ECE 45 – Circuits and Systems
Winter 2025

Homework #3

Due: January 23 at 11:59pm, submitted via GradeScope.

You can make multiple upload attempts to experiment with the system and the best way to upload. You must correctly mark the answers to the problems in GradeScope, e.g. problem 1, problem 2, problem 3, to get full credit. Note that you must tag your problems when uploading to GradeScope or they will not be graded and you will not receive credit. Any regrade requests must be placed through GradeScope within one week of the return of the homework.

Remember, discussion of homework questions is encouraged. Please be absolutely sure to submit your own independent homework solution.

1. Simplify the following expressions. Be mindful of the cases where the expression simplifies to a signal, simplifies to a number, or does not simplify.

   (a) $\left(1+v+v^2\right)\delta(v)$

   Solution: Recall that $\delta(v)$ is the Dirac delta function, which is zero everywhere except at $v=0$. Thus, any term multiplied by $\delta(v)$ is evaluated at $v=0$. Therefore,
   $$\left(1+v+v^2\right)\delta(v) = 1\cdot\delta(v).$$
   The simplified result is:
   $$\delta(v).$$

   (b) $\int_{-\infty}^{\infty}\delta(t-1)x(1-t)dt$ where $x(t)$ is a signal

   Solution: Using the sifting property of the Dirac delta function:
   $$\int_{-\infty}^{\infty}\delta(t-c)f(t)\,dt = f(c),$$
   we set $c=1$. Therefore, the delta function evaluates $x(1-t)$ at $t=1$:
   $$x(1-t)\big|_{t=1} = x(1-1) = x(0).$$
   The simplified result is:
   $$x(0).$$

   (c) $\int_{-\infty}^{\infty}\delta(t-4)\frac{\sin(\pi t^2)}{\pi t^2}dt$

   Solution: Using the sifting property of the Dirac delta function, the argument of the delta function evaluates the expression $\frac{\sin(\pi t^2)}{\pi t^2}$ at $t=4$:
   $$\left.\frac{\sin(\pi t^2)}{\pi t^2}\right|_{t=4} = \frac{\sin(\pi\cdot 4^2)}{\pi\cdot 4^2} = \frac{\sin(16\pi)}{16\pi}.$$

Since $\sin(16\pi) = 0$, the result is:

$$0.$$

(d) $\sum_{n=0}^{10}(t+1)^n\delta(t)$

Solution: Using the sifting property of the Dirac delta function, the entire summation is evaluated at $t = 0$:

$$\sum_{n=0}^{10}(t+1)^n\delta(t) = \left(\sum_{n=0}^{10}(1)^n\right)\delta(t) = 11\delta(t).$$

(e) $f(t) = \int_{-\infty}^{t}\delta(\tau - 3)d\tau$

Solution: The integral accumulates the value of the delta function. The property of the delta function states:

$$\int_{-\infty}^{t}\delta(\tau - a)\,d\tau = \begin{cases} 0 & t < a, \\ 1 & t \geq a. \end{cases}$$

Therefore,

$$f(t) = \begin{cases} 0 & t < 3, \\ 1 & t \geq 3. \end{cases}$$

(f) $\sin(2\pi t)\delta(1/2 - 2t)$

Solution: Using the sifting property of the delta function:

$$\delta(a(t - t_0)) = \frac{1}{|a|}\delta(t - t_0)$$

we first solve $\frac{1}{2} - 2t = 0 \implies t_0 = \frac{1}{4}$ and $a = -2$. Hence:

$$\delta\left(\frac{1}{2} - 2t\right) = \frac{1}{2}\delta\left(t - \frac{1}{4}\right).$$

Substituting into the expression:

$$\sin(2\pi t)\delta\left(\frac{1}{2} - 2t\right) = \sin\left(2\pi \cdot \frac{1}{4}\right) \cdot \frac{1}{2}\delta\left(t - \frac{1}{4}\right).$$

