Lab #5 Feedback Stabilization: Stick Balancing

This lab is based upon the problems in Section 11.1, "Feedback Stabilization: Stick Balancing," in the course text *Computer Explorations in SIGNALS AND SYSTEMS* by Buck, Daniel, Singer.

This exercise considers the problem of stabiiizing the unstable system that governs the balancing of an object like a stick in your hand.

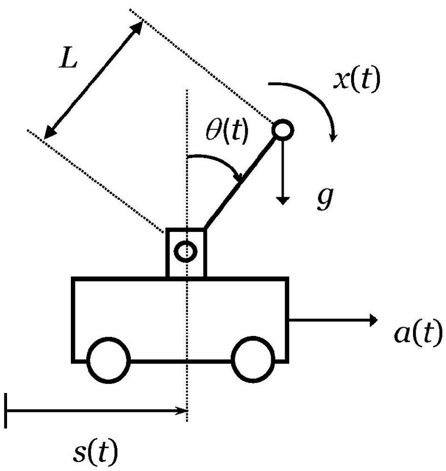


Fig. 1

Consider the systems shown in Fig. 1. The cart, whose position is given by $s(t)$, can move forward or backward. The acceleration of the cart follows as $d^2s/dt^2 = a(t)$ A stick of length L is attached to the cart with a hinge so that it can only move in the same direction as the cart. The position of the stick is given by the angle $\theta(t)$. Assume that all of the mass of the stick is concentrated in a ball at the end of the stick. Also shown is an angular acceleration $x(t)$ imparted on the stick by external forces, such as the wind. In order to balance the stick, the cart must be moved with an appropriate acceleration $a(t)$.

Balancing forces on the mass along the direction perpendicular to the rod, the differential equation relating $\theta(t)$, $a(t)$, and $x(t)$ is

$$L\frac{d^2\theta(t)}{dt^2} = g\sin(\theta(t)) - a(t)\cos(\theta(t)) + Lx(t).$$

This is not a linear differential equation. However, we can linearize it for small $\theta(t)$ to examine the dynamics when the stick is nearly vertical, which is the region of interest when balancing the stick. Using the small angle approximations $\sin(\theta(t)) \approx \theta(t)$ and $\cos(\theta(t)) \approx 1$ for $|\theta(t)| \ll \pi$, the linearized differential equation governing the system is

$$L\frac{d^2\theta(t)}{dt^2} = g\theta(t) - a(t) + Lx(t).$$

In the numerical problems solved with MATLAB below, assume that $L = 1$m and $g = 9.8$ m/sec^2.

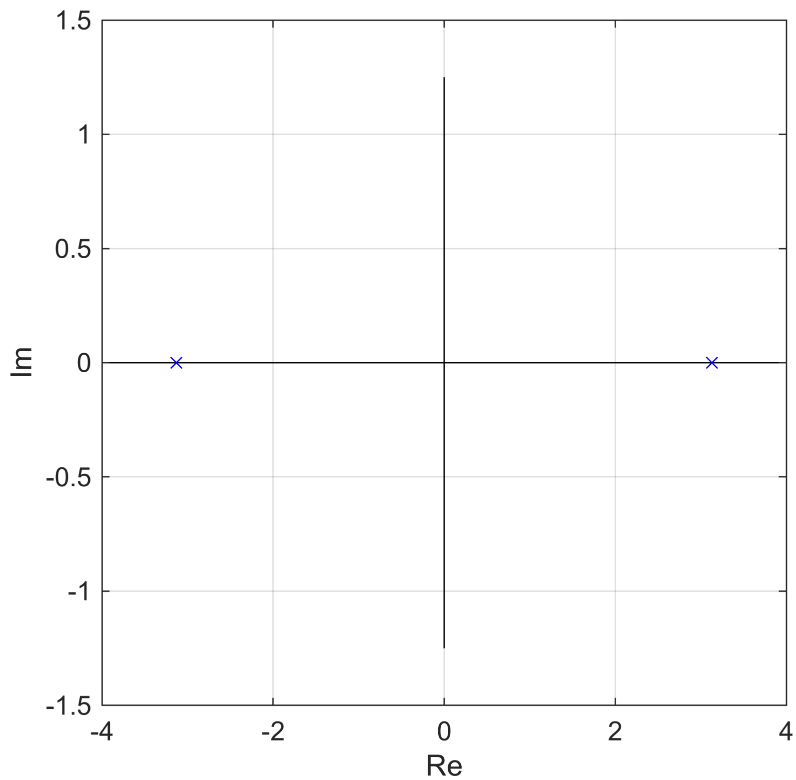
Task 1. Assuming the cart is stationary with zero acceleration, $a(t) = 0$, derive the system function $H(s)$ relating the input (disturbance) $x(t)$ to the output $\theta(t)$. Determine the poles and zeros of the system. Using your result, explain why the system is unstable. Use `plotpz` to numerically determine the poles and zeros, confirming your analysis, and to make a pole-zero plot for this system. (The function `plotpz` is included at the end of this script.)

