ECE 101 Linear Systems

Problem Set 3 Solutions

Problem 1 - Synthesize:

1.

The nonzero Fourier series coefficients for $x[n]$ are: $a_0 = 2$, $a_2 = a^*_{-2} = 2e^{j\frac{\pi}{6}}$, $a_4 = a^*_{-4} = e^{j\frac{\pi}{3}}$. Since, the period is $N = 5$, we need a_k from $k = 0$ to 4 which are obtained as follows:

- $a_0 = 2$ from the problem statement
- $a_1 = a_{-4+5} = a_{-4} = e^{-j\frac{\pi}{3}}$ from the periodicity of the DTFS coefficients
- $a_2 = 2e^{j\frac{\pi}{6}}$ from the problem statement
- $a_3 = a_{-2+5} = a_{-2} = 2e^{-j\frac{\pi}{6}}$ from the periodicity of the DTFS coefficients
- $a_4 = e^{j\frac{\pi}{3}}$ from the problem statement

Using the synthesis equation we have

$$\begin{aligned}
x[n] &= 2 + e^{(1)j\frac{2\pi}{N}n} \cdot e^{-j\frac{\pi}{3}} + e^{(4)j\frac{2\pi}{N}n} \cdot e^{j\frac{\pi}{3}} + e^{(3)j\frac{2\pi}{N}n} \cdot 2e^{-j\frac{\pi}{6}} + +e^{(2)j\frac{2\pi}{N}n} \cdot 2e^{j\frac{\pi}{6}} && (1)\\
&= 2 + e^{(-4)j\frac{2\pi}{N}n} \cdot e^{-j\frac{\pi}{3}} + e^{(4)j\frac{2\pi}{N}n} \cdot e^{j\frac{\pi}{3}} + e^{(-2)j\frac{2\pi}{N}n} \cdot 2e^{-j\frac{\pi}{6}} + +e^{(2)j\frac{2\pi}{N}n} \cdot 2e^{j\frac{\pi}{6}} && (2)\\
&= 2 + 2\cos\left(\frac{8\pi}{N}n + \frac{\pi}{3}\right) + 4\cos\left(\frac{4\pi}{N}n + \frac{\pi}{6}\right) && (3)\\
&= 2 + 2\sin\left(\frac{8\pi}{5}n + \frac{5\pi}{6}\right) + 4\sin\left(\frac{4\pi}{5}n + \frac{2\pi}{3}\right). && (4)
\end{aligned}$$

2(a)

Periodicity is $N = 8$. Using synthesis equation and the coefficients given in the problem we have

$$\begin{aligned}
x[n] &= \sum_{k=0}^{6} \sin(\frac{\pi k}{3}) e^{jk\frac{\pi}{4}n} && (5)\\
&\sum_{k=0}^{6} \frac{\left(e^{\frac{j\pi k}{3}} - e^{-\frac{j\pi k}{3}}\right)}{2j} e^{jk\frac{\pi}{4}n} && (6)\\
&= \frac{1}{2j} \sum_{k=0}^{6} \left(e^{jk\pi(\frac{n}{4}+\frac{1}{3})} - e^{jk\pi(\frac{n}{4}-\frac{1}{3})}\right), && (7)
\end{aligned}$$

now using geometric series sum formula we have

$$x[n] = \frac{1}{2j}\left(\frac{1-e^{j7\pi(\frac{n}{4}+\frac{1}{3})}}{1-e^{j\pi(\frac{n}{4}+\frac{1}{3})}} - \frac{1-e^{j7\pi(\frac{n}{4}-\frac{1}{3})}}{1-e^{j\pi(\frac{n}{4}-\frac{1}{3})}}\right) \tag{8}$$

$$= \frac{1}{2j}\left(e^{j\frac{3\pi n}{4}}\frac{\sin(\frac{7}{2}\pi(\frac{n}{4}-\frac{1}{3}))}{\sin(\frac{1}{2}\pi(\frac{n}{4}-\frac{1}{3}))} - e^{j\frac{3\pi n}{4}}\frac{\sin(\frac{7}{2}\pi(\frac{n}{4}+\frac{1}{3}))}{\sin(\frac{1}{2}\pi(\frac{n}{4}+\frac{1}{3}))}\right). \tag{9}$$

