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Chapter 4
Continuous-time Fourier transform

Signals and Systems

Fourier transform

Key points

- Define Fourier transform
- Determine the Fourier transforms of CT and DT signals

Fourier transform

- For a signal $x(t)$, the Fourier transform (FT) $X(j\omega)$ is

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- The notation $\mathcal{F}\{\cdot\}$ means to take the Fourier transform of the function inside the brackets

- In some books, $X(\omega)$ is used instead of $X(j\omega)$

Interpreting the Fourier transform

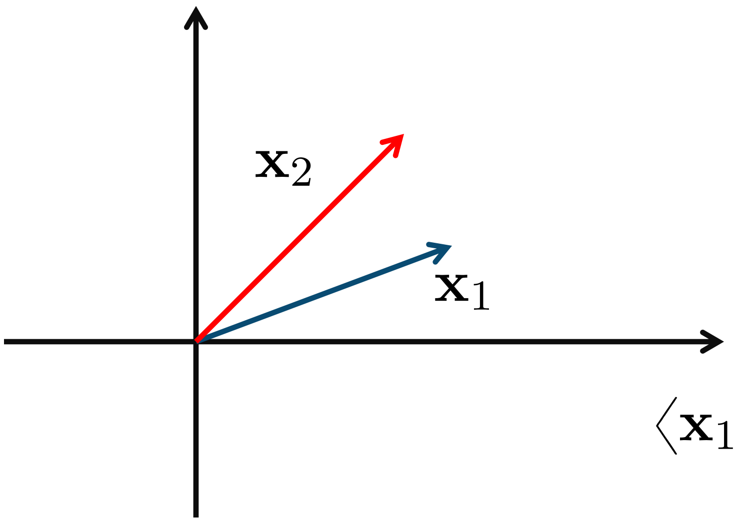
- The Fourier transform can be written

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \langle x(t), e^{j\omega t} \rangle$$

Inner product

How much does $x(t)$ look like complex sinusoid $e^{j\omega t}$ with frequency ω?



$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ Measures how much the vectors "line up" with each other

Why do we care about complex sinusoids anyways?

- For an LTI system, with input $e^{j\omega t}$ it is easy to compute the output

$$e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega)e^{j\omega t}$$

- If a signal can be represented as a sum of sinusoids, then it is possible to compute the output of an LTI system

without convolution

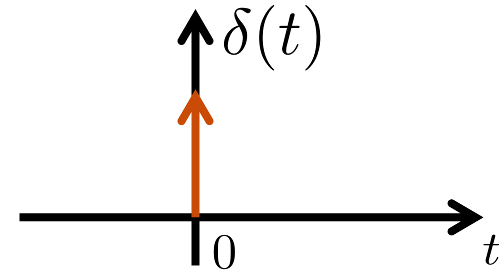
Fourier transform!!

- Further notice that

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

Example – Unit impulse

- Consider the signal $\boxed{x(t) = \delta(t)}$



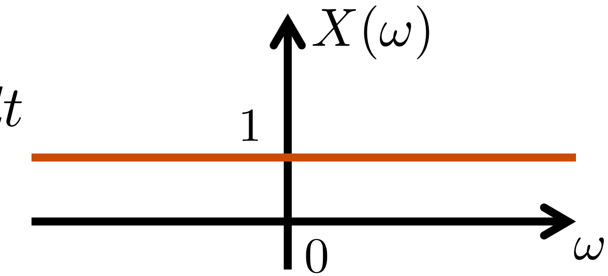
- Its FT is given by

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega 0} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) dt$$

$$= 1$$



Example - Triangle

- Compute the Fourier transform of $$x(t) = \begin{cases} 1 - t & t \in [0, 1] \\ 0 & \text{else} \end{cases}$$

- Solution:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_0^1 (1 - t) e^{-j\omega t} dt \\ &= \int_0^1 e^{-j\omega t} dt - \int_0^1 t e^{-j\omega t} dt \end{aligned}$$

Example - Triangle (continued)

- Compute the Fourier transform of $$x(t) = \begin{cases} 1 - t & t \in [0,1] \\ 0 & \text{else} \end{cases}$$

- Solution:

$$\begin{aligned}
X(j\omega) &= \int_0^1 e^{-j\omega t} dt - t\frac{1}{-j\omega} e^{-j\omega t}\Big|_0^1 + \int_0^1 \frac{1}{-j\omega} e^{-j\omega t} dt \\
&= \int_0^1 e^{-j\omega t} dt - t\frac{1}{-j\omega} e^{-j\omega t}\Big|_0^1 + \left(\frac{1}{j\omega}\right)^2 e^{-j\omega t}\Big|_0^1 \\
&= \frac{1}{j\omega} - \frac{1}{j\omega} e^{-j\omega} + \frac{1}{j\omega} e^{-j\omega} - \frac{1}{\omega^2} e^{-j\omega} + \frac{1}{\omega^2} \\
&= \frac{1}{j\omega} - \frac{1}{\omega^2} e^{-j\omega} + \frac{1}{\omega^2}
\end{aligned}$$

Inverse Fourier transform

- Given the frequency response $X(j\omega)$, $x(t)$ is given by the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

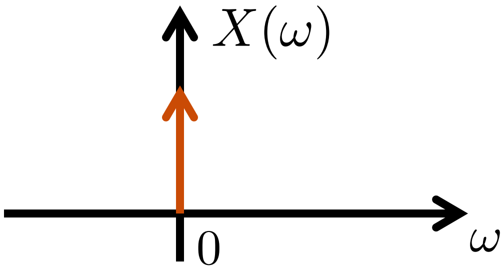
$$= \mathcal{F}^{-1}\{X(j\omega)\}$$

- Transform looks similar to direct transform except
  - Sign of the exponential is different
  - Scaling factor in front (results from using radians and not Hertz)

Similarity will lead to the concept of duality

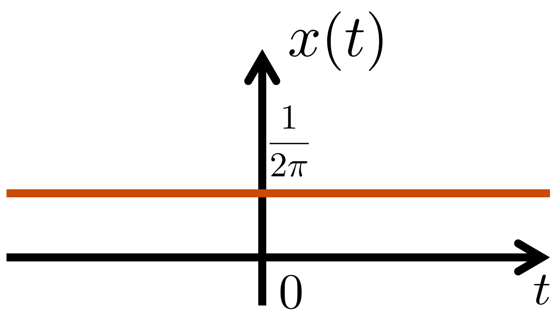
Example – Unit impulse in frequency

- Consider the signal $\boxed{X(j\omega) = \delta(\omega)}$



- Its FT is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}d\omega$$
$$= \frac{1}{2\pi}$$



Notice the duality here in each domain

Example – Shifted delta in frequency

- Consider an impulse in the frequency domain $X(j\omega) = \delta(\omega - \omega_0)$

- Its inverse FT is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega$$

$$= \frac{e^{j\omega_0 t}}{2\pi} \underbrace{\int_{-\infty}^{\infty} \delta(\omega - \omega_0) d\omega}_{1}$$

$$= \frac{e^{j\omega_0 t}}{2\pi}$$

Example – A few deltas in frequency

- Use the Fourier transform synthesis equation to determine the inverse Fourier transform of

$$X(\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$$

- Solution

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)\right) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\left(2\pi e^{j0t} + \pi e^{j4\pi t} + \pi e^{-j4\pi t}\right) \\ &= 1 + \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} \\ &= 1 + \cos(4\pi t) \end{aligned}$$

Summarizing the Fourier transform and its inverse

Fourier transform (analysis)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform (synthesis)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\boxed{x(t) \leftrightarrow X(j\omega)}$$

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Note: An alternative formulation that is common uses Hertz rather than radians/sec

Fourier transform in Hertz

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \quad f \text{ in Hz}$$

Inverse Fourier transform from X(f) (note lack of 1/2π)

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

In EE 313 we will use $X(j\omega)$ (in EE 471C we use above notation)

Existence of the Fourier transform

Key points

- Identify sufficient conditions for the “integrals to work”
- Define the Fourier transforms of CT and periodic CT signals

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Sufficient condition: Finite energy

- Consider the class of signals that have finite energy $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$
- For such signals the Fourier transform exists (is finite)
- Further there is zero-energy in the error in the sense that

$$\int_{-\infty}^{\infty} \left| x(t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right|^2 = 0$$

FT gives zero squared error for finite energy signals

Sufficient condition: satisfies Dirichlet

If true

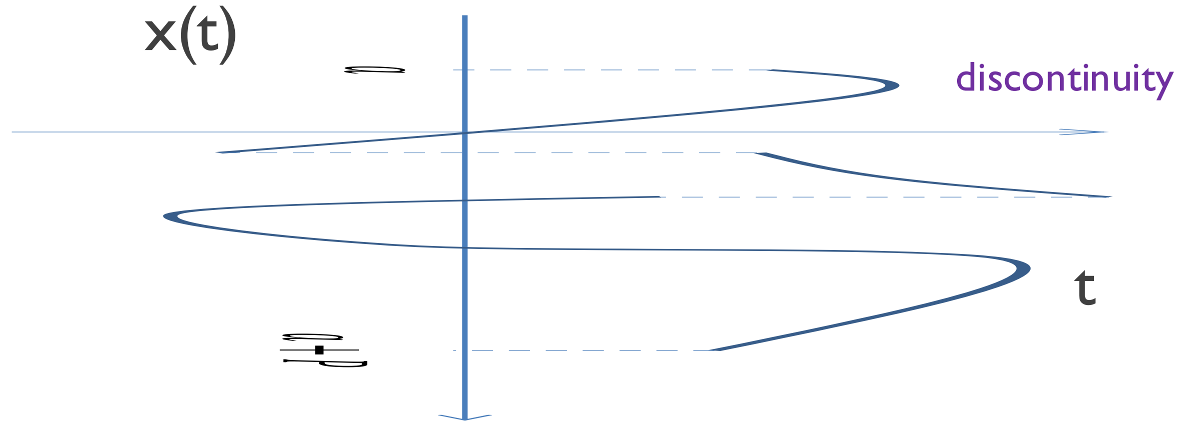
- An aperiodic signal $x(t)$ that satisfies the following conditions
  - (1) Absolute integrability $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
  - (2) Finite number of minima and maxima over a finite interval
  - (3) Finite number of discontinuities over a finite interval

Then

$x(t)$ and $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ are equal except at a discontinuity

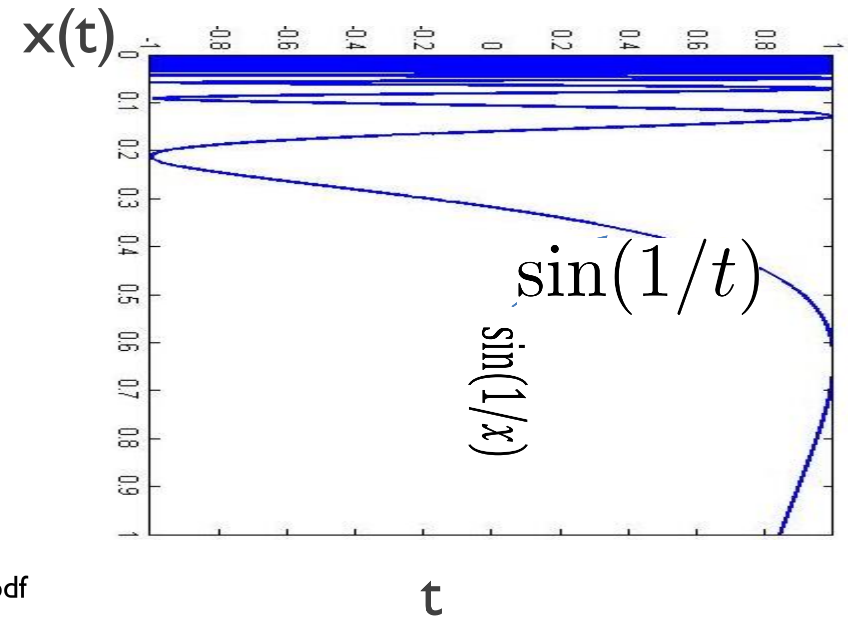
Example

Satisfied



number of maxima and minima tends to infinity as t approaches zero

Not satisfied



http://volkov.eng.ua.edu/ME501/2017-Fall-ME501-05-FourierAnalysis.pdf

What about periodic signals?

- Periodic signals do not satisfy either sufficient condition because

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |x(t)| dt = \infty$$

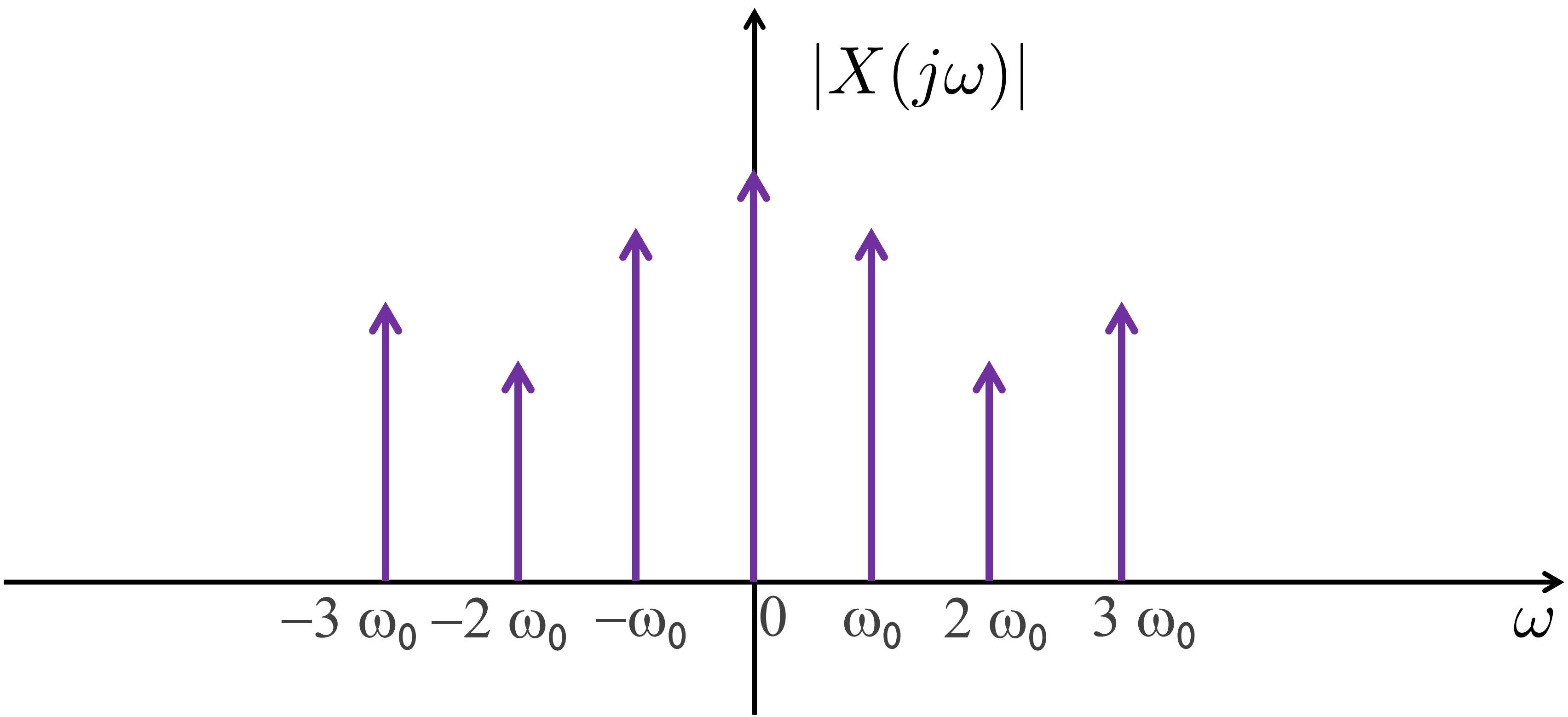
- If Dirac delta is acceptable $\delta(t)$ then can define the FT from FS
- Consider a periodic signal $x(t) \leftrightarrow \{a_k\}$ then the FT of $x(t)$ is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Fourier transform

FS coefficients

Periodic signals have a "line" spectrum at the harmonics of the fundamental frequency, as we have seen



Example – Shifted delta in frequency (again)

- Consider the signal

$$x(t) = \frac{1}{2\pi} e^{j\omega_0 t}$$

- The signal is periodic with fundamental frequency ω_0

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \underbrace{\frac{1}{2\pi}}_{a_1} e^{j\omega_0 t}$$

k=1 in terms of FS

$$\frac{1}{2\pi} e^{j\omega_0 t} \leftrightarrow \delta(\omega - \omega_0)$$

more generally

$$\boxed{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

Fourier transform examples

Key points

- Use the Fourier transform synthesis and analysis equations
- Learn and use the transforms for common signals

Fourier transform of a causal exponential

- Consider the signal $\boxed{x(t) = e^{-at}u(t), \quad \mathrm{Re}\{a\} > 0}$

- Its FT is given by

$$X(j\omega) = \underbrace{\int_0^{\infty}}_{\text{b/c of } u(t)} \underbrace{e^{-at}}_{x(t)} e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

$$= \frac{1}{a+j\omega}$$

Fourier transform of a cosine

- Consider the signal $\boxed{x(t) = \cos \omega_0 t}$

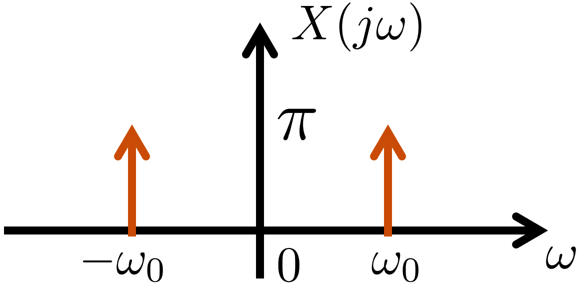
- This is a periodic signal, can find its FT using the FS

$$x(t) = \cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

- Leveraging the frequency impulse results

$$X(j\omega) = \frac{1}{2}\left(2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right)$$

$$= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier transform of a sine

- Consider the signal $\boxed{x(t) = \sin \omega_0 t}$

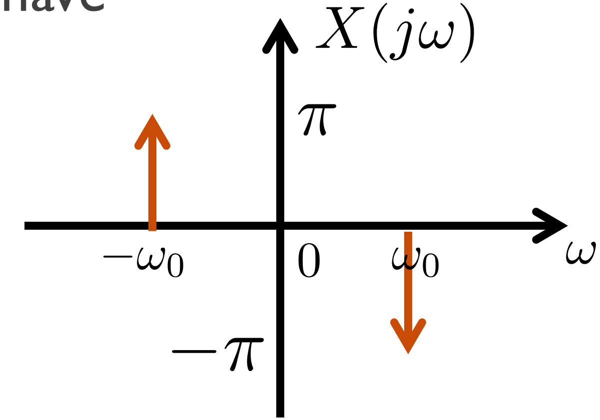
- To get its FT, we note that

$$x(t) = \sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

- Leveraging the frequency impulse results, we have

$$X(j\omega) = \frac{1}{2j}\left(2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)\right)$$

$$= \pi j\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right)$$



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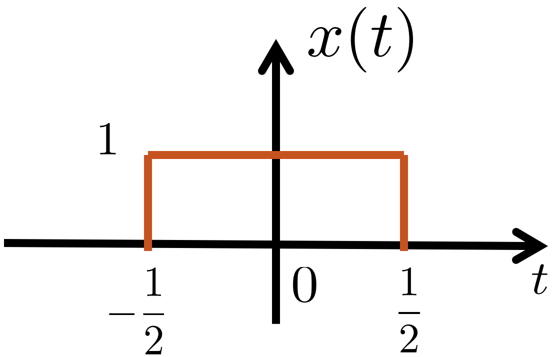
FT of a rectangle function is an important example

Fourier transform of a rectangle function

- Consider the signal

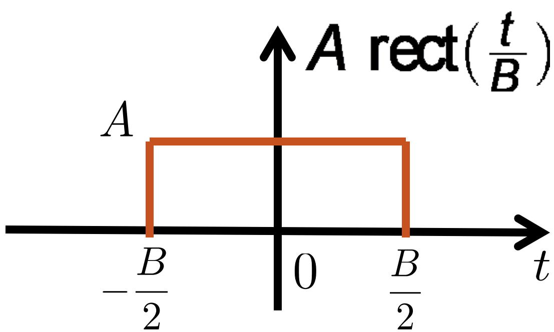
$$\boxed{x(t) = \text{rect}(t) = \Pi(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leqslant \frac{1}{2} \end{cases}}$$

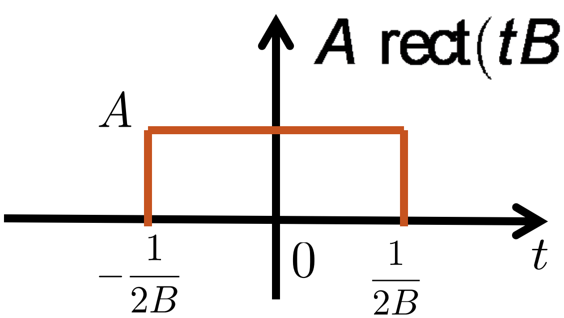


- Note that

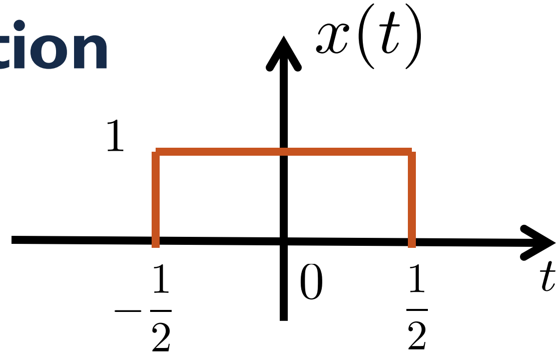
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = 1$$

- Can create other related shapes





Fourier transform of a rectangle function



- Its FT is given by

$$X(j\omega) = \int_{-\infty}^{\infty} \mathrm{rect}(t) e^{-j\omega t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{j\omega} (e^{-\frac{j\omega}{2}} - e^{\frac{j\omega}{2}})$$

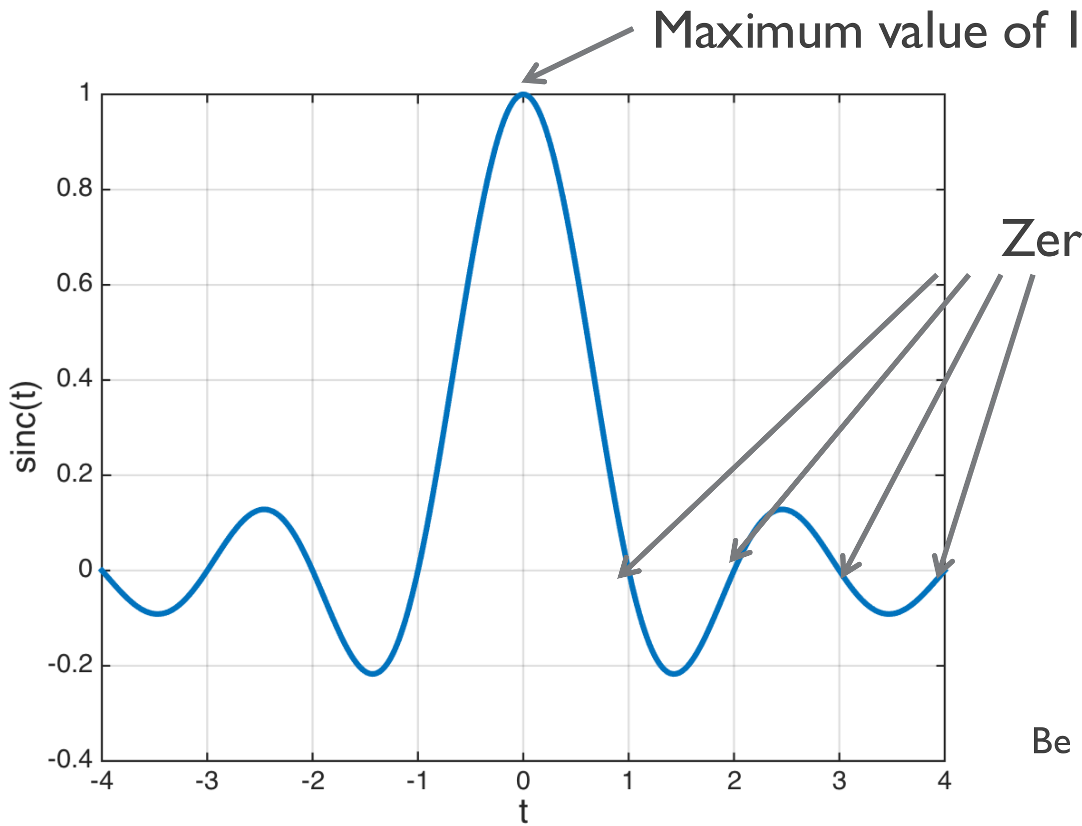
$$= \frac{2}{\omega} \cdot \frac{1}{2j} (e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}})$$

$$= \frac{2}{\omega} \sin \frac{\omega}{2}$$

$$= \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$$

$$= \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

About the sinc function



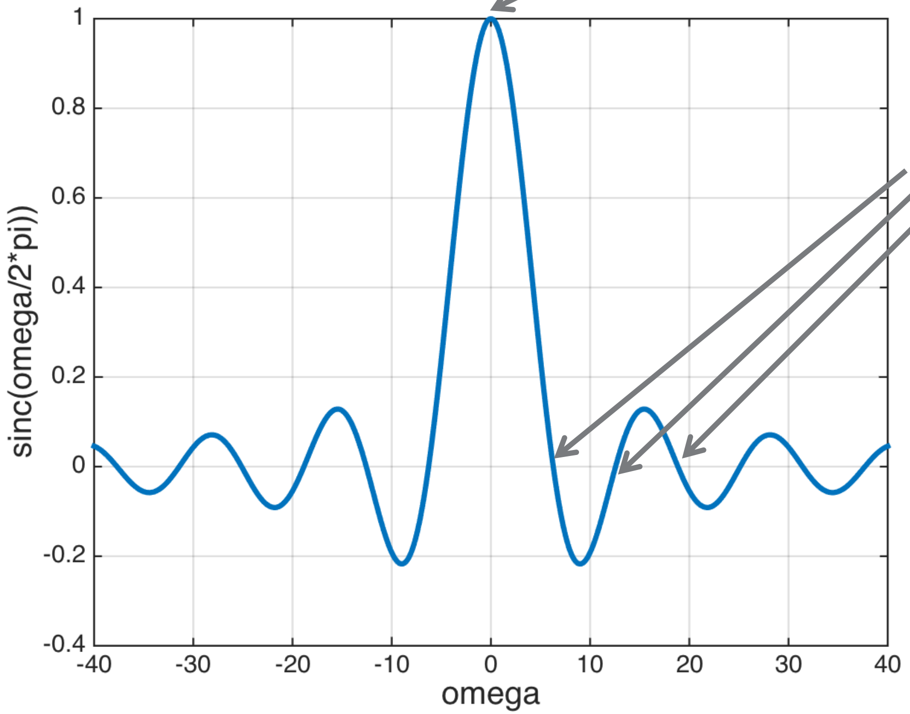
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Be aware, sometimes sinc is defined like this

$$\mathrm{sinc}(t) = \frac{\sin(t)}{t}$$

Fourier transform of rectangle function

Maximum value of 1

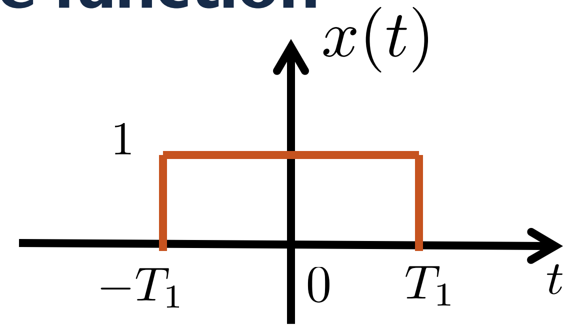


Zero crossings at +/- 2π,+/- $4\pi\ldots$

$$X(j\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

Fourier transform of a scaled rectangle function

- Its FT is given by



$$X(j\omega) = \int_{-\infty}^{\infty} \mathrm{rect}(t/2T_1) e^{-j\omega t} dt$$

$$= \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}(e^{-j\omega T_1} - e^{j\omega T_1})$$

