# Lecture 5

Impulse response and convolution

# Preview of today's lecture

- Impulse response of an LTI system
  - Define the impulse response of a continuous-time system
  - Connect the impulse response to the convolution integral
- Convolution
  - Determine the output of an LTI system using the convolution
  - Compute the continuous-time convolution between two signals

# Time invariant (TI)

A system is time invariant if behaves in the same way regardless of the current time

Formally: consider the system

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If

$$x(t - t_o) \longrightarrow \boxed{\text{System}} \longrightarrow y(t - t_o)$$

For all $t_0$ then the system is time invariant otherwise it is time varying.

# Linear systems

A system is linear if scaling is preserved and superposition holds

Consider two different inputs and outputs same

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

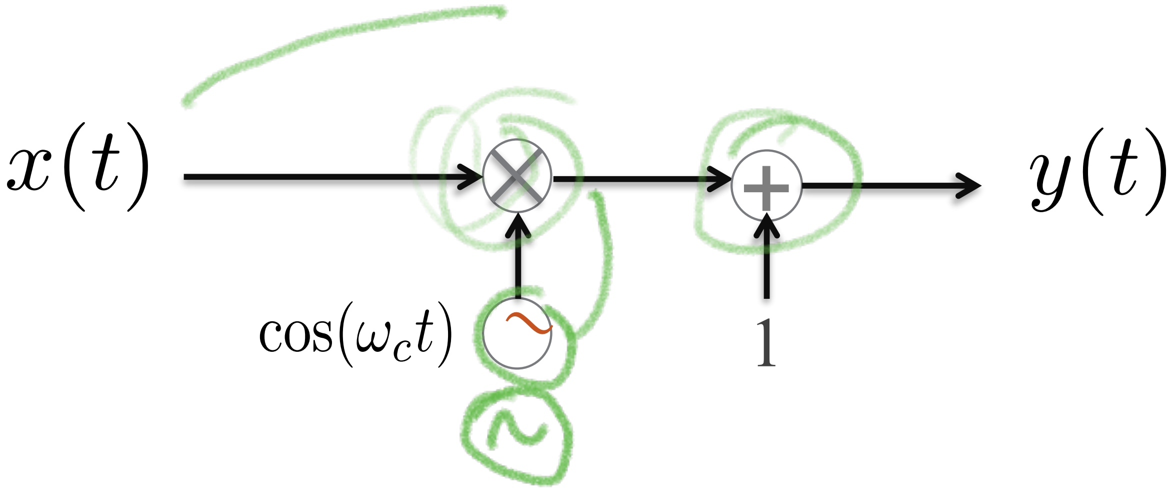
$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

If the following holds for any scalar A and B then the system is linear

$$A x_1(t) + B x_2(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow A y_1(t) + B y_2(t)$$

# Example

- Determine if the following systems are time-invariant and/or linear
- System 1




- System 2

$$y(t) = \int_0^1 x(t-q)dq$$

$$y(t) = 1 + \cos(\omega_c t)\, x(t)$$

$y(t)$ ↑ output    $x(t)$ ↑ input

TI    Let $x_1(t) \to y_1(t) = 1 + \cos(\omega_c t)\, x_1(t)$

$x_2(t) \to y_2(t) = 1 + \cos(\omega_c t)\, x_2(t)$

$x_2(t) = x_1(t - t_0)$

$= 1 + \cos(\omega_c t)\, x_1(t - t_0)$

Check does $y_1(t - t_0) \stackrel{?}{=} y_2(t)$ ? $\forall t_0$

$y_1(t - t_0) = 1 + \cos(\omega_c (t - t_0))\, x_1(t - t_0)$

$y_2(t) = 1 + \cos(\omega_c t)\, x_1(t - t_0)$

NOT TI

$y(t) = x^2(t)$    $x(t^2)$

Linearity

$$y(t) = (1+\cos(\omega_c t))\,x(t)$$

Check scaling

$$Ax(t) \rightarrow Ay(t)$$

$$x_1(t) = Ax(t) \rightarrow y_1(t) = (1+\cos(\omega_c t))\,x_1(t)$$

$$= (1+\cos(\omega_c t))\,Ax(t)$$

$$= (1+A\cos(\omega_c t))\,x(t)$$

$$Ay(t) = (A + A\cos(\omega_c t))\,x(t)$$

$$\neq (1+A\cos(\omega_c t))\,x(t)$$

Scaling does not hold

## Counterexample

Let $x(t) = 0 \quad \forall t$

Then by scaling property

$y(t) = 0 \quad \forall t$ But

in this case

$$y(t) = 1 + \cos(\omega_c t)\, x(t)$$

$$= 1 + \cos(\omega_c t) \cdot 0$$

$$= \underline{1} \neq 0$$

# Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

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| | Date | Theme | Topic | Readings | Out | In |
|---|---|---|---|---|---|---|
| 1 | 1/7 | Signals | Signals, systems, circuits and phasors | 1.1 | HW1 | |
| 2 | 1/9 | Signals | Rectangle, step functions, signal transformations, periodic, even and odd | 1.2 | HW2 | HW1 |
| 3 | 1/14 | Signals | Exponential, sinusoids, complex exponentials, Dirac Delta | 1.3 - 1.4 | | |
| 4 | 1/16 | LTI in time | Systems, linearity, time invariance | 1.6.5, 1.6.6 | HW3 | HW2 |
| 5 | 1/21 | LTI in time | Impulse response and convolution | 2.2 | | |
| 6 | 1/23 | LTI in time | Convolution with a sinusoid, connection to phasors | 2.2 | HW4 | HW3 |
| 7 | 1/28 | LTI in time | Convolution properties | 2.3 | | |
| 8 | 1/30 | Fourier series | Fourier series | 3.1 - 3.3 | HW5 | HW4 |
| | 2/4 | | **Midterm 1** | | | |
| 9 | 2/6 | Fourier series | Fourier series convergence and properties | 3.4 | HW6 | HW5 |
| 10 | 2/11 | Fourier series | Fourier series properties | 3.5 | | |
| 11 | 2/13 | LTI in frequency | Frequency response of LTI systems | 3.9 | HW7 | HW6 |
| 12 | 2/18 | LTI in frequency | Filters, bode plots | 3.10, 6.2.3 | | |
| 13 | 2/20 | Fourier transform | Fourier transform | 4.1-4.2 | HW8 | HW7 |
| | 2/25 | | **Midterm 2** | | | |
| 14 | 2/27 | Fourier transform | Fourier transform properties | 4.3 | HW9 | HW8 |
| 15 | 3/4 | Fourier transform | Rectangle and sinc functions | 4.3 | | |
| 16 | 3/6 | Fourier transform | Convolution property | 4.4 | HW10 | |
| 17 | 3/11 | Fourier transform | Multiplication property | 4.5 | | |
| 18 | 3/13 | Sampling | Sampling theorem | 7.1 | | HW10 |
| | 3/19 | | **Final exam Tuesday 3-6pm** | | | |

# Impulse response of an LTI system

## Learning objectives

- Define the impulse response of a continuous-time system
- Connect the impulse response to the convolution integral

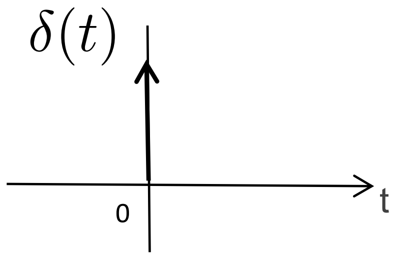
# System impulse response

$$x(t) = \delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t)$$

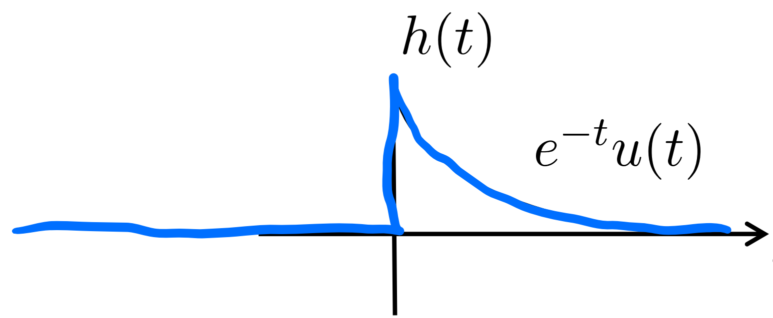
- Consider the input signal $\delta(t)$
- The output corresponding to this input is the **impulse response**
  - The resulting sequence is usually called $h(t)$
- All systems have an impulse response, but:
  - The impulse response is special <u>only</u> for LTI systems
  - Focus on LTI systems throughout this course

# Example (typical "first-order" differential system)

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t) = e^{-t}u(t)$$




delta function input




Exponential function out

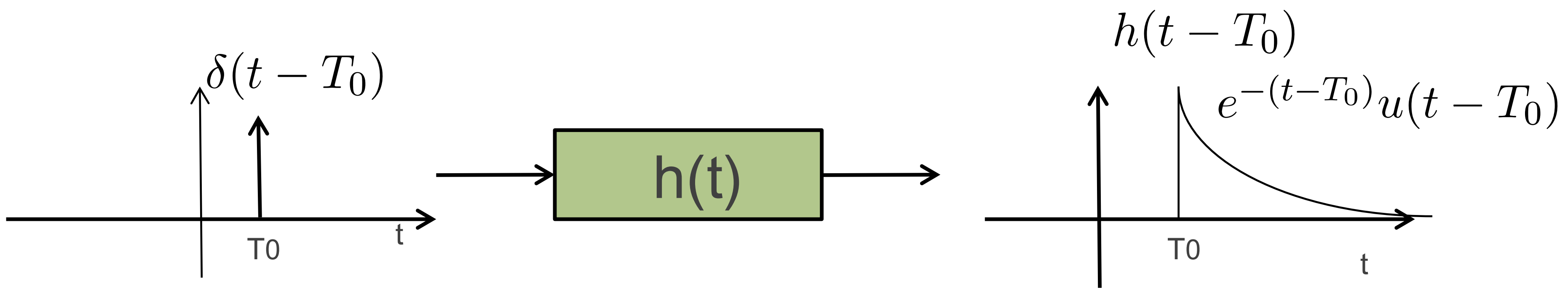
# Time invariance

- Because of time invariance

$$\delta(t-\underbrace{T_0}_{\text{fixed}}) \longrightarrow \boxed{h(t)} \longrightarrow h(t-T_0)$$

When written like this, implies an LTI system with impulse response h(t).