$$H(s) = \frac{L}{Ls^2 - g}$$

Setting $L = 1$, $g = 9.8$, we get $H(s) + \frac{1}{s^2 - 9.8}$.

The poles are at $s = \pm\sqrt{9.8} \approx \pm 3.1305$. Since there is a pole with positive real part, the system is not stable.

```
[p,z]=plotpz(1,[1 0 -9.8]);
```



Task 2. Consider stabilizing the system with proportional (P) feedback using an acceleration of the cart which is proportional to the angle $\theta(t)$, i.e., $a(t) = k\theta(t)$. Determine the system function for the system with proportional feedback. Let k range over the values k=linspace(0,20,11). Determine the maximum magnitude of the real part $MaxR$ and the minimum magnitude of the real part $MinR$ among all of the poles for all k in this

range. Then compute the maximum magnitude of the imaginary part $MaxI$ and the minimum magnitude of the imaginary part $MinI$ among all of the poles for all k in this range.

$$H(s) = \frac{L}{Ls^2 + (k-g)}.$$

Setting $L = 1$, $g = 9.8$, we get $H(s) = \dfrac{1}{s^2 + (k-9.8)}$

For $k < 9.8$, the poles are at $s = \pm\sqrt{9.8-k}$. For $K > 9.8$, the poles are at $s = \pm j\sqrt{k-9.8}$.

The maximum real part of any pole $MaxR = \sqrt{9.8} \approx 3.1305$, corresponding to $k = 0$.

The maximum imaginary part of any pole $MaxI = \sqrt{20-9.8} \approx 3.1937$, corresponding to $k = 20$.

```
k=linspace(0,20,11)
```

```
k = 1×11
     0     2     4     6     8    10    12    14    16    18    20
```

```
MaxR=max(abs((9.8-k(k<9.8))).^0.5)
```

```
MaxR = 3.1305
```

```
MinR=min(abs((9.8-k(k<9.8))).^0.5)
```

```
MinR = 1.3416
```

```
MaxI=max(abs((9.8-k(k>9.8))).^0.5)
```

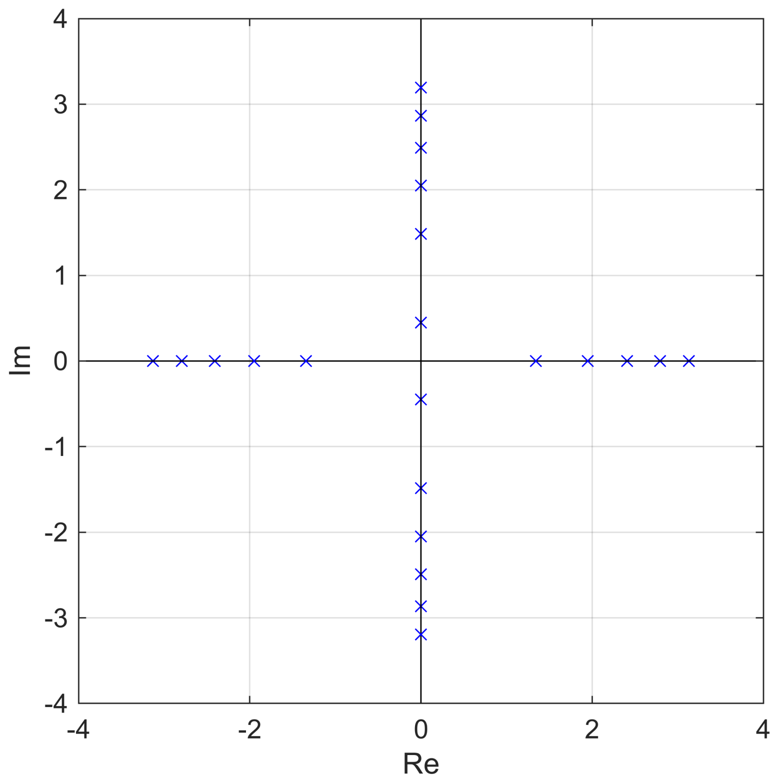
```
MaxI = 3.1937
```

```
MinI=min(abs((9.8-k(k>9.8))).^0.5)
```

```
MinI = 0.4472
```

