

Lecture 17

Generalizing Bode plots

Preview of today's lecture

- ◆ Bode plot for a general frequency response
 - ✦ Decompose the frequency response into a product of terms
 - ✦ Plot the low-pass and high-pass terms
 - ✦ Identify the breakpoints and slopes for all of the lines

Frequency response of a low-pass RC filter

- ◆ The frequency response of this system is

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}} \qquad \omega_0 = \frac{1}{RC}$$

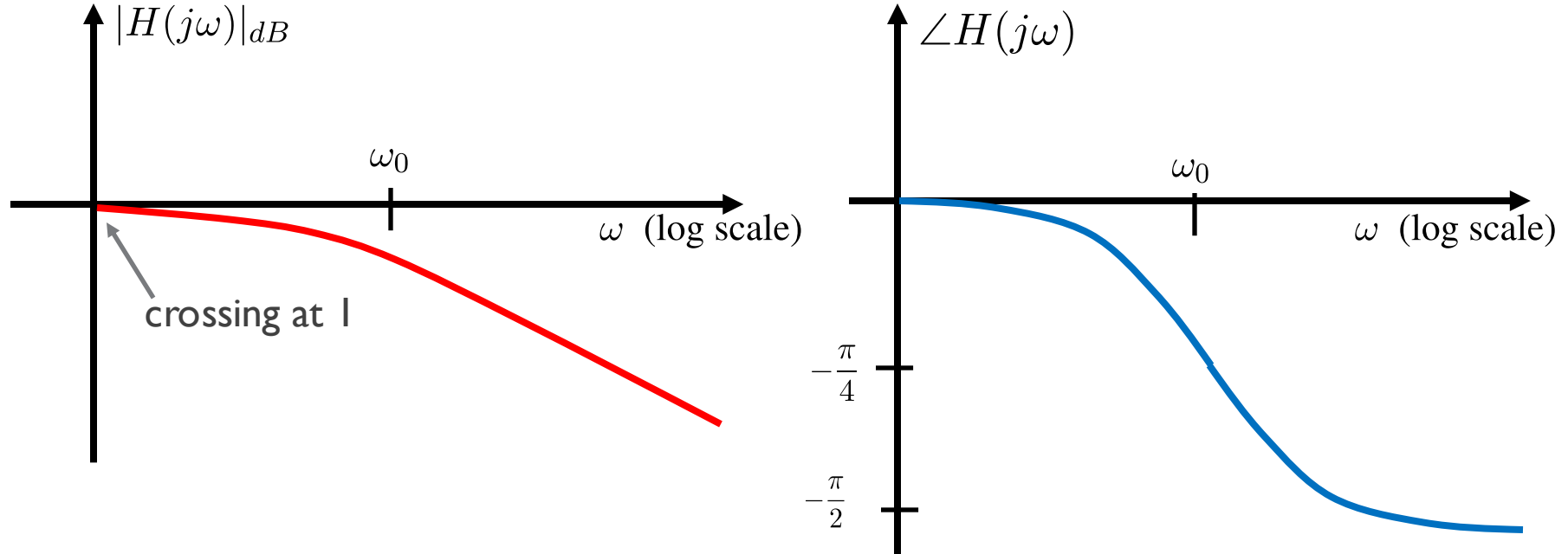
- ◆ The magnitude response in decibels is

$$\begin{aligned} |H(j\omega)|_{\text{dB}} &= 20 \log_{10} |H(j\omega)| \\ &= -20 \log_{10} \sqrt{1 + (\omega/\omega_0)^2} \end{aligned}$$

- ◆ The phase response in rad/s is

$$\angle H(j\omega) = -\tan^{-1} \left(\frac{\omega}{\omega_0} \right)$$

Plot of the frequency response of a low-pass RC filter



Bode approximation of the frequency response

- ◆ The frequency response of this system is

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}} \quad \omega_0 = \frac{1}{RC}$$

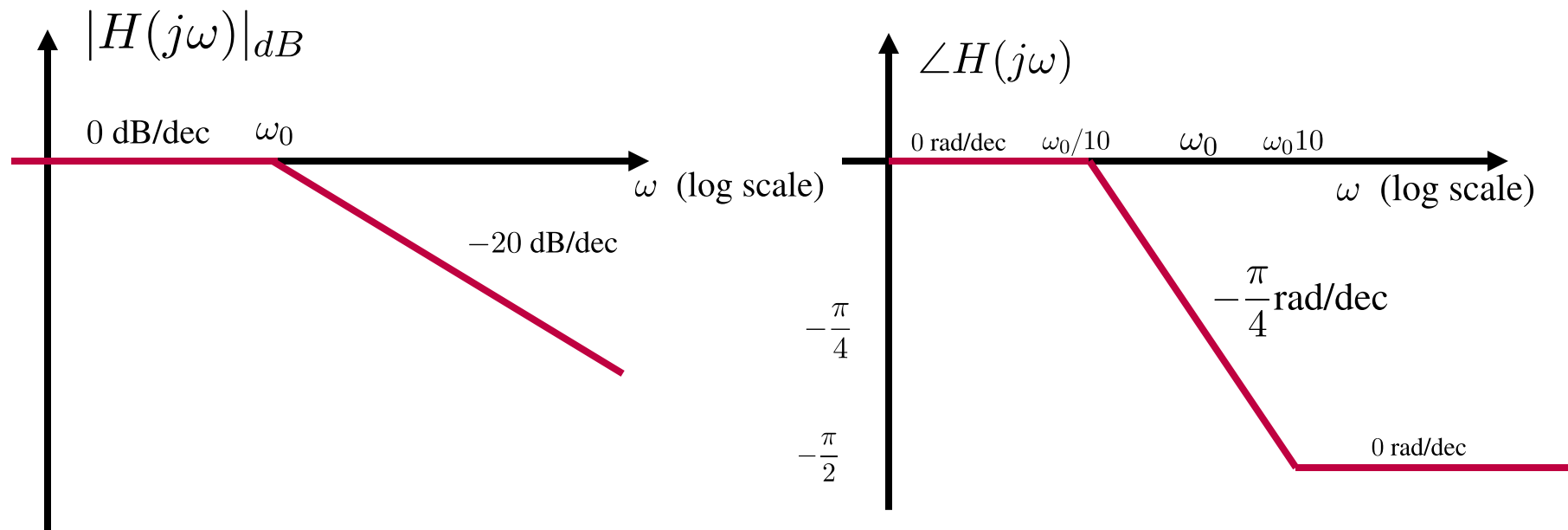
- ◆ The magnitude response is approximated as

$$|H(j\omega)|_{dB} \approx \begin{cases} 0 \text{ dB} & \omega < \omega_0 \\ -20 \log_{10} \omega + 20 \log_{10} \omega_0 & \omega \geq \omega_0 \end{cases}$$

