UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2021
Linear Systems Fundamentals

SOLUTIONS TO MIDTERM EXAM

You are allowed one 2-sided sheet of notes.

No books, no other notes, no calculators.

PRINT YOUR NAME ______________________________

Signature ______________________________

Your signature confirms that you have completed this exam on your own and in accordance with the ECE 101 Academic Integrity Agreement.

Student ID Number ______________________________

Problem	Weight	Score
1	36 pts	
2	32 pts	
3	32 pts	
Total	100 pts	

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Useful facts and Tables 3.1 and 3.2 from the textbook are attached to the exam.

Good luck!

Name/Student ID: ____

Problem 1 (36 points)

A discrete-time (DT) system S is described by

$$y[n] = \sum_{k=n-1}^{n+1} (x[k])^2.$$

(a) (18 points)

Check the appropriate box indicating whether or not the system S satisfies the specified property. **Justify your answers.**

True	**False**	
☐	☒	Invertible
☐	☒	Causal
☒	☐	Stable

Invertible?

$x[n]$=1 and $x[n] = -1$ both produce the output $y[n] = 3$. So S is **not invertible**.

Causal?

$y[n]$ depends on $x[n+1]$. So S is **not causal**.

Stable?

If $\left|x[n]\right| \leq B$, then $\left|y[n]\right| = \left|\sum_{k=n-1}^{n+1}(x[n])^2\right| \leq \sum_{k=n-1}^{n+1}\left|(x[k])^2\right| \leq 3B^2$. So S is **stable**.

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Problem 1 (cont.)

(b) (18 points)

Consider the same system S as in part (a). Check the appropriate box indicating whether or not the system S satisfies the specified property. **Justify your answers.**

True	**False**	
☒	☐	Time-invariant
☐	☒	Scalable
☐	☒	LTI

Time-invariant?

$x_1[n] = x[n-n_0] \longrightarrow y_1[n] = \sum_{k=n-1}^{n+1}(x_1[k])^2 = \sum_{k=n-1}^{n+1}(x[k-n_0])^2.$

So, $y_1[n] = (x[n-n_0-1])^2 + (x[n-n_0])^2 + (x[n-n_0+1])^2$.
$y[n-n_0] = \sum_{k=n-n_0-1}^{n-n_0+1}(x[k])^2 = (x[n-n_0-1])^2 + (x[n-n_0])^2 +$
$(x[n-n_0+1])^2 = y_1[n].$
So, S is **time-invariant**.

Scalable?

$x_1[n] = ax[n] \longrightarrow y_1[n] = \sum_{k=n-1}^{n+1}(x_1[k])^2 = \sum_{k=n-1}^{n+1}(ax[k])^2 = a^2\sum_{k=n-1}^{n+1}(x[k])^2.$

$ay[n] = a\sum_{k=n-1}^{n+1}(x[k])^2$. So, $ay[n] \neq y_1[n]$ for at least some choice of $x[n]$ and $a \in \mathbb{C}$, such as $x[n] = 1$ and $a = 2$. So, S is **not scalable**.

LTI?

S is not scalable and, therefore, not linear. Therefore, S is **not LTI**.

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Problem 2 (32 points)[8 points each part]

Let S be the DT LTI system with impulse response

$$h[n] = \delta[n] + \frac{1}{4}\delta[n-2].$$

(a) Determine a difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

that describes S.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (\delta[n] + \frac{1}{4}\delta[n-2]) \\ &= x[n] + \frac{1}{4}x[n-2] \end{aligned}$$

(b) Determine the transfer function $H(z)$ of S.

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ &= 1 + \frac{1}{4}z^{-2} \end{aligned}$$

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Problem 2 (cont.)

(c) Determine the output $y[n]$ of the system S for input signal $x[n] = (\frac{1}{2})^n$.

Apply the eigenfunction propert: $x[n] = z^n$ produces $y[n] = H(z)z^n$.

$$y[n] = H(\frac{1}{2})\left(\frac{1}{2}\right)^n = \left(1 + \frac{1}{4}\left(\frac{1}{2}\right)^{-2}\right)\left(\frac{1}{2}\right)^n = 2\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n-1}.$$

Alternatively, apply the difference equation from part (a):

$$\begin{aligned} y[n] &= x[n] + \frac{1}{4}x[n-2] \\ &= \left(\frac{1}{2}\right)^n + \frac{1}{4}\left(\frac{1}{2}\right)^{n-2} \\ &= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n \\ &= 2\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n-1} \end{aligned}$$

(d) Determine the output $y[n]$ of the system S for the input signal

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Express your answer in the simplest form you can.