Task 3. Modify plotpz by commenting out the statements setting `MaxR` and `MaxI` . Add statements setting the values of `MaxR` and `MaxI` to the integer ceiling of $MaxR$ and $MaxI$ found in Task 2. Now use plotpz in a `for` loop to plot the pole locations for the system using proportional feedback for all values in k in a single pole-zero plot..

```
figure;
for k=linspace(0,20,11)
    plotpz(1,[1 0 k-9.8]);
end
```



Task 4. Using the result of Task 3, determine if the system can be stabilized using only proportional feedback. Explain your answer. Find the smallest and largest values of k in the array k such that the stick location will oscillate back and forth indefinitely when $x(t) = \delta(t)$. Explain your answer.

For $k < 9.8$, there is a pole on the positive real axis, so the system is unstable.

For $k > 9.8$, the poles are purely imaginary (i.e., lie on the $j\omega$-axis). The impulse response corresponding to $H(s) = \dfrac{1}{s^2 + (k - 9.8)}$ will be oscillatory, with impulse response $h(t) = \dfrac{1}{\omega_0}\sin(\omega_0 t)u(t)$, where $\omega_0 = \sqrt{k - 9.8}$, with period $T = \dfrac{2\pi}{\omega_0}$ (see Table 9.2 in the text). The smallest and largest values in k for which this occurs are $k_{\min} = 10$ and $k_{\max} = 20$.

```
k=linspace(0,20,11);
r=k((9.8-k)<0);
maxk=max(r)
```

```
maxk = 20
```