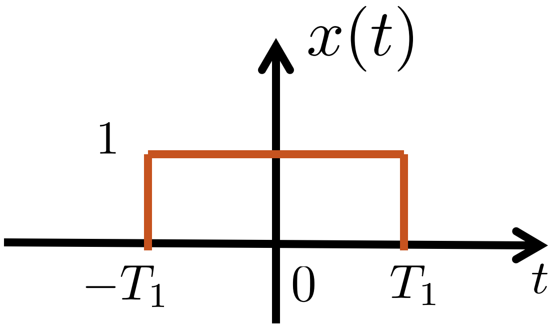
$$= \frac{2}{\omega} \cdot \frac{1}{2j}(e^{j\omega T_1} - e^{-j\omega T_1})$$

$$= \frac{2}{\omega} \sin \omega T_1$$

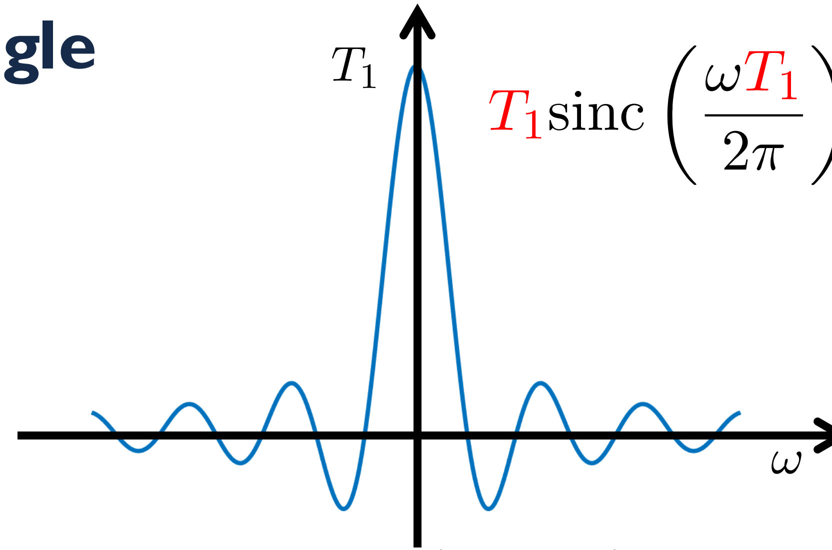
$$= T_1 \frac{\sin \frac{\omega T_1}{2}}{T_1 \frac{\omega}{2}}$$

$$= T_1 \mathrm{sinc}\left(\frac{\omega T_1}{2\pi}\right)$$

Fourier transform of rectangle



$$\overset{F}{\longleftrightarrow}$$



Zeros at $\pm\pi/T_1, \pm\pi/T_1, \ldots$

- Crossings at $\pm\pi/T_1, \pm\pi/T_1, \ldots$
- For $T_1 \to \infty$, FT is $\delta($! $)$
- So
  - $T_1 \downarrow$, pulse narrow, sinc wide
  - $T_1 \uparrow$, sinc becomes narrow, pulse wide

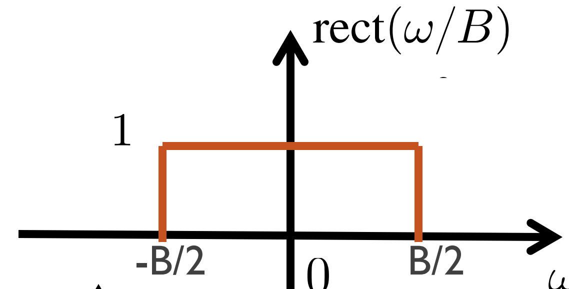
Narrow pulse in time is broad in frequency

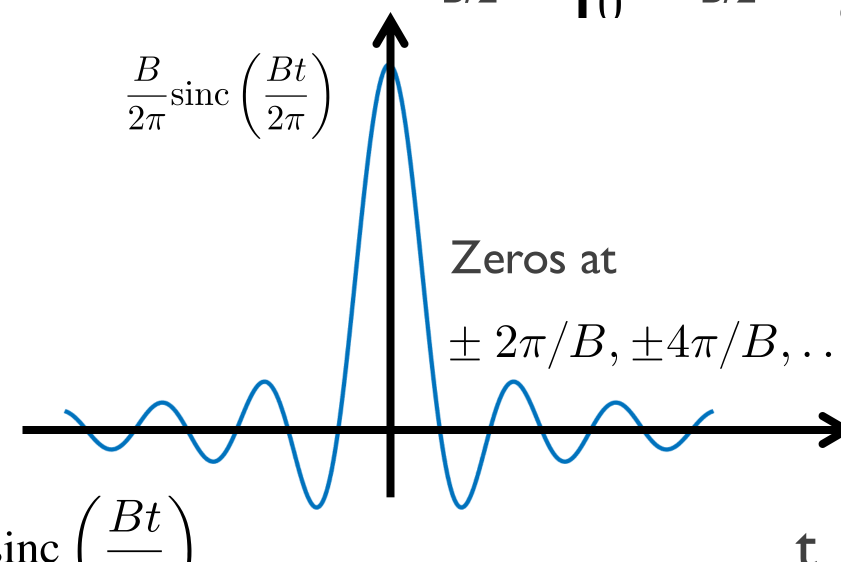
Inverse Fourier transform of the rectangle function

- Can compute the FT directly

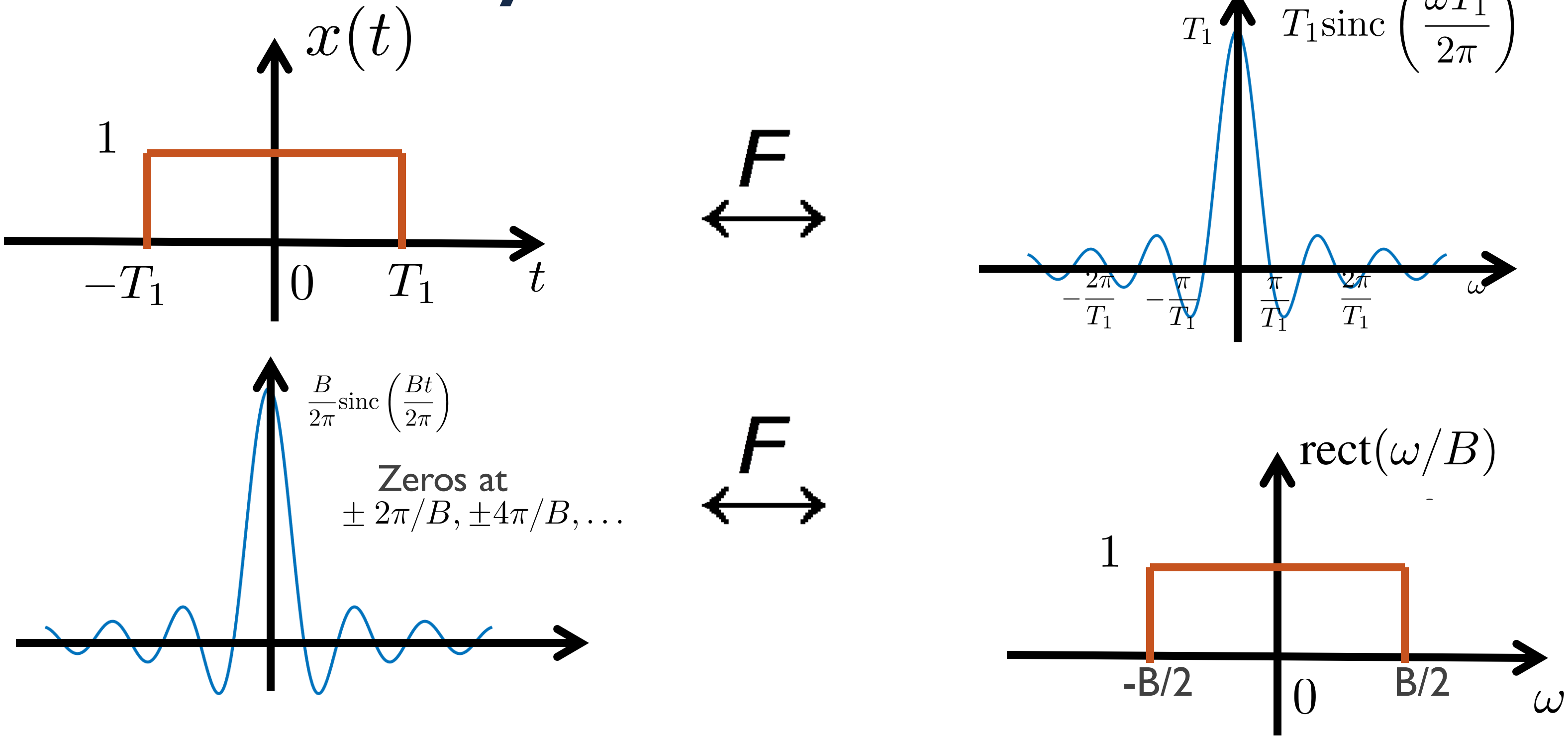
$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\frac{B}{2}}^{\frac{B}{2}} e^{j\omega t}d\omega \\
&= \frac{1}{j2\pi t}e^{j\omega t}\Big|_{-\frac{B}{2}}^{\frac{B}{2}} \\
&= \frac{1}{j2\pi t}\left(e^{\frac{jB}{2}t} - e^{-\frac{jB}{2}t}\right) \\
&= \frac{1}{\pi t}\sin(Bt/2) \\
&= \frac{B/2\pi}{\pi Bt/2\pi}\sin\left(\frac{Bt\pi}{2\pi}\right)
\end{aligned}
$$

$$\Rightarrow \frac{B}{2\pi}\text{sinc}\left(\frac{Bt}{2\pi}\right)$$





Preview of duality



It is possible to find one set of transforms from the other

Essential Fourier transforms

Key points

- Know these important FT pairs

(more pairs in the book)

Basic Fourier transform pairs 1/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Delta	$\delta(t)$	2π
Constant	$\frac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\frac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega - \omega_0)$
Causal exponential	$e^{-at}u(t)$ $\mathrm{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$

(more pairs in the book)

Basic Fourier transform pairs 2/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Cosine	$\cos \omega_0 t$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
Sine	$\sin \omega_0 t$	$\pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

(more pairs in the book)

Basic Fourier transform pairs 3/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Rectangle	$\mathrm{rect}(t)$	$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
Scaled rectangle	$\mathrm{rect}\left(\frac{t}{2T_1}\right)$	$2T_1\frac{\sin(\omega)}{\omega} = 2T_1\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$
Sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
Scaled sinc	$\frac{B}{2\pi}\mathrm{sinc}\left(\frac{Bt}{2\pi}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$

Fourier transform properties

Key points

- Use FT properties to simplify calculation & build intuition
- Analyze problems that include FT properties

Fourier transform properties I $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

	Time domain	Fourier transform
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(j\omega)$

Fourier transform properties 2

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

	Time domain	Fourier transform
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert} x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$	

Fourier transform properties 3

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

$$h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(j\omega)$$

	Time domain	Fourier transform
Convolution in time	$y(t) = h(t) * x(t)$	$Y(j\omega) = H(j\omega)X(j\omega)$
Multiplication in time	$y(t) = h(t)x(t)$	$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$

Linearity

- If $$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega), \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$
- Then $$ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$$

Sums in time lead to sums in frequency

Linearity example

- Consider

$$\cos t \leftrightarrow \pi[\delta(\omega - 1) + \delta(\omega + 1)]$$

$$\sin t \leftrightarrow \pi j\,[\delta(\omega + 1) - \delta(\omega - 1)]$$

- By linearity

$$\cos t + j\sin t \leftrightarrow \pi\delta(\omega - 1) + \underbrace{\pi\delta(\omega + 1) - \pi\delta(\omega + 1)}_{0} + \pi\delta(\omega - 1)$$

$$= 2\pi\delta(\omega - 1)$$

$$= F\{e^{jt}\}$$

Time shifting

- If
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then
$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

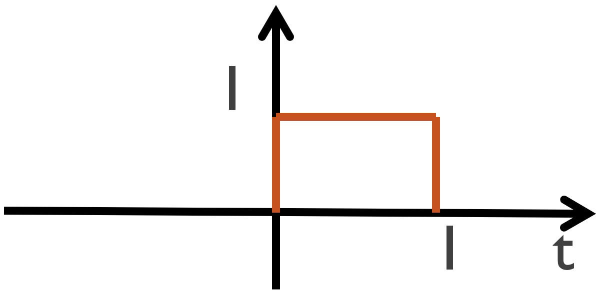
- Time shifting does not change the magnitude in the freq. domain
$$|X(j\omega)e^{-j\omega t_0}| = |X(j\omega)||e^{-j\omega t_0}|$$

- Phase changes are linear with frequency $!$ and shift t_0
$$\angle(X(j\omega)e^{-j\omega t_0}) = \angle X(j\omega) - \omega t_0$$

Shift in time leads to linear phase shift in frequency

Time shifting example

- Find the Fourier transform of



- This signal is just a shifted rectangle function $\mathrm{rect}(t-1/2)$

- Using the time shifting property

$$\mathrm{rect}(t-1/2) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega/2}\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

Differentiation

- If
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
- Then
$$\frac{dx}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$
- Proof
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$
$$\underbrace{\frac{dx}{dt}}_{\text{new func}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}(e^{j\omega t})d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega X(j\omega)}_{\text{new FT}} e^{j\omega t}d\omega$$

Differentiation example

- What is the FT of the system characterized by

$$\frac{dy}{dt} + ay(t) = x(t)$$

- Solution:
  - Take FT of both sides

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$
$$(j\omega + a)Y(j\omega) = X(j\omega)$$

  - Therefore $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{1}{j\omega + a}$

Integration

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

DC component

Time scaling

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Time expansion |a|<1 leads to frequency compression

Time compression |a|>1 leads to frequency expansion

Frequency scaling

- If
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

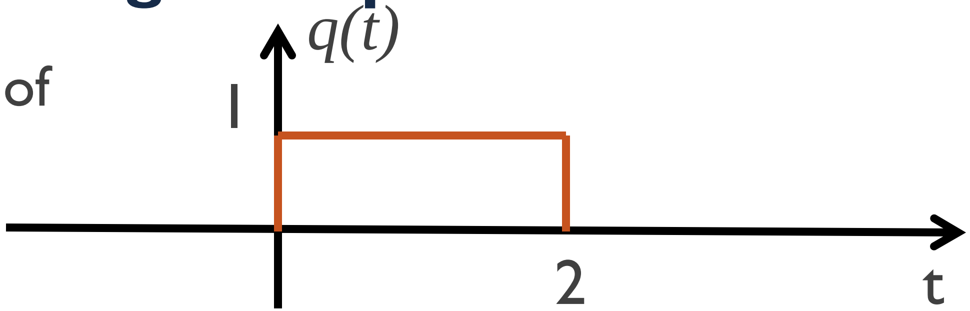
- Then
$$\frac{1}{|b|}x\left(\frac{t}{b}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(jb\omega)$$

Frequency expansion |b|<1 leads to time compression

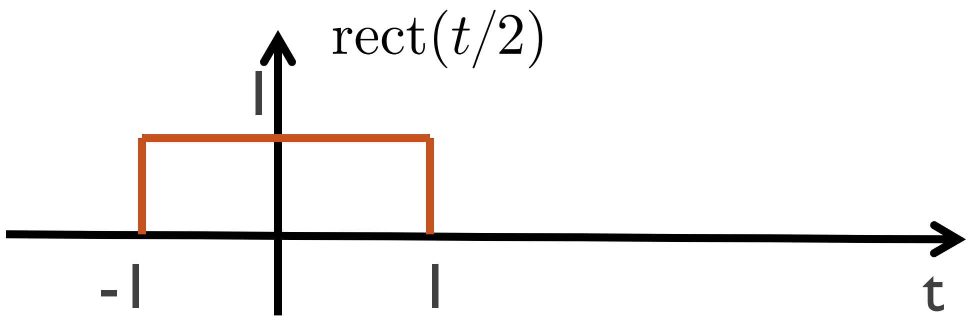
Frequency compression |b|>1 leads to time expansion

Shift and scaling example

- Find the FT of



- To start, notice that



- Shifting gives our function $q(t) = \mathrm{rect}\left(\frac{t-1}{2}\right) = \mathrm{rect}\left(\frac{t}{2} - \frac{1}{2}\right)$

Shift and scaling example (continued)

- From the scaling property

$$\mathrm{rect}\left(\frac{t}{2}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\mathrm{sinc}\left(\frac{2\omega}{2\pi}\right) = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

- From the shift property

$$\mathrm{rect}\left(\frac{t-1}{2}\right) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega} 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

Another scaling example

- What is the inverse Fourier transform of $\text{sinc}(\omega)$?
- We know that
  - From the rect-sinc Fourier pair $$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\frac{\omega}{2\pi}\right)$$
  - From the scaling law $$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$
- Using the scaling property $$\text{rect}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\text{sinc}\left(\omega\right)$$
- Therefore using linearity $$\frac{1}{2\pi}\text{rect}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\omega\right)$$

Inversion

- Find the FT of $x(-t)$

- This is just a special case of time and frequency scaling

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- With $a = -1$

$$x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$$

Example using scaling and time shift

- Determine the Fourier transform of $\mathrm{sinc}(1-2t)$

$$
\begin{aligned}
x(t) &= \mathrm{sinc}(1-2t) \\
&= y(2t) \\
y(t) &= \mathrm{sinc}(1-t) \\
&= \mathrm{sinc}(-(t-1)) \\
&= z(t-1) \\
z(t) &= \mathrm{sinc}(-t) \\
&= \mathrm{sinc}(t) \\
\mathrm{rect}(t) &\overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}(\omega/2\pi)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{sinc}(t/2\pi) &\overset{\mathcal{F}}{\longleftrightarrow} 2\pi\mathrm{rect}(-\omega) = 2\pi\mathrm{rect}(\omega) \\
\mathrm{sinc}(t) &\overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}(\omega/(2\pi)) \\
Z(j\omega) &= \mathrm{rect}(\omega/(2\pi)) \\
Y(j\omega) &= e^{-j\omega}Z(j\omega) \\
X(j\omega) &= \frac{1}{2}Y\left(j\frac{\omega}{2}\right) \\
&= \frac{1}{2}e^{-j\omega/2}\mathrm{rect}(\omega/(4\pi))
\end{aligned}
$$

Frequency shifting

- If $$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
- Then $$x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

  this is called modulation

- Corollary $$x(t)\cos\omega_0 t \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$$

Modulate in time leads to shift in frequency

Example combining shift and scaling

- Determine the inverse Fourier transform of

$$X(j\omega) = \frac{2\sin\left(3(\omega - 2\pi)\right)}{(\omega - 2\pi)}$$

Example (continued)

- Given

$$X(j\omega) = \frac{2\sin\left(3(\omega - 2\pi)\right)}{(\omega - 2\pi)}$$

- Use the following fact

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$$

- For convenience let $R(j\omega) = \dfrac{\sin(\omega/2)}{\omega/2}$

- Rewrite as another shifted function

$$X(j\omega) = Y(j(\omega - 2\pi))$$

$$Y(j\omega) = \frac{2\sin(3\omega)}{\omega}$$

Example (continued)

- Rewrite again as

$$Y(j\omega) = 3\frac{2\sin(6\omega/2)}{6\omega/2}$$
$$= 6R(j6\omega)$$

- Using the scaling property

$$Y(j\omega) = 6R(j6\omega) \Rightarrow y(t) = \mathrm{rect}(t/6)$$

- Using the shift property

$$X(j\omega) = Y(j(\omega - 2\pi)) \Rightarrow x(t) = e^{j2\pi t}\,\mathrm{rect}(t/6)$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Energy of the signal in the time domain

Energy of the signal in the frequency domain

- This is a result of conservation of energy
- Scaling factor is because of radians

Example usign Parseval's theorem

- If the signal $x(t)$ has the FT below $X(j\omega)$

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2 \qquad X(j\omega) = \begin{cases} \frac{j}{2\pi}, & -2 \leqslant \omega < 0 \\ -\frac{j}{2\pi}, & 0 \leqslant \omega \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$$

- Calculate

$$A = \int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

Example with Parseval's theorem (cont.)

$$\int_{-\infty}^{\infty} t^2 \left(\frac{\sin(t)}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi^3}$$

Duality in the Fourier transform

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$X(jt) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

If you know one Fourier pair then you know the other Fourier pair

Applications of duality

- Reproving frequency shift
  - Consider $$x(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$
  - Then $$x(t)e^{j\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

- Impulse in time and frequency
  - Consider $$\delta(t) \stackrel{F}{\longleftrightarrow} 1$$
  - Then $$1 \stackrel{F}{\longleftrightarrow} 2\pi\delta(\omega)$$

Duality example

- Consider

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

- Then

$$\mathrm{sinc}\left(\frac{t}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \cdot \mathrm{rect}(-\omega)$$

$$= 2\pi \cdot \mathrm{rect}(\omega)$$

Since rect is an even function.

UC San Diego

Symmetry, even, and odd

Key points

- Connect signal properties like symmetric, even and odd in both time and frequency domains

Even and odd

- Can decompose a signal into even and odd components

$$x(t) = e(t) + o(t)$$

- Even part is

$$e(t) = \frac{1}{2}(x(t) + x^*(-t))$$

- Odd part is

$$o(t) = \frac{1}{2}(x(t) - x^*(-t))$$

- Can similarly decompose

$$X(j\omega) = E(j\omega) + O(j\omega)$$

Connecting the properties

$$x(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

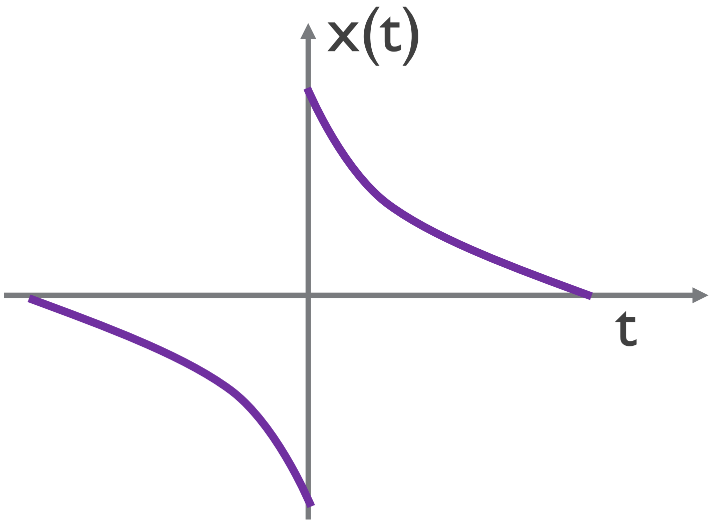
$$X(j\omega) = \mathrm{Re}\{E(j\omega)\} + j\mathrm{Im}\{E(j\omega)\} + \mathrm{Re}\{O(j\omega)\} + j\mathrm{Im}\{O(j\omega)\}$$

- Implications
  - Conjugate symmetry $x(t)$ is real $\rightarrow X(j\omega) = X^*(-j\omega)$
  - Real signals are even in amplitude since $|X(j\omega)| = |X(-j\omega)|$

Even / odd example

$$x(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

$$X(j\omega) = \cancel{\mathrm{Re}\{E(j\omega)\}} + \cancel{j\mathrm{Im}\{E(j\omega)\}} + \cancel{\mathrm{Re}\{O(j\omega)\}} + j\mathrm{Im}\{O(j\omega)\}$$



A real signal x(t) has the graph above

Answer the following questions:

$\mathrm{Re}\left\{X(j\omega)\right\} = 0$ Yes

$\mathrm{Im}\left\{X(j\omega)\right\} = 0$ No

Summarizing symmetry

$$x(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

$$X(j\omega) = \mathrm{Re}\{E(j\omega)\} + j\mathrm{Im}\{E(j\omega)\} + \mathrm{Re}\{O(j\omega)\} + j\mathrm{Im}\{O(j\omega)\}$$

- Key symmetry equation relates real, imaginary, even, and odd in the time and frequency domains
- Can determine signal characteristics in one domain by inspecting the other domain

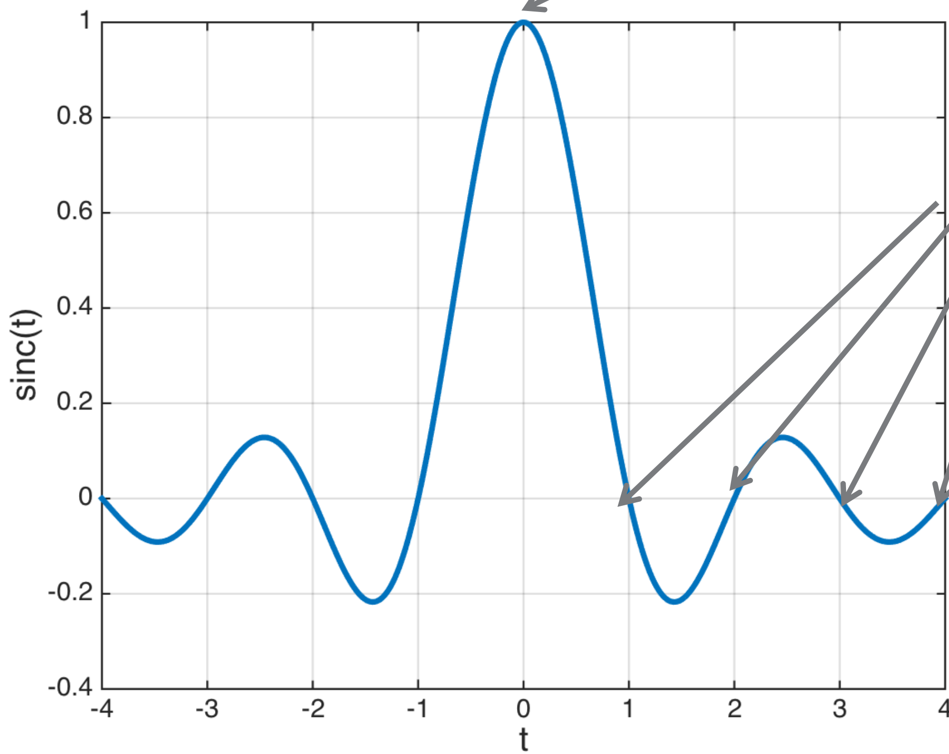
Practical implications of the sinc and rect functions

Key points

- Explain the connection between the sinc function and ideal lowpass filters
- Explain the application of sinc in communication systems

Sinc function

Maximum value of 1, i.e. $\mathrm{sinc}(0) = 1$



Zero crossings at +/-1, +/- 2, ….

