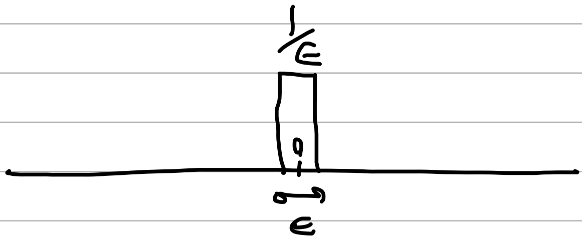
Discussion Session 2: Dirac Delta Function, linear, and time invariant systems.

The dirac delta function: limit of a sequence of functions that become increasing narrow and tall while maintaining an area of 1 under curve. Common approximation of the Dirac Delta function is rectangular pulse:



$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \, \mathrm{Rect}\left(\frac{t}{\epsilon}\right)$$

properties:

1) zero every where except at $t = 0$: $\delta(t) = 0$, for all $t \neq 0$

2) Integral equal one: $\int_{-\infty}^{\infty} \delta(t) \, dt = 1$

3) Shifting Effect: $\int_{-\infty}^{\infty} f(t) \, \delta(t - t_0) \, dt = f(t_0)$

Importance:

The input $\delta(t)$ to a linear System yields the system's impulse response which characterizes the system completely.

Example:

a) $\int_{-\infty}^{\infty} (2t+3) \, \delta(t-2) \, dt = \int_{-\infty}^{\infty} (2 \times 2 + 3) \, \delta(t-2) \, dt = 7 \int_{-\infty}^{\infty} \delta(t-2) \, dt = 7$

b) $\int_{-\infty}^{\infty} \cos(2\pi t) \left[\delta(t-1) + \delta(t+1) \right] = \int_{-\infty}^{\infty} \cos(2\pi t) \, \delta(t-1) \, dt + \int_{-\infty}^{\infty} \cos(2\pi t) \, \delta(t+1) \, dt$

$= \cos(2\pi 1) + \cos(2\pi(-1)) = 1 + 1 = 2$

c) $\int_{-\infty}^{\infty} \delta(at)\,dt \Rightarrow u = at ,\quad du = a\,dt \Rightarrow dt = \frac{du}{a}$

for $a > 0$: $= \int_{-\infty}^{\infty} \delta(u)\frac{du}{a} = \frac{1}{a}$

for $a < 0$, $= \int_{\infty}^{-\infty} \delta(u)\frac{du}{a} = \frac{-1}{a}\int_{-\infty}^{\infty} \delta(u)\,du = \frac{1}{-a} = \frac{1}{|a|}$

$\Rightarrow \delta(a.x) = \frac{1}{|a|}\delta(x)$

d) $\int_{-\infty}^{\infty} e^{-t^2}\delta(3t-6)\,dt = \int_{-\infty}^{\infty} e^{-t^2}\delta(3(t-2))\,dt$

$= \int_{-\infty}^{\infty} e^{-t^2}\frac{1}{3}\delta(t-2)\,dt = \frac{1}{3}e^{-4}$

Time - Invariant vs. Time - Variant Systems:

A system is time-invariant if its behavior does not change over time.

Mathematically $y(t) = \text{System}[x(t)]$ implies $y(t-t_0) = \text{System}[x(t-t_0)]$

Shifting the input $x(t)$ by t_0 results in the output $y(t)$ also being shifted by t_0.

A system is time-variant if its behavior depend on time, meaning a time shift in the input does not result in the same shift in the output.

How to check?

1) Shift the input signal $x(t)$ by t_0: Replace $x(t)$ with $x(t-t_0)$
2) Pass the shifted input through the system and find new output $y'(t)$
3) Shift the original output $y(t)$ to get $y(t-t_0)$
4) Compare $y'(t)$ and $y(t-t_0)$

Linear vs. Non Linear Systems:

A system is linear if it satisfies two properties.

1) Additivity: the response to sum of inputs equals the sum of the response to each input:

If $y_1(t) = System[x_1(t)]$ and $y_2(t) = System[x_2(t)]$
then $System[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$

2) Homogenity: The response to a scaled input is the scaled response:
$y(t) = System[x(t)]$, then $System[a \cdot x(t)] = a \cdot y(t)$

How to check?

1) Additivity: if $y_1(t) = System[x_1(t)]$ and $y_2(t) = System[x_2(t)]$, then $S[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$

2) Homogeneity (Scaling): If $y(t) = S[x(t)]$, then $S[a \cdot x(t)] = a \cdot y(t)$

Example:

a) $y(t) = x(2-t) + 1$

time-invariance:

$$\left.\begin{cases} y'(t) = x(2 - t - t_0) + 1 = x(2 - t - t_0) + 1 \\ y(t-t_0) = x(2 - (t - t_0)) + 1 = x(2 - t + t_0) + 1 \end{cases}\right\} \Rightarrow \text{Time-variant}$$

Linearity:

$$\left.\begin{array}{l} y'(t) = a\,x(2-t) + 1 \\ a\,y(t) = a\,x(2-t) + a \end{array}\right\} \longrightarrow \text{not linear}$$

b) $y(t) = t\sin(x(t))$

time-invariance:

$$\left.\begin{cases} y'(t) = t \sin(x(t-t_0)) \\ y(t-t_0) = (t-t_0) \sin(x(t-t_0)) \end{cases}\right\} \longrightarrow \text{Time-variant}$$

Linearity:

$$S[x_1(t)+x_2(t)] = t\sin(x_1(t)+x_2(t)) = t[\sin(x_1(t))\cos(x_2(t)) + \sin(x_2(t))$$
$$\cos(x_1(t))]$$

$$S[x_1(t)] + S[x_2(t)] = t\sin(x_1(t)) + t\sin(x_2(t))$$

c) $$y(t) = \int_t^{t+2} x(T)\,dT$$

time invariance

$$\begin{cases} y'(t) = \int_t^{t+2} x(T-t_0)\,dT = \int_{t-t_0}^{t+2-t_0} x(z)\,dz & (\text{take } z = T-t_0) \\ y(t-t_0) = \int_{t-T_0}^{t-T_0+2} x(T)\,dT & \Rightarrow (\text{time invariant}) \end{cases}$$

You can show it is linear.

d) $$y(t) = \int_{t-1}^{t+1} (t-T)\,x(T)\,dT$$

$$y'(t) = \int_{t-1}^{t+1} (t-T)\,x(T-t_0)\,dT$$

$$y(t-T_0) = \int_{t-t_0-1}^{t-t_0+1} (t-t_0-T)\,x(T)\,dT$$

$$= \int_{t-T_0-1}^{t-t_0+1} (t-(T+t_0))\,x(T)\,dT \qquad z = T+t_0 \rightarrow T = z-t_0, \quad dz = dT$$

$$= \int_{t-1}^{t+1} (t-z)\,x(z-t_0)\,dz \Rightarrow \text{Time - Invariant}$$