Simplifying:

$$\sin(2\pi t)\delta\left(\frac{1}{2} - 2t\right) = 1 \cdot \frac{1}{2}\delta\left(t - \frac{1}{4}\right) = \frac{1}{2}\delta\left(t - \frac{1}{4}\right).$$

(g) $\int_{-\infty}^{\infty}(du(t)/dt - \mathrm{rect}(t))\,dt$

Solution: The function $u(t)$ is the step function, so we calculate the integral of the derivative $\frac{du(t)}{dt}$. The derivative of the step function is the Dirac delta function $\delta(t)$, and the integral over all time is:

$$\int_{-\infty}^{\infty}\frac{du(t)}{dt}dt = \int_{-\infty}^{\infty}\delta(t)dt = 1$$

Next, the integral of the rectangular function $\text{rect}(t)$ over all time:

$$\int_{-\infty}^{\infty} \text{rect}(t)dt = 1$$

Therefore, the overall result is:

$$1 - 1 = 0$$

2. Determine (with justification) whether the following systems are (i) time invariant and (ii) linear. In the following $x(t)$ is the input to the system and $y(t)$ is the output of the system.

   (a) $y(t) = 2x(t-3)$

   Solution:

   i. Time Invariance: Let the input be time-shifted as $x'(t) = x(t - t_0)$. Then the output of the system for this input is:

   $$y'(t) = 2x'(t-3) = 2x((t-t_0)-3) = 2x(t-3-t_0).$$

   The time-shifted output of the original system is $y(t-t_0) = 2x((t-t_0)-3)$. Since $y'(t) = y(t-t_0)$, the system is **time invariant**.

   ii. Linearity: Consider two inputs $x_1(t)$ and $x_2(t)$ with corresponding outputs $y_1(t) = 2x_1(t-3)$ and $y_2(t) = 2x_2(t-3)$. For a linear combination $ax_1(t) + bx_2(t)$, the output is:

   $$y'(t) = 2[ax_1(t-3) + bx_2(t-3)] = ay_1(t) + by_2(t).$$

   Thus, the system is **linear**.

   (b) $y(t) = \int_{-\infty}^{t} x(\gamma)d\gamma$

   Solution:

   i. Time Invariance: Let the input be time-shifted as $x'(t) = x(t - t_0)$. Then the output of the system for this input is:

   $$y'(t) = \int_{-\infty}^{t} x'(u)\,du = \int_{-\infty}^{t} x(u-t_0)\,du.$$

   Perform the change of variable $\gamma = u - t_0$, so $u = \gamma + t_0$, and the limits shift accordingly:

   $$y'(t) = \int_{-\infty}^{t-t_0} x(\gamma)\,d\gamma.$$

   The time-shifted output of the original system is $y(t-t_0) = \int_{-\infty}^{t-t_0} x(\gamma)\,d\gamma$. Since $y'(t) = y(t-t_0)$, the system is **time invariant**.

   ii. Linearity: Consider two inputs $x_1(t)$ and $x_2(t)$ with corresponding outputs:

   $$y_1(t) = \int_{-\infty}^{t} x_1(\gamma)\,d\gamma, \quad y_2(t) = \int_{-\infty}^{t} x_2(\gamma)\,d\gamma.$$

   For a linear combination $ax_1(t) + bx_2(t)$, the output is:

   $$y(t) = \int_{-\infty}^{t} [ax_1(\gamma) + bx_2(\gamma)]\,d\gamma = ay_1(t) + by_2(t).$$

   Thus, the system is **linear**.