- ◆ The phase response is approximated as

$$\angle H(j\omega) \approx \begin{cases} 0 & \omega < \omega_0/10 \\ -\frac{\pi}{4} \log_{10} \omega + \frac{\pi}{4} (\log_{10} \omega_0 - 1) & \frac{\omega_0}{10} < \omega < 10\omega_0 \\ -\frac{\pi}{2} & \omega > 10\omega_0 \end{cases}$$

Bode plot of the frequency response for the low-pass



True responses approximated with a sequence of lines

Frequency response of a high-pass RC filter

- ◆ The frequency response of this system is

$$H(j\omega) = 1 + j\frac{\omega}{\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

- ◆ The magnitude response in decibels is

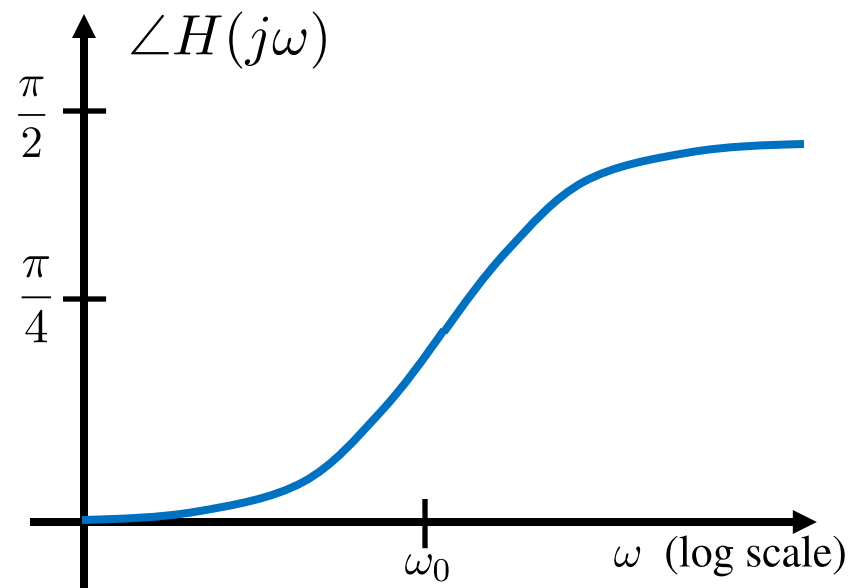
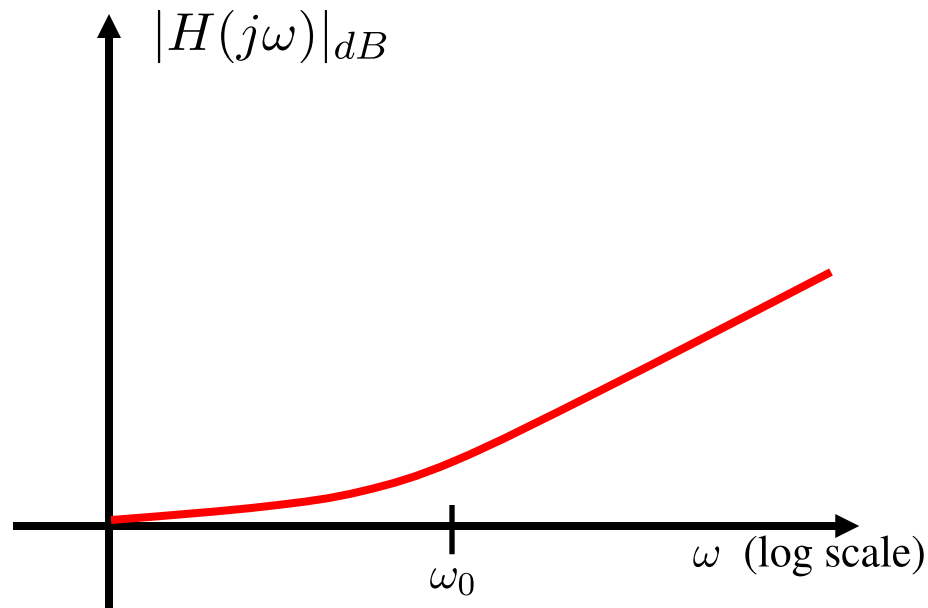
$$|H(j\omega)|_{\text{dB}} = 20 \log_{10} \sqrt{1 + (\omega/\omega_0)^2}$$

- ◆ The phase response in rad/s is

$$\angle H(j\omega) = \tan^{-1} \left(\frac{\omega}{\omega_0} \right)$$

Note sign change versus the low-pass RC filter

Plot of the frequency response of a low-pass RC filter



Bode approximation of the frequency response

- ◆ The frequency response of this system is

$$H(j\omega) = 1 + j\frac{\omega}{\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

