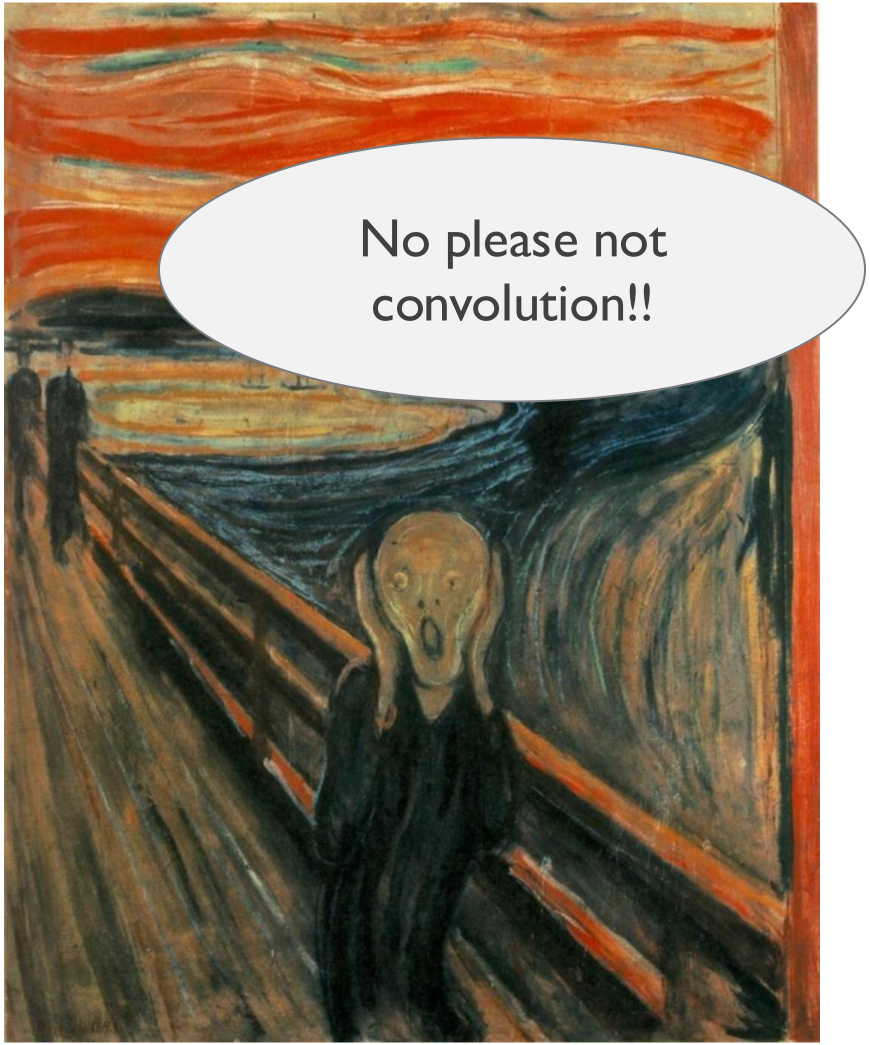
Chapter 2
Linear time-invariant systems

ECE 301 Linear Systems

Discrete-time system response (LTI systems)

Learning objectives

- Determine the output of the system in terms of its input
- Explain the impulse response properties of LTI systems



https://commons.wikimedia.org/wiki/File:The_Scream.jpg with additions by Prof. Heath

Computing the output of a system

- Consider the following system

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n] = f(x[n], \text{the system})$$

- Want to determine a precise relationship between y[n] and x[n]
- Focus on linear and time invariant (LTI) systems
  - Models many realistic systems quite well
  - Has a special and precise relationship between the output & the input

System impulse response

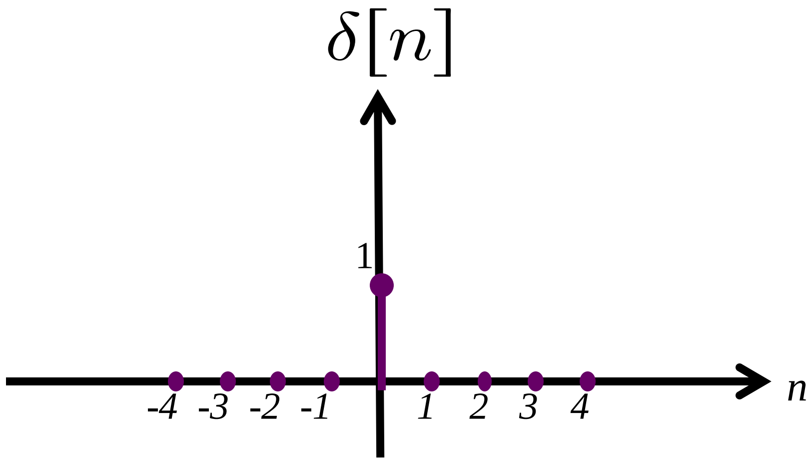
The output of a system when the input is a delta function is given a special name

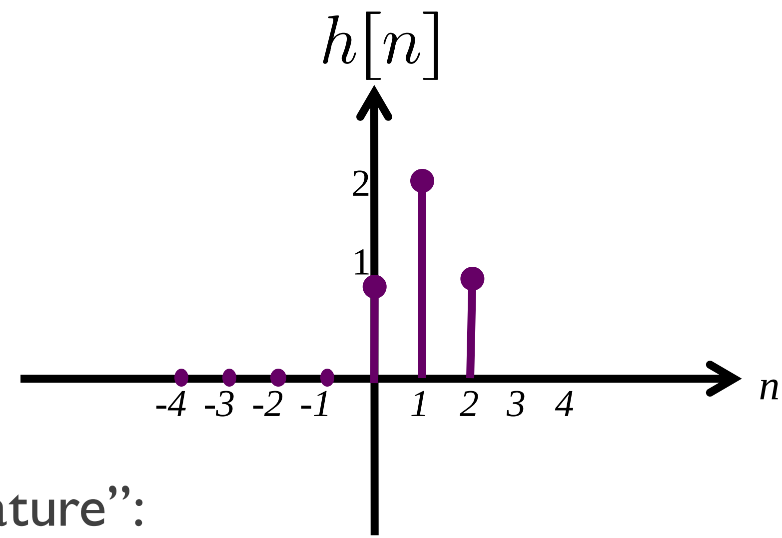
$$\delta[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n] \triangleq h[n]$$

- Consider the input signal $\delta[n]$
- The output corresponding to this input is the impulse response
  - The resulting output sequence is given special label: h[n]
- All systems have an impulse response, however, …

The impulse response is special <u>only</u> for linear, time invariant (LTI) systems

Example





In this case, the system has this “signature”:

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

Now we develop the output for any input

- If the system has an impulse response $h[n]$, then

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

- Because the system is linear

$$A\delta[n] \longrightarrow \boxed{h[n]} \longrightarrow Ah[n]$$

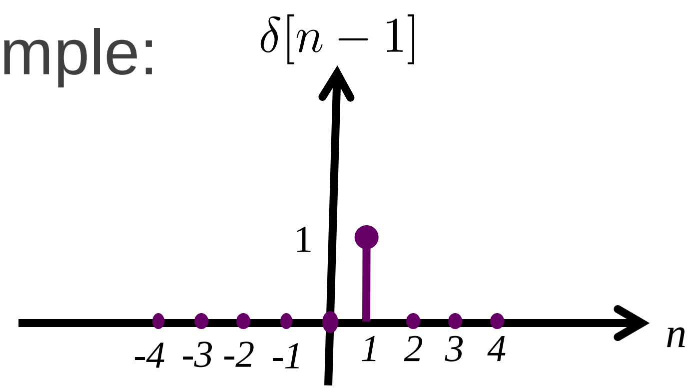
For LTI systems we write the impulse response here in the box

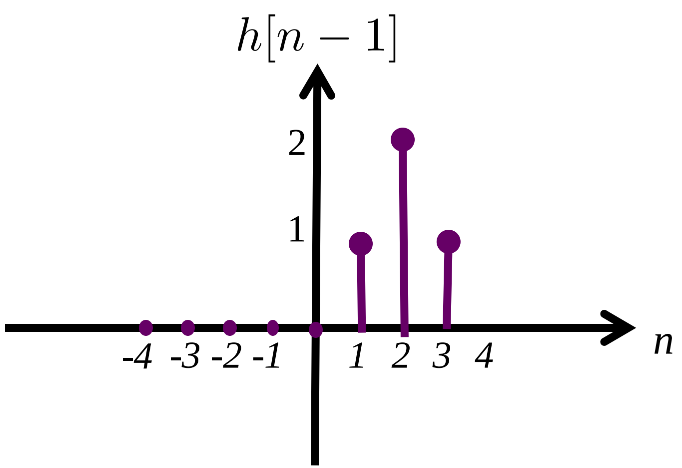
Now we develop the output for any input (cont)

- Because the system is time-invariant

$$x[n] = \delta[n-1] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n-1]$$

Example:





Now we develop the output for any input (cont)

- Combining linearity and time invariance

$$a_0\delta[n] + a_1\delta[n-1] \longrightarrow \boxed{h[n]} \longrightarrow a_0 h[n] + a_1 h[n-1]$$

- Extending to a sequence of shifted delta functions

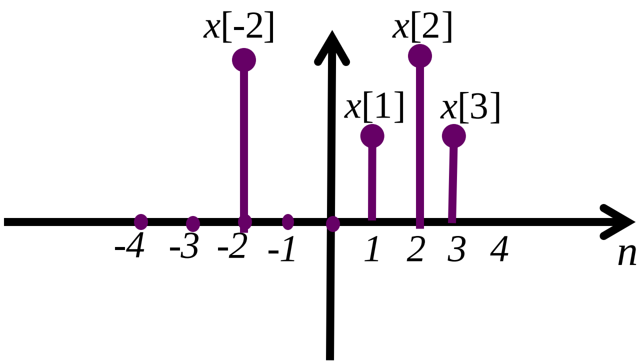
$$\sum_{k=-\infty}^{\infty} a_k \delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow \sum_{k=-\infty}^{\infty} a_k h[n-k]$$

This is a key result but why?

Uncovering the fundamental relationship

- Self-referenced definition of x[n]

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



- Compute the output for this special input to find

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

recall

$$\sum_{k=-\infty}^{\infty} a_k\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow \sum_{k=-\infty}^{\infty} a_k h[n-k]$$

This sum has a special name: Convolution sum

Time variable for output sequence

Shorthand notation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

Sum index variable, also corresponds to time

- Key fact (can prove by setting m = n – k and some algebra):

$$h[n] * x[n] = x[n] * h[n]$$

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Key Fact: Commutative property

- Can rewrite the convolution to flip and slide either signal
  (can prove by setting m = n – k and some algebra)

$$h[n] * x[n] = x[n] * h[n]$$

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

This is useful as it may be easier to compute the convolution in one form or the other – always think first which is best

Key Fact: Convolution with the delta

- Convolution with the delta is easy

$$\boxed{\begin{aligned} x[n] * \delta[n] &= x[n] \\ h[n] * \delta[n] &= h[n] \end{aligned}}$$

$$\begin{aligned} x[n] * \delta[n] &= \sum_{k=-\infty}^{\infty} x[k] \underbrace{\delta[n-k]}_{\text{impulse at } k=n} \\ &= \sum_{k=-\infty}^{\infty} x[n]\delta[n-k] \\ &= x[n] \sum_{k=-\infty}^{\infty} \delta[n-k] \\ &= x[n] \cdot 1 = x[n] \end{aligned}$$

Key Fact: Unique and complete description

Since $y[n] = x[n] * h[n]$ for any input
→ $h[n]$ is a unique and complete description of the LTI system

If you have an LTI system and you know h[n], then you know everything there is to know about the system

Any LTI system has only one impulse response

Summary of the discrete-time system response

- Impulse response of a system is

$$\delta[n] \longrightarrow \boxed{\begin{matrix}\text{System}\\ h[n]\end{matrix}} \longrightarrow y[n] \triangleq h[n]$$

- Output of an LTI system is

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\begin{matrix}\text{System}\\ h[n]\end{matrix}} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution connects the input & system to the output in an LTI system

Computing discrete-time convolution

Learning objectives

- Calculate the convolution of two signals
- Determine LTI systems' output for different inputs

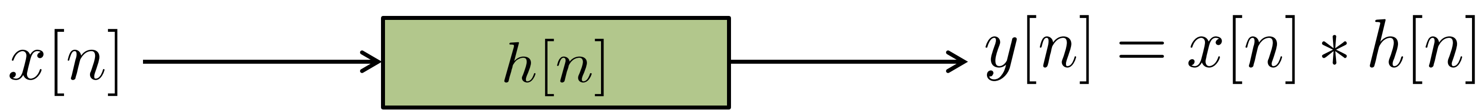
Discrete-time convolution

- Given input $x[n]$ into an LTI system with impulse response $h[n]$, the output is given by the **convolution** of $x[n]$ and $h[n]$ defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

convolution sum

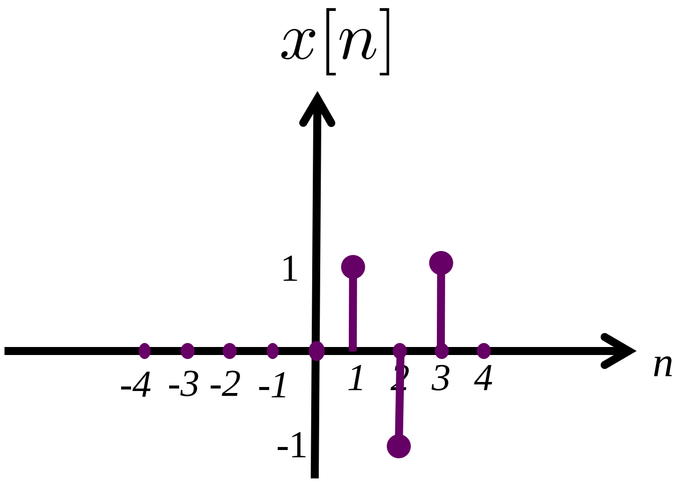
Shorthand notation

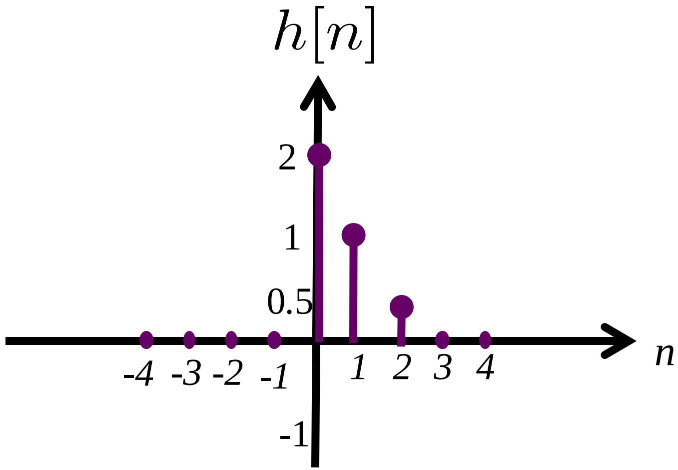


This is an example that shows the how to calculate the output of the convolution using the LTI property. This approach typically works well for simple cases where one signal has just a few values.

Convolution example 1

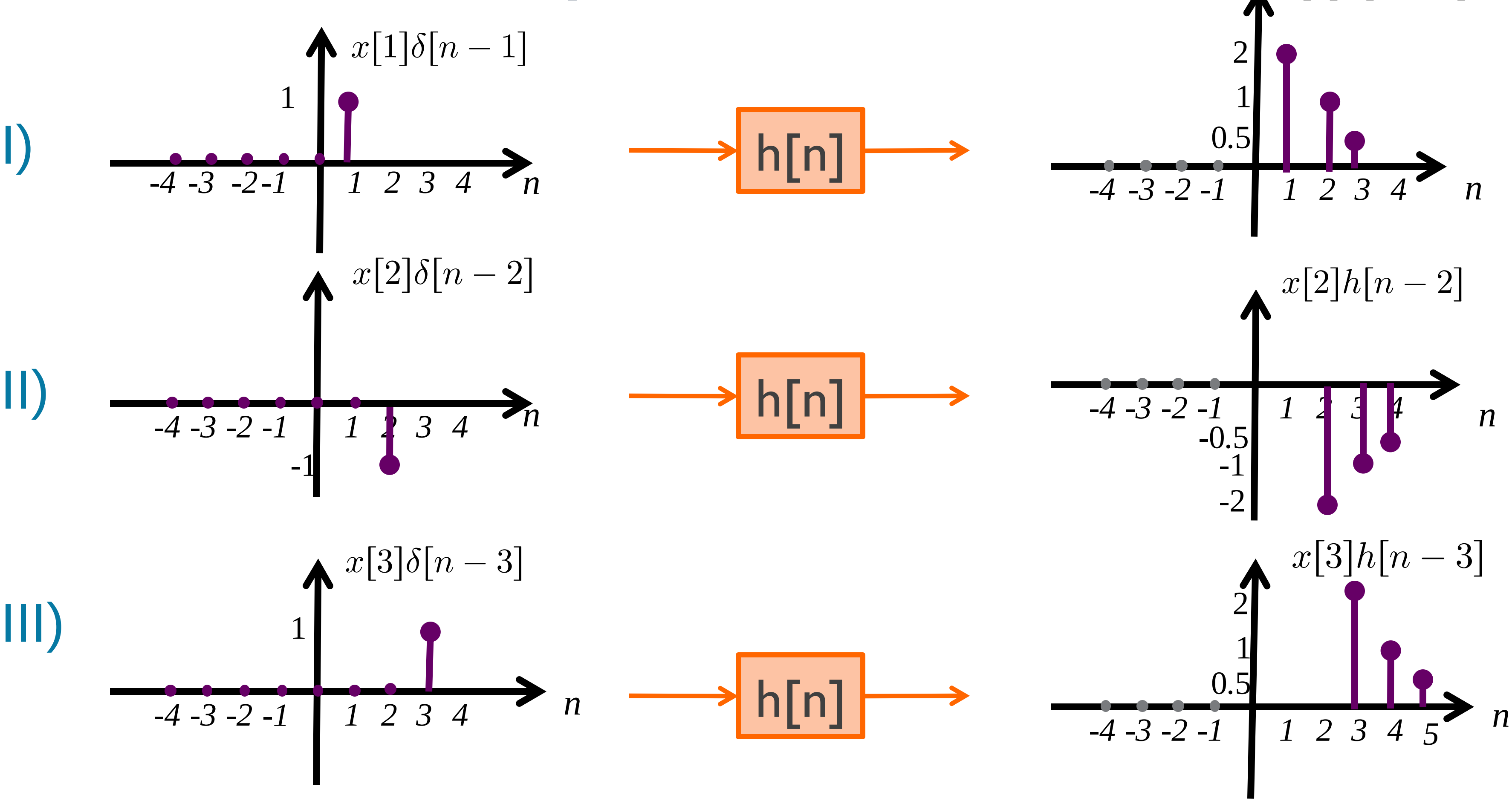
- Graphically perform the convolution





- Find the output of the system?
  - Let's do that step by step
  - For each input sample, find the output and then sum the outputs (LTI)

Convolution example I

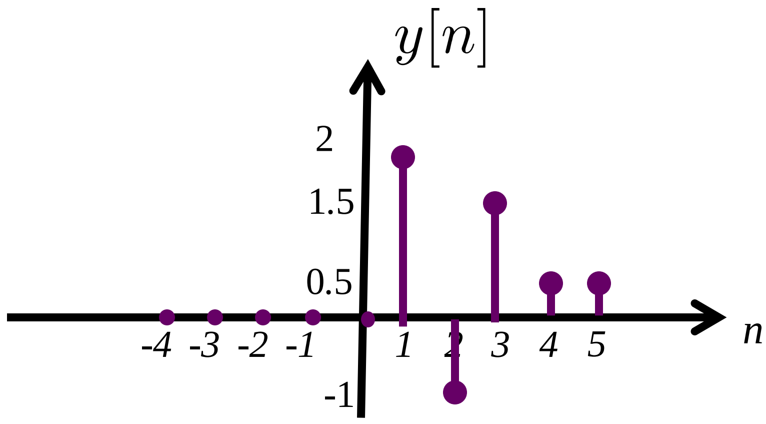


Convolution example I

- Output is

$$y[n] = \sum_k x[k]h[n-k]$$
$$= x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3]$$

So, the final output is adding I), II), and III)



Convolution example 2

This is an example that shows the how to calculate the output of the convolution using the graphical flip and slide method. The graphical approach should be your default approach. It is quite helpful in determining the boundary regions for the convolutions.

- Consider this input and impulse response:

$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$
$$h[n] = u[n] - u[n-5]$$

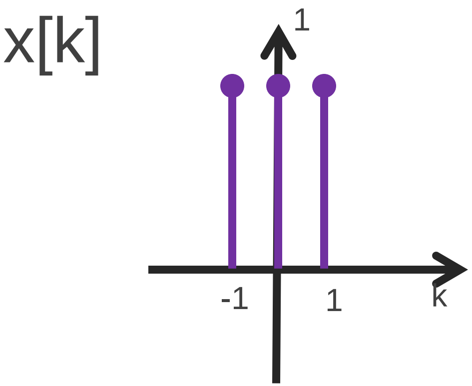
- Determine the output y[n] using the "Flip and slide" graphical method

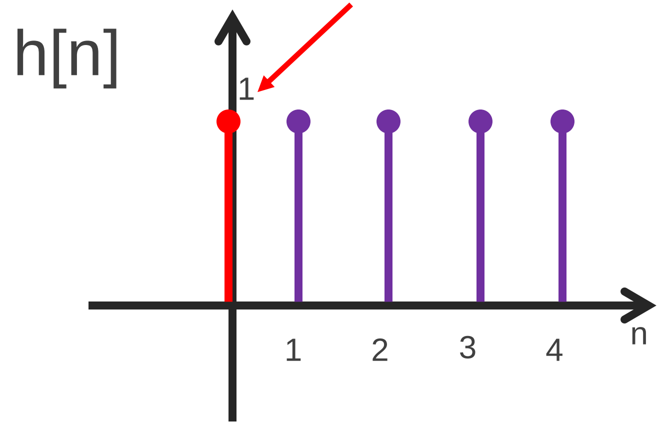
Convolution example 2 solution

Convolution two rectangles is an important example. You should be able to do this with rectangles of different widths and different shifts.

- Plot the two signals (aside note that x[n] = u[n+1] – u[n-2])

We color code this so that we can follow the sample when we flip and slide the signal





$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

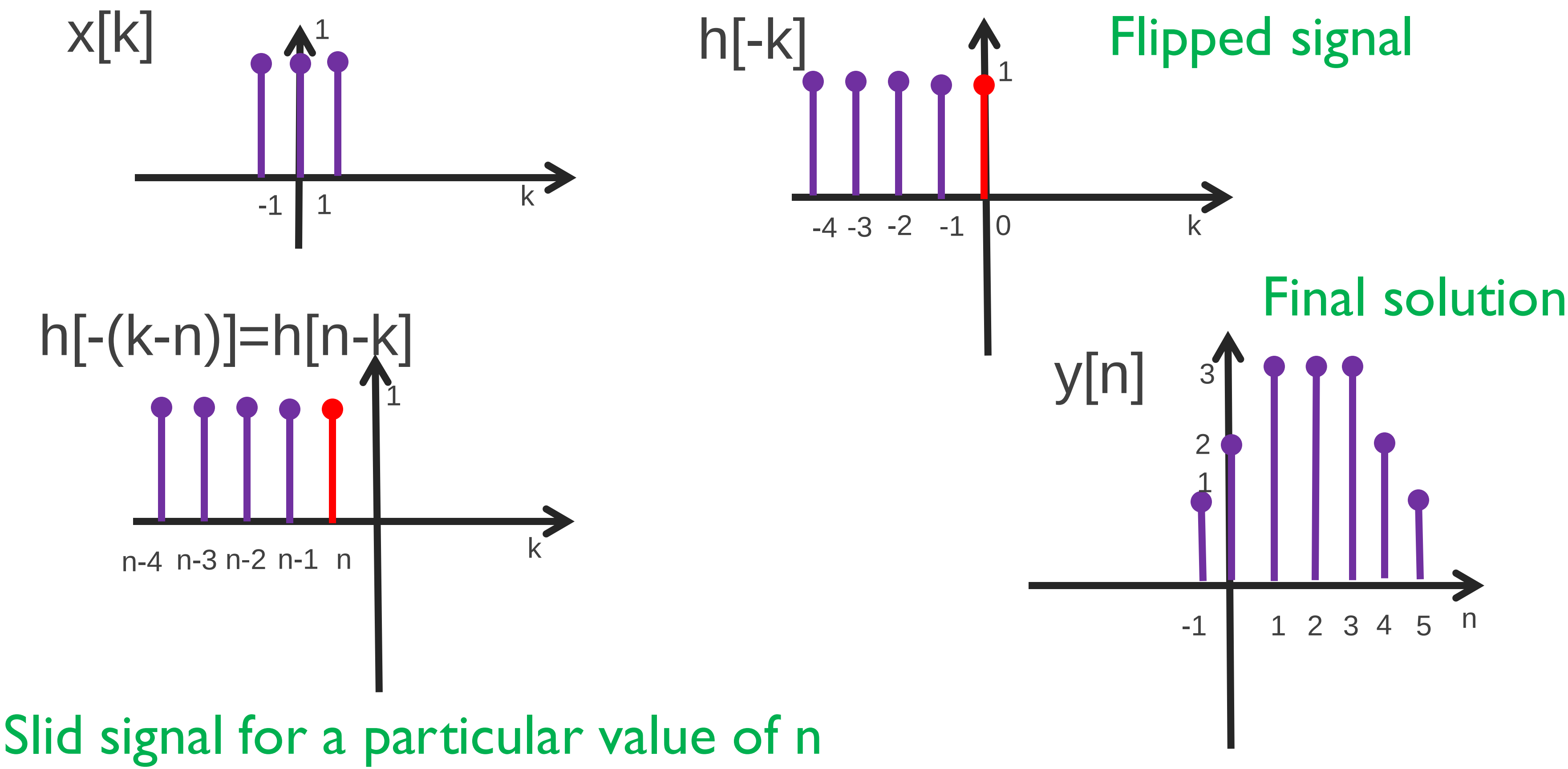
$$h[n] = u[n] - u[n-5]$$

Convolution example 2 solution (flip and slide)