- Example




Shifts in the input shift the output

# Linearity

- Because of the homogenous property

$$\alpha\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow \alpha h(t)$$

**Scaling the input scales the output**

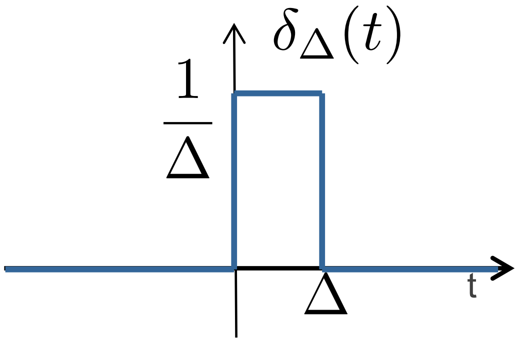
- Because of the additive property

$$\alpha\delta(t) + \beta\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow \overbrace{\alpha h(t)}^{\alpha} + \overbrace{\beta h(t - T_0)}^{\beta}$$

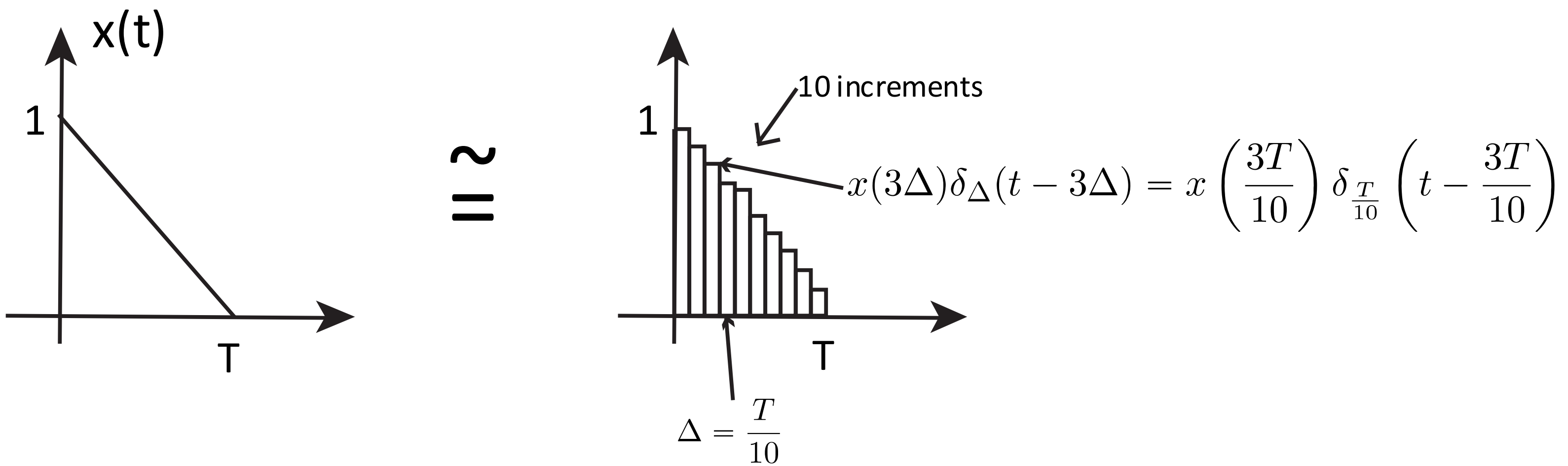
**A sum of inputs leads to a sum of outputs**

# Stair step approximation of an input signal

- Consider the rectangle function




- Suppose that we approximate a signal using this function

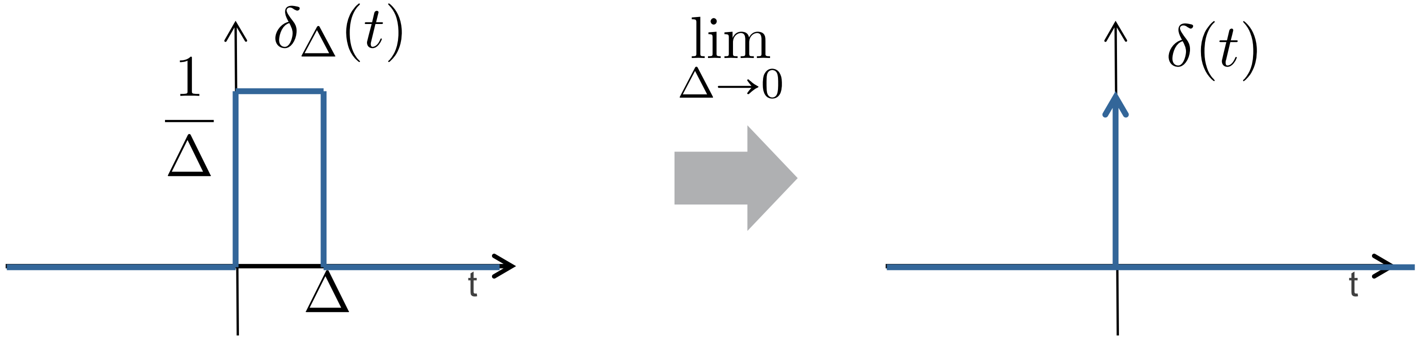


# Stair step approximation of an input signal

- Write the stair case approximation of a function as

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

- Recall from the derivation of the delta function



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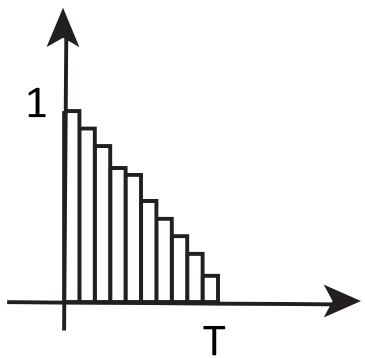
# Stair step approximation of an input signal

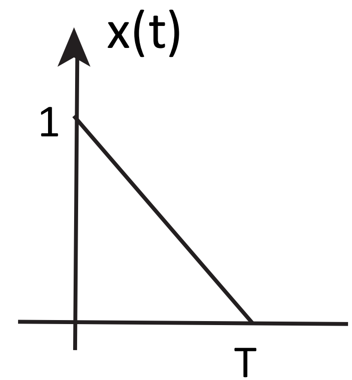
- Now taking the limit

$$\lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$
$$= x(t)$$

Easy to see via sifting property that this must be true




$\lim_{\Delta \to 0}$




**Any signal can be written as an integral of itself with shifted deltas**

# Back to the LTI system

- What if we put in

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{h(t)} \longrightarrow x(\tau)h(t-\tau)$$

LTI

- How about

$$x(\tau_1)\delta(t-\tau_1) + x(\tau_2)\delta(t-\tau_2) \longrightarrow \boxed{h(t)} \longrightarrow$$

$$x(\tau_1)h(t-\tau_1) + x(\tau_2)h(t-\tau_2)$$

# Uncovering the convolution

- Now putting x(t) in the integral format

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{h(t)} \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

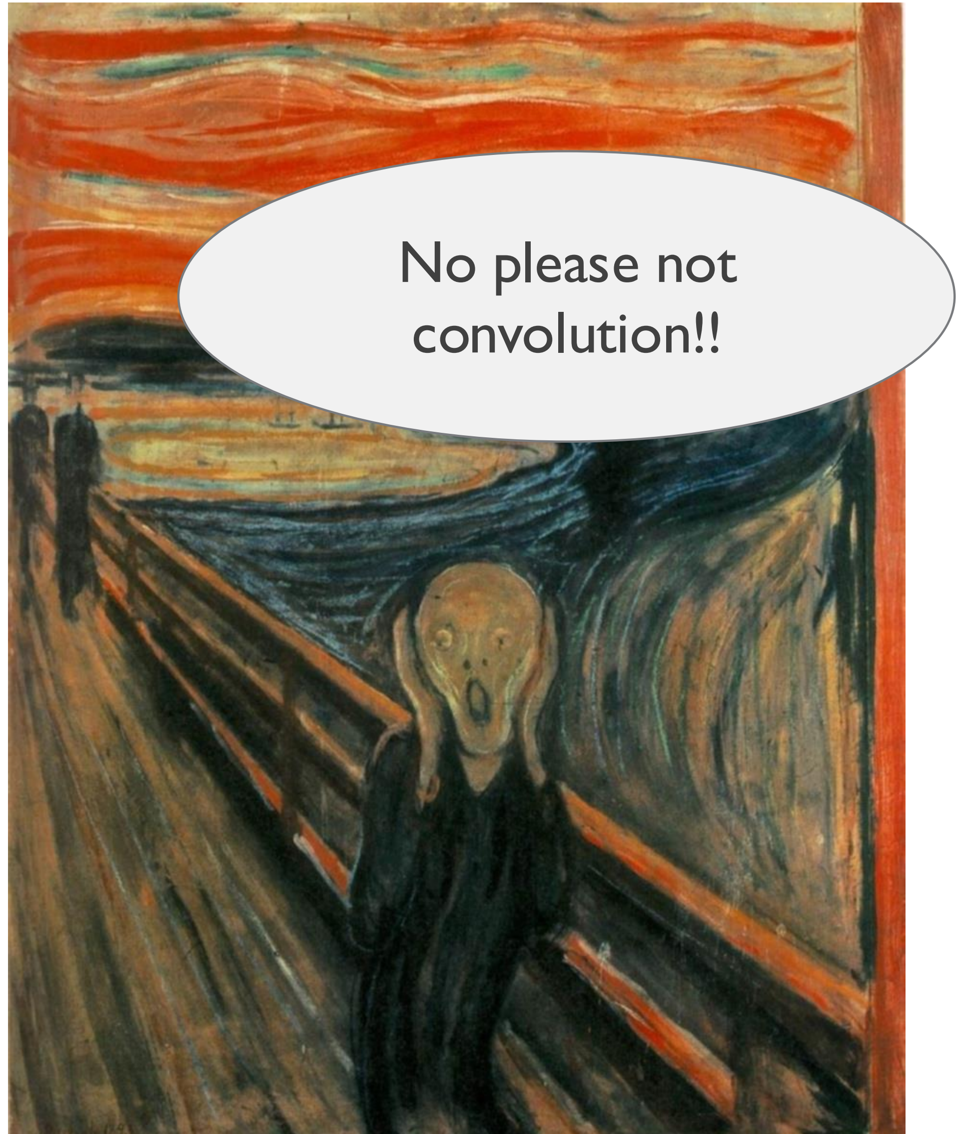
- Thus, the input and output of an LTI system are related via the **convolution integral:**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

# Convolution

## Learning objectives

- Determine the output of an LTI system using the convolution
- Compute the continuous-time convolution between two signals




https://commons.wikimedia.org/wiki/File:The_Scream.jpg with additions by Prof. Heath

# Uncovering the convolution

- Consider an LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- The output can be computed from the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

# Basic convolution properties

- Commutative

Shorthand notation

$$y(t) = x(t) * h(t) = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau$$

Choose option that makes it easy!