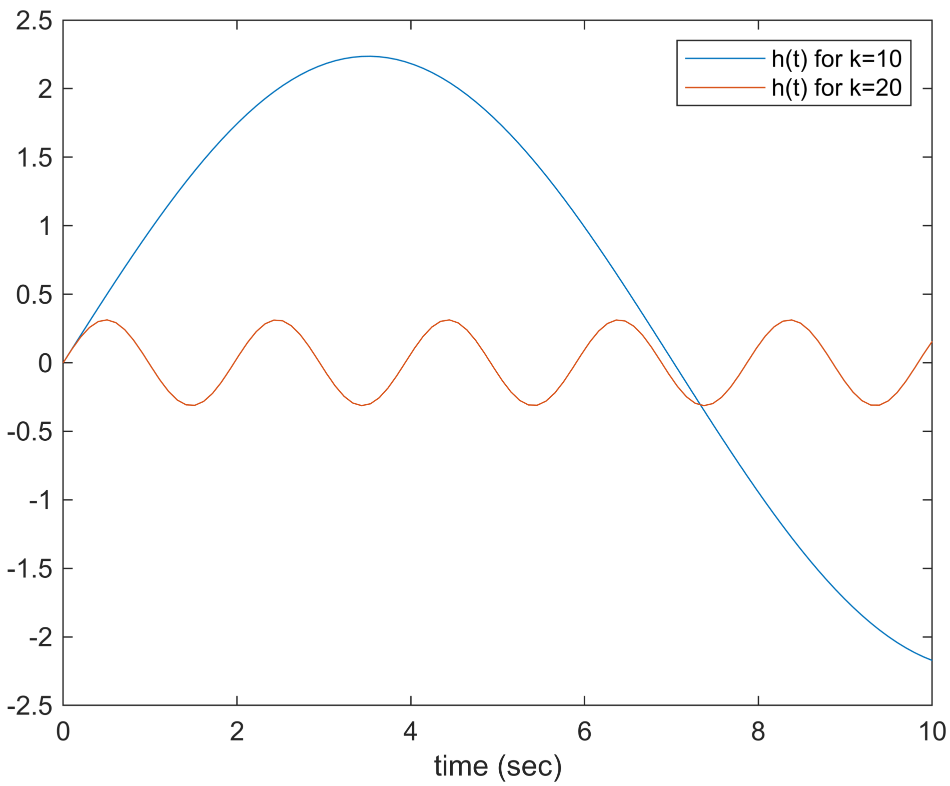
```
mink=min(r)
```

```
mink = 10
```

Task 5. Use `impulse` with the values of k from Task 4 to simulate the impulse response of the corresponding proportional feedback system over the time interval `t=linspace(0,10,100)`, storing the simulated values for $\theta(t)$ in arrays `th1` and `th2`. Make a plot showing `th1` and `th2` over the time interval `t`, and label the plot appropriately.

```
figure;
t=linspace(0,10,100);
th1=impulse(1,[1 0 0.2],t);
th2=impulse(1,[1 0 10.2],t);
```

```
plot(t,th1,t,th2);
xlabel('time (sec)');
legend('h(t) for k=10','h(t) for k=20')
```



Next, you will consider using proportional-plus-derivative (PD) feedback to stabilize the system, i.e., feedback of the form

$$a(t) = k_1\theta(t) + k_2\frac{d\theta(t)}{dt}.$$

Task 6. Analytically determine the system function for the system with PD feedback. Show that you can find values for k_1 and k_2 so that the system function has the form

$$H(s) = \frac{1}{s^2 + 4s + 4}.$$

(This corresponds to a critically dampled second-order system with damping ratio $\zeta = 1$ and undampled natural frequency $\omega_n = 2$ rad/sec.)

$$H(s) = \frac{L}{Ls^2 + k_2 s + (k_1 - g)}$$

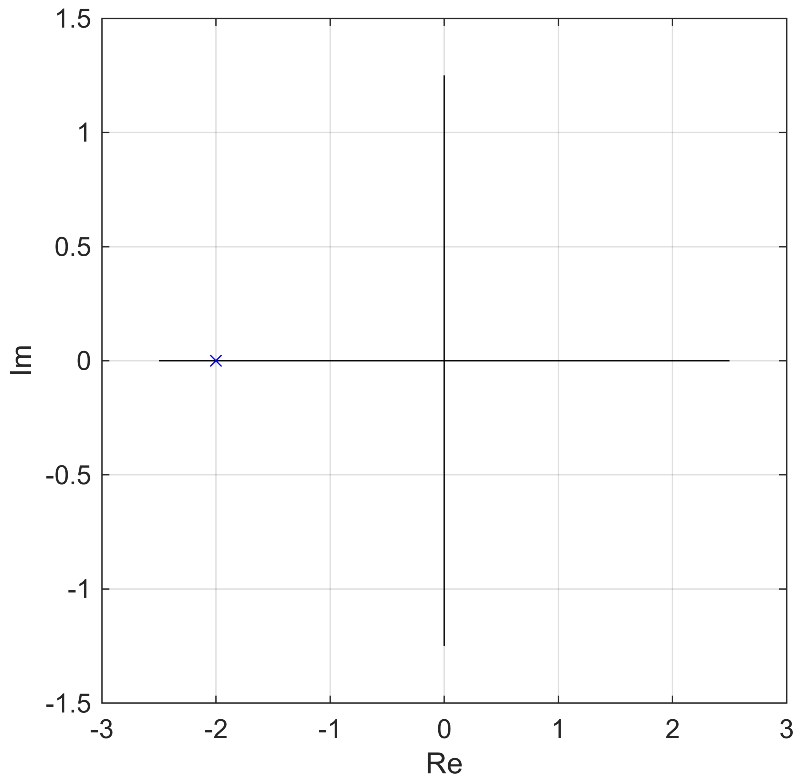
With $L = 1$ and $g = 9.8$, the values of k_1 and k_2 that give the specified systems are $k_1 = 13.8$ and $k_2 = 4$.