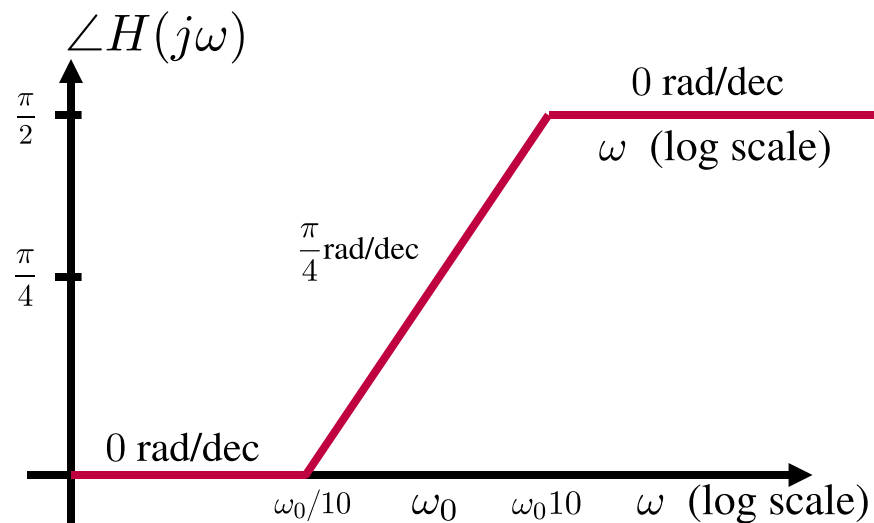
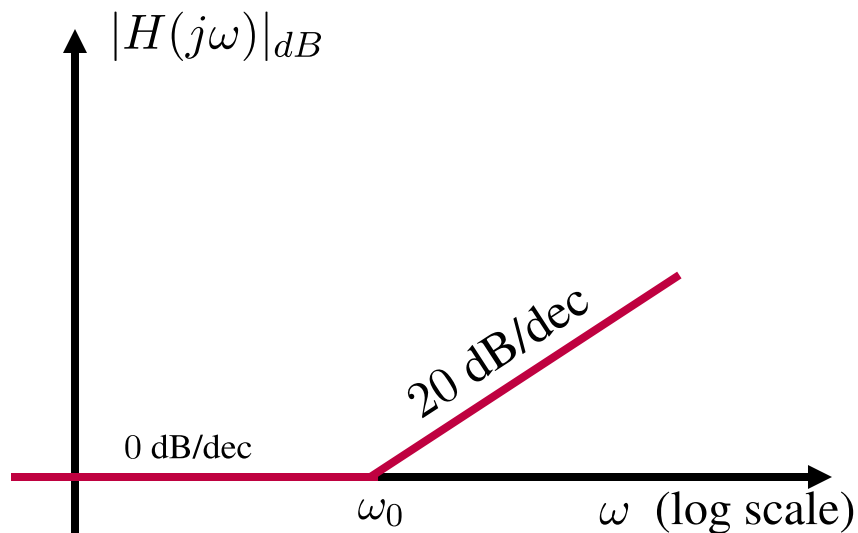
- ◆ The magnitude response is approximated as

$$|H(j\omega)|_{dB} \approx \begin{cases} 0 \text{ dB} & \omega < \omega_0 \\ 20 \log_{10} \omega - 20 \log_{10} \omega_0 & \omega \geq \omega_0 \end{cases}$$

- ◆ The phase response is approximated as

$$\angle H(j\omega) \approx \begin{cases} 0 & \omega < \omega_0/10 \\ \frac{\pi}{4} \log_{10} \omega - \frac{\pi}{4} (\log_{10} \omega_0 - 1) & \frac{\omega_0}{10} < \omega < 10\omega_0 \\ \frac{\pi}{2} & \omega > 10\omega_0 \end{cases}$$

Bode plot of the frequency response for the high-pass



True responses approximated with a sequence of lines

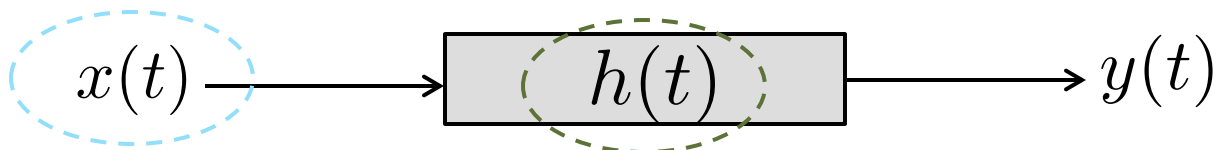
See how the sign difference manifests here vs the low-pass case

What is missing?

- ◆ So far only considered a simple RC filter
 - ✦ But general circuits can result in higher-order differential equations
- ◆ Approach
 - ✦ Decompose the higher order frequency response into terms that look like simple high- and low-pass filters
 - ✦ Build the composite frequency response using the examples of the low and high pass responses
 - ✦ Main task will be to find break points and slopes

Connections back to ECE 45

Lectures 2 - 3 working with signals



Lectures 4 - 7 LTI systems in the time domain

Lectures 16, 17 LTI systems in the frequency domain



Lectures 8 - 10 Fourier series

Lectures 11 - 15 Fourier transform

Fourier


General Bode plots

Key points

- Generalize Bode plots for higher order filters building upon the foundations of the low-pass and high-pass filters
- Process the plots step-by-step, starting from the left-most breakpoint

Generalizing the frequency response

- ◆ A LTI system described by a LCCDE can in general be written in a factored form as

$$H(j\omega) = \frac{A \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \dots}{(j\omega)^n \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right) \dots}$$


maybe in the numerator or denominator with different integer values of n

Example

- ◆ Find the factored form of the following LCCDE system

$$20\frac{d^2}{dt^2}y(t) + 100\frac{d}{dt}y(t) + 120y(t) = \frac{d^2}{dt^2}x(t) + \frac{d}{dt}x(t)$$

- ◆ Hint: $20(z^2 + 5z + 6) = 20(z + 2)(z + 3)$

Example (continued)

Refactoring the frequency response

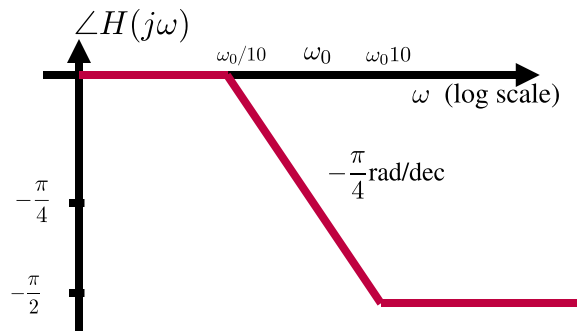
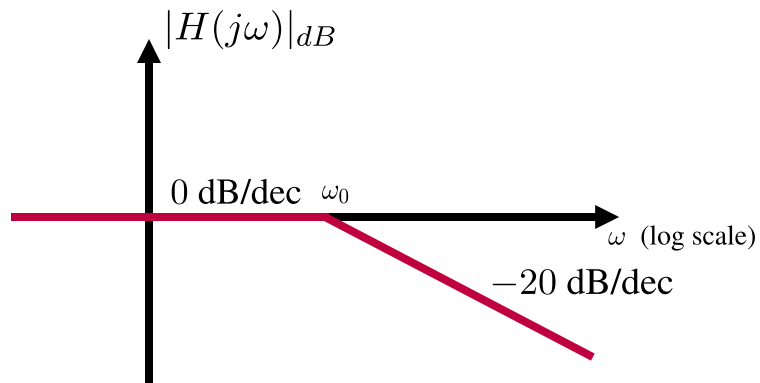
$$\begin{aligned}
 H(j\omega) &= \frac{A \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \dots}{(j\omega)^n \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right) \dots} \\
 &= A \frac{1}{(j\omega)^n} \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \dots \frac{1}{\left(1 + \frac{j\omega}{p_1}\right)} \cdot \frac{1}{\left(1 + \frac{j\omega}{p_2}\right)} \\
 &= A(j\omega)^{-n} \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right) \dots \left(1 + \frac{j\omega}{p_1}\right)^{-1} \left(1 + \frac{j\omega}{p_2}\right)^{-1} \dots \\
 &= A(j\omega)^{-n} \prod_{i=1}^N \left(1 + \frac{j\omega}{\omega_i}\right)^{a_i}, \quad a_i = \pm 1
 \end{aligned}$$