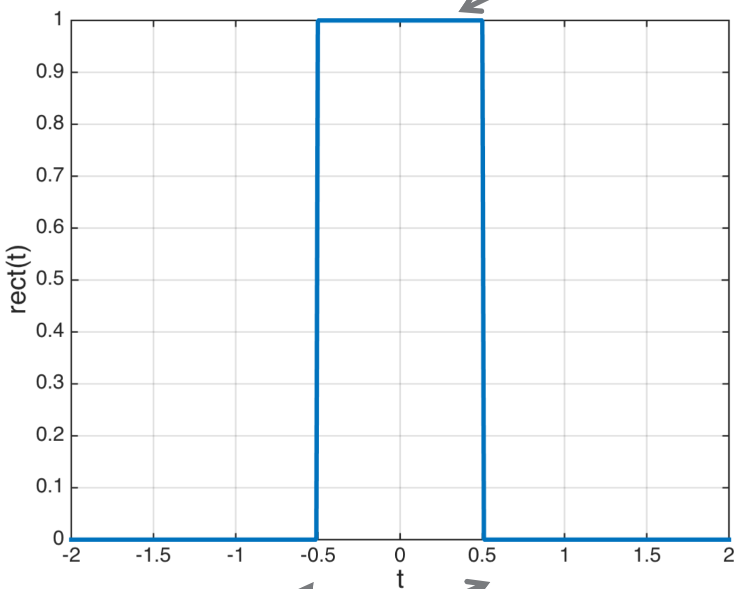
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Be aware, sometimes sinc is defined like this

$$\mathrm{sinc}(t) = \frac{\sin(t)}{t}$$

Rect function

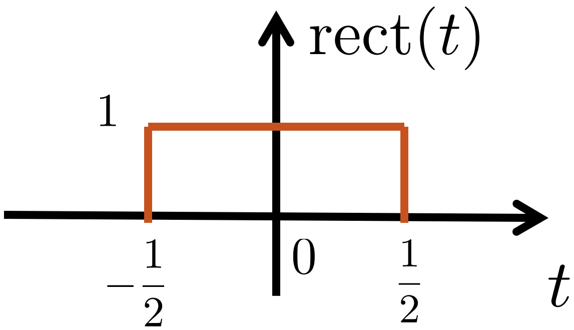
Maximum value of 1, i.e.



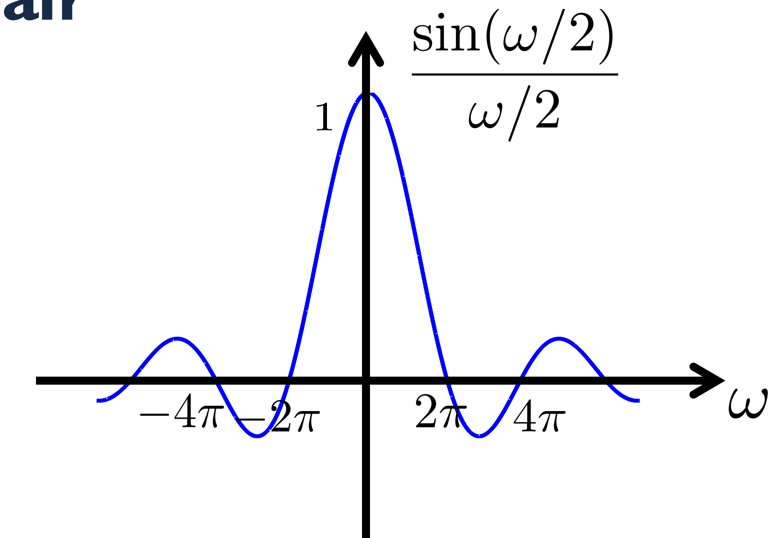
$$\boxed{\mathrm{rect}(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}}$$

Goes to zero at +/- 1/2

Rect – Sinc Fourier transform pair



$$\overset{F}{\longleftrightarrow}$$



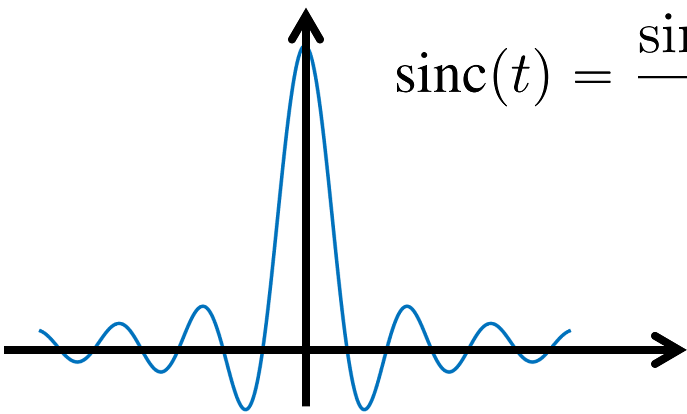
$$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \frac{\sin\frac{1}{2\pi}\pi\omega}{\frac{1}{2\pi}\cdot\pi\omega} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$\boxed{\mathrm{rect}(t) \overset{F}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)}$$

Aside: if using Hertz instead of radians/s

$$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \frac{\sin\frac{2\pi f}{2}}{\frac{2\pi f}{2}} = \mathrm{sinc}(f), \quad f \text{ in Hz}$$

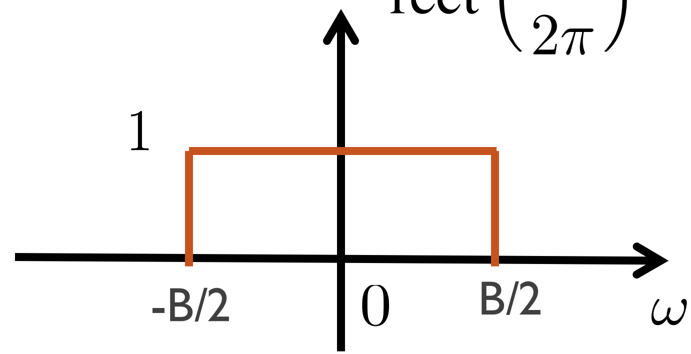
Sinc - Rect Fourier transform pair



$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

$$\xleftrightarrow{F}$$

$$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$



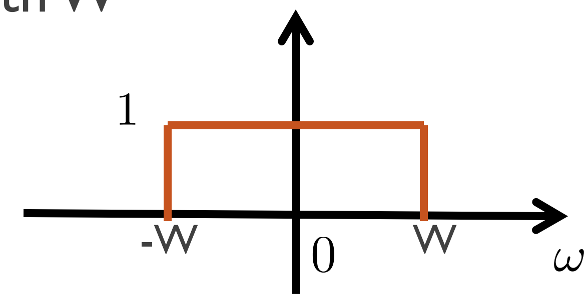
Zero crossings at +/-1, +/- 2, ….

$$\boxed{\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2\pi}\right)}$$

Connection to low pass filter design

- Consider an ideal lowpass filter with bandwidth W

$X(j\omega) = \mathrm{rect}(\omega/2W)$



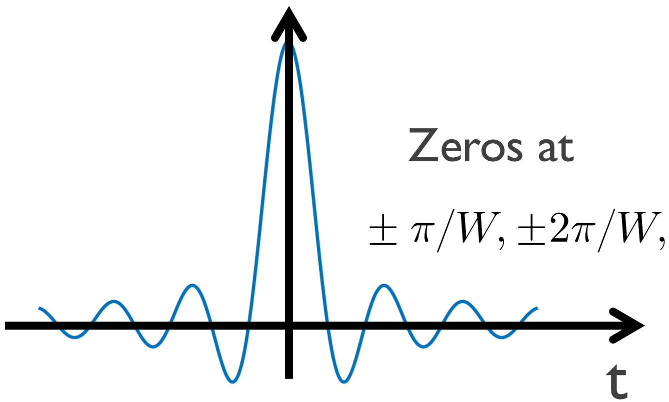
$$X(j\omega) = \begin{cases} 1 & |\omega| \leqslant W \\ 0 & |\omega| > W \end{cases}$$

$$\mathrm{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$ pair

$$\frac{1}{|b|} x\left(\frac{t}{b}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(jb\omega)$$ scaling

$$\frac{2W}{2\pi} x\left(t\frac{2W}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} X\left(j\omega\frac{2\pi}{2W}\right)$$ substitute

$$\frac{W}{\pi} \mathrm{sinc}\left(\frac{tW}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2W}\right)$$ simplify



Implications on filter design

- The ideal lowpass filter is a sinc function with impulse response

$$h(t) = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$

- The ideal lowpass filter is non-causal and infinite

- Can not be readily implemented using a LCCDE

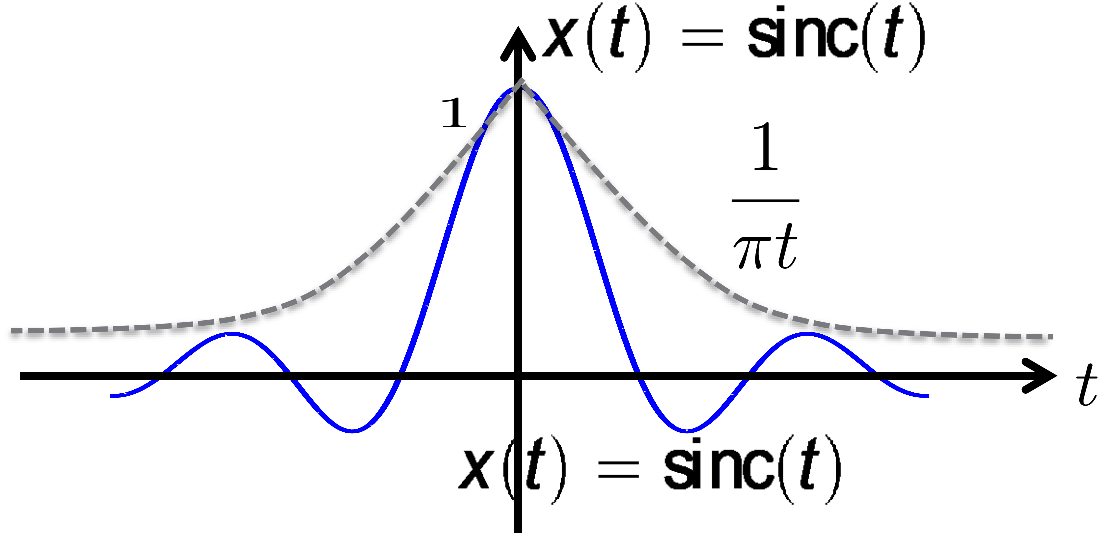
$$h(t) = \frac{b_N}{a_N}\delta(t) + \frac{1}{a_N}\left[b_N\frac{d^N y_n(t)}{dt^N} + b_{N-1}\frac{d^{N-1} y_n(t)}{dt^{N-1}} + \cdots + b_1\frac{dy_n(t)}{dt} + b_0 y_n(t)\right]u(t)$$

Has terms like $C_k e^{\lambda_k t}$

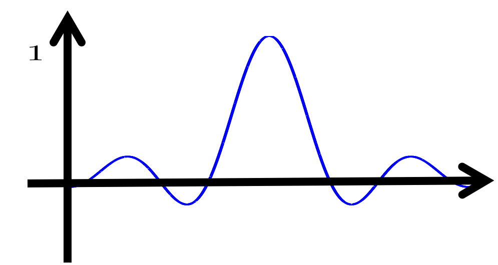
Causal

Working with the sinc function

$$\boxed{\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}}$$

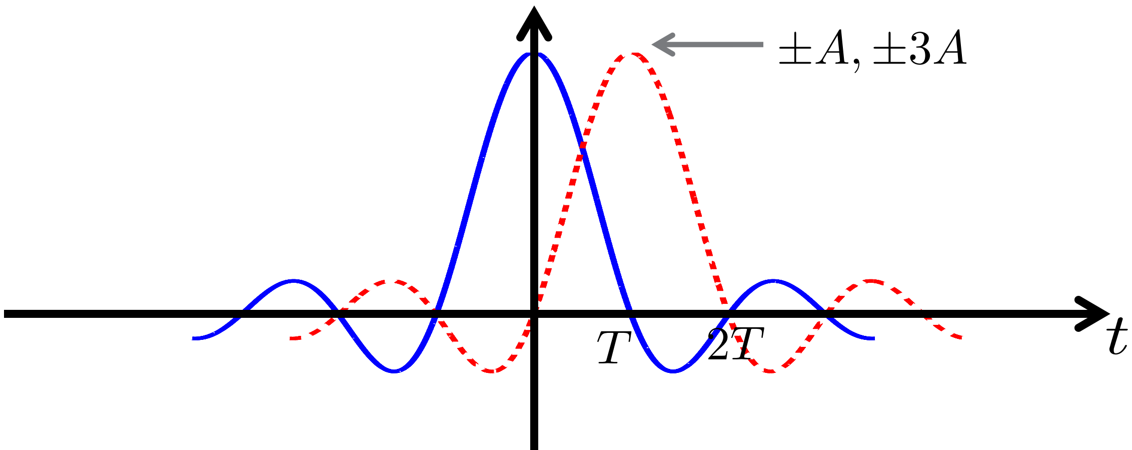


- Everlasting, non-causal time domain signal
- Truncating the sinc function
  - After 20 crossings, less than 5% of peak value
- Making the sinc causal
  - Shift to make approximately causal with delay based on truncation

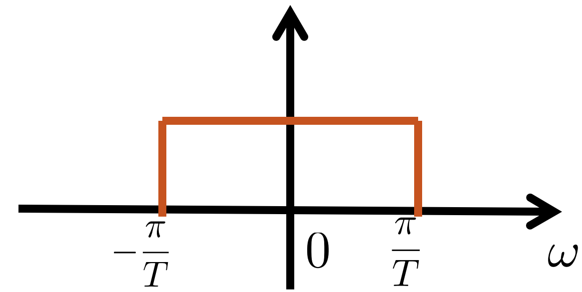


Connection to communications 1/2

- If we used the sinc to send a pulse $\pm 1, \pm 3$ every T seconds



Spectrum has the form



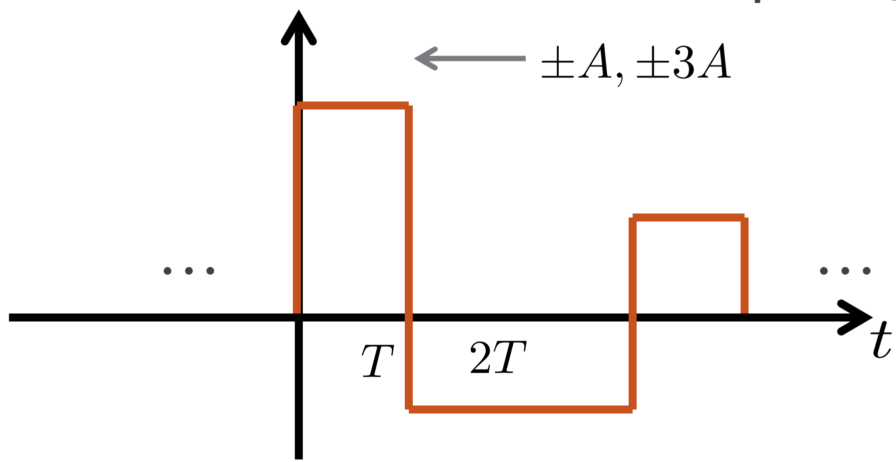
$$\underbrace{\mathrm{sinc}\left(\frac{t}{T}\right)}_{u(t)} \perp \underbrace{\mathrm{sinc}\left(\frac{t \pm kT}{T}\right)}_{v(t)} \qquad \forall k$$

$$\int u(t)v(t)dt = 0$$

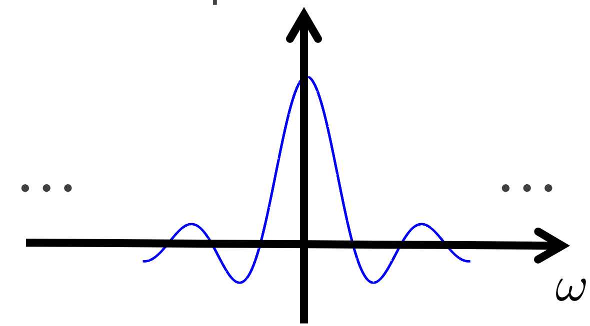
Sincs are good choices to carry data because shifted sinc functions are orthogonal

Connection to communications 2/2

- If we used the rect to send a pulse $\pm 1, \pm 3$ every T seconds



Spectrum has the form

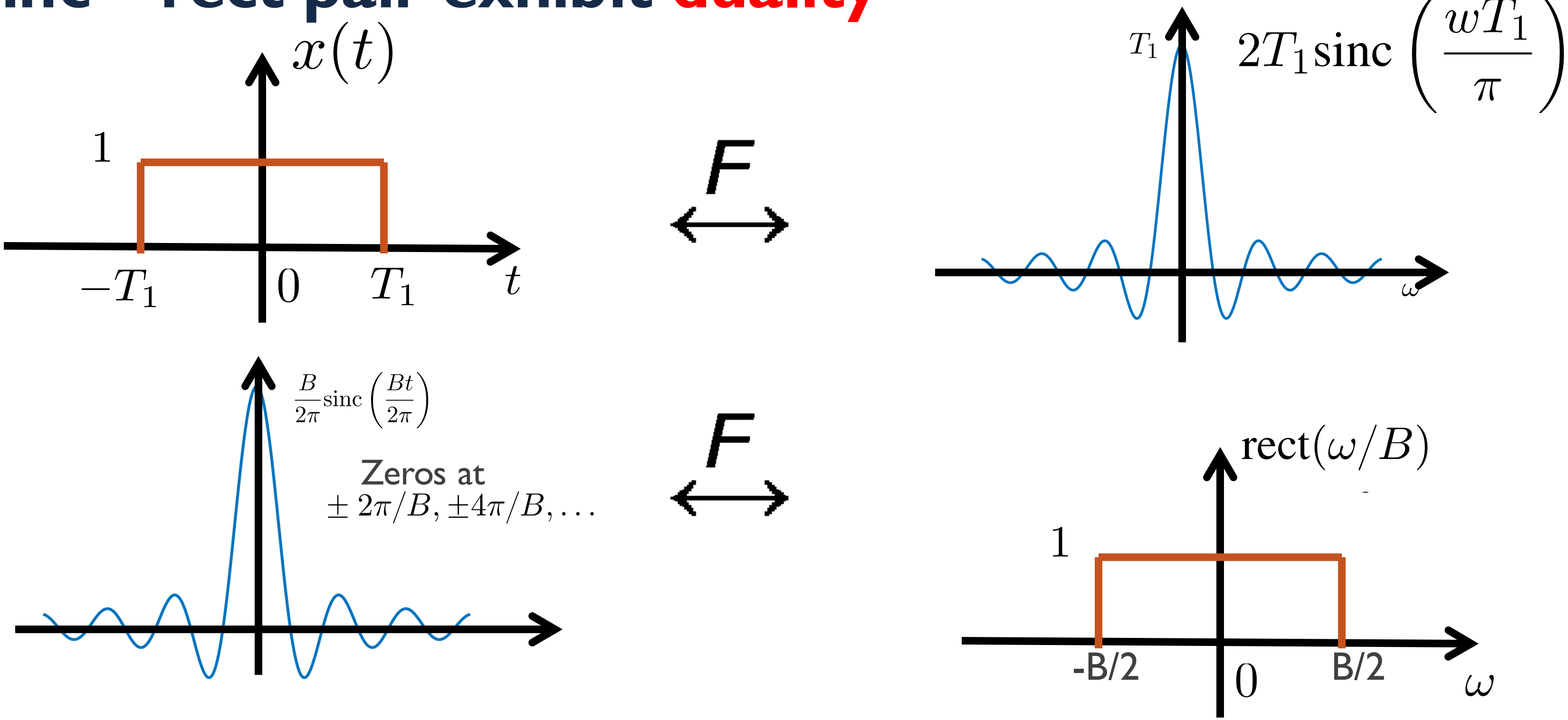


Zero crossings at

$\pm 2\pi/T, \pm 4\pi/T, \ldots$

- Rectangle pulse uses infinite bandwidth!
- Sinc pulses or variations are used extensively in communications (have fixed, minimum bandwidth – very efficient)
  - Spectrum is expensive, more than $1000/Hz in the US for prime broadband spectrum (so a 10 MHz channel is about $10b)

Sinc – rect pair exhibit duality



It is possible to figure out one set of transforms from the other

This is an alternative version of the the sinc and rectangle function slides that organizes the material a bit different, just provided for reference

Practical implications of the sinc and rect functions

Key points

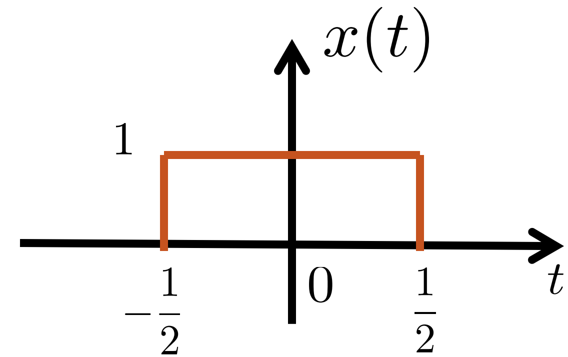
- Explain the connection between the sinc function and ideal lowpass filters
- Explain the application of sinc in communication systems

Fourier transform of a rectangle function

Very important example with practical implications that will be discussed in later sections

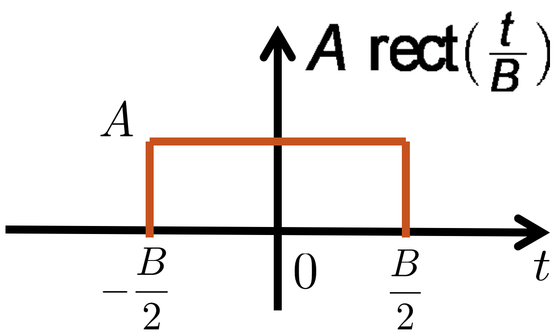
- Consider the signal

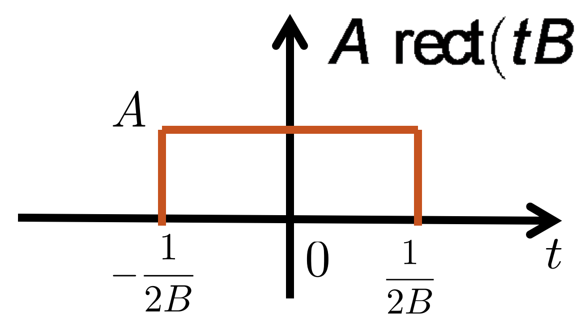
$$\boxed{x(t) = \mathrm{rect}(t) = \Pi(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leqslant \frac{1}{2} \end{cases}}$$



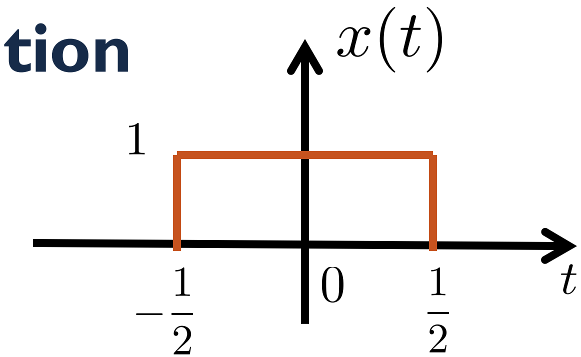
- Note that $\displaystyle\int_{-\infty}^{\infty} |x(t)|^2 dt = 1$

- Can create other related shapes





Fourier transform of a rectangle function



- Its FT is given by

$$X(j\omega) = \int_{-\infty}^{\infty} \mathrm{rect}(t) e^{-j\omega t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{j\omega}(e^{-\frac{j\omega}{2}} - e^{\frac{j\omega}{2}})$$

$$= \frac{2}{\omega} \cdot \frac{1}{2j}(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}})$$

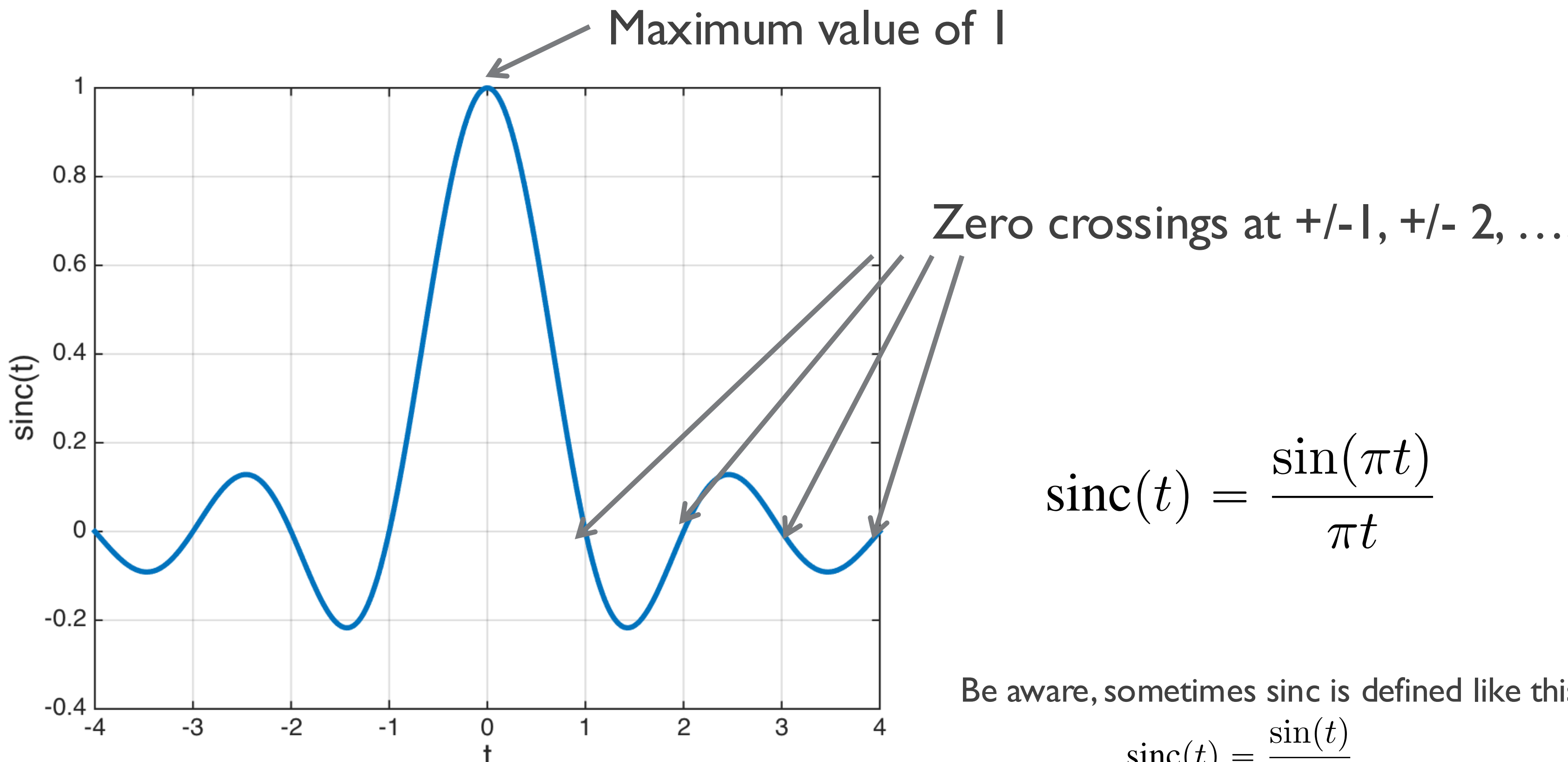
$$= \frac{2}{\omega} \sin \frac{\omega}{2}$$

$$= \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$$

$$= \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

You should practice plotting the sinc function, focusing on getting the zero crossing correct.