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

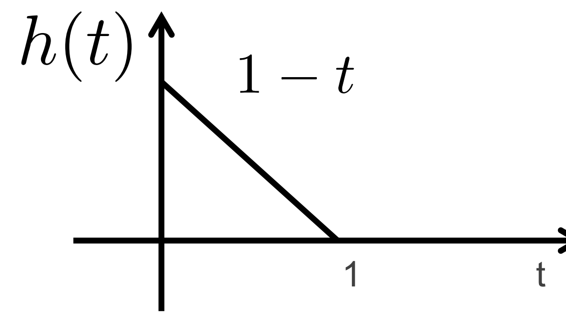
**Use properties to simplify convolutions (more next lecture)**

# CT convolution example #1

This is a typical example using two finite length signals. Rectangles and triangle functions are common in examples / HW as they give results that are easy to integrate. It is important here to understand the different **intervals** in the convolutions.

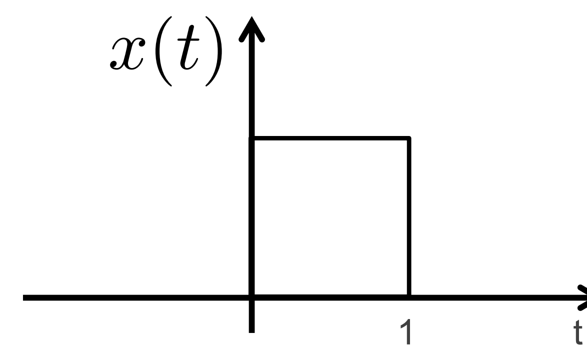
- Find the output of a system with impulse response

$$h(t) = (1-t)[u(t) - u(t-1)]$$



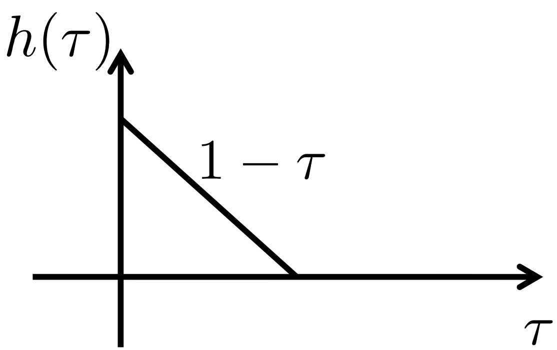

for the input

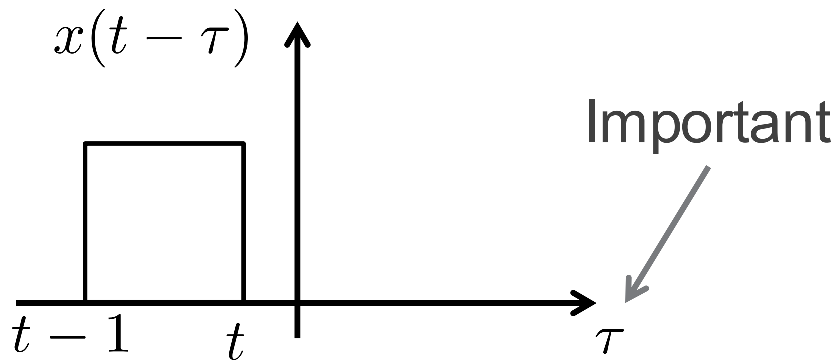
$$x(t) = u(t) - u(t-1)$$



# CT convolution example #1: Graphical solution

- Plot one signal versus τ
- Flip the second signal and shift it by *t*
  - Here, plot it to the left of h(τ)
  - So plotted *t* has a negative value, usually



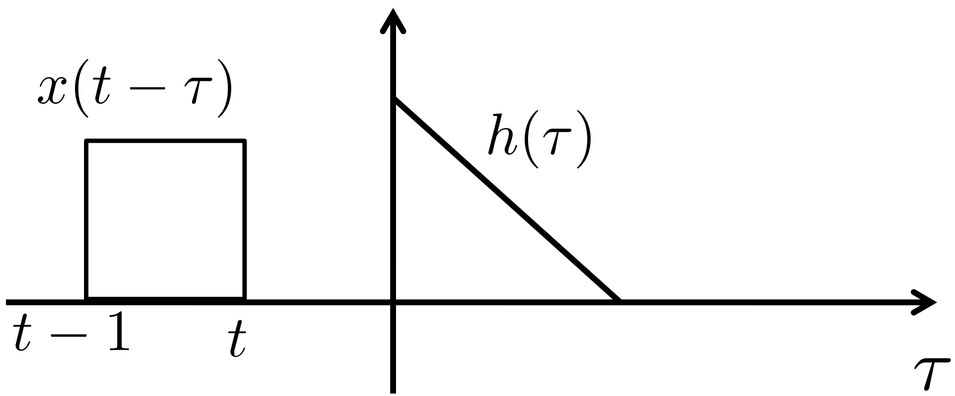




- There are 4 intervals (why?):

$$t < 0 \qquad 0 \leqslant t \leqslant 1 \qquad 1 \leqslant t \leqslant 2 \qquad 2 < t$$

# CT convolution example #1: First interval

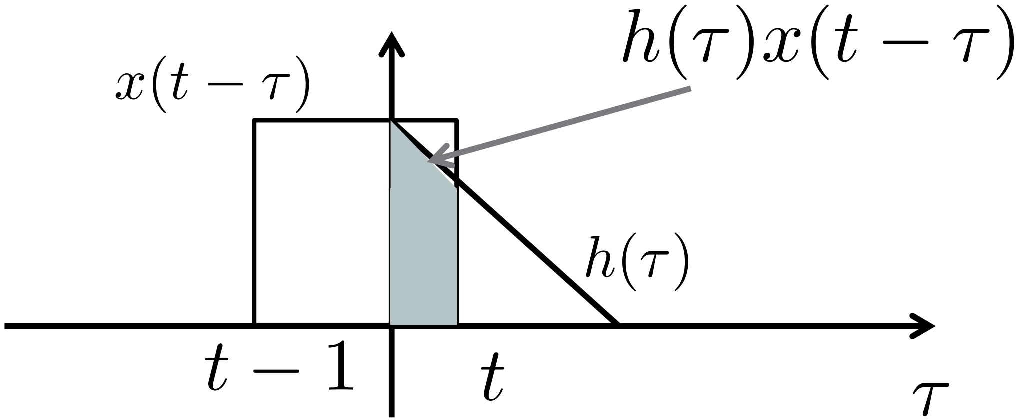
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$




- First interval: there is no overlap!

$$t < 0 \qquad h(\tau)x(t-\tau) = 0 \quad \Rightarrow \quad y(t) = 0$$

# CT convolution example #1: Second interval




- Second interval $\quad 0 \leqslant t \leqslant 1$

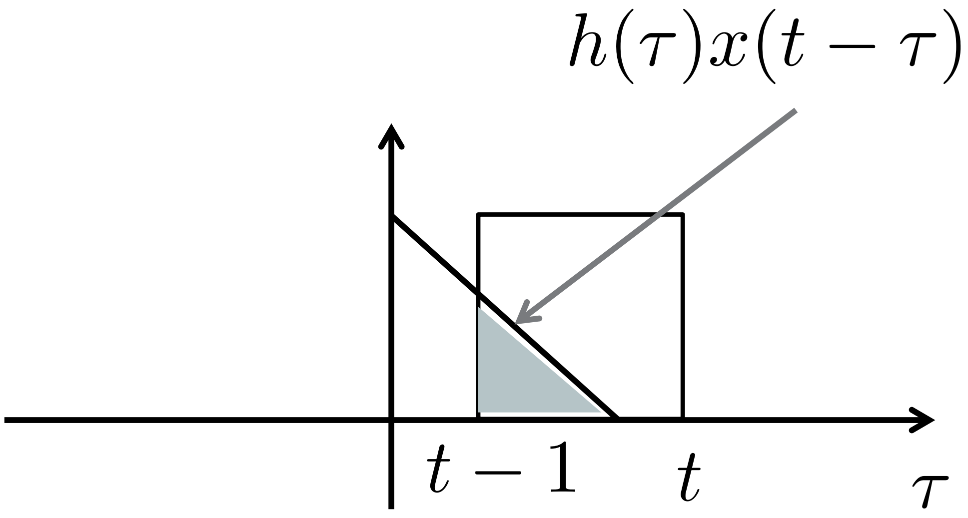
$$y(t) = \int_0^t (1-\tau)d\tau = \tau - \frac{\tau^2}{2}\Bigg]_0^t$$

$$= t - \frac{t^2}{2}$$

# CT convolution example #1: Third interval

- Third interval $1 \leqslant t \leqslant 2$

$$
\begin{aligned}
y(t) &= \int_{t-1}^{1} (1-\tau)d\tau \\
&= \tau - \frac{\tau^2}{2}\bigg]_{t-1}^{1} \\
&= 1 - \frac{1}{2} - \left(t - 1 - \frac{(t-1)^2}{2}\right) \\
&= \frac{t^2}{2} - 2t + 2
\end{aligned}
$$