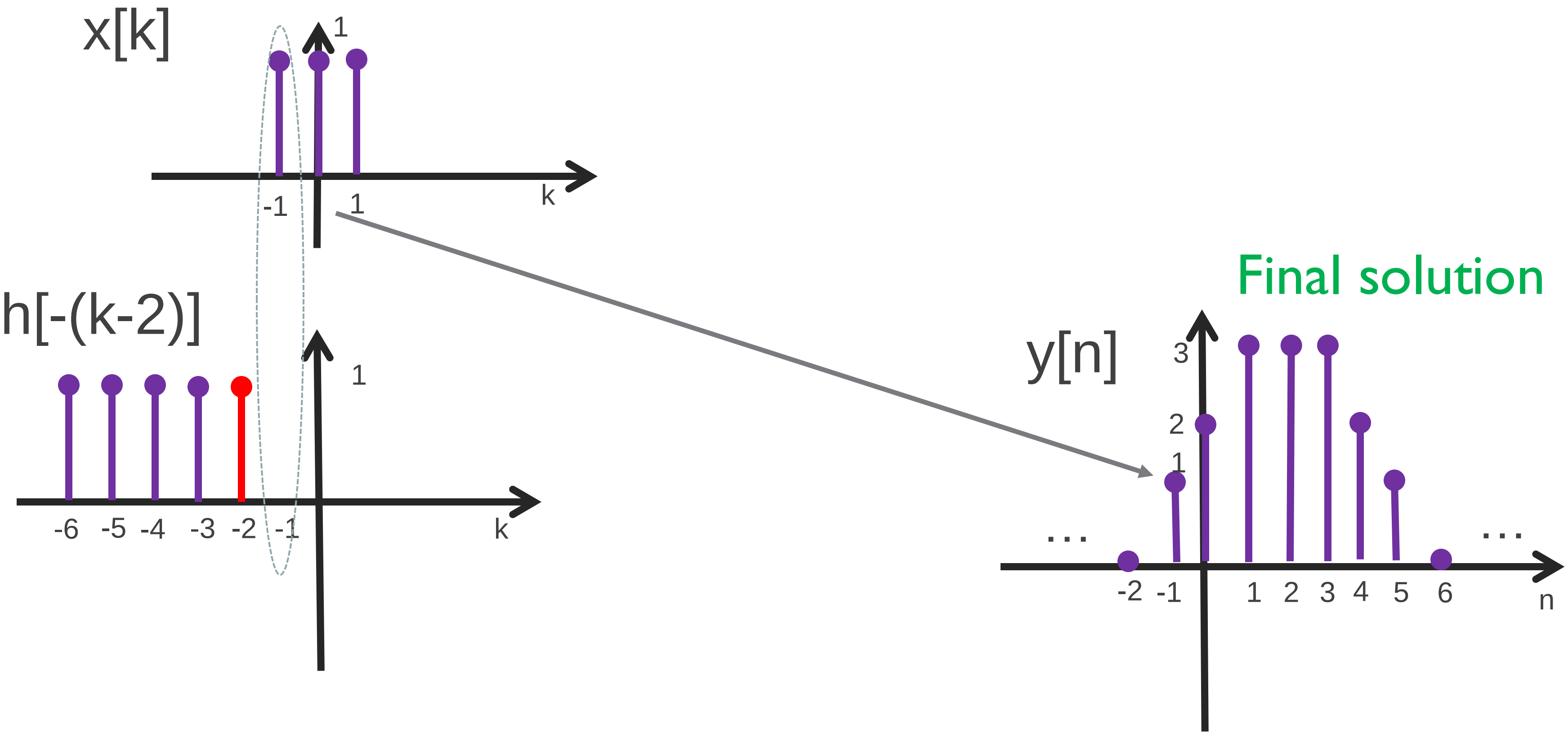
- General approach for the flip and slide method
  - Plot both signals
  - Select one signal to flip and slide
  - Plot the non-flipped signal in terms of k
  - Plot the flipped signal in terms of –(k-n)
  - Slide for different values of n and sum to compute y[n]

- In this specific example
  - Easier to flip and slide h[n]
  - Gives

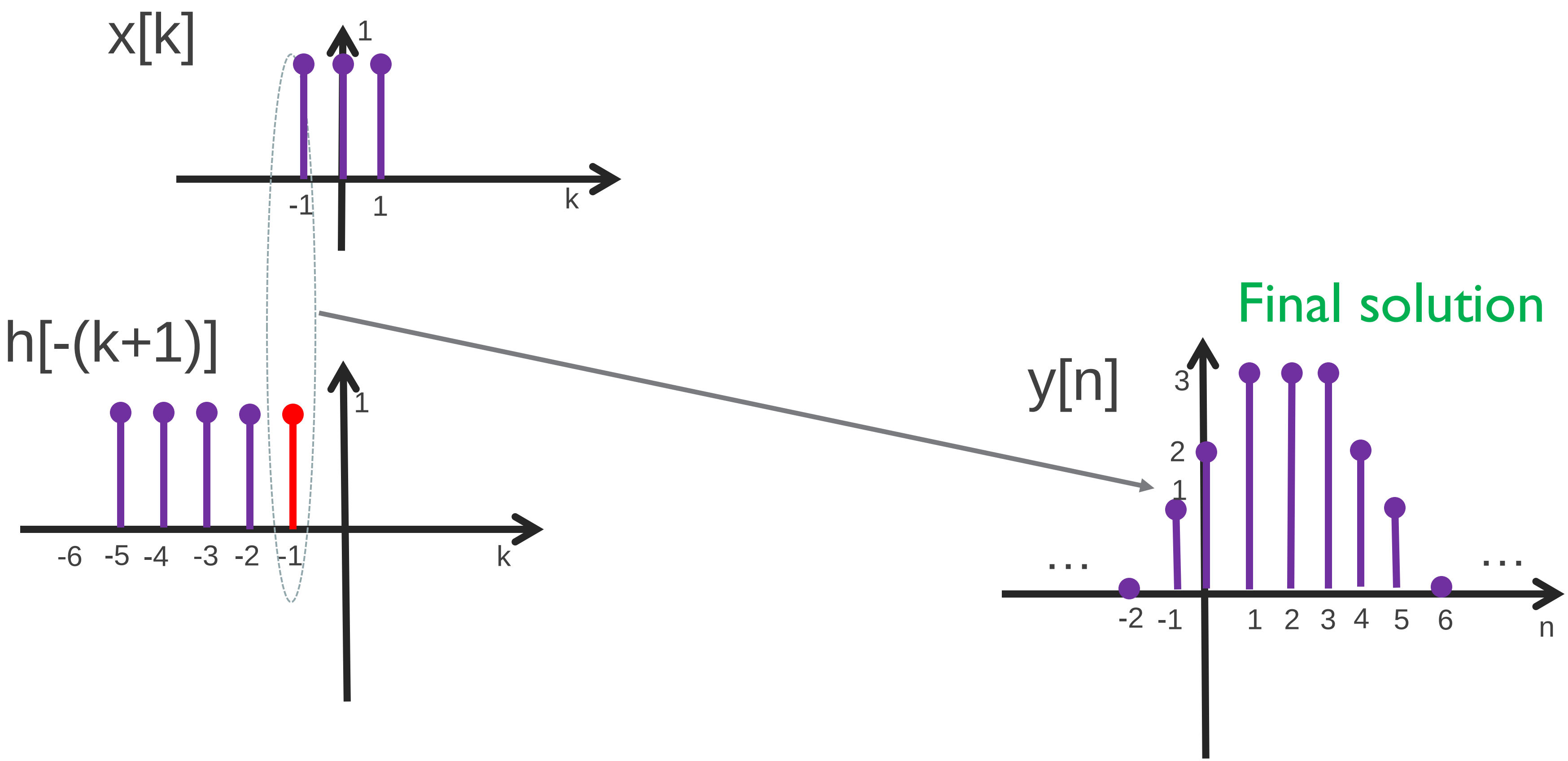
Convolution example 2 solution (flip and slide)



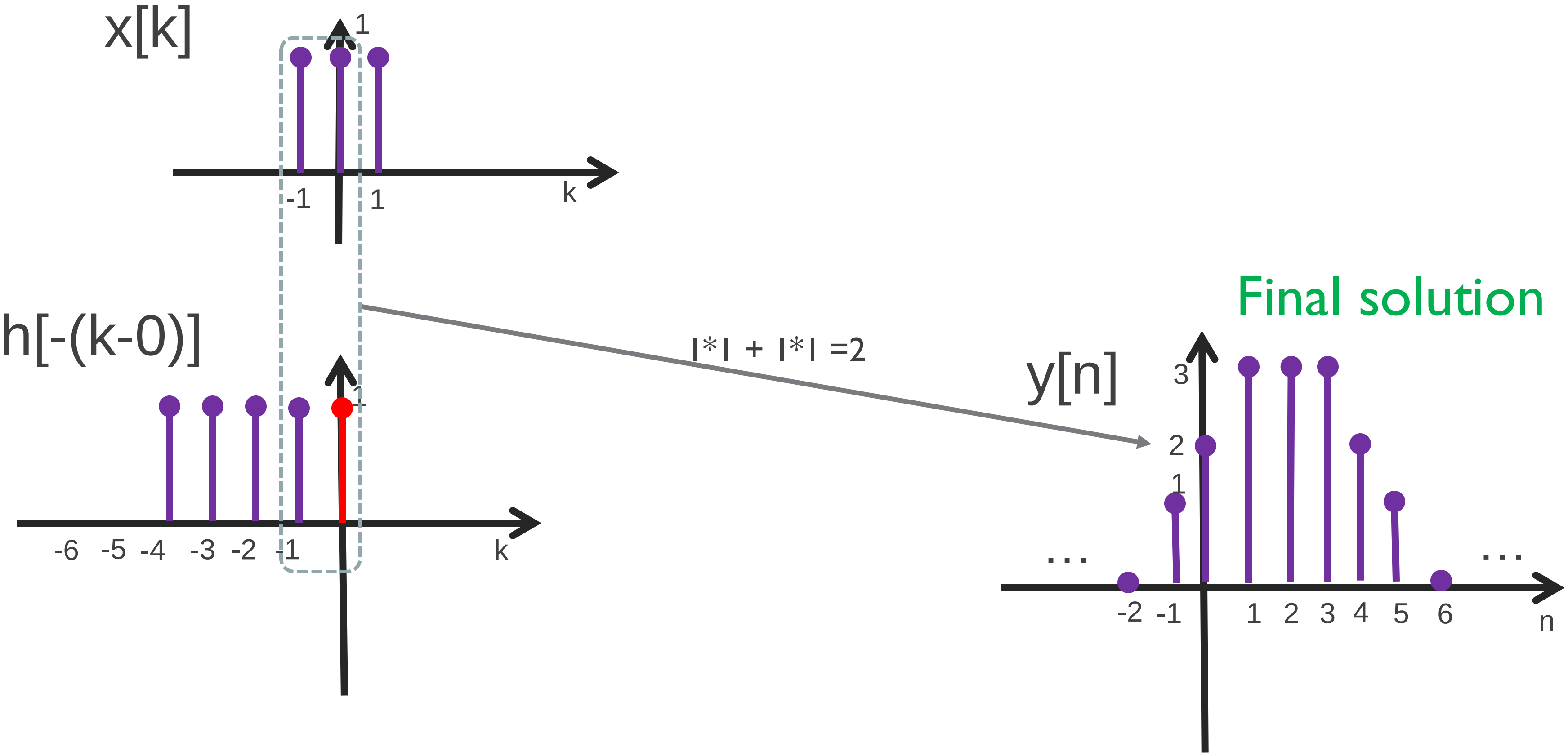
Convolution example 2 solution (flip and slide)



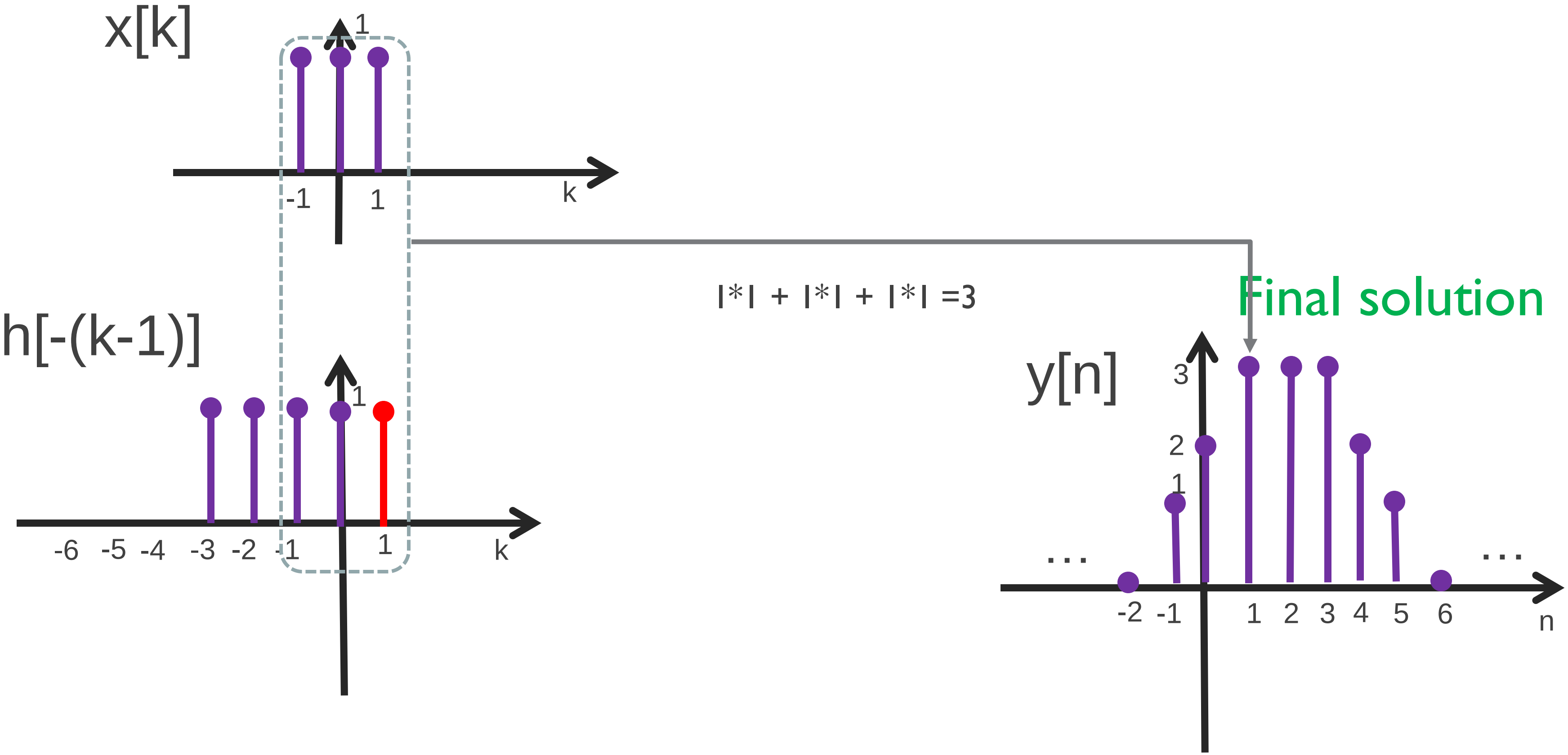
Convolution example 2 solution (flip and slide)



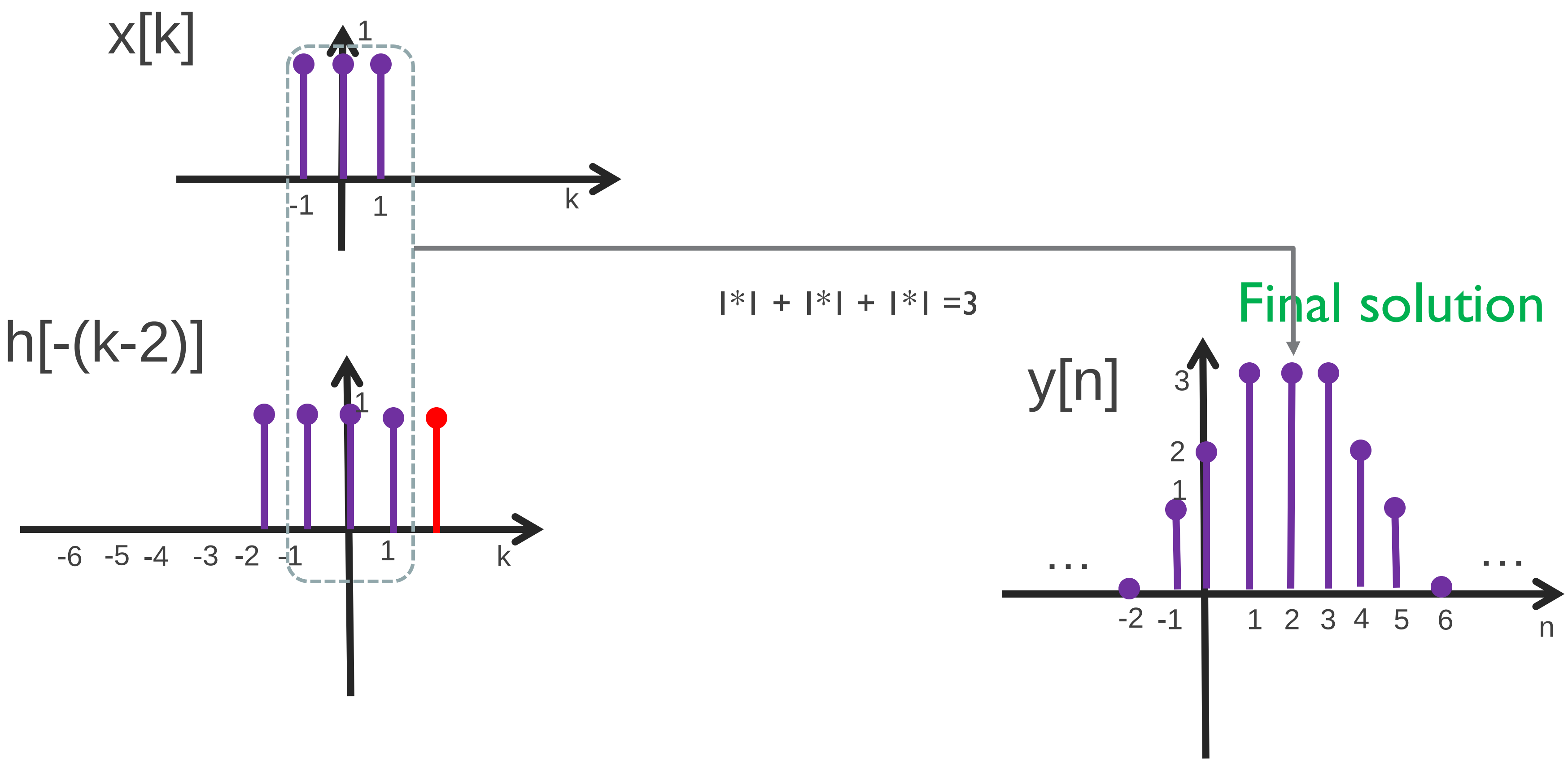
Convolution example 2 solution (flip and slide)



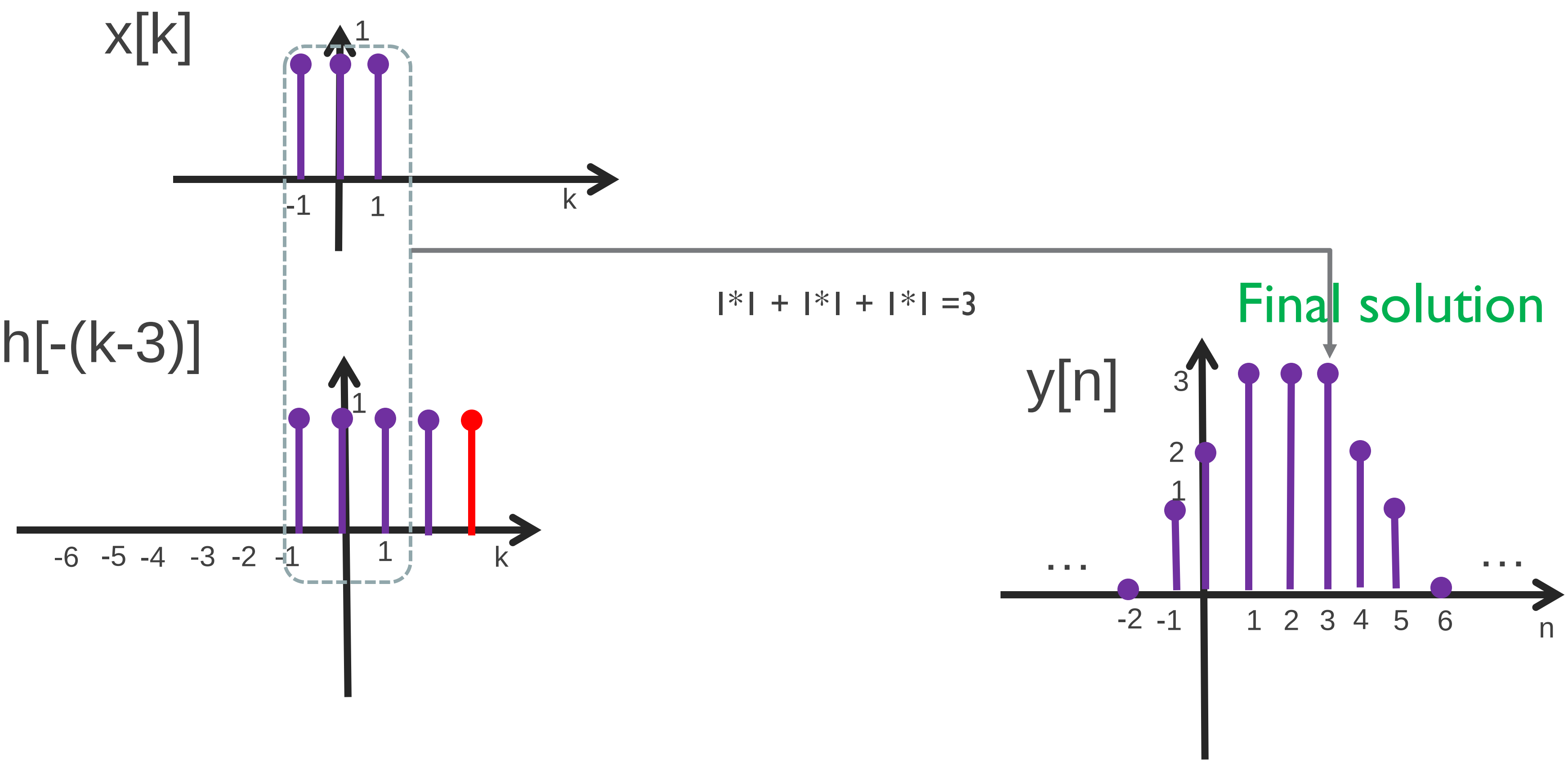
Convolution example 2 solution (flip and slide)



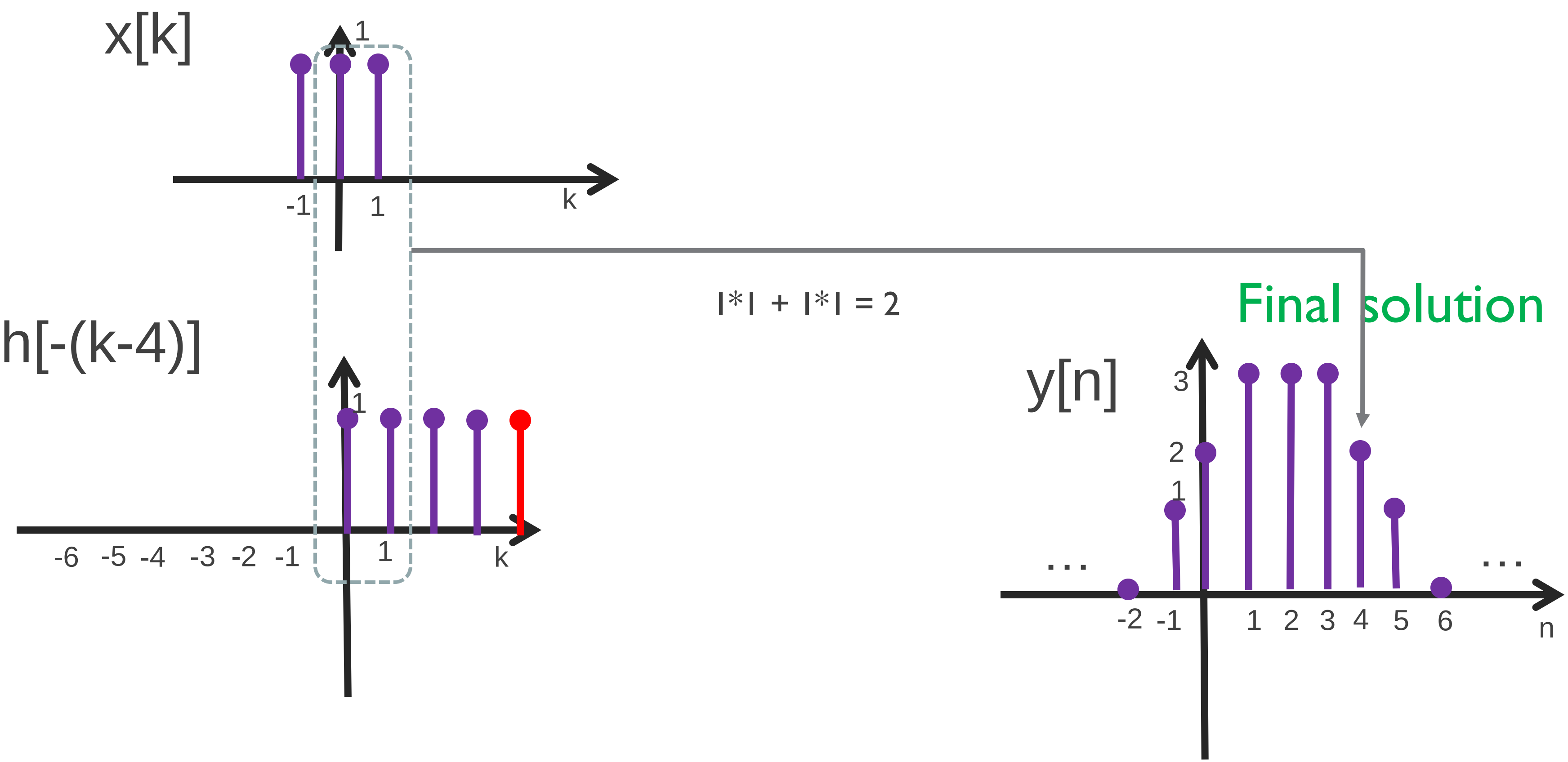
Convolution example 2 solution (flip and slide)