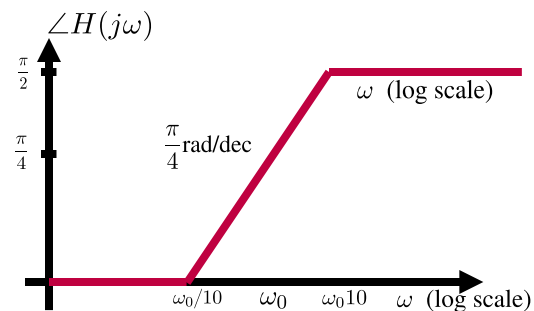
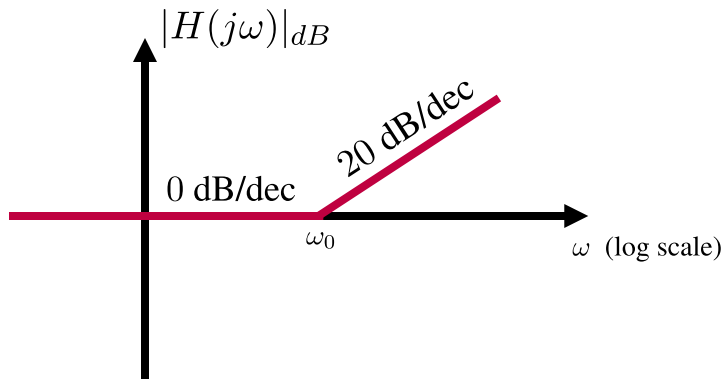
low-pass filter or high-pass filter depending on the factor

Key factors

$a_i = -1$
low-pass



$a_i = 1$
high-pass



Magnitude response

overall gain

$$|H(j\omega)|_{dB} = 20 \log_{10} A - 20n \log_{10} \omega$$

can be high or low-pass
in this case low-pass

high-pass
terms

$$+ 20 \log_{10} \left(1 + \frac{j\omega}{z_1} \right) + 20 \log_{10} \left(1 + \frac{j\omega}{z_2} \right) + \dots$$

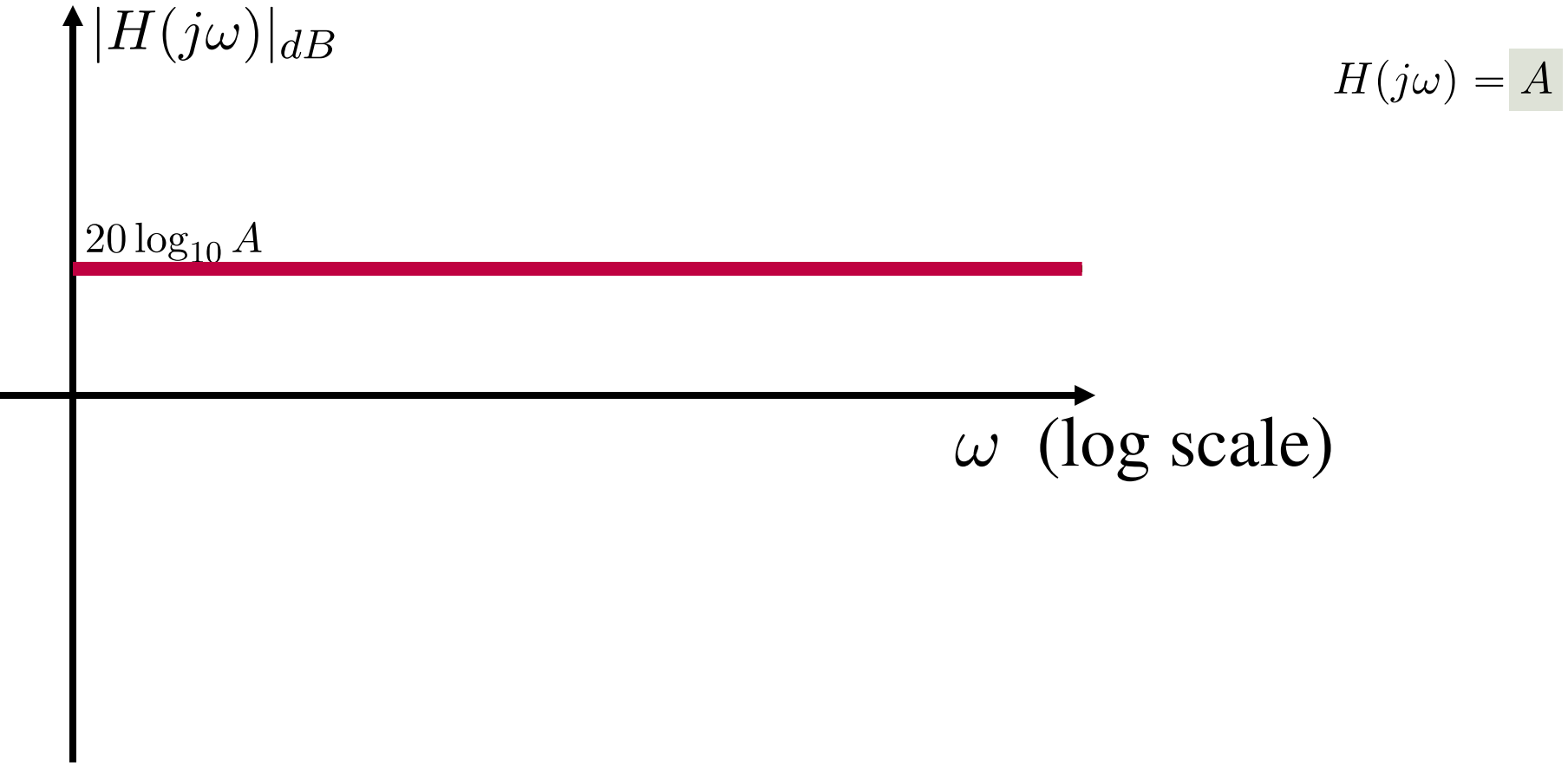
low-pass
terms

$$- 20 \log_{10} \left(1 + \frac{j\omega}{p_1} \right) - 20 \log_{10} \left(1 + \frac{j\omega}{p_2} \right) + \dots$$