About the sinc function



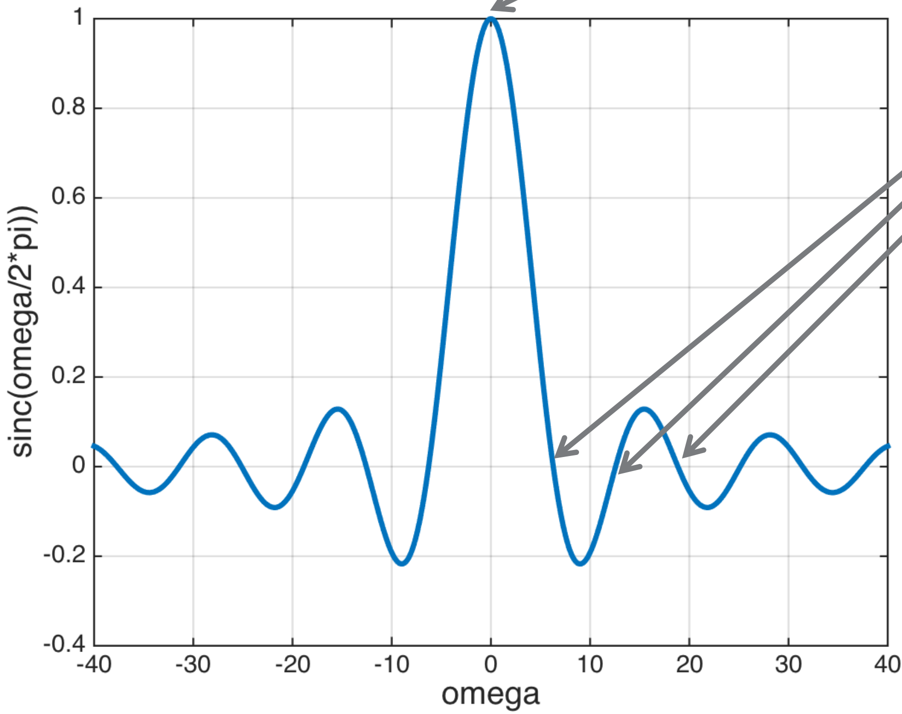
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Be aware, sometimes sinc is defined like this

$$\mathrm{sinc}(t) = \frac{\sin(t)}{t}$$

Fourier transform of rectangle function

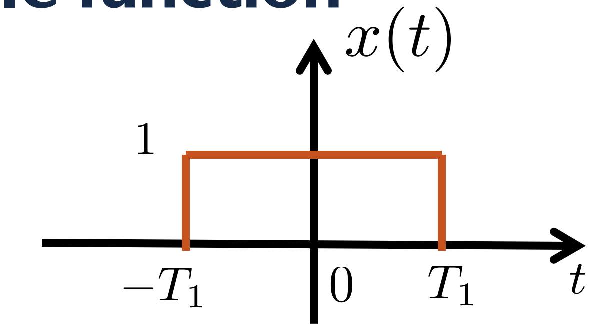
Maximum value of 1



Zero crossings at +/- 2π,+/- $4\pi\ldots$

$$X(j\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

Fourier transform of a scaled rectangle function

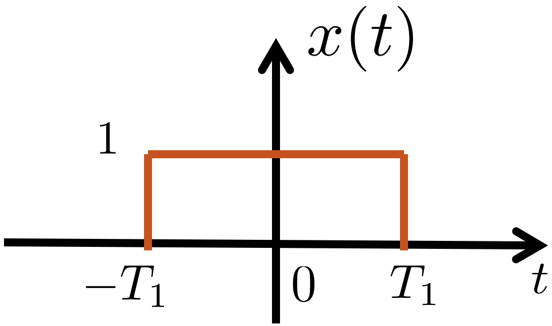


- Its FT is given by

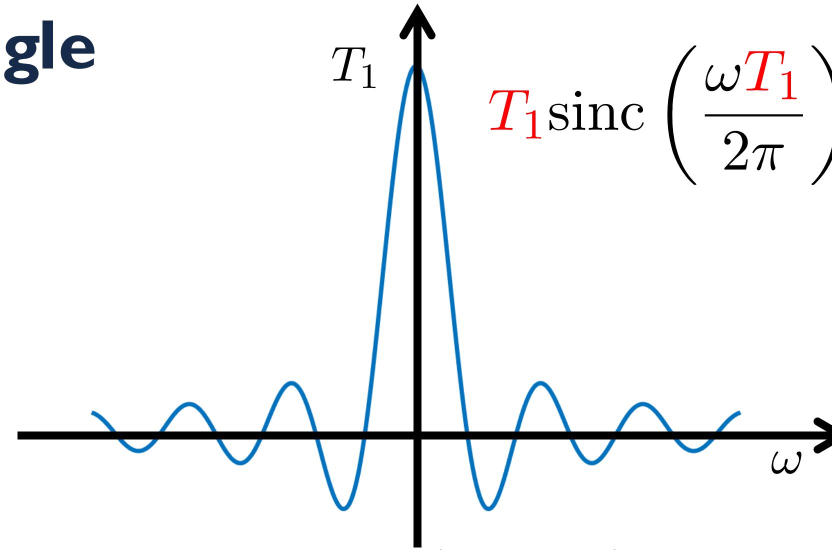
$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t/2T_1) e^{-j\omega t} dt \\
&= \int_{-T_1}^{T_1} e^{-j\omega t} dt \\
&= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} \\
&= -\frac{1}{j\omega}(e^{-j\omega T_1} - e^{j\omega T_1})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\omega} \cdot \frac{1}{2j}(e^{j\omega T_1} - e^{-j\omega T_1}) \\
&= \frac{2}{\omega} \sin \omega T_1 \\
&= T_1 \frac{\sin \frac{\omega T_1}{2}}{T_1 \frac{\omega}{2}} \\
&= T_1 \mathrm{sinc}\left(\frac{\omega T_1}{2\pi}\right)
\end{aligned}
$$

Fourier transform of rectangle



$\overset{F}{\longleftrightarrow}$



Zeros at $\pm\pi/T_1, \pm\pi/T_1, \ldots$

- Crossings at $\pm\pi/T_1, \pm\pi/T_1, \ldots$
- For $T_1 \rightarrow \infty$, FT is $\delta(\omega)$
- So
  - $T_1 \downarrow$, pulse narrow, sinc wide
  - $T_1 \uparrow$, sinc becomes narrow, pulse wide

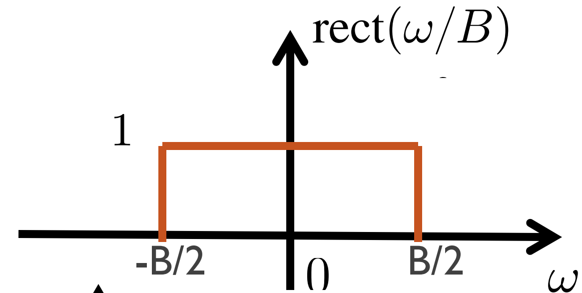
Narrow pulse in time is broad in frequency

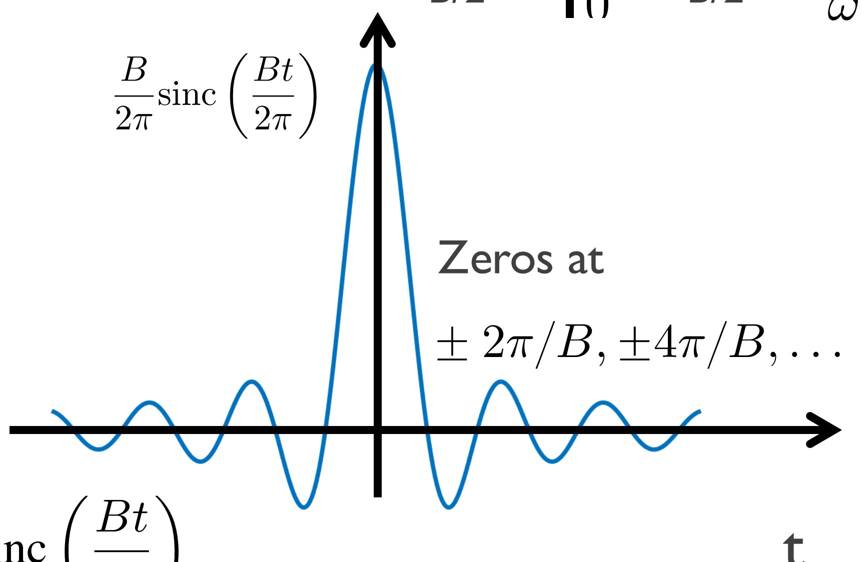
Inverse Fourier transform of the rectangle function

- Can compute the FT directly

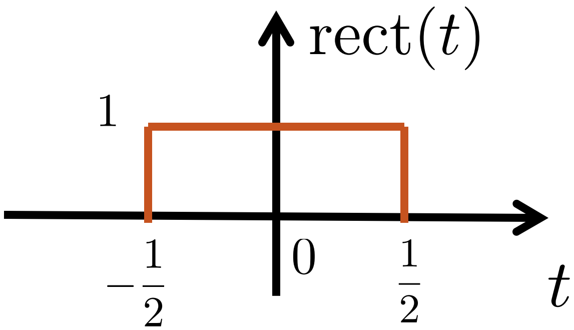
$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\frac{B}{2}}^{\frac{B}{2}} e^{j\omega t}d\omega \\
&= \frac{1}{j2\pi t}e^{j\omega t}\Big|_{-\frac{B}{2}}^{\frac{B}{2}} \\
&= \frac{1}{j2\pi t}(e^{\frac{jB}{2}t} - e^{-\frac{jB}{2}t}) \\
&= \frac{1}{\pi t}\sin(Bt/2) \\
&= \frac{B/2\pi}{\pi Bt/2\pi}\sin\left(\frac{Bt\pi}{2\pi}\right)
\end{aligned}
$$

$$\frac{B}{2\pi}\text{sinc}\left(\frac{Bt}{2\pi}\right)$$

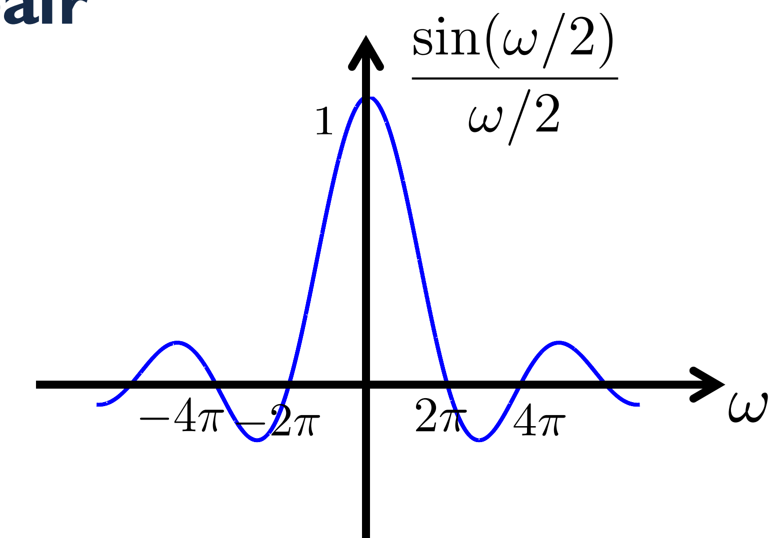




Rect – Sinc Fourier transform pair



$$\xleftrightarrow{F}$$



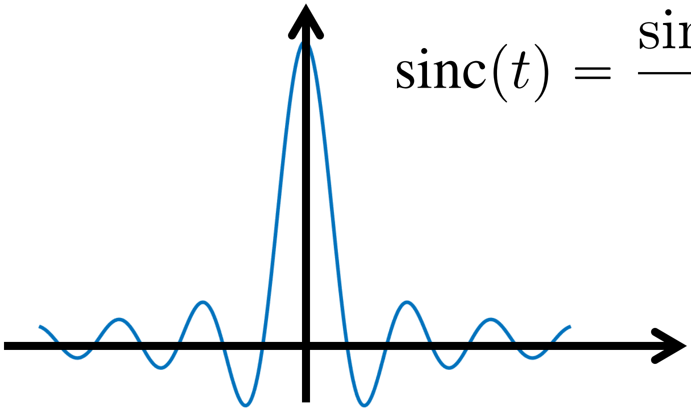
$$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \frac{\sin\frac{1}{2\pi}\pi\omega}{\frac{1}{2\pi}\cdot\pi\omega} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$\boxed{\mathrm{rect}(t) \xleftrightarrow{F} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)}$$

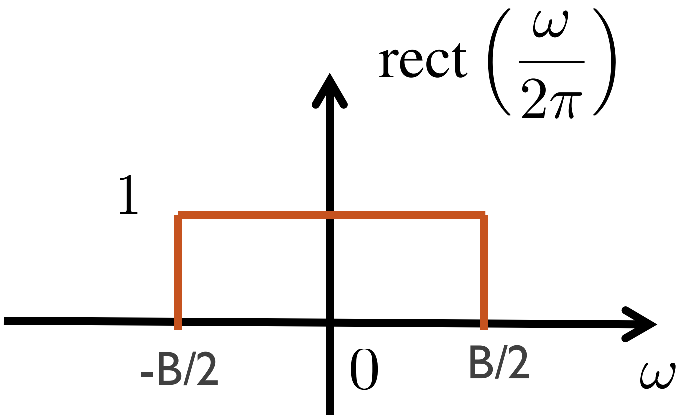
Aside: if using Hertz instead of radians/s

$$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \frac{\sin\frac{2\pi f}{2}}{\frac{2\pi f}{2}} = \mathrm{sinc}(f), \quad f \text{ in Hz}$$

Sinc - Rect Fourier transform pair



$\xleftrightarrow{F}$



Zero crossings at +/-1, +/- 2, ….

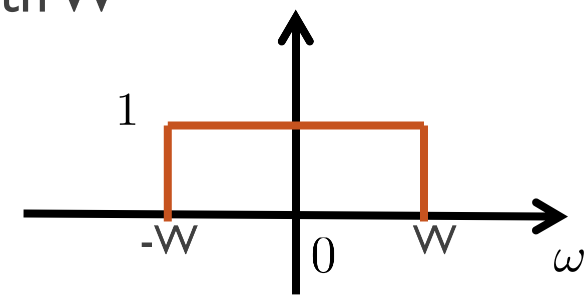
$$\boxed{\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2\pi}\right)}$$

Connection to low pass filter design

- Consider an ideal lowpass filter with bandwidth W

$$X(j\omega) = \begin{cases} 1 & |\omega| \leqslant W \\ 0 & |\omega| > W \end{cases}$$

$X(j\omega) = \mathrm{rect}(\omega/2W)$

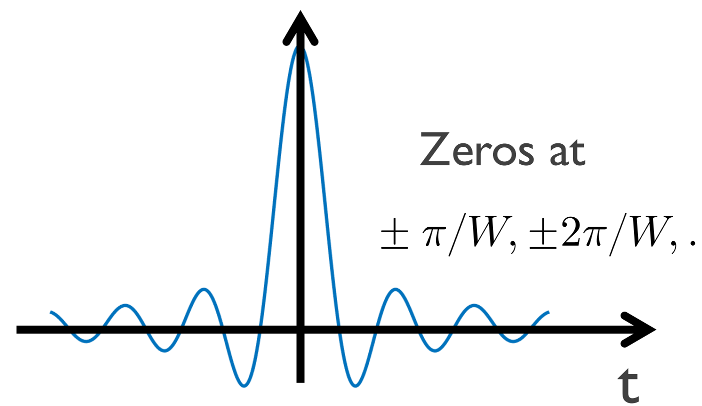


$$\mathrm{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$ pair

$$\frac{1}{|b|} x\left(\frac{t}{b}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(jb\omega)$$ scaling

$$\frac{2W}{2\pi} x\left(t\frac{2W}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} X\left(j\omega\frac{2\pi}{2W}\right)$$ substitute

$$\frac{W}{\pi}\mathrm{sinc}\left(\frac{tW}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2W}\right)$$ simplify



Implications on filter design

- The ideal lowpass filter is a sinc function with impulse response

$$h(t) = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$

- The ideal lowpass filter is non-causal and infinite

- Can not be readily implemented using a LCCDE

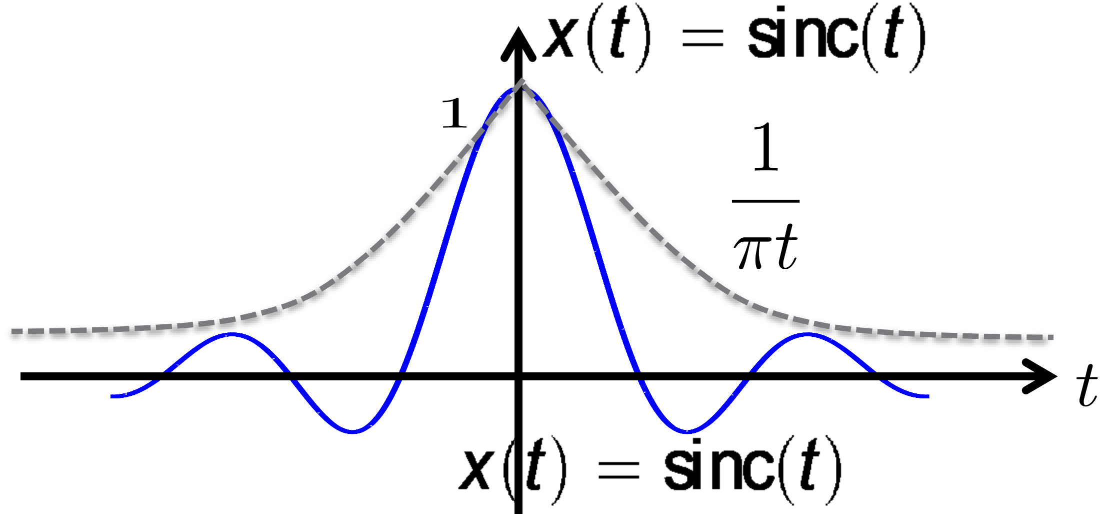
$$h(t) = \frac{b_N}{a_N}\delta(t) + \frac{1}{a_N}\left[b_N\frac{d^N y_n(t)}{dt^N} + b_{N-1}\frac{d^{N-1}y_n(t)}{dt^{N-1}} + \cdots + b_1\frac{dy_n(t)}{dt} + b_0 y_n(t)\right]u(t)$$

Has terms like $C_k e^{\lambda_k t}$

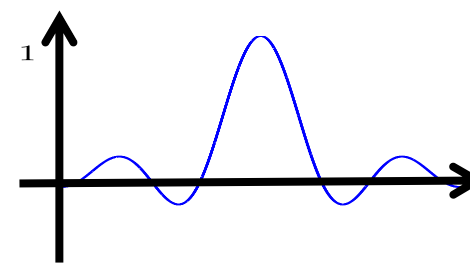
Causal

Working with the sinc function

$$\boxed{\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}}$$

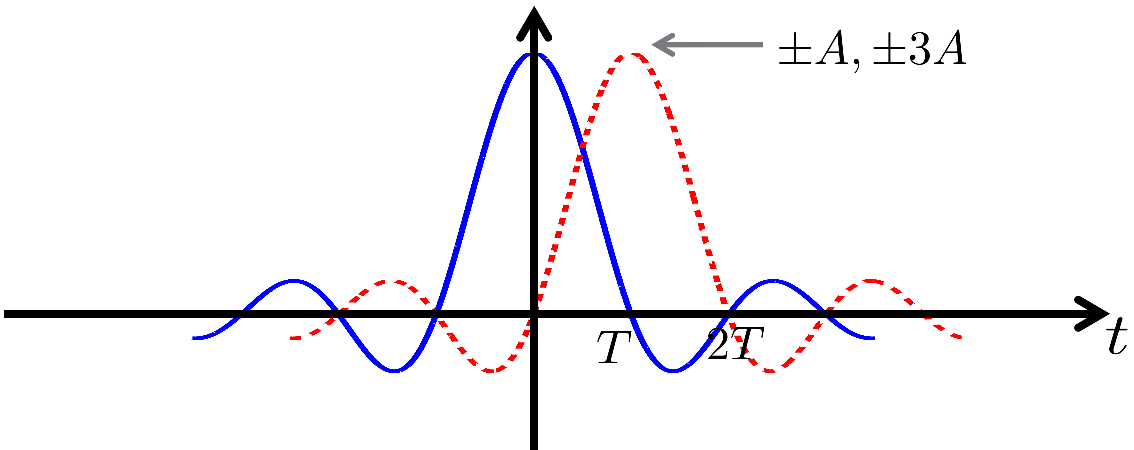


- Everlasting, non-causal time domain signal
- Truncating the sinc function
  - After 20 crossings, less than 5% of peak value
- Making the sinc causal
  - Shift to make approximately causal with delay based on truncation

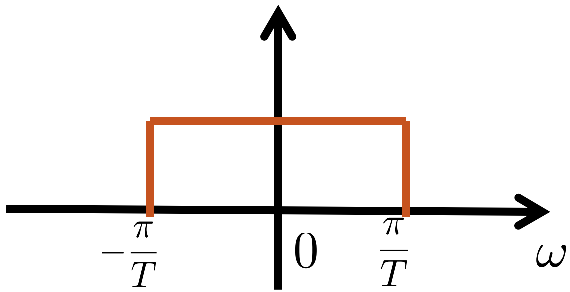


Connection to communications 1/2

- If we used the sinc to send a pulse $\pm 1, \pm 3$ every T seconds



Spectrum has the form



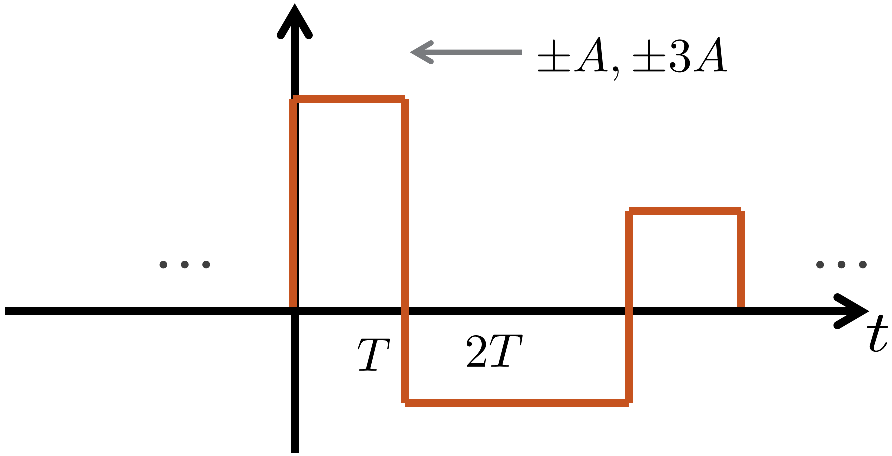
$$\underbrace{\operatorname{sinc}\left(\frac{t}{T}\right)}_{u(t)} \perp \underbrace{\operatorname{sinc}\left(\frac{t \pm kT}{T}\right)}_{v(t)} \quad \forall k$$

$$\int u(t)v(t)dt = 0$$

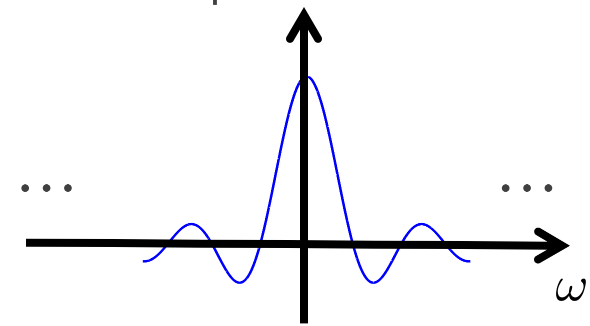
Sincs are good choices to carry data because shifted sinc functions are orthogonal