(c) $y(t) = \text{Re}\{x(t)\}$

Solution:

i. Time Invariance: Let the input be time-shifted as $x'(t) = x(t - t_0)$. Then the output of the system for this input is:

$$y'(t) = \text{Re}\left\{x'(t)\right\} = \text{Re}\left\{x(t - t_0)\right\}.$$

The time-shifted output of the original system is $y(t - t_0) = \text{Re}\{x(t - t_0)\}$. Since $y'(t) = y(t - t_0)$, the system is **time invariant**.

ii. Linearity: Consider two inputs $x_1(t)$ and $x_2(t)$ with corresponding outputs:

$$y_1(t) = \text{Re}\{x_1(t)\}, \quad y_2(t) = \text{Re}\{x_2(t)\}.$$

For a linear combination $ax_1(t) + bx_2(t)$, the output is:

$$y(t) = \text{Re}\{ax_1(t) + bx_2(t)\}.$$

If a and b are real numbers, this becomes:

$$y(t) = a\text{Re}\{x_1(t)\} + b\text{Re}\{x_2(t)\} = ay_1(t) + by_2(t),$$

and the system is linear in this case. However, if a or b are complex numbers, then:

$$y(t) = \text{Re}\{ax_1(t)\} + \text{Re}\{bx_2(t)\},$$

which is not equal to $ay_1(t)+by_2(t)$ because $\text{Re}\{ax(t)\} = \text{Re}\{a\}\text{Re}\{x(t)\}-\text{Im}\{a\}\text{Im}\{x(t)\}$ in general. Thus, the system is **not linear when the scaling factor is complex**.

(d) $y(t) = x(t - 2) + x(2 - t)$

Solution:

i. Time Invariance: Let the input be time-shifted as $x'(t) = x(t - t_0)$. The output for this input is:

$$y'(t) = x'(t-2)+x'(2-t) = x((t-2)-t_0)+x((2-t)-t_0) = x(t-2-t_0)+x(2-t-t_0).$$

The time-shifted output of the original system is:

$$y(t - t_0) = x((t - t_0) - 2) + x(2 - (t - t_0)) = x(t - 2 - t_0) + x(2 - t + t_0).$$

Since $y'(t) \neq y(t - t_0)$, the system is **time variant**.

ii. Linearity: Consider two inputs $x_1(t)$ and $x_2(t)$ with corresponding outputs:

$$y_1(t) = x_1(t - 2) + x_1(2 - t), \quad y_2(t) = x_2(t - 2) + x_2(2 - t).$$

For a linear combination $ax_1(t) + bx_2(t)$, the output is:

$$y(t) = (ax_1(t) + bx_2(t) - 2) + (ax_1(2 - t) + bx_2(2 - t)).$$

Expanding, this becomes:

$$y(t) = a[x_1(t - 2) + x_1(2 - t)] + b[x_2(t - 2) + x_2(2 - t)] = ay_1(t) + by_2(t).$$

Thus, the system is **linear**.

(e) $y(t) = \log_2(1 + |x(t)|^2)$

Solution:

i. Time Invariance: Let the input be time-shifted as $x'(t) = x(t - t_0)$. The output for this input is:
$$y'(t) = \log_2(1 + |x'(t)|^2) = \log_2(1 + |x(t - t_0)|^2).$$
The time-shifted output of the original system is:
$$y(t - t_0) = \log_2(1 + |x(t - t_0)|^2).$$
Since $y'(t) = y(t - t_0)$, the system is **time invariant**.

ii. Linearity: Consider two inputs $x_1(t)$ and $x_2(t)$ with corresponding outputs:
$$y_1(t) = \log_2(1 + |x_1(t)|^2), \quad y_2(t) = \log_2(1 + |x_2(t)|^2).$$
For a linear combination $ax_1(t) + bx_2(t)$, the output is:
$$y(t) = \log_2\left(1 + |ax_1(t) + bx_2(t)|^2\right).$$
This is not equal to $ay_1(t) + by_2(t)$ because of the nonlinear operations $\log_2$ and $|\cdot|^2$. Thus, the system is **not linear**.

