that interact

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https://www.tibco.com/blog/2015/05/05/lego-learning-the-building-blocks-of-data-visualization/
http://www.brickowl.com/catalog/lego-city-corner-set-7641

In summary

- A signal is a function
  - Representation of a value as a function of some index (usually time)
  - May be in continuous or discrete time
  - Discrete-time often created from sampling continuous-time
  - Derived from measurements or models

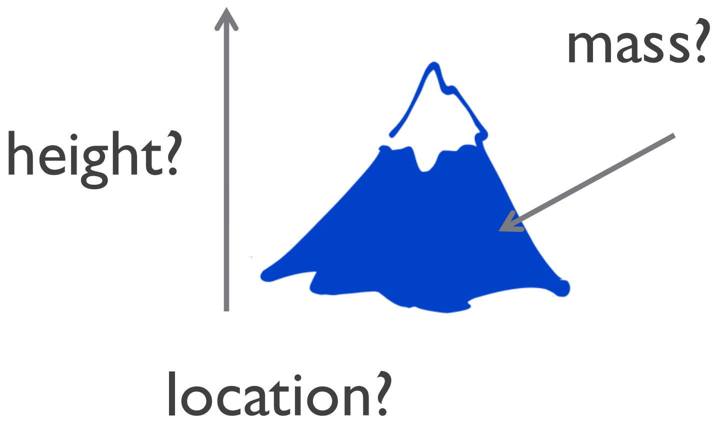
- A system is an operation on an input signal to generate an output
  - Abstracted by block diagrams
  - Each block represents a certain functions/process on its input
  - Characterized (later) based on their properties

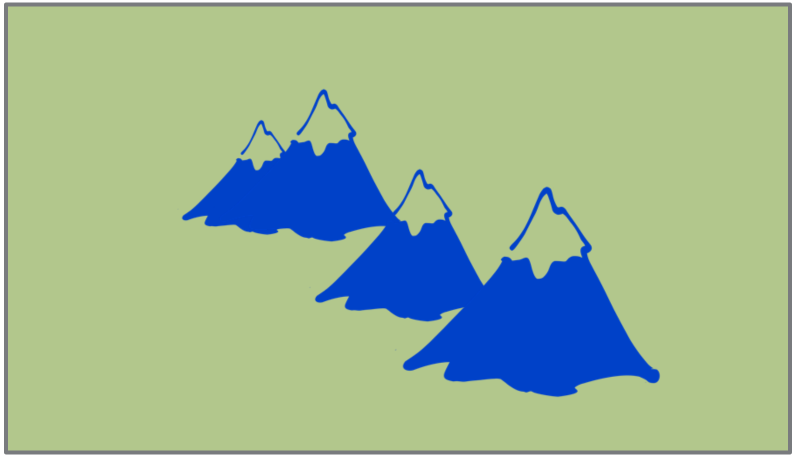
Basic signal transformations

Learning objectives

- Apply different transformations on continuous and discrete signals
- Create new continuous and discrete signals from these transformations

Moving mountains (function is over space here)



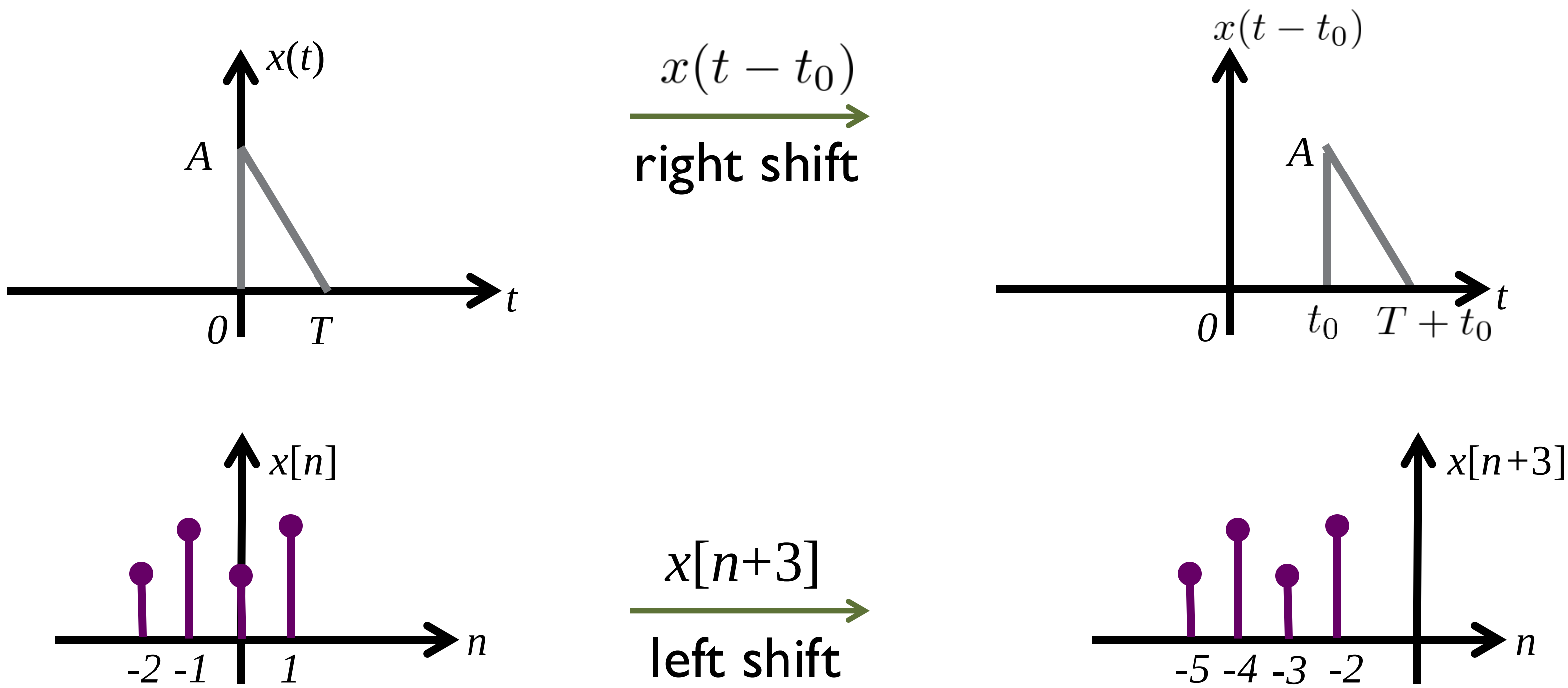


- Use mountain description to build a map
  - Where are the mountains located?
  - What are their size?

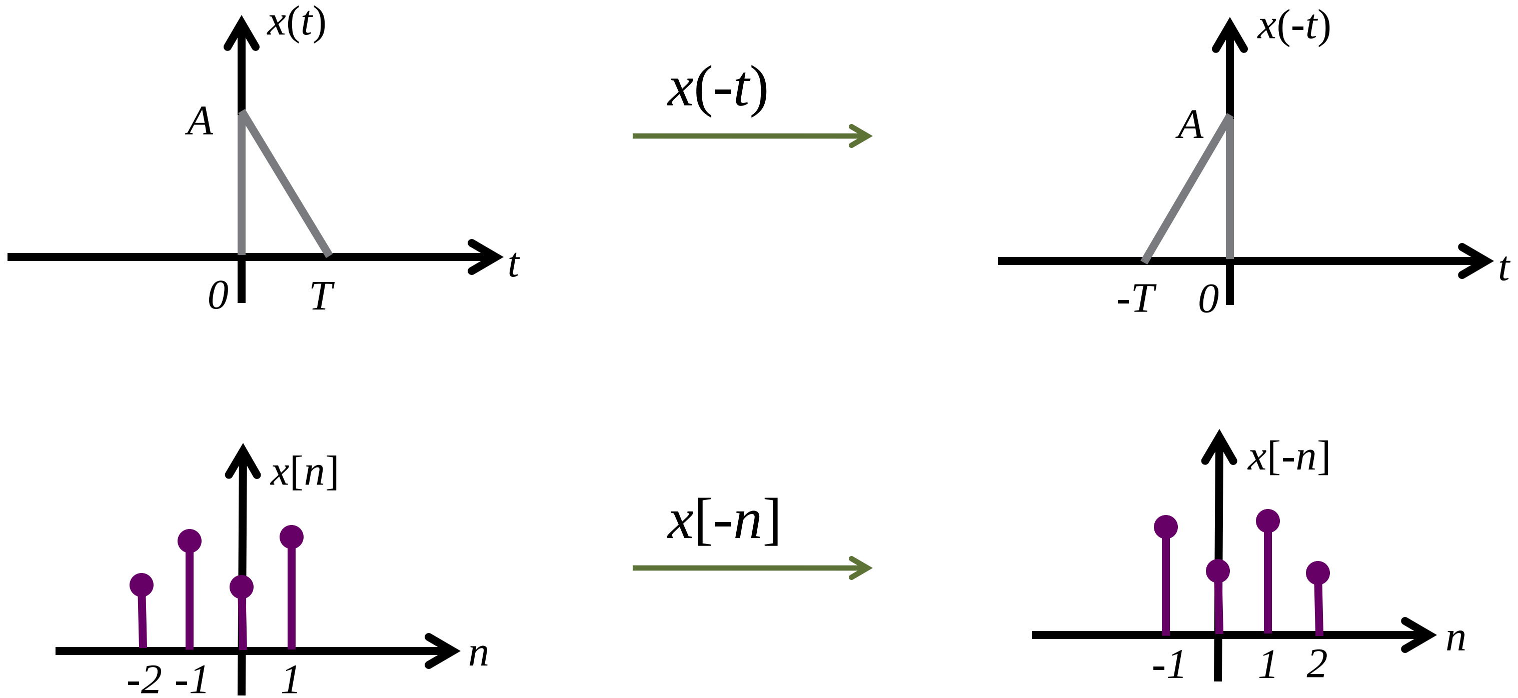
Signal transformations are ways to describe and manipulate signals

from clipartpanda.com

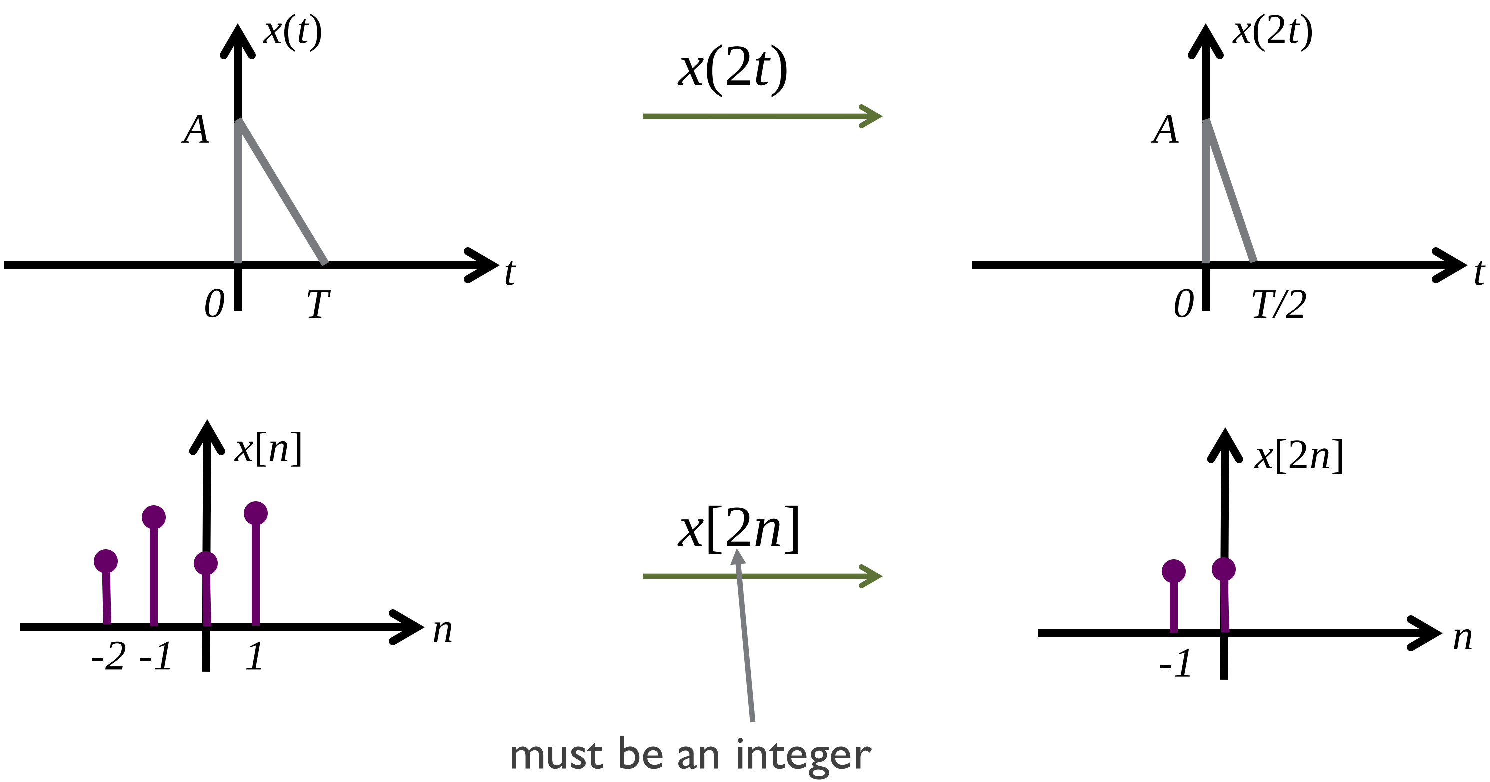
Example: time shift



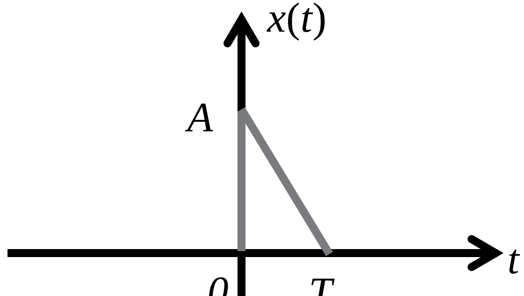
Example: reflection or time reversal



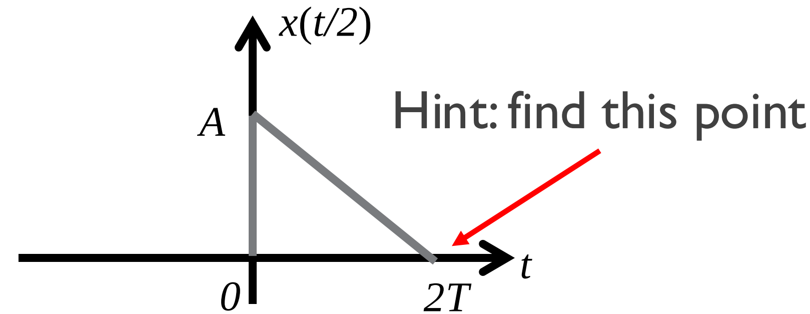
Example: time compression / downsampling

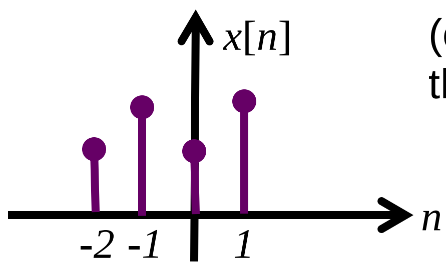


Example: time expansion (upsampling)

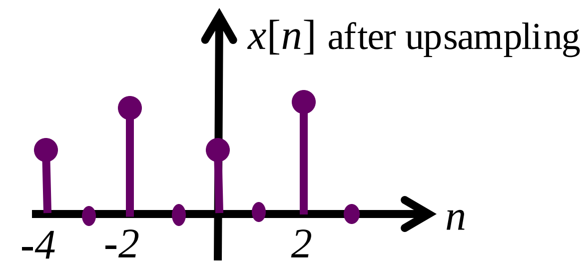


$x(t/2)$





(complicated operation, but this is the correct notion)



Example

- Let $x[n]$ be a signal with $x[n]=0$ for n < -2 and n > 4. For the signal below, determine the values of n for which it is guaranteed to be zero

$$x\left[-n-2\right]$$

- Solution

- Hint. Always do operations in this order:
  - Shift
  - Flip (or not)
  - Scale

Signal transformation summary

- There are different ways to transform a signal
  - This section focused on transformations of the independent variable

- Transformations
  - Time shifting changes the starting point of a signal
  - Time scaling changes how fast the signal is "played"
  - Time reversal flips a signal
  - Compression / expansion change the "speed" of a signal

Signal characteristics: periodic, even, and odd

Learning objectives

- Distinguish between periodic and aperiodic signals
- Compute the period of a periodic signal
- Compute even and odd parts of an arbitrary signal

Periodic signals

- Periodic signals satisfy for some finite non-zero T or N

$$x(t) = x(t+T)$$ periodic with period T for all t

$$x[n] = x[n+N]$$ periodic with period N (integer)

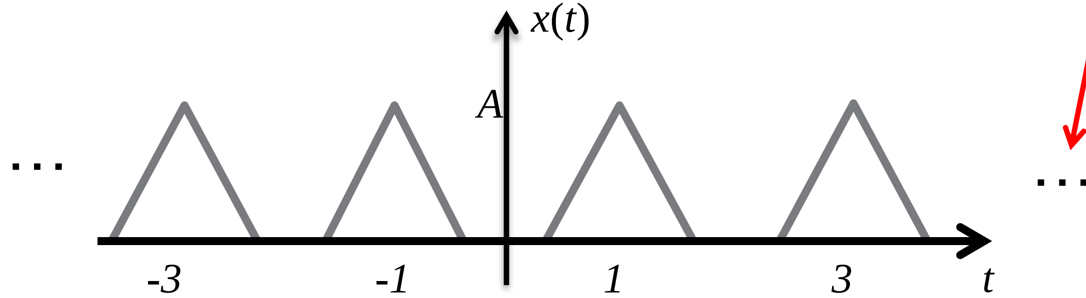
- Period is the smallest non-zero solution
  - This is called the fundamental period
  - Normally period means fundamental period

Periodic signals can be treated with special mathematical tools

Examples of periodic signals

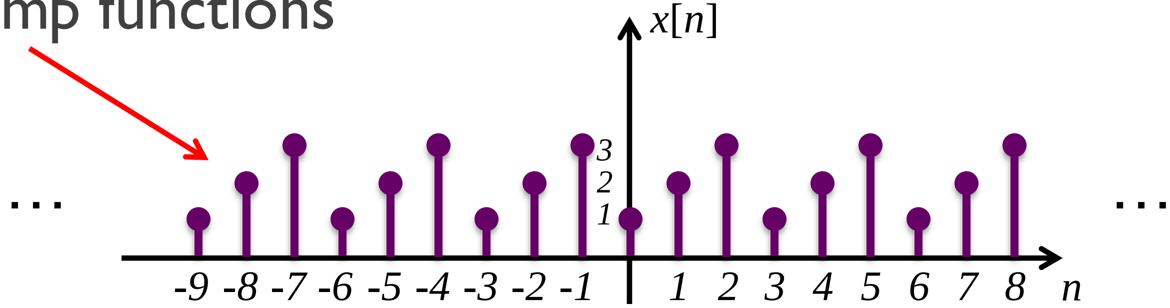
Dots indicate that the signal continues

pulse train of triangles



period?

train of ramp functions



period?