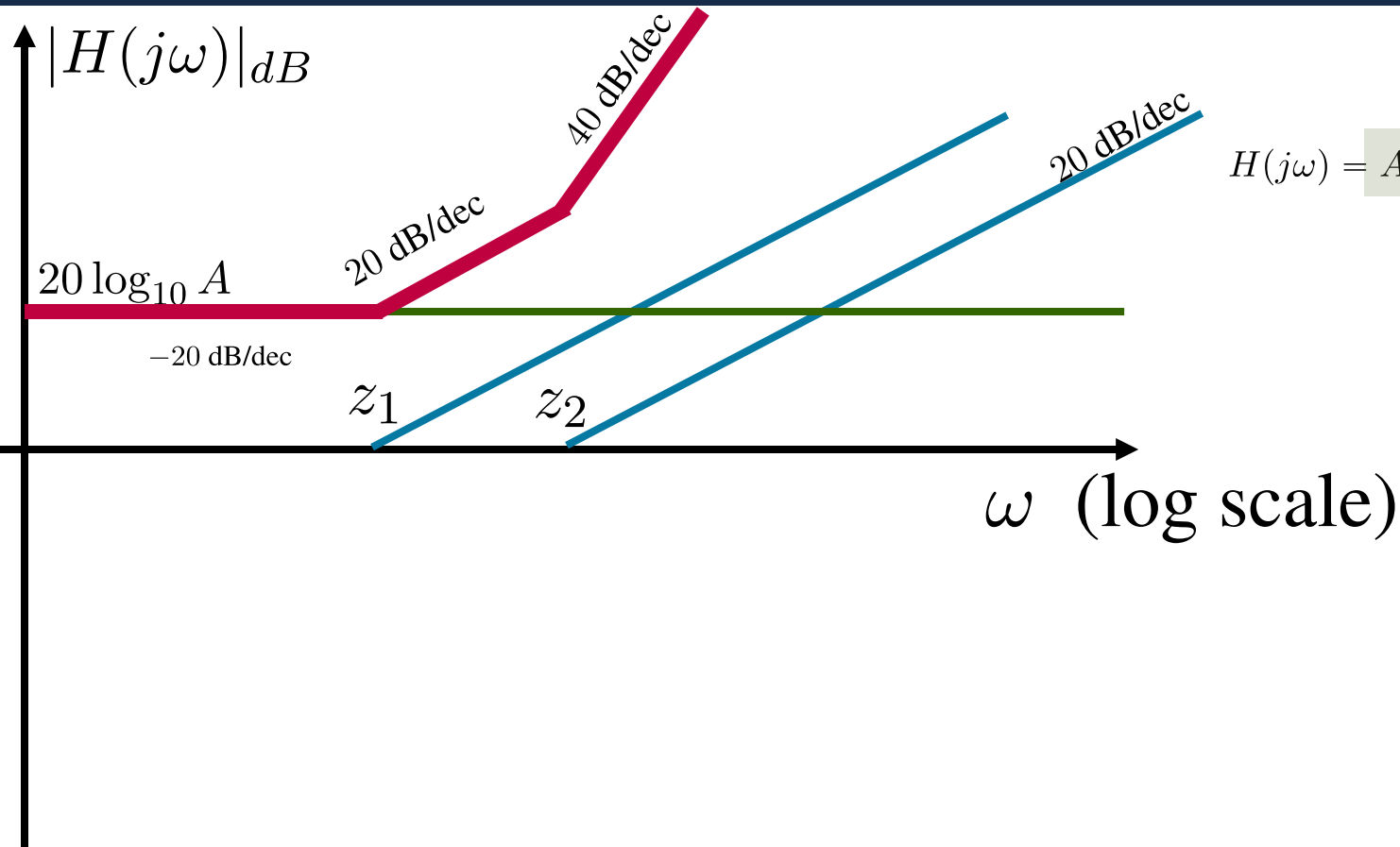
Sketching the magnitude response

1. Identify all break-points
2. Plot the individual Bode plot for each term
3. Sum the result
4. Label all break points, slopes, and transition points