Connection to communications 2/2

- If we used the rect to send a pulse $\pm 1, \pm 3$ every T seconds



Spectrum has the form

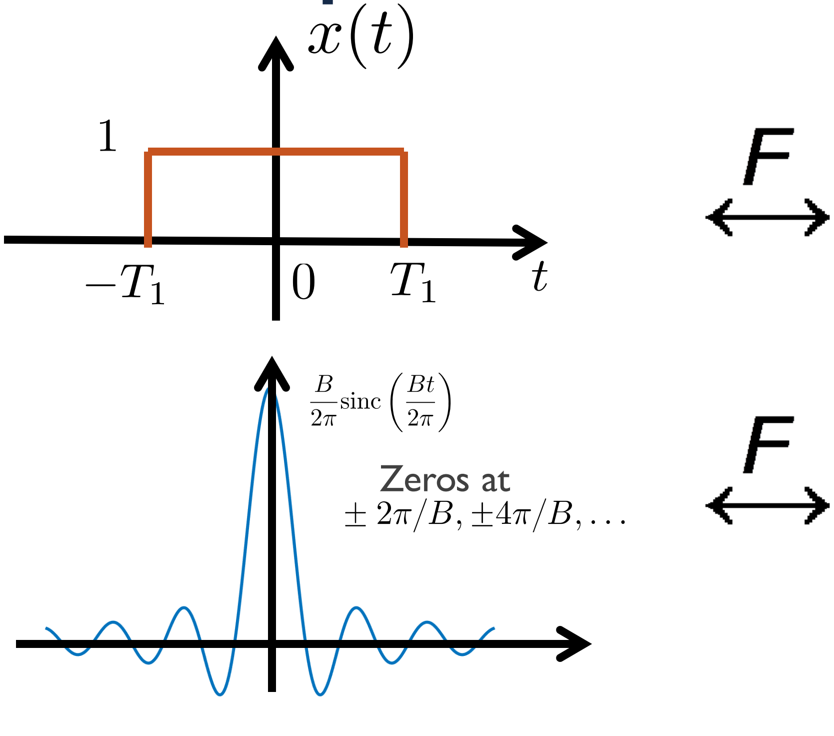


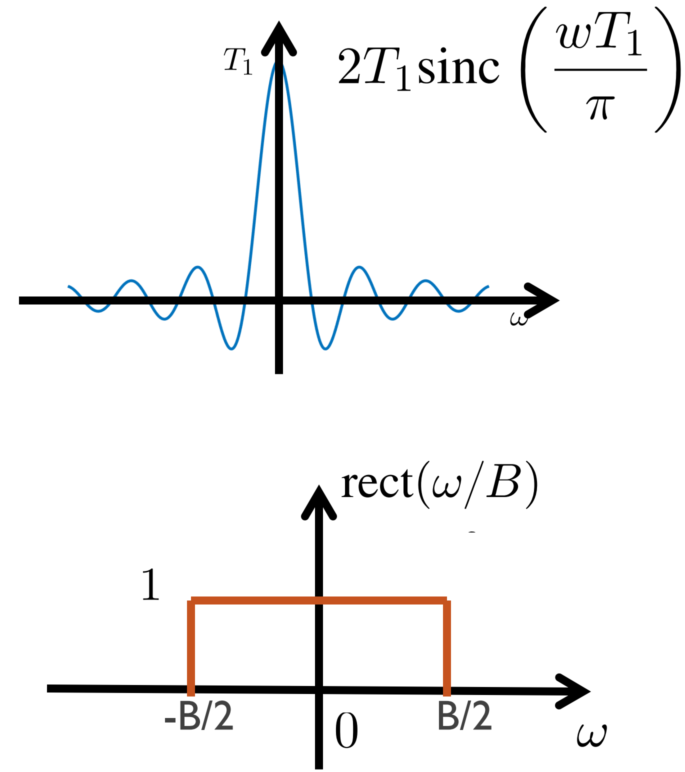
Zero crossings at

$\pm 2\pi/T, \pm 4\pi/T, \ldots$

- Rectangle pulse uses infinite bandwidth!
- Sinc pulses or variations are used extensively in communications (have fixed, minimum bandwidth – very efficient)
  - Spectrum is expensive, more than \$1000/Hz in the US for prime broadband spectrum (so a 10 MHz channel is about \$10b)

Sinc – rect pair exhibit duality





It is possible to figure out one set of transforms from the other

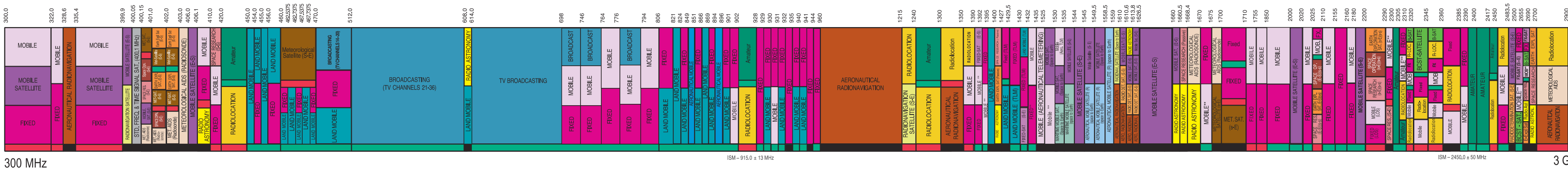
Application to communication systems

Key points

- Summarize some key principles of communication systems
- Explain the connection between communication and the frequency domain

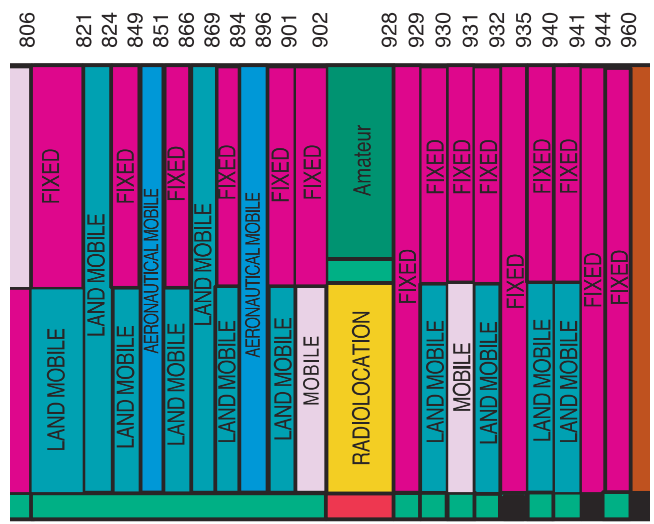
Spectrum allocation

Numbers are the frequency given in MHz, multiply by 2 π to get Mrad/s



UHF band

Cellular is called "Land mobile"



ISM – 915.0 ± 13 MHz

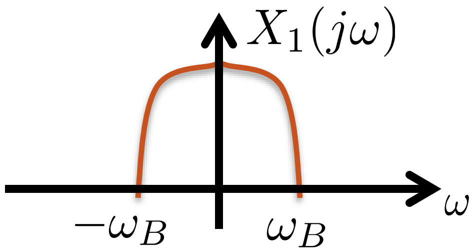
ISM band here is used for cordless phones, sensors

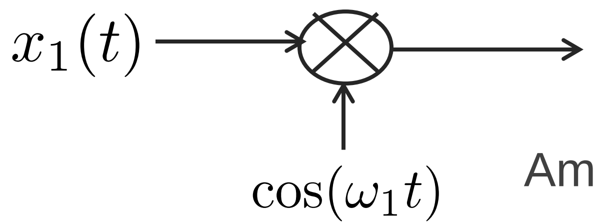
Practical application – Modulation

- Modulation is a concept widely used in communication systems
- While the "modulation" can mean different things, in communications it usually refers using a sinusoid to carry information
- Many types of modulation
  - Amplitude modulation (AM)
  - Frequency modulation (FM)
  - Phase modulation (PM)
- In this lecture: focus on an example with amplitude modulation

Practical application – Amplitude modulation

- One classically type of modulation is AM





$$x(t) = x_1(t)\cos(\omega_1 t)$$

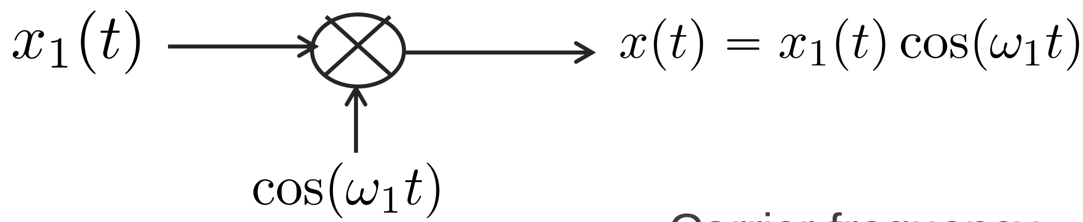
Amplitude of cosine is modulated

- What happens in the frequency domain?

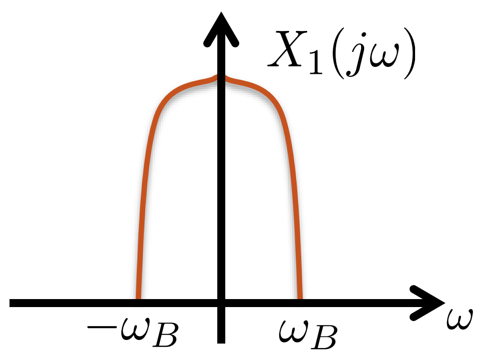
$$x_1(t)\cos(\omega_1 t) = x_1(t)\frac{1}{2}e^{j\omega_1 t} + x_1(t)\frac{1}{2}e^{-j\omega_1 t}$$

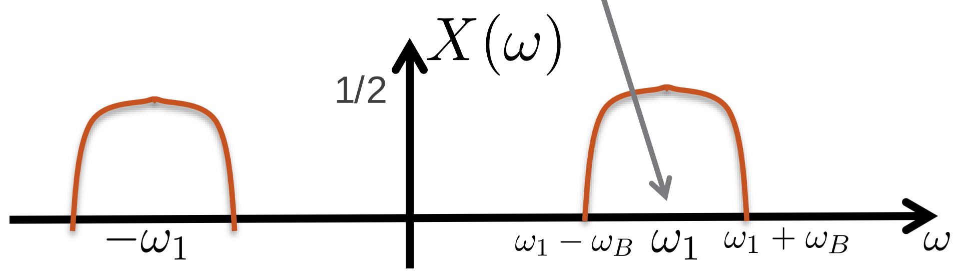
$$\mathcal{F}\left\{x_1(t)\cos(\omega_1 t)\right\} = \frac{1}{2}X_1(j(\omega-\omega_1)) + \frac{1}{2}X_1(j(\omega+\omega_1))$$

Practical application – Amplitude modulation (cont.)



Carrier frequency





Spectrum of the information signal
(aka baseband signal)

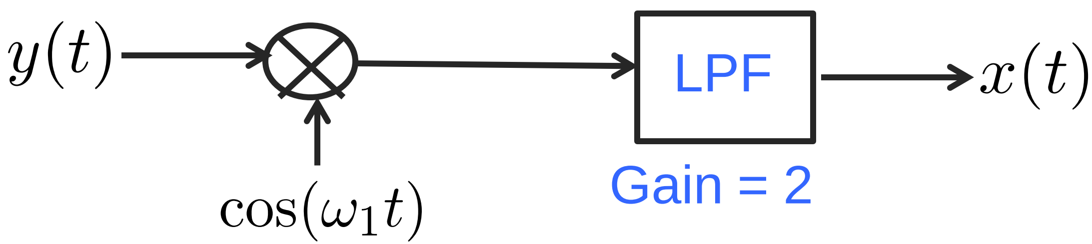
Spectrum of the modulated signal
(aka the passband signal)

Practical application – Demodulation

- The reverse of modulation is called demodulation
  - Exploit the fact that

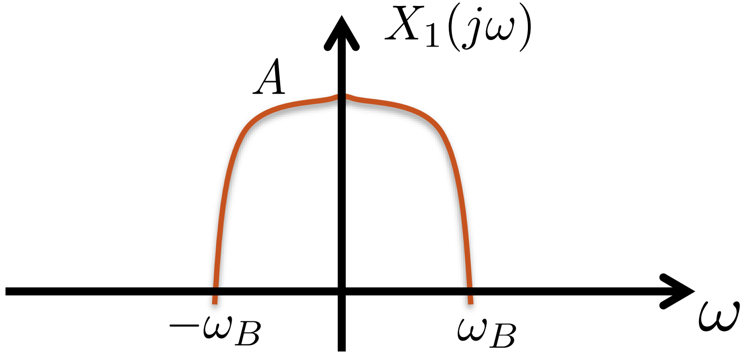
$$y(t) = x(t)\cos^2(\omega_1 t) = x(t)\frac{1}{2}(1 + \cos(2\omega_1 t))$$

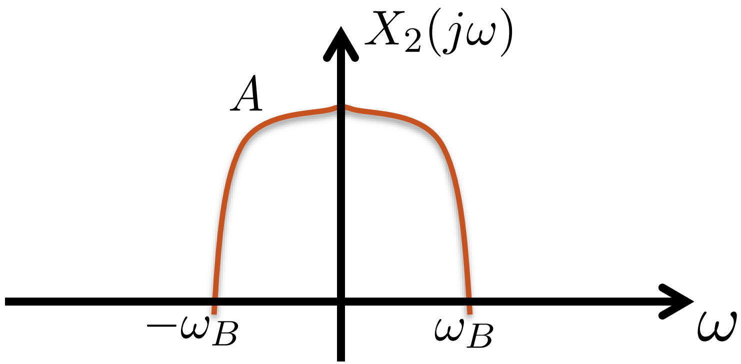
Can eliminate this with a lowpass filter!



Practical application – Frequency division multiplexing

- Consider a communication system with two users
- Suppose that the signal for each user has frequency response

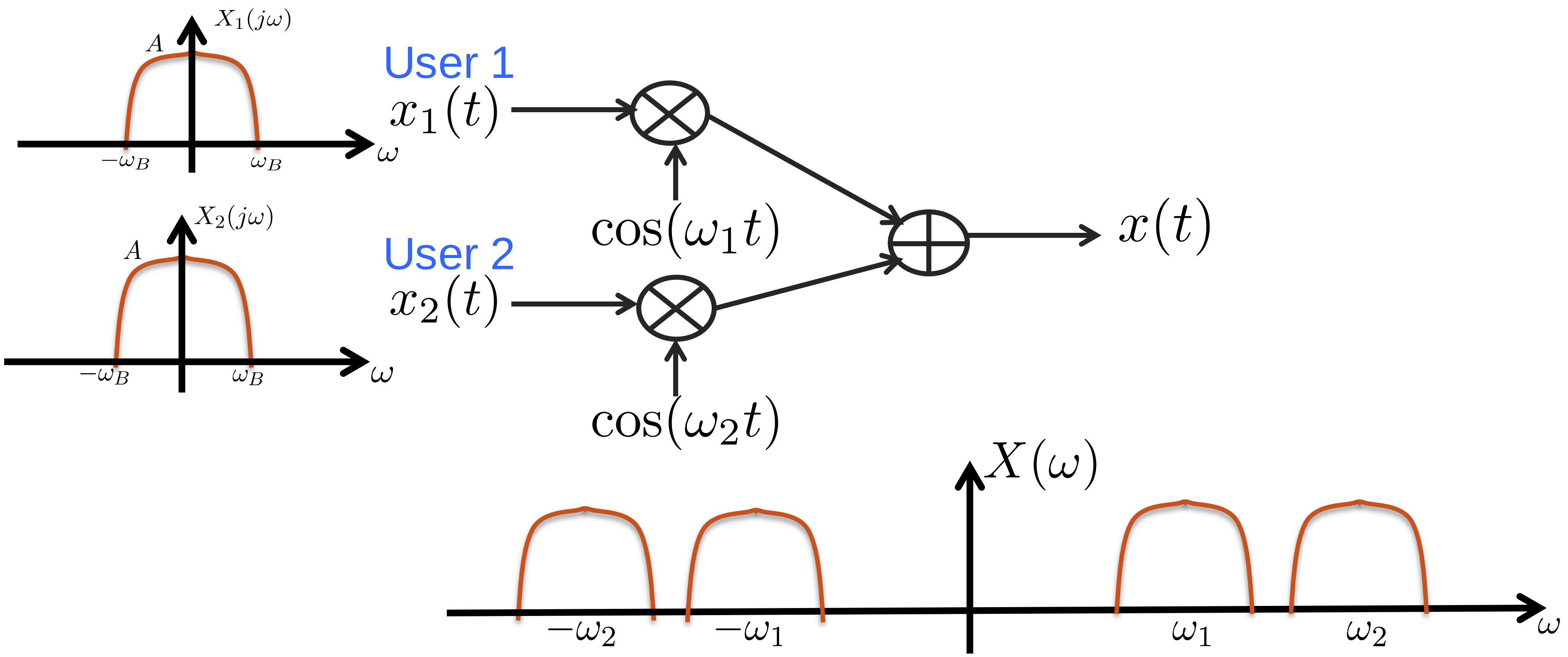




- Objective: Create a communication signal that carries the information contained in each users' signal

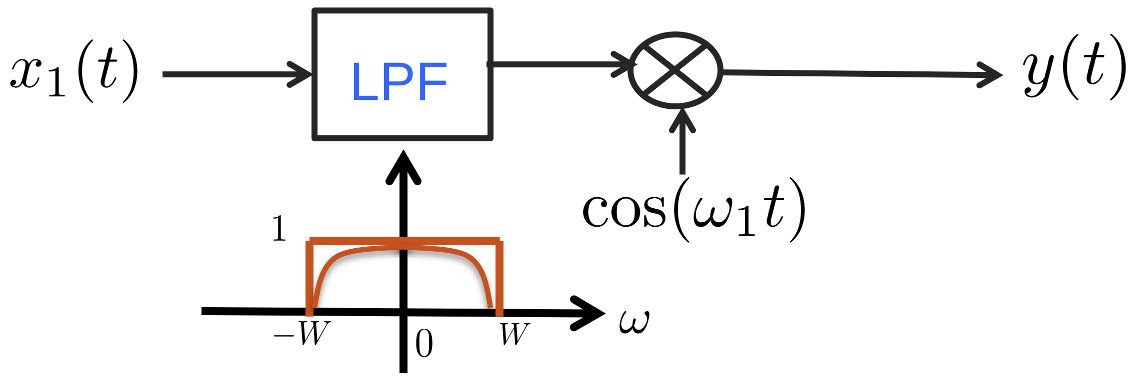
Practical application – Frequency division multiplexing (cont.)

- One possible solution is AM with different carrier frequencies

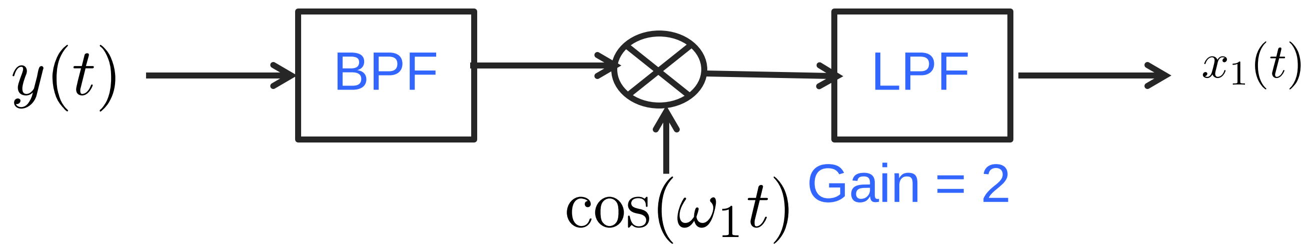


Practical application – Frequency division multiplexing (cont.)

- How to prevent possible overlap in the signals?
  - At the transmitter

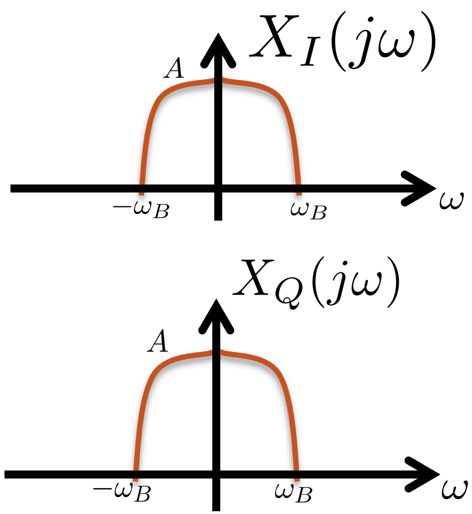


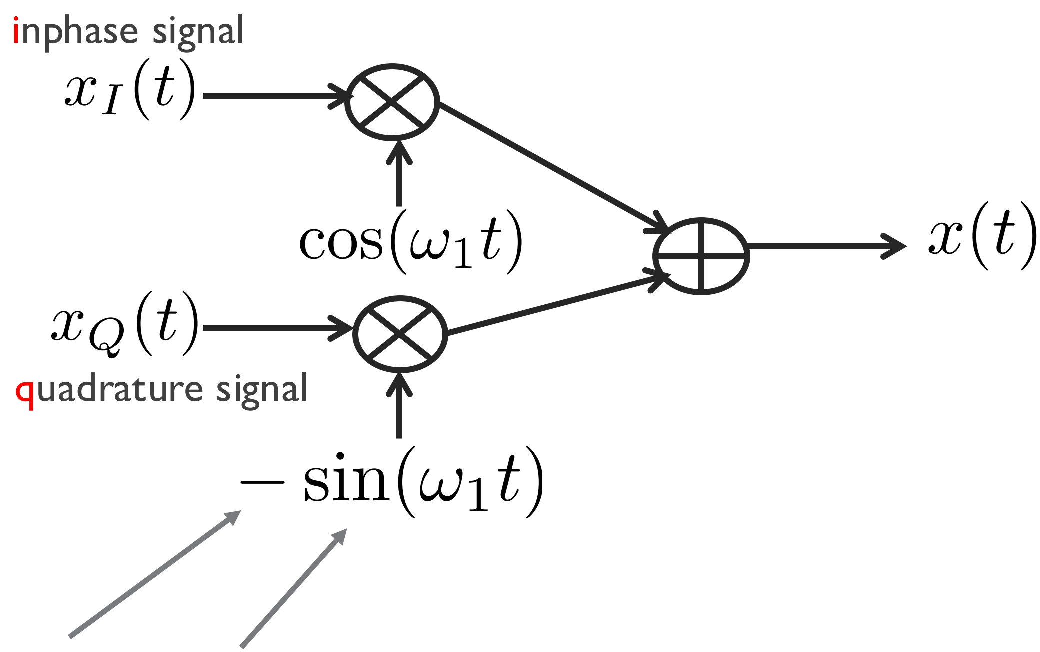
  - At the receiver



Practical application – Inphase and quadrature

- What if two information signals are sent as follows?





note the sign and sine here, which will make sense shortly

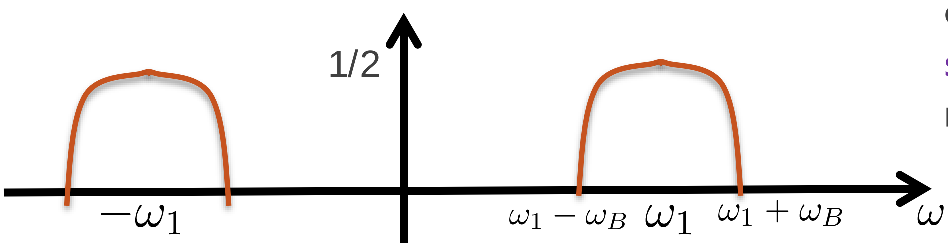
Practical application – Inphase and quadrature (cont.)

- What happens in the frequency domain?
  - Inphase term

$$\mathcal{F}\left\{x_I(t)\cos(\omega_1 t)\right\} = \frac{1}{2}X_I(j(\omega-\omega_1)) + \frac{1}{2}X_I(j(\omega+\omega_1))$$

  - Quadrature term

$$\mathcal{F}\left\{-x_Q(t)\sin(\omega_1 t)\right\} = \frac{j}{2}X_Q(j(\omega-\omega_1)) - \frac{j}{2}X_Q(j(\omega+\omega_1))$$



mixture of inphase and quadrature terms but not the same mixture at positive and negative frequencies

Practical application – Inphase and quadrature (cont.)

- What about demodulation?
  - Trig identities

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$
$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$
$$\sin u \cos v = \frac{1}{2}\left[\sin(u-v) + \sin(u+v)\right]$$

Can recover both inphase and quadrature!

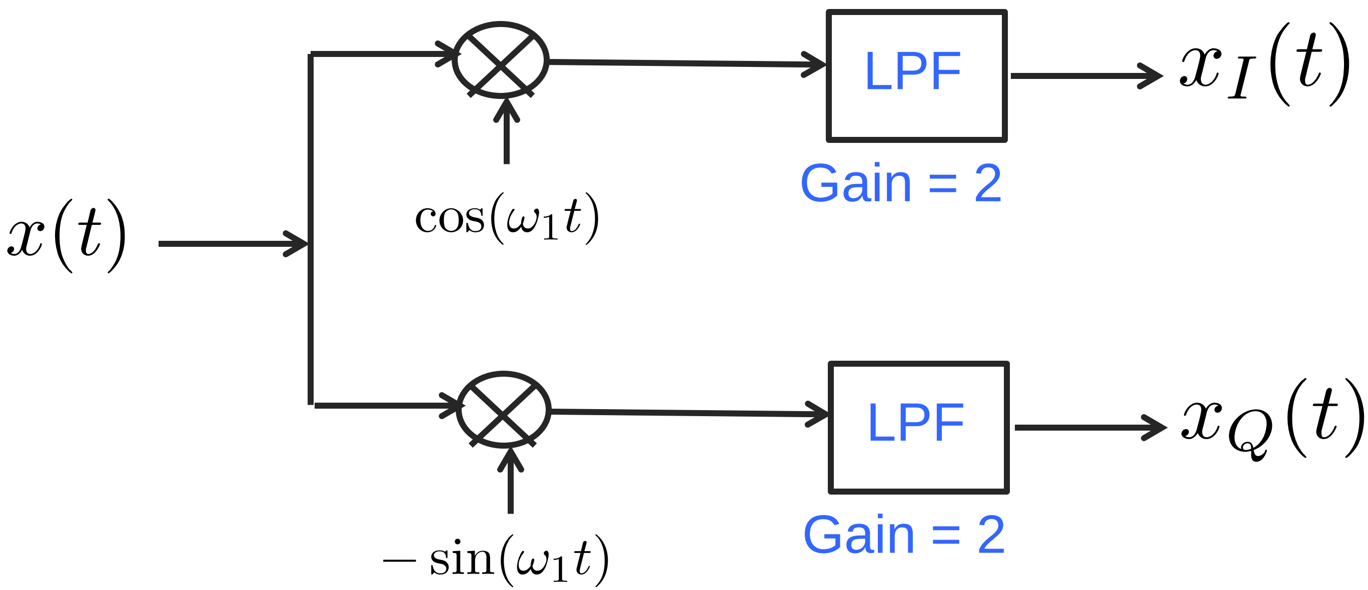
  - Applying the identities

filter out

$$x(t)\cos(\omega_1 t) = \frac{1}{2}x_I(t) + \frac{1}{2}x_I(t)\cos(2\omega_1 t) - \frac{1}{2}x_Q(t)\sin(2\omega_1 t)$$
$$x(t)\sin(\omega_1 t) = -\frac{1}{2}x_Q(t) + \frac{1}{2}x_Q(t)\cos(2\omega_1 t) + \frac{1}{2}x_I(t)\sin(2\omega_1 t).$$

Practical application – Inphase and quadrature (cont.)