As an aside, the correspondence between k_1 and k_2 and the damping ratio and natural frequency can be found by using the relations $\omega_n^2 = k_1 - 9.8$ and $2\xi\omega_n = k_2$. This implies $\omega_n = \sqrt{k_1 - 9.8}$ and $\xi = \dfrac{k_2}{2\sqrt{k_1 - 9.8}}$. Setting $\xi = 1$ and $\omega_n = 2$, we retrieve the values $k_1 = 13.8$ and $k_2 = 4$.

Task 7. Restore the original calculation of `MaxI` and `MaxR` in `plotpz`. Use plotpz to plot a pole-zero diagram for the system using the values for k_1 and k_2 you found in Task 6. What is the order of the poles and zeros?

The system has a second-order pole at $s = -2$. There are no zeros.

```
figure;
[p,z]=plotpz(1,[1 4 4]);
```

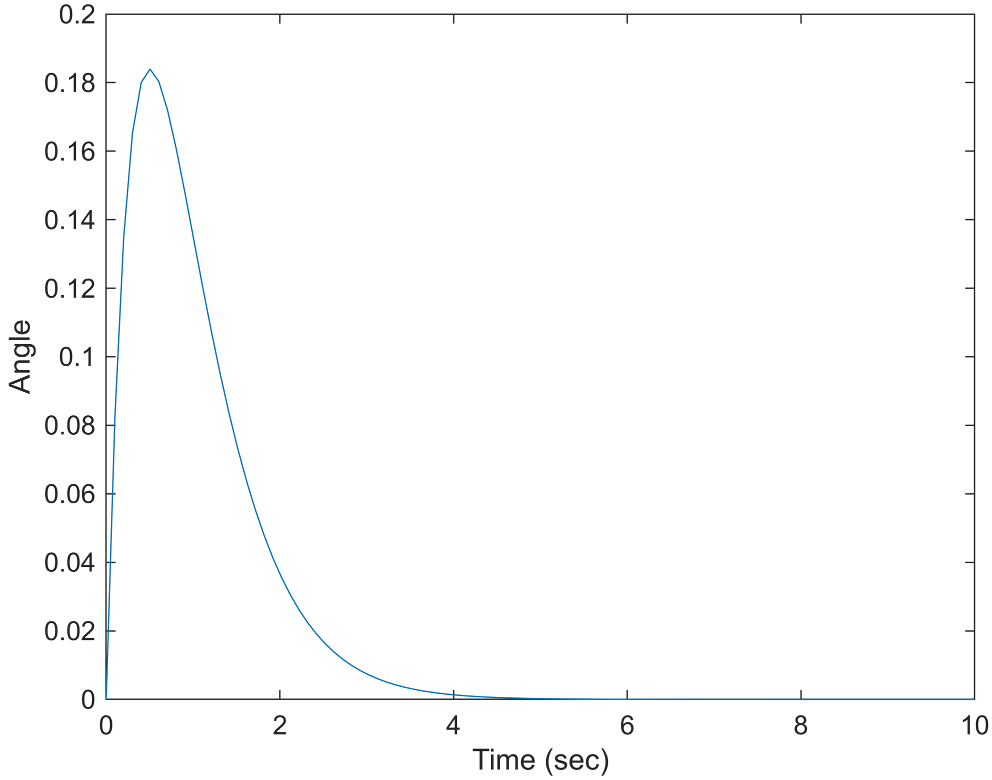


Task 8. Use `impulse` to simulate the impulse response of the system with PD feedback from Task 6 over the time interval `t=linspace(0,10,100)`, storing the simulated values for $\theta(t)$ in `th3`. Make a plot showing `th3` over the time interval `t`, and label the plot appropriately. Determine the impulse response of the causal system $H(s)$ analytically and confirm that it matches your simulation result. Does the system with PD feedback stabilize the stick in response to an impulse disturbance?

$$h(t) = te^{-4t}u(t) \leftrightarrow H(s) = \frac{1}{(s+2)^2}$$

Yes, the system stabilizes the stick.