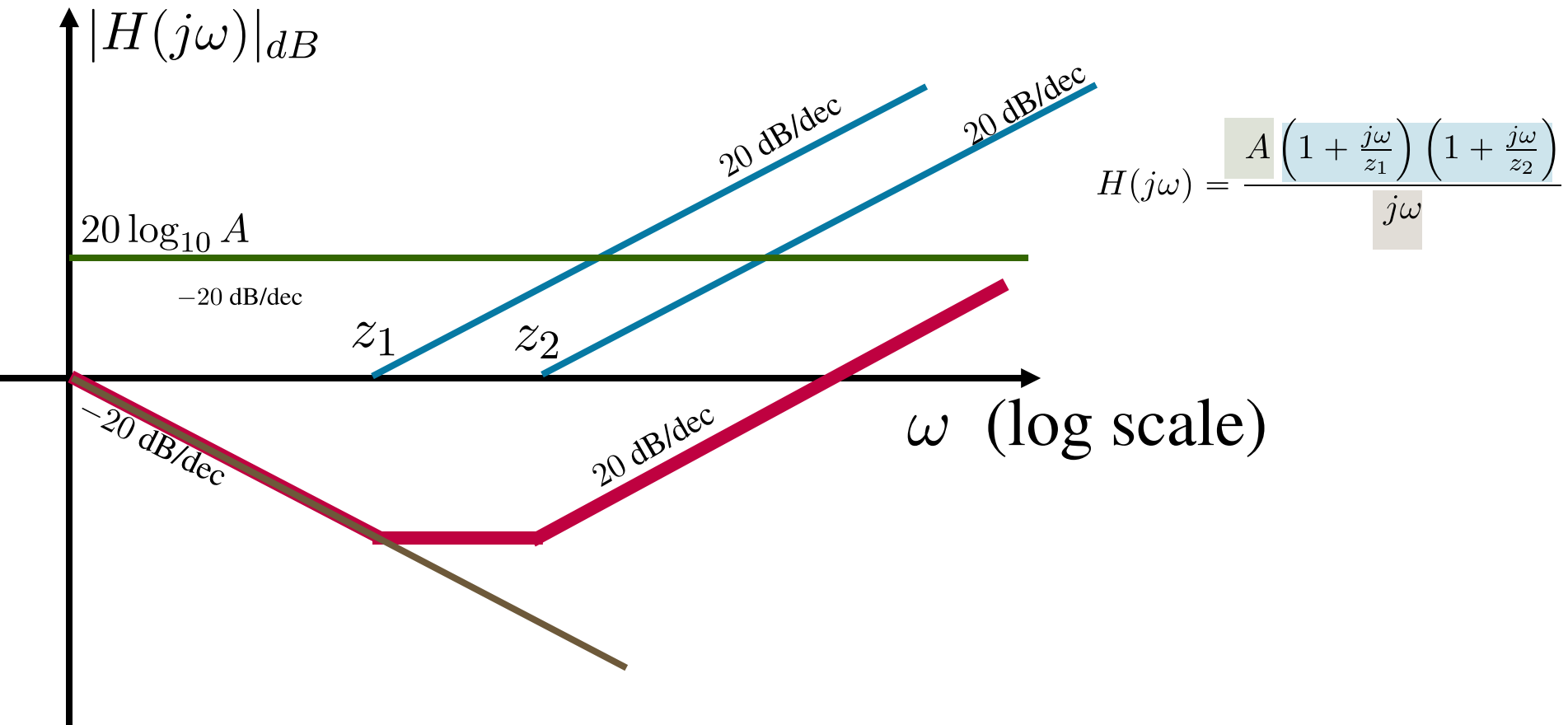
It is easiest to work from left to right, using the slopes of the lines to create the final plot

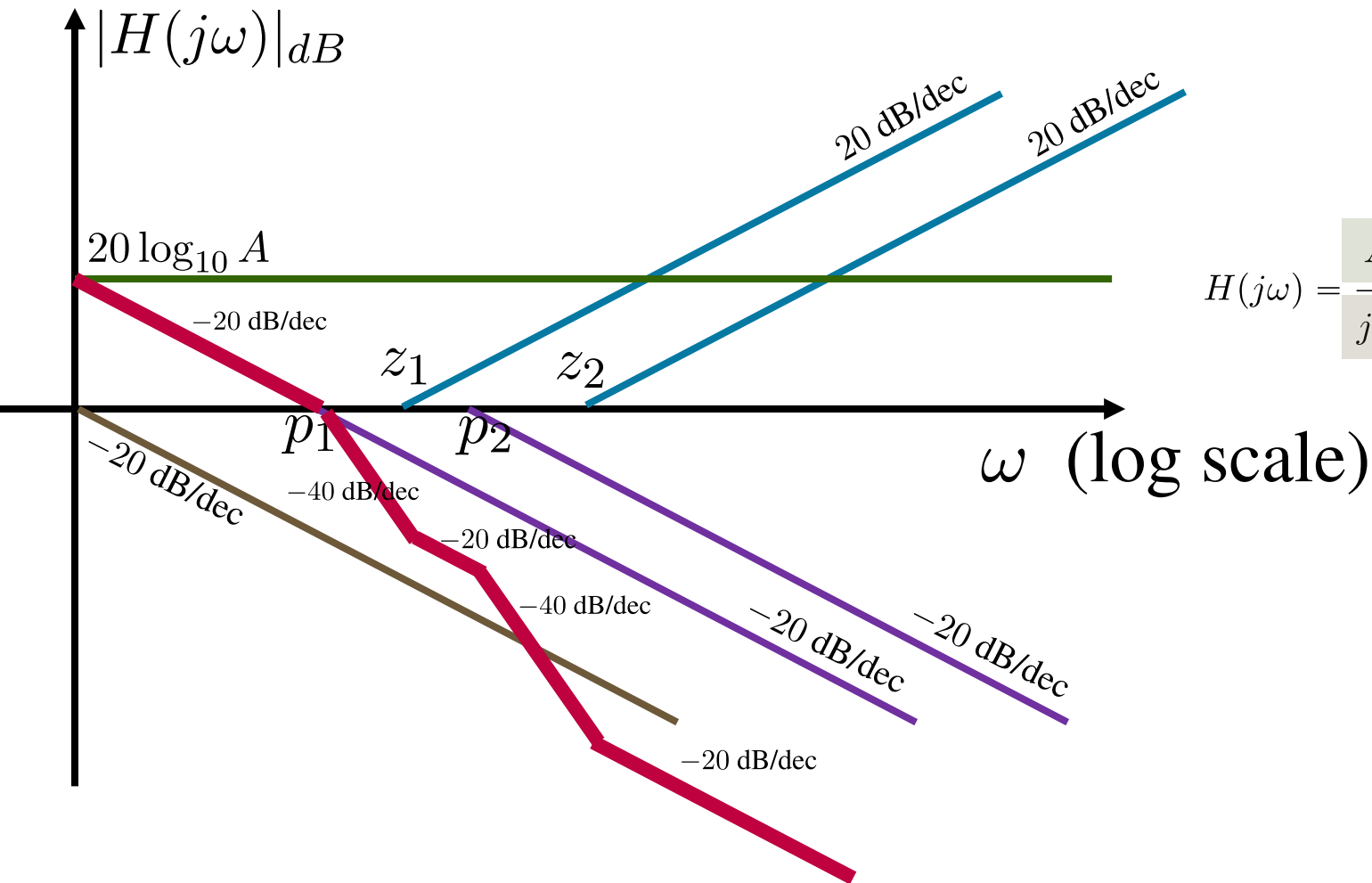


Note: when given actual numbers, also label the y-axis



$$H(j\omega) = A \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right)$$





$$H(j\omega) = \frac{A \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right)}{j\omega \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right)}$$

