CSE 167 (WI 2025) Exercise 3

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Exercise 3.1

Problem

Consider an orthonormal basis $(\vec{r}, \vec{u}, \vec{b})$, where the right vector $\vec{r}$ points from left to right, the up vector $\vec{u}$ points from bottom to top, and the back vector $\vec{b}$ points from front to back.

The rotation operator $R_{\vec{r},\theta}$ rotates this basis by an angle θ about the right vector $\vec{r}$, producing a new orthonormal basis $(\vec{r}', \vec{u}', \vec{b}')$. The transformation is defined as:

$$R_{\vec{r},\theta}\vec{r} = \vec{r}', \quad R_{\vec{r},\theta}\vec{u} = \vec{u}', \quad R_{\vec{r},\theta}\vec{b} = \vec{b}'.$$

(a) Write down the rotation matrix R in terms of θ and trigonometric functions.
(b) Given a world basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, suppose $\vec{r}, \vec{u}, \vec{b}$ have coefficients in the world basis:

$$\vec{r} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the transformation matrix M such that:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = M \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Solution

(a) The rotation matrix around the right vector $\vec{r}$ is derived as follows:
The standard rotation matrix for a rotation by angle θ around the x-axis (right vector) is:

$$R_{\vec{r},\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

This matrix ensures that the $\vec{r}$ vector remains unchanged, while $\vec{u}$ and $\vec{b}$ are rotated in the plane perpendicular to $\vec{r}$.

(b) To transform the rotated object into world coordinates:
The given basis $(\vec{r}, \vec{u}, \vec{b})$ is represented in the world basis as matrix R:

$$R = \begin{bmatrix} \vec{r} & \vec{u} & \vec{b} \end{bmatrix}.$$

The transformation matrix M is obtained as:

$$M = \begin{bmatrix} \vec{r} & \vec{u} & \vec{b} \end{bmatrix} \cdot R_{\vec{r},\theta}.$$

Exercise 3.2

Problem

Find the quaternion $q \in \mathbb{H}$ satisfying:

$$qiq^{-1} = j, \quad qjq^{-1} = k, \quad qkq^{-1} = i, \quad \mathrm{Re}(q) > 0.$$

Solution

To solve for the quaternion q, we analyze the rotation properties:
1. The given transformations correspond to a 120-degree rotation about the axis $(1, 1, 1)$.

2. The quaternion representation of a rotation by angle θ about a unit vector $\hat{v}$ is:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(v_x i + v_y j + v_z k).$$

3. For a 120-degree rotation ($\theta = 120^\circ$) around $(1, 1, 1)$, the unit vector is:

$$\hat{v} = \frac{1}{\sqrt{3}}(i + j + k).$$

4. Substituting $\theta = 120^\circ$:

$$q = \cos 60^\circ + \sin 60^\circ \frac{(i + j + k)}{\sqrt{3}}.$$

5. Simplifying using $\cos 60^\circ = \frac{1}{2}$ and $\sin 60^\circ = \frac{\sqrt{3}}{2}$:

$$q = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k.$$

Thus, the required quaternion is:

$$q = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k.$$