- IQ demodulator



Practical application – Inphase and quadrature (cont.)

- Why do we use complex signals?

This is called the complex baseband signal

$$x_{bb}(t) = x_I(t) + jx_Q(t)$$

$\mathrm{Re}\{x_{bb}(t)\}$

$\mathrm{Im}\{x_{bb}(t)\}$

Complex signals become a convenient way to work with inphase and quadrature together, avoiding the need for matrix notation

Convolution property

Key points

- Convolution in time is multiplication in frequency
- Use this fact to compute convolutions

Convolution property

- If $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$ $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$
- Then

$$\boxed{y(t) = h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)}$$

Convolution in time is multiplication in frequency

Proof of the convolution property

$$Y(j\omega) = \mathcal{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t}dt$$

Apply definition

$$= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dtd\tau$$

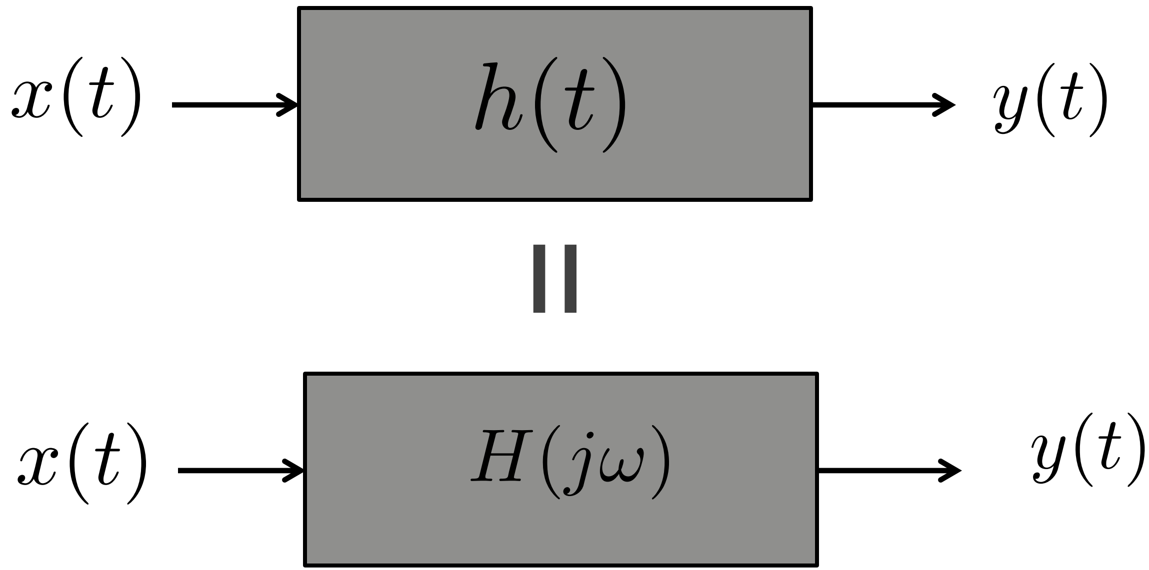
Exchange order of integration

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau$$

Time shift property

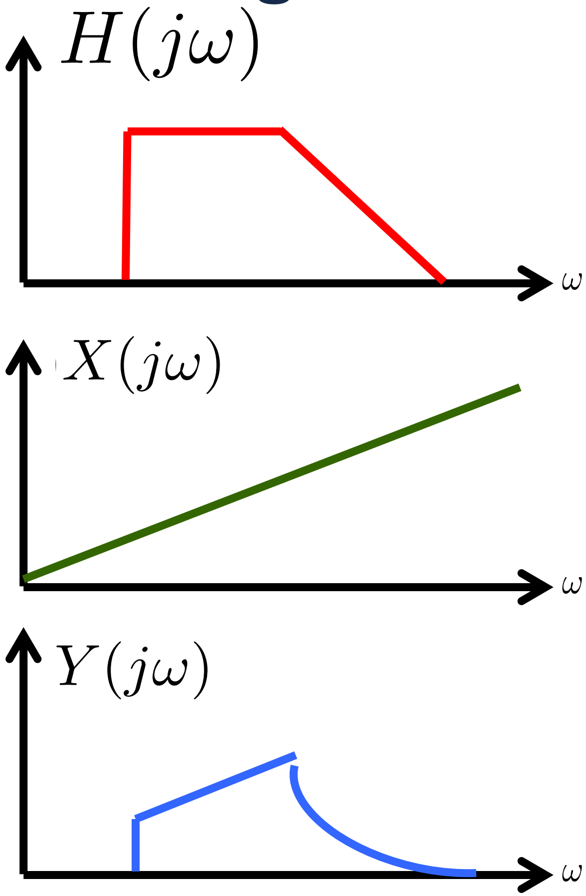
$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \quad = H(j\omega)X(j\omega)$$

Block diagrams



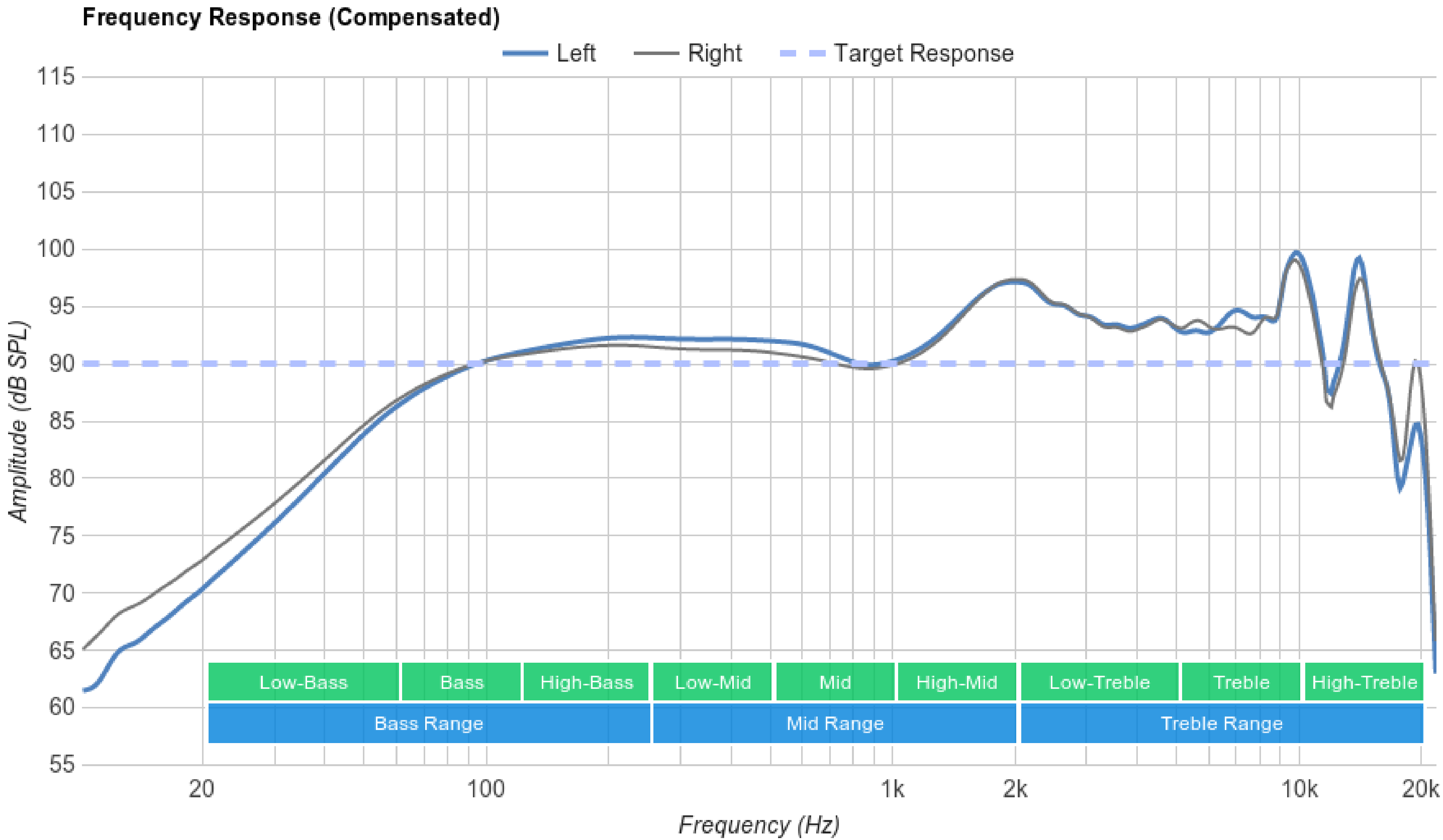
From a notational perspective, an LTI system may be described by the impulse response in the time or frequency domains

Visualizing the convolution property

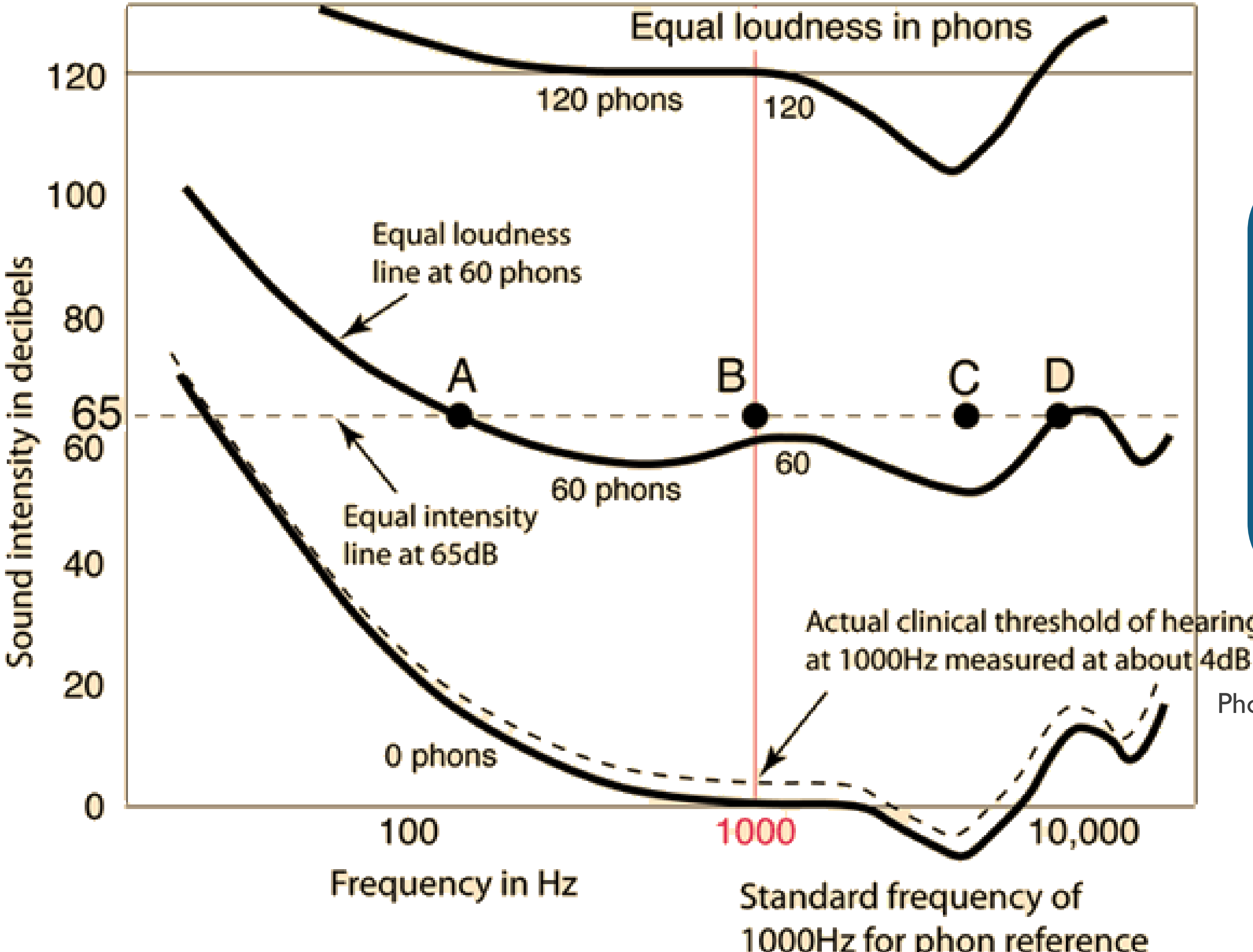


$$Y(j\omega) = H(j\omega)X(j\omega)$$

Direct multiplication at each frequency



Contributed by EE 313 student Erte Bablu from http://www.rtings.com/headphones/reviews/apple/wireless-airpods



Perceptual response to sound intensity is not uniform in frequency

Phon is sound referenced to 1kHz

http://hyperphysics.phy-astr.gsu.edu/hbase/Sound/earcrv.html

Using the convolution property to do convolutions

- Compute the following convolution

$$y(t) = h(t) * x(t)$$

- Convert the two signals into the frequency domain

$$H(j\omega) = \mathcal{F}\left\{h(t)\right\}$$
$$X(j\omega) = \mathcal{F}\left\{x(t)\right\}$$

- Compute the product

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Go from frequency domain back into the time domain

$$y(t) = \mathcal{F}^{-1}\left\{Y(j\omega)\right\}$$

Double sinc example

- Given where $\omega_i > 0$ and $\omega_c > 0$

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \qquad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

- Find

$$y(t) = h(t) * x(t)$$

Double sinc example (continued)

- Solve by going into the frequency domain
- First find

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Need to compute

$$\mathcal{F}\left\{\frac{\sin(\omega_i t)}{\pi t}\right\}\mathcal{F}\left\{\frac{\sin(\omega_c t)}{\pi t}\right\}$$

- But note that

$$\text{sinc}\left(\frac{t}{2\pi}\right) = \frac{\sin(t/2)}{t/2} \quad \text{and} \quad \text{sinc}\left(\frac{t}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\text{rect}(\omega)$$

Double sinc example (continued)

- Using the scaling property

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- Write

$$\frac{\sin(\omega_i t)}{\pi t} = \frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}$$

- It follows that

$$\begin{aligned}\mathcal{F}\left\{\frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}\right\} &= 2\pi \frac{\omega_i}{\pi} \frac{1}{|2\omega_i|} \mathrm{rect}(\omega/2\omega_i) \\ &= \mathrm{rect}(\omega/2\omega_i)\end{aligned}$$

Double sinc example (continued)

- The convolution is then

$$Y(j\omega) = \mathrm{rect}(\omega/2\omega_i)\mathrm{rect}(\omega/2\omega_c)$$

$$= \mathrm{rect}(\omega/2\min(\omega_c, \omega_i))$$

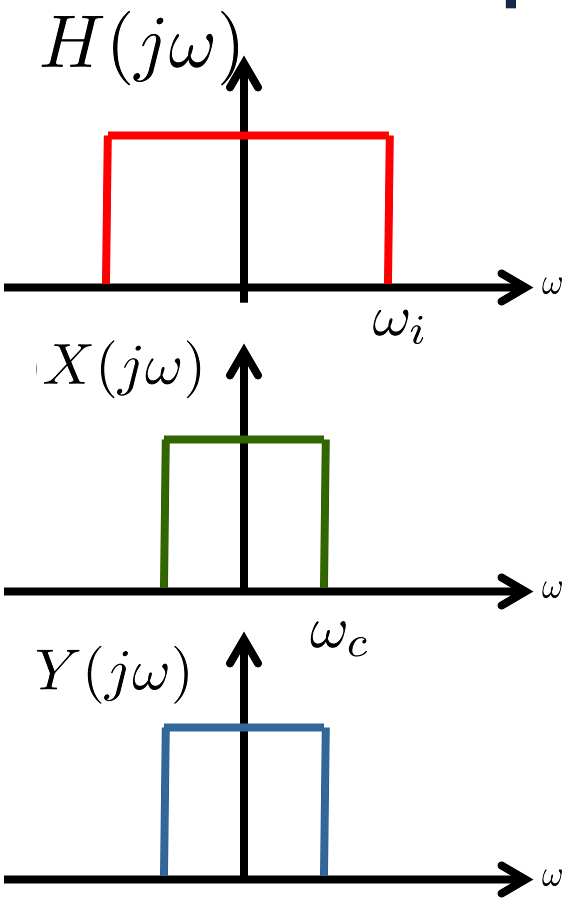
- Back in the time domain

$$y(t) = \frac{\sin(\min(\omega_i, \omega_c)t)}{\pi t}$$

This is a general result that sinc convolved with sinc gives sinc

Double sinc example (concluded)

Visualizing the effect in the frequency domain



$$Y(j\omega) = H(j\omega)X(j\omega)$$

Example where ω_i is bigger than ω_c

Summarizing the convolution property

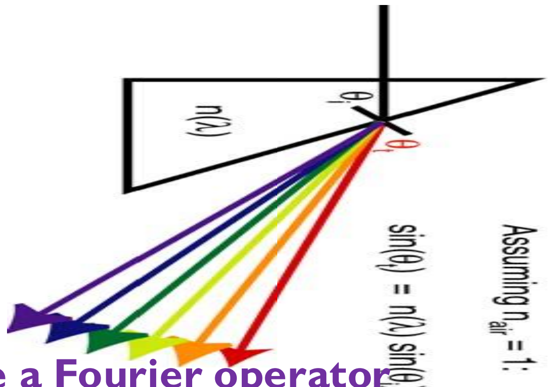
- Convolution between two signals in time becomes the product of the Fourier transforms of those signals in the frequency domain
- Convolutions are easy to do in the frequency domain as they involve a simple point-wise multiplication
- The convolution property explains how the frequency response of a system directly effects the frequencies of the input signal to create the output signal

Fourier in practice

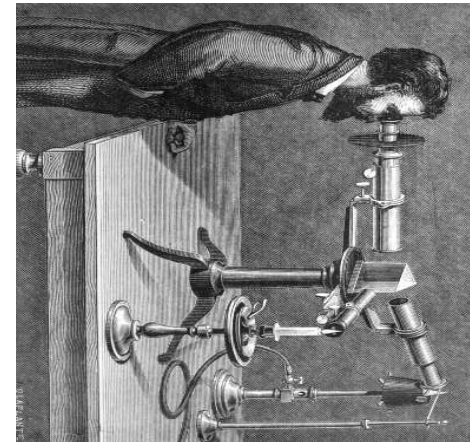
Key points

- Fourier concepts show up everywhere

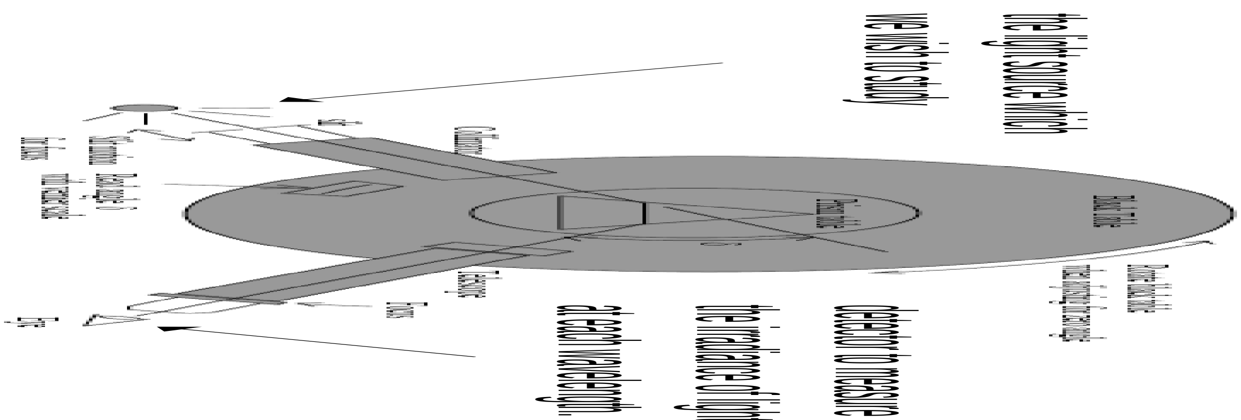
Spectrometer



Prism is like a Fourier operator

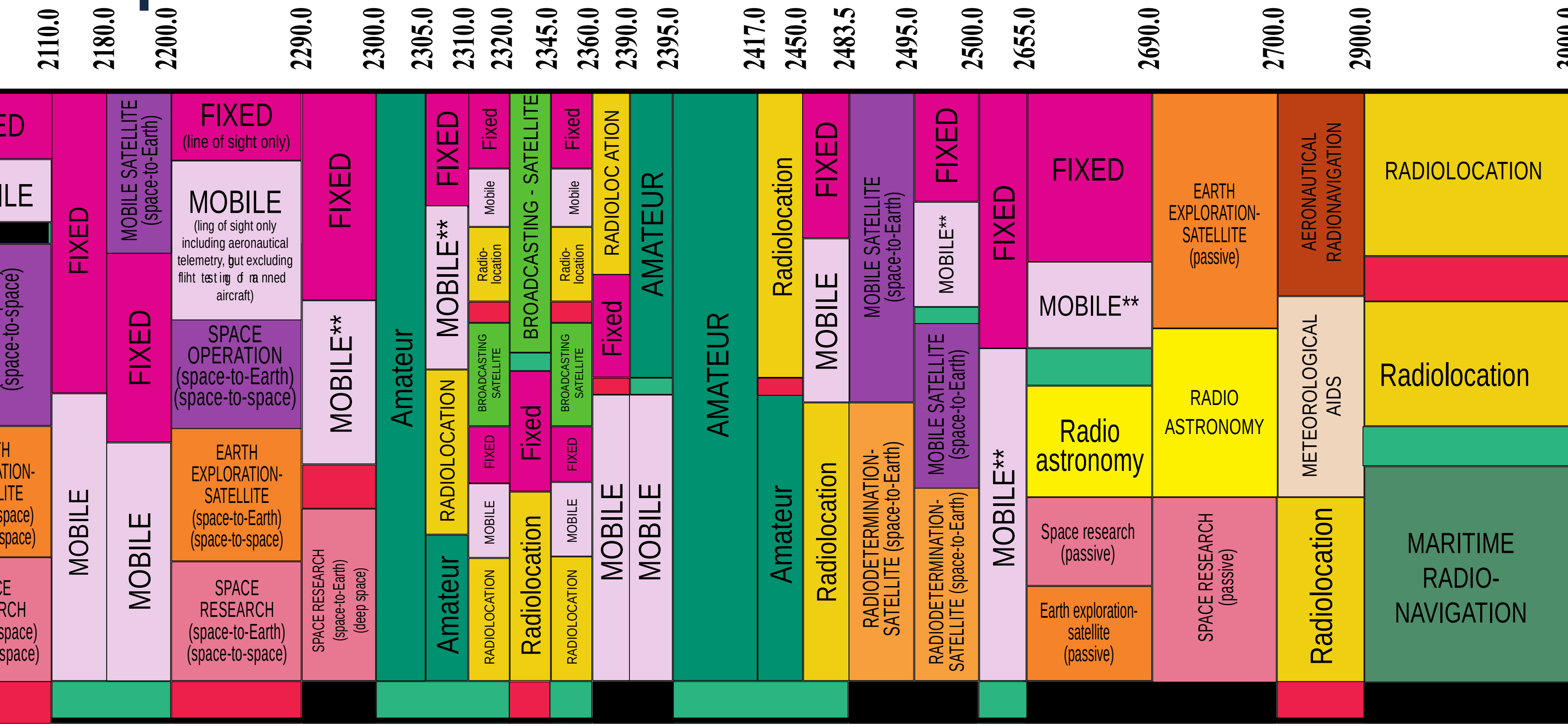


Robert Bunsen, 1859



Figures from https://www.brown.edu/research/labs/mittleman/sites/brown.edu.research.labs.mittleman/files/uploads/lecture19_0.pdf

Spectrum allocation



ISM – 2450.0± .50 MHz

3 GHz

https://www.ntia.doc.gov/files/ntia/publications/january_2016_spectrum_wall_chart.pdf

Communications

Spectrum for in-band signal

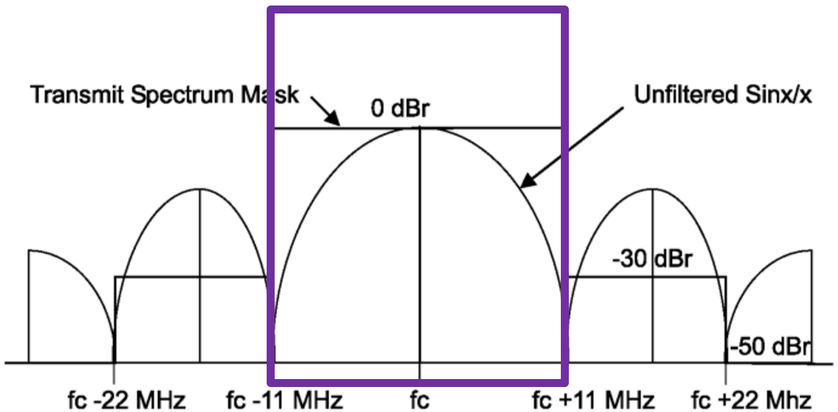
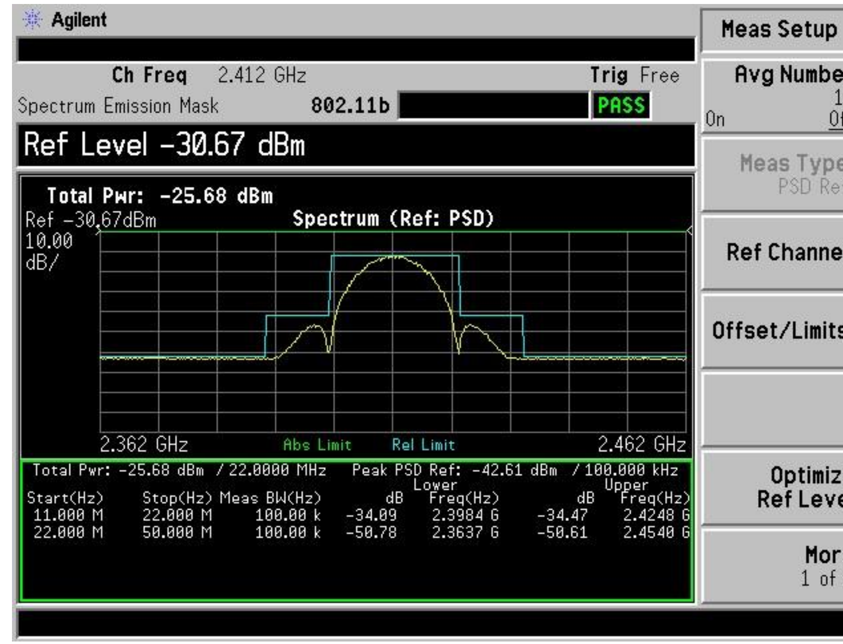