Establishing periodicity

- To prove a CT signal is periodic
  - Direct: find a $T>0$ such that $x(t)=x(t+T)$ for all t
  - Indirect: show that $x(t)$ is in a known class of periodic signals
- To prove a CT signal is aperiodic
  - Direct: Show that there is no $T>0$ such that $x(t)=x(t+T)$ for all t
- Procedure is similar for a DT signal but N must be an integer
- Remember the fundamental period is the smallest non-zero T or N

Example of direct method

- Determine whether or not the following signal is periodic? If it is periodic, determine its fundamental period.

$$x(t) = \sin(2t) + \cos(4t + \pi/2)$$

- Want to find T such that $x(t) = x(t+T)$

$$x(t+T) = \sin(2(t+T)) + \cos(4(t+T) + \pi/2)$$ shifted signal

$$\sin(2(t+T)) = \sin(2t) \text{ for } T = k\pi \text{ where } k \in \mathbb{Z}$$ multiple solutions

$$\cos(4(t+T) + \pi/2) = \cos(4t + \pi/2) \text{ for } T = k\pi/2 \text{ where } k \in \mathbb{Z}$$ multiple solutions

$$\boxed{T = \pi}$$ fundamental period

Example of indirect method

- Determine whether or not the following signal is periodic? If it is periodic, determine its fundamental period.

$$x(t) = [\cos(2t - \pi/3)]^2$$

- Recall that $\cos^2(x) = \frac{1}{2}(1 + \cos 2x)$

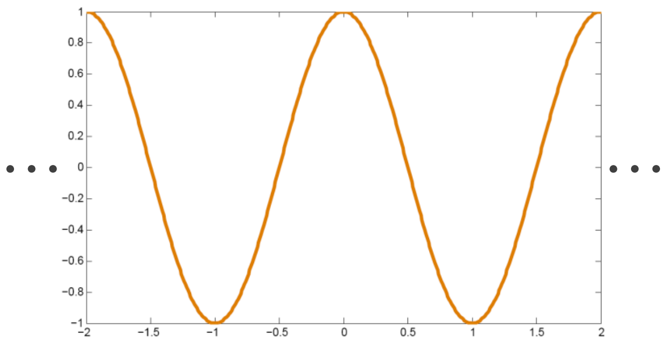
$x(t) = [1 + \cos(4t - 2\pi/3)]/2$. Periodic, period $= 2\pi/(4) = \pi/2$.

- By showing this simplifies to a known periodic function, we can also conclude it is period and find the frequency

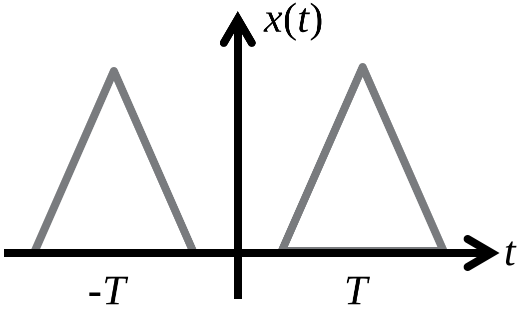
Related definition: even signals (real)

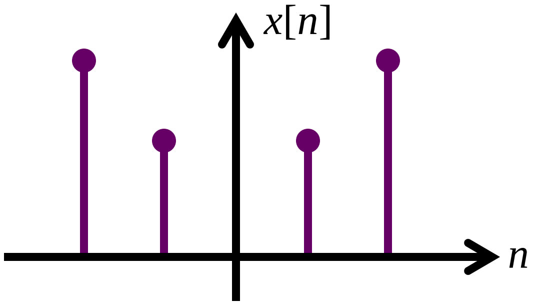
- Even signals satisfy $$x(-t) = x(t)$$ $$x[-n] = x[n]$$

- Examples (need not be periodic)



$\cos(2\pi f t)$





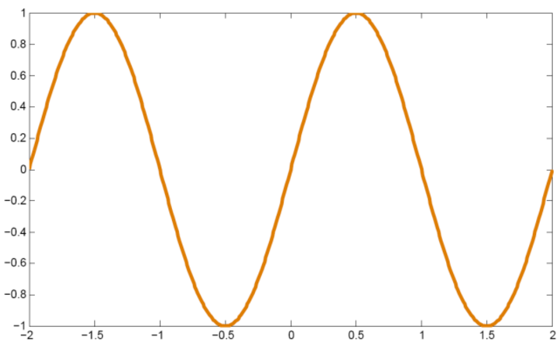
Related definition: odd signals (real)

- Odd signals satisfy

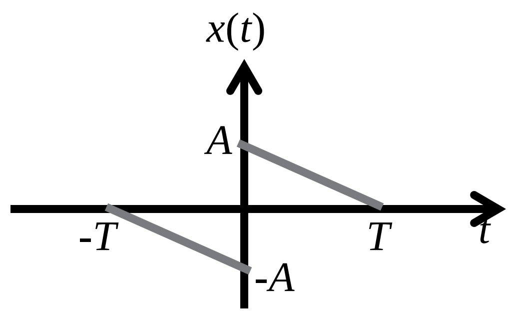
$$x(-t) = -x(t)$$

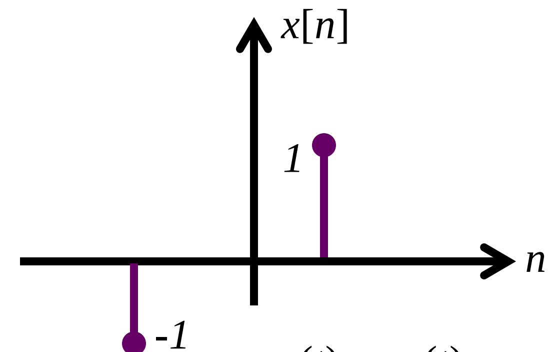
$$x[-n] = -x[n]$$

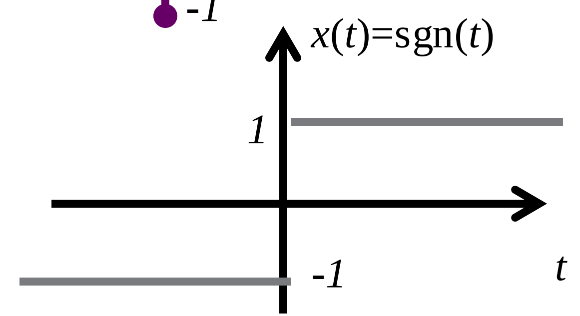
- Examples



$\sin(2\pi f t)$







Interesting facts about even and odd functions

- Any function can be written in terms of its even and odd parts

$$e(t) = \frac{1}{2}[f(t) + f(-t)]$$
$$o(t) = \frac{1}{2}[f(t) - f(-t)]$$
$$e(-t) = \frac{1}{2}[f(-t) + f(t)] = e(t)$$
$$o(-t) = \frac{1}{2}[f(-t) - f(t)] = -o(t)$$
$$f(t) = e(t) + o(t)$$

Interesting facts about even and odd functions

- Integration properties

$$\int_{-\infty}^{\infty} e(t)dt = 2\int_{0}^{\infty} e(t)dt$$

$$\int_{-\infty}^{\infty} o(t)dt = 0 = \int_{-T_0}^{T_0} o(t)dt$$

- Multiplications of two odd or two even functions → even function

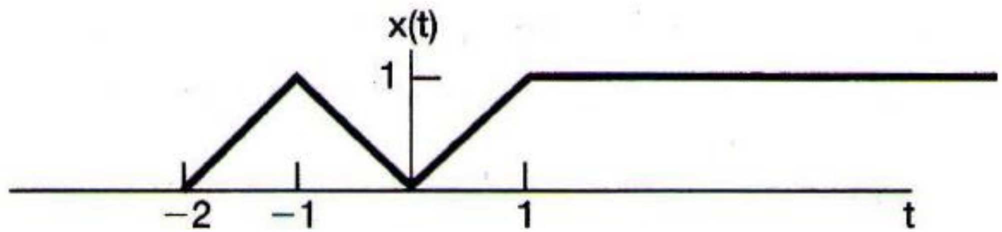
$$y(t) = o_1(t)o_2(t) = e(t)$$

$$y(t) = e_1(t)e_2(t) = \text{even function}$$

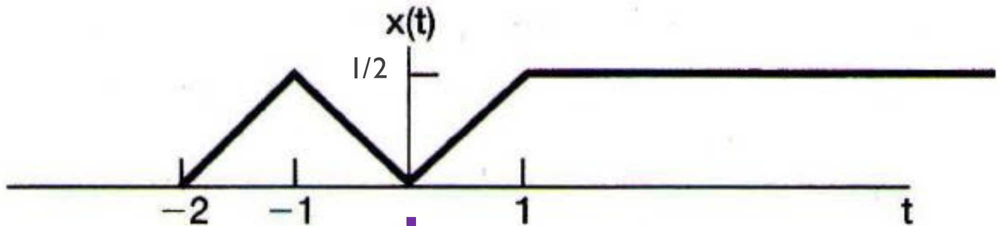
$$y(t) = o(t)e(t) = \text{odd function}$$

Example

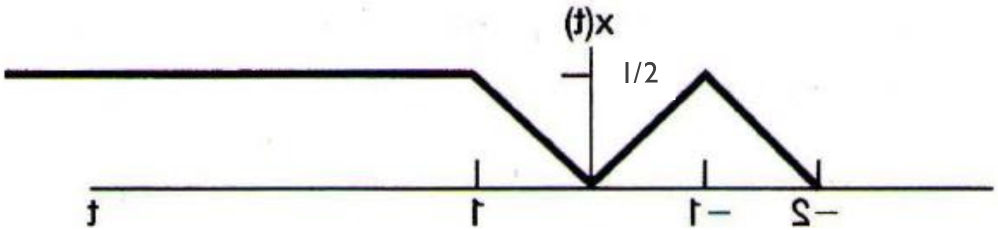
- Determine and sketch the even and odd parts of the signal



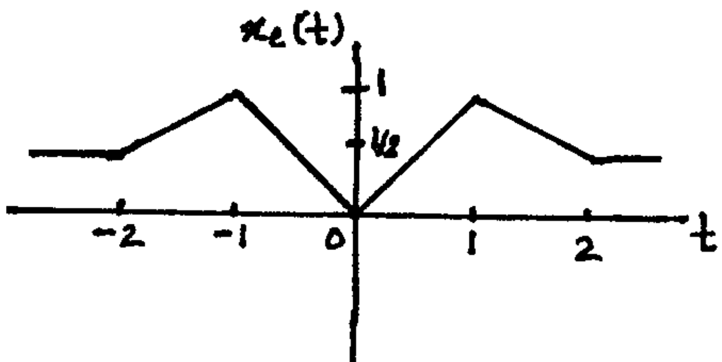
Sketching the even solution



plus

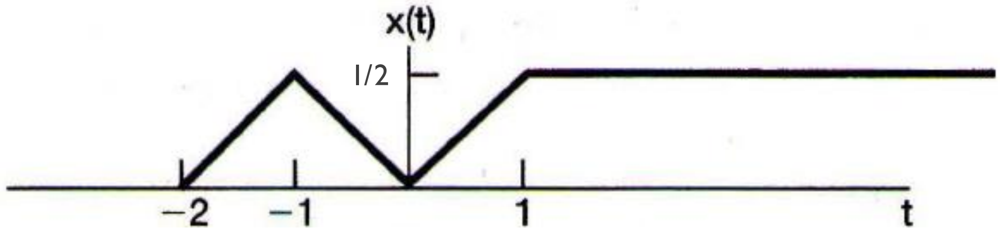


$e(t) = ½ (x(t) + x(-t))$

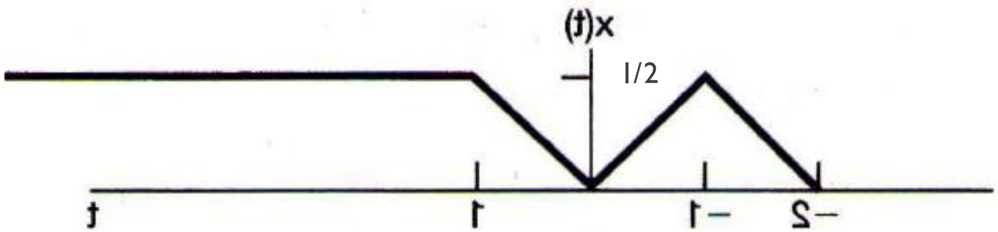


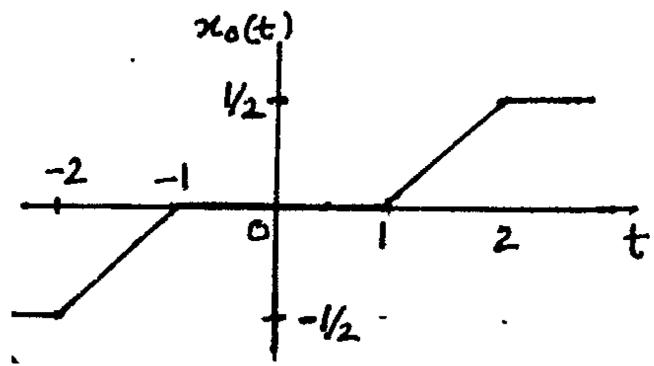
Sketching the odd solution

o(t) = ½ (x(t) - x(-t))



minus





Signal characteristics summary

- Periodic signals
  - Special type of signals that repeat
  - Need to determine if a signal is periodic and its period

- Even and odd signals
  - Signals with symmetry about the y axis
  - Fact used to simplify computations and derive intuition

- These special signal structures will be used in Fourier analysis

CT sinusoids and exponentials

Learning objectives

- Determine the key parameters of a complex exponential
- Sketch a complex exponential based on its form

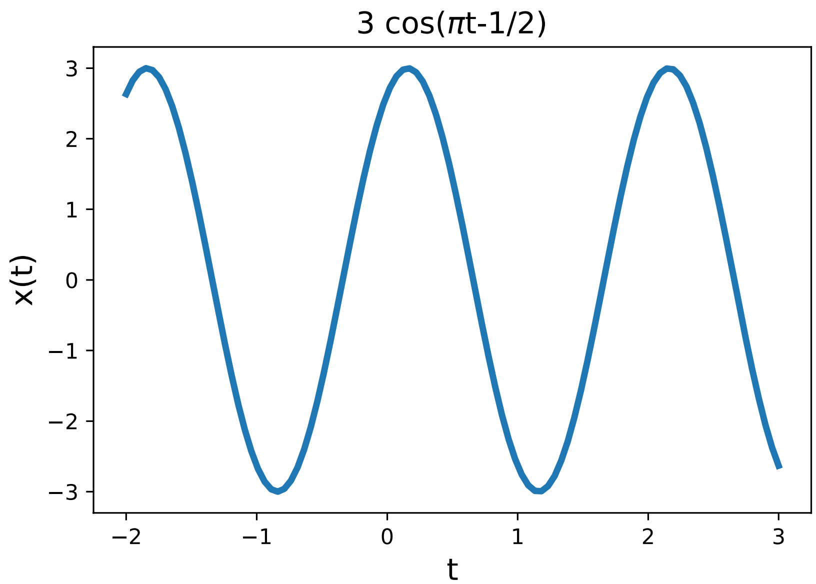
CT real sinusoid

amplitude > 0 phase

$$x(t) = C\cos(\omega_0 t + \theta)$$

frequency in radians/s

Periodic with period $T = \dfrac{2\pi}{\omega_0}$



$C = 3$

$\omega_0 = \pi$

$\theta = -1/2$

Complex numbers

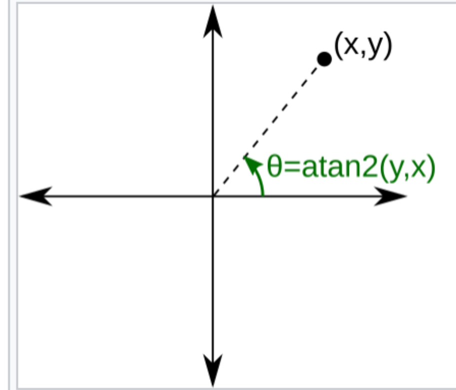
$$C = x + jy \qquad \text{Cartesian}$$

$$= |C|\, e^{j\theta} \qquad \text{Polar}$$

Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$C = \underbrace{|C|\cos(\theta)}_{\mathrm{Re}\left\{C\right\}} + j\underbrace{|C|\sin(\theta)}_{\mathrm{Im}\left\{C\right\}}$$



atan2(y, x) returns the angle θ between the ray to the point (x, y) and the positive x-axis, confined to ($-\pi$, π].