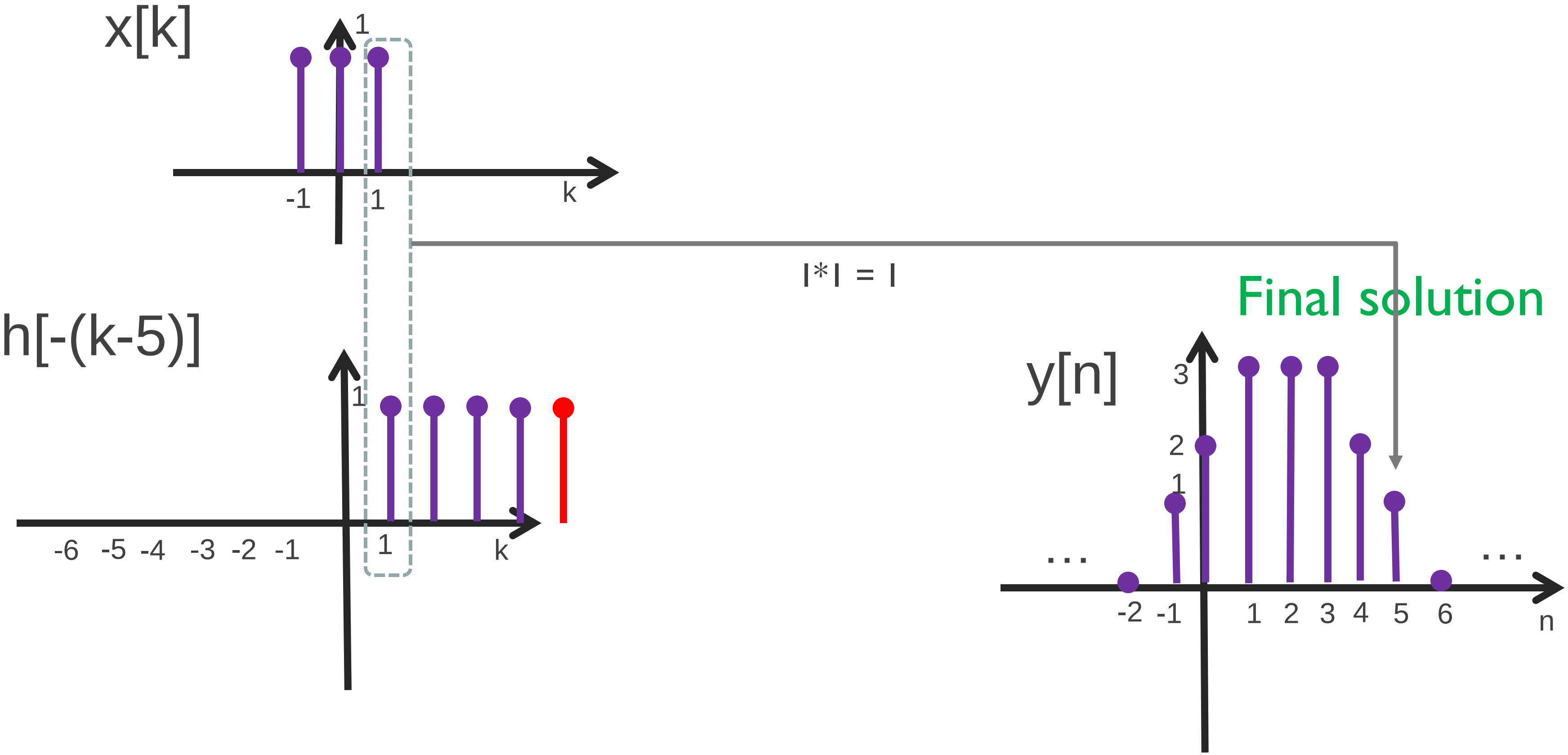
Convolution example 2 solution (flip and slide)



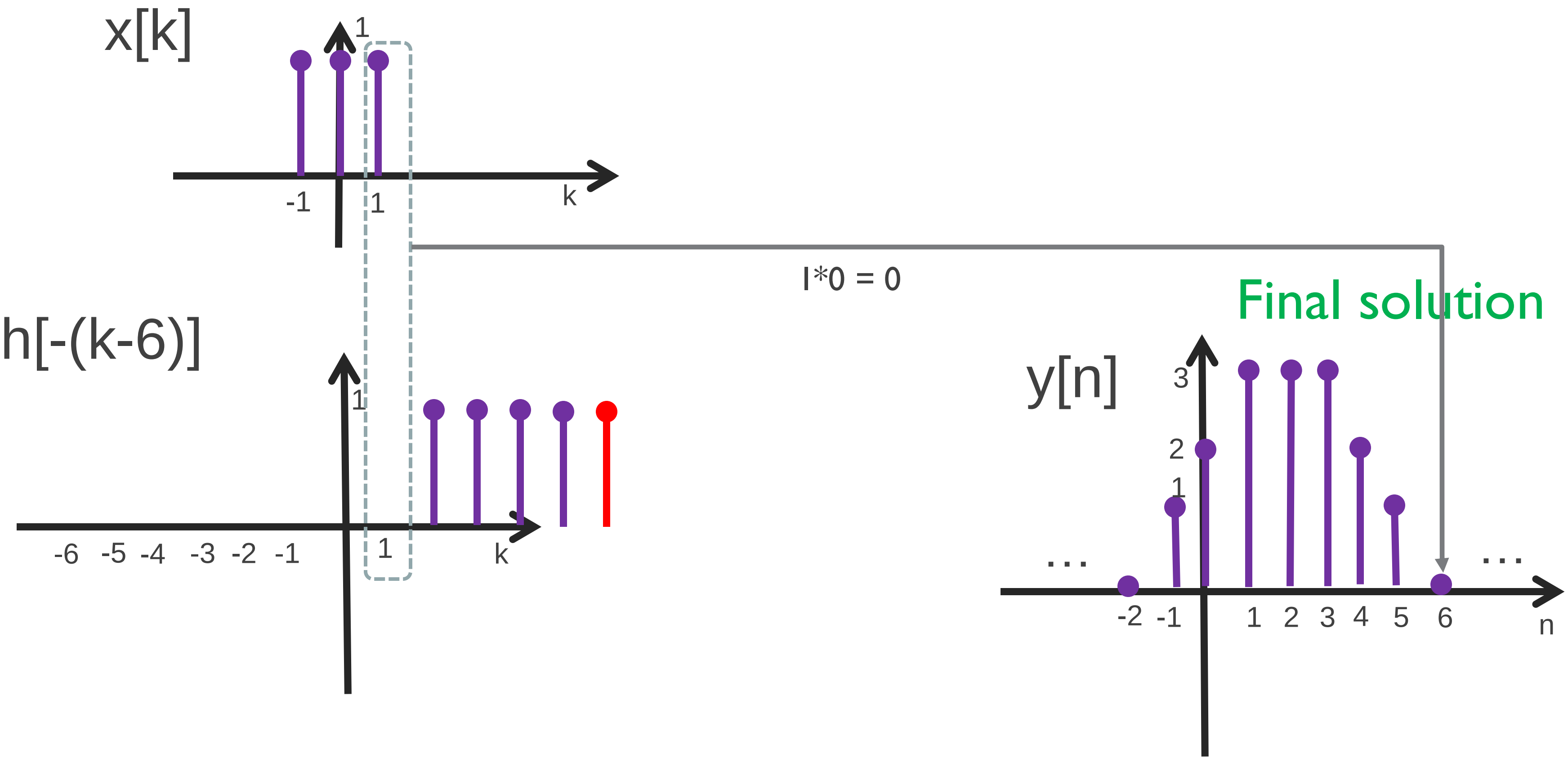
Convolution example 2 solution (flip and slide)



Convolution example 2 solution (flip and slide)

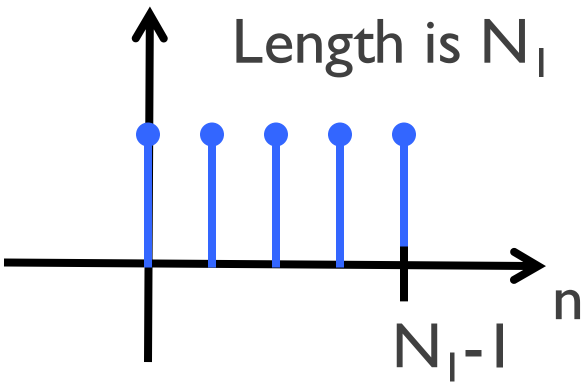


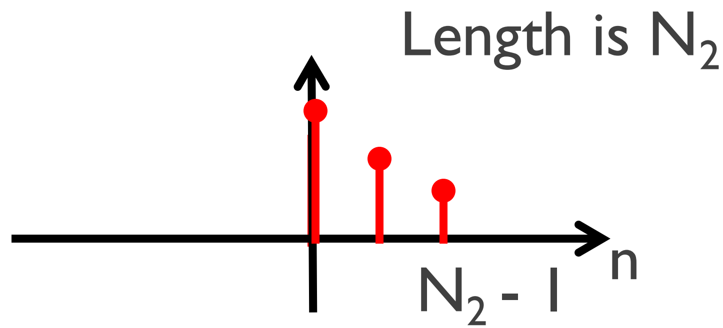
Convolution example 2 solution (flip and slide)

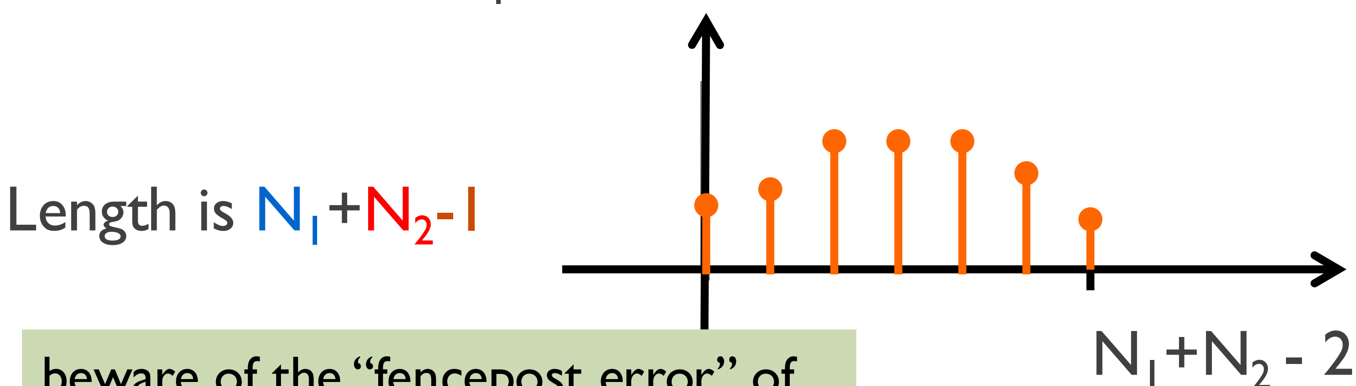


This can be important for checking your work. How do you extend the result if one or both signals do not start at zero?

Key Fact: Length of a convolution in discrete time







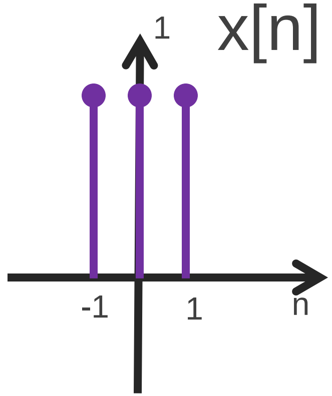
beware of the "fencepost error" of counting the gaps, not samples!

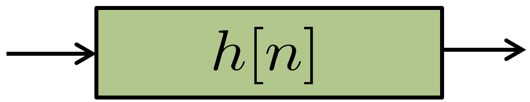
Convolution example 2 Redux: sliding tape method

n	0	0	0	0	X[-2]=0	X[-1]=1	X[0]=1	X[1]=1	X[2]=0	0	0	0	0	0	Y[n]
-2	1	1	1	1	1										0
-1		1	1	1	1	1									1
0			1	1	1	1	1								2
1				1	1	1	1	1							3
2					1	1	1	1	1						3
3						1	1	1	1	1					3
4							1	1	1	1	1				3
5								1	1	1	1	1			2
6									1	1	1	1	1		1
7										1	1	1	1	1	0

This is a way to implement the flip and slide method without drawing the pictures, each row is like a slide piece of a tape (measure)

Convolution example 2 Redux^2 (impulse property)





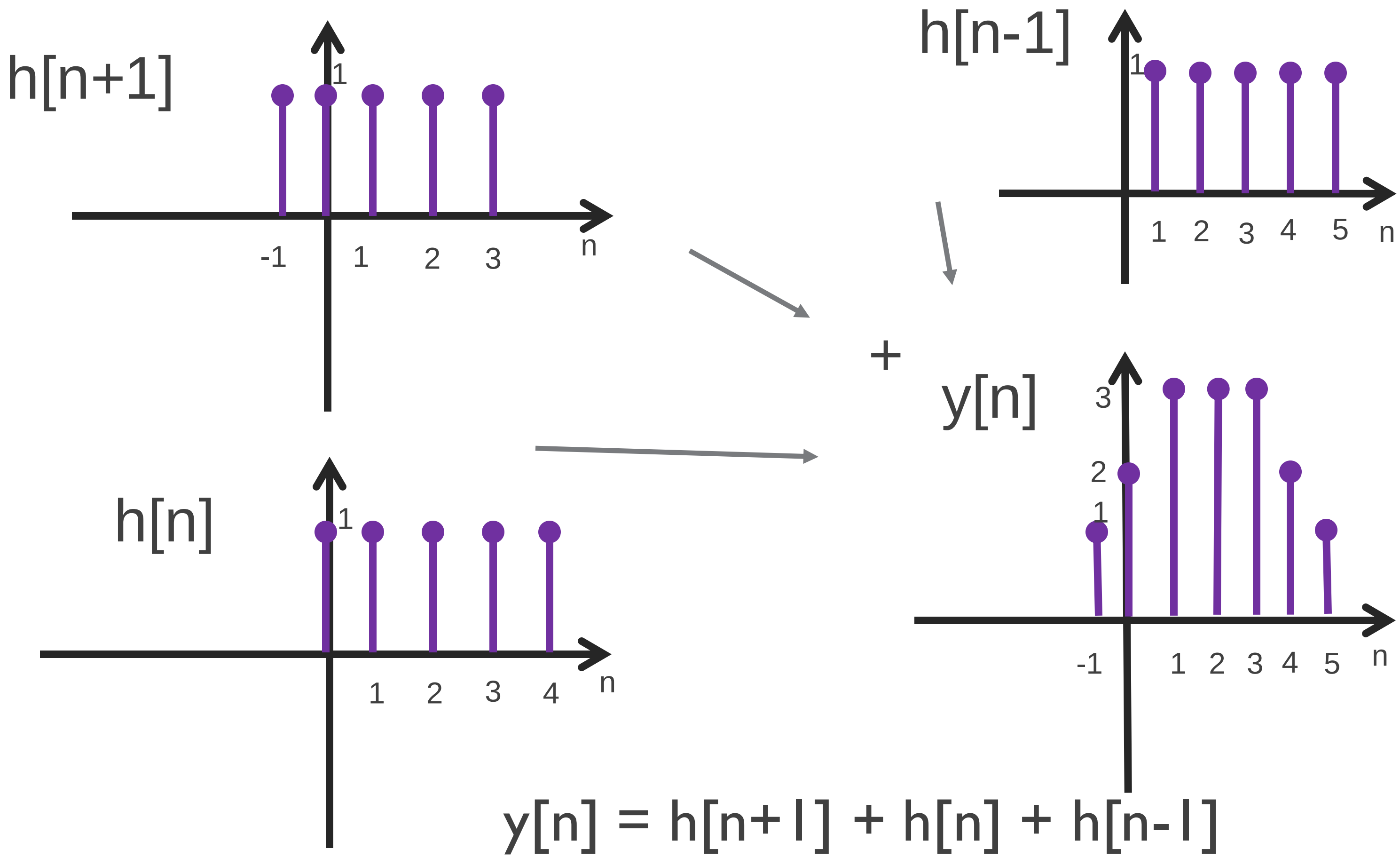
$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} (\delta[k+1] + \delta[k] + \delta[k-1])h[n-k] \\ &= \underbrace{\sum_{k=-\infty}^{\infty} \delta[k+1]h[n-k]}_{h[n+1]} \\ &+ \underbrace{\sum_{k=-\infty}^{\infty} \delta[k]h[n-k]}_{h[n]} \\ &+ \underbrace{\sum_{k=-\infty}^{\infty} \delta[k-1]h[n-k]}_{h[n-1]} \end{aligned}$$

- Brute force method
  - Each input sample triggers the impulse response
  - Thus y[n] = h[n+1] + h[n] + h[n-1]
  - Plot these 3 functions and manually sum them up

This alternative approach works when there are just a few impulses in the convolution. It can be faster at times.

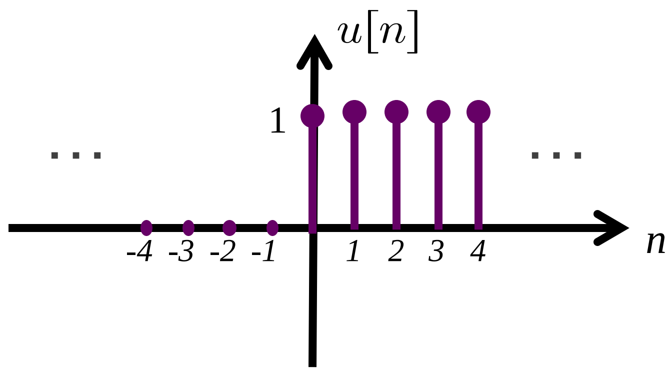
Convolution example 2 Redux² (impulse property)

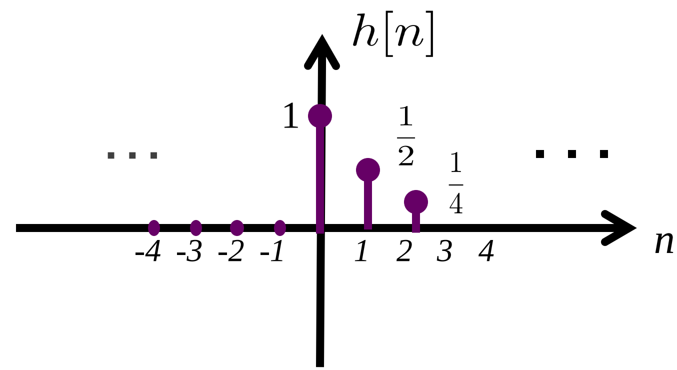


$$y[n] = h[n+1] + h[n] + h[n-1]$$

Convolution example 2 Redux² (impulse property)

$$x[n] = u[n] \qquad h[n] = \left(\frac{1}{2}\right)^n u[n]$$





- Find the output of the system

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

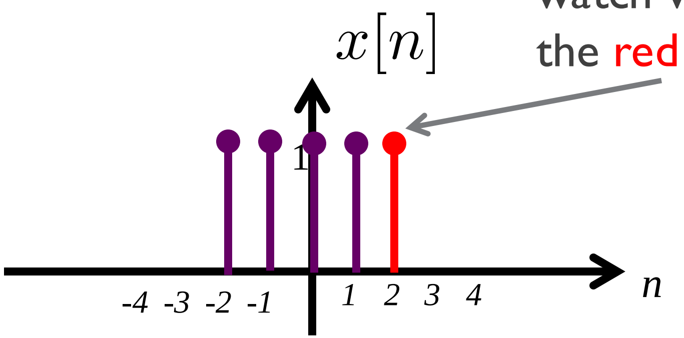
Convolution example 3

This is a comprehensive example that shows how to use the graphical approach to compute the analytical solution. It is quite useful when you have signals of infinite or doubly infinite duration. The idea of **intervals** is important here.

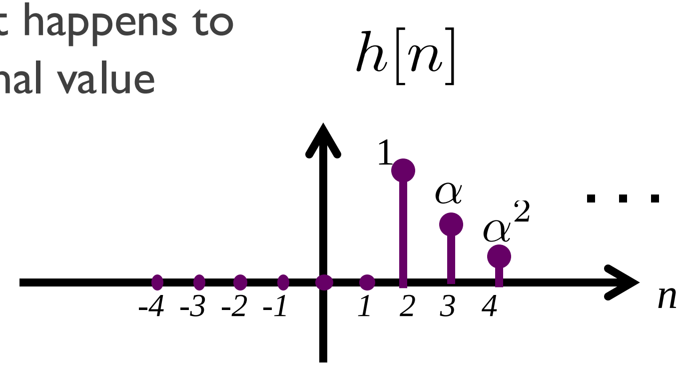
- One finite duration signal with one infinite-duration signal

$$x[n] = \begin{cases} 1 & |n| \le 2 \\ 0 & \text{else} \end{cases}$$

$$h[n] = \square^{n-2}u[n-2] \quad (\text{assume } |\square| < 1)$$



watch what happens to the red signal value



- Find the output of the system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Convolution example 3

- Let's solve it graphically
  - Plot one signal versus k
  - Reverse the second signal and shift it with n
- Note that convolution is commutative

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

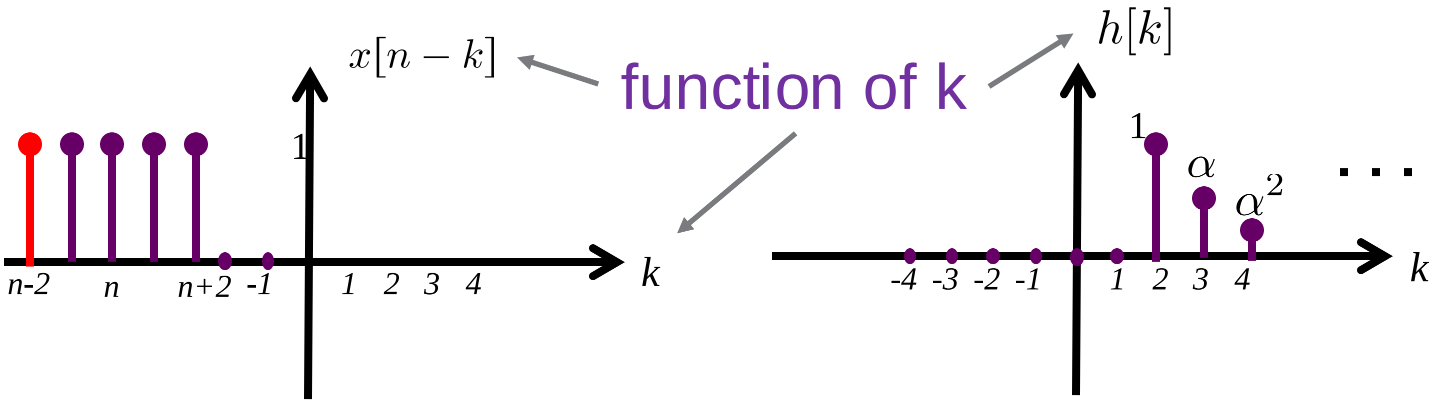
can flip and slide either!

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Which one is more suitable here? → The finite-duration signal is easier to shift, so we will use the second sum

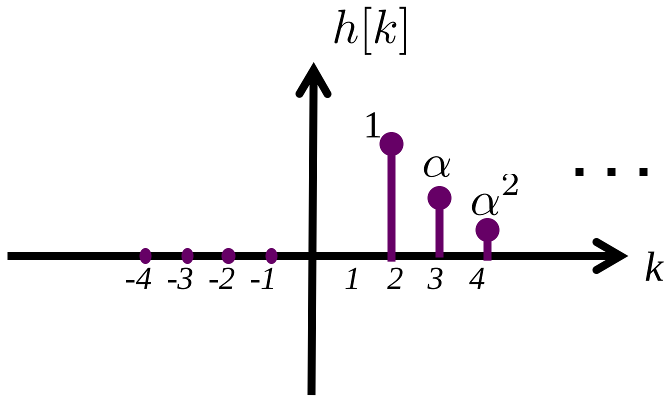
Convolution example 3

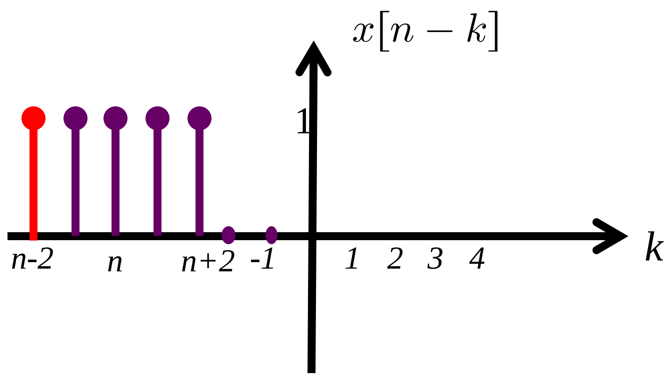
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$



- We have 3 different intervals: $n < 0$ $\quad 0 \leqslant n \leqslant 4$ $\quad n \geqslant 5$
- We will get an expression for the output at each interval