2(b)

Periodicity is $N = 8$. Using synthesis equation, the coefficients given in the problem and using the fact that $e^{j\pi n} = (-1)^n$ we have

$$x[n] = \sum_{k=0}^{7} a_k e^{jk\frac{\pi}{4}n} \tag{10}$$

$$= 1 + e^{jn\frac{\pi}{4}} + e^{jn\frac{3\pi}{4}} + (-1)^n + e^{jn\frac{5\pi}{4}} + e^{jn\frac{7\pi}{4}} \tag{11}$$

$$= 1 + e^{jn\frac{\pi}{4}} + e^{jn\frac{3\pi}{4}} + (-1)^n + e^{jn\frac{5\pi}{4}-2\pi n} + e^{jn\frac{7\pi}{4}-2\pi n} \tag{12}$$

$$= 1 + (-1)^n + \left(e^{jn\frac{\pi}{4}} + e^{-jn\frac{\pi}{4}}\right) + \left(e^{jn\frac{3\pi}{4}} + e^{-jn\frac{3\pi}{4}}\right) \tag{13}$$

$$= 1 + (-1)^n + 2\cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{3\pi}{4}n\right). \tag{14}$$

Problem 2 - Analyze:

a(a) Given the period $N = 7$, hence using the analysis equation and geometric series sum formula for $k \neq 0$ we have

$$a_k = \frac{1}{7}\sum_{N=0}^{6} x[n]e^{-j\frac{2\pi}{7}kn} = \frac{1}{7}\sum_{N=0}^{4} e^{-j\frac{2\pi}{7}kn} = \frac{1}{7}\frac{1-e^{-j\frac{10\pi}{7}k}}{1-e^{-j\frac{2\pi}{7}k}} = \frac{1}{7}\frac{e^{-j\frac{5\pi}{7}k}}{e^{-j\frac{\pi}{7}k}}\frac{\sin(\frac{5\pi}{7}k)}{\sin(\frac{\pi}{7}k)} \tag{15}$$

$$= \frac{1}{7}e^{-j\frac{4\pi}{7}k}\frac{\sin(\frac{5\pi}{7}k)}{\sin(\frac{\pi}{7}k)}. \tag{16}$$

For $k = 0$, we have $a_0 = \frac{5}{7}$.

2(b)

Given the signal is periodic with $N = 12$, hence using the analysis equation and geometric series sum formula for k we have

$$a_k = \frac{1}{12}\sum_{N=0}^{11}\left(1 - \sin(\frac{n\pi}{4})\right)e^{-j\frac{\pi}{6}kn} \tag{17}$$

$$= \frac{1}{12}\left(1 + (1-\frac{1}{\sqrt{2}})e^{-j\frac{\pi}{6}k} + (1-\frac{1}{\sqrt{2}})e^{-j\frac{3\pi}{6}k} + e^{4\frac{\pi}{6}k} + (1+\frac{1}{\sqrt{2}})e^{-j\frac{5\pi}{6}k} + \right. \tag{18}$$

$$\left. + 2e^{-jk\pi} + (1+\frac{1}{\sqrt{2}})e^{-j\frac{7\pi}{6}k} + e^{8\frac{\pi}{6}k} + (1-\frac{1}{\sqrt{2}})e^{-j\frac{9\pi}{6}k} + (1-\frac{1}{\sqrt{2}})e^{-j\frac{11\pi}{6}k}\right), \tag{19}$$