(f) $y(t) = \cos(x(t))$

Solution:

i. Time Invariance: Let the input be time-shifted as $x'(t) = x(t - t_0)$. The output for this input is:
$$y'(t) = \cos(x'(t)) = \cos(x(t - t_0)).$$
The time-shifted output of the original system is:
$$y(t - t_0) = \cos(x(t - t_0)).$$
Since $y'(t) = y(t - t_0)$, the system is **time invariant**.

ii. Linearity: Consider two inputs $x_1(t)$ and $x_2(t)$ with corresponding outputs:
$$y_1(t) = \cos(x_1(t)), \quad y_2(t) = \cos(x_2(t)).$$
For a linear combination $ax_1(t) + bx_2(t)$, the output is:
$$y(t) = \cos(ax_1(t) + bx_2(t)).$$
This is not equal to $ay_1(t) + by_2(t)$ because the cosine function is nonlinear. Thus, the system is **not linear**.

(g) $y(t) = \begin{cases} 0, & x(t) < 1 \\ \int_0^1 x(t - \tau)d\tau & x(t) \geq 1 \end{cases}$

Solution:

i. Time Invariance: Let the input be time-shifted as $x'(t) = x(t - t_0)$. The output for this input is:

$$y'(t) = \begin{cases} 0, & x'(t) < 1 \\ \int_0^1 x'(t-\tau)\, d\tau, & x'(t) \geq 1 \end{cases}.$$

Substituting $x'(t) = x(t - t_0)$:

$$y'(t) = \begin{cases} 0, & x(t-t_0) < 1 \\ \int_0^1 x((t-\tau)-t_0)\, d\tau, & x(t-t_0) \geq 1 \end{cases}.$$

Perform a time shift for the original output, $y(t - t_0)$:

$$y(t-t_0) = \begin{cases} 0, & x(t-t_0) < 1 \\ \int_0^1 x((t-t_0)-\tau)\, d\tau, & x(t-t_0) \geq 1 \end{cases}.$$

Comparing $y'(t)$ with $y(t - t_0)$, we see that $y'(t) = y(t - t_0)$. Thus, the system is **time invariant**.

ii. Linearity: Consider two inputs $x_1(t)$ and $x_2(t)$ with corresponding outputs:

$$y_1(t) = \begin{cases} 0, & x_1(t) < 1 \\ \int_0^1 x_1(t-\tau)\, d\tau, & x_1(t) \geq 1 \end{cases}, \quad y_2(t) = \begin{cases} 0, & x_2(t) < 1 \\ \int_0^1 x_2(t-\tau)\, d\tau, & x_2(t) \geq 1 \end{cases}.$$

For a linear combination $ax_1(t) + bx_2(t)$, the output is:

$$y(t) = \begin{cases} 0, & ax_1(t)+bx_2(t) < 1 \\ \int_0^1 \big(ax_1(t-\tau)+bx_2(t-\tau)\big)\, d\tau, & ax_1(t)+bx_2(t) \geq 1 \end{cases}.$$

In the first case (when $ax_1(t) + bx_2(t) < 1$), the output is 0 regardless of the individual outputs $y_1(t)$ and $y_2(t)$. Thus, the system does not satisfy additivity when the condition $x(t) \geq 1$ is not met. Therefore, the system is **not linear**.

iii. **Counterexample:** Let $x(t) = 2$, which satisfies $x(t) \geq 1$. The output for this input is:

$$y(t) = \int_0^1 x(t-\tau)\, d\tau = \int_0^1 2\, d\tau = 2.$$

Now, let $a = \frac{1}{3}$ and consider $x'(t) = ax(t) = \frac{2}{3}$. Since $x'(t) < 1$, the output for this input is:

$$y'(t) = 0.$$

Comparing, we find:

$$y'(t) \neq ay(t) \quad (\text{since } 0 \neq \frac{1}{3} \cdot 2).$$

Therefore, the system is **not linear**.

3. Correct your previous week's homework using a colored pen (or annotation) so it's obvious what you've corrected. If you got a problem exactly right, just use a red check mark to indicate as such.