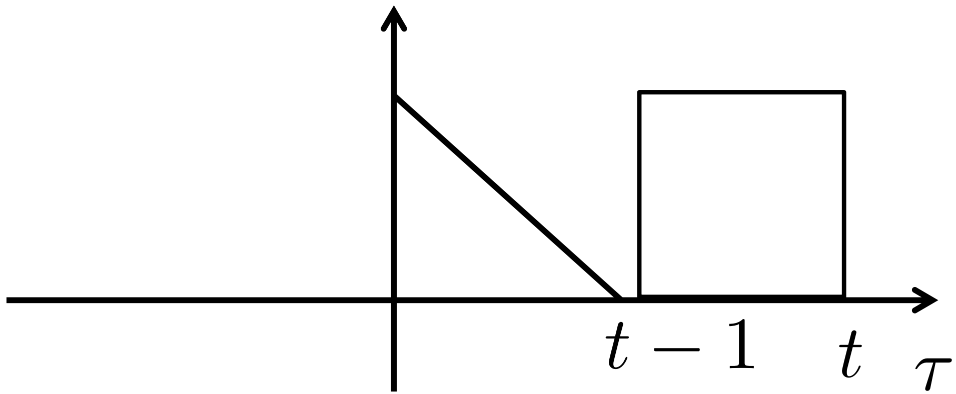



Integrating the same function, but with different integration limits

# CT convolution example #1: Fourth and final interval

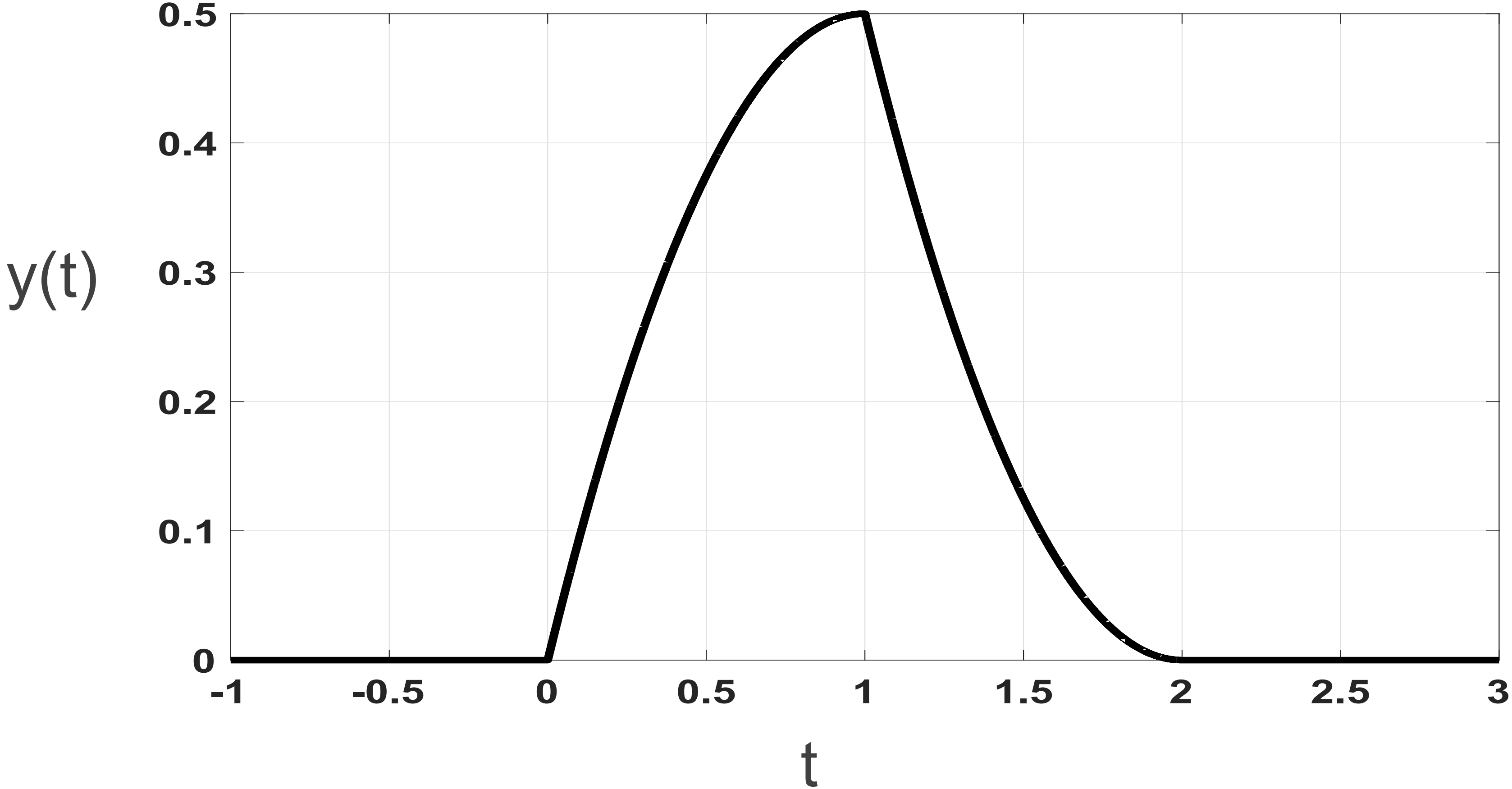
- Fourth interval $2 < t$

$$y(t) = 0$$



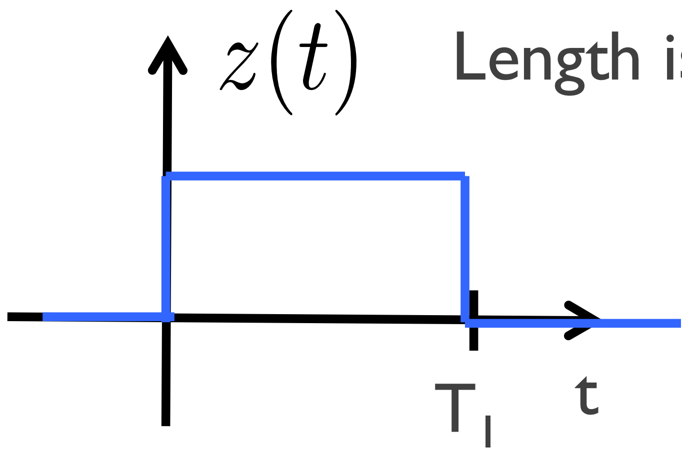

- Sanity checks
  - Check at $t=1$, the output should be the same for intervals 2 & 3 in order for it to be continuous
  - Same for t = 0 and t = 2 (should be zero there)
  - Duration of output should be $T_1 + T_2 = 1+1 = 2$