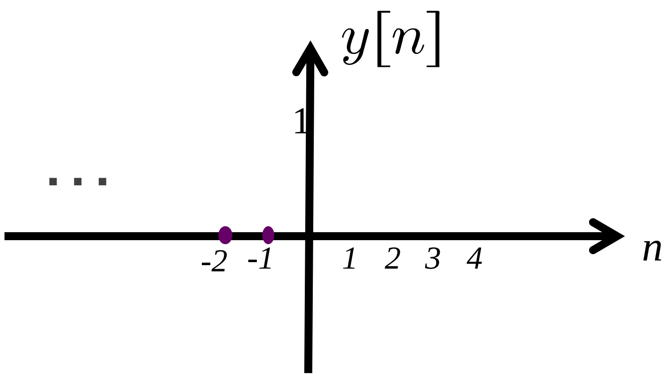
Convolution example 3



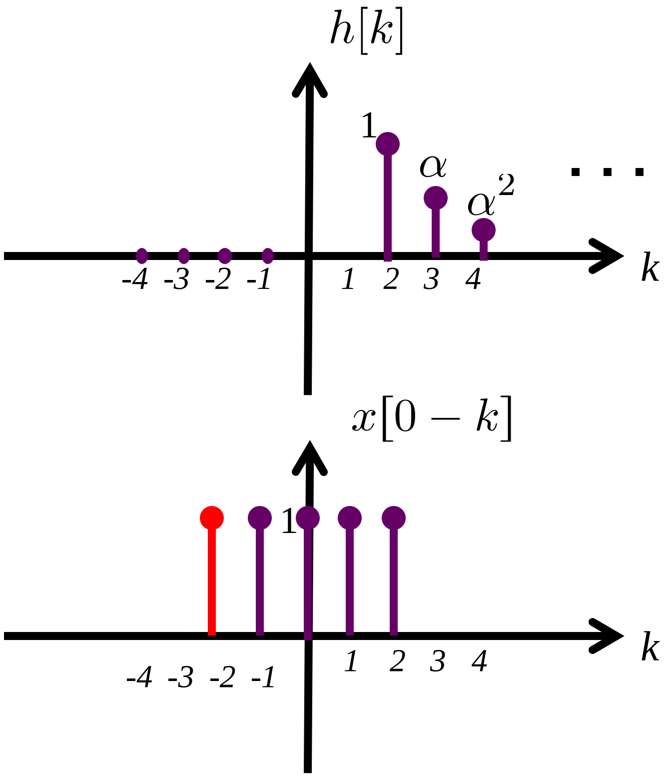


1st interval: $n+2<2$

$$n<0 \longrightarrow y[n]=0$$

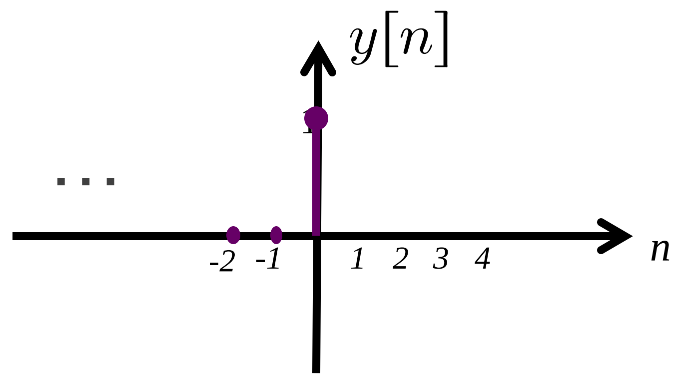


Convolution example 3

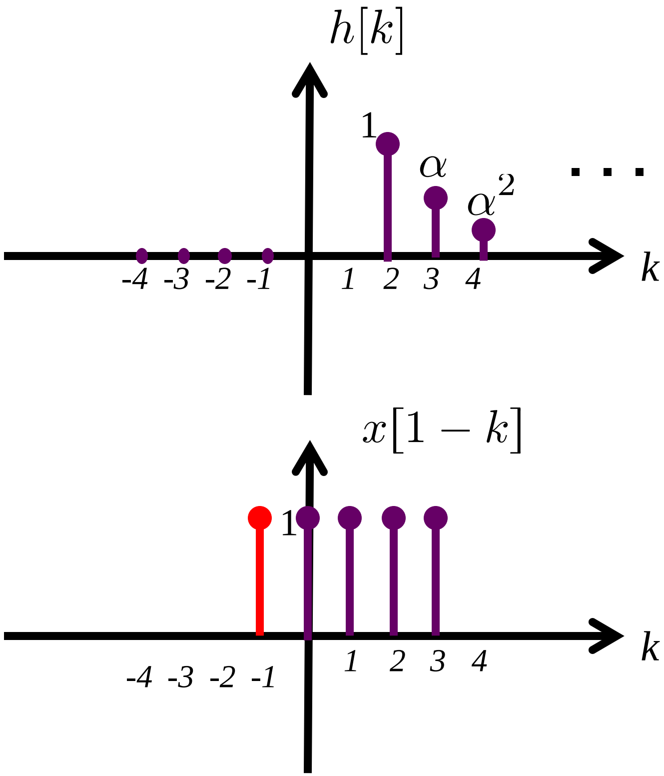


2nd interval: $0 \leqslant n \leqslant 4$

$$y[0] = 1 \cdot 1 = 1$$

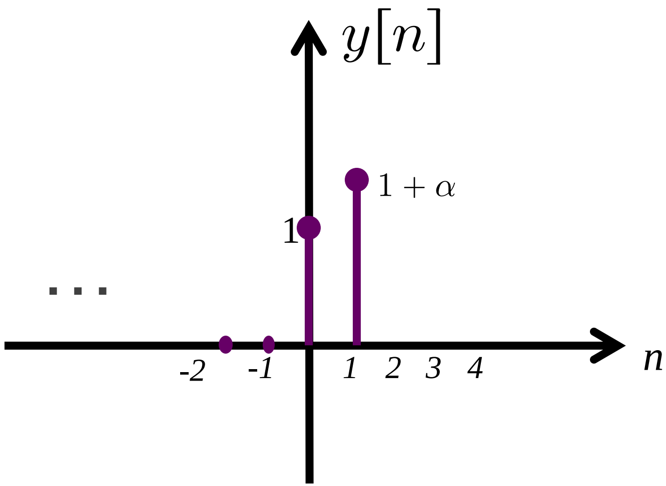


Convolution example 3

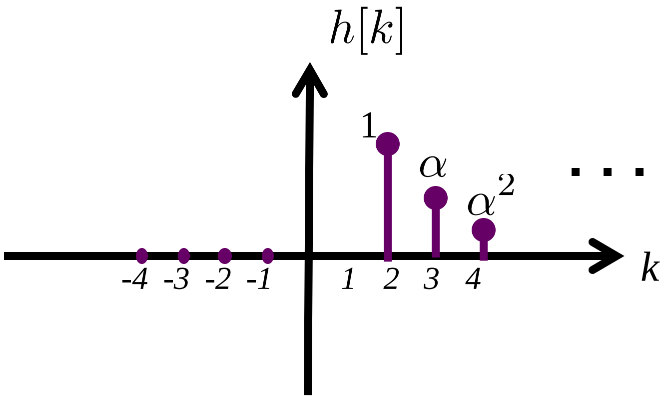


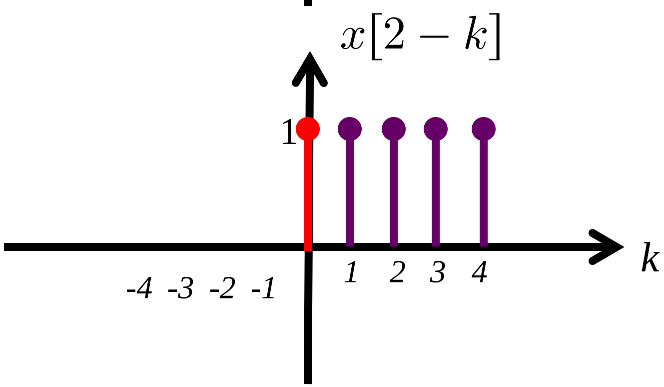
<u>2nd interval</u>: $0 \leqslant n \leqslant 4$

$$y[1] = 1 \cdot 1 + 1 \cdot \alpha = 1 + \alpha$$



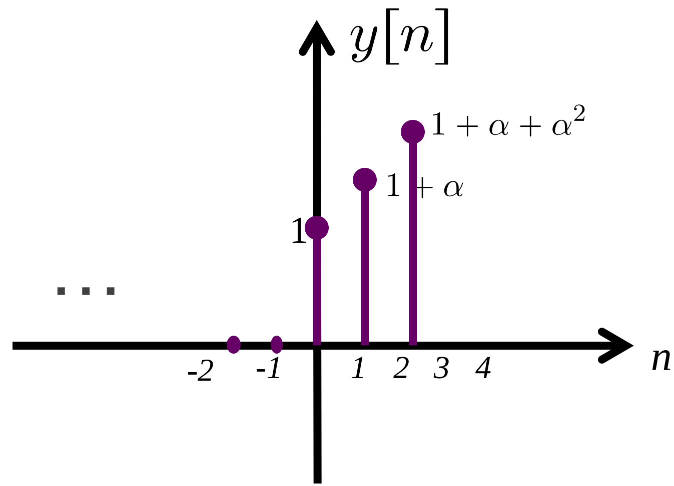
Convolution example 3



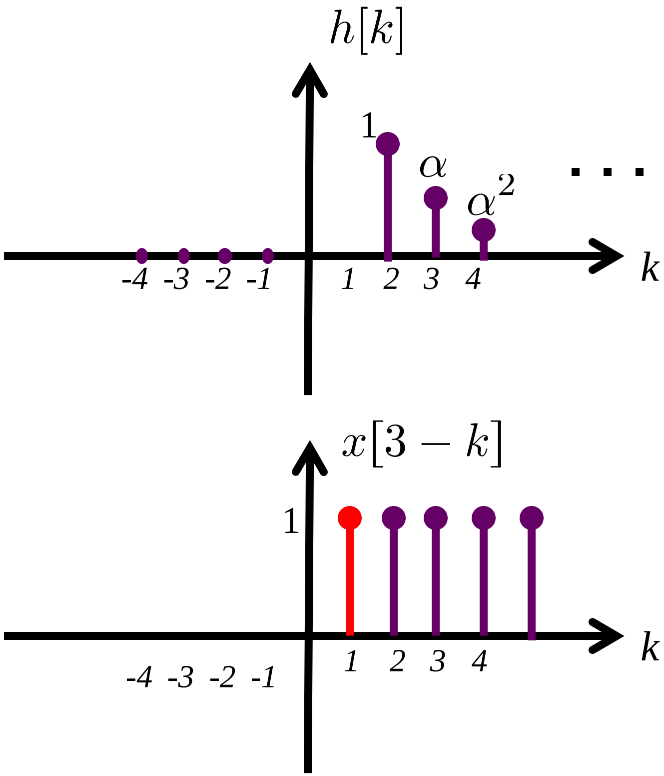


2nd interval: $0 \leqslant n \leqslant 4$

$$y[2] = 1 + \alpha + \alpha^2 = 1 + \alpha + \alpha^2$$

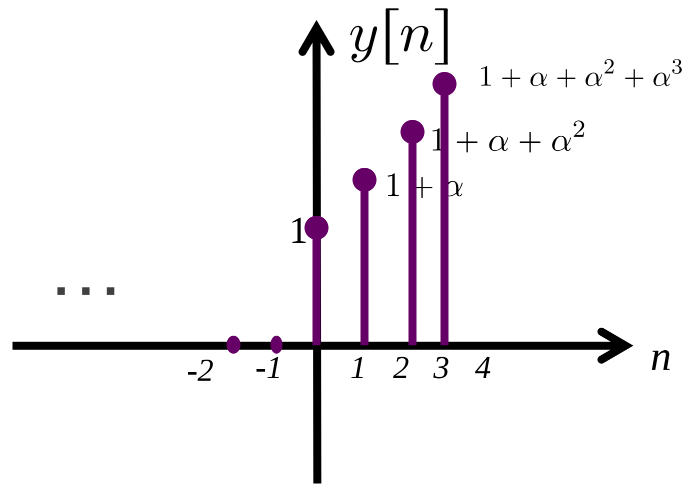


Convolution example 3



2nd interval: $0 \leqslant n \leqslant 4$

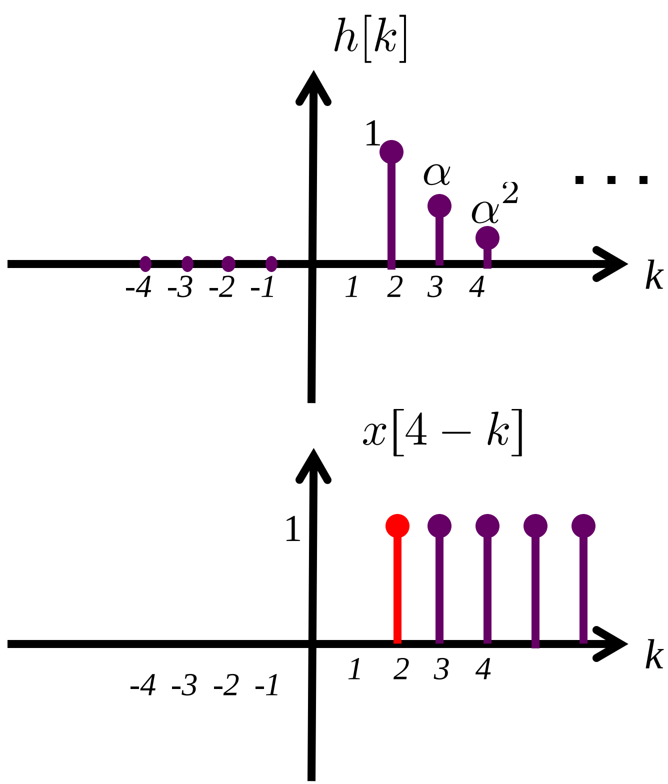
$$y[3] = 1 + \alpha + \alpha^2 + \alpha^3$$

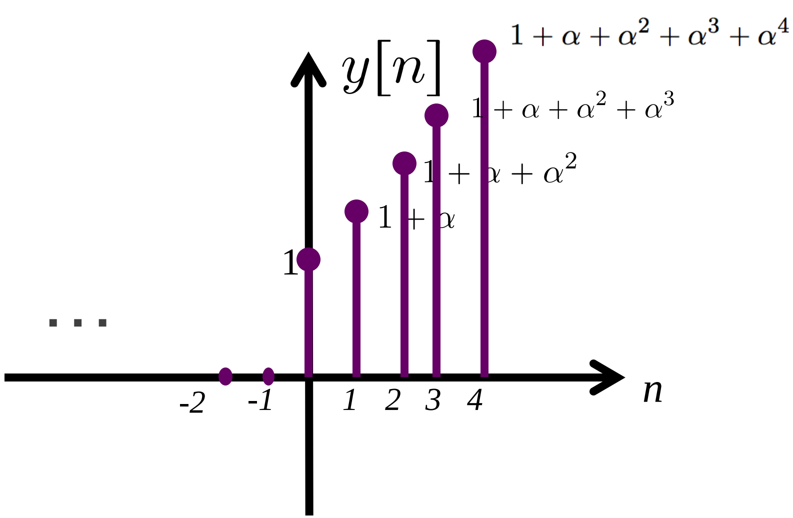


Convolution example 3

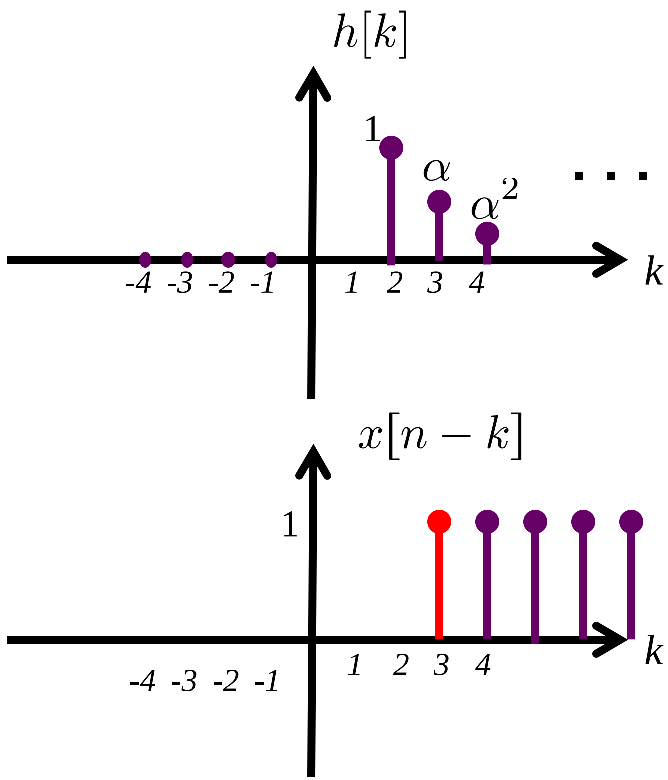
General form for the 2nd interval: $0 \leqslant n \leqslant 4$

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$





Convolution example 3



3rd interval: $n \geqslant 5$

$$y[n] = \sum_{k=n-2}^{n+2} \alpha^{k-2} = \sum_{k=n-4}^{n} \alpha^k$$

$$= \sum_{k=0}^{4} \alpha^{k+n-4} = \sum_{k=0}^{4} \alpha^k \cdot \alpha^{n-4}$$

$$= \alpha^{n-4} \sum_{k=0}^{4} \alpha^k = \alpha^{n-4} \cdot \frac{1-\alpha^5}{1-\alpha}$$

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Convolution example 3

Complete solution

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$= \begin{cases} 0 & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha} & n \in [0,4] \\ \alpha^{n-4} \cdot \frac{1-\alpha^5}{1-\alpha} & n \geqslant 5 \end{cases}$$

Convolution example 4

This example shows how to compute the convolution directly from the formula using properties of the unit step function. Normally it is easier though to do this leveraging the graphical method.

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[k] \left(\frac{1}{2}\right)^{n-k} u[n-k]$$

$$= u[n] \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{-k}$$

$$= u[n] \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} 2^{k}$$

u[k] means sum starts at k=0.

u[n-k] vs. k is a unit step that stops at k=n

Convolution example 4

- Using the finite sum formula (be sure you know this)

$$\sum_{k=0}^{n} 2^k = \begin{cases} \frac{1-2^{n+1}}{1-2} & n \geqslant 0 \\ 0 & n < 0 \end{cases}$$

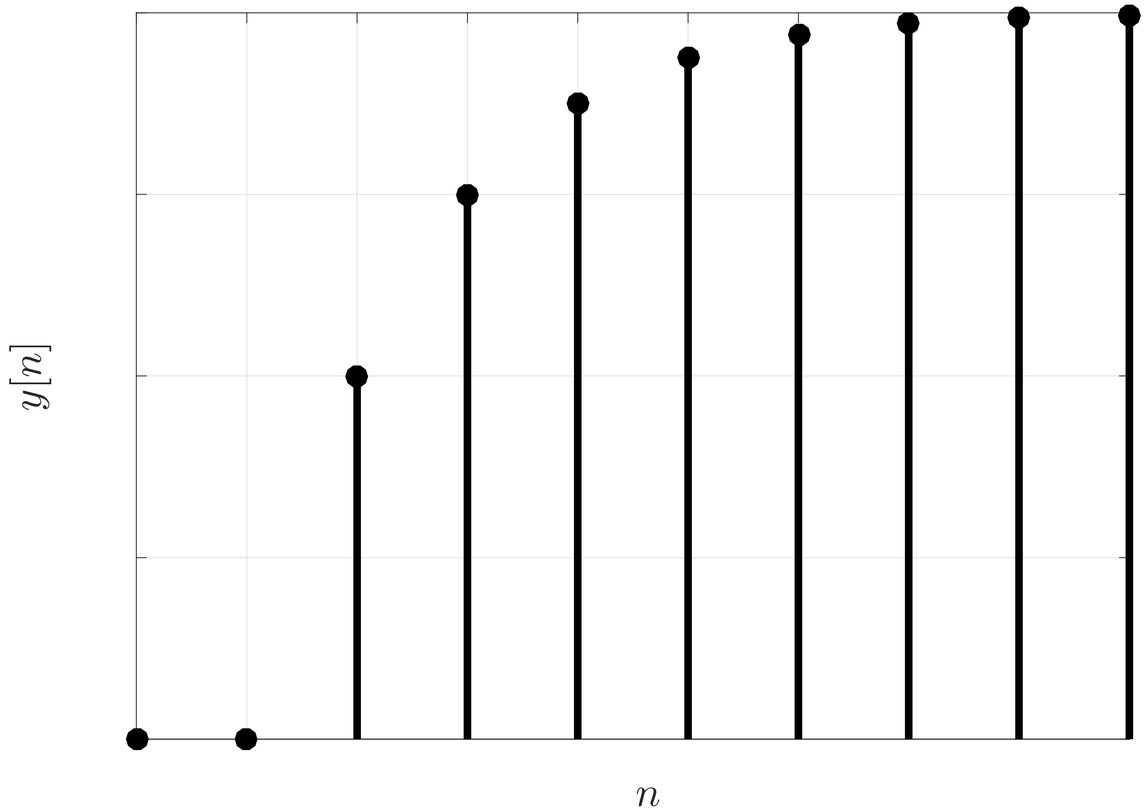
$$= (2^{n+1} - 1)u[n]$$

- The output of the system is then

$$y[n] = \left(\frac{1}{2}\right)^n (2^{n+1} - 1)u[n]$$

$$= \left[2 - \left(\frac{1}{2}\right)^n\right] u[n]$$

Solution y[n] for convolution example 4



Convolution example 5

Try to work this example yourself, at least to set up the intervals, and then use calculus to find the results. Example with mathematica is included at the end.

- Consider the following problem

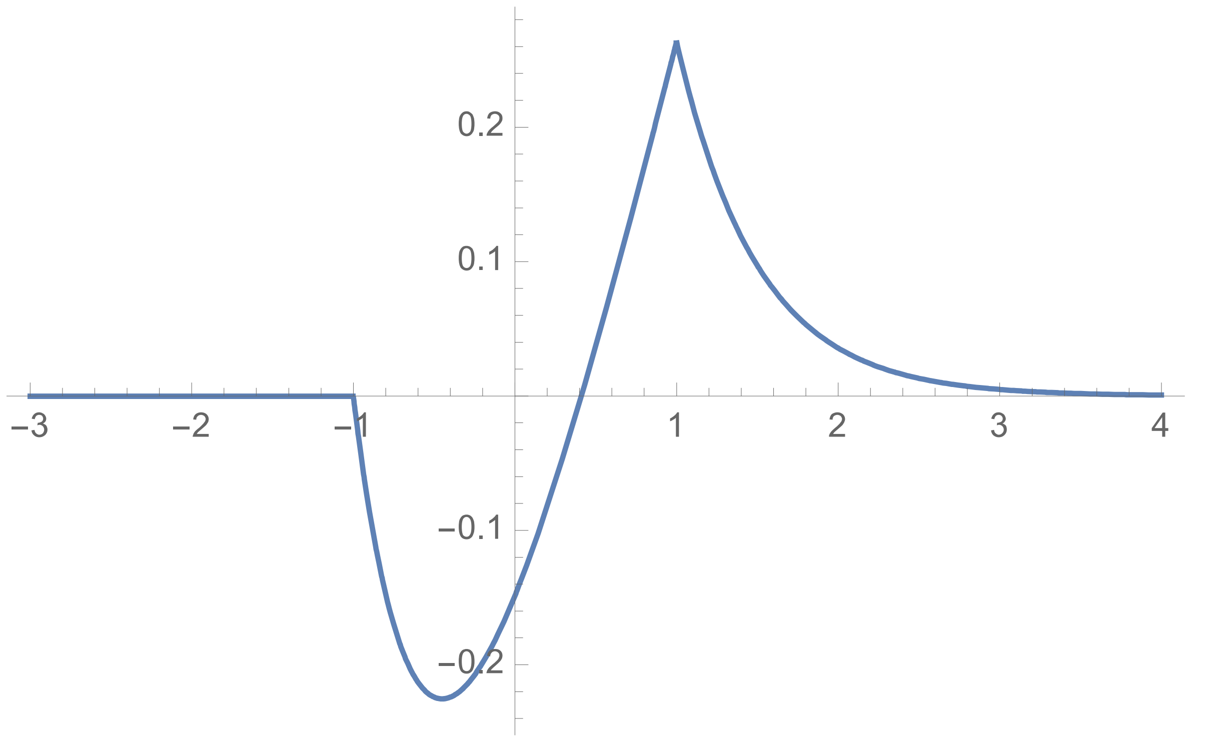
$$h(t) = e^{-2t}u(t)$$
$$x(t) = t\,\mathrm{rect}(t/2)$$
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

- Recall integration by parts

$$\int u dv = uv - \int v du$$

Solution

$$\begin{cases} \frac{1}{4} e^{-2(t+1)} \left(3 + e^{4}\right) & t > 1 \\ \frac{1}{4} \left(2t + 3 e^{-2(t+1)} - 1\right) & -1 < t \leq 1 \end{cases}$$



For reference, computed in Mathematica

```
In[9]:= x[t_] := t UnitBox[t/2]

In[10]:= h[t_] := Exp[-2 t] UnitStep[t]

In[17]:= Convolve[x[tau], h[tau], tau, t]
```

Out[17]=
$$\begin{cases} \frac{1}{4} e^{-2(1+t)} \left(3 + e^4\right) & t > 1 \\ \frac{1}{4} \left(-1 + 3 e^{-2(1+t)} + 2 t\right) & -1 < t \le 1 \\ 0 & \text{True} \end{cases}$$

In[18]:=
$$y[t\_] := \begin{cases} \frac{1}{4} e^{-2(1+t)} \left(3 + e^4\right) & t > 1 \\ \frac{1}{4} \left(-1 + 3 e^{-2(1+t)} + 2 t\right) & -1 < t \le 1 \\ 0 & \text{True} \end{cases}$$

Convolution animation examples

- It is important to be able to visualize convolution in order to build intuition on how systems process signals

- Some examples you can use include:
  - Discrete convolution Matlab demos:
    - http://users.ece.gatech.edu/mcclella/matlabGUIs/
    - http://dspfirst.gatech.edu/

Convolution summary

- The input and output of an LTI system are related through the convolution sum

$$y[n] = \sum_{k} x[k]h[n-k]$$

- You should practice performing convolutions
  - Graphically – key is to plot signals with respect to k
  - Analytically – manipulate the sum, account for special cases
  - Using MATLAB – leverage the provided example, good for finite length signals, but symbolic approach needed when infinite

Continuous-time convolution

Learning objectives

- Define the impulse response of a continuous-time system
- Determine the output of an LTI system using the convolution
- Compute the continuous-time convolution between two signals

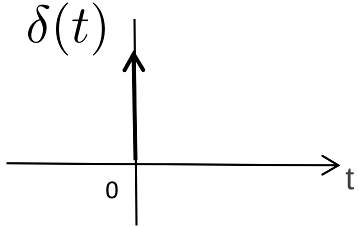
System impulse response

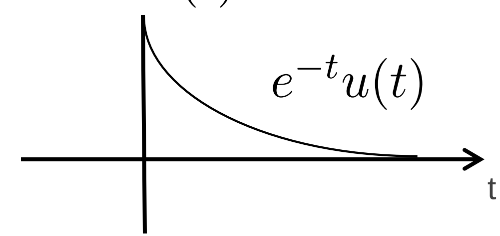
$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t)$$

- Consider the input signal $\delta(t)$
- The output corresponding to this input is the **impulse response**
  - The resulting sequence is usually called $h(t)$
- All systems have an impulse response, but:
  - The impulse response is special <u>only</u> for LTI systems
  - Focus on LTI systems throughout this course

Example (typical "first-order" differential system)

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t)$$





delta function input

Exponential function out

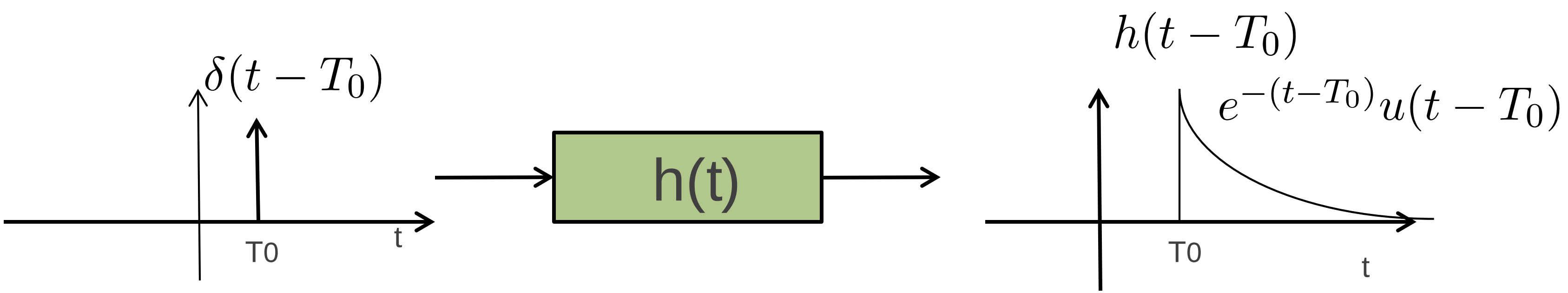
Time invariance

- Because of time invariance

$$\delta(t-T_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t-T_0)$$

When written like this, implies an LTI system with impulse response h(t).