https://en.wikipedia.org/wiki/Atan2

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

CT complex sinusoid

complex frequency in radians/s

$$x(t) = Ce^{j\omega_0 t} \quad \text{period } T = \frac{2\pi}{\omega_0}$$

$$= |C|e^{j\theta}e^{j\omega_0 t}$$

$$= \boxed{|C|e^{j(\omega_0 t+\theta)}}$$

amplitude

phase

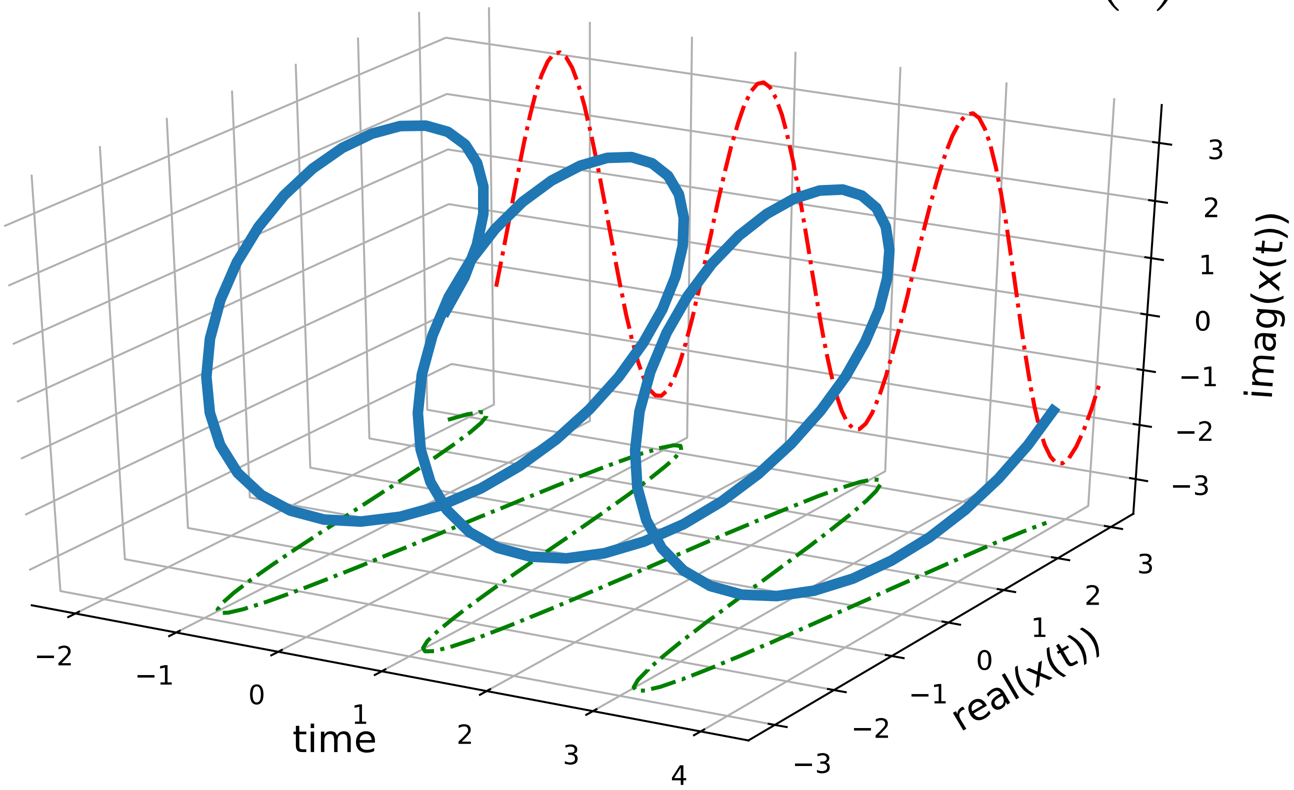
$$= |C|\cos(\omega_0 t + \theta) + j|C|\sin(\omega_0 t + \theta)$$

real and imaginary portions related through Eulers

Note: if the amplitude C is negative, then we could simply compensate for the negative by shifting the phase by pi

Visualizing a complex sinusoid

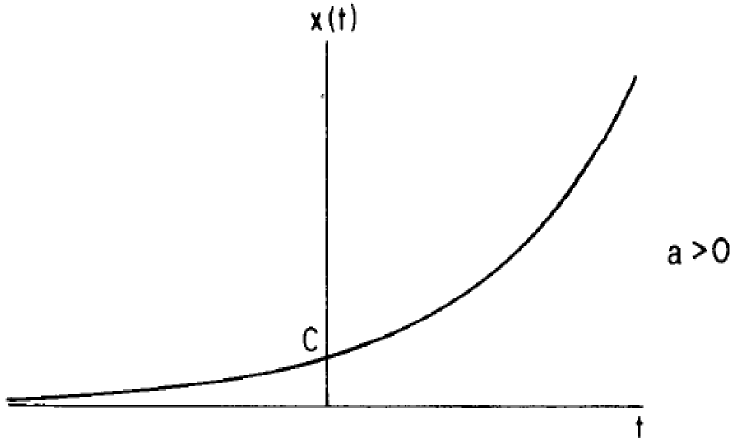
$$x(t) = 3e^{j(\pi t - \frac{1}{2})}$$



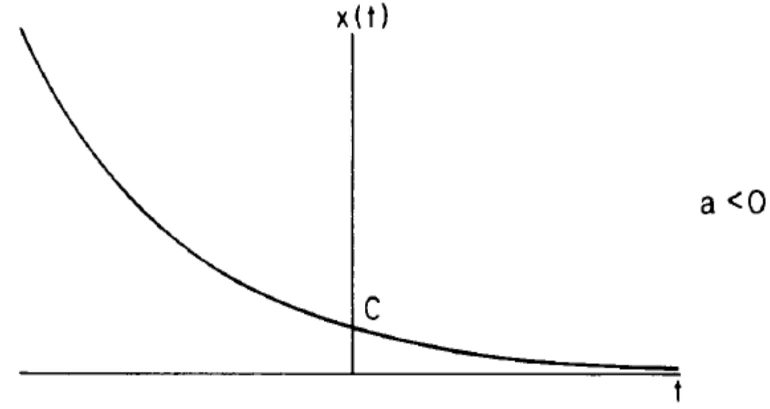
CT real exponential signal

$$x(t) = Ce^{at}$$

real   real



Increasing exponential
(a > 0, C > 0)



Decaying exponential
(a < 0, C > 0)

CT complex exponential: general case

$$x(t) = Ce^{at}$$

complex complex

$$C = c_\sigma + jc_\omega, \quad \text{cartesian}$$

$$= |C|e^{j\theta}, \quad \text{polar}$$

$$a = r + j\omega_0, \quad \text{cartesian}$$

$$x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t}$$

$$\boxed{x(t) = |C|e^{rt}e^{j(\omega_0 t+\theta)}}$$

General case includes real exponential, real sinusoid, and complex sinusoid as special cases

Visualizing CT complex exponentials

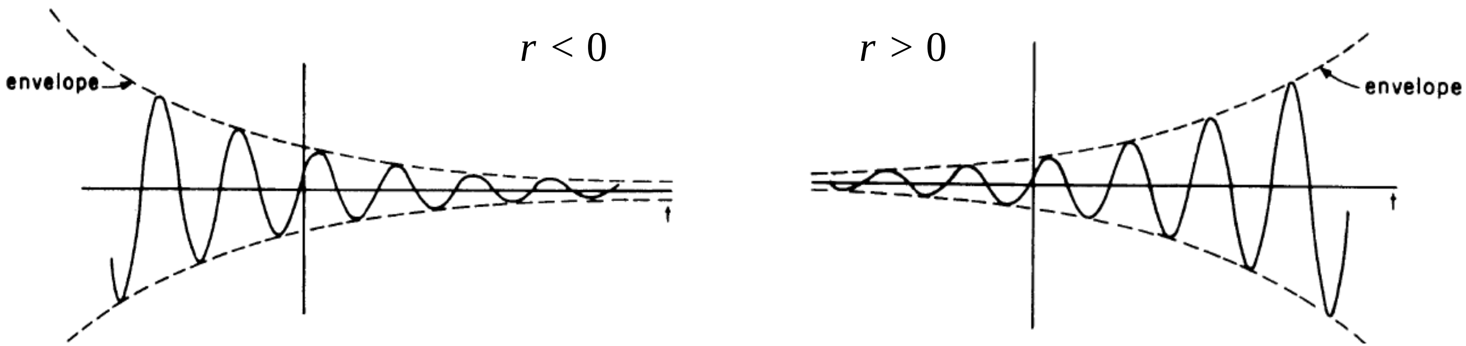
$$x(t) = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

Increasing or decaying exponential "envelope"

Complex sinusoid

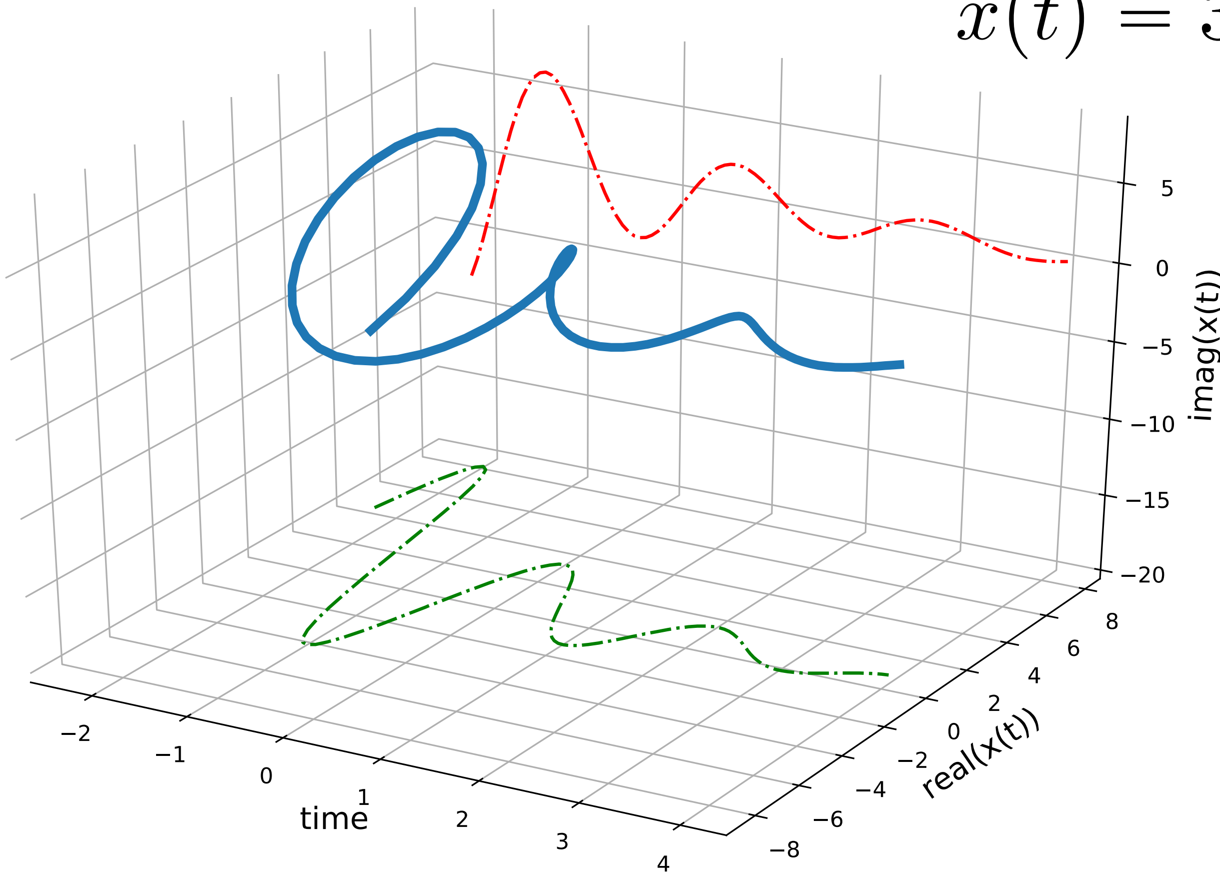
$$Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t + \theta)$$
$$Im\{x(t)\} = |C|e^{rt}\sin(\omega_0 t + \theta)$$

$$e^{rt} \text{ is the "damping" term} \longrightarrow \begin{cases} r > 0 \rightarrow & \text{blows up} \\ r < 0 \rightarrow & \text{decays to zero} \end{cases}$$



Visualizing CT complex exponentials in 3D

$$x(t) = 3e^{-t/2}e^{j(\pi t - 1/2)}$$



Example: Sketching a complex exponential

- Consider a complex exponential with the following values

$$x(t) = Ce^{at}$$

$$C = 3 - j, \;\; a = 1 + 10j$$

- The problem:
  - Express x(t) in terms of its envelope and complex sinusoidal parts
  - Express the real and imaginary parts of x(t)
  - Express the magnitude of x(t)
  - Plot all of these from time t = 0 to t = 3, showing the envelope

Solution I

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

- Convert $C = 3 - j$ to polar form

$$\begin{aligned} C &= \sqrt{3^1 + 1} e^{j \operatorname{atan2}(-1,3)} \\ &= \sqrt{10} e^{-j0.3218} \end{aligned}$$

- Substitute to get the general form (recall $a = 1 + 10j$)

$$\begin{aligned} x(t) &= Ce^{at} \\ &= \sqrt{10} e^{-j0.3218} e^{t(1+10j)} \\ &= \sqrt{10} e^{t} e^{j(10t - 0.3218)} \end{aligned}$$

See more about the two argument arc tangent here https://en.wikipedia.org/wiki/Atan2

Solution 2

- Apply Euler's to get real and imaginary parts:

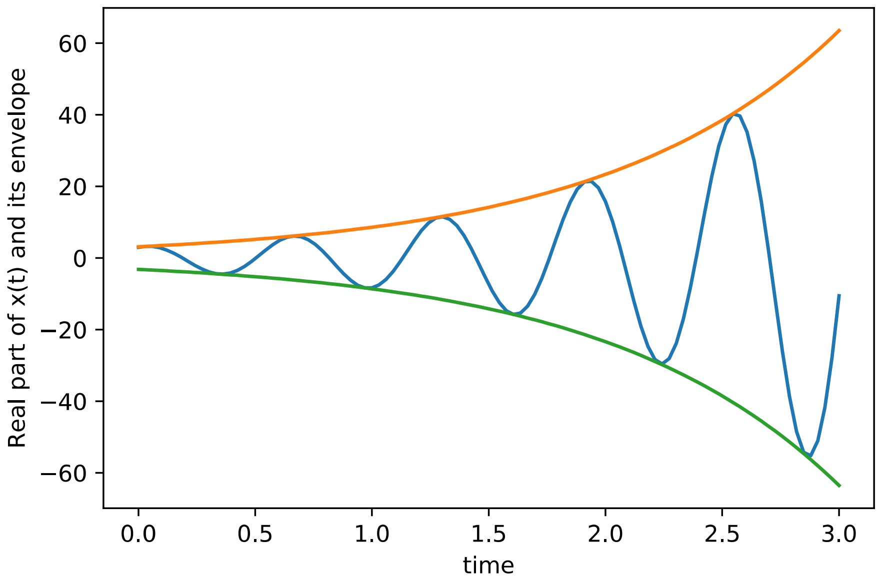
$$Re\{x(t)\} = \sqrt{10}e^t \cos(10t - 0.3218)$$
$$Im\{x(t)\} = \sqrt{10}e^t \sin(10t - 0.3218)$$

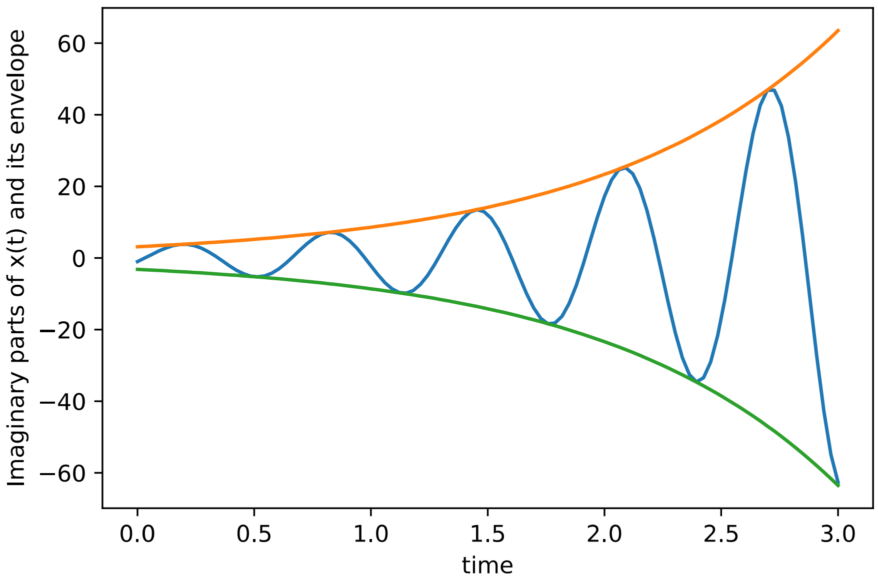
- Use the general form to find the magnitude

$$|x(t)| = \sqrt{10}e^t$$

- Sketch the signal behavior

Real and imaginary parts are just π/2 shifts (magnitude is just the positive envelope)





$$Re\{x(t)\} = \sqrt{10}e^t \cos(10t - 0.3218)$$

$$Im\{x(t)\} = \sqrt{10}e^t \sin(10t - 0.3218)$$

MATLAB to create these plots

```
C = 3-j;
a = 1+10j;
t = 0:.01:3; % equivalent to linspace (0,3,101)
x = C*exp(a*t);
env = abs(C)*exp(real(a) *t);

figure(1);
plot(t,real(x),t,env,t,-env);
xlabel('time');
ylabel('Real part of x(t) and its envelope');

figure(2);
plot(t,imag(x),t,env,t,-env);
xlabel('time');
ylabel('Imaginary parts of x(t) and its envelope');
```