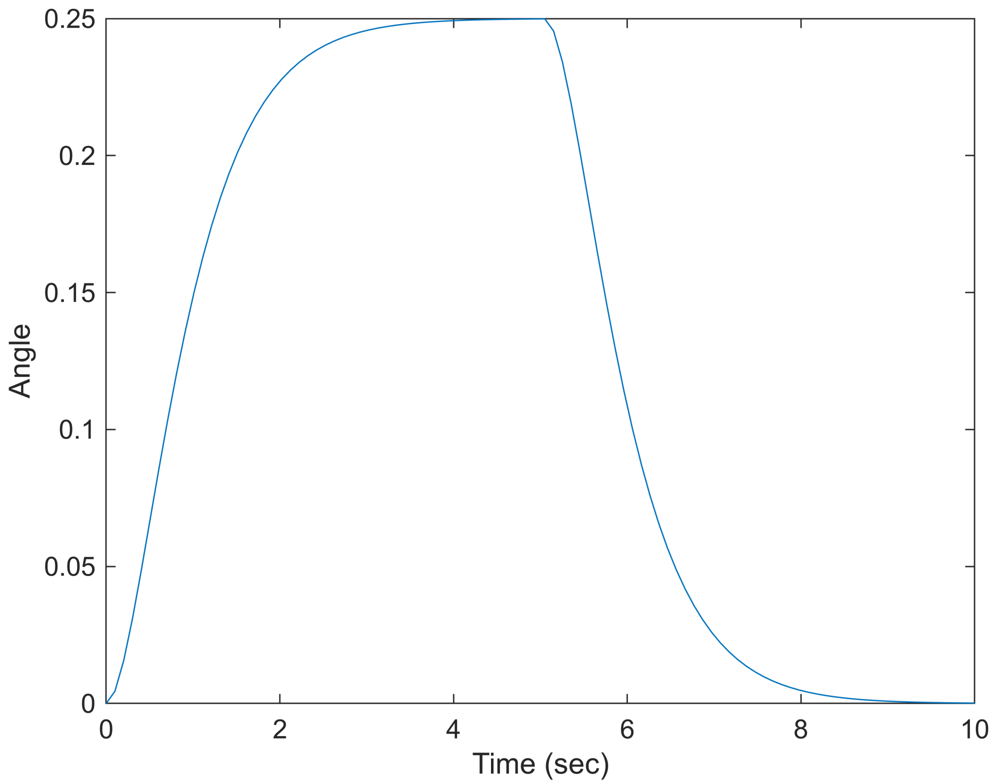
```
figure;
t=linspace(0,10,100);
num=[1];
den=[1 4 4];
[h]=impulse(num,den,t);
plot(t,h)
xlabel('Time (sec)')
ylabel('Angle')
```



Task 9. Create a disturbance $x(t)$ that takes the value 1 over the time interval $0 \leq t \leq 5$ using `x=[ones(1,50) zeros(1,50)]`. Use `lsim` to simulate the response of the system with PD feedback from Task 6 to the input $x(t)$ over the time interval `t`, storing the resulting simulated values for $\theta(t)$ in `th4`. Make a plot showing `th4` over the time interval `t`, and label the plot appropriately. Does the system recover from this disturbance and stabilize the stick?

Yes, the system recovers from the disturbance and the stick returns to $\theta = 0$.