- Example



Shifts in the input shift the output

Linearity

- Because of the homogenous property

$$\alpha\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow \alpha h(t)$$

Scaling the input scales the output

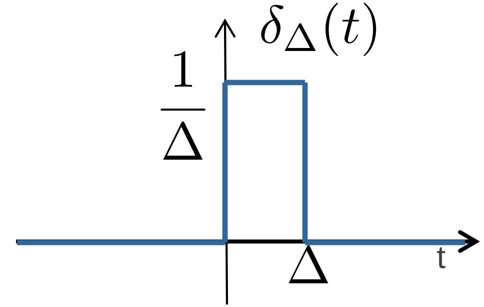
- Because of the additive property

$$\alpha\delta(t) + \beta\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow \alpha h(t) + \beta h(t - T_0)$$

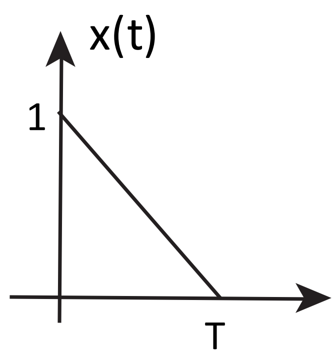
A sum of inputs leads to a sum of outputs

Stair step approximation of an input signal

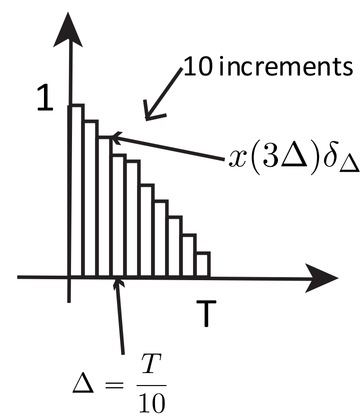
- Consider the rectangle function



- Suppose that we approximate a signal using this function



$\simeq$



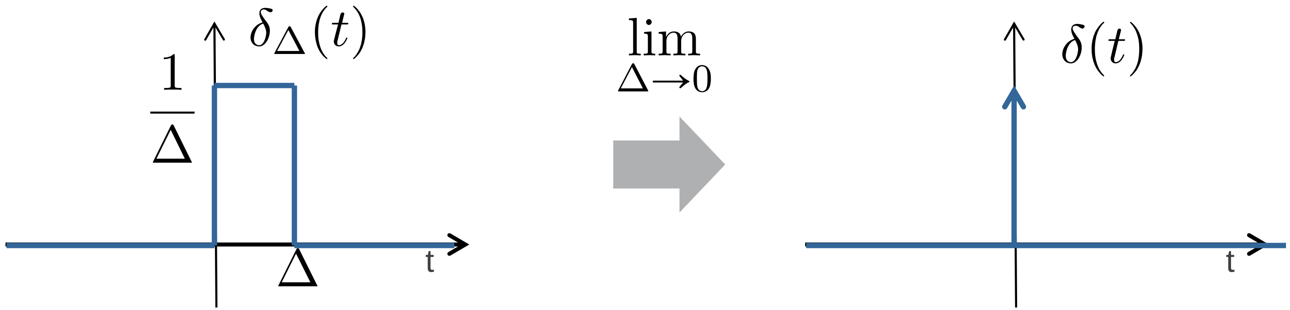
$$x(3\Delta)\delta_\Delta(t-3\Delta) = x\left(\frac{3T}{10}\right)\delta_{\frac{T}{10}}\left(t-\frac{3T}{10}\right)$$

Stair step approximation of an input signal

- Write the stair case approximation of a function as

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

- Recall from the derivation of the delta function

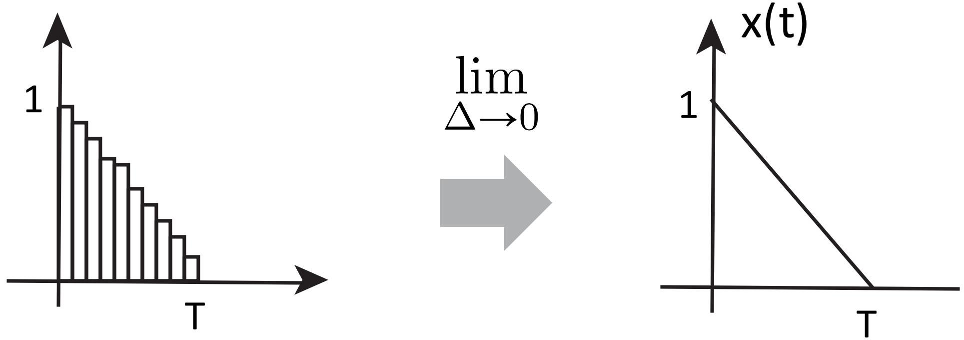


Stair step approximation of an input signal

- Now taking the limit

$$\lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$
$$= x(t)$$

Easy to see via sampling property that this must be true



Any signal can be written as an integral of itself with shifted deltas

Back to the LTI system

- What if we put in

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{h(t)} \longrightarrow x(\tau)h(t-\tau)$$

- How about

$$x(\tau_1)\delta(t-\tau_1) + x(\tau_2)\delta(t-\tau_2) \longrightarrow \boxed{h(t)} \longrightarrow$$

$$x(\tau_1)h(t-\tau_1) + x(\tau_2)h(t-\tau_2)$$

Uncovering the convolution

- Now putting x(t) in the integral format

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{h(t)} \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Thus, the input and output of an LTI system are related via the convolution integral:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

Basic convolution properties

- Commutative

$$y(t) = x(t) * h(t) \qquad = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) \qquad = \int h(\tau)x(t-\tau)d\tau$$

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

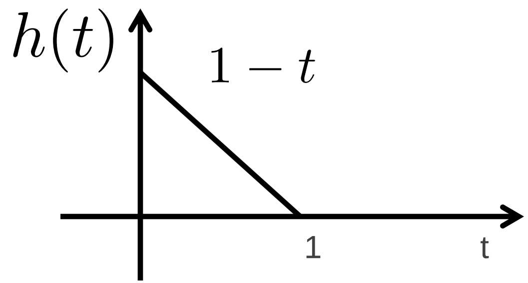
Same properties hold in DT case as well

CT convolution example #1

This is a typical example using two finite length signals. Rectangles and triangle functions are common in examples / HW as they give results that are easy to integrate. It is important here to understand the different **intervals** in the convolutions.

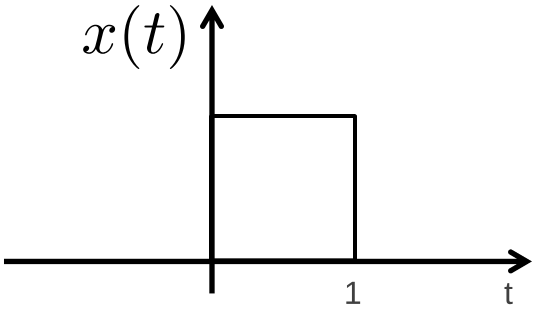
- Find the output of a system with impulse response

$$h(t) = (1-t)[u(t) - u(t-1)]$$



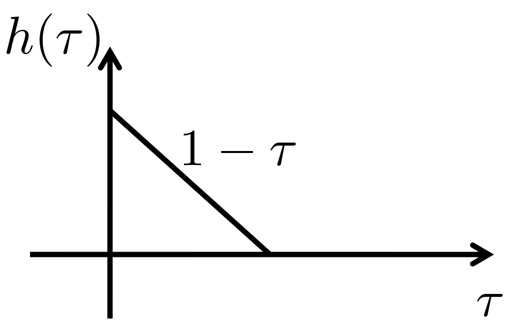
for the input

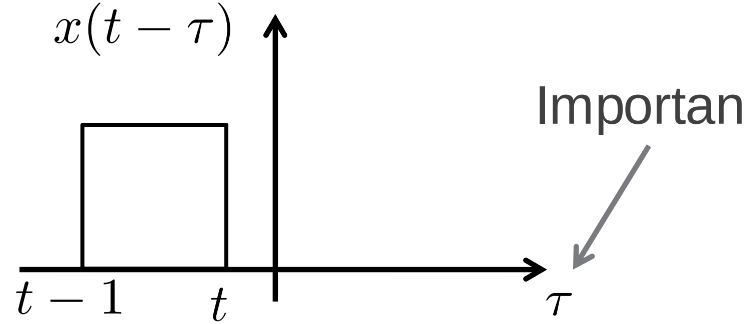
$$x(t) = u(t) - u(t-1)$$



CT convolution example #1: Graphical solution

- Plot one signal versus τ
- Reverse the second signal and shift it by t
  - Here, plot it to the left of $h(\tau)$
  - So plotted t has a negative value, usually



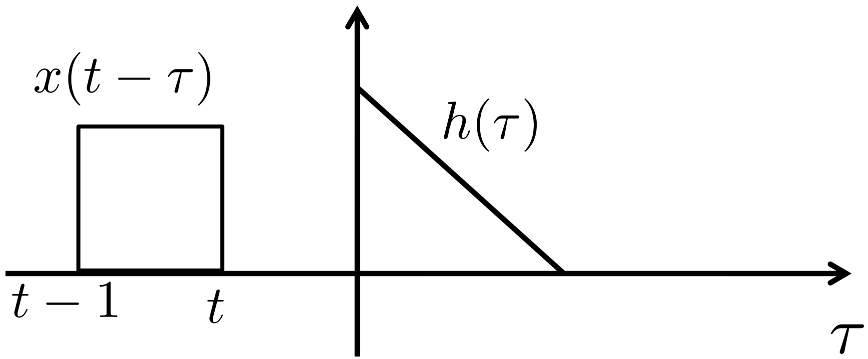


- There are 4 intervals (why?):

$t < 0 \qquad 0 \leqslant t \leqslant 1 \qquad 1 \leqslant t \leqslant 2 \qquad 2 < t$

CT convolution example #1: First interval

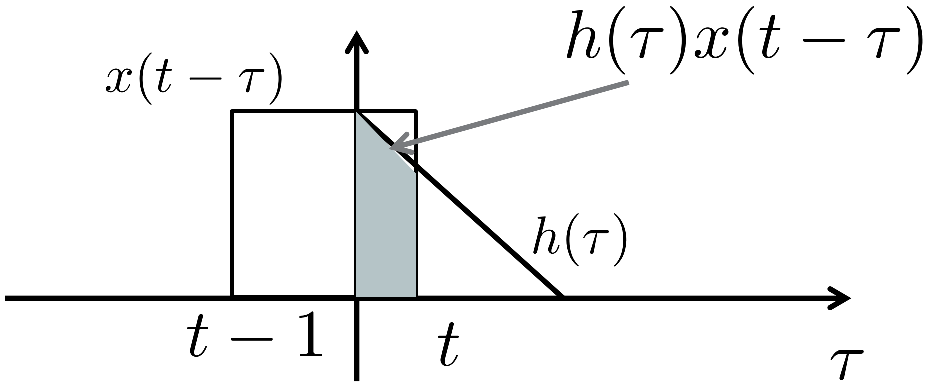
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



- First interval: there is no overlap!

$$t < 0 \qquad h(\tau)x(t-\tau) = 0 \quad \Rightarrow \quad y(t) = 0$$

CT convolution example #1: Second interval



- Second interval $0 \leqslant t \leqslant 1$

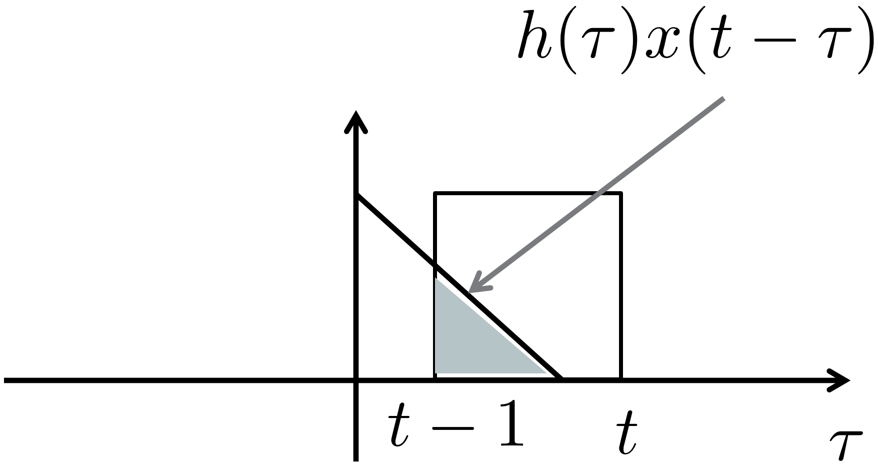
$$y(t) = \int_0^t (1-\tau)d\tau = \tau - \frac{\tau^2}{2}\Big]_0^t$$

$$= t - \frac{t^2}{2}$$

CT convolution example #1: Third interval

- Third interval $1 \leqslant t \leqslant 2$

$$
\begin{aligned}
y(t) &= \int_{t-1}^{1} (1-\tau)d\tau \\
&= \tau - \frac{\tau^2}{2}\Big]_{t-1}^{1} \\
&= 1 - \frac{1}{2} - \left(t - 1 - \frac{(t-1)^2}{2}\right) \\
&= \frac{t^2}{2} - 2t + 2
\end{aligned}
$$

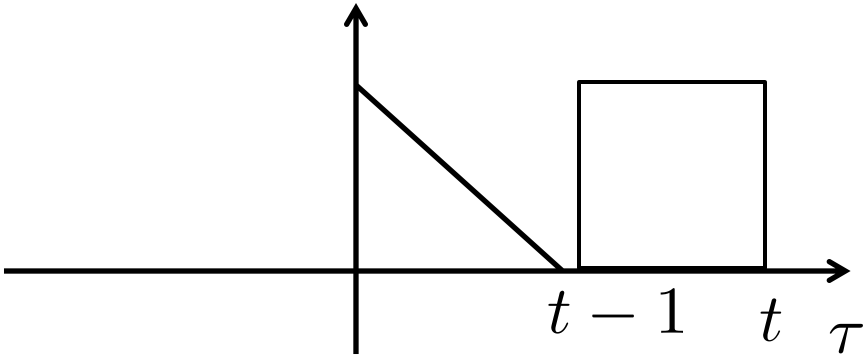


Integrating the same function, but with different integration limits

CT convolution example #1: Fourth and final interval

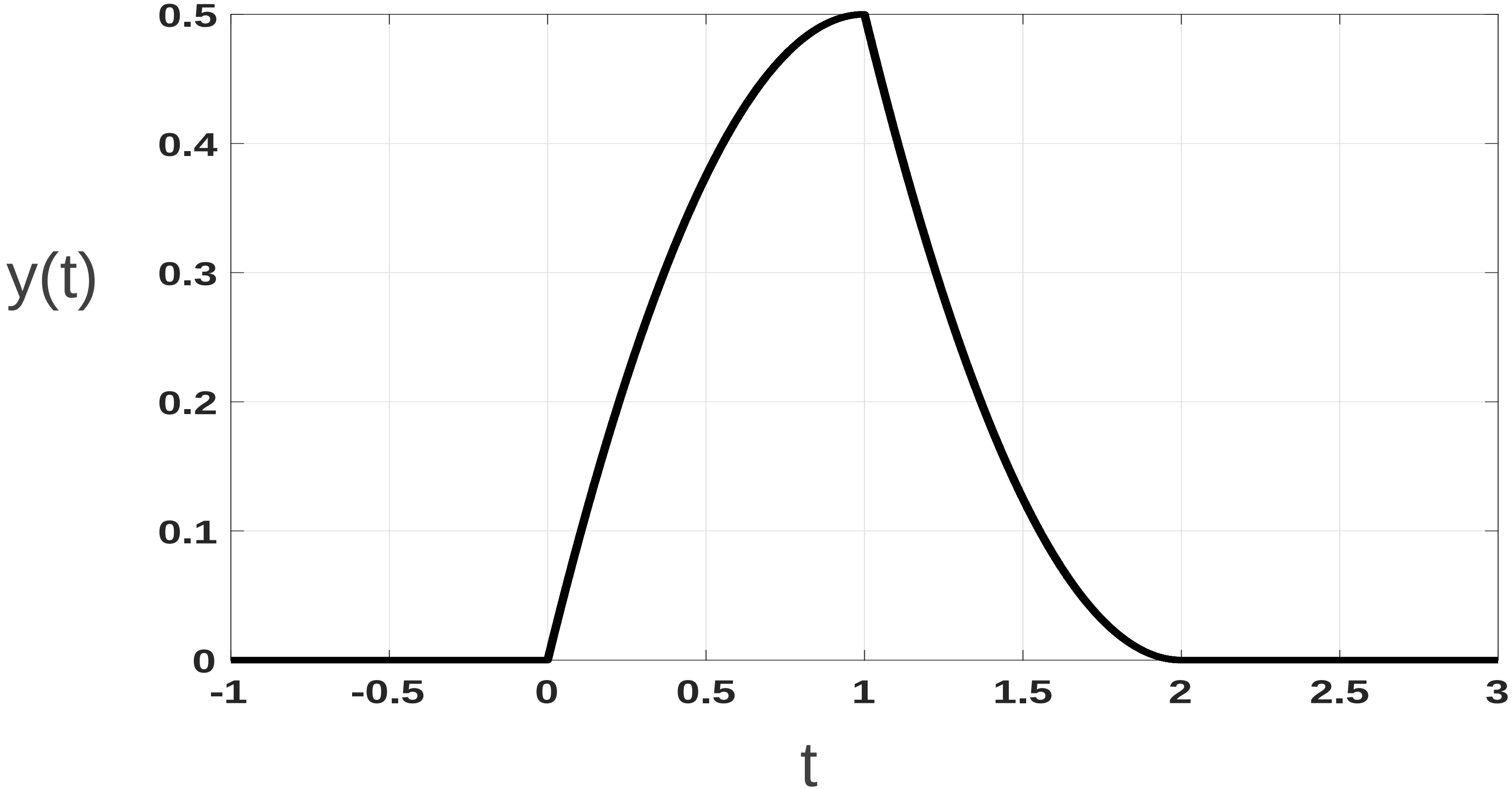
- Fourth interval $2 < t$

$$y(t) = 0$$

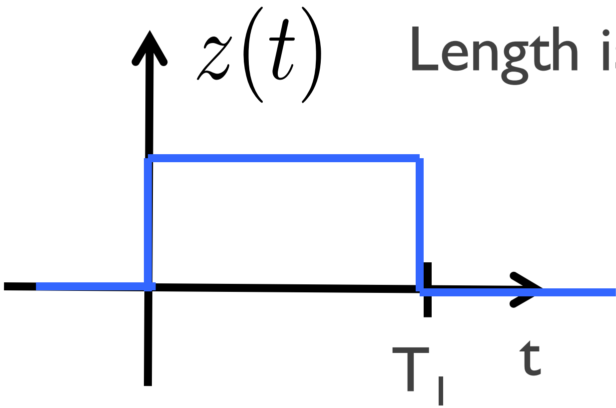


- Sanity checks
  - Check at $t=1$, the output should be the same for intervals 2 & 3 in order for it to be continuous
  - Same for t = 0 and t = 2 (should be zero there)
  - Duration of output should be $T_1 + T_2 = 1+1 = 2$

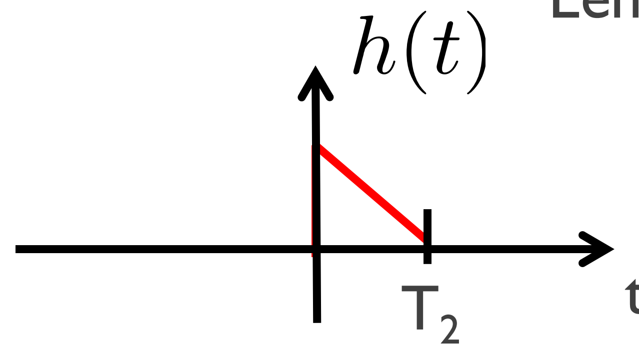
CT convolution example #1: Plot of y(t)



Length / duration of a convolution in continuous time

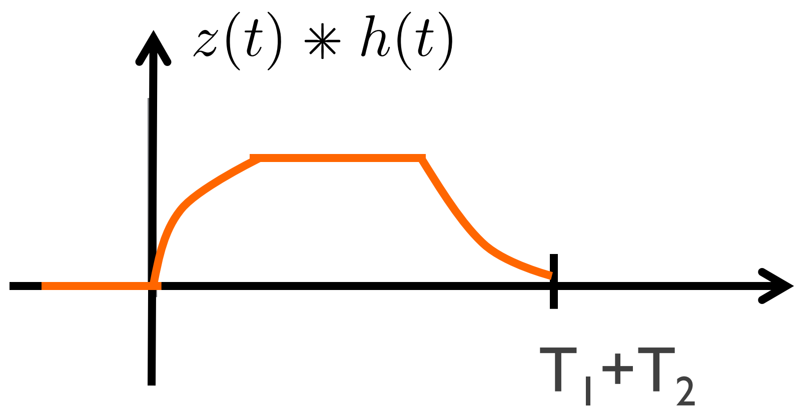


Length is T_1

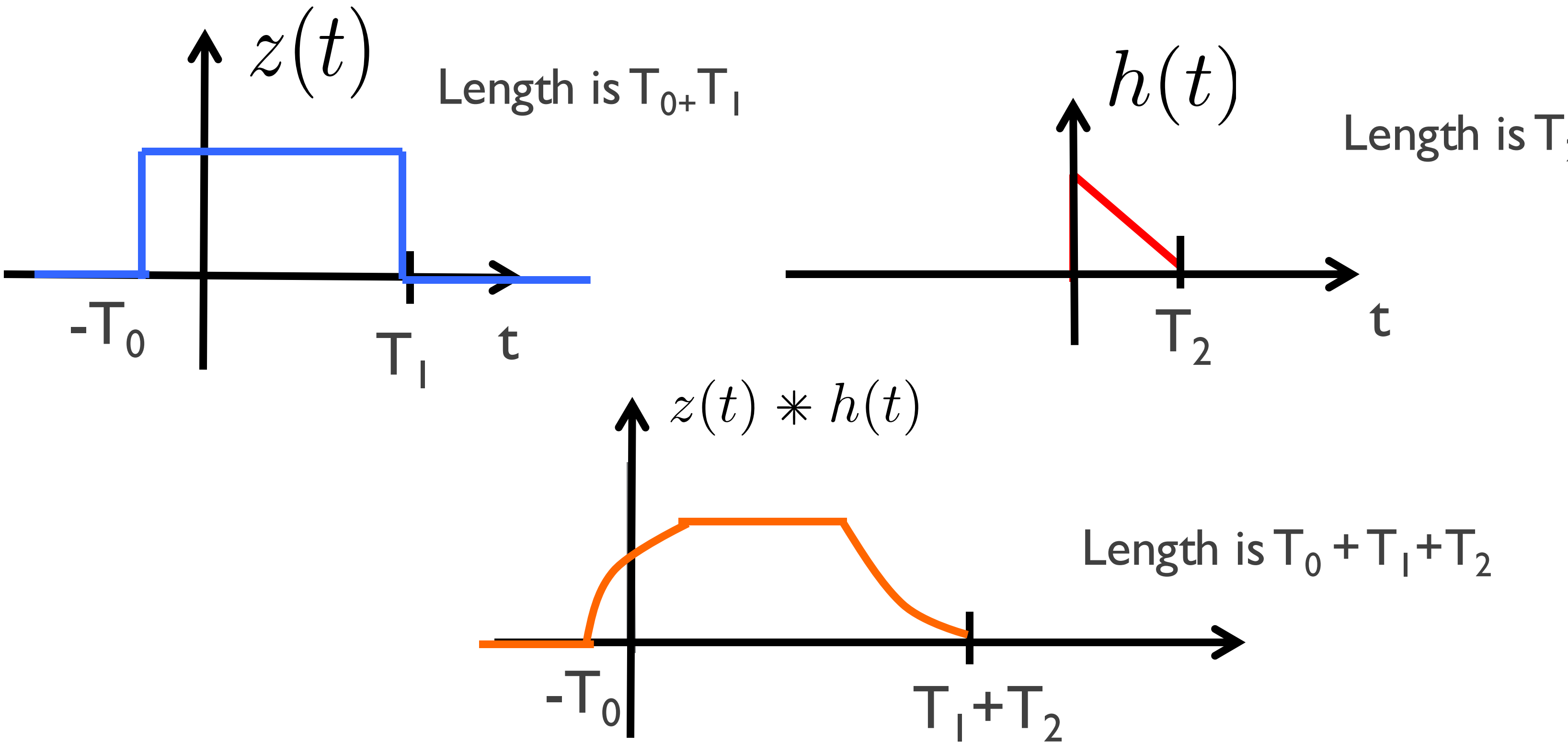


Length is T_2

Length is T_1+T_2



Length / duration of a convolution in continuous time



Animation example

http://www.cse.yorku.ca/~asif/spc/ConvolutionIntegral_Final3.swf

CT convolution example #2

This is an example with a simple input that consists of a few delta functions. The convolution is easy to compute using the LTI property of convolution and the definition of impulse response.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Determine and sketch the convolution of the following input

$$x(t) = \delta(t) + 2\delta(t-1)$$

and system with impulse response

$$h(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1, \\ 2-t, & 1 < t \leqslant 2, \\ 0, & \text{elsewere} \end{cases}$$

CT convolution example #2: Solution approach

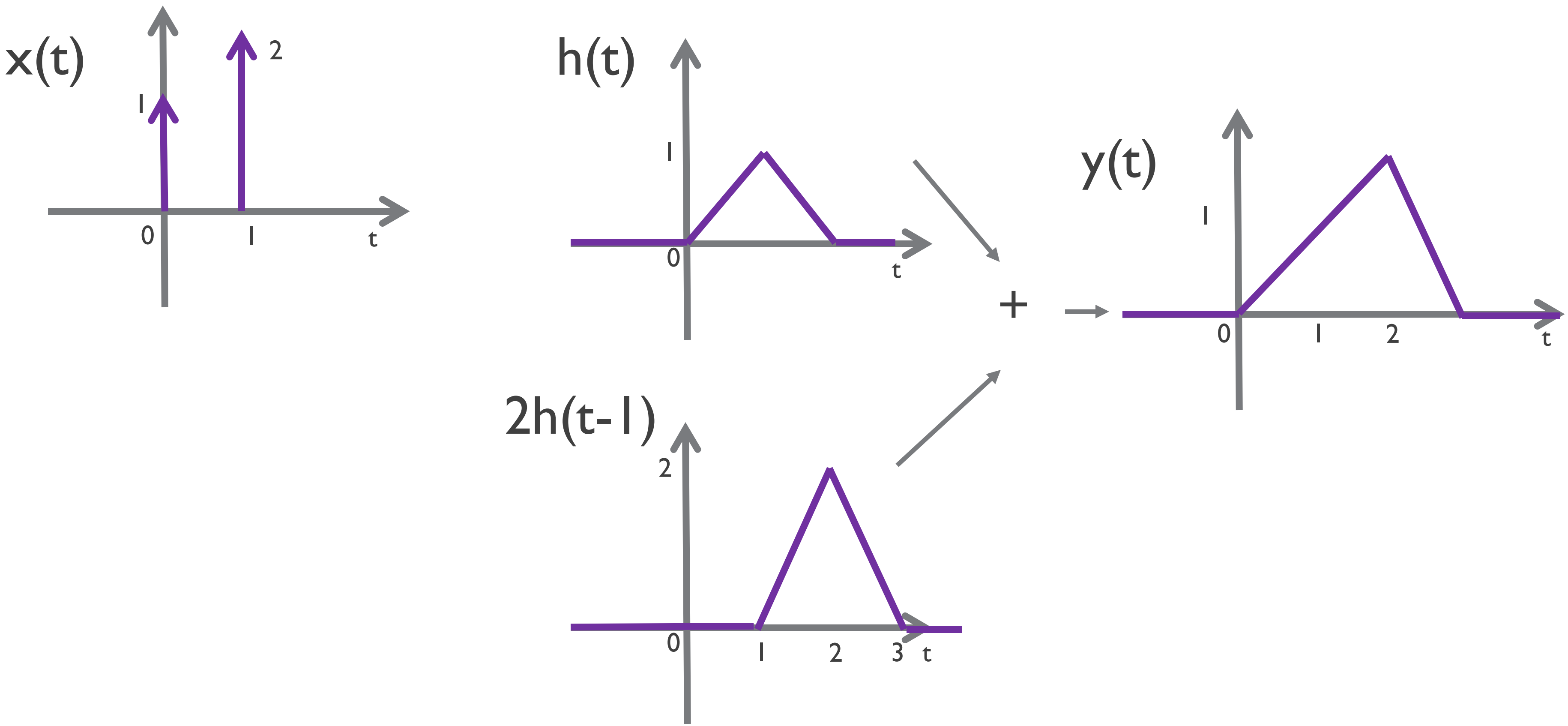
- Recall the property

$$\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t - T_0)$$

- Therefore

$$x(t) = \delta(t) + 2\delta(t-1) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) + 2h(t-1)$$

CT convolution example #2: Solution sketch

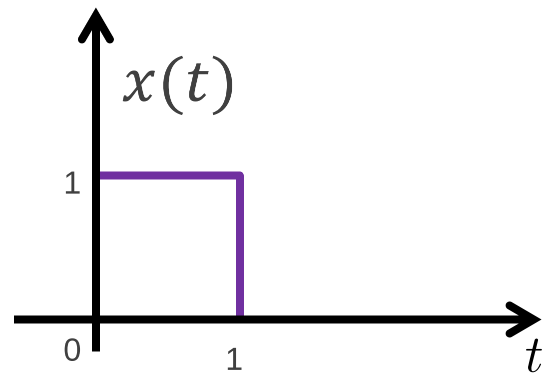


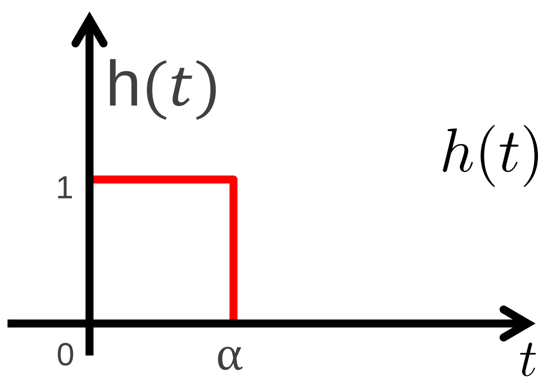
CT convolution example #3

This is an important example involving the convolution of two rectangles of different widths. The solution will be different depending on the widths of the rectangles. Notice what happens in each interval.