CT exponential and sinusoidal signals in summary

- Complex exponentials and sinusoids
  - Important building blocks for future lectures
  - Can be understood through Euler's identity
  - Related to real exponentials and real sinusoids
  - Will become close friends by the end of the course ☺

- You should be able to
  - Identify complex exponentials and complex sinusoids
  - Determine the period of a complex sinusoid
  - Plot a complex exponential sketch and in Python

DT exponential and sinusoidal signals

Learning objectives

- Explain the properties of exponential and sinusoidal signals
- Analyze problems that include exponential and sinusoidal signals

DT real sinusoid

amplitude > 0 phase

$$x[n] = C\cos(\omega_0 n + \theta)$$

frequency in radians (not per second)

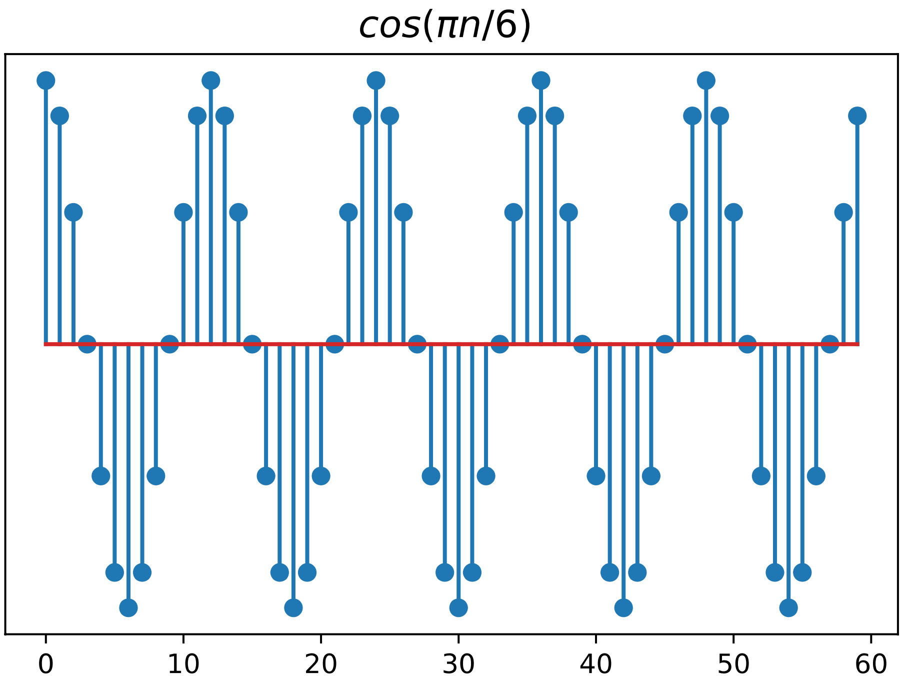
Before finding the period, consider an example

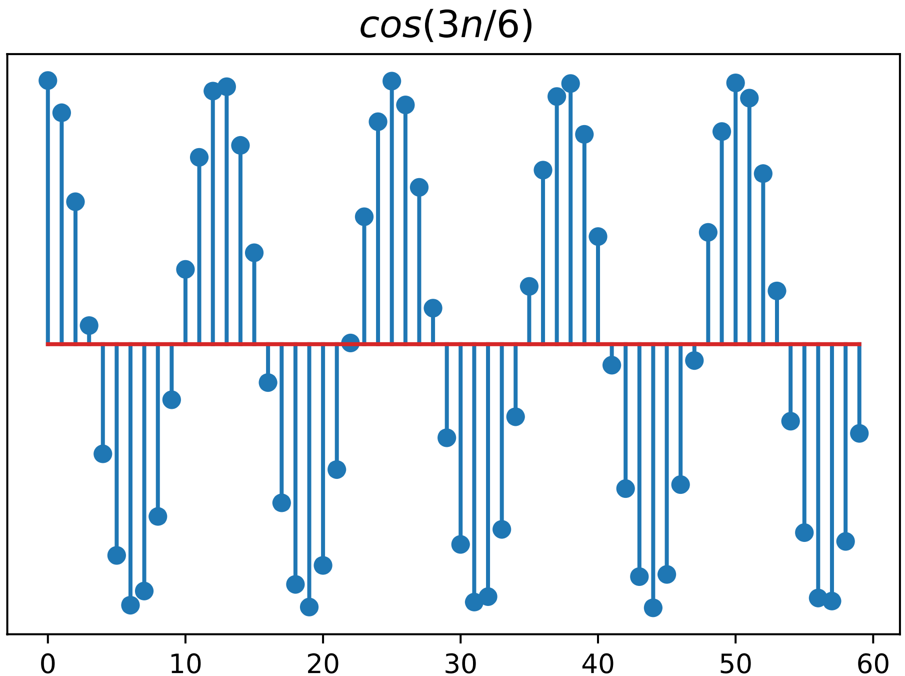
$$\cos\left(\frac{\pi}{6}n\right) \quad \text{vs} \quad \cos\left(\frac{3}{6}n\right)$$

Note: if the amplitude C is negative, then we could simply compensate for the negative by shifting the phase by pi

Periodicity is not guaranteed in discrete-time

Period of 12





periodic

not periodic

MATLAB code for reference

screenshot from MATLAB editor

```
%
n_range = 0:60;
omega_0 = pi/6;
omega_1 = 3/6;

figure(1);
stem(n_range, cos(omega_0*n_range),'LineWidth',3, 'MarkerSize',9);
title(['cos(\pi n /6 ) '],'FontSize',28);
set(gca,'ytick',[]);
print -f1 -depsc lecture3Fig1;

figure(2);
stem(n_range, cos(omega_1*n_range),'LineWidth',3, 'MarkerSize',9);
title(['cos(3 n /6 ) '],'FontSize',28);
set(gca,'ytick',[]);
print -f2 -depsc lecture3Fig2;
```

Python code for reference

screenshot from Spyder editor

```
# initializations
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl

# here are the main plots
n_range = np.arange(0,60,1) # produces 60 points, not exaclty same as MATLAB
omega_0 = np.pi/6
omega_1 = 3/6

plt.figure(1)
plt.stem(n_range, np.cos(omega_0*n_range),linewidth=3, markersize=9)
plt.title('$cos(\pi n /6 )$ ',fontsize=14) # dollar signs are used to make the pi with LaTeX
plt.yticks([]) # get rid of y axis labels and ticks for lecture purposes
plt.savefig('lecture3f1.eps',   # the file ending determs figure type
            bbox_inches='tight',# reduces the bounding box
            transparent=True)   # makes the figure transparent, good for powerpoint
plt.show()

plt.figure(2)
plt.stem(n_range, np.cos(omega_1*n_range),linewidth=3, markersize=9)
plt.title('$cos(3 n /6 )$ ',fontsize=14) # dollar signs are used to make the pi with LaTeX
plt.yticks([]) # get rid of y axis labels and ticks for lecture purposes
plt.savefig('lecture3f2.eps',   # the file ending determs figure type
            bbox_inches='tight',# reduces the bounding box
            transparent=True)   # makes the figure transparent, good for powerpoint
plt.show()
```

MATLAB code for reference

```
n_range = 0:60;
omega_0 = pi/6;
omega_1 = 3/6;

figure(2);
stem(n_range, cos(omega_0*n_range),'LineWidth',3, 'MarkerSize',9);
title(['cos(\pi n /6 ) '],'FontSize',28);
set(gca,'ytick',[]);
print -f2 -depsc lecture3Fig2;

figure(3);
stem(n_range, cos(omega_1*n_range),'LineWidth',3, 'MarkerSize',9);
title(['cos(3 n /6 ) '],'FontSize',28);
set(gca,'ytick',[]);
print -f3 -depsc lecture3Fig3;
```

Python code for reference

Python code

```
# here are the main plots
n_range = np.arange(0,60,1) # produces 60 points, not exaclty same as MATLAB
omega_0 = np.pi/6
omega_1 = 3/6

plt.figure(1)
plt.stem(n_range, np.cos(omega_0*n_range),linewidth=3, markersize=9)
plt.title('$cos(\pi n /6 )$ ',fontsize=14) # dollar signs are used to make the pi with LaTeX
plt.yticks([]) # get rid of y axis labels and ticks for lecture purposes
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plt.show()

plt.figure(2)
plt.stem(n_range, np.cos(omega_1*n_range),linewidth=3, markersize=9)
plt.title('$cos(3 n /6 )$ ',fontsize=14) # dollar signs are used to make the pi with LaTeX
plt.yticks([]) # get rid of y axis labels and ticks for lecture purposes
plt.savefig('lecture3f2.eps',   # the file ending determins figure type
            bbox_inches='tight',# reduces the bounding box
            transparent=True)   # makes the figure transparent, good for powerpoint
plt.show()
```

Explaining DT periodicity

- Find the period of

$$x[n] = C\cos(\omega_0 n + \theta)$$

- Need to find N such that

$$\begin{aligned} x[n+N] &= x[n] \\ &= C\cos(\omega_0(n+N)+\theta) \\ &= C\cos(\omega_0 n + \theta + \omega_0 N) \end{aligned}$$

Smallest such k that makes N an integer

$$\boxed{N = \frac{2\pi k}{\omega_0}}$$

- Must have $\omega_0 N = 2\pi k$ for some positive integer k (if possible!)

For arbitrary ω_0 a discrete-time sinusoid is not periodic

DT complex sinusoid

complex frequency in radians/s

$$
\begin{aligned}
x[n] &= Ce^{j\omega_0 n} \\
&= |C|e^{j\theta}e^{j\omega_0 n} \\
&= |C|e^{j(\omega_0 n+\theta)} \\
&= |C|\cos(\omega_0 n + \theta) + j|C|\sin(\omega_0 n + \theta)
\end{aligned}
$$

phase

amplitude

Periodic with period N for the smallest value of k such that $N = \dfrac{2\pi k}{\omega_0}$
(aperiodic if there is no such value of k)

Example

- Determine the fundamental period of the periodic signal

$$x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$$

- Solution

$$k\frac{2\pi}{4\pi/7} = k\frac{1}{2/7} \rightarrow 7 \qquad k\frac{2\pi}{2\pi/5} = k\frac{1}{1/5} \rightarrow 5$$

  periods 1

- Signal is a sum of three periodic signals
  - Need an integer multiple of each signal's period
  - To find N, find smallest integers a, b, c such that

$$N = a \cdot 1 = b \cdot 7 = c \cdot 5$$ 35

DT complex sinusoids – high & low frequency

- Observe the following fact for integers k and n (n is the "time")

$$\begin{aligned} e^{j((\omega_0+2\pi k)n+\theta)} &= e^{j(\omega_0 n+\theta+2\pi kn)} \\ &= e^{j(\omega_0 n+\theta)} e^{j2\pi kn} \rightarrow 1 \\ &= e^{j\omega_0 n+\theta} \end{aligned}$$

- This means that frequencies $\omega_0 + 2\pi k$ are equivalent!!
  - We normally report the smallest value as the frequency
  - Range $\omega_0 \in [0, 2\pi]$ low is near 0 or 2π , high is near π
  - Range $\omega_0 \in [-\pi, \pi]$ low is near 0, high is near π or $-\pi$

Illustration of DT frequency

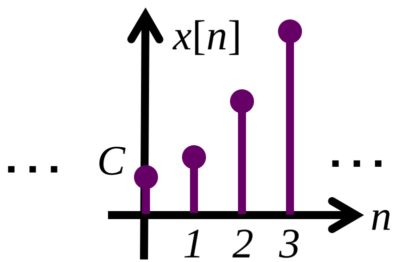
MATLAB

```
figure(1);
n_range = 0:60;
increment = 0.05;
omega = 2*pi*increment;
for k=1:100
    % Note extra formatting below to make fonts and lines bigger
    stem(n_range, cos(omega*k*n_range),'LineWidth',3, 'MarkerSize',9);
    xlabel('n');
    title(['cos(2\pi',num2str(increment*k,4), 'n)'],'FontSize',28);
    set(gca,'FontSize',14)
    pause; % used for lecture to show successive plots, press enter
end; %k
```

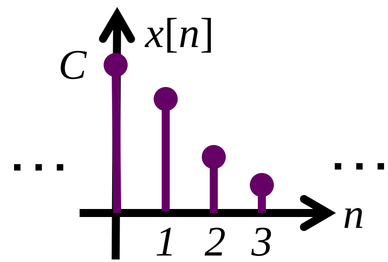
DT real exponential signal

$$x[n] = C\alpha^n, \; C \text{ and } \alpha \text{ are real}$$

$\alpha > 1$ C>0

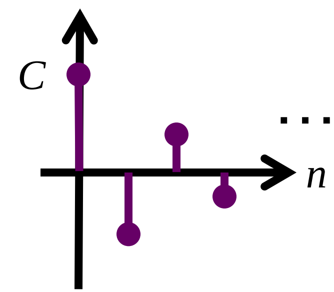


$0 < \alpha < 1$ C>0



…

$-1 < \alpha < 0$ C>0



DT exponentials are increasing or decreasing sequences, and may alternate

DT complex exponentials in general

- General form is $x[n] = C\alpha^n$

complex complex

- Rewriting using the polar form

$$x[n] = \underbrace{|C|e^{j\theta}}_{C}(\underbrace{Re^{j\omega_0}}_{\alpha})^n$$

$$= |C|R^n e^{j\theta}e^{j\omega_0 n}$$

$$= |C|R^n e^{j(\omega_0 n+\theta)}$$

- Note: $|C|R^n$ is complex envelope, ω_0 is frequency and θ is the phase

DT exponential and sinusoidal signals in summary

- DT complex sinusoids
  - Are only periodic for special values of the frequency
  - High and low frequency are not uniquely defined
  - Forms the foundation of DT signal analysis in the Fourier domain

- DT complex exponentials
  - Important building blocks for future lectures
  - Used for the solution of difference equations

- You should be able to
  - Determine the period of a DT complex sinusoid (if it has one)
  - Plot a DT complex exponential sketch on paper and in MATLAB

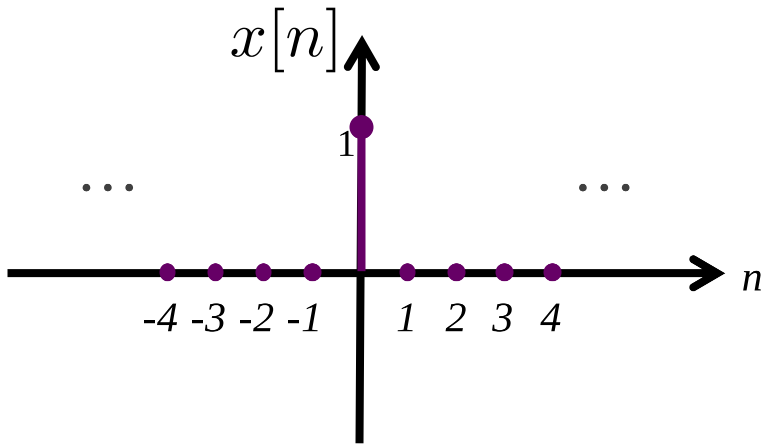
Discrete-time unit–step and unit-impulse functions

Learning objectives

- Explain the properties of unit-impulse and unit-step functions
- Understand how they can be used
- Analyze problems that include unit-impulse and unit-step functions