# CT convolution example #1: Plot of y(t)

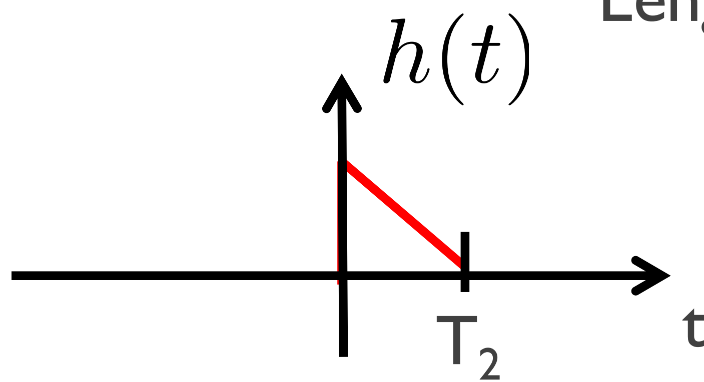


# Length / duration of a convolution in continuous time

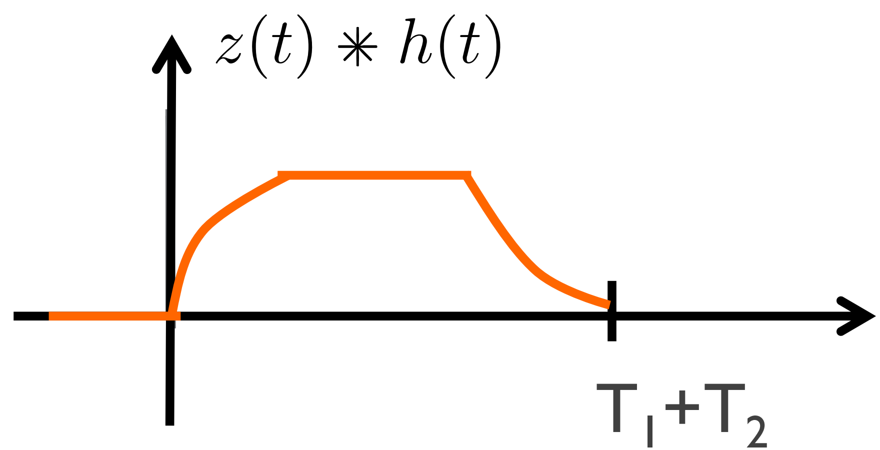
Length is $T_1$



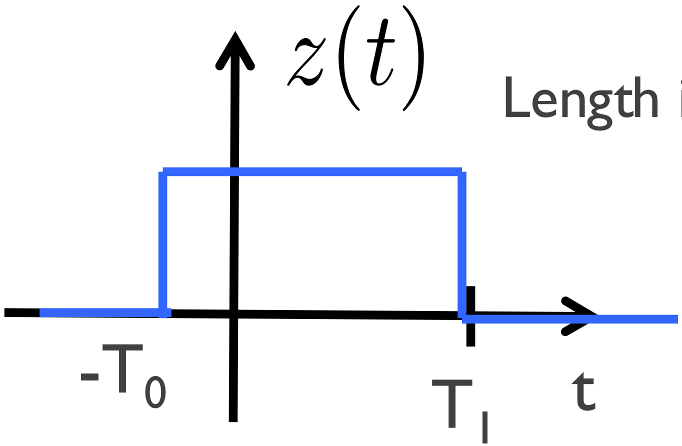

Length is $T_2$



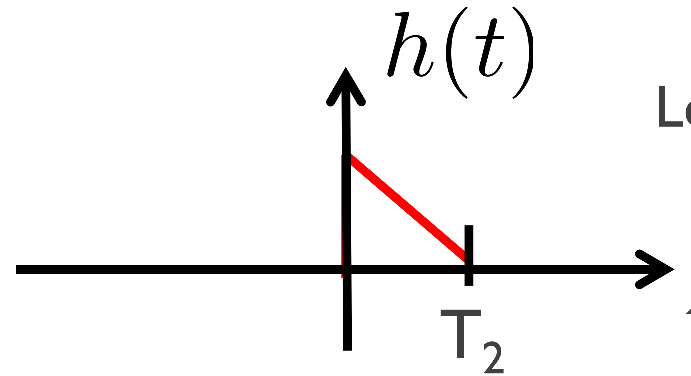

Length is $T_1+T_2$



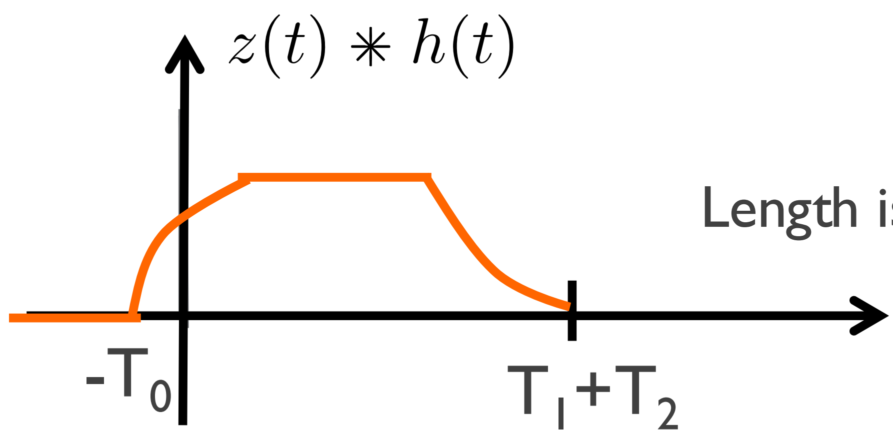
# Length / duration of a convolution in continuous time




Length is $T_0+T_1$




Length is $T_2$




Length is $T_0 + T_1 + T_2$

# Animation example

http://www.cse.yorku.ca/~asif/spc/ConvolutionIntegral_Final3.swf

# CT convolution example #2

This is an example with a simple input that consists of a few delta functions. The convolution is easy to compute using the LTI property of convolution and the definition of impulse response.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Determine and sketch the convolution of the following input

$$x(t) = \delta(t) + 2\delta(t-1)$$

and system with impulse response

$$h(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1, \\ 2-t, & 1 < t \leqslant 2, \\ 0, & \text{elsewere} \end{cases}$$

# CT convolution example #2: Solution approach

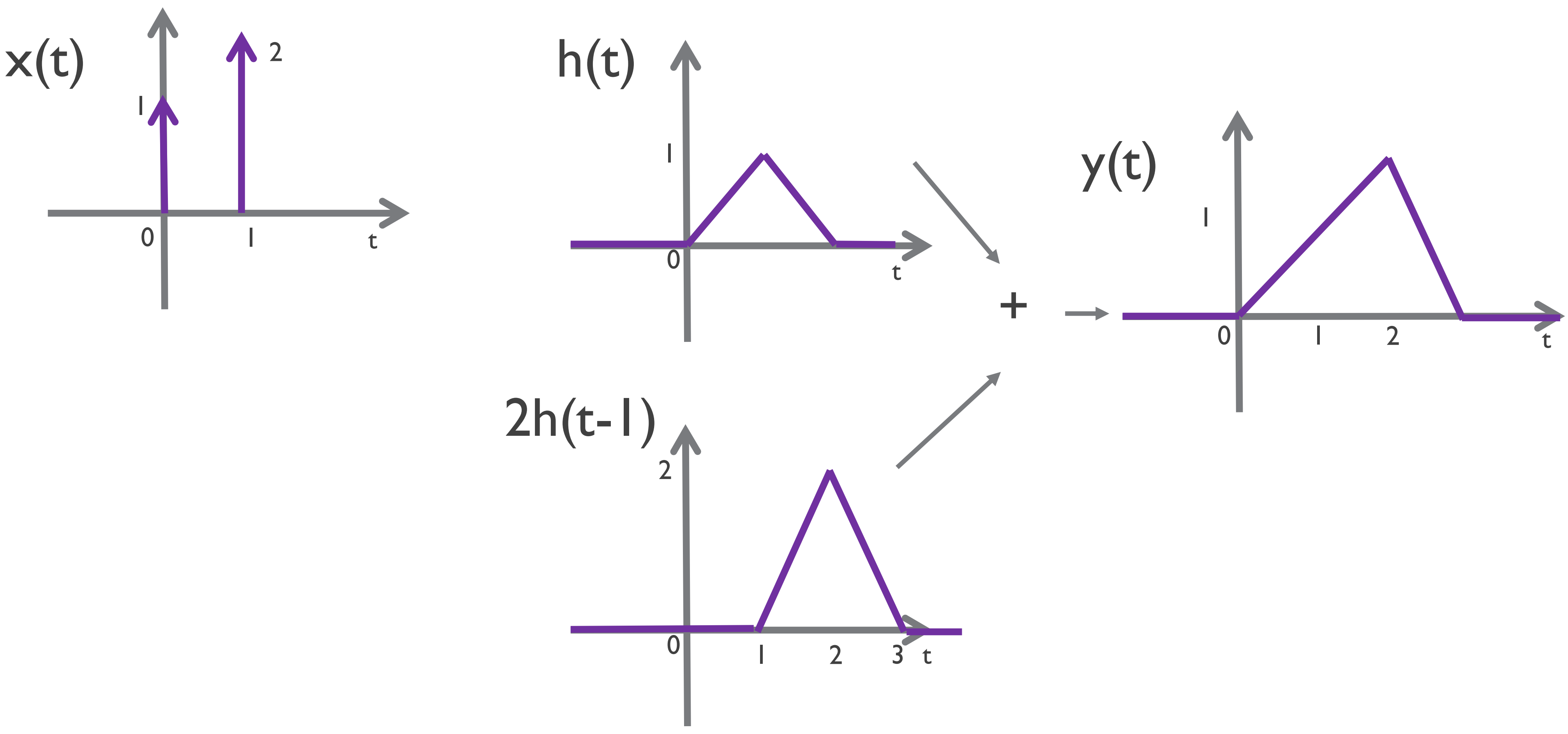
- Recall the property

$$\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t - T_0)$$

- Therefore

$$x(t) = \delta(t) + 2\delta(t-1) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) + 2h(t-1)$$

# CT convolution example #2: Solution sketch

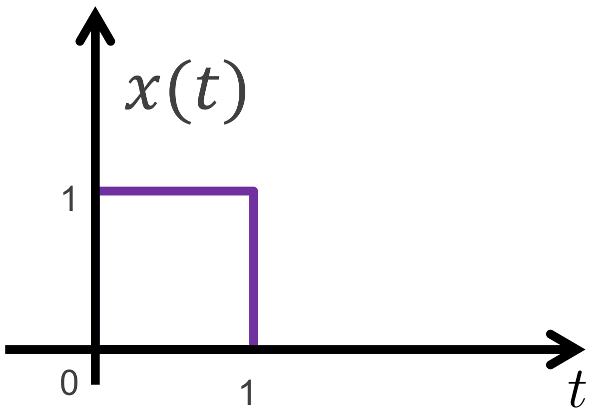


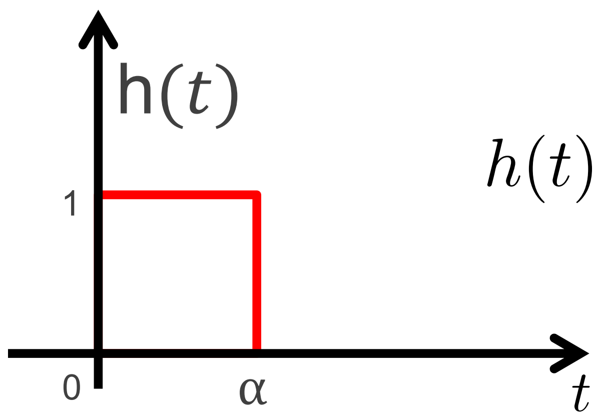
# CT convolution example #3

This is an important example involving the convolution of two rectangles of different widths. The solution will be different depending on the widths of the rectangles. Notice what happens in each interval.

- Determine and sketch $y(t) = x(t) * h(t)$, where

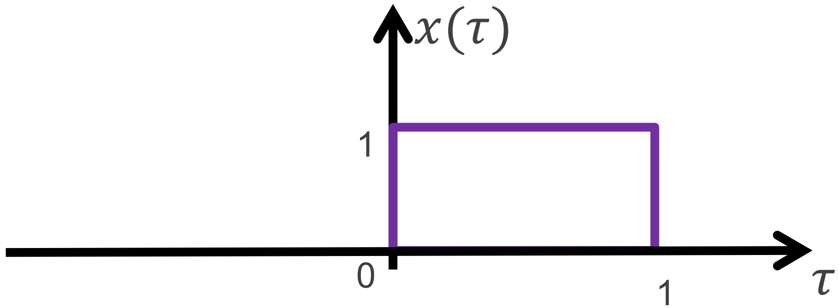
$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