- Determine and sketch $y(t) = x(t) * h(t)$, where

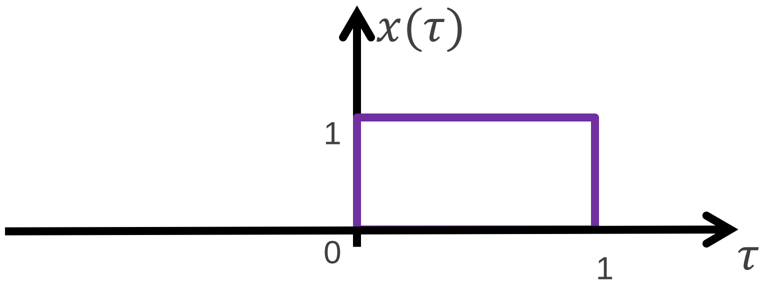
$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

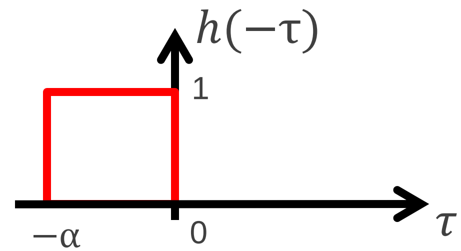




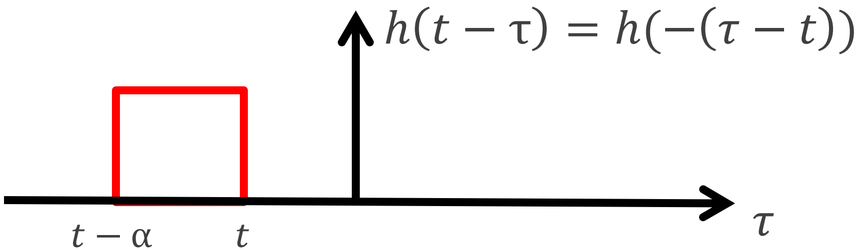
$$h(t) = x(t/\alpha) \qquad 0 < \alpha \leq 1$$

CT convolution example #3: Solution



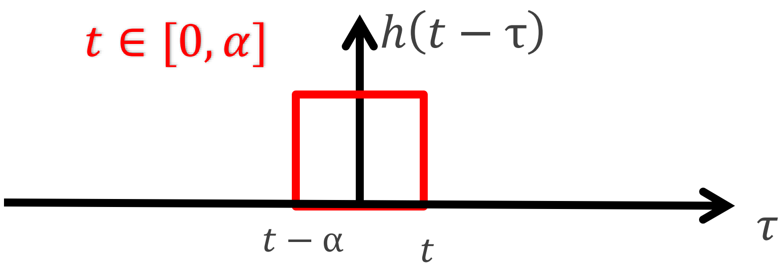


$t < 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0 \quad t < 0$$

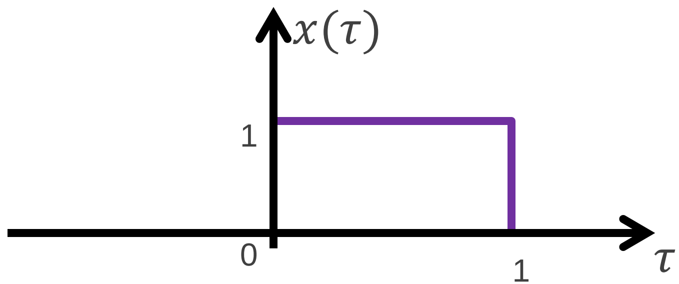
$t \in [0, \alpha]$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} 1 d\tau \quad t \in [0, \alpha]$$

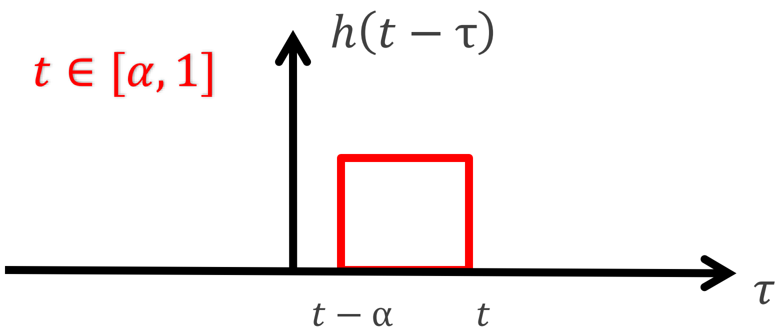
$$= t \quad t \in [0, \alpha]$$

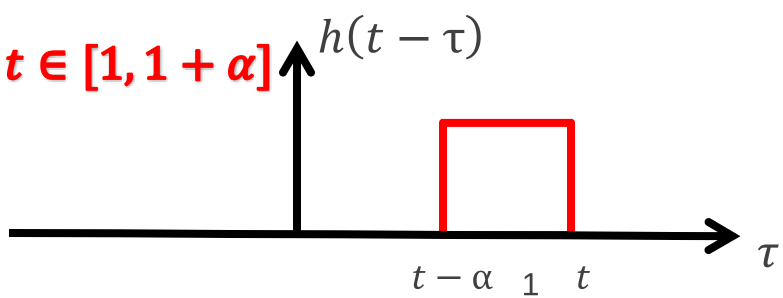
CT convolution example #3: Solution



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{t} 1 d\tau \quad t \in [\alpha, 1]$$

$$= \alpha \quad t \in [\alpha, 1]$$

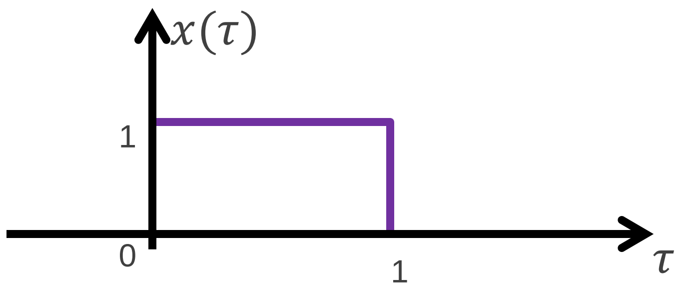




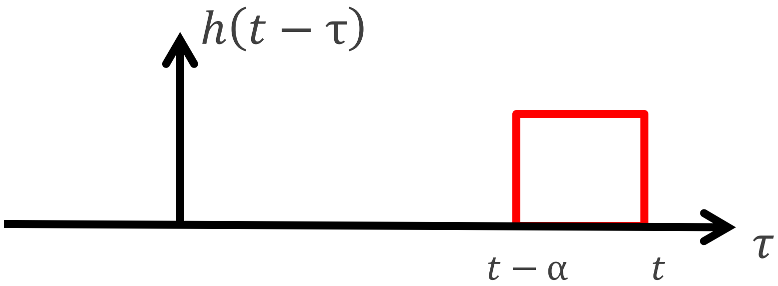
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{1} 1 d\tau \quad t \in [1, 1+\alpha]$$

$$= 1-(t-\alpha) \quad t \in [1, 1+\alpha]$$

CT convolution example #3: Solution



$t \in [1+\alpha, \infty]$

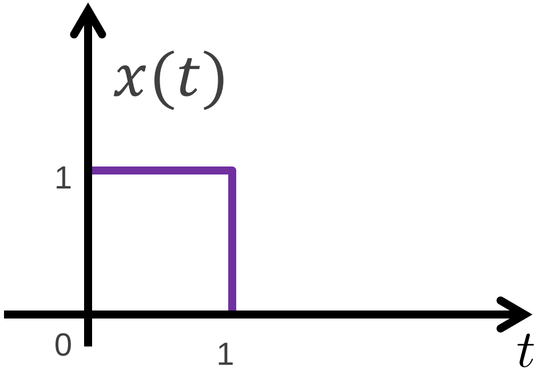


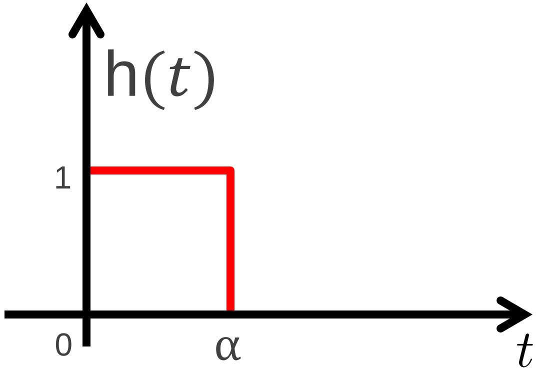
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{1}^{t} 0 d\tau \quad t \in [\alpha+1, \infty]$$

$$= 0 \quad t \in [\alpha+1, \infty]$$

CT convolution example #3: Solution (summary)

$$y(t) = x(t) * h(t)$$





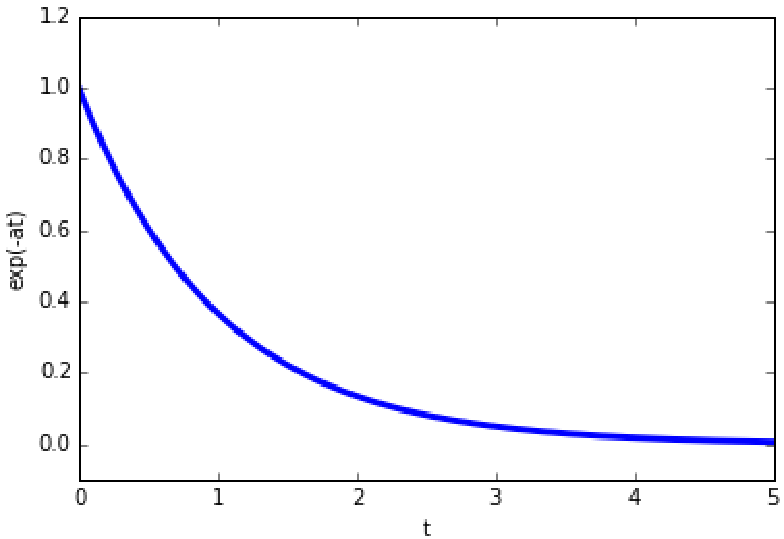
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} t < 0 & 0 \\ t \in [0,\alpha] & t \\ t \in [\alpha, 1] & \alpha \\ t \in [1, 1+\alpha] & 1-(t-\alpha) \\ t > 1+\alpha & 0 \end{cases}$$

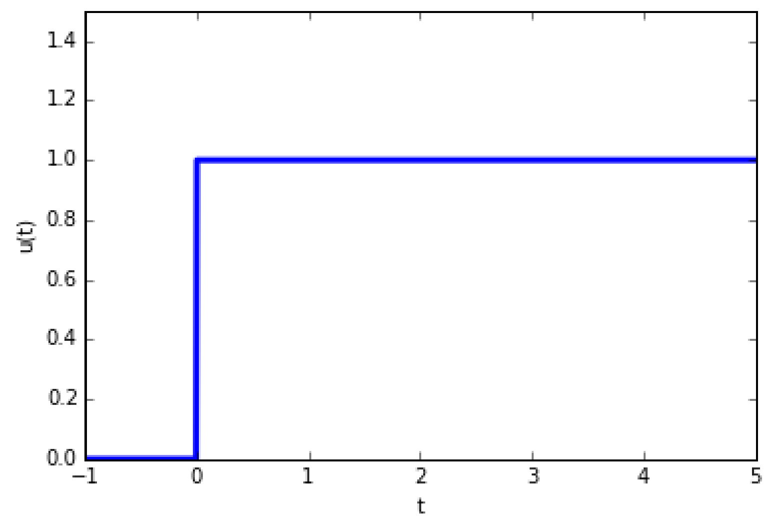
CT convolution example #4

This is an important example of convolution to causal signals together. It will be solved by using the graphical approach to help set up the integrals.

- Determine and sketch $y(t) = x(t) * h(t)$, where

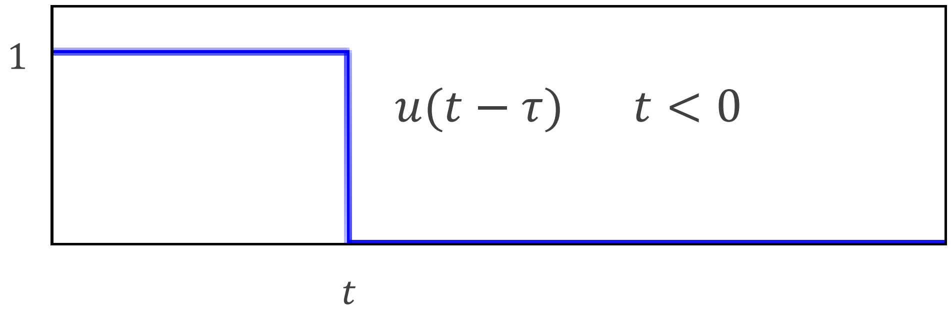
$$x(t) = e^{-at}u(t), \qquad a > 0 \qquad\qquad h(t) = u(t)$$

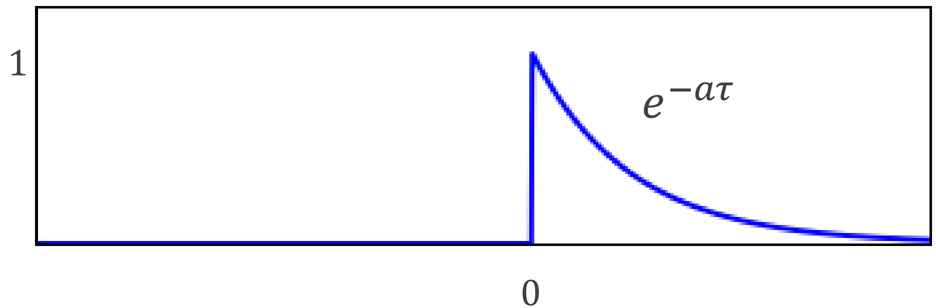




CT convolution example #4: Solution

$$y(t) = x(t) * h(t)$$

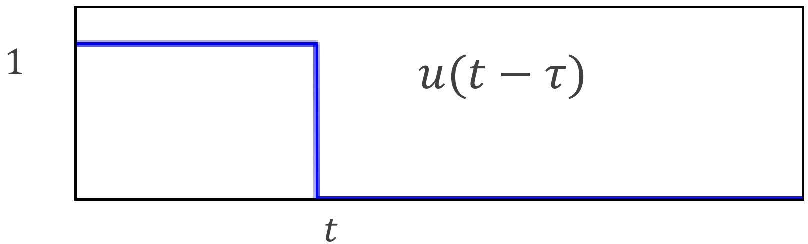


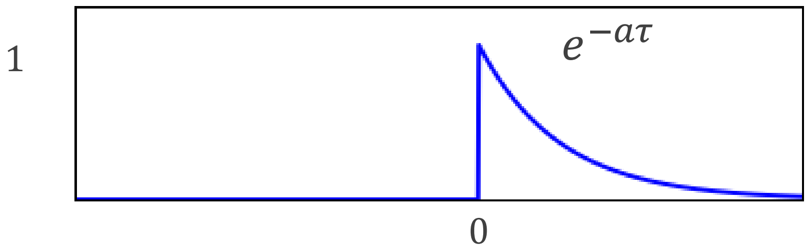


$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-a\tau}u(t-\tau)d\tau \end{aligned}$$

CT convolution example #4: Solution

For $t < 0$



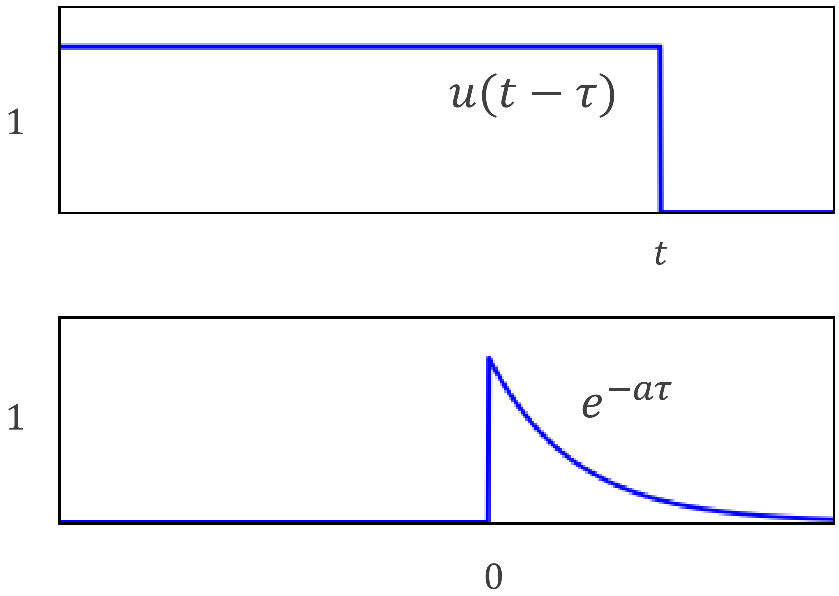


$$x(t) * h(t) = \int_{-\infty}^{\infty} 0 d\tau$$

$$= 0$$

CT convolution example #4: Solution

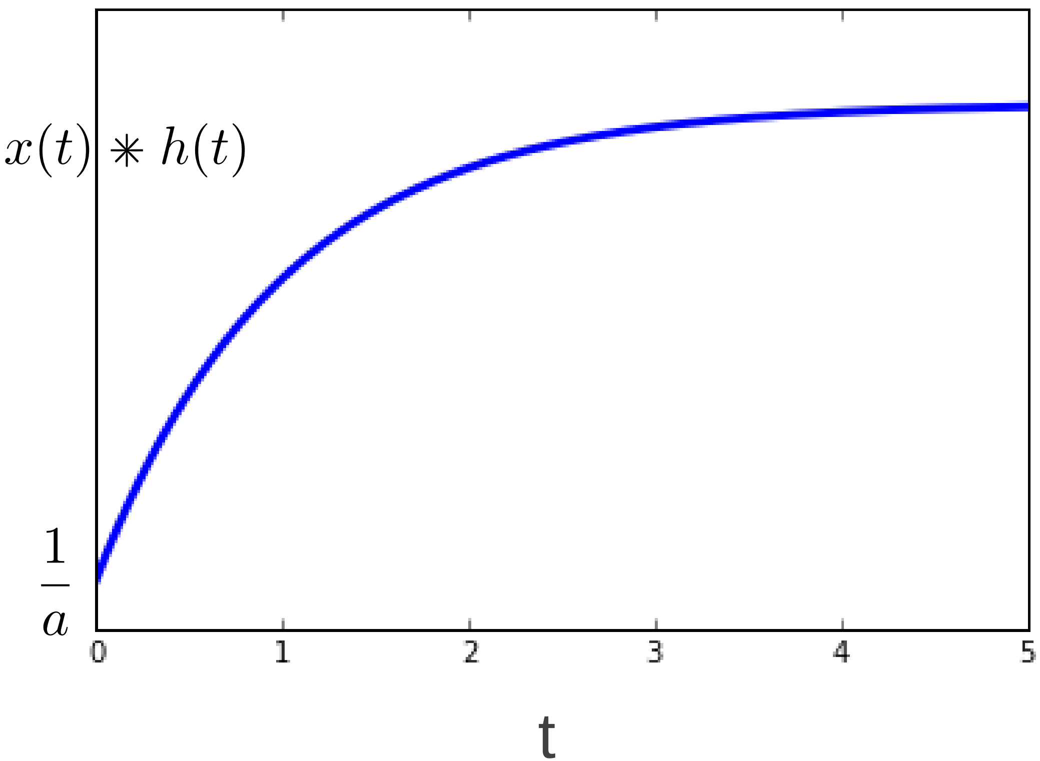
For $t \geq 0$



$$x(t) * h(t) = \int_0^t e^{-a\tau} u(t-\tau) d\tau$$

$$= \frac{1 - e^{-at}}{a}$$

CT convolution example #4: Solution



1/a (e.g. for a>0)

$$x(t) * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1-e^{-at}}{a} & t \geqslant 0 \end{cases}$$

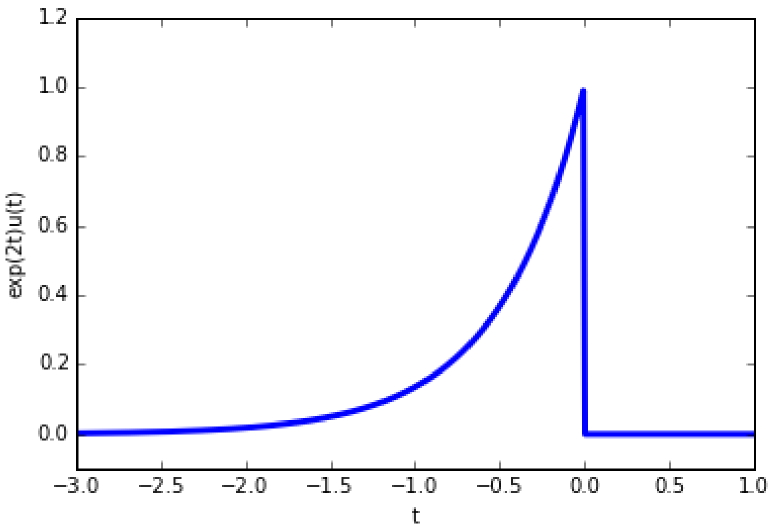
$$= \frac{1 - e^{-at}}{a} u(t)$$

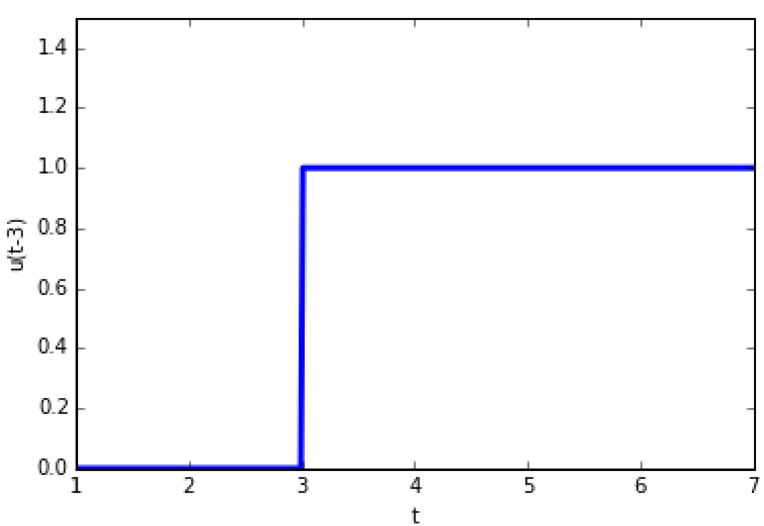
CT convolution example #5

This is an example where an anti-causal and a causal signal are convolved together, unlike the previous examples. Notice the difference intervals in this case.