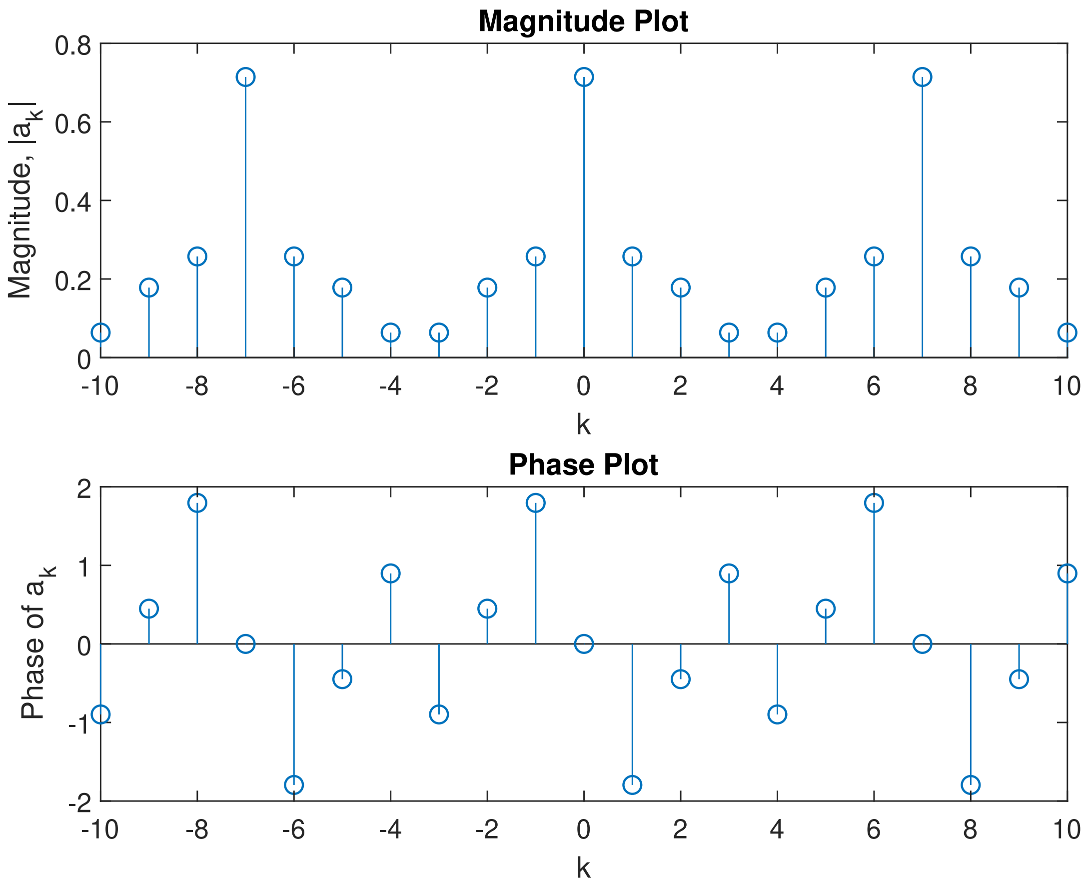


Figure 1: Magnitude and phase of a_k in problem 3. 28 (a)

again the using the periodicity property of the coefficients and some rearranging of terms we have

$$a_k = \frac{1}{12}\Bigg(1 + (1 - \frac{1}{\sqrt{2}})(e^{j\frac{\pi}{6}k} + e^{-j\frac{\pi}{6}k}) + (1 - \frac{1}{\sqrt{2}})(e^{j\frac{\pi}{2}k} + e^{-j\frac{\pi}{2}k}) + (e^{j\frac{2\pi}{3}k} + e^{-j\frac{2\pi}{3}k}) + \tag{20}$$

$$+(1 + \frac{1}{\sqrt{2}})(e^{j\frac{5\pi}{6}k} + e^{-j\frac{5\pi}{6}k}) + 2(-1)^k\Bigg) \tag{21}$$

$$= \frac{1}{12}\Bigg(1 + 2(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{6}) + 2(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}) + 2\cos(\frac{2\pi k}{3}) + \tag{22}$$

$$+2(1 + \frac{1}{\sqrt{2}})\cos(\frac{5\pi k}{6}) + 2(-1)^k\Bigg) \tag{23}$$

Problem 3 - Deduce:

- Part (a): Given input $x[n] = \frac{1}{2^n}u[n]$ and output $y[n] = 4^n u[-n]$ satisfy the following linear time invariant difference equation:

$$y[n] - \frac{1}{4}y[n+1] = \delta[\text{n}] = x[n] - \frac{1}{2} x[n-1]$$

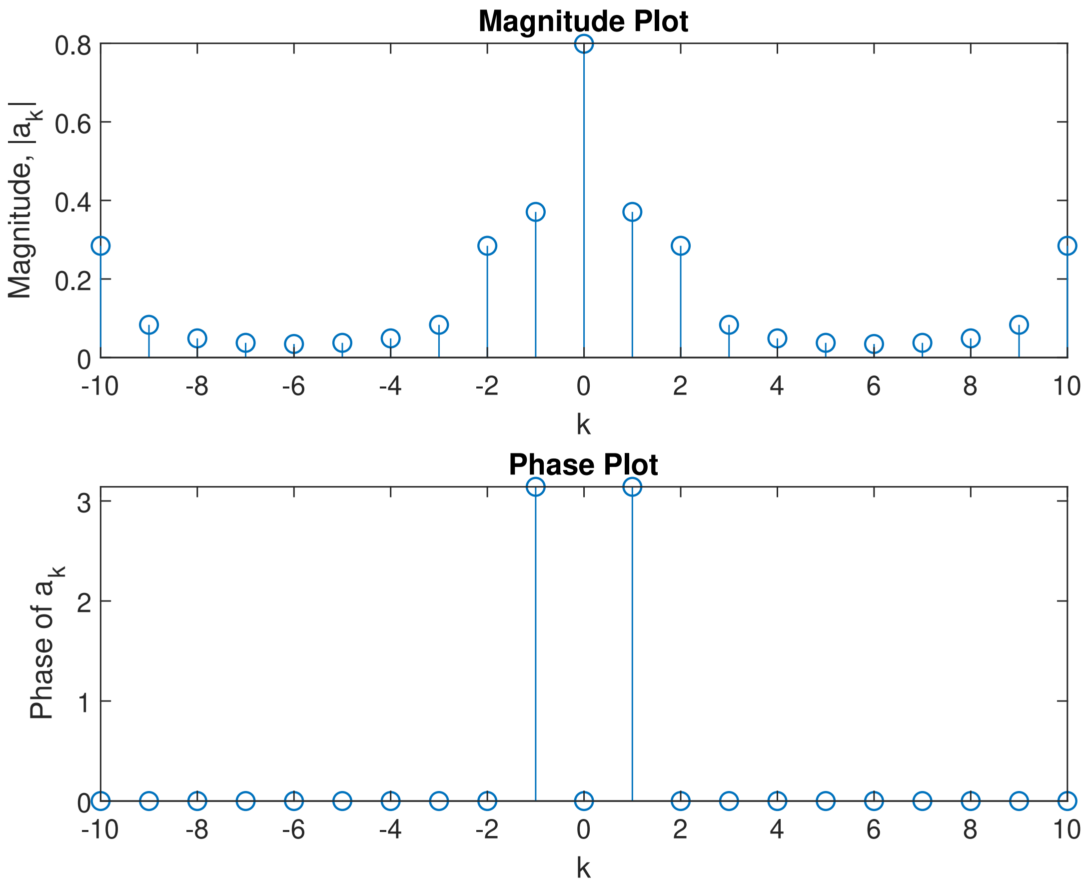


Figure 2: Magnitude and phase of a_k in problem 3. 28 (d)

Now using the eigenfunction property of $e^{j\omega}$ we have

$$e^{j\omega n}H(e^{j\omega}) - \frac{1}{4} \quad e^{j\omega(n+1)}H(e^{j\omega}) = e^{j\omega n} - \frac{1}{2} \quad e^{j\omega(n-1)}$$

$$H(e^{j\omega})\left(1 - \frac{1}{4}e^{j\omega} \quad \right) = 1 - \frac{1}{2}e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega}}{1 - \frac{1}{4}e^{j\omega}}.$$

This system is LTI and unique (although we do not know enough at this point to prove it).