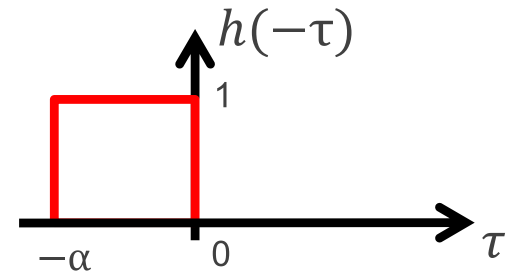


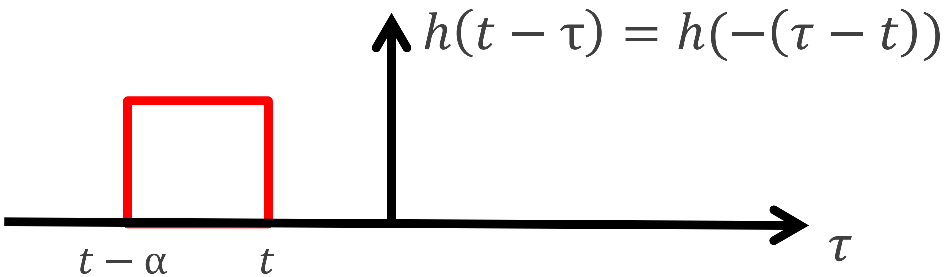

$$h(t) = x(t/\alpha) \qquad 0 < \alpha \leq 1$$

# CT convolution example #3: Solution



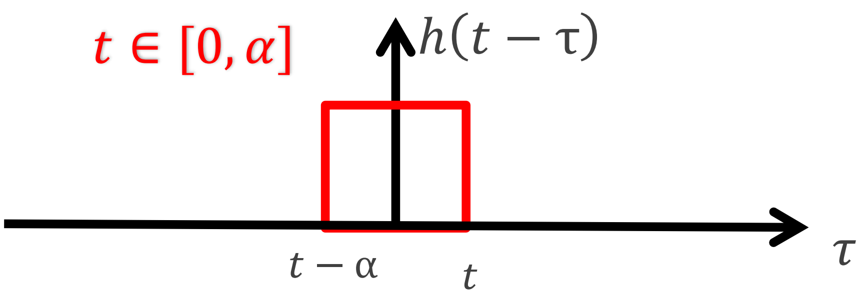




$t < 0$




$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0 \quad t < 0$$

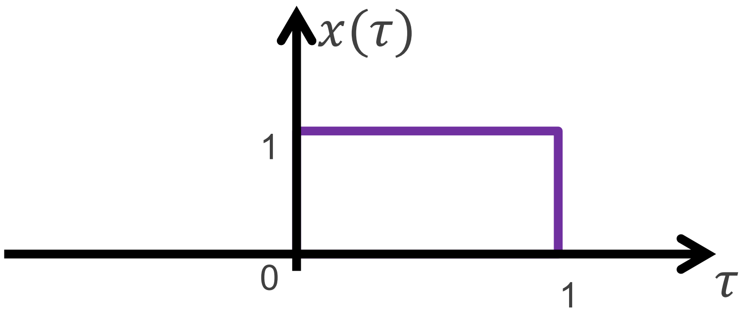
$t \in [0, \alpha]$



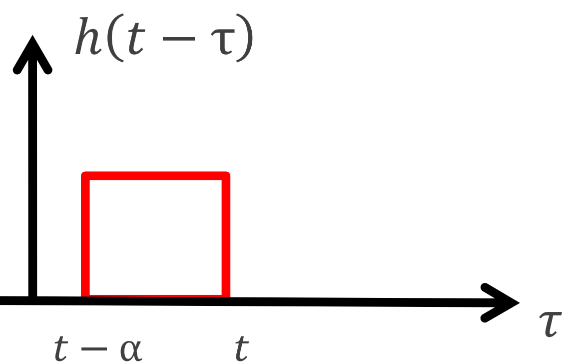

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} 1 d\tau \quad t \in [0, \alpha]$$

$$= t \quad t \in [0, \alpha]$$

# CT convolution example #3: Solution



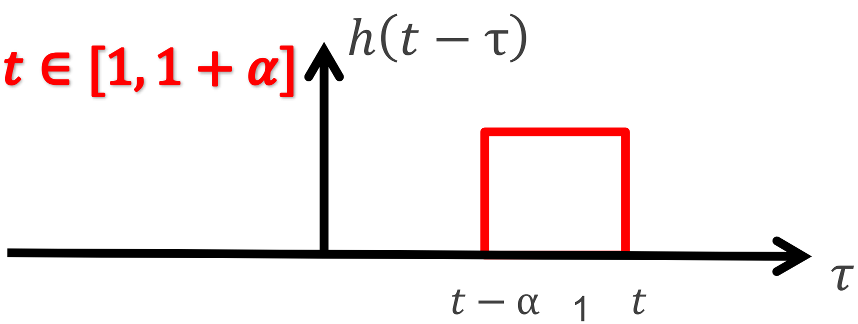

$t \in [\alpha, 1]$




$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{t} 1 d\tau \quad t \in [\alpha, 1]$$

$$= \alpha \quad t \in [\alpha, 1]$$

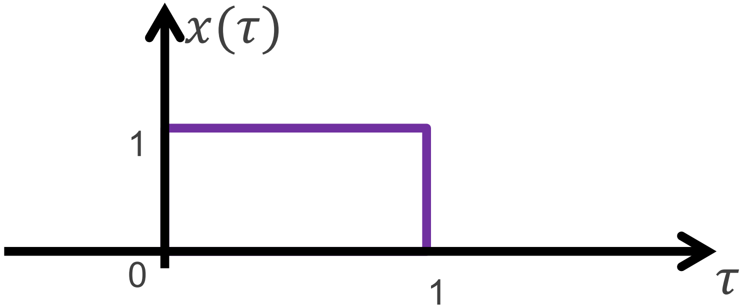
**$t \in [1, 1+\alpha]$**



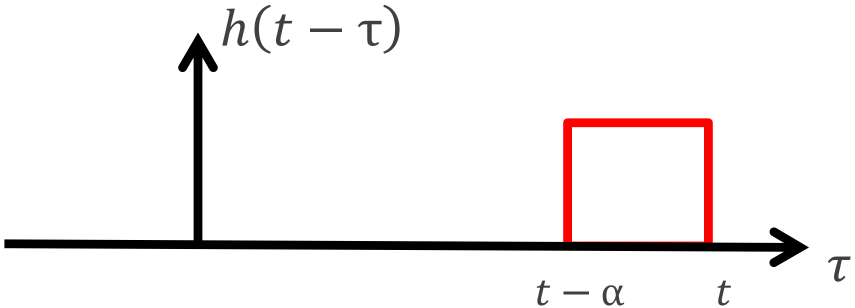

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{1} 1 d\tau \quad t \in [1, 1+\alpha]$$

$$= 1 - (t-\alpha) \quad t \in [1, 1+\alpha]$$

# CT convolution example #3: Solution




$t \in [1 + \alpha, \infty]$

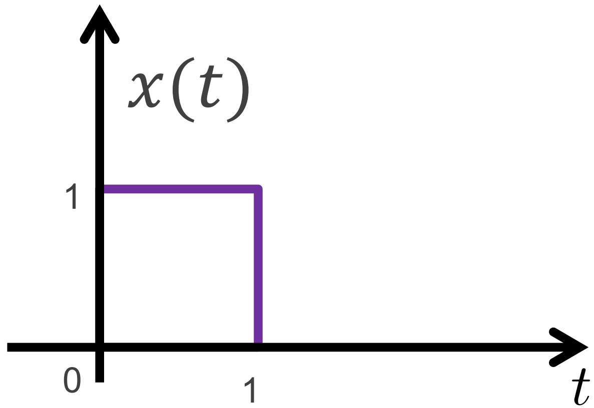


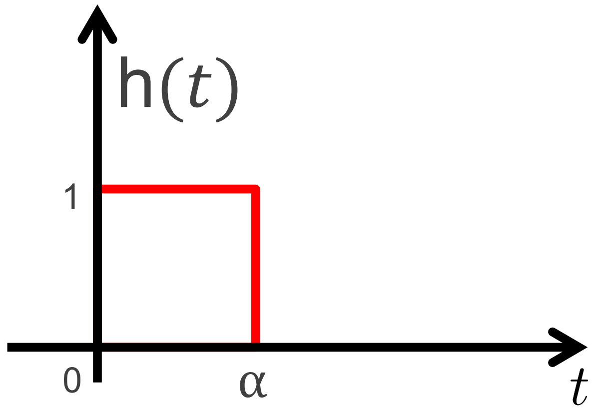

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{1}^{t} 0 d\tau \quad t \in [\alpha + 1, \infty]$$

$$= 0 \quad t \in [\alpha + 1, \infty]$$

# CT convolution example #3: Solution (summary)

$$y(t) = x(t) * h(t)$$



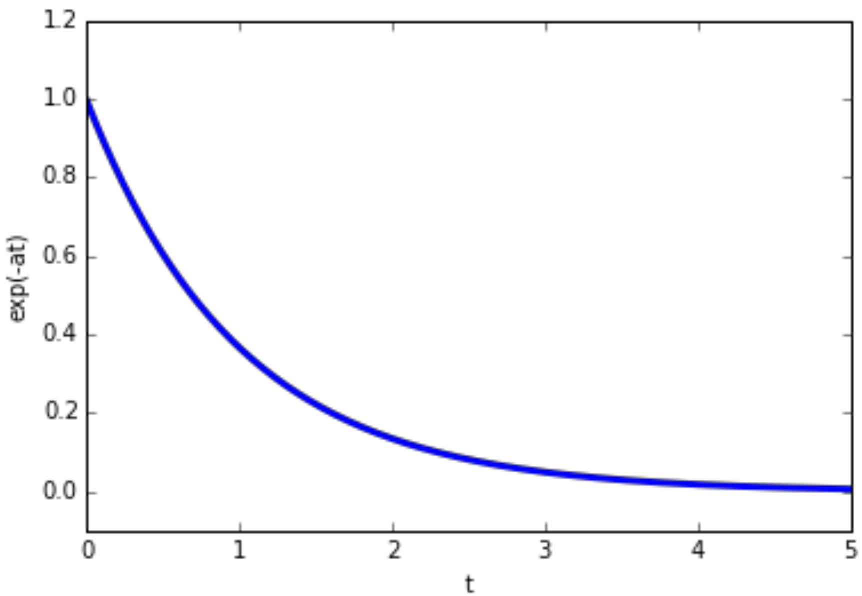

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} t < 0 & 0 \\ t \in [0,\alpha] & t \\ t \in [\alpha, 1] & \alpha \\ t \in [1, 1+\alpha] & 1-(t-\alpha) \\ t > 1+\alpha & 0 \end{cases}$$

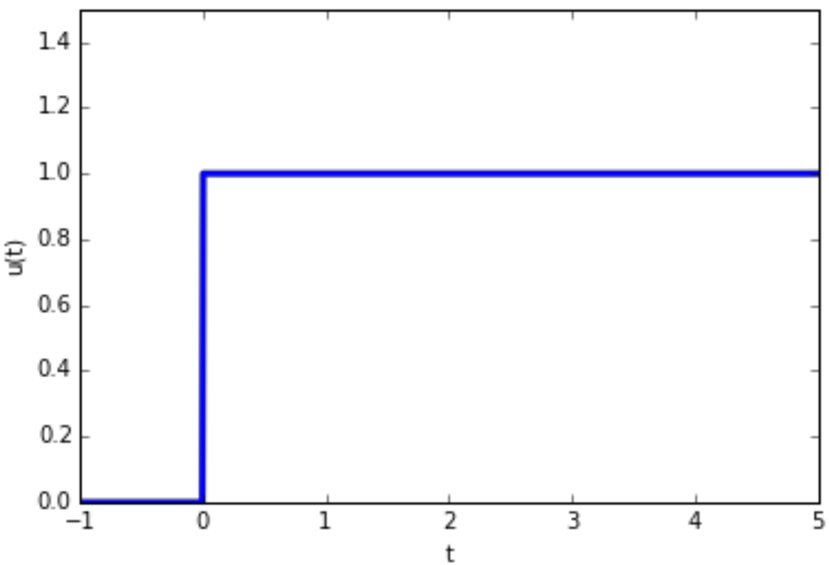
# CT convolution example #4

This is an important example of convolution to causal signals together. It will be solved by using the graphical approach to help set up the integrals.