Note: when given actual numbers, also label the y-axis

Phase response

$$\angle H(j\omega) = \overset{\text{zero}}{\angle A} - \angle (j\omega)^n$$

constant offset can be $-\frac{\pi}{2}, -\pi, \frac{\pi}{2}, 0$
(depending on value of n)

high-pass
terms

$$+ \angle \left(1 + \frac{j\omega}{z_1} \right) + \angle \left(1 + \frac{j\omega}{z_1} \right) + \dots$$

low-pass
terms

$$- \angle \left(1 + \frac{j\omega}{p_1} \right) - \angle \left(1 + \frac{j\omega}{p_2} \right) - \dots$$

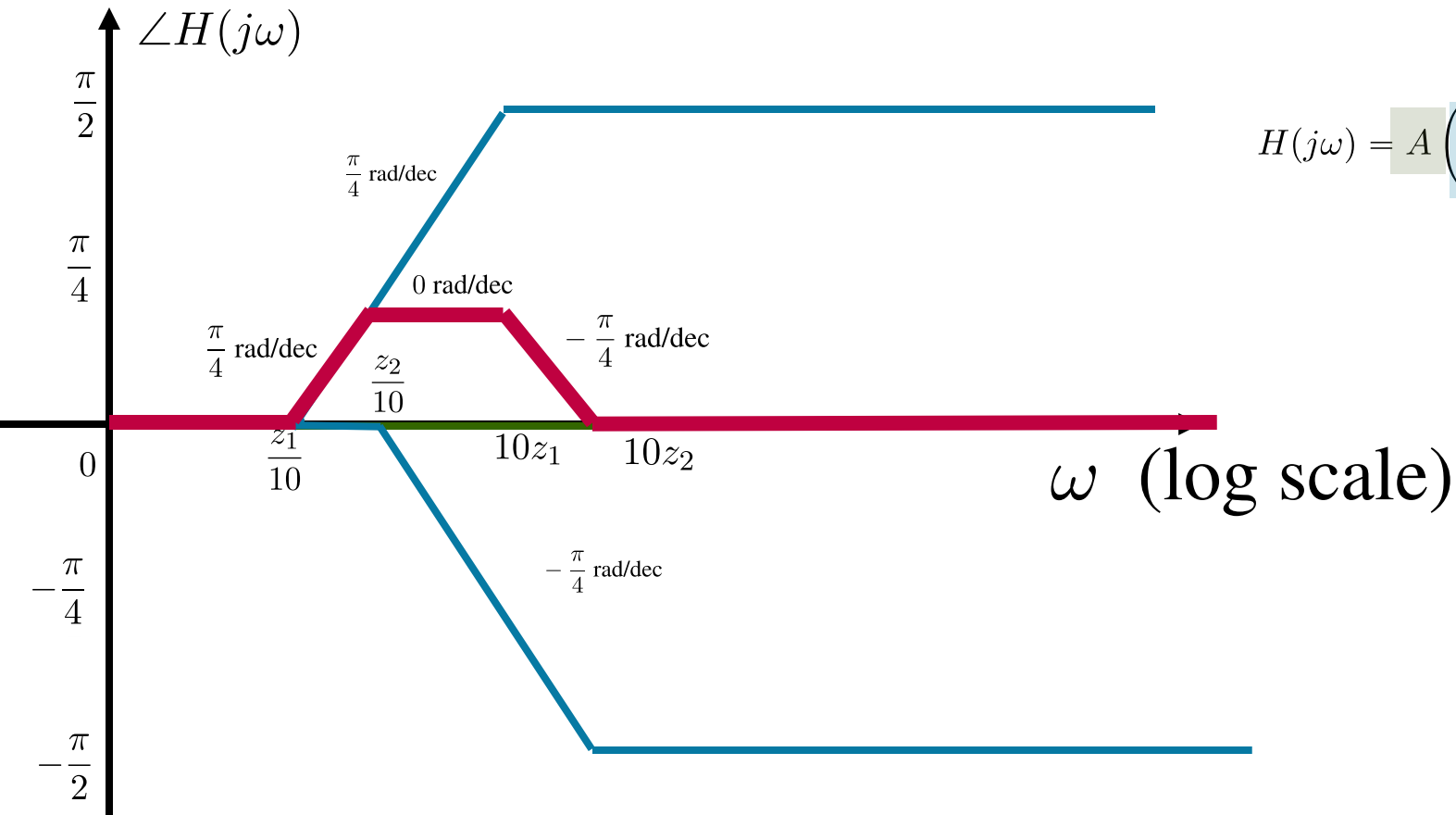
Sketching the phase response

1. Identify all break-points
2. Plot the individual Bode plot for each term
3. Sum the result
4. Label all break points, slopes, and transition points

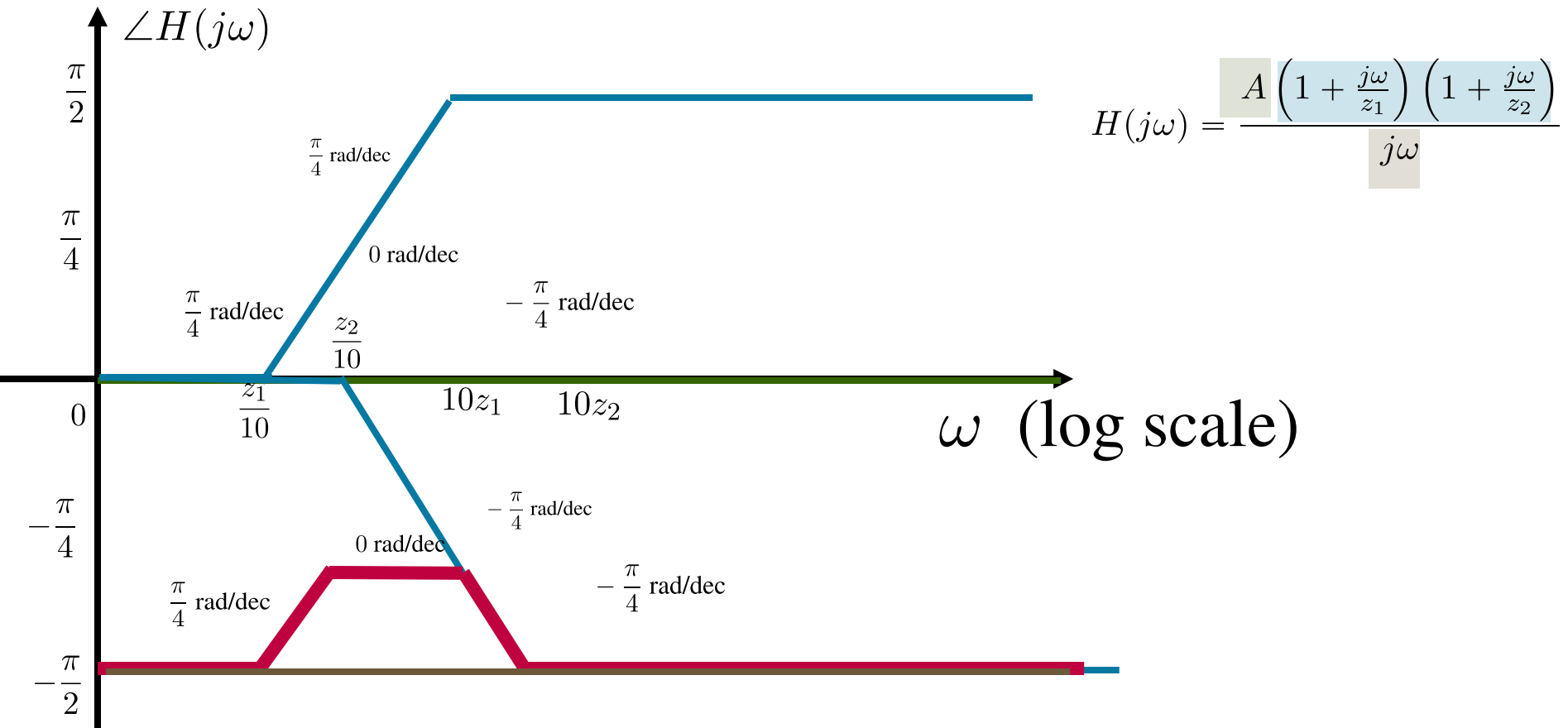
Be mindful that there are more break points