Figure 15-10—Transmit spectrum mask



Allowed out-of-band leakage

Transmit spectrum mask from IEEE 802.11-2016, 15.4.5.5 WiFi!
https://www.keysight.com/us/en/lib/resources/user-manuals/transmit-spectrum-mask-332766.html

Multiplication property

Key points

- Multiplication in time is convolution in frequency
- Use this fact to explain windowing

Multiplication property

- If $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$ $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

- Then

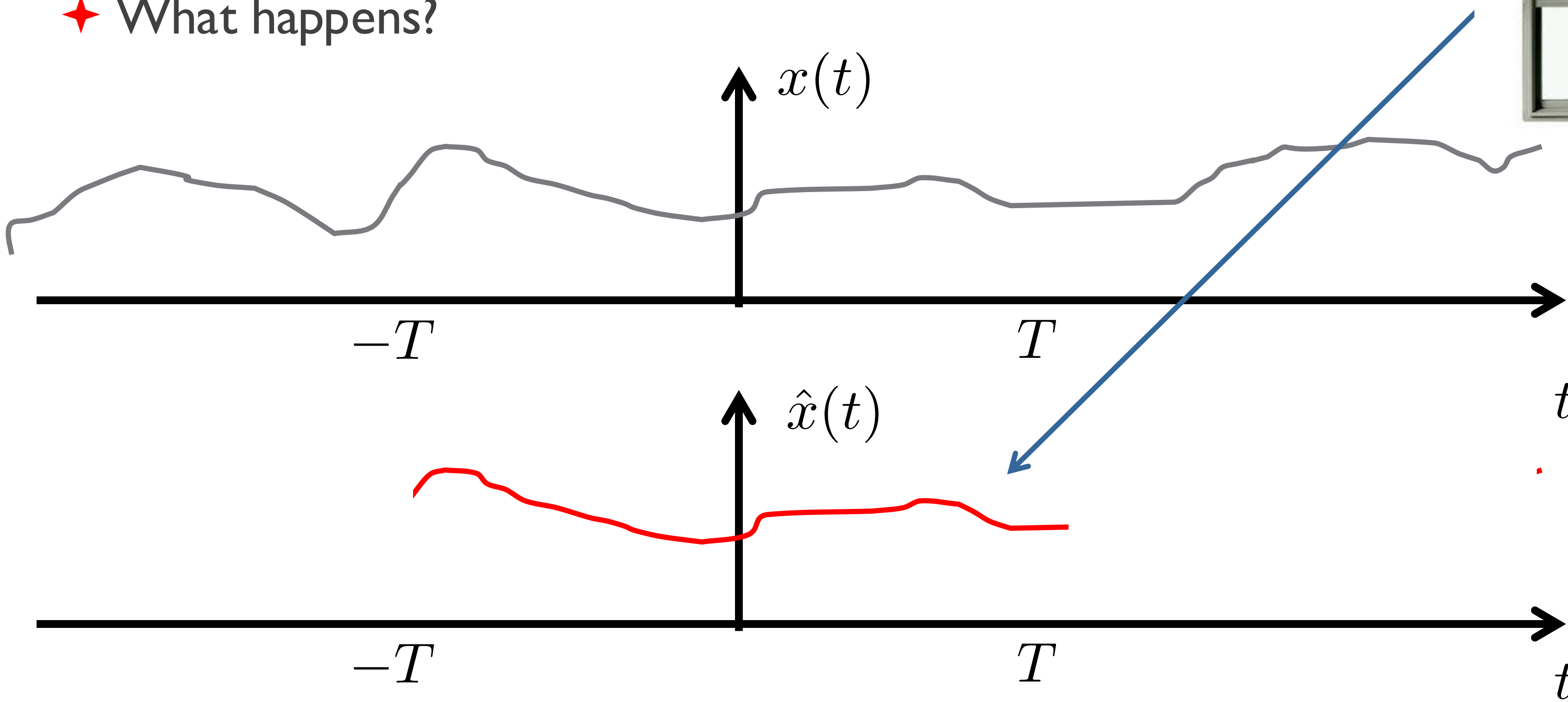
$$\boxed{y(t) = h(t)x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta}$$

Product in time is convolution in frequency

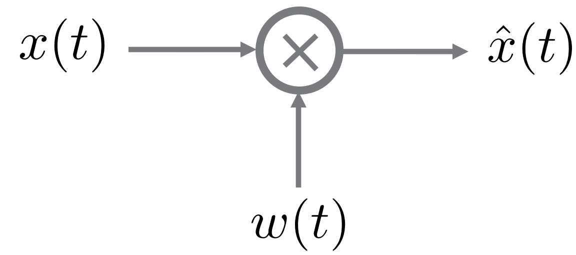
Implication of product property

- Suppose that you have a signal x(t)
  - But you only measure x(t) from –T….T
  - What happens?

Have only a window of the data



Windowing the spectrum



- The observed signal can be written as

$$\hat{x}(t) = \underbrace{\mathrm{rect}(t/(2T))}_{w(t)} x(t)$$

- In the frequency domain

$$\hat{X}(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2T\mathrm{sinc}\left(\frac{2T\theta}{2\pi}\right) X(j(\omega-\theta))d\theta$$

Spectrum is filtered by the sinc function

Example – windowing a cosine

$$x(t) = \cos(\omega_c t) \qquad \hat{x}(t)$$

$$w(t) = \mathrm{rect}(t/(2T))$$

$$X(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

$$W(j\omega) = 2T\mathrm{sinc}\left(\frac{2T\omega}{2\pi}\right)$$

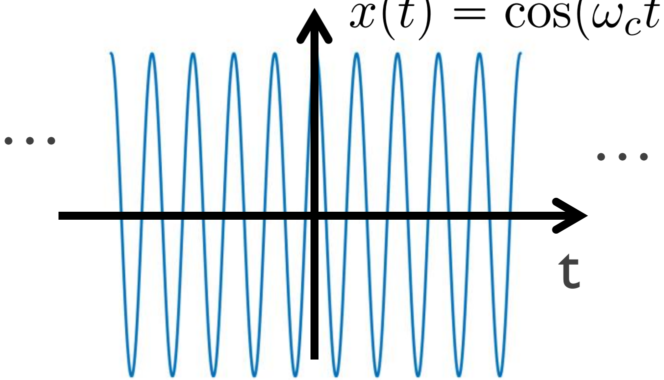
$$\hat{X}(j\omega) = \frac{T}{\pi}\mathrm{sinc}\left(\frac{2T(\omega - \omega_c)}{2\pi}\right) + \frac{T}{\pi}\mathrm{sinc}\left(\frac{2T(\omega + \omega_c)}{2\pi}\right)$$

Impulses get smeared due to windowing

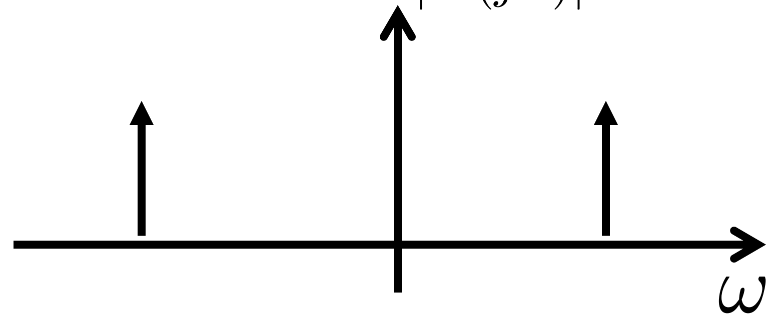
Intuition on windowing

$x(t) = \cos(\omega_c t)$

cosine

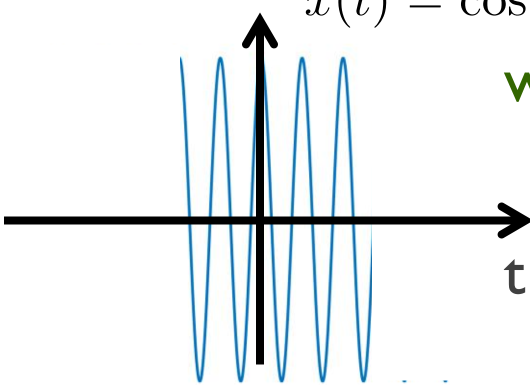


$|X(j\omega)|^2$

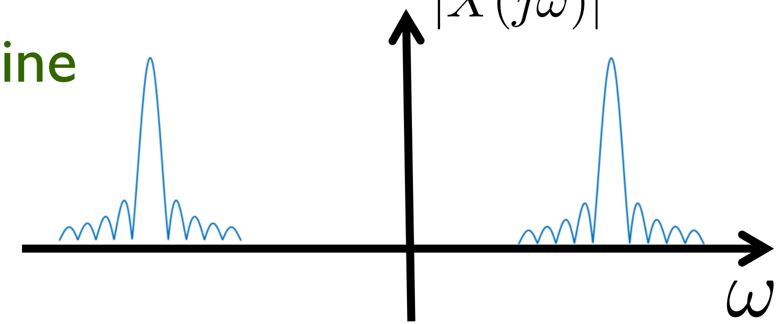


$\hat{x}(t) = \cos(\omega_c t)\,\mathrm{rect}(t/2T)$

windowed cosine



$|\hat{X}(j\omega)|^2$



Impact of windowing on resolution

- Suppose that we window a sum of two cosines

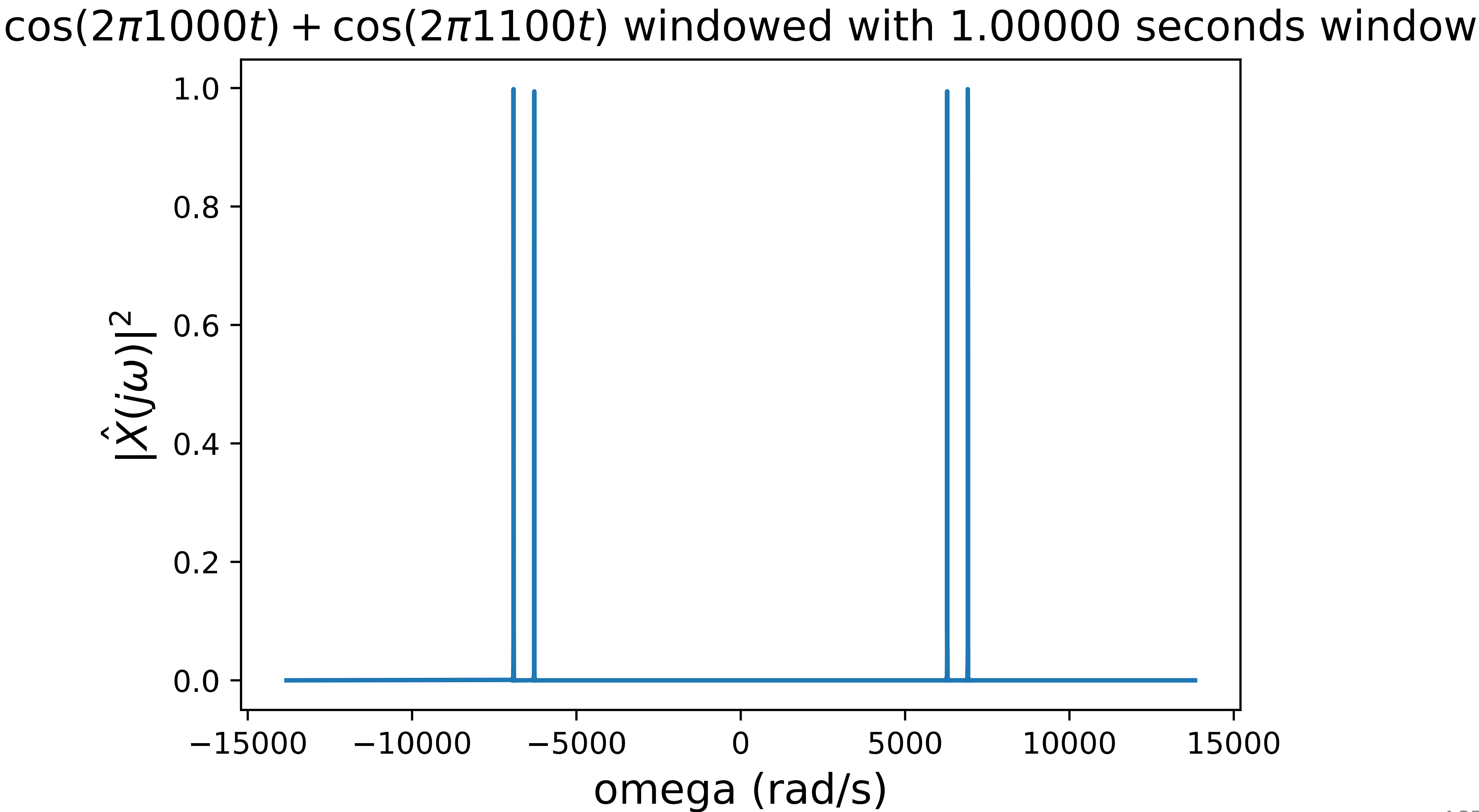
$$\cos(2\pi 1000 t) + \frac{1}{2}\cos(2\pi 1100 t) \longrightarrow \otimes \longrightarrow \hat{x}(t)$$

$$\mathrm{rect}(t/(2T))$$

1000 Hz or 6,280 rad/s

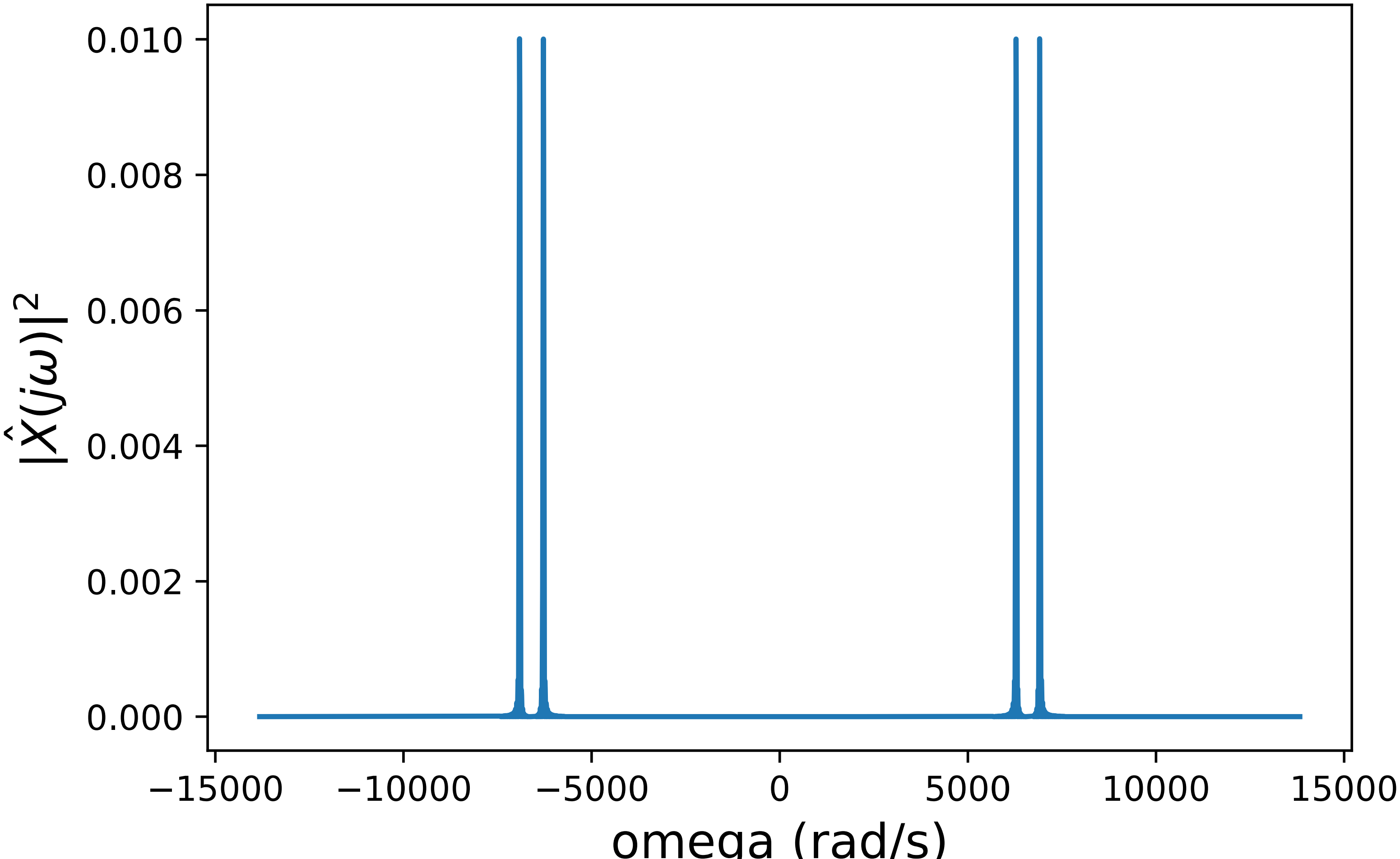
1100 Hz or 6,911 rad/s

What is the impact of the window size T on the ability to resolve the sinusoids?

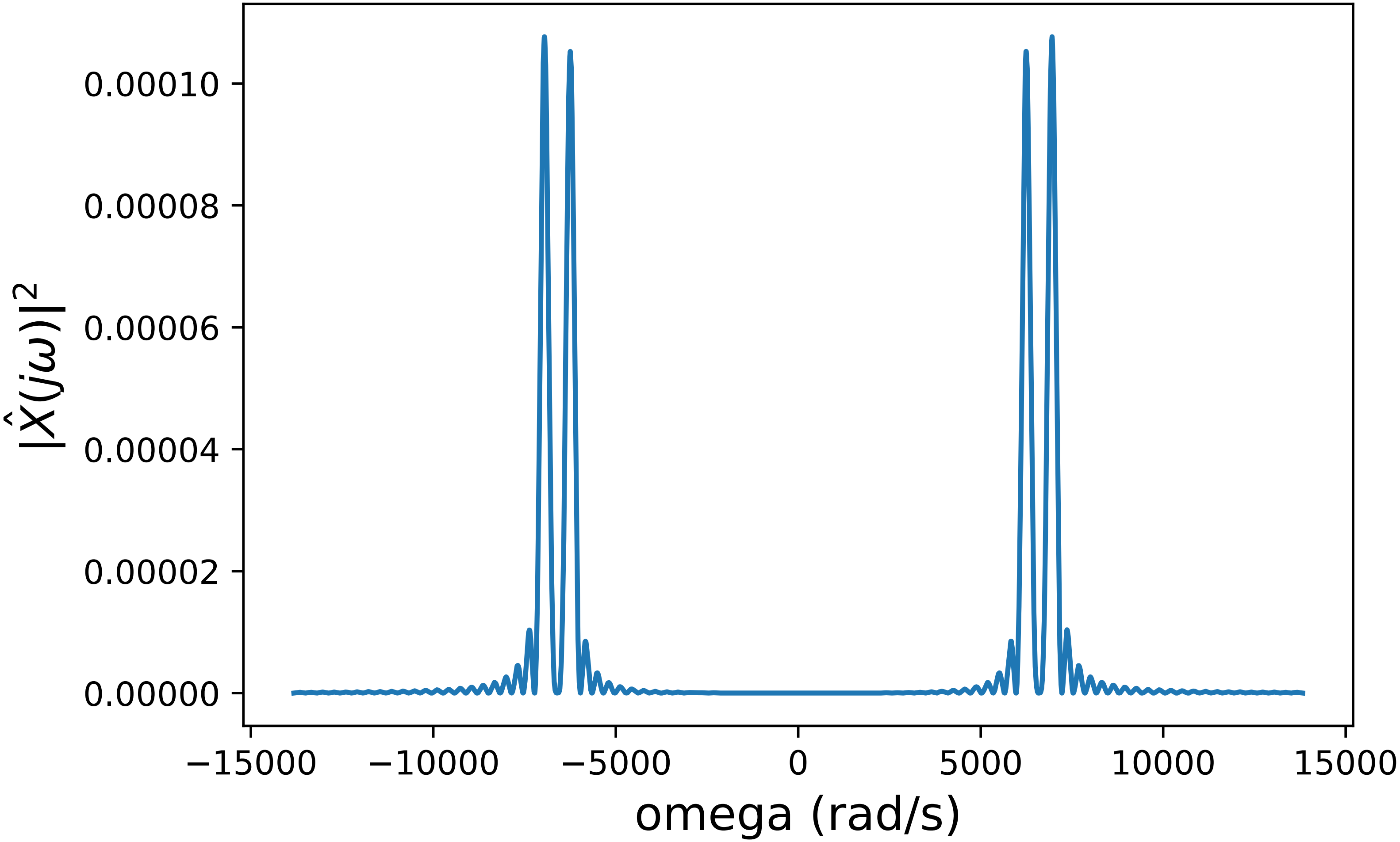

cos(2π1000t) + cos(2π1100t) windowed with 1.00000 seconds window
1.0
0.8
0.6
0.4
0.2
0.0
|X̂(jω)|²
−15000
−10000
−5000
0
5000
10000
15000
omega (rad/s)

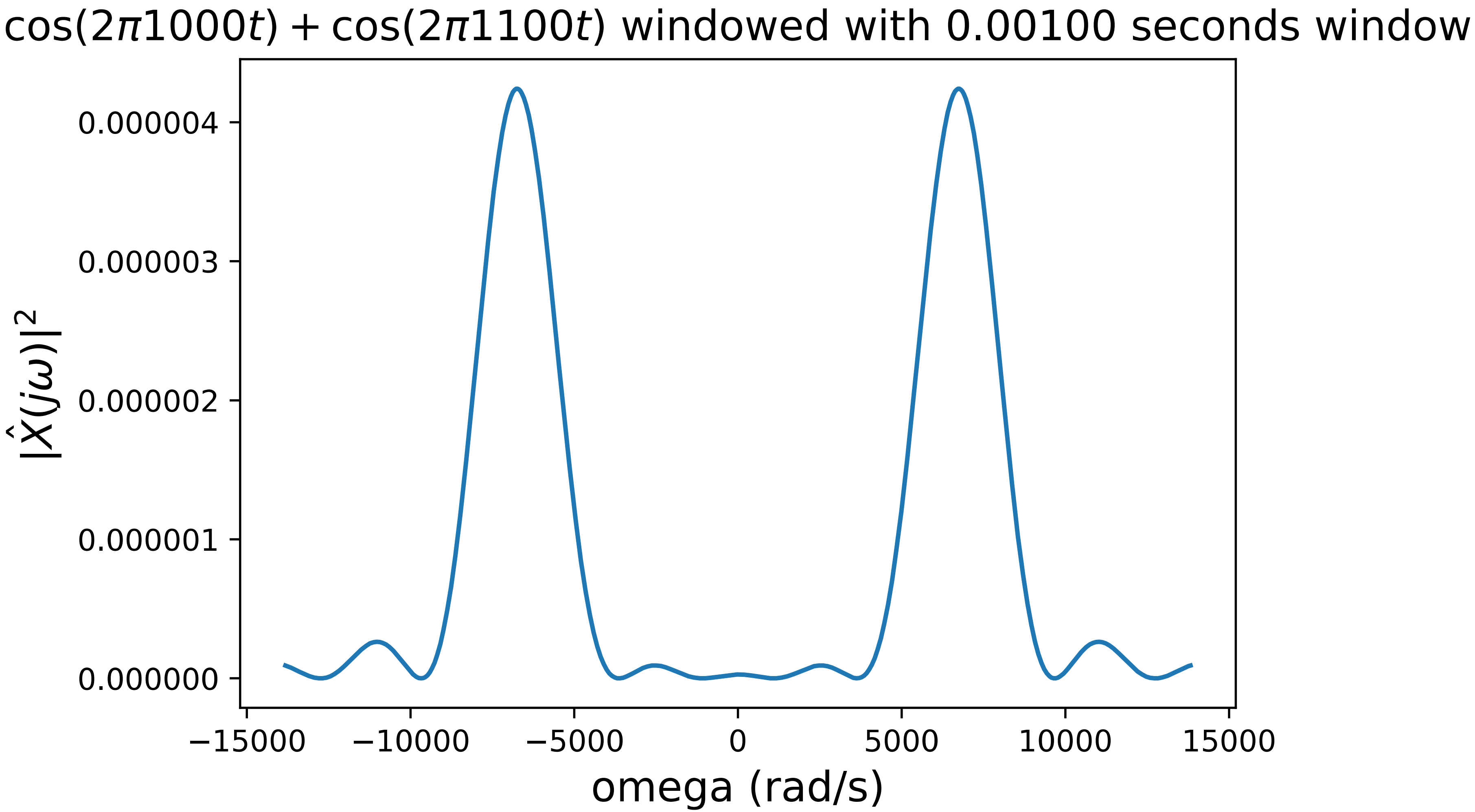
UC San Diego

$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 0.10000 seconds window

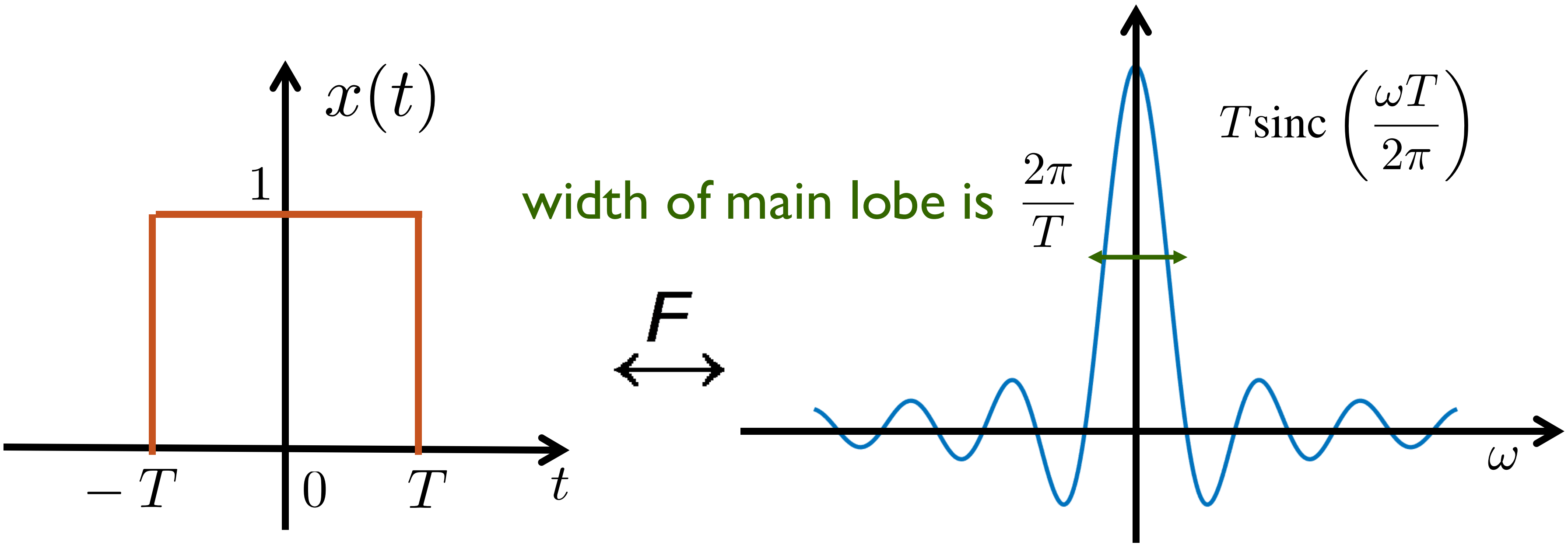


$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 0.01000 seconds window




$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 0.00100 seconds window
$|\hat{X}(j\omega)|^2$
0.000004
0.000003
0.000002
0.000001
0.000000
−15000
−10000
−5000
0
5000
10000
15000
omega (rad/s)

How much time is needed to resolve these cosines?