DT unit-impulse function

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



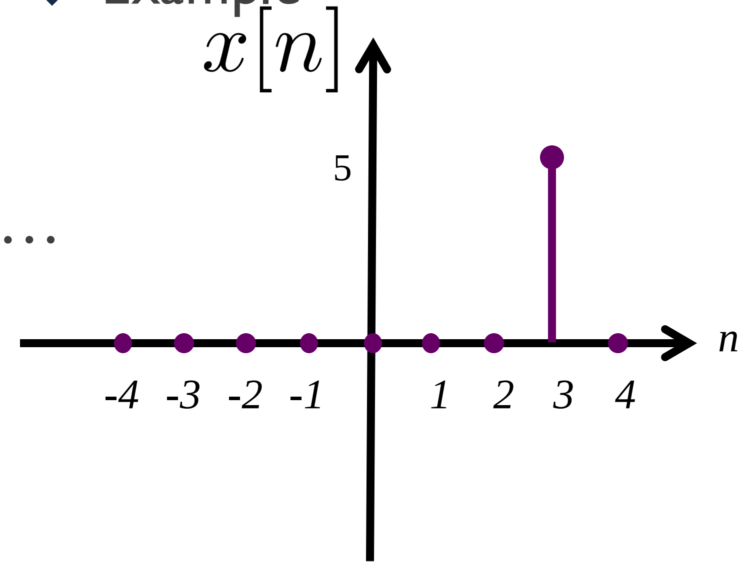
Also known as the Kronecker delta function

note

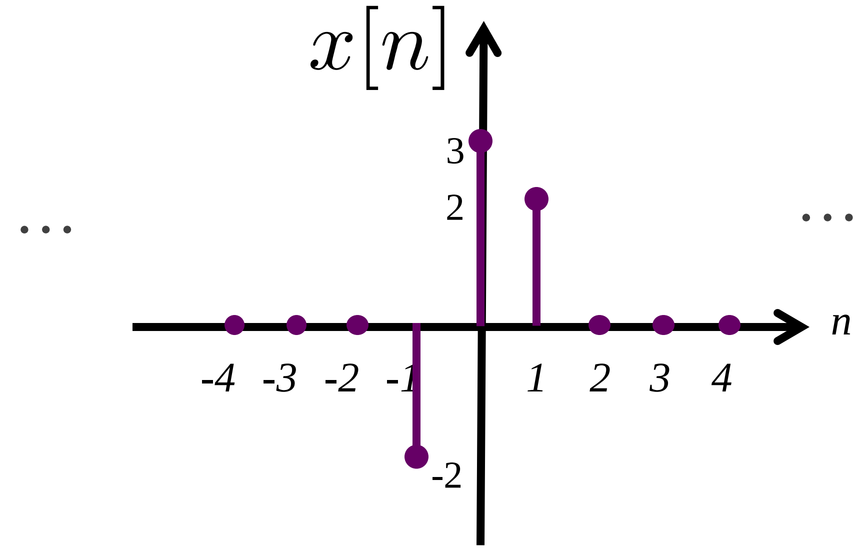
$$\sum_{n=-\infty}^{\infty} \delta[n] = 1$$

Can build any DT sequence using the unit-impulse

- Example

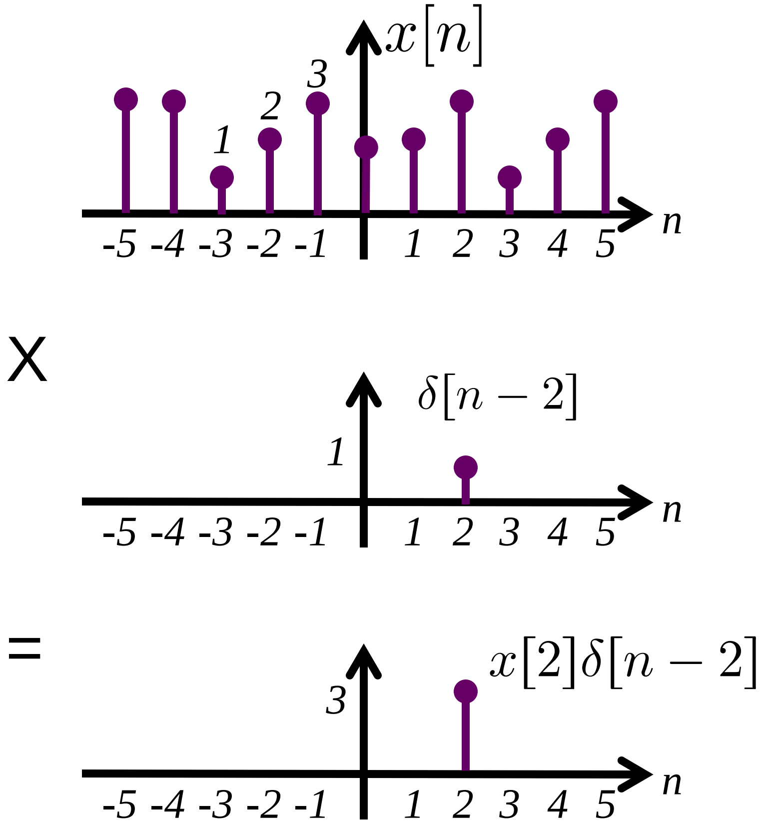


$$5\delta[n-3]$$



$$3\delta[n] - 2\delta[n+1] + 2\delta[n-1]$$

Sifting property



Sifting property

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

Examples

$$\boxed{x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]}$$

- Simplify the following expressions

$$x[n]\delta[n+1]$$

$$x[-1]\delta[n+1]$$

signal

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n+1]$$

$$\sum_{n=-\infty}^{\infty} x[-1]\delta[n+1] = x[-1]$$

value

$$(\cos(\pi n/4)+1)\delta[n-1]$$

$$(\cos(\pi 1/4)+1)\delta[n-1]$$

signal

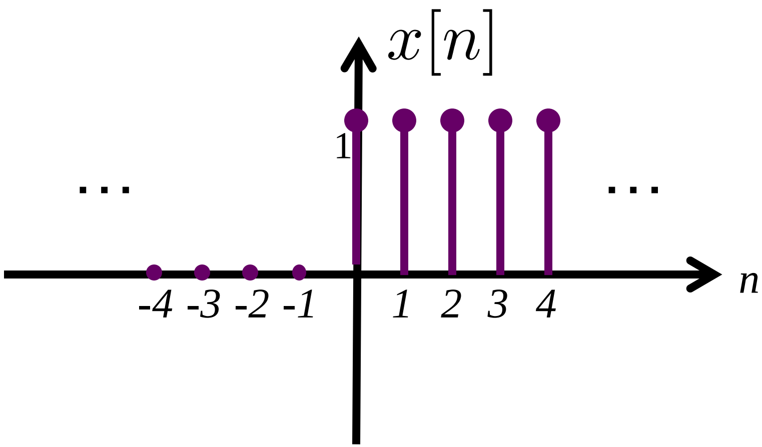
$$\sum_{n=-\infty}^{\infty} (\cos(\pi n/4)+1)\delta[n-1]$$

$$\sum_{n=-\infty}^{\infty} (\cos(\pi 1/4)+1)\delta[n-1] = (\cos(\pi 1/4)+1)$$

value

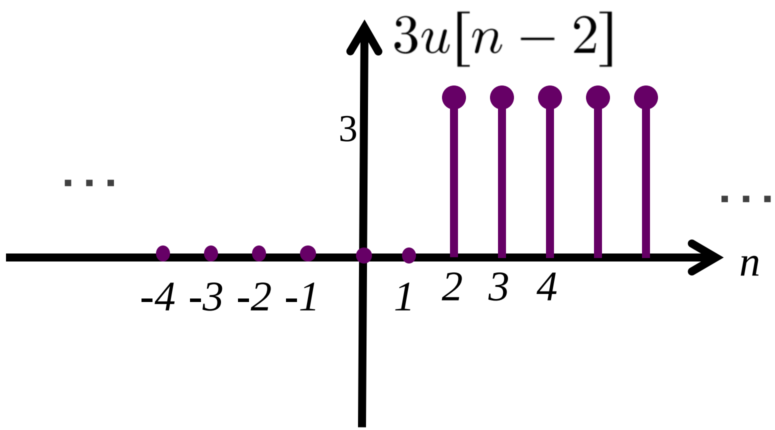
DT unit-step function

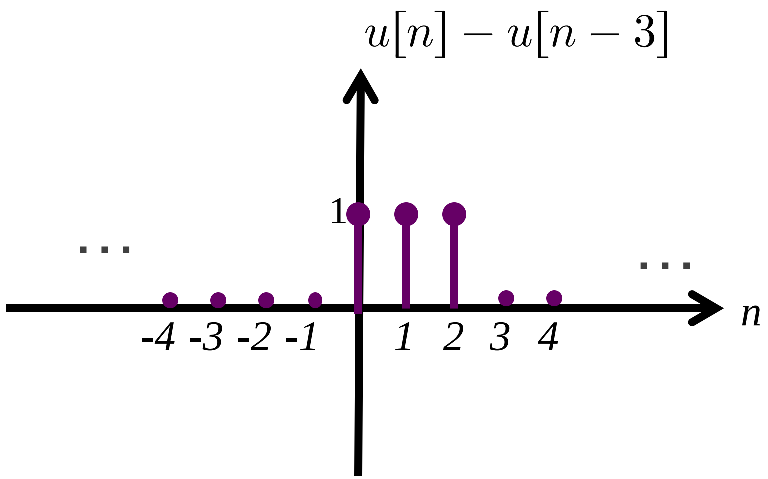
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



DT unit-step function

- Example





Note: $u[n]-u[n-3]=\delta[n]+\delta[n-1]+\delta[n-2]$

remember this trick for later

Using the sifting property with unit step functions

- Consider

$$\begin{aligned} x[n] &= \alpha^n u[n], \quad \alpha = 2 \\ &= 2^n u[n] \end{aligned}$$

- Simplify

$$\begin{aligned} y[n] &= x[n]\delta[n-10] \\ &= x[10]\delta[n-10] \\ &= 2^{10}u[10]\delta[n-10] \\ &= 1024 \cdot \delta[n-10] \end{aligned}$$

Connections between impulse and step functions

- Important relations to remember

$$\delta[n] = u[n] - u[n-1]$$

$$\delta[n-n_0] = u[n-n_0] - u[n-(n_0+1)]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

$$u[n] = \sum_{m=0}^{\infty} \delta[n-m]$$

Also

$$\delta[n]\delta[n] = \delta[n]$$

$$u[n]u[n] = u[n]$$

DT impulse and unit step signals in summary

- DT unit-impulse or Kronecker delta in important basic signal
  - Kronecker delta is a signal, i.e. a function of n
  - Sifting a signal with a Kronecker delta function gives another Kronecker delta function with a new signal value
  - Summing over a Kronecker delta extracts the value of the signal

- DT unit-step function
  - Another important building block signal
  - Can be built from Kronecker delta functions

- DT unit-impulse and unit-step can be written in terms of each other

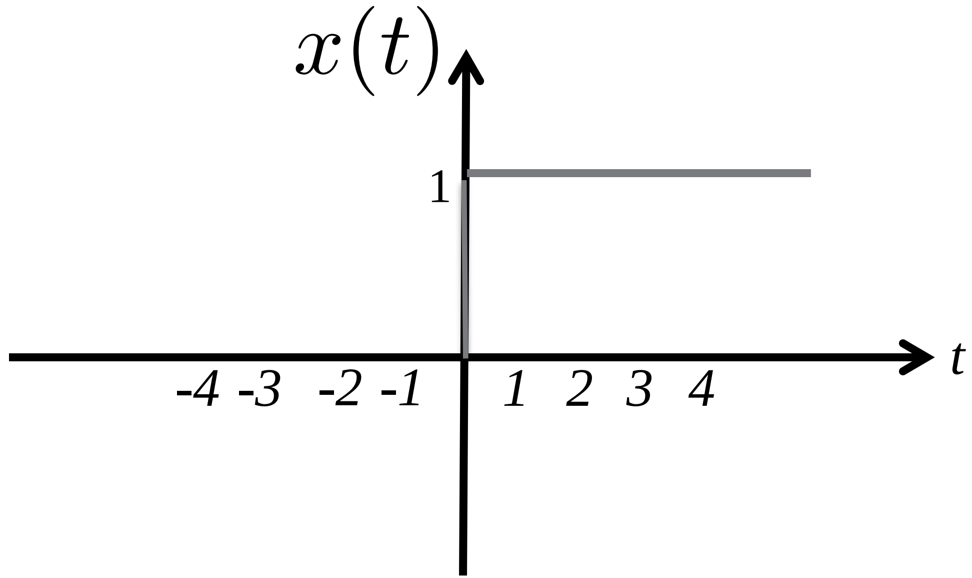
Unit–step and unit-impulse functions

Learning objectives

- Explain the properties of unit-impulse and unit-step functions
- Understand how they can be used
- Analyze problems that include unit-impulse and unit-step functions

CT unit step function

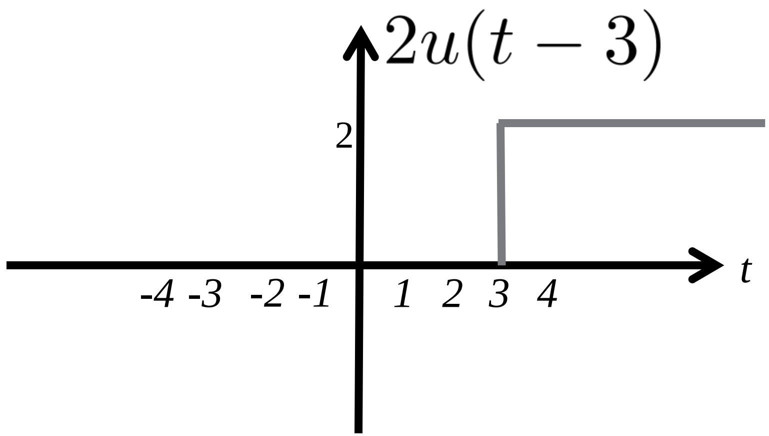
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

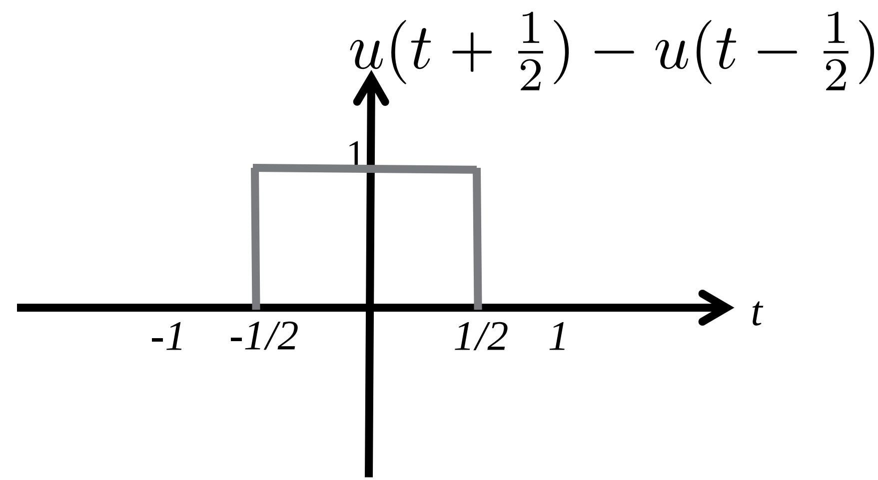


- At t=0, u(t) may be either 0, 1, or ½ depending on the book
  - The specific choice is only important in a mathematical analysis class

CT unit step function examples

- Examples





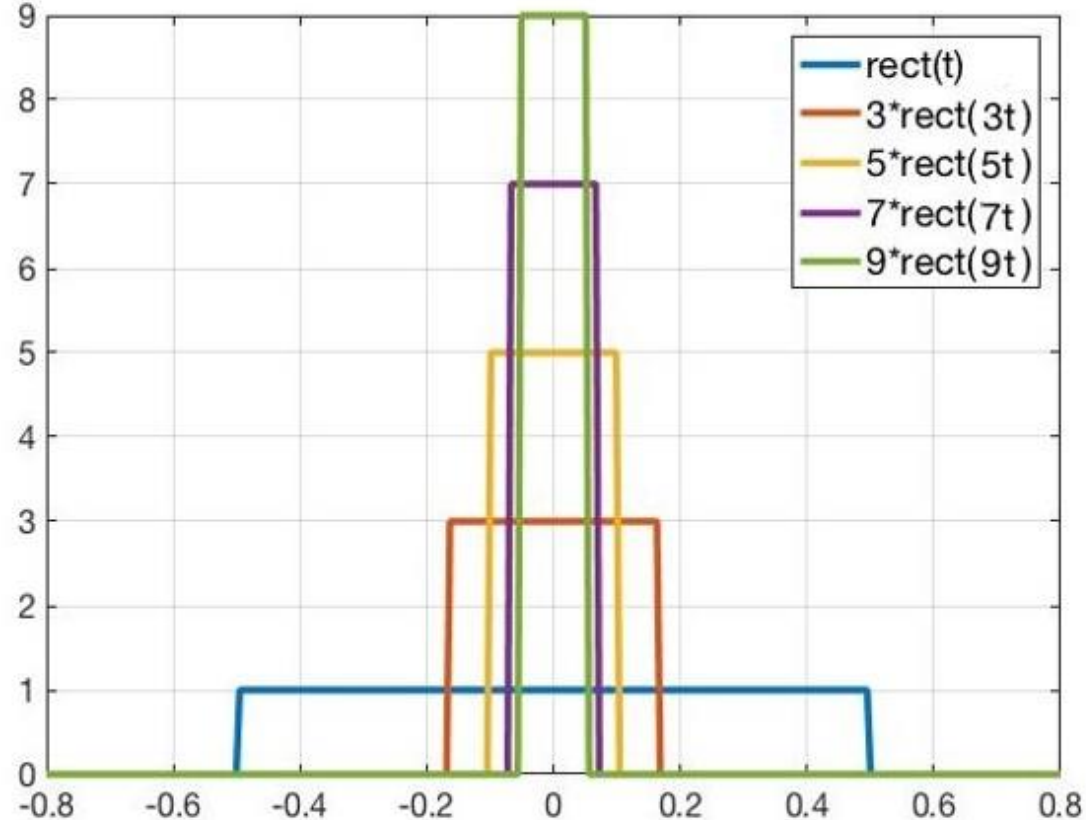
$$u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2}) = \mathrm{rect}(t)$$

This is another common function

CT unit-impulse "delta" function $\delta(t)$

Unit area rectangle function of width D and height 1/D

As D goes to 0 this becomes $\delta(t)$



Think of the delta function as an extremely short burst of energy or as a disturbance, like hitting a table

Putting a delta into a system leads to the impulse response, which is an important way to characterize a system