- Part (b): Given input $x[n] = j^n$ and output $y[n] = 2j^n(1-j)$. In other words, $x[n] = e^{j\frac{\pi}{2}n}$ and $y[n] = 2e^{j\frac{\pi}{2}n}(1-e^{j\frac{\pi}{2}})$. This system is LTI but not unique since we only require $H(e^{j\frac{\pi}{2}}) = 2(1-e^{j\frac{\pi}{2}})$.

- Part (c): Given input $x[n] = \cos(\pi\frac{n}{3})$ and output $y[n] = \cos(\pi\frac{n}{3}) + \sqrt{3}\sin(\pi\frac{n}{3})$. In other words, $x[n] = \frac{e^{j\pi\frac{n}{3}} + e^{-j\pi\frac{n}{3}}}{2}$ and $y[n] = 2\Big(\frac{1}{2}\cos(\pi\frac{n}{3}) + \frac{\sqrt{3}}{2}\sin(\pi\frac{n}{3})\Big) = 2\cos(\pi\frac{n}{3} - \frac{\pi}{3}) = e^{j\pi\frac{n}{3}}e^{-j\frac{\pi}{3}} +$

$e^{-j\pi\frac{n}{3}}e^{j\frac{\pi}{3}}$. We have $\omega_0 = \frac{\pi}{3}$, $a_1 = a_{-1} = \frac{1}{2}$. If this system is LTI then, from output $y[n]$ we have $b_1 = e^{-j\frac{\pi}{3}} = a_1 H(e^{j\frac{\pi}{3}})$ and $b_{-1} = e^{j\frac{\pi}{3}} = a_{-1}H(e^{-j\frac{\pi}{3}})$. This system is LTI but not unique since we require only $H(e^{j\frac{\pi}{3}}) = 2e^{-j\frac{\pi}{3}}$ and $H(e^{-j\frac{\pi}{3}}) = 2e^{j\frac{\pi}{3}}$.

- Part (d): Note that both $x[n]$ and $y_1[n]$ are periodic with fundamental periods $N = 12$ and 6. Hence, it is possible to have an LTI system without violating the eigenvalue property like in the part (g). Similar, to part (g), $H(e^{j\omega})$ needs to be of a specific value only for $\omega = k\omega_0$ where $k = 0$, $1, \dots, 11$ for $\omega_0 = \frac{\pi}{6}$.

Problem 4 - Filter:

The fundamental period of the signal is $N = 4$ and fundamental frequency is $\frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$. Since the signal is non-sinusoidal we will use the analysis equation and hence for all k we have

$$\begin{aligned}
a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n} \\
&= \frac{1}{4}\sum_{n=0}^{3}\sum_{i=-\infty}^{\infty} \delta(n-4i)e^{-jk\omega_0 n} \\
&\overset{(a)}{=} \frac{1}{4}\sum_{i=-\infty}^{\infty}\delta(-4i) + \frac{1}{4}\sum_{i=-\infty}^{\infty}\delta(1-4i)e^{-jk\omega_0} + \frac{1}{4}\sum_{i=-\infty}^{\infty}\delta(2-4i)e^{-2jk\omega_0} + \frac{1}{4}\sum_{i=-\infty}^{\infty}\delta(3-4i)e^{-2jk\omega_0} \\
&= \frac{1}{4},
\end{aligned}$$

where (a) is obtained by noting that $\sum_{i=-\infty}^{\infty}\delta(-4i) = 1$ for $i = 0$ and $\sum_{i=-\infty}^{\infty}\delta(1-4i)e^{-jk\omega_0} = 0$ since there is no integer i that can make $1-4i = 0$, hence no integer i for which $\delta(1-4i) = 1$. Use similar logic for rest of the terms.

The frequency response is obtained as

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\
&\overset{(a)}{=} \left(e^{-j\omega 0} + e^{-j\omega} + e^{-2j\omega}\right) + (-1)\left(e^{j\omega} + e^{2j\omega}\right) \\
&= -e^{j\omega} - e^{2j\omega} + 1 + e^{-j\omega} + e^{-2j\omega},
\end{aligned}$$

where (a) is obtained by substituting values $h[n]$ for $n = -2, -1, 0, 1, 2$.