Apply the convolution property of LTI systems:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \left(\frac{1}{2}\right)^n u[n] + \frac{1}{4}\left(\frac{1}{2}\right)^{n-2} u[n-2] \\ &= \left(\frac{1}{2}\right)^n (u[n] + u[n-2]) \\ &= \delta[n] + \frac{1}{2}\delta[n-1] + \left(\frac{1}{2}\right)^{n-1} u[n-2] \end{aligned}$$

$$y[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ \left(\frac{1}{2}\right)^{n-1}, & n > 1 \end{cases}$$

Equivalently, apply the difference equation to get the same result.

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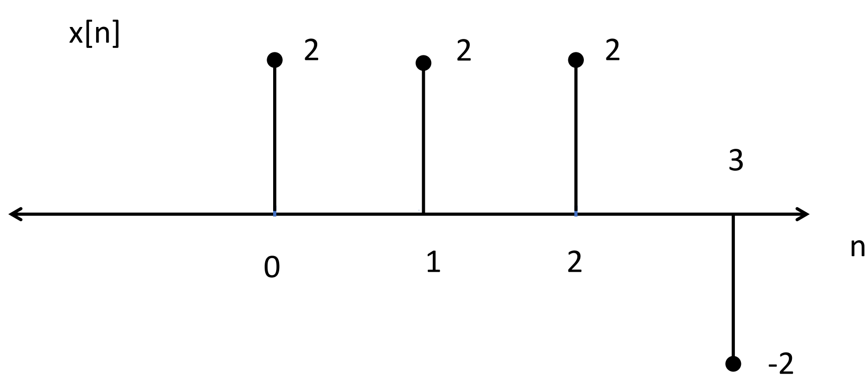
Problem 3 (32 points)[16 points each part]

(a) Let $x[n]$ be a DT signal with fundamental period $N = 4$ and DTFS $a_0 = 1, a_1 = -j, a_2 = 1, a_3 = j$. Determine and sketch one period of $x[n]$, for $n = 0, 1, 2, 3$.

From the synthesis equation:

$$\begin{aligned} x[n] &= \sum_{k=0}^{3} a_k e^{jk\frac{\pi}{2}n} \\ &= 1 + (-j)e^{j\frac{\pi}{2}n} + e^{j\pi n} + je^{j\frac{3\pi}{2}n} \\ &= 1 + -j(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}) + e^{j\pi n} \\ &= 1 - j(2j\sin(\frac{\pi}{2}n)) + (-1)^n \\ &= 1 + (-1)^n + 2\sin(\frac{\pi}{2}n) \end{aligned}$$

So, $x[0] = 2, x[1] = 2, x[2] = 2, x[3] = -2$.



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Problem 3 (cont.)

(b) Let $x[n]$ be a DT signal with fundamental period 6 and Fourier series coefficients $a_0 = 1, a_1 = \frac{1}{2j}, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = -\frac{1}{2j}$. Consider the DT LTI system whose frequency response in the interval $[-\pi, \pi]$ is given by

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{2} \leq |\omega| \leq \pi \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the signal $x[n]$ is the input to this system.

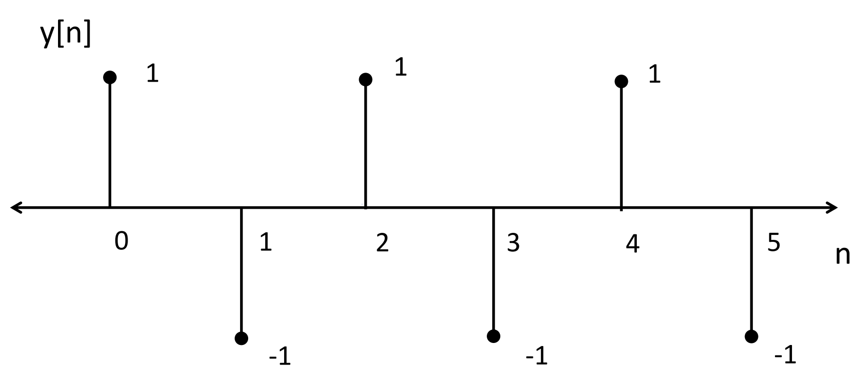
Determine and sketch precisely the output $y[n]$ in the interval $[0, \ldots, 5]$.

$$\omega_0 = 2\pi/6$$

$$y[n] = \sum_{k=0}^{5} a_k H(e^{jk2\pi/6}) e^{jk(2\pi/6)n}$$

$$H(e^{jk2\pi/6}) = \begin{cases} 1 & \text{for } k = 2, 3, 4 \\ 0 & \text{for } k = 0, 1, 5. \end{cases}$$

$$\begin{aligned} y[n] &= 0 \cdot e^{j(4\pi/6)n} + 1 \cdot e^{j(6\pi/6)n} + 0 \cdot e^{j(8\pi/6)n} \\ &= e^{j\pi n} \\ &= (-1)^n \end{aligned}$$



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