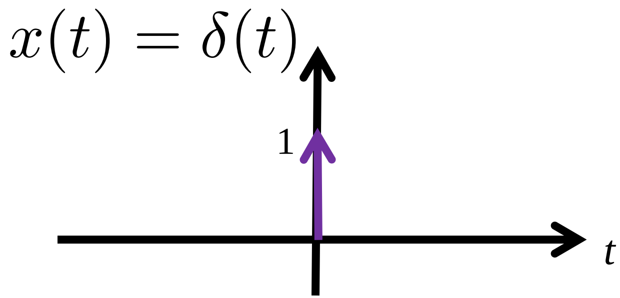
Important relationships

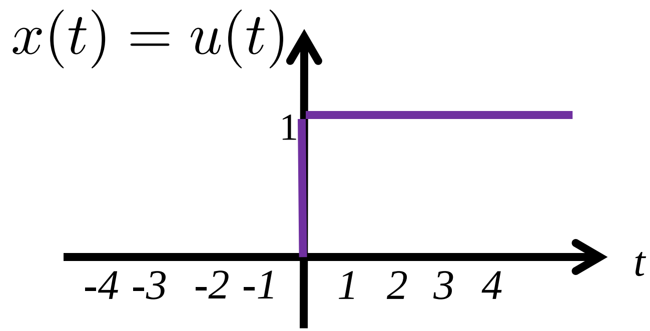
Unit area

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$



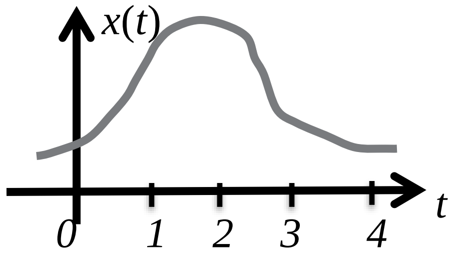


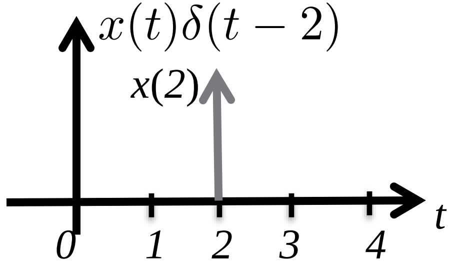
Important properties of the delta function

- **Sifting** (or sampling) property

Avoid common error: be sure to leave in delta function!

$$x(t)\delta(t) = x(0)\delta(t)$$
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$





- **Integration** property

$$\int x(\tau)\delta(\tau)d\tau = x(0)$$

$$\int x(\tau)\delta(t-\tau)d\tau = x(t)$$

$$\int x(\tau)y(\tau)z(\tau)\delta(t-\tau)d\tau = x(t)y(t)z(t)$$

Example with the sifting property

- Consider the signal

$$x(t) = 2t$$

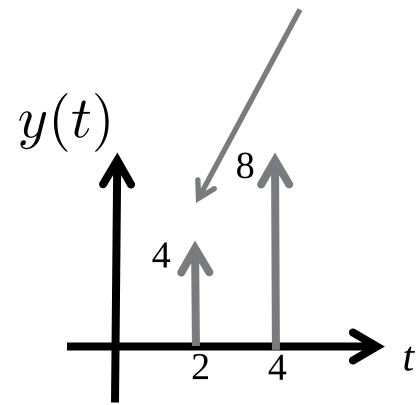
- Simplify

$$y(t) = x(t)\left(\delta(t-2) + \delta(t-4)\right)$$

- Applying the sifting property

$$y(t) = 4\delta(t-2) + 8\delta(t-4)$$

We draw deltas like this



Examples with the integration property

- Simplify the following expressions

$$\int_{-\infty}^{\infty} \cos\left(\frac{\pi t^2}{2}\right)\delta(t+2)dt$$

$$\int_{-\infty}^{\infty} \cos\left(\pi\frac{(-2)^2}{2}\right)\delta(t+2)dt = \cos(2\pi) = 1$$

$$\int_{-\infty}^{\infty} \delta(t-2)\delta(t+2)dt$$

$$0$$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$x(t)$$

Cautionary notes on the CT delta function

- The CT unit-impulse function, also called the Direct delta function, is really a *generalized function*
  - It does not really behave like a normal function
  - It is either zero or infinity
  - Note also that unlike discrete time case, here $\delta(t)\delta(t) = \infty$.
- We should technically only be using $\delta(t)$ under the integral sign
  - It is well defined in the integral sign
  - Some Professors will complain if not in the integral sign (but not me)
  - Take real analysis in the math department for further enlightenment
- Despite these quirks, <u>the delta function is extremely useful</u> for modeling and understanding signals and systems

  "All models are wrong, but some are useful" – George Box (statistician)

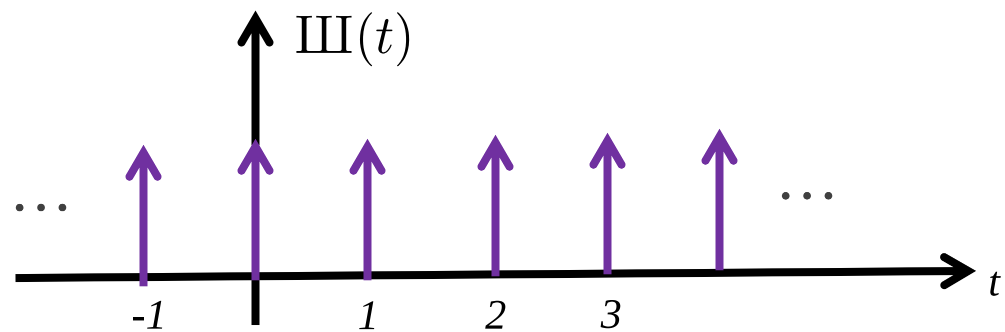
Example leveraging derivative property

- Consider a periodic signal with a period of 2. On (0,2) it is:

$$x(t) = \begin{cases} 1 & 0 \leqslant t < 1 \\ -2 & 1 \leqslant t < 2 \end{cases}$$

- Relate the derivative of this signal to the impulse train

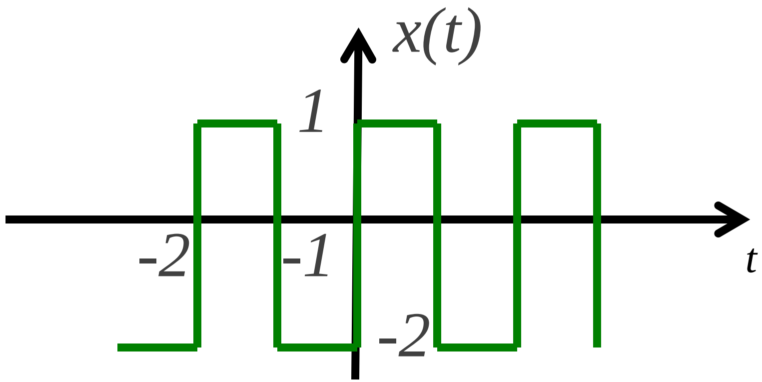
$$Ш(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



"shah" function, also known as the comb function

Solution

- Plot the function

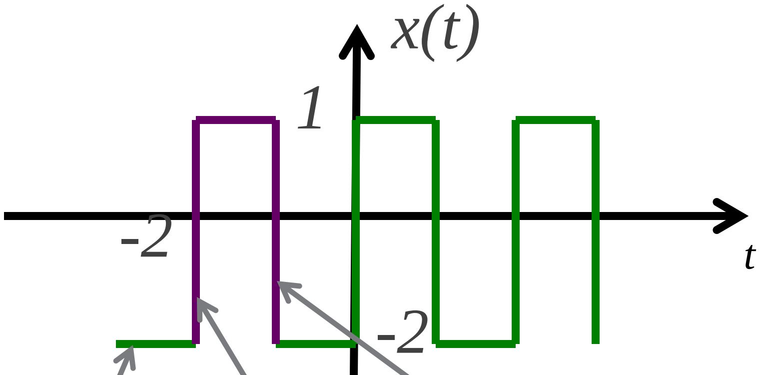


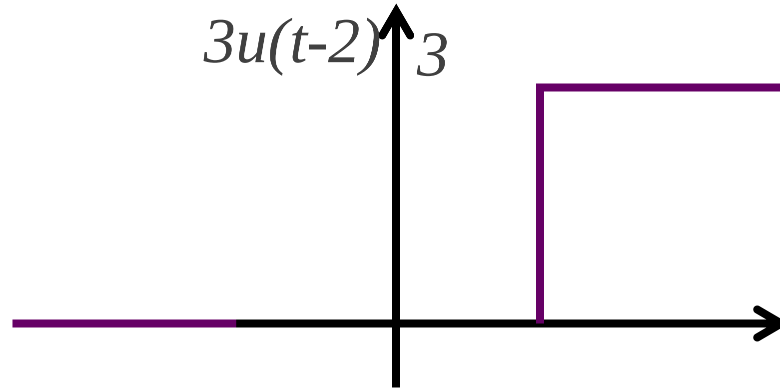
- Use the property

$$\frac{du(t)}{dt} = \delta(t)$$

Solution

- Key is to be able to deduce that:





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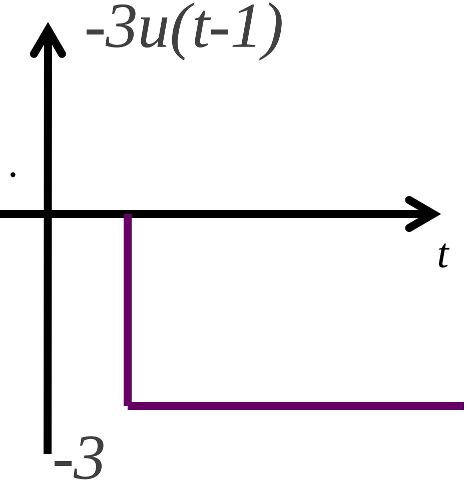


$$x(t) = -2 + \ldots + 3u(t+2) - 3u(t+1) + 3u(t) - 3u(t-1) + \ldots$$

$$= -2 + \sum_{n=-\infty}^{\infty} 3u(t-2n) - \sum_{n=-\infty}^{\infty} 3u(t-2n-1)$$

Solution

- Using the property

$$\frac{du(t)}{dt} = \delta(t)$$

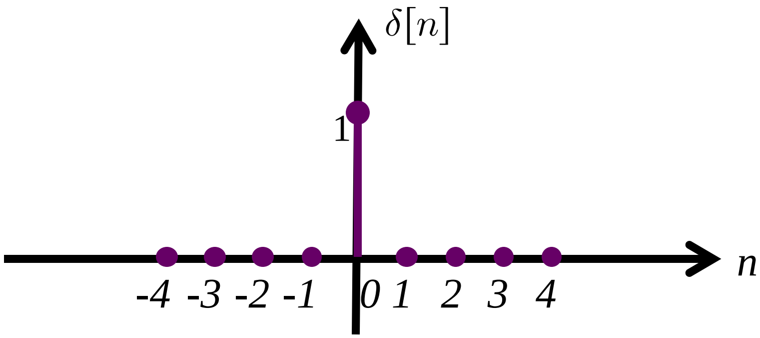
(and assuming we can move the derivative inside the sum)

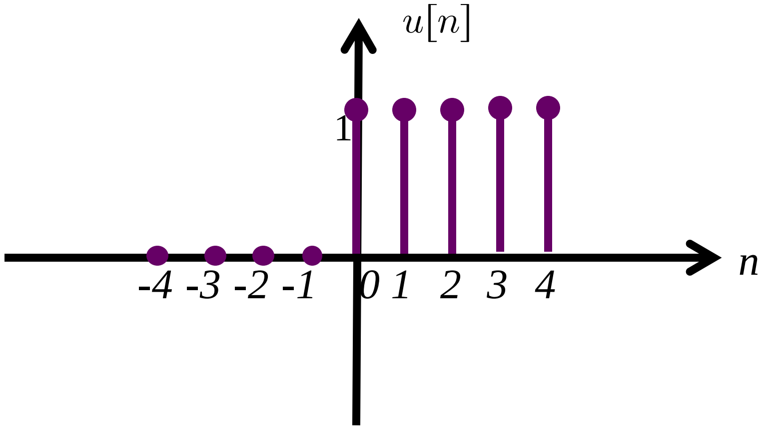
$$x(t) = -2 + \sum_{n=-\infty}^{\infty} 3u(t-2n) - \sum_{n=-\infty}^{\infty} 3u(t-2n-1)$$

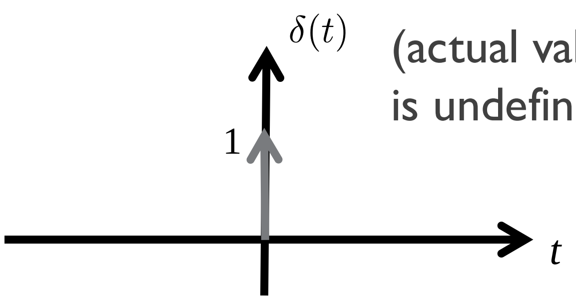
$$g(t) = \frac{dx(t)}{dt}, \text{where}$$

$$g(t) = \sum_{n=-\infty}^{\infty} 3\delta(t-2n) - \sum_{n=-\infty}^{\infty} 3\delta(t-2n-1)$$

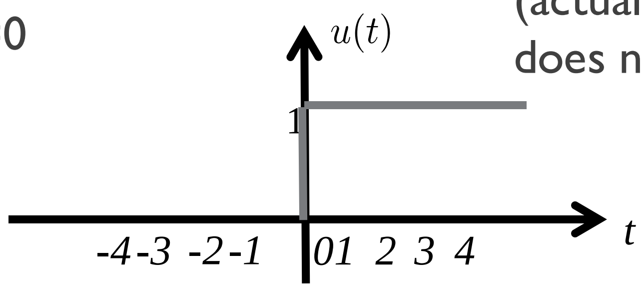
Comparing DT and CT delta and unit-step functions







(actual value at t=0 is undefined)



(actual value at t=0 does not matter)

CT impulse and unit step signals in summary

- CT unit-impulse or Dirac delta in important basic signal
  - Generalized function of time
  - Sifting a signal with a Dirac delta function gives another Dirac delta function with a new signal value
  - Integrating over a Dirac delta extracts the value of the signal
  - Need to use caution when evaluation the Dirac delta, only evaluate under the integral

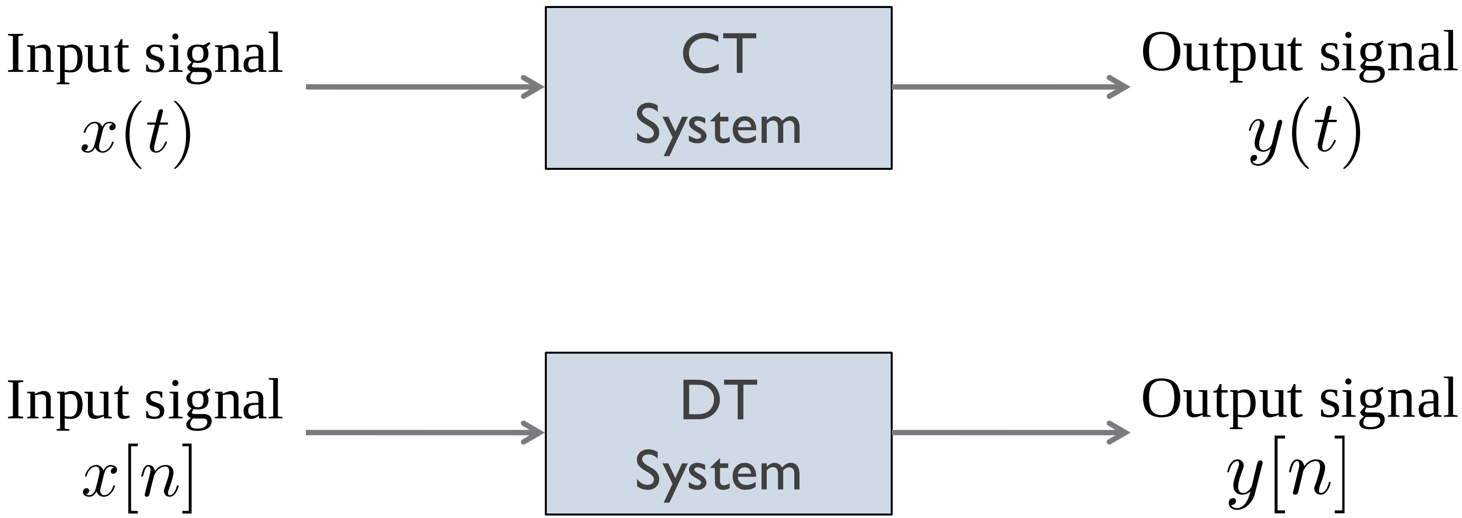
- CT unit-step function
  - Another important building block signal
  - Related to the delta through the derivative operation

Introduction to Systems

Learning objectives

- Describe the output of the systems in terms of their inputs
- Provide CT and DT system examples

Continuous and discrete time systems



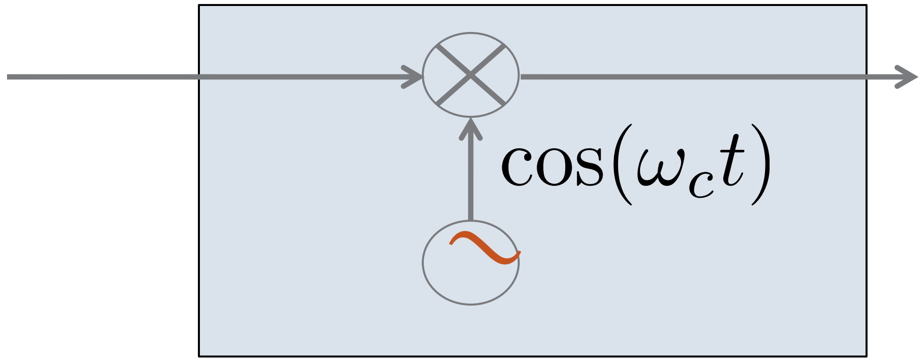
Amplifier (or all-pass amplifier)

$$x(t) \longrightarrow \boxed{\text{A}} \longrightarrow y(t) = Ax(t)$$

- A is the gain
- Passes all input frequencies equally (makes more sense w/ Fourier)
  - In our parlance, it does not *filter* x(t)