Therefore, $b_k = a_k H(e^{jk\omega_0}) = \frac{1}{4}\left(-e^{jk\frac{\pi}{2}} - e^{2jk\frac{\pi}{2}} + 1 + e^{-jk\frac{\pi}{2}} + e^{-2jk\frac{\pi}{2}}\right) = \frac{1}{4}\left(-e^{jk\frac{\pi}{2}} + 1 + e^{-jk\frac{\pi}{2}}\right)$.

Problem 5 -Properties:

Let $x[n]$ be a periodic signal with period $N = 8$ and Fourier series coefficients satisfying $a_k = -a_{k-4}$. The **frequency shifting property** of DTFS states that

$$\text{if} \quad x[n] \overset{F.S.}{\longleftrightarrow} a_k \quad \text{then} \quad e^{jM\frac{2\pi}{N}} n x[n] \overset{F.S.}{\longleftrightarrow} a_{k-M}.$$

So, setting $M = 4$, we have $e^{j4\frac{2\pi}{8}n} x[n] = e^{j\pi n} x[n] = (-1)^n x[n] \overset{F.S.}{\longleftrightarrow} a_{k-4}$.

Then $(-1)^{n+1} x[n] \longleftrightarrow -a_{k-4}$. Now, since $a_k = -a_{k-4}$, we conclude $x[n] = (-1)^{n+1} x[n]$ which implies

$$x[n-1] = (-1)^n x[n-1]. \tag{1}$$

We are told that

$$y[n] = \left(\frac{1 + (-1)^n}{2} \right) x[n-1]$$

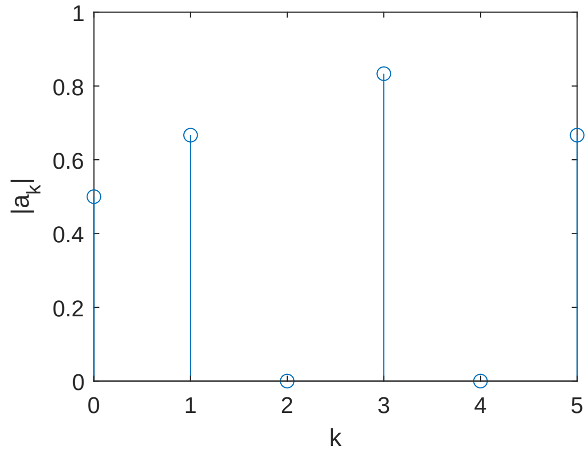
with Fourier series coefficients b_k. Substituting from equation (1), we get $y[n] = x[n-1]$.

The **time-shifting property**, states that

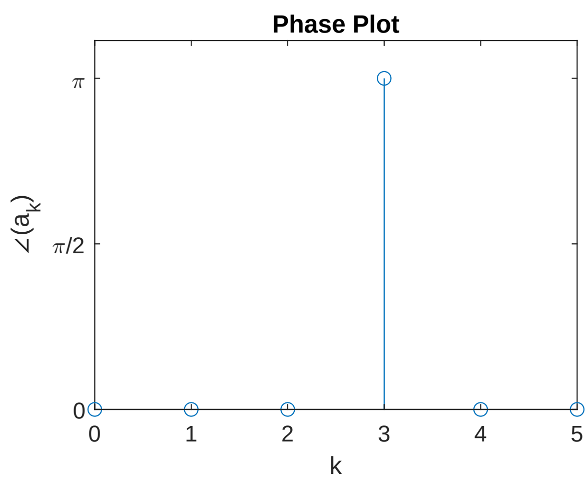
$$x[n - n_0] \longleftrightarrow a_k e^{-jk\frac{2\pi}{N} n_0}$$

so, setting $n_0 = 1$, we conclude

$$b_k = a_k e^{-jk\frac{2\pi}{8}} = e^{-jk\frac{\pi}{4}}.$$



1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
k
|a_k|



Phase Plot
π
π/2
0
0
1
2
3
4
5
k
∠(a_k)