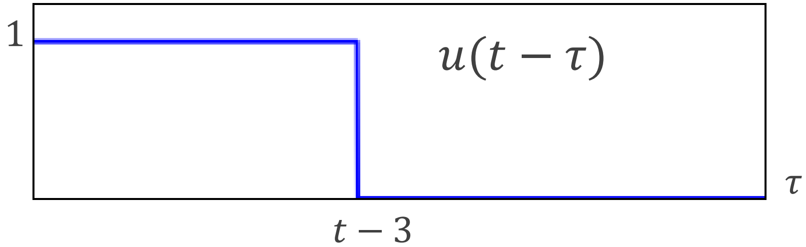
- Determine and sketch $y(t) = x(t) * h(t)$, where

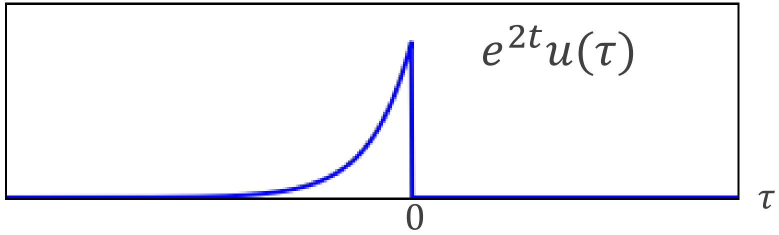
$$x(t) = e^{2t}u(-t) \qquad h(t) = u(t-3)$$





CT convolution example #5: Solution



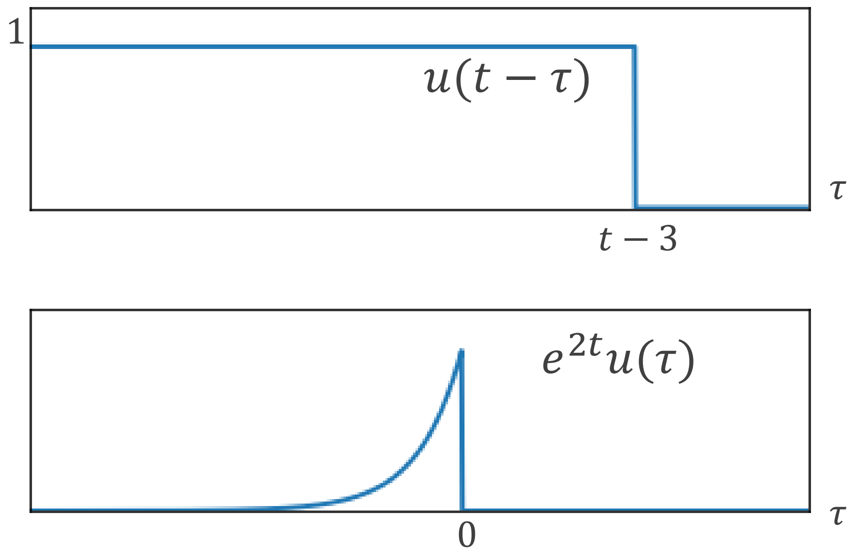


For t-3 < 0

$$x(t) * h(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau$$

$$= \frac{e^{2(t-3)}}{2}$$

CT convolution example #5: Solution



For t-3 > 0

$$x(t) * h(t) = \int_{-\infty}^{0} e^{2\tau} d\tau$$

$$= \frac{1}{2}$$

CT convolution example #5: Solution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \begin{cases} \frac{e^{2(t-3)}}{2} & t < 3 \\ \frac{1}{2} & t \geqslant 3 \end{cases}$$

Words of wisdom on convolution

- Convolution is a fact of life
  - All real world LTI systems generate outputs in the time domain given by the convolution sum or integral (yes RLC circuits are doing convolution for you)
- You need to practice CT and DT convolutions
  - There are many examples in the book and videos online
- In future lectures, we will cover tools and ideas that allow us to avoid computing convolutions (most of the time)
  - But you still need to learn how to do it in the time domain (don't wait)
- Today: Can infer much about the system properties from inspecting the impulse response

LTI systems properties in terms of the impulse response

Learning objectives

- Relate system properties to impulse response characteristics
- Determine system properties from the impulse response

Memoryless

An LTI system is memoryless if the impulse response that is simply a scaled impulse

- For discrete-time systems

$$h[n] = k\delta[n]$$

$$y[n] = h[n] * x[n] = kx[n]$$

- For continuous-time systems

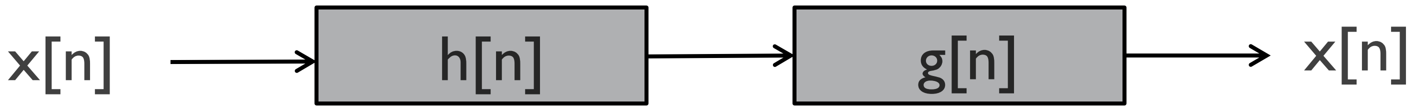
$$h(t) = k\delta(t)$$
$$y(t) = kx(t)$$

- Yawn…can't do very much with a memoryless system!

Invertibility

An LTI system is invertible if there exists $g[n]$ such that

$$h[n] * g[n] = \delta[n]$$



inverting system (often called an equalizer)

- The definition is similar for continuous-time
- Finding the correct g[n] or g(t) can be a challenge
- Question: If h[n] is causal <u>and</u> has memory, will g[n] be causal?

Invertibility example 1

In this example, we use our understanding of convolving deltas to show invertibility by construction.

- Is this system defined by the impulse response below invertible?

$$h(t) = \delta(t - ✎, \quad ✎ \; 0$$

- Yes, it is invertible because

$$g(t) = \delta(t + ✎, \quad \text{(shift back)}$$

satisfies

$$h(t) * g(t) = \delta(t)$$

Invertibility example 2

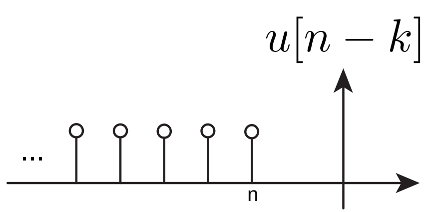
In this example, we try to rewrite the equation for the output y[n] as a function of the input x[n]. There is an interesting trick here when we expand the sum.

- Consider the system defined by the impulse response $h[n] = u[n]$
- The output of the system is given by

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k], \quad \text{(definition of conv)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \quad \underbrace{u[n-k]}_{\text{stops at time } k=n}$$



$$= \sum_{k=-\infty}^{n} x[k]$$

Invertibility example 2 (continued)

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- This function is called an "accumulator"
- Key is to note that the output can be written recursively as

$$y[n] = x[n] + \underbrace{x[n-1] + x[n-2] \cdots}_{=y[n-1]}$$

- Therefore, $x[n]$ can be recovered from $y[n]$ as

$$x[n] = y[n] - y[n-1]$$

Invertibility example 2 (continued)

- The inverting system's impulse response is therefore

$$g[n] = \delta[n] - \delta[n-1]$$

- Check:

$$h[n] * g[n] \stackrel{?}{=} \delta[n]$$

$$\begin{aligned} u[n] * (\delta[n] - \delta[n-1]) &= u[n] - u[n-1] \\ &= \delta[n] \end{aligned}$$

Causality in discrete-time

- Consider the output of a discrete-time LTI system

$$\sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{-1} h[k]x[n-k] + \sum_{k=0}^{\infty} h[k]x[n-k]$$

depends on x[n+1], x[n+2], etc (future values)

depends on x[n], x[n-1], etc (past values)

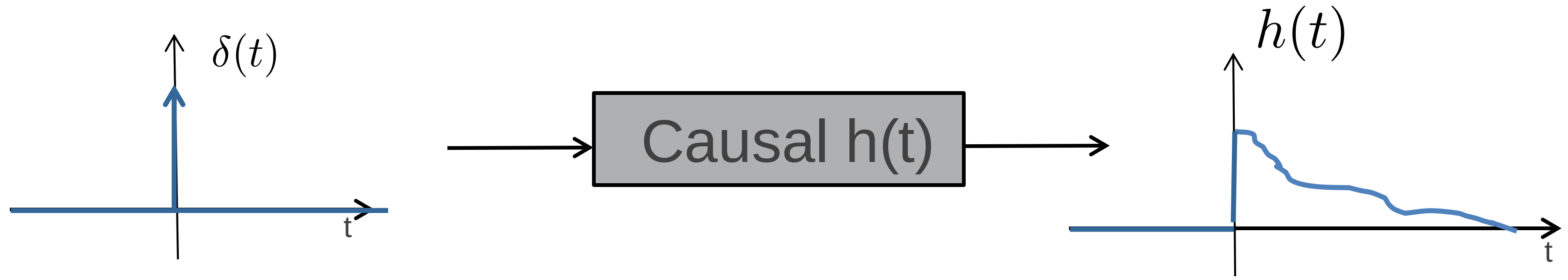
A discrete-time system is causal if h[n]=0 for n < 0

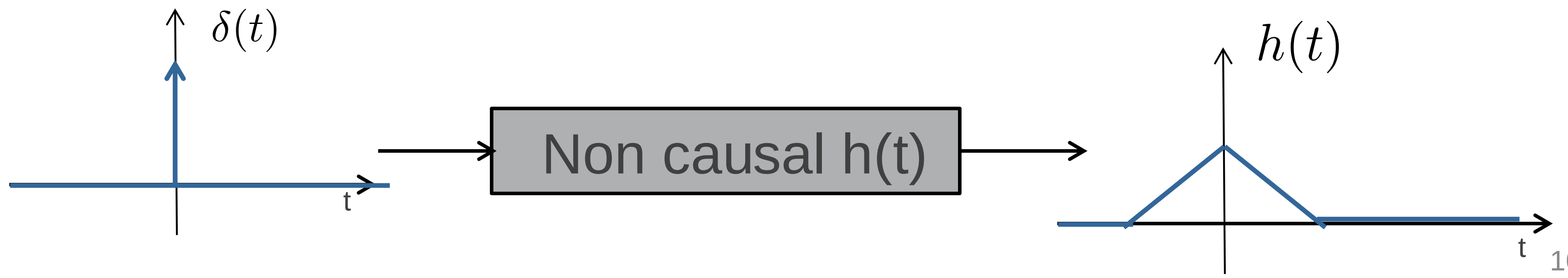
Intuitive: an impulse at time 0 should not cause any response before it actually happens!

Causality in continuous-time

A continuous-time system is causal if h(t)=0 for t < 0

- An impulse at time 0 should not cause a system response to occur before the impulse actually happens!





Stability

- LTI system is BIBO stable if and only if its impulse response is absolutely summable (or integrable) to a finite value

For discrete-time systems

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

For continuous-time systems

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Why absolute summability for stability?

- Condition ensures that the output of a bounded input is bounded
  - Consider $|x[n]| < B, \forall n$, where B is a constant
  - The output, using the Cauchy-Schwarz inequality, will be

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$\leqslant \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$|y[n]| \leqslant B \underline{\sum_{k=-\infty}^{\infty} |h[k]|}$$

If this term is bounded then, the output is bounded

Note we have proven here that if the impulse response is bounded then it is stable but can also show that if at LTI system is BIBO stable then the impulse response is absolutely summable

Systems example

- Determine if the following system is (a) causal and/or (b) stable

$$h[n] = n\left(\frac{1}{3}\right)^n u[n-1]$$

- Solution:
  - Causal?
  - Stable?

this formula might be useful:

$$\sum_{n=1}^{\infty} n a^{n-1} = \frac{1}{(1-a)^2} \text{ for } |a| < 1$$

Systems example – sketch of solution

- Determine if the following system is (a) causal and/or (b) stable

$$h[n] = n\left(\frac{1}{3}\right)^n u[n-1]$$

- Solution:
  - Causal? Yes because $h[n] = 0$ for n < 0
  - Stable? Yes because

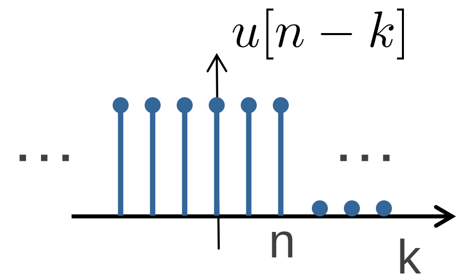
$$\begin{aligned}\sum_{n=-\infty}^{\infty}|h[n]| &= \sum_{n=-\infty}^{\infty} n\left(\frac{1}{3}\right)^n u[n-1] \\ &= \sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n \\ &= \frac{1}{\left(1-\frac{1}{3}\right)^2} \; < \; \infty\end{aligned}$$

Step response: DT systems

- Step response s[n] is the output of the system when x[n] = u[n]

$$u[n] \longrightarrow \boxed{\text{System}} \longrightarrow s[n] \triangleq h[n] * u[n]$$

$$\sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$



- Important response conceptually, what happens when I "turn the input on to a constant value and leave it on"?
- Note: $h[n] = s[n] - s[n-1]$

DT step response calculation example

- Determine the step response of the system with impulse response

$$h[n] = \alpha^n u[n]$$

- Convolve the impulse response with the step function

$$s[n] = \sum_{k=-\infty}^{n} \alpha^k u[k] = \sum_{k=0}^{n} \alpha^k$$

$$= \frac{1-\alpha^{n+1}}{1-\alpha}$$

- Typically s[n] either approaches an asymptote such as $(1-\alpha)^{-1}$ here, or it goes to infinity (for an unstable system)

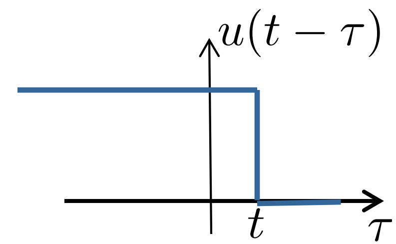
The step response is much easier to measure directly in practice compared with the impulse response.

Step response – CT systems

- The output of the system when the input is a unit-step function

$$s(t) \triangleq h(t) * u(t) = \int_{-\infty}^{\infty} h(\square)u(t-\square)d\square$$

$$= \int_{-\infty}^{t} h(\square)d\square$$



- Note: $\dfrac{d}{dt}s(t) = \dfrac{d}{dt}\int_{-\infty}^{t} h(\square)d\square = h(t)$

CT step response calculation example I

- What is the impulse response of the system with the step response

$$s(t) = t^2 u(t)$$

- Impulse response:

$$h(t) = \frac{d}{dt} s(t)$$

$$= 2tu(t) = \underline{2r(t)}$$

Unit-ramp function

CT step response calculation example 2

- Determine and plot the step response of a system with a real impulse response given by

$$h(t) = Ae^{-bt}u(t),\ A > 0,\ b > 0$$

CT step response calculation example 2: solution

- Determine and plot the step response of a system with a real impulse response given by

$$h(t) = Ae^{-bt}u(t),\ A > 0,\ b > 0$$

Recall that

$$\begin{aligned} s(t) &\triangleq u(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} h(\tau)d\tau \end{aligned}$$

Now compute

$$\begin{aligned} s(t) &= u(t)\int_0^t Ae^{-b\tau}d\tau \\ &= u(t)\frac{A}{b}(1-e^{-bt}) \end{aligned}$$

Introduction to continuous-time systems as differential equations

Learning objectives

- Define a linear constant coefficient differential equation
- Formulate differential equations for circuits problems

Systems described with differential equations

- Many practical systems are described by differential equations
  - RLC circuits and filters
  - Mechanical systems
  - Heat transfer systems
  - Chemical systems

Linear <u>constant coefficient</u> differential equations have many connections to LTI systems

A simple differential equation example

Constant coefficients

$$a_0\, y(t) + a_1 \frac{d}{dt} y(t) = b_0\, x(t) + b_1 \frac{d}{dt} x(t)$$

output

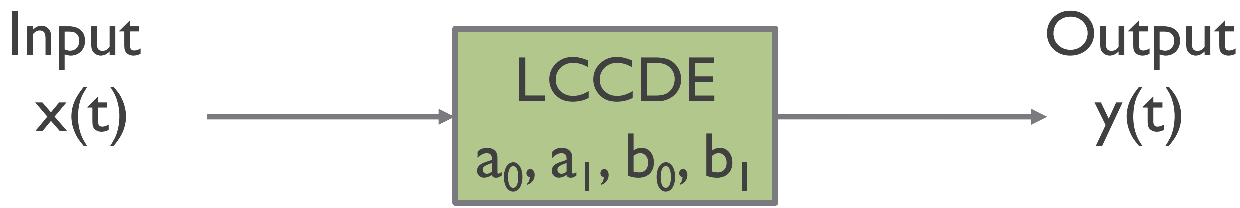
input

derivative of output

derivative of input

Example of a linear constant coefficient differential equation

Connecting differential equations to systems



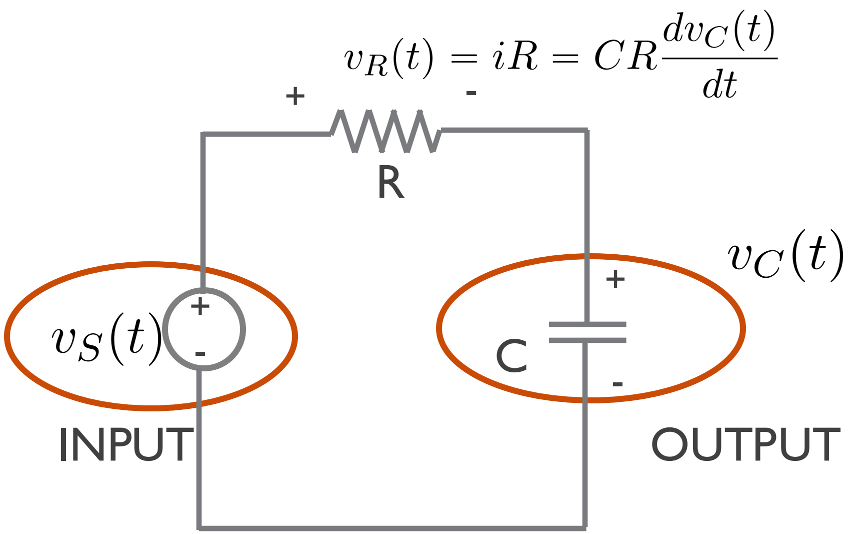
- Input and output are related through the linear constant coefficient differential equation (LCCDE)

$$a_0\, y(t) + a_1\, \frac{d}{dt} y(t) = b_0\, x(t) + b_1\, \frac{d}{dt} x(t)$$

- Solution to the differential equation provides a formula for the output y(t) as a function of the input x(t) and system parameters

Circuit examples

- RC lowpass filter as a differential equation
  - Source voltage as the input
  - Capacitor voltage as the output



Current through capacitor

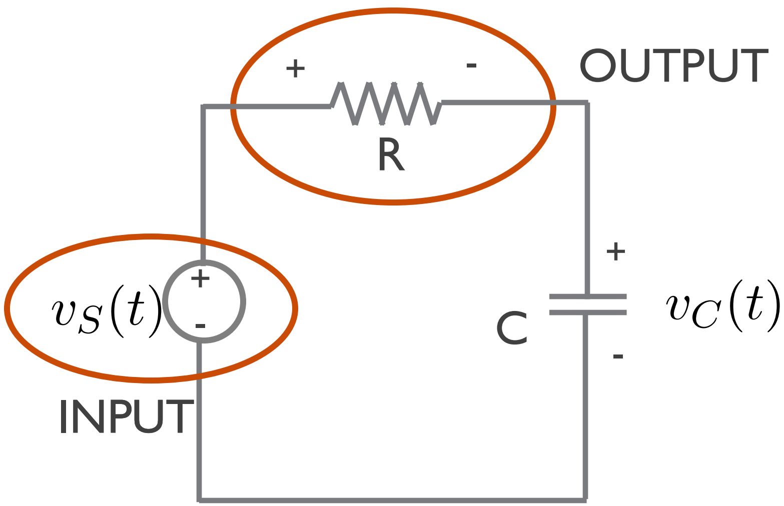
$$i = C\frac{dv_C(t)}{dt}$$

Resulting differential equation

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

Circuit examples

- RC highpass filter
  - Source voltage as the input as a differential equation
  - Resistor voltage as the output



$$v_R(t) = iR = RC\frac{dv_C(t)}{dt}$$

$$\frac{1}{RC}\int v_R(t)dt = v_C(t)$$

$$v_R(t) + \frac{1}{RC}\int v_R(t)dt = v_S(t)$$

$$\frac{dv_R(t)}{dt} + \frac{1}{RC}v_R(t) = v_S(t)$$

Resulting differential equation

$$RC\frac{dv_R(t)}{dt} + v_R(t) = RCv_S(t)$$

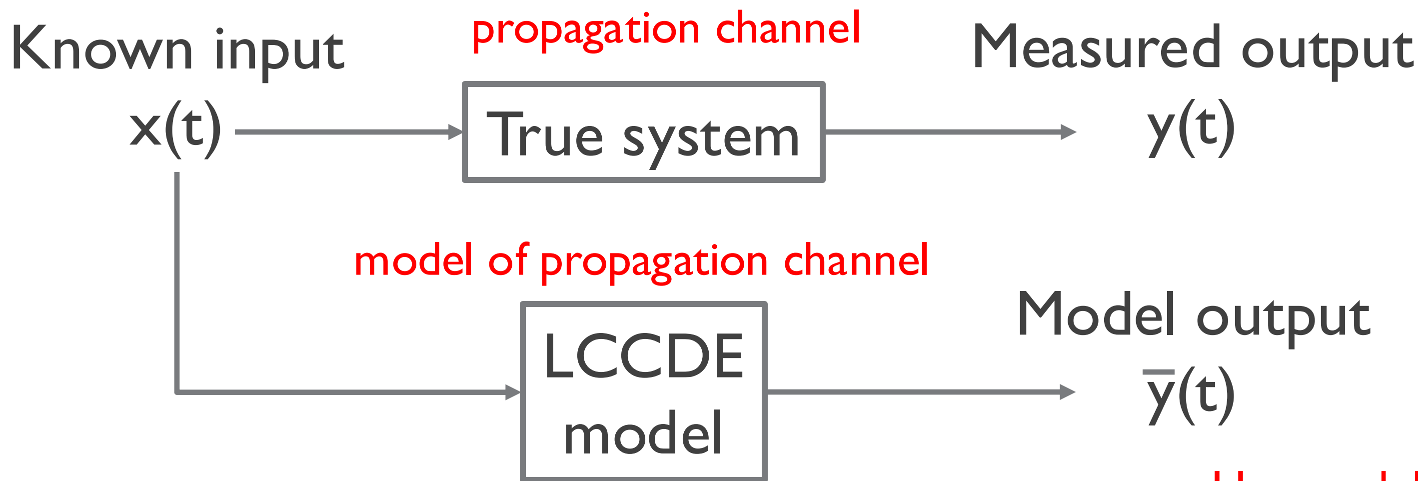
Applications of DEs in Electrical Engineering

- DE may result directly from mathematical fundamentals
  - Electromagnetics, e.g. Maxwell´s equations
  - Passive circuits (RC, RLC examples)

- DE may be used as to model observed phenomena
  - Attenuation on a wire or cable
  - Wireless propagation channels
  - Spectrum utilization
  - Control systems

A common modeling problem

In red: how this works in wireless



Use model to equalize and recover transmitted data

Find the coefficients of the LCCDE such that the model output is a good approximation of the measured output

Why LCCDE's as models?

- Describe a range of phenomena with a few coefficients
- Provide a convenient way to represent LTI systems with long impulse responses (under certain conditions that will explained)
- Can be realized using passive circuits or op-amps
- Solutions to LCCDEs are well understood

Solving differential equations

Learning objectives

- Solve a linear constant coefficient differential equation
- Compute the zero-input, impulse-response, and particular solution

General LCCDE relating input and output

$$a_N \frac{d^N y}{dt^N} + a_{N-1} \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x}{dt^M} + b_{M-1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x(t)$$

- Coefficients on left-hand side (LHS) are related to the state polynomial
- Coefficients on right-hand side (RHS) are related to the input polynomial
  - State polynomial is relatively the more important of the two
- Order of the DE is *max(N,M)*
  - In ECE 301, usually $N > M$, $N = 1$ or 2 and often $M = 0$

You are assumed to remember differential equations ☺

Solution of a LCCDE has a general form

$$y(t) = y_h(t) + y_p(t)$$

- $y_h(t)$ is the natural or homogeneous or zero-input response
  - Solution with RHS = 0 $(x(t) = 0)$
  - Solution will be a function of the initial conditions
  - Does not depend on the input or input polynomial

$$y(t_0) = A_0$$

$$\frac{dy(t_0)}{dt} = A_1$$

- $y_p(t)$ is the particular or driven or zero-state response
  - Solution with the initial conditions equal to zero $(x(t) \rightarrow y_p(t))$
  - Does not depend on the initial conditions
  - Depends only on the system's response to the input

Recipe for finding the solution to a LCCDE

- There are many ways to solve differential equations
  - Most approaches follow several steps in a prescribed recipe
  - Different recipes are used in different courses
  - We follow a specific recipe that is well served with linear systems

- ECE 301 "recipe" for solving LCCDEs

  (1) Find the homogenous response

  (2) Find the impulse response

  (3) Find the particular solution

  (4) Combine homogenous and particular to find the total solution

Step 1: Finding the homogenous response $y_h(t)$

- Find the characteristic polynomial

$$Q(\lambda) = a_N \lambda^N + a_{N-1}\lambda^{N-1} + \cdots + a_1\lambda + a_0, \quad (\text{derivatives} \rightarrow \lambda)$$

- Solve for $Q(\lambda) = 0 \rightarrow N$ roots $\lambda_1, \lambda_2, \cdots, \lambda_N$
  - Non-repeated root contribute a characteristic mode of the form

$$C_k e^{\lambda_k t}$$

  - Repeated real roots contribute a characteristic mode of the form

$$(c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda_1 t}$$

  - For real equations, complex roots come in pairs: $\lambda_1 = \alpha + j\beta, \lambda_2 = \alpha - j\beta$

$$\tfrac{c}{2}e^{j\theta}e^{(\alpha+j\beta)t} + \tfrac{c}{2}e^{-j\theta}e^{(\alpha-j\beta t)} = ce^{\alpha t}\cos\left(\beta t + \theta\right)$$

$$C_1 = \tfrac{c}{2}e^{j\theta}$$

Sum the contributions from each mode

Step 1: Finding the homogenous response $y_h(t)$

- Example for real and distinct roots the solution looks like

$$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

- Solve for the constants using initial conditions, e.g.

$$y_h(t_0) = A_0$$

Time value is t_0, often 0

Need N initial conditions

$$\frac{dy_h(t_0)}{dt} = A_1$$

These are values that are given

$$\vdots$$

$$\frac{d^{N-1}y_h(t_0)}{dt^{N-1}} = A_{N-1}$$

Example 1 – finding the homogenous solution

We will build on this example as each step of the LCCDE recipe is explained.

- Consider a first order differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

- Find the zero-input response given $y(0) = 2$

Example 1 – finding the homogenous solution

- Consider a first order differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

- Find the zero-input response given $y(0) = 2$ this is the initial condition

solve characteristic equation $(\lambda + 2) = 0$

find characteristic root $\lambda = -2$

characteristic mode $y_h(t) = Ce^{-2t}$

solve for the unknown constant

$$\begin{aligned} y_h(0) &= 2 \\ &= Ce^{-2\cdot 0} \end{aligned}$$

final homogeneous solution $y_h(t) = 2e^{-2t}$

Step 2: Calculating the impulse response

- The impulse response of a LCCDE is (assuming M<=N) *

$$h(t) = \frac{b_N}{a_N}\delta(t) + \frac{1}{a_N}\left[b_N \frac{d^N y_n(t)}{dt^N} + b_{N-1}\frac{d^{N-1}y_n(t)}{dt^{N-1}} + \cdots + b_1 \frac{dy_n(t)}{dt} + b_0 y_n(t)\right] u(t)$$

- Where
  - b_N is non-zero only if N=M
  - $y_n(t)$ looks like zero-input response $y_h(t)$ but is found for special initial conditions

$$y_n(0) = \frac{dy_n(0)}{dt} = \cdots = \frac{d^{N-2}y_n(0)}{d^{N-2}t} = 0, \frac{d^{N-1}y_n(0)}{d^{N-1}t} = 1$$

- If M > N then the form is different, not required for this course

* you <u>do not</u> need to know the derivation of this

Example 1 – finding the impulse response

- Consider a first order differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

- Find the impulse response corresponding to this system

Example 1 – finding the impulse response

- Consider a first order differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

- Find the impulse response corresponding to this system

form of natural response $y_n(t) = Ce^{-2t}$

solve for unknown constant
$$y_n(0) = 1$$
$$= Ce^{-2\cdot 0}$$
$$\Rightarrow C = 1$$

final form of natural response $y_n(t) = e^{-2t}$

impulse response $h(t) = e^{-2t}u(t)$

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Step 3: Computing the particular solution

- The particular solution is the convolution of the output & input

$$y_p(t) = x(t) * h(t)$$

Step 4: Computing the total solution

- Total solution is the sum of the homogeneous and particular solutions

$$y(t) = y_h(t) + y_p(t)$$

When do LCCDE describe LTI systems?