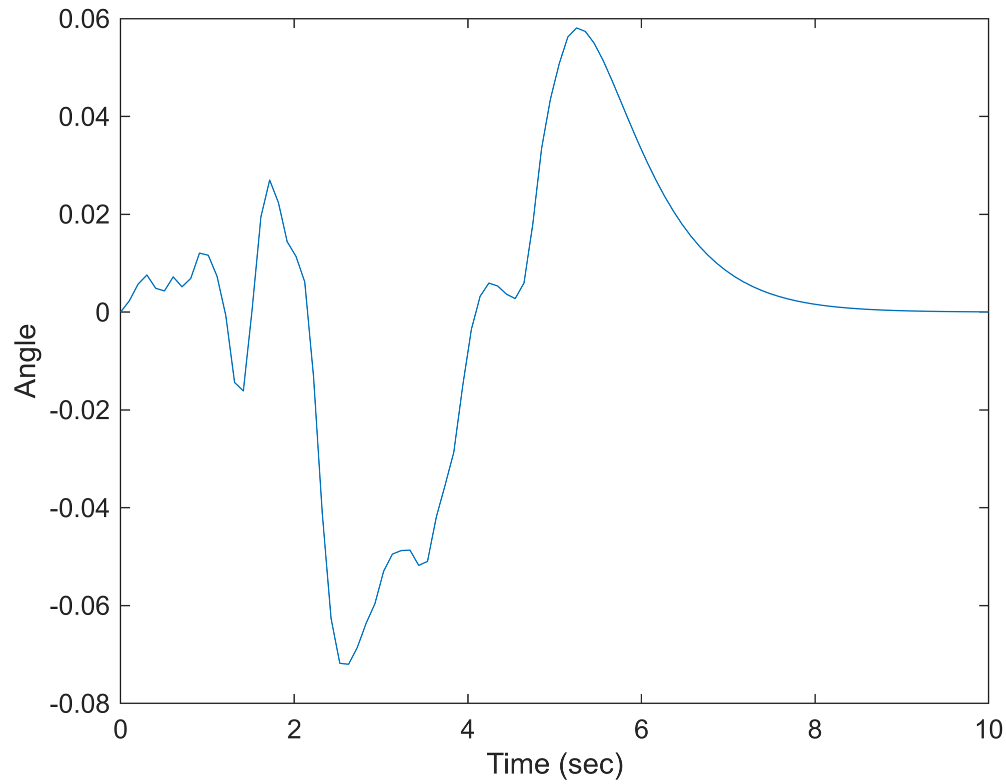
```
figure;
x1=ones(1,50);
x2=zeros(1,50);
x=[x1,x2];
th4=lsim(num,den,x,t);
plot(t,th4)
xlabel('Time (sec)');
ylabel('Angle');
```



Task 10. Create a *random* disturbance $x(t)$ over the time interval $0 \leq t \leq 5$ using `x=[randn(1,50) zeros(1,50)]`. Use `lsim` to simulate the response of the system with PD feedback from Task 6 to the input $x(t)$ over the time interval `t`, storing the resulting simulated values for $\theta(t)$ in `th5`. `Make a plot showing th5` over the time interval `t`, and label the plot appropriately. Does the system recover from the random disturbance and stabilize the stick?

Yes, the system recovers from the disturbance and the stick returns to $\theta = 0$.

```
%
% The command rng(1) can be used to set the seed of the random number
% generator if you want to replicate the random experiment
%
figure;
x1=randn(1,50);
x2=zeros(1,50);
x=[x1,x2];
th5=lsim(num,den,x,t);
plot(t,th5)
xlabel('Time (sec)');
ylabel('Angle');
```