Amplitude modulation

$x(t)$



$y(t) = x(t)\cos(\omega_c t)$

Information "rides" the carrier

Amplitude Modulation transmitter (AM radio)

- At the receiver:

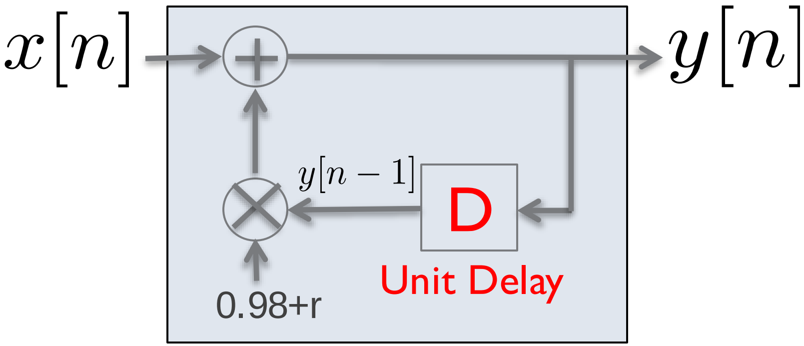
$$
\begin{aligned}
(x(t)\cos(\omega_c t))\cos(\omega_c t) &= \cos^2(\omega_c t)x(t) \\
&= \frac{1}{2}(1 + \cancel{\cos(2\omega_c t)})x(t) \\
&\longrightarrow x(t)
\end{aligned}
$$

Filter out high frequencies

Note: The filter operation only works for certain bandlimited signals x(t), not true in general (we will cover concepts of filtering and bandlimited in subsequent lectures)

Accumulating credit card balance

- Input x[n] is expenses in month n
- Output y[n] is balance in month n
- Setup of problem
  - Pay 2% of the balance each month
  - Interest rate is *r* per month
- What is governing equation and block diagram for this "system"?



$$y[n] = (1+r)y[n-1] - 0.02y[n-1] + x[n]$$
$$= (0.98+r)y[n-1] + x[n]$$

Note: assumes balance computed at end of month (T = 1 month)

Output depends on current input and past outputs (causal)

RC circuit example

Remember KCL:

$$v_r(t) = i(t)R$$

$$v_{\text{out}}(t) = \int_{-\infty}^{t} i(t)\frac{1}{C}dt$$

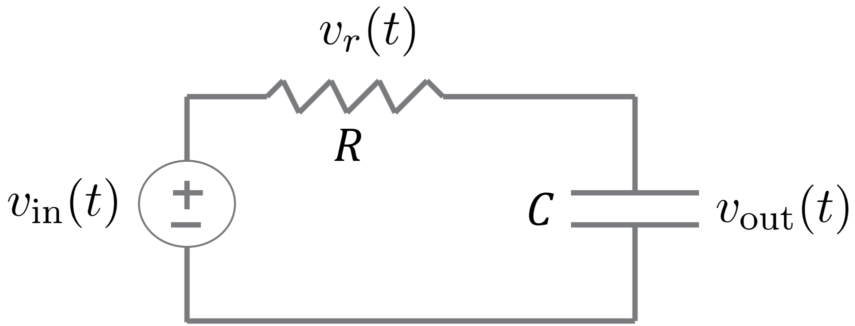
$$i(t) = C\frac{dv_{\text{out}}}{dt}$$

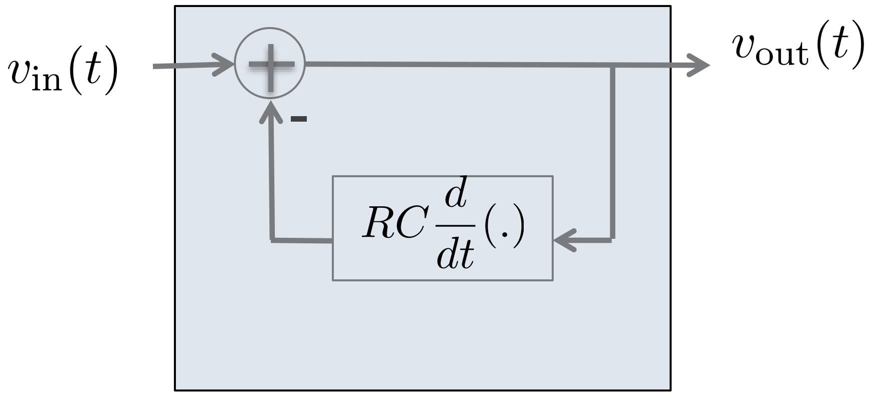
$$v_{\text{out}}(t) + v_r(t) = v_{\text{in}}(t)$$

$$v_{\text{out}}(t) + i(t)R = v_{\text{in}}(t)$$

Solution involves solving the DE

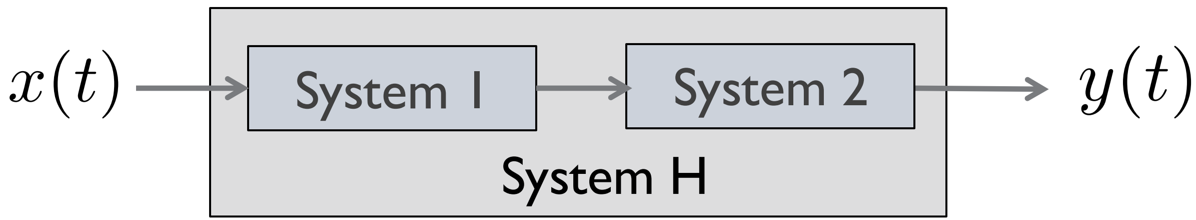
$$v_{\text{out}}(t) + RC\frac{dv_{\text{out}}}{dt} = v_{\text{in}}(t)$$



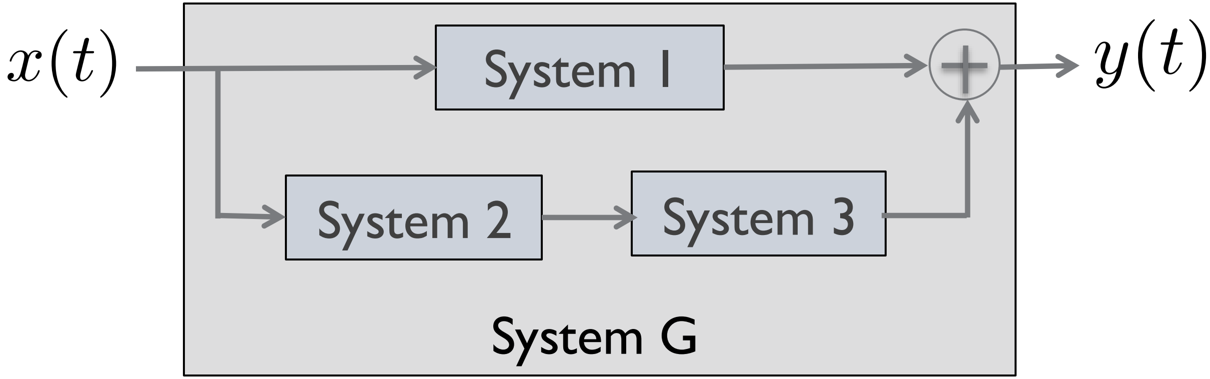


Inter-connected systems – "systems-of-systems"

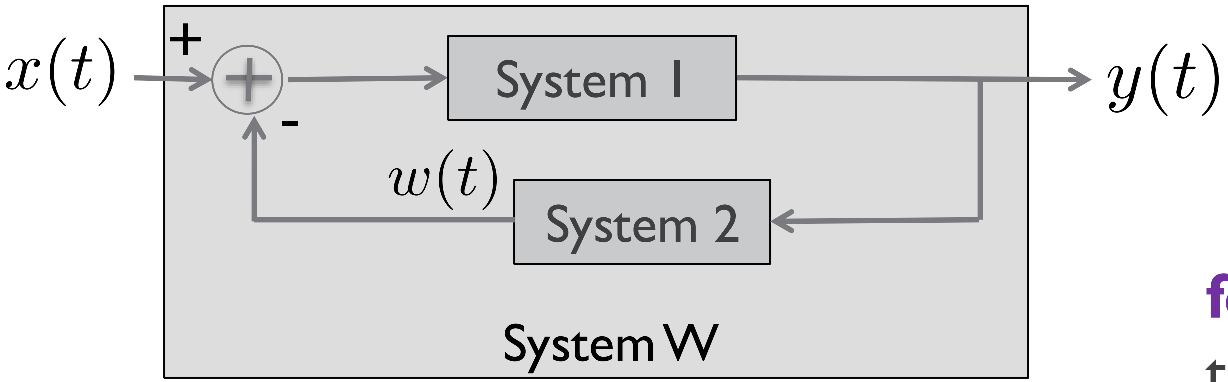
Serial



Parallel



Interconnected systems with feedback



feedback is when the output goes back into the input

Feedback is used in control systems

System introduction in summary

- A system is a functional unit that relates an input signal to an output signal
- Systems can be described in many ways including mathematically, via block diagram, or a code snippet
- CT system has a CT signal as an input and output
- DT system has a DT signal as an input and output
- Mixed signal systems do exist and will be covered briefly when discussing sampling and reconstruction

Simple system properties

Learning objectives

- Classify systems based on their properties
- Determine if a system is memoryless, causal, and/or invertible

Memoryless systems

A system is **memoryless** if the
output depends only on the value of the current input
otherwise, it is said to have memory

Informally: does the system depend on any values of the input besides the current value?

General approach for checking: see if there is any dependence on past or future inputs

Memoryless examples based on AM radio variations

- Modulation with a cosine

$$y(t) = x(t)\cos\omega_c t$$ memoryless

- Modulation with a shifted cosine

$$y(t) = x(t)\cos\omega_c(t-3)$$ memoryless

- Shifting and modulating

$$y(t) = x(t-3)\cos\omega_c(t-3)$$ memory

Causal system

A system is causal if
it only depends on the current and past inputs
otherwise, it is noncausal

Informally: does the system know the future?

General approach for checking: see if there is any dependence on future inputs

Causal system examples

- Product of two signals

$$y(t) = (t+3)x(t-3)$$

Causal

- Time reverse and shift

$$y(t) = x(-t+1)$$

Noncausal

UC San Diego

Invertible system

A system is invertible if
distinct inputs lead to distinct outputs

Informally: can the input be recovered from the output ?

General approach for checking: Can any input be computed from the output? Or find a counter-example.

Invertible system example

- A system that performs integration

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

- This is invertible because there exists an inverse system such that

$$x(t) \longrightarrow \boxed{\int (.)dt} \longrightarrow \boxed{\frac{d}{dt}(.)} \longrightarrow x(t)$$

differentiator

Example

- Downsampler $$y[n] = x[2n]$$

- Is it
  - Memoryless? No
  - Causal? No
  - Invertible? No

Example

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow \log|x(t)|$$

- Is it
  - Memoryless? Yes
  - Causal? Yes
  - Invertible? No because the system:
    - Loses the sign of x(t)
    - Loses the real and imaginary parts

Examples

$$y(t) = x(t/3)$$

- Is it
  - Memoryless? No
  - Causal? No
  - Invertible? Yes

Example

$$y(t) = \cos(3t)x(t)$$

- Is it
  - Memoryless? Yes
  - Causal? Yes
  - Invertible? No
    - Can't divide by cos(3t) since you'll divide by zero twice a period.
    - Can't usually filter this out either (that integrates x(t) as well, and/or we can't separate positive and negative frequency components)

Example

$$y(t) = x(t/3)$$

- Is it
  - Memoryless? No (consider t = 9)
  - Causal? No (consider t = -9)
  - Invertible? Yes (z(t) = y(3t))

Simple system properties in summary

A system is memoryless if the output depends only on the value of the current input (otherwise it has memory)

Check if there is any dependence on t or n other than the current instant of time

A system is causal if it only depends on the current and past inputs, otherwise it is noncausal

Check if there is dependence on t or n ahead of the current instant of time

A system is invertible if distinct inputs lead to distinct outputs, otherwise it is noninvertible

Check the mathematical description of the system, can the input be recovered from the output?

Stability

Learning objectives

- Determine if a system is stable, specifically if it is "bounded input bounded output" (BIBO) stable

BIBO stable systems

A system is BIBO stable if finite inputs result in finite outputs

Formally: A system is BIBO stable if for any **bounded** input

$$|x(t)| < B$$

then the output is also **bounded**

$$|y(t)| < C$$

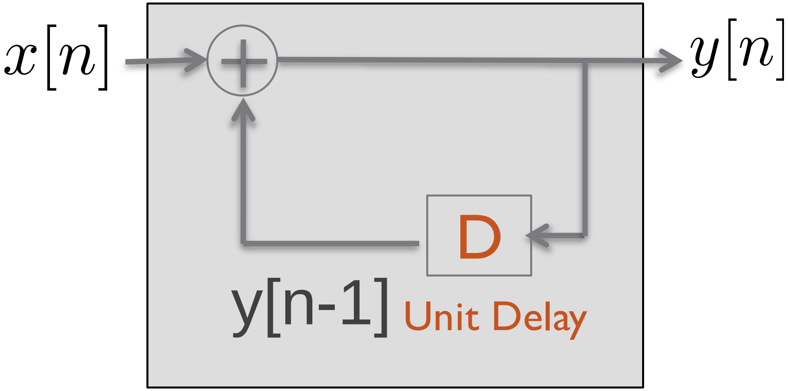
where B and C are positive non-infinite constants

Note: There are other notions of stability explored in courses like automatic control

Stability example with positive feedback

- The output for this system is

$$y[n] = x[n] + y[n-1]$$



Stability example with positive feedback - solution

$$y[n] = x[n] + y[n-1]$$

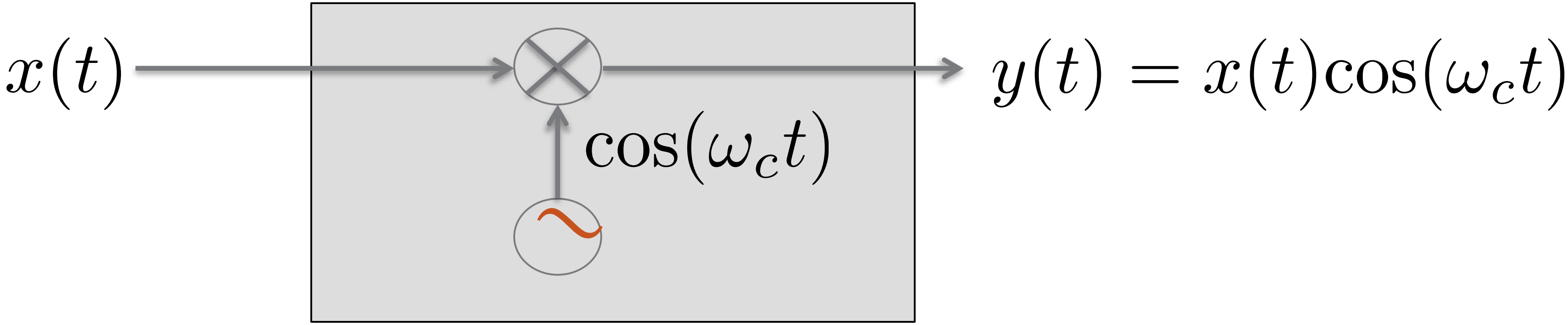
- Consider input $x[n] = u[n]$
  - This input is bounded since $|x[n]| \leqslant 1$
- The output for this input is

$$\begin{aligned} y[n] &= x[n] + y[n-1] \\ &= \sum_{m=0}^{\infty} x[n-m] \\ &= \sum_{m=0}^{\infty} u[n-m] \\ &= nu[n] \end{aligned}$$

Not BIBO stable

since $\lim_{n \to \infty} |y[n]| \to \infty$

Stability example with amplitude modulation

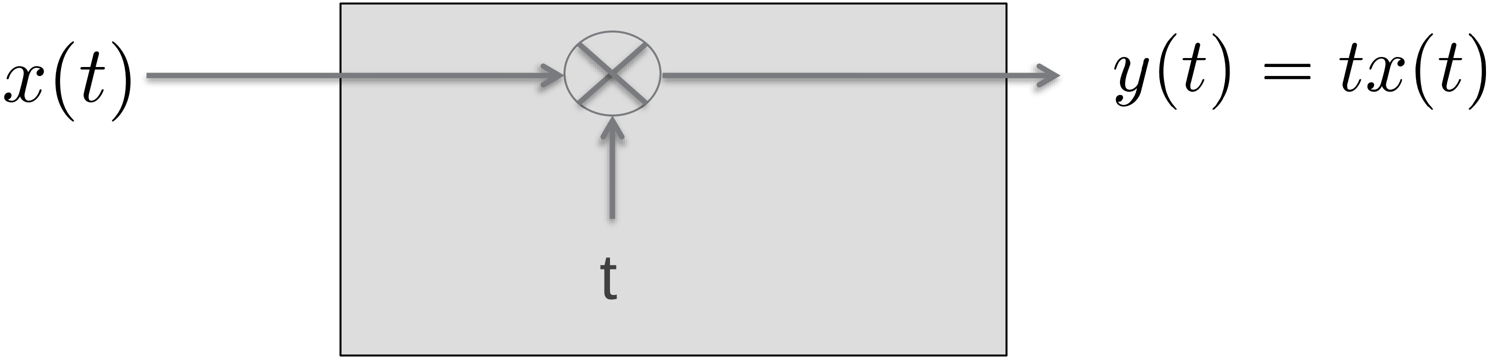


- Suppose that the input is bounded with $|x(t)| \leqslant B$
- Then the output is bounded since

$$\begin{aligned} |y(t)| &= |x(t)\cos(\omega_c t)| \\ &= |x(t)||\cos(\omega_c t)| \\ &\leqslant |x(t)| \\ &\leqslant B \end{aligned}$$

Note: in this example, y(t) happens to be bounded with the same constant as x(t)

Stability example with time scaling



Suppose that $|x(t)| < B$ then $|y(t)| = |tx(t)|$

$$= |t||x(t)|$$

But $\lim_{t \to \infty} |y(t)| = \infty$ therefore **Unstable**

Stability summary

- A system is BIBO stable if the output is bounded when the input is bounded
- One of many types of stability that may be applied to systems, though the only one of interest in this course
- Direct approach is to suppose a bounded input and show that there exists a constant such that the output is bounded
- Indirect approach is to find an input that makes the system output infinite

Time invariance

Learning objectives

- Determine if a system is time invariant or time varying

Time invariant (TI)

A system is time invariant if behaves in the same way regardless of the current time

Formally: consider the system

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If

$$x(t - t_o) \longrightarrow \boxed{\text{System}} \longrightarrow y(t - t_o)$$

For all t_0 then the system is time invariant otherwise it is time varying.