Zeros at $\pm\pi/T, \pm\pi/T, \ldots$

One main lobe of separation

$$\frac{2\pi}{T} = |\omega_1 - \omega_2| \quad \Rightarrow \quad T = \frac{2\pi}{|\omega_1 - \omega_2|} = \frac{2\pi}{2\pi 100} = 0.01\,\text{s}$$

Summarizing the multiplication property

- Product between two signals in time becomes the (scaled) convolution of the Fourier transforms of those signals in the frequency domain

- Truncating a real signal for analysis, called windowing, leads to a distortion of the original signal's Fourier transform

- The ability to resolve different frequencies in a signal improves as the observation window grows longer

Bandwidth

Key points

- Finite duration signals have infinite bandwidth
- Different measures of bandwidth are used in practice

Isolation in time and frequency

- From the windowing theorem

sinc has infinite duration

$$x(t)\text{rect}(t/2T) \Rightarrow \frac{2T}{2\pi}\left[X(j\omega) * \text{sinc}(2T\omega/2\pi)\right]$$

Finite duration in time

Infinite duration in frequency

- From the convolution theorem

sinc has infinite duration

$$X(j\omega)\text{rect}(\omega/2B) \Rightarrow x(t) * 2B\text{sinc}(2Bt/2\pi)$$

Finite duration in frequency

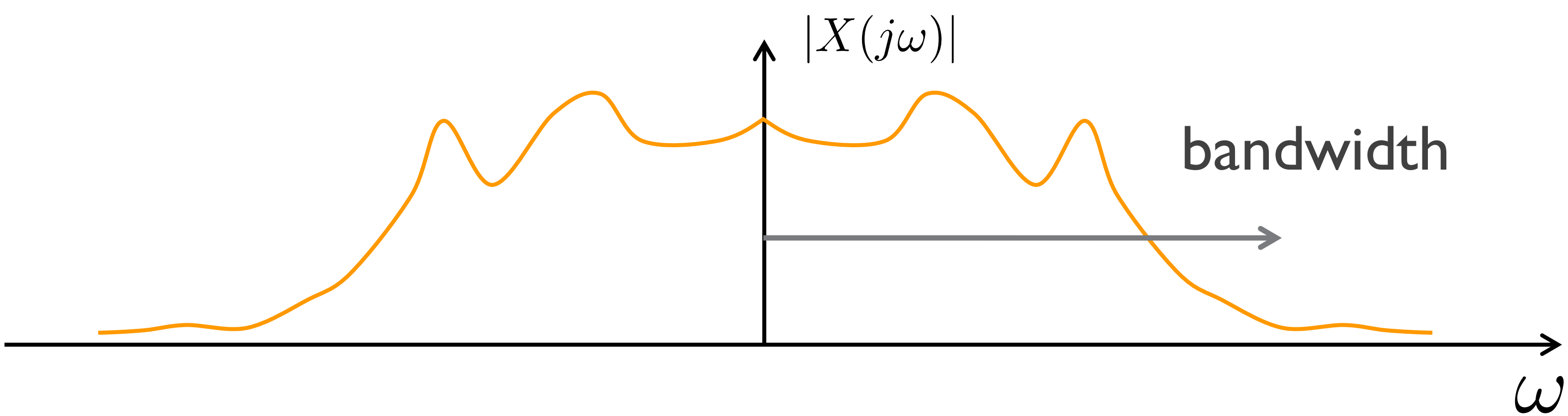
Infinite duration in time

Windowing and convolution have impact on the spectrum of practical signals (infinite) and the impulse response of an ideal low-pass filter (infinite)

Bandwidth of a practical signal

- If time duration is finite → bandwidth is infinite
  - For any practical signal, the absolute bandwidth is infinite

- Define a "bandwidth" to measure the extent of frequency content



Common definitions of bandwidth

- Fractional containment bandwidth
  - Bandwidth such that a fraction of energy is contained
  - Solve for ω_B such that

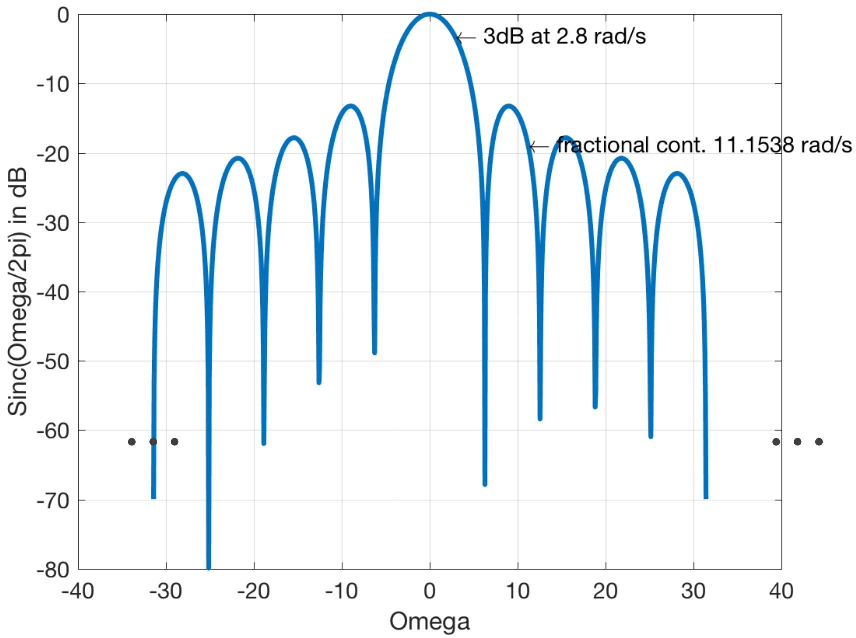
$$\int_{-\omega_B}^{\omega_B} |X(j\omega)|^2 d\omega \geqslant (1-\epsilon) \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- 3dB bandwidth (or half-power bandwidth)
  - Bandwidth where the signal achieves half the peak value
  - Makes the most sense with simple filters

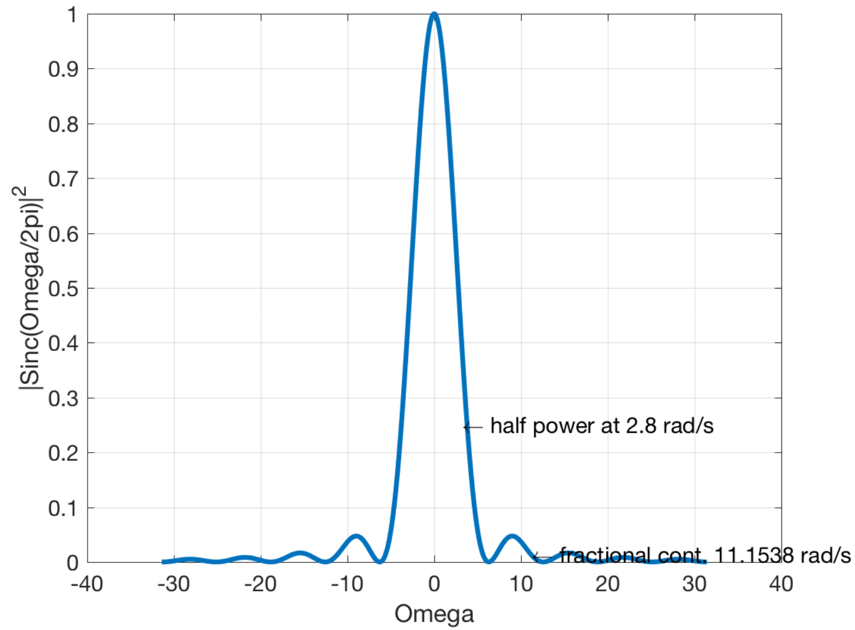
$$|X(j\omega_B)|^2 = \frac{1}{2} \max_{\omega} |X(j\omega)|^2$$

Examples of bandwidth

In dB



In linear scale (magnitude squared)



- Bandwidth of $\mathrm{sinc}(\omega/2\pi)$ with ½ power or 95% containment

Example fractional containment calculation

- Consider the following facts about Gaussian signals (proof of these facts is beyond the scope of this course)

1) Gaussian is its own Fourier transform $$e^{-t^2} \overset{\mathcal{F}}{\longleftrightarrow} \sqrt{\pi} e^{-\frac{\omega^2}{4}}$$

2) Integral of tail is "known" $$\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt = Q(x) = \frac{1}{2} \mathrm{erfc}(x/\sqrt{2})$$

3) Unit area $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-t^2/2} dt = 1$$

Gaussian distribution is a big part of probability and statistics

Example fractional containment calculation (cont.)

- Consider signal $x(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$
- Find an expression for the fractional containment bandwidth

$$\int_{-\omega_B}^{\omega_B} |X(j\omega)|^2 d\omega \geqslant (1-\epsilon) \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- Note that

$$|X(j\omega)| = e^{-\frac{\omega^2}{4}}$$

$$|X(j\omega)|^2 = e^{-\frac{\omega^2}{2}}$$

Example fractional containment calculation (cont.)

$$\int_{-\omega_B}^{\omega_B} e^{-\frac{\omega^2}{2}} d\omega = (1-\epsilon) \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega$$

- For the RHS note that

$$\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega = 1$$

- For the LHS

$$\begin{aligned}\int_{-\omega_B}^{\omega_B} e^{-\frac{\omega^2}{2}} d\omega &= \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \int_{\omega_B}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \int_{-\infty}^{-\omega_B} e^{-\frac{\omega^2}{2}} d\omega \\ &= \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - 2\int_{\omega_B}^{\infty} e^{-\frac{\omega^2}{2}} d\omega \\ &= 1 - 2Q(\omega_B)\end{aligned}$$

Example fractional containment calculation (cont.)

- Simplifying, we need to solve

$$1 - 2Q(\omega_B) = 1 - \epsilon$$

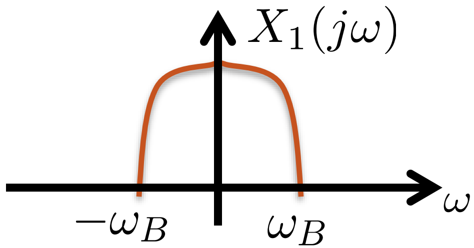
- Rearranging terms

$$Q(\omega_B) = \epsilon/2$$

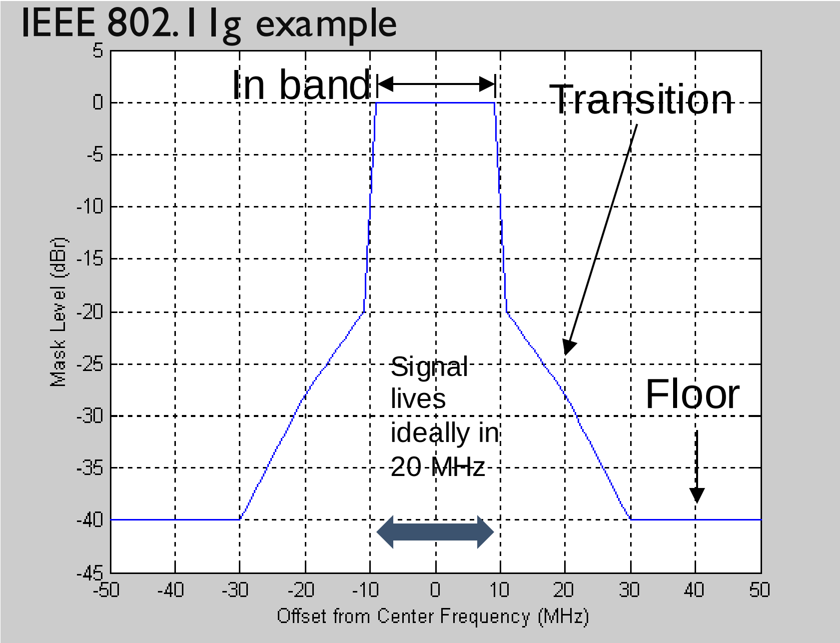
$$\Rightarrow \quad \omega_B = Q^{-1}(\omega_B)$$

inverse Q function, available in some form (maybe with a different name) in Excel, Python, MATLAB, etc.

Spectrum masks



Since communication spectrum is not exactly band limited, the allowed profile is called a spectrum mask



In Band: encompasses the desired signal
Transition: bounds adjacent channel interference
Floor: bounds other channel interference

Summarizing bandwidth

- Bandwidth is a measure of the extent of the non-zero frequency components present in a signal
- Practical signals always have infinite bandwidth due to be generated in a finite amount of time, a result of the windowing property
- There are different ways to define the bandwidth of a practical signal based on determining when the frequencies are sufficiently small

Human eye & the Fourier transform

Human eye

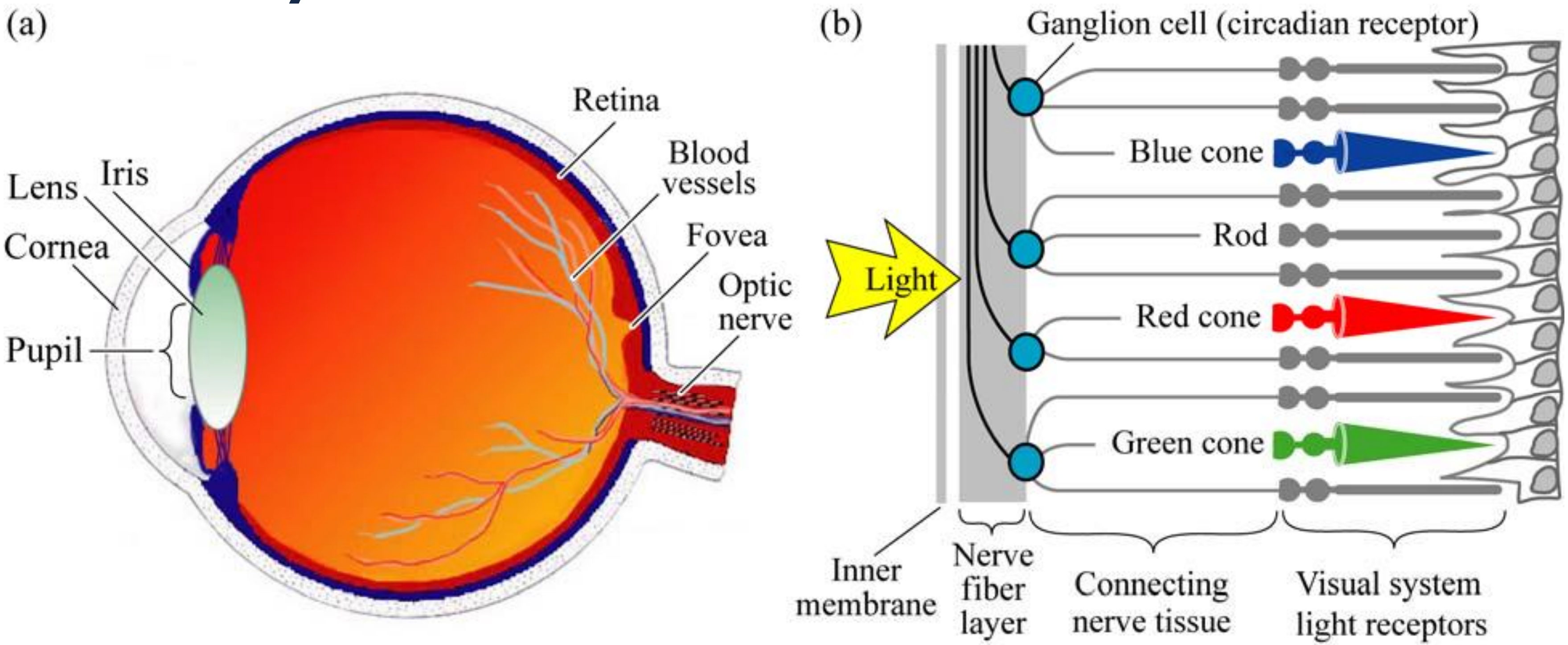


Fig. 16.1. (a) Cross section through a human eye. (b) Schematic view of the retina including rod and cone light receptors (adapted from Encyclopedia Britannica, 1994).

From http://www.ecse.rpi.edu/~schubert/Light-Emitting-Diodes-dot-org/chap16/chap16.htm

Vision regimes

Dim light, only see **grays**

Bright light, can see in **color**

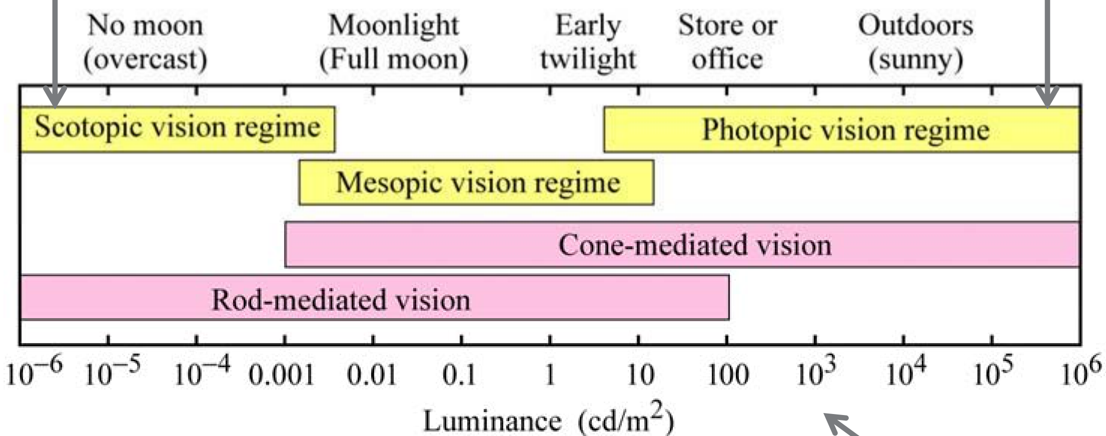


Fig. 16.2. Approximate ranges of vision regimes and receptor regimes (after Osram Sylvania, 2000).

Note log scale here

From http://www.ecse.rpi.edu/~schubert/Light-Emitting-Diodes-dot-org/chap16/chap16.htm

Frequency response of the eye

Normalized frequency response (power)

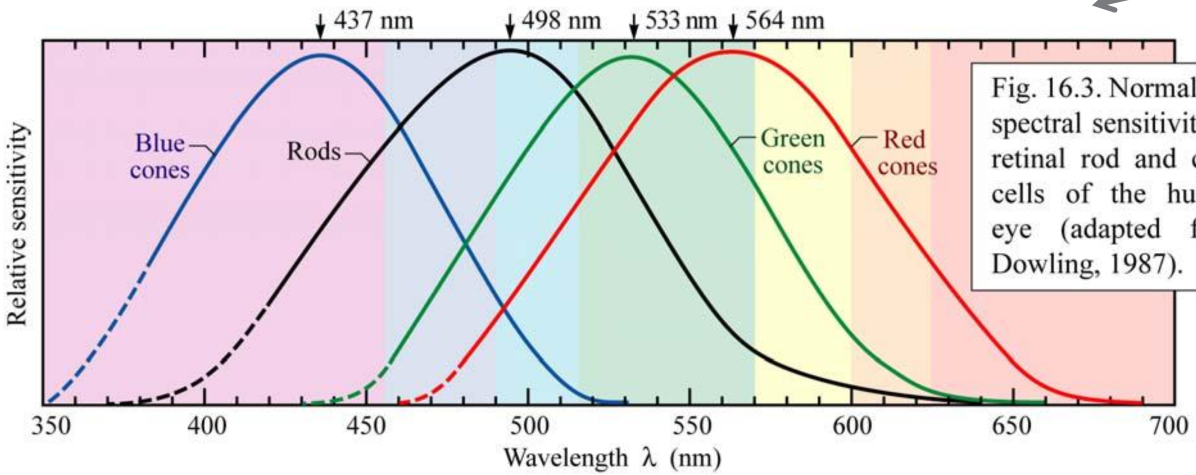
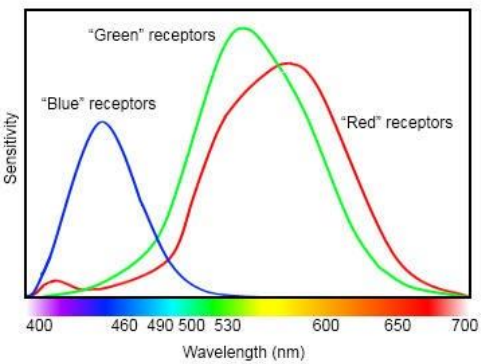


Fig. 16.3. Normalized spectral sensitivity of retinal rod and cone cells of the human eye (adapted from Dowling, 1987).

Non-normalized frequency response (power)

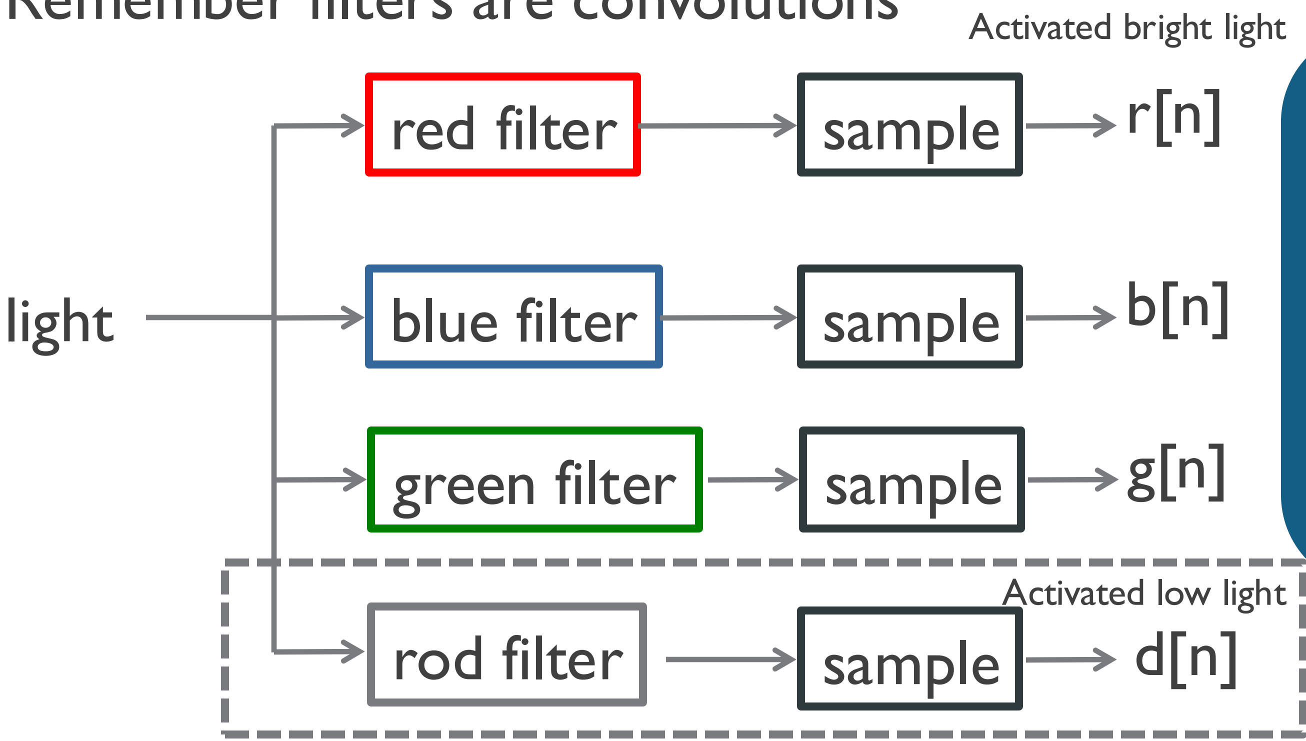


- Significant overlap between red and green
- Only three primary colors due to three cones

Top from http://www.ecse.rpi.edu/~schubert/Light-Emitting-Diodes-dot-org/chap16/chap16.htm
Right from http://obsessive-coffee-disorder.com/wp-content/uploads/2014/11/eye5.jpeg

From a signal processing perspective

Remember filters are convolutions



Humans perceive complete spectrum of light as a linear combination of three primary colors

Note human eye sampling rate is in the range of 48 Hz to 70 Hz