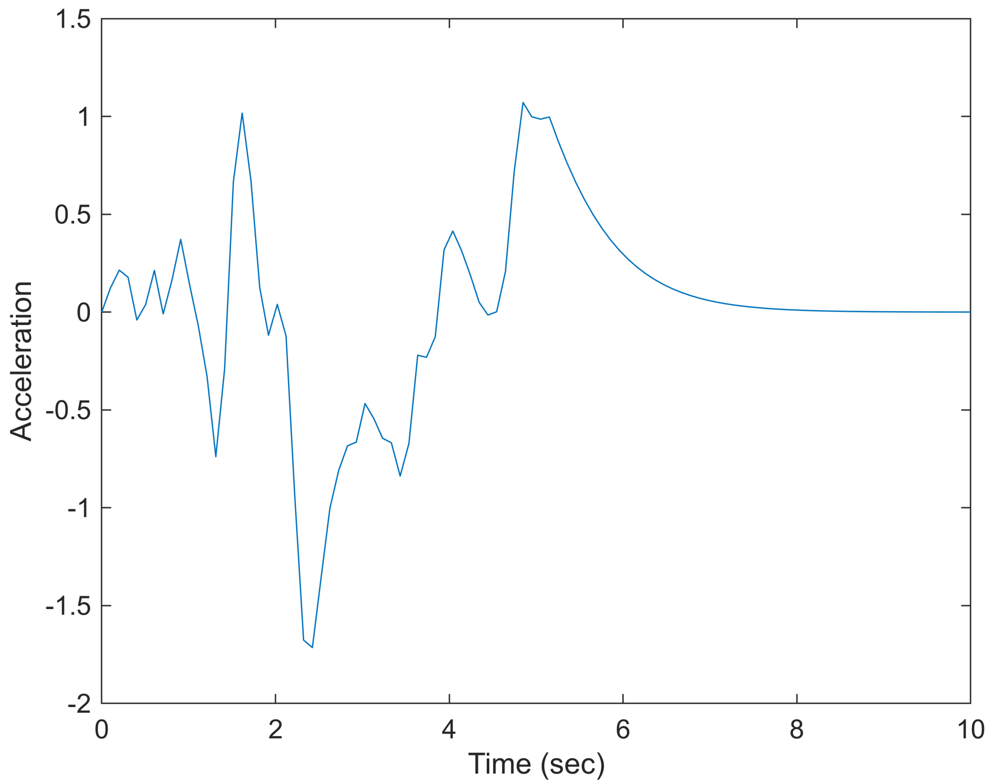
Task 11. Plot the acceleration of the cart $a(t)$ used to balance the stick in response to the disturbance in Task 10. Since the feedback is PD, you will have to use both $\theta(t)$ and $\frac{d\theta(t)}{dt}$ to calculate $a(t)$. You can approximate $\frac{d\theta(t)}{dt}$ from $\theta(t)$ by using a backwards Euler approximation based on the limiting definiton of the derivative

$$\frac{d\theta(t)}{dt} = \lim_{\Delta t \to 0} \frac{\theta(t) - \theta(t - \Delta t)}{\Delta t}$$

which translates in the sampled domain to `dth[n]=(th[n]-th[n-1])/dt`, where `dt` is the time separation of the values in `t`. For a cart of mass m, the force required to generate acceleration $a(t)$ would be $F = ma(t)$.

The acceleration qualitatively tracks the disturbance and the angle of the stick.

```
figure;
dt=10/99;
dth=(th5'-[0 th5(1:length(th5)-1)'])/dt;
a = 13.8*th5'+4*dth;
plot(t,a)
xlabel('Time (sec)')
ylabel('Acceleration')
```




```
0;
%--------------------------------------------------
%
%  function [h]=impulse(num,den,t)
%
%  t   = row vector of time samples where impulse response is computed
%  num = numerator of rational system function  (row vector of high-order to
%  low-order coefficients)
%  den = denominator of rational system function  (row vector of high-order to
%  low-order coefficients)
%
%
%-------------------------------------------------
```

```
function [p,z]=plotpz(b,a)

%--------------------------------------------------------------
% copyright 1996, 2001, by John Buck, Michael Daniel, and Andrew Singer.
% For use with the textbook "Computer Explorations in Signals and
% Systems using MATLAB", Prentice Hall, 1997, 2002.
%--------------------------------------------------------------
```

```
p=roots(a); % determine poles
z=roots(b); % determine zeros
p=p(:); % make into column vector
z=z(:); % make into column vector

% For plotting, determine maximum real part of all poles, zeros, and 1
MaxR= max(abs(real([p;  z; 1])));
% For plotting, determine maximum imaginary part of all poles, zeros, and j
MaxI= max(abs(imag([p; z; j])));
%MaxR=
%MaxI=
hold on
plot(1.25*[-MaxR MaxR],[0 0],'k')     % Plot the real axis
plot([0 0],1.25*[-MaxI MaxI],'k')     % Plot the imag axis
plot(real(z),imag(z),'ro')   % Plot zeros
plot(real(p),imag(p),'bx')   % Plot poles
xlabel('Re');
ylabel('Im');
%
box on
axis('square');                      % Make square aspect ratio
grid on
hold off
end
```