- Determine and sketch $y(t) = x(t) * h(t)$, where

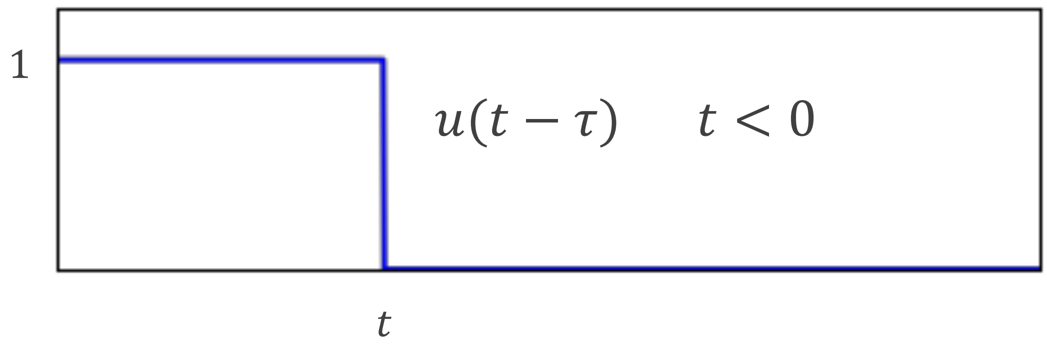
$$x(t) = e^{-at}u(t), \qquad a > 0 \qquad\qquad h(t) = u(t)$$

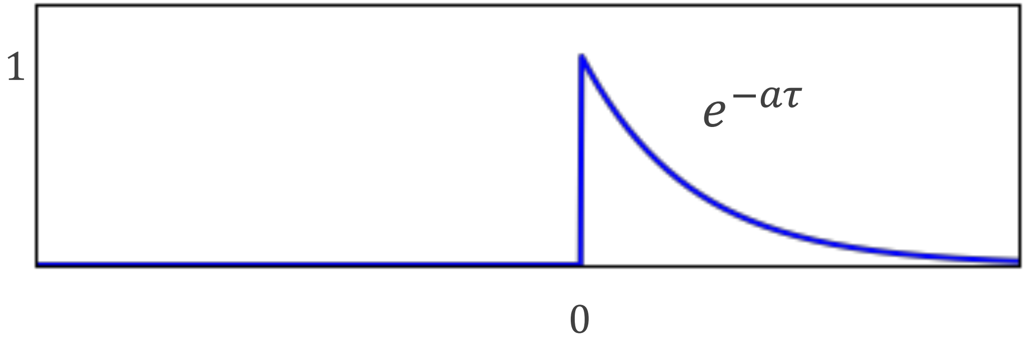


# CT convolution example #4: Solution

$$y(t) = x(t) * h(t)$$

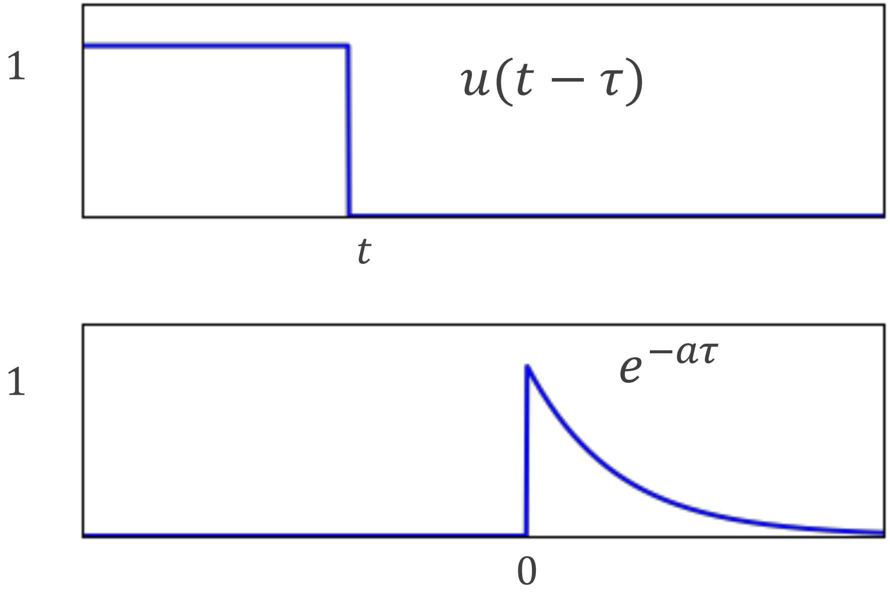



$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-a\tau}u(t-\tau)d\tau \end{aligned}$$