How to check if the system is time-invariant?

- Method #1 (direct approach)
  - Consider the system

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

  - Apply the shifted input $x_2(t) = x_1(t - t_0)$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

  - Does the $y_2(t) = y_1(t - t_0)$? If yes, the system is TI

How to check if the system is time-invariant?

- Method #2 (counter example)

  If we suspect the system is time variant, find an example where time invariance fails "a counterexample"

  - This method is often quicker

- Note:
  - If you can not find a counter example, then you have to use Method 1
  - The counter example is just a simple way to **disprove** TI

- Hint: Usually, but not always, if the output includes any function of time other than x(t), it is time-varying

Time invariance example I

System described by $y(t) = 3tx(t-3)$

$$y_1(t) = 3tx_1(t-3)$$ Output for generic input $x_1(t)$

$$y_2(t) = 3tx_2(t-3)$$ Output for generic input $x_2(t)$

Now, let: $x_2(t) = x_1(t-t_0)$

$$y_2(t) = 3tx_1(t-t_0-3)$$

$$y_1(t-t_0) = 3(t-t_0)x_1(t-t_0-3)$$

As $y_2(t)$ does not match $y_1(t-t_0)$ $\longrightarrow$ **time-variant system**

Time invariance example 2

System described by $y(t) = x(t)x(t-1)$

$$y_1(t) = x_1(t)x_1(t-1)$$
$$y_2(t) = x_2(t)x_2(t-1)$$

Let $x_2(t) = x_1(t-t_0)$

$$y_2(t) = x_1(t-t_0)x_1(t-t_0-1)$$
$$= y_1(t-t_0)$$

The system is **time invariant (TI)**

(Is it Causal? Memoryless? Invertible? Stable?)

Time invariance example 3

System with AM modulation $y(t) = x(t)\cos(\omega_c t)$

Counterexample, let $x_1(t) = \delta(t), x_2(t) = \delta(t - \frac{\pi}{2\omega_c})$

$$y_1(t) = \delta(t)$$

$$y_2(t) = \delta\left(t - \frac{\pi}{2\omega_c}\right)\cos(\omega_c t) = 0$$

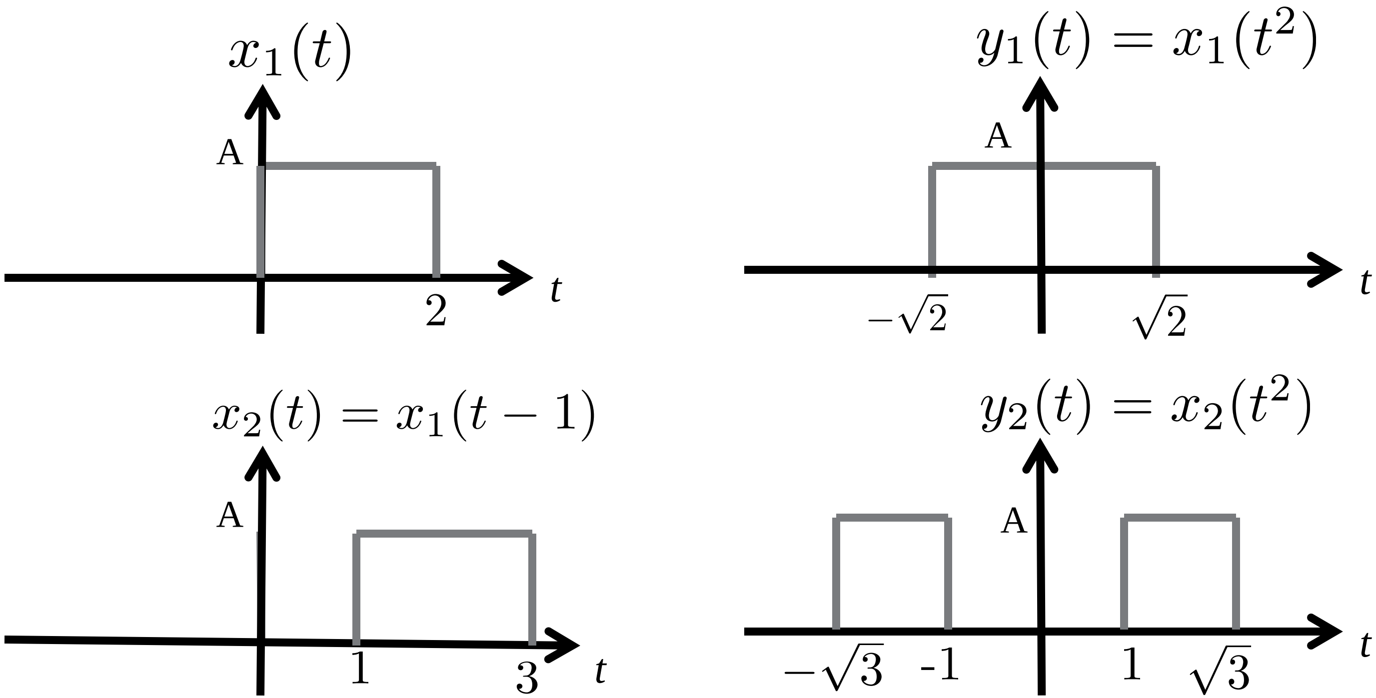
$$y_2(t) \neq y_1\left(t - \frac{\pi}{2\omega_c}\right)$$

$$= \delta\left(t - \frac{\pi}{2\omega_c}\right)$$

Time-variant

Time invariance example 4

$$y(t) = x(t^2)$$



Time-variant

(hint didn't work, Method 1 also tricky to see)

Time invariance example 5

- Consider a time-invariant system with input *x(t)* and output *y(t)*. Show that if *x(t)* is periodic with period *T*, then *y(t)* is also periodic with period *T*.

This is an important fact about periodicity that will be used later in the course

Output of system $x(t) \rightarrow y(t)$

Because it is TI $x(t-t_0) \rightarrow y(t-t_0)$

Also works for T $x(t-T) \rightarrow y(t-T)$

Since $x(t) \rightarrow y(t)$ it follows that y(t) is periodic w/ period T

Time invariance summary

- A system is time invariant if behaves in the same way regardless of the current time
- Time invariant systems are much easier to design and analyze compared to time varying systems
- The main way to check time invariance is to shift the input and see if the output is always shifted
- A counter example is sufficient to show a system is not time invariant

Linearity

Learning objectives

- Determine if a system is linear or nonlinear
- Understand sub-properties of superposition and scaling

Scaling the input

If a system obeys the scaling property, then scaled inputs lead to scaled outputs

Consider the following system:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If the scaling property is satisfied then for any scalar value A

$$Ax(t) \longrightarrow \boxed{\text{System}} \longrightarrow Ay(t)$$

Superposition property

If a system obeys the superposition property, then system acts in the same way onto each system

If for two different inputs:

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

then superpositon holds if for input $x(t) = x_1(t) + x_2(t)$ then:

$$x_1(t) + x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t) + y_2(t)$$

Linear systems

A system is linear if scaling is preserved and superposition holds

Consider two different inputs and outputs

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

If the following holds for any scalar A and B then the system is linear

$$Ax_1(t) + Bx_2(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow Ay_1(t) + By_2(t)$$

How to check if the system linear?

- Direction approach
  - Check that scaling holds $Ax(t) \Rightarrow Ay(t)$
  - Check the superposition holds $x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)$
- Slightly faster direct approach
  - Check that $Ax_1(t) + Bx_2(t) \Rightarrow Ay_1(t) + By_2(t)$
- Find a counter example
  - One that may be useful (from the scaling property)

$$0 \longrightarrow \boxed{\text{System}} \longrightarrow 0$$

  - If a system generates a non-zero output to a signal that is zero for all time then it is non-linear

Linearity example I

$$y(t) = x(t)x(t-1)$$

Let us check if the scaling property holds $x_1(t) = Ax(t)$

$$\begin{aligned} y_1(t) &= x_1(t)Ax_1(t-1) \\ &= Ax(t)Ax(t-1) \\ &= A^2x(t)x(t-1) \\ &\neq Ay(t) \end{aligned}$$

Scaling fails → Nonlinear

Linearity example I – alternative solution

Consider inputs

$$y(t) = x(t)x(t-1)$$

$$x_1(t) \to y_1(t) = x_1(t)x_1(t-1)$$

$$x_2(t) \to y_2(t) = x_2(t)x_2(t-1)$$

$$x_3(t) = Ax_1(t) + Bx_2(t)$$

Note that

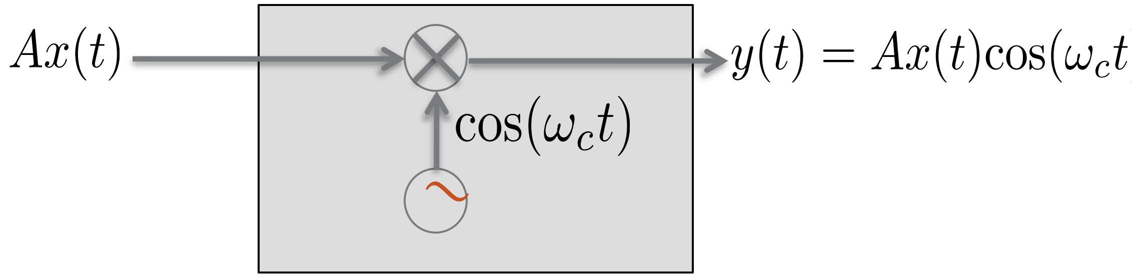
Nonlinear

$$\begin{aligned} y_3(t) &= x_3(t)x_3(t-1) \\ &= \left(Ax_1(t) + Bx_2(t)\right)\left(Ax_1(t-1) + Bx_2(t-1)\right) \\ &= A^2x_1(t) + B^2x_2(t) + ABx_1(t)x_2(t-1) + ABx_1(t-1)x_2(t) \\ &\neq Ay_1(t) + By_t(t) = Ax_1(t)x_1(t-1) + Bx_2(t)x_2(t-1) \end{aligned}$$

Linearity example 2

System with AM modulation

Scaling is preserved



What about superposition?

$$y_1(t) = x_1(t)\cos\omega_c t$$

$$y_2(t) = x_2(t)\cos\omega_c t$$

$$x(t) = x_1(t) + x_2(t) \longrightarrow y(t) = (x_1(t) + x_2(t))\cos\omega_c t$$

$$= x_1(t)\cos\omega_c t + x_2(t)\cos\omega_c t$$

$$= y_1(t) + y_2(t)$$

Linear system

Linearity example 3

- Consider the affine system

$$y[n] = 2x[n] + 1$$

- Suppose that

$$x_1[n] \rightarrow y_1[n]$$

$$x_2[n] \rightarrow y_2[n] \text{ where } x_2[n] = Ax_1[n]$$

- Now observe that

$$\begin{aligned} y_2[n] &= 2x_2[n] + 1 \\ &= 2Ax_1[n] + 1 \\ &\neq Ay_1[n] \end{aligned}$$

$\Rightarrow$ Scaling does not hold therefore is a **nonlinear system**

Linearity summary

- A system linear if *scaled inputs* lead to *scaled outputs* and the *sum of inputs* leads to a *sum of outputs* if the inputs were applied separately

- Linear systems are easier to design and analyze

- Many systems in practice are nonlinear but are designed to be as linear as possible, or are only used with inputs where they behave in a linear fashion

- Need to check both the scaling and superposition properties to prove that a system is linear

Reference example with details

- Is the following system stable? linear? Time-invariant?

$$y(t) = t^2 x(t-1)$$

Reference example – checking stability

Consider a bounded input

$$|x(t)| < B$$

To produce output

$$y(t) = t^2 x(t-1)$$

Find the magnitude of the output

Not BIBO stable!

$$|y(t)| = |t^2||x(t-1)|$$

Is not bounded (grows with t)

Reference example – checking time invariance

Consider the output to input $x_1(t)$

$$y_1(t) = t^2 x_1(t-1)$$

Define a new input

$$x_2(t) = x_1(t - t_0)$$

Compute the output

$$\begin{aligned} y_2(t) &= t^2 x_2(t-1) \\ &= t^2 x_1(t - 1 - t_0) \end{aligned}$$

Not time invariant!

Compare with a shifted version of the first output

$$y_1(t - t_0) = (t - t_0)^2 x_1(t - 1 - t_0) \neq y_2(t)$$

Reference example – checking linearity

Consider the inputs and outputs

$$x_1(t) \rightarrow y_1(t) = t^2 x_1(t-1)$$

$$x_2(t) \rightarrow y_2(t) = t^2 x_2(t-1)$$

Define a new input

$$x_3(t) = a x_1(t) + b x_2(t)$$

Compute the output

Linear!

$$\begin{aligned} y_3(t) &= t^2 x_3(t-1) \\ &= t^2 (a x_1(t-1) + b x_2(t-1)) \\ &= a y_1(t) + b y_2(t) \end{aligned}$$

Useful reference material

Trigonometric identities

Euler's theorem: $e^{\pm ju} = \cos u \pm j \sin u$

$\cos u = \frac{1}{2}(e^{ju} + e^{-ju})$

$\sin u = (e^{ju} - e^{-ju})/2j$

$\sin^2 u + \cos^2 u = 1$

$\cos^2 u - \sin^2 u = \cos 2u$

$2 \sin u \cos u = \sin 2u$

$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$

$\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$

$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$

$\sin u \cos v = \frac{1}{2}[\sin(u - v) + \sin(u + v)]$

From Signals and Systems: Continuous and Discrete by Ziemer, Tranter, and Fannin

Indefinite integrals

$$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos(ax)\, dx = \frac{1}{a}\sin(ax)$$

$$\int \sin^2(ax)\, dx = x/2 - \sin(2ax)/4a$$

$$\int \cos^2(ax)\, dx = x/2 + \sin(2ax)/4a$$

$$\int x \sin(ax)\, dx = [\sin(ax) - ax\cos(ax)]/a^2$$

$$\int x \cos(ax)\, dx = [\cos(ax) + ax\sin(ax)]/a^2$$

$$\int x^m \sin(x)\, dx = -x^m \cos(x) + m\int x^{m-1}\cos(x)\, dx$$

$$\int x^m \cos(x)\, dx = x^m \sin(x) - m\int x^{m-1}\sin(x)\, dx$$

$$\int \sin(ax)\sin(bx)\, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$$

$$\int \sin(ax)\cos(bx)\, dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right], \qquad a^2 \neq b^2$$

$$\int \cos(ax)\cos(bx)\, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$$

$$\int e^{ax}\, dx = e^{ax}/a$$

$$\int x^m e^{ax}\, dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1} e^{ax}\, dx$$

$$\int e^{ax}\sin(bx)\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)]$$

$$\int e^{ax}\cos(bx)\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)]$$

From Signals and Systems: Continuous and Discrete by Ziemer, Tranter, and Fannin

Definite integrals

$$\int_0^\infty \frac{a\,dx}{a^2+x^2} = \pi/2, \qquad a>0$$

$$\int_0^{\pi/2} \sin^n(x)\,dx = \int_0^{\pi/2} \cos^n(x)\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots(n)}\,\dfrac{\pi}{2}, & n \text{ even, } n \text{ an integer} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots(n)}, & n \text{ odd} \end{cases}$$

$$\int_0^\pi \sin^2(nx)\,dx = \int_0^\pi \cos^2(mx)\,dx = \pi/2, \qquad n \text{ an integer}$$

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = \int_0^\pi \cos(mx)\cos(nx)\,dx = 0, \quad m \neq n,\ m \text{ and } n \text{ integer}$$

$$\int_0^\pi \sin(mx)\cos(nx)\,dx = \begin{cases} 2m/(m^2-n^2), & m+n \text{ odd} \\ 0, & m+n \text{ even} \end{cases}$$

$$\int_0^\infty \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2}, \qquad a>0$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty e^{-a^2x^2}\,dx = \sqrt{\pi}/2a, \qquad a>0$$

$$\int_0^\infty x^n e^{-ax}\,dx = n!/a^{n+1}, \qquad n \text{ an integer and } a>0$$

$$\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}, \qquad a>0$$

$$\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2}, \qquad a>0$$

$$\int_0^\infty e^{-a^2x^2}\cos(bx)\,dx = \frac{\sqrt{\pi}}{2a}\,e^{-b^2/4a^2}$$

From Signals and Systems: Continuous and Discrete by Ziemer, Tranter, and Fannin