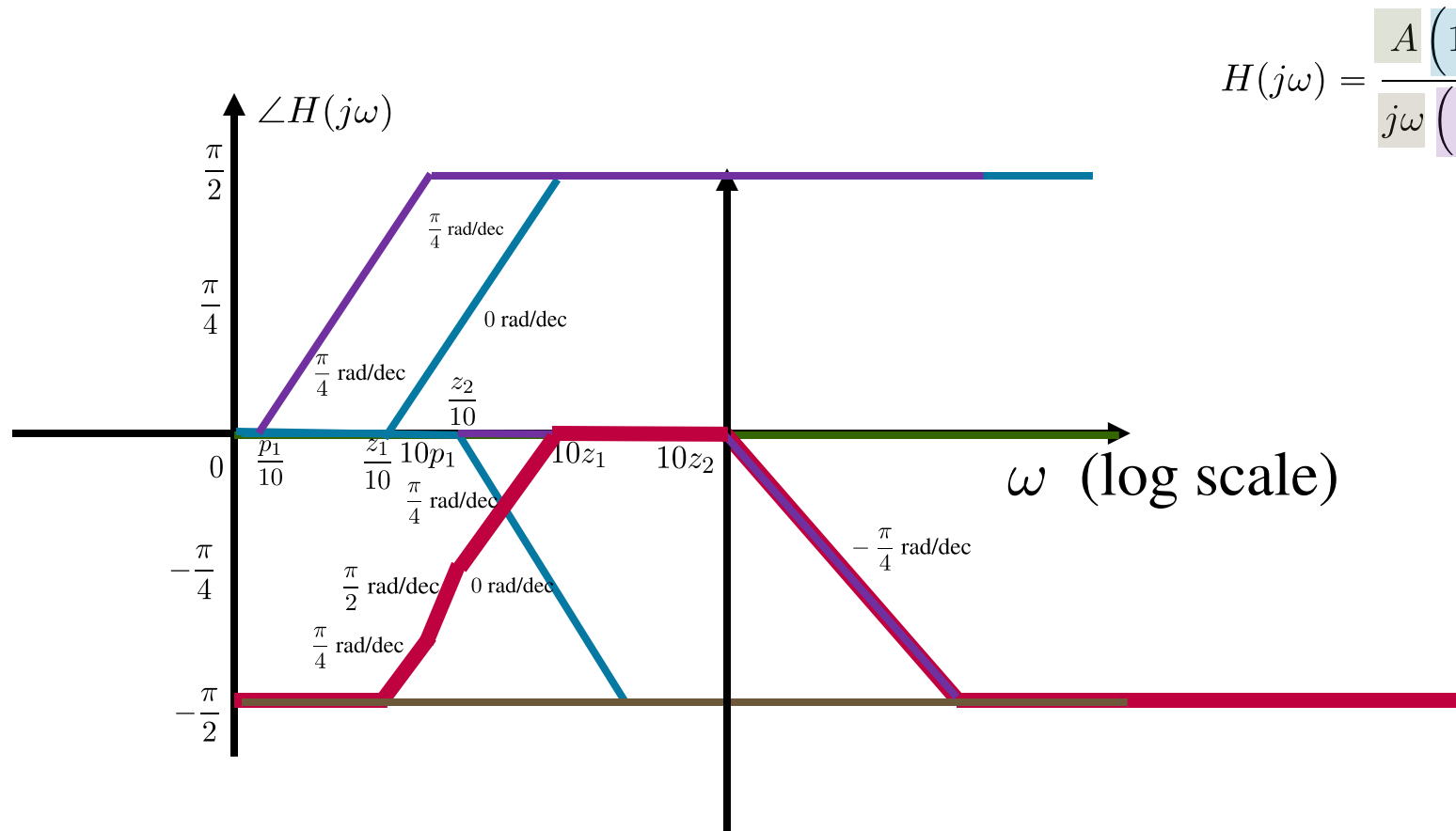


$$H(j\omega) = A$$



$$H(j\omega) = A \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right)$$





$$H(j\omega) = \frac{A \left(1 + \frac{j\omega}{z_1}\right) \left(1 + \frac{j\omega}{z_2}\right)}{j\omega \left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right)}$$

Example

$$H(j\omega) = \frac{20 \left(j \frac{\omega}{200} + 1 \right)}{j\omega \left(j \frac{\omega}{10} + 1 \right)}$$

Summary

- ◆ Bode plots are a way of visualizing the frequency response of a LTI system described by a LCCDE
- ◆ Bode plots are made by writing the frequency response in a factored form and identifying the contributions of the low-pass and the high-pass components
- ◆ Bode magnitude and phase plots involve identifying and labeling all break points and slopes