# CT convolution example #4: Solution

For $t < 0$

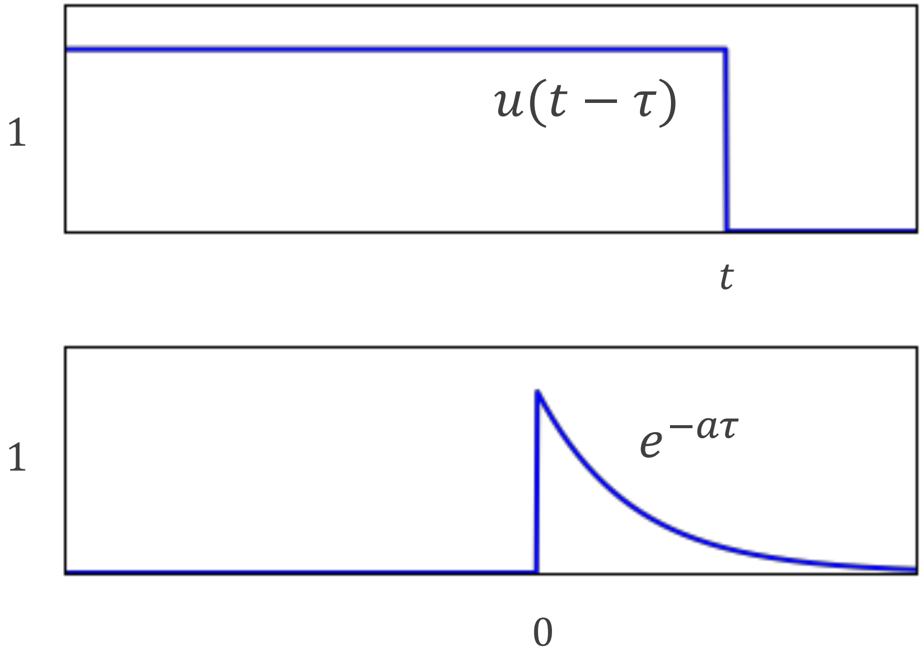



$$x(t) * h(t) = \int_{-\infty}^{\infty} 0 d\tau$$

$$= 0$$

# CT convolution example #4: Solution

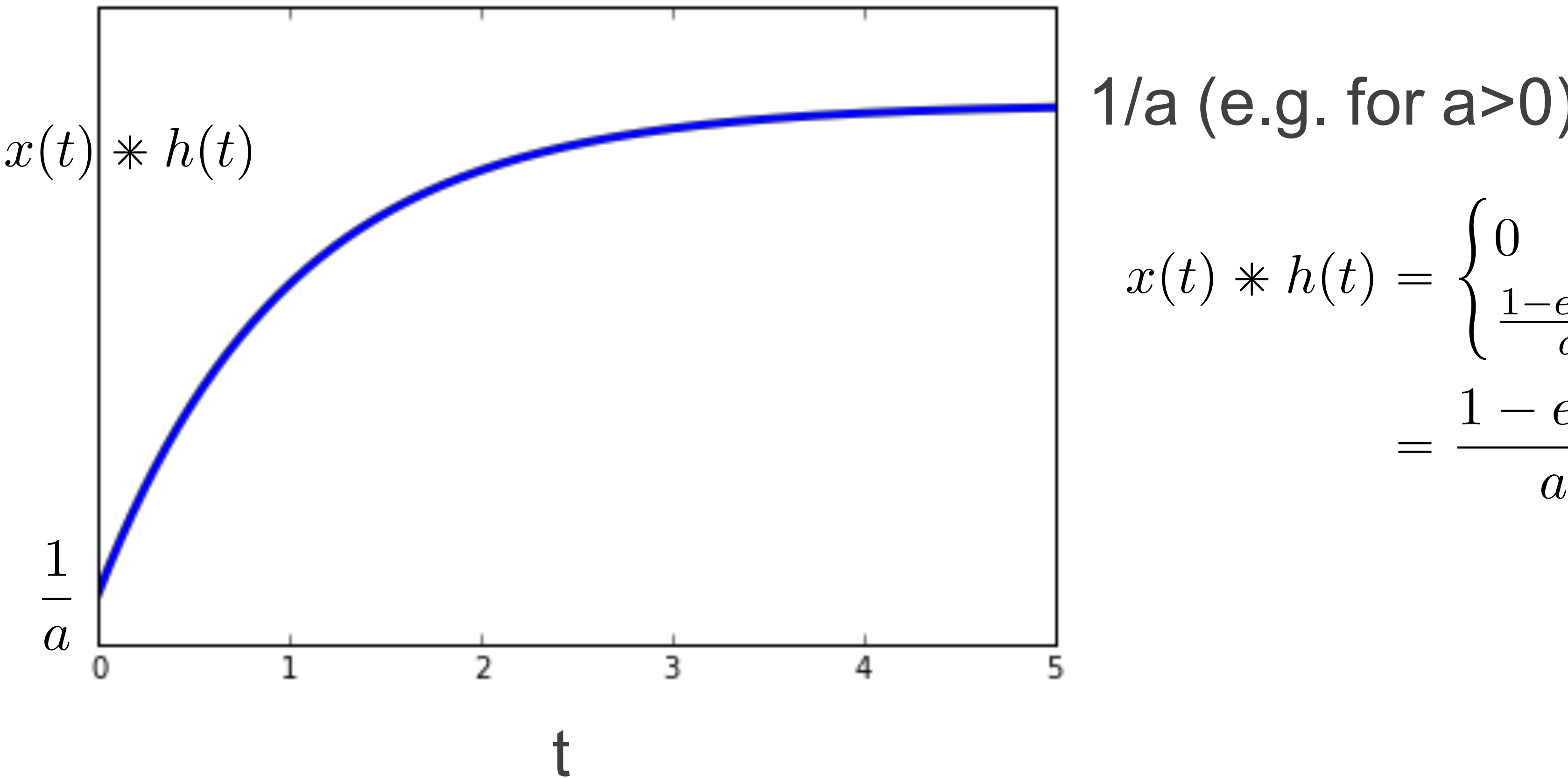
For $t \geq 0$




$$x(t) * h(t) = \int_0^t e^{-a\tau} u(t-\tau) d\tau$$

$$= \frac{1 - e^{-at}}{a}$$

# CT convolution example #4: Solution




$$x(t) * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1-e^{-at}}{a} & t \geqslant 0 \end{cases}$$

$$= \frac{1 - e^{-at}}{a} u(t)$$

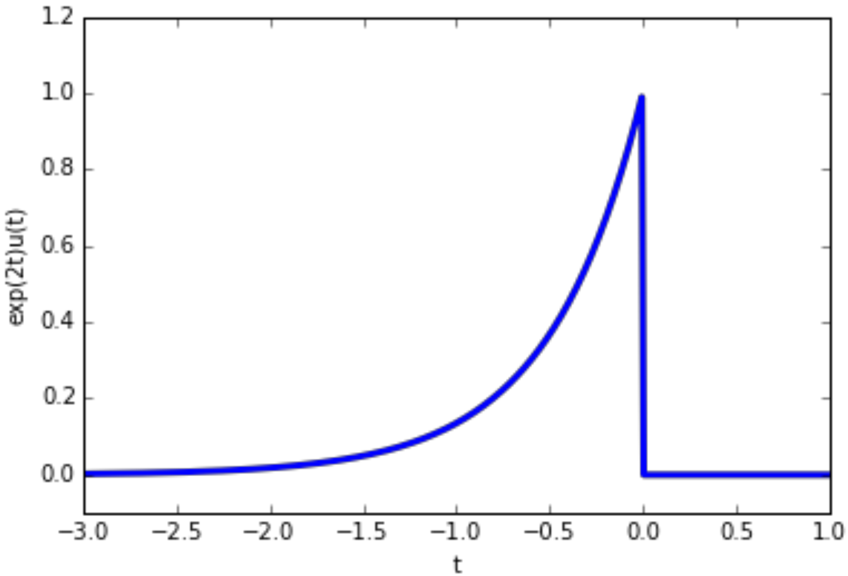
# CT convolution example #5

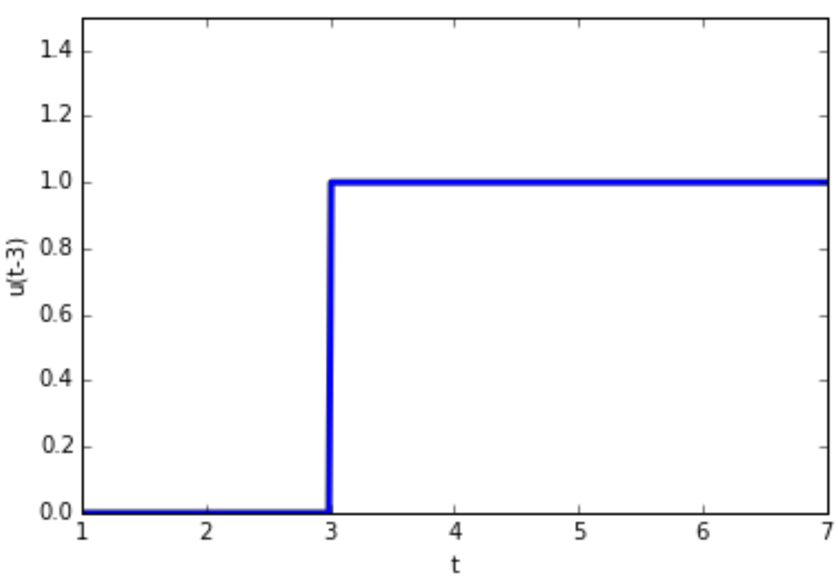
This is an example where an anti-causal and a causal signal are convolved together, unlike the previous examples. Notice the difference intervals in this case.

- Determine and sketch $y(t) = x(t) * h(t)$, where

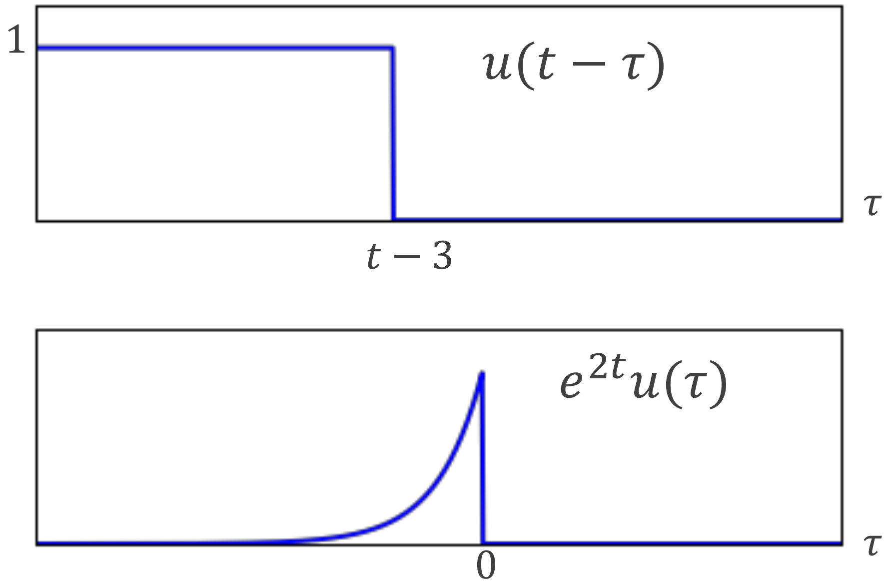
$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t-3)$$



# CT convolution example #5: Solution

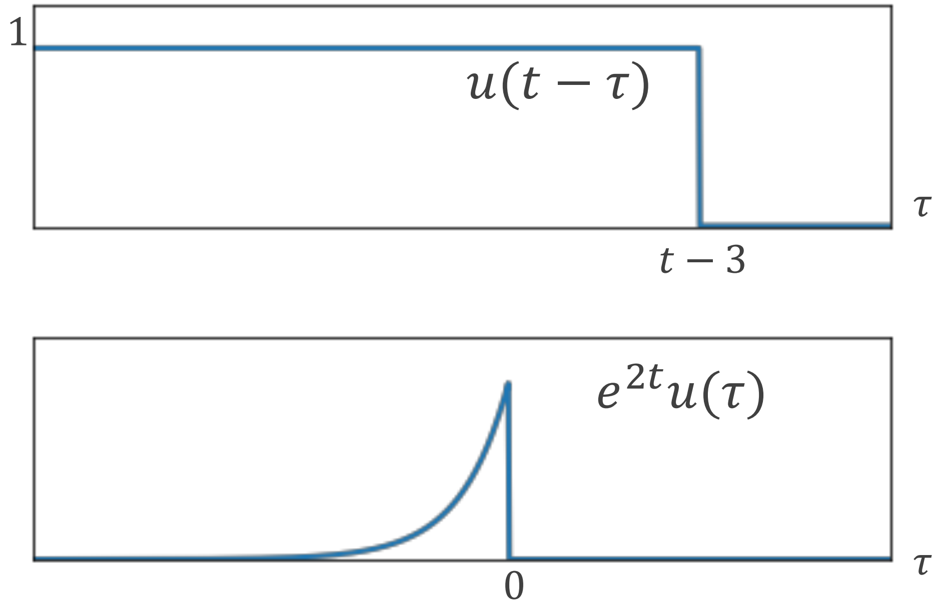



For t-3 < 0

$$x(t) * h(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau$$

$$= \frac{e^{2(t-3)}}{2}$$

# CT convolution example #5: Solution




For t-3 > 0

$$x(t) * h(t) = \int_{-\infty}^{0} e^{2\tau} d\tau$$

$$= \frac{1}{2}$$

# CT convolution example #5: Solution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \begin{cases} \frac{e^{2(t-3)}}{2} & t < 3 \\ \frac{1}{2} & t \geqslant 3 \end{cases}$$

# Basic convolution properties

- Commutative

$$y(t) = x(t) * h(t) = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau$$

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

Use properties to simplify convolutions

# Words of wisdom on convolution

- Convolution is a fact of life
  - All real world LTI systems generate outputs in the time domain given by the convolution sum or integral (yes RLC circuits are doing convolution for you)
- You need to practice convolutions
  - There are many examples in the book and videos online
- In future lectures, we will cover tools and ideas that allow us to avoid computing convolutions (most of the time)
  - But you still need to learn how to do it in the time domain (don't wait)