IMPORTANT

- A LCCDE described by

$$a_N \frac{d^N y}{dt^N} + a_{N-1} \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t)$$
$$= b_M \frac{d^M x}{dt^M} + b_{M+1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x(t)$$

- Is **also** an LTI system if the initial conditions are zero

$$y(t) = \cancel{y_h(t)} + y_p(t)$$

  - System must be at rest to be LTI
  - Makes sense because $y_p(t) = x(t) * h(t)$

At rest is another term for zero initial conditions

Example 1 – computing the particular solution

- Consider a first order differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

- Find the particular solution (zero-state response) given input

$$x(t) = e^{3t}u(t)$$

Example 1 – computing the particular solution

$$h(t) * x(t) = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau) e^{3(t-\tau)} u(t-\tau) d\tau$$

$$= e^{3t} \int_{-\infty}^{\infty} e^{-5\tau} u(\tau) u(t-\tau) d\tau$$

$$= e^{3t} \int_{0}^{\infty} e^{-5\tau} u(t-\tau) d\tau$$

$$= \begin{cases} e^{3t} \int_0^t e^{-5\tau} d\tau & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

$$= \frac{1}{5} e^{3t} (1 - e^{-5t}) u(t)$$

Example 1 – computing the total solution

- Consider a first order differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

- Find the total solution given

$$x(t) = e^{3t}u(t) \qquad\qquad y(0) = 2$$

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= 2e^{-2t} + \frac{1}{5}e^{3t}u(t) - \frac{1}{5}e^{-2t}u(t) \end{aligned}$$

- Some books include a unit step for the homogeneous response

Example 2

Reference example with a derivative on both the input and the output. This makes computing the impulse response more challenging as we have to use that complicated formula.

- Consider a modification of Example 1 with $y(0) = 2 \quad x(t) = e^{3t}u(t)$

$$\frac{d}{dt}y(t) + 2y(t) = x(t) + 2\frac{d}{dt}x(t)$$

- The homogenous solution is the solution to

$$\frac{d}{dt}y(t) + 2y(t) = 0$$

Subject to the initial conditions $y(0) = 2$

- Same as Example 1! $y_h(t) = 2e^{-2t}$

Example 2 (continued)

Provided for completeness

- Characteristic polynomial $\lambda + 2$
- Characteristic equation $\lambda + 2 = 0$
- Characteristic roots $\lambda_1 = -2$
- Characteristic modes e^{-2t}
- Homogeneous response

$$y_h(t) = Ce^{-2t} \xrightarrow{\text{Using initial conditions}} y_h(t) = 2e^{-2t}$$

Example 2 (continued)

- Now find the impulse response using the formula

$$h(t) = \frac{b_N}{a_N}\delta(t) + \frac{1}{a_N}\left[b_N \frac{d^N y_n(t)}{dt^N} + b_{N-1}\frac{d^{N-1} y_n(t)}{dt^{N-1}} + \cdots + b_1 \frac{dy_n(t)}{dt} + b_0 y_n(t)\right] u(t)$$

- First find $y_n(t)$
  - It has the form $y_n(t) = Ce^{-2t}$
  - Solve with special initial conditions $y_n(0) = 1$
  - This leads to $y_n(t) = e^{-2t}$

- Same as Example 1 again
  - But we have not yet found the impulse response…

Example 2 (continued)

- Plugging in to the general impulse response formula

$$h(t) = \frac{b_N}{a_N}\delta(t) + \frac{1}{a_N}\left[b_N \frac{d^N y_n(t)}{dt^N} + b_{N-1}\frac{d^{N-1}y_n(t)}{dt^{N-1}} + \cdots + b_1 \frac{dy_n(t)}{dt} + b_0 y_n(t)\right] u(t)$$

- With the values from the equation

$$\begin{aligned} h(t) &= \frac{2}{1}\delta(t) + \left[2\frac{d}{dt}e^{-2t} + e^{-2t}\right] u(t) \\ &= 2\delta(t) + \left[2(-2)e^{-2t} + e^{-2t}\right] u(t) \\ &= 2\delta(t) - 3e^{-2t}u(t) \end{aligned}$$

- Not the same as Example 1!!

Example 2 (continued)

- The particular solution is $y_p(t) = h(t) * x(t)$

with $x(t) = e^{3t}u(t)$

$$
\begin{aligned}
h(t) * x(t) &= \int_{-\infty}^{\infty} e^{3\tau}u(\tau)\left(2\delta(t-\tau) - 3e^{-2(t-\tau)}u(t-\tau)\right)d\tau \\
&= 2e^{3t}u(t) - 3\int_{-\infty}^{\infty} e^{3\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau \\
&= 2e^{3t}u(t) - 3\cdot\frac{1}{5}\left(e^{3t} - e^{-2t}\right)u(t) \\
&= 2e^{3t}u(t) - \frac{3}{5}\left(e^{3t} - e^{-2t}\right)u(t)
\end{aligned}
$$

We computed this in Example 1

Example 2 (concluded)

- Find the total solution

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = 2e^{-2t} + 2e^{3t}u(t) - \frac{3}{5}\left(e^{3t} - e^{-2t}\right)u(t)$$

Example 3: Solve the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = x(t) \qquad x(t) = 2e^{-2t}u(t)$$

Initial conditions : $y(0) = 3; y'(0) = -4$

- Find the homogeneous solution
- Find the impulse response
- Find the particular solution
- Find the total solution

Reference example with two different roots. This makes it more challenging to find the coefficients for the homogeneous response and for the impulse response.

Example 3: Homogenous solution

- Characteristic polynomial

$$Q(\lambda) = \lambda^2 + 4\lambda + 4$$

- Characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0$$

- Characteristic roots

$$\lambda = -2, -2$$

Example 3: Homogenous solution

- Homogeneous response and its derivative

$$y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$
$$y_h'(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$$

- Solving for the unknowns

$$\begin{aligned} y_h(0) &= c_1 e^{-2\cdot 0} + c_2 \cdot 0 \cdot e^{-2\cdot 0} \\ &= c_1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} y_h'(0) &= -2c_1 e^{-2\cdot 0} - 2c_2 \cdot 0 e^{-2\cdot 0} + c_2 e^{-2\cdot 0} \\ &= -2c_1 + c_2 \\ &= -6 + c_2 \\ &= -4 \\ &\implies c_2 = 2 \end{aligned}$$

$$\boxed{y_h(t) = 3e^{-2t} + 2te^{-2t}}$$

Example 3: Impulse response

- The formula for the impulse response in this case

$$h(t) = \frac{b_2}{a_2}\delta(t) + \left[\frac{b_2}{a_2}\frac{d^2 y_n(t)}{dt^2} + \frac{b_1}{a_2}\frac{dy_n(t)}{dt} + \frac{b_0}{a_2}y_n(t)\right]u(t)$$

- Recall that we use the following response

$$y_n(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

with different initial conditions

$$y_n(0) = y_n'(0) \cdots = y_n^{(m-2)}(0) = 0 \quad \text{and} \quad y_n^{(m-1)}(0) = 1$$

Example 3: Impulse response

- Solving for the unknowns

$$
\begin{aligned}
y_n(t) &= c_1 e^{-2t} + c_2 t e^{-2t} \\
y_n(0) &= c_1 + 0 \cdot c_2 \\
&= 0 \\
&\implies c_1 = 0
\end{aligned}
$$

$$
\begin{aligned}
y_n'(0) &= -2c_1 e^{-2\cdot 0} - 2c_2 \cdot 0 e^{-2\cdot 0} + c_2 e^{-2\cdot 0} \\
&= -2c_1 + c_2 \\
&= c_2 \\
&= 1 \\
&\implies c_2 = 1
\end{aligned}
$$

Example 3: Impulse response

- Inserting into the impulse response formula and simplifying

$$\begin{aligned}
h(t) &= \frac{b_2}{a_2}\delta(t) + \left[\frac{b_2}{a_2}\frac{d^2 y_n(t)}{dt^2} + \frac{b_1}{a_2}\frac{dy_n(t)}{dt} + \frac{b_0}{a_2}y_n(t)\right]u(t) \\
&= 0 \cdot \delta(t) + \left[0 \cdot \frac{d^2 y_n(t)}{dt^2} + 0 \cdot \frac{dy_n(t)}{dt} + y_n(t)\right]u(t) \\
&= (1)(te^{-2t})u(t) \\
&= te^{-2t}u(t)
\end{aligned}$$

Example 3: Particular solution

- The particular solution is

$$
\begin{aligned}
y_p(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} 2e^{-2(t-\tau)} u(t-\tau) \tau e^{-2\tau} u(\tau) d\tau \\
&= u(t) e^{-2t} \int_0^{\infty} 2\tau u(t-\tau) d\tau
\end{aligned}
$$

unit step $\rightarrow$ $1 \text{ for } \tau > 0$

unit step $\rightarrow$ $1 \text{ for } \tau < t$

Example 3: Particular solution

- Solving for $t \geqslant 0$

$$\begin{aligned} x(t) * h(t) &= e^{-2t} \int_0^t 2\tau d\tau \\ &= e^{-2t} t^2 \end{aligned}$$

- Solving for $t \leqslant 0$

$$\begin{aligned} x(t) * h(t) &= e^{-2t} \int_t^0 0 d\tau \\ &= 0 \end{aligned}$$

unit step function is used to write the answer in a simpler form

$$\Rightarrow \quad y_p(t) = x(t) * h(t) = e^{-2t} t^2 u(t)$$

Example 3: Total response

- Total response is the sum of the homogenous and particular solutions

$$\begin{aligned} y(t) &= y_p(t) + y_h(t) \\ &= e^{-2t}t^2u(t) + 3e^{-2t} + 2te^{-2t} \end{aligned}$$

Discrete-time systems as difference equations

Learning objectives

- Solve a difference equation recursively
- Produce the general solution of a difference equation

Relating differences and derivatives

- Recall the definition of derivative from calculus

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

T is the sample period

- Suppose we sample the continuous-time signal $x[n] = x(nT)$

$$\lim_{\Delta t \to 0} \frac{x(Tn + \Delta t) - x(Tn)}{\Delta t}$$

- If we want to do this calculation in discrete-time, the smallest Δt is T

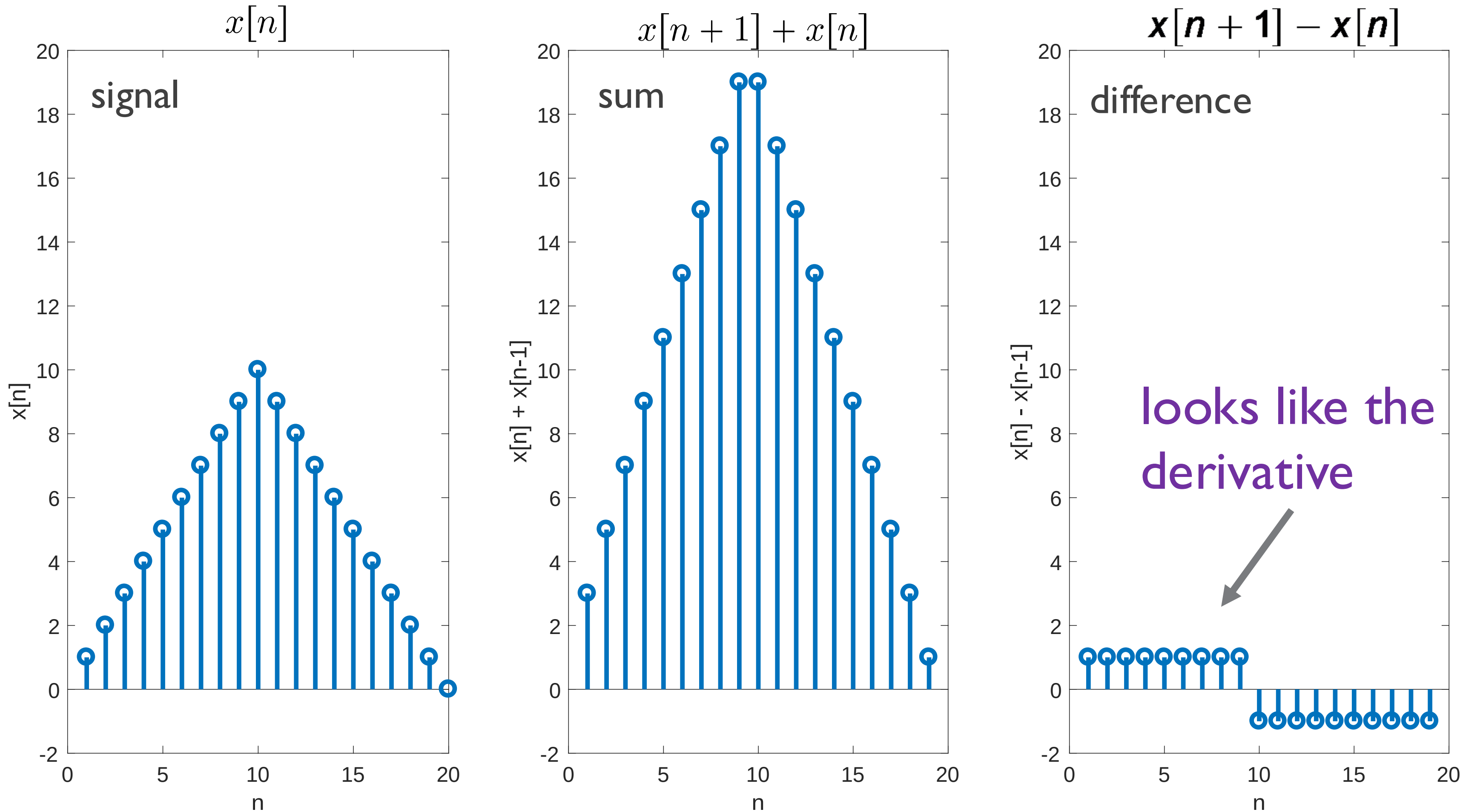
$$\lim_{\Delta t \to 0} \frac{x(Tn + \Delta t) - x(Tn)}{\Delta t} \approx \frac{x(Tn + T) - x(Tn)}{T}$$

$$= \frac{x[n+1] - x[n]}{T}$$

means they are equal

difference

Differences are like derivatives in discrete-time



Forms of linear constant coefficient difference equations

order is max(N,M)

- **Delay** format

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

coefficients

- **Advance** format

$$\sum_{k=0}^{N} a_{N-k} y[n+k] = \sum_{k=0}^{M} b_{N-k} x[n+k]$$

The two formats describe the same system and have the same solution

Places where difference equations show up

- As approximations to solve large differential equations
  - Computational electromagnetics (e.g. finite difference time domain method), mechanics, bio, petroleum, etc.
  - Quantitative finance
- Through recursive relationships between input and output
  - Bank balances, credit card payments
- To model or design LTI systems with a few parameters
  - Communication, image processing, sonar, radar, etc.

Solving a difference equation recursively

Learning objectives

- Write the output of a difference equation in terms of past outputs and current and past inputs
- Solve difference equations recursively

Population example

- A population is growing at 2% a year. How long for it to double?
  - Modeling

$$y[n] = \underbrace{y[n-1]}_{\text{}} + x[n] - z[n]$$

population at year n   new births/immigrants   deaths

  - We are not told how many births/immigrants or deaths but we know

$$y[n] = 1.02y[n-1]$$

Population example (continued)

- In terms of $y[0]$

$$y[n] = 1.02y[n-1]$$
$$y[1] = 1.02y[0]$$
$$y[2] = (1.02)^2 y[0]$$
$$\cdots$$
$$y[n] = (1.02)^n y[0]$$

- So the population at least doubles, solve for n then round up

$$\frac{y[n]}{y[0]} = (1.02)^n = 2 \rightarrow n \ln(1.02) = \ln 2 \rightarrow \boxed{n = \frac{\ln 2}{\ln(1.02)} = 35}$$

note there should be the ceiling here but in this case they divide almost exactly into 35

Rule of 72

$r = p/100$

- Doubling rate $\approx \frac{72}{p}$ where p is the % increase in time
- Where does this come from?

$$n = \frac{\ln 2}{\ln(1+r)}$$

- Using the first order Taylor series for small r

$$\ln(1+r) \approx r$$

$$p = 100r$$

$$\frac{\ln 2}{r} = \frac{100 \ln 2}{p}$$

$$\approx \frac{69}{p}$$

Why use 72 then?
It has more divisors (2, 3, 4, 6, 8, 9, 12…)
http://en.wikipedia.org/wiki/Rule_of_72

Recursive solution to a difference equation

- Consider the following general difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Apply the input at time n
- Initial conditions typically take the form

$$y[-1] = A_0, \quad y[-2] = A_1, \quad \cdots \quad y[-N] = A_{N-1}$$

- It is possible to recursively solve by writing

$$y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0} \sum_{k=1}^{N} a_k y[n-k]$$

current and past inputs past outputs

Recursive solution to a difference equation

- Inserting the initial conditions gives the recursive solution

$$
\begin{aligned}
y[0] &= \frac{1}{a_0} \sum_{k=0}^{M} b_k x[0-k] - \frac{1}{a_0} \sum_{k=1}^{N} a_k y[0-k] \\
&= \frac{1}{a_0} b_0 x[0] - \frac{1}{a_0} \left(a_1 y[-1] + a_2 y[-2] + \cdots + a_N y[-N]\right)
\end{aligned}
$$

$$
\begin{aligned}
y[1] &= \frac{1}{a_0} \sum_{k=0}^{M} b_k x[1-k] - \frac{1}{a_0} \left(a_1 y[0] - \sum_{k=2}^{N} a_k y[1-k] \right) \\
&= \frac{1}{a_0} \left(b_0 x[1] + b_1 x[0]\right) - \frac{1}{a_0} a_1 y[0] - \frac{1}{a_0} \left(a_2 y[-1] + a_3 y[-2] + \cdots + a_N y[-N+1]\right)
\end{aligned}
$$

$$\vdots$$

Credit card example

- Your credit card bill
  - Starting balance of \$1000
  - Minimum monthly payment = \$20
  - Interest rate: 18% per year → r=0.015/month

How long to pay the min. monthly amount with no new expenses?

- Solution
  - Modeling

$$y[n] = \underbrace{y[n-1]}_{\text{balance}} + \underbrace{ry[n-1]}_{\text{interest}} - \underbrace{x[n]}_{\text{payment}}$$

Credit card example (continued)

- Write the output and use the recursive approach

$$
\begin{aligned}
y[n] &= (1+r)y[n-1] - x[n] \\
&\triangleq \gamma y[n-1] - xu[n], \quad (x = 20, \gamma = 1+r) \\
y[1] &= \gamma y[0] - x \\
y[2] &= \gamma y[1] - x \\
&= \gamma(\gamma y[0] - x) - x \\
&= \gamma^2 y[0] \underbrace{-\gamma x - x}_{-x(1+\gamma)} \\
y[n] &= \gamma^n y[0] - x \sum_{k=0}^{n-1} \gamma^k \quad = \gamma^n y[0] - x\left(\frac{1-\gamma^n}{1-\gamma}\right)
\end{aligned}
$$

Credit card example (continued)

- Now, solving for n with $x = 20, y[0] = 1000, \gamma = 1.015$

$$n = \left\lceil \ln\left(\frac{\frac{x}{1-\gamma}}{y[0] + \frac{\gamma}{1+\gamma}}\right) \Big/ \ln\gamma \right\rceil = 93 \text{ months}$$

Don't make the minimum payment!!

Recursive solution summary

- Rewrite the output as a function of current and past

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0}\sum_{k=1}^{N} a_k y[n-k]$$

- Use the initial conditions to start the recursion

- Recursive approach is a useful computational solver
  - Applies to more general difference equations as well
  - Has a role to play in finding the general solution

General solution of a difference equation

Learning objectives

- Solve linear constant coefficient difference equations in general
- Compute the homogenous solution
- Compute the impulse response
- Compute the particular solution

Linear constant coefficient difference equations

- Suppose that the difference equation is written in delay format

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

left hand side (LHS) right hand side (LHS)

- Solving involves finding an expression for y[n] that depends on the coefficients $\{a_k\}$, $\{b_k\}$, the initial conditions, and the input

- The initial conditions often have the form need N of them!

$$y[-1] = A_0, \quad y[-2] = A_1, \quad \cdots \quad y[-N] = A_{N-1}$$

Form of the solution of a LCCDE

$$y[n] = y_h[n] + y_p[n]$$

- $y_h[n]$ is the natural or homogeneous or zero-input response
  - Solution with RHS = 0
  - Solution will be a function of the initial conditions
  - Does not depend on the input

- $y_p[n]$ is the particular or driven or zero-state response
  - Solution with prior output values equal to zero (causal solution)
  - Does not depend on the initial conditions

Step 1: Finding the homogenous response $y_h[n]$

- Find the "characteristic polynomial"

$$Q(r) = a_0 r^N + a_1 r^{N-1} + \cdots + a_N$$

- Solve for $Q(r) = 0$ roots are $\gamma_1, \cdots, \gamma_N$
  - Non-repeated root contribute a function of the form

$$C_k \gamma_k^n$$

functions of n

  - Repeated real roots contribute a function of the form

$$\left(C_1 + nC_2 + \cdots + n^{R-1} C_R\right) \gamma_1^n$$

the root that is repeated

  - For real equations, complex roots come in pairs $\gamma_1 = |\gamma_1| e^{j\beta}, \quad \gamma_1^* = |\gamma_1| e^{-j\beta}$

$$C_1 \gamma_1^n + C_1^* (\gamma_1^*)^n = C_1 |\gamma_1|^n e^{j\beta n} + C_1^* |\gamma_1|^n e^{-j\beta n} = c|\gamma_1|^n \cos(\beta n + \theta)$$

$C_1 = \frac{c}{2} e^{j\theta}$

Sum the contributions from each root

Step 1: Finding the homogenous response $y_h[n]$

- Example for real and distinct roots

$$y_h[n] = C_1\gamma_1^n + \cdots + C_N\gamma_N^n$$

- Solve for the constants using initial conditions, e.g.

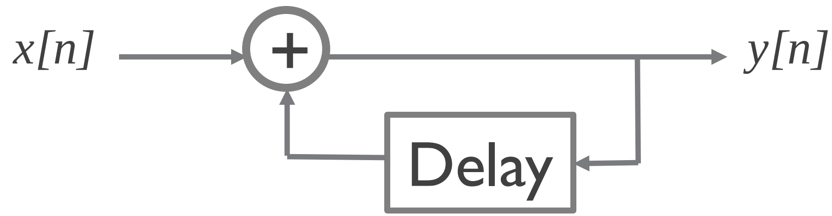
Need N initial conditions

$$y[-1] = A_0, \quad y[-2] = A_1, \quad \cdots \quad y[-N] = A_{N-1}$$

May be given other values of y[n]

Example will be continued for each step

Example 1



- Consider the following difference equation

$$y[n] - \gamma y[n-1] = x[n]$$

- Find the homogeneous solution given $y[-1]$

- In this example, we find a general solution that depends on the initial condition *y[-1]* and the parameter γ

Example 1 (continued)

$$y[n] - \gamma y[n-1] = 0$$

- Characteristic polynomial

$$Q(r) = (r - \gamma)$$

- Characteristic equation

$$(r - \gamma) = 0$$

- Characteristic root

$$r = \gamma$$

- Characteristic mode

$$C\gamma^n$$

Example 1 (continued)

$$y[n] - \gamma y[n-1] = 0$$

- Homogeneous response is

$$y_h[n] = C\gamma^n$$

- Solve for the unknown coefficients using the initial conditions

$$y_h[-1] = C\gamma^{-1} \Rightarrow C = y_h[-1]\gamma$$

$$\boxed{\begin{aligned} y_h[n] &= y_h[-1]\gamma\gamma^n \\ &= y_h[-1]\gamma^{n+1} \end{aligned}}$$

$h[n]$

Step 2: Calculating the impulse response

- One approach is to solve for the impulse response recursively
- Let the input be an impulse

$$x[n] = \delta[n]$$

- Assume zero initial conditions

$$h[-1] = 0, \;\; h[-2] = 0, \;\; \cdots \; h[-N] = 0$$

- Find a recursive formula from $\mathrm{h[n]}$

- This approach helps in finding the general solution explained next

$h[n]$

Step 2: Calculating the impulse response (cont.)

- For $M \leqslant N$ there is a closed form solution for the impulse response

impulsive feedthrough

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

$y_c[n]$ looks like $y_h[n]$ but solve for the coefficients using

$$h[0], h[1], \cdots, h[N-1]$$

- If not given, obtain $h[0], h[1], \cdots, h[N-1]$ from recursive solution

Step 2: Calculating the impulse response (cont.)

$h[n]$

- Using recursion

$$h[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k \delta[n-k] - \frac{1}{a_0} \sum_{k=1}^{N} a_k h[n-k]$$

- For n = 0

$$\begin{aligned} h[0] &= \frac{1}{a_0} b_0 \delta[0] - \frac{1}{a_0} \left(a_1 h[-1] + a_2 h[-2] + \cdots + a_N h[-N] \right) \\ &= \frac{b_0}{a_0} \end{aligned}$$

- For n = 1

$$\begin{aligned} h[1] &= \frac{1}{a_0} b_1 - \frac{1}{a_0} \left(a_1 h[0] + a_2 h[-1] + \cdots + a_N h[-N+1] \right) \\ &= \frac{1}{a_0} b_1 \delta[0] - \frac{1}{a_0} \left(a_1 h[0] \right) \end{aligned}$$

Example 1 (continued)

- Consider again the following difference equation

Note: $a_0=1$ $a_1=-\gamma$ $b_0=1$ $b_1=0$

$$y[n] - \gamma y[n-1] = x[n]$$

- Find the impulse response

Example 1 (continued)

$a_0=1$ $a_1=-\gamma$ $b_0=1$ $b_1=0$

$$y[n] - \gamma y[n-1] = x[n]$$

- From the general impulse formula solution

$$h[n] = \cancelto{0}{\frac{b_1}{a_1}}\delta[n] + y_c[n]u[n]$$

- It follows that

$$h[n] = C_1 \gamma^n u[n]$$

Example 1 (continued)

$$y[n] - \gamma y[n-1] = x[n]$$

- From the recursion equation

$$h[n] = \delta[n] + \gamma h[n-1]$$

- Solving for n=0

$$\begin{aligned} h[0] &= \delta[0] + \gamma h[-1] \\ &= 1 + \gamma \cdot 0 \\ &= 1 \\ &= C_1 \gamma^0 u[0] \end{aligned} \Rightarrow \quad C_1 = 1$$

$$\boxed{h[n] = \gamma^n u[n]}$$

Step 3: Computing the particular solution

- The particular solution is the convolution of the output & input

$$y_p[n] = x[n] * h[n]$$

Example 1 (continued)

- Consider again the following difference equation

$$y[n] - \gamma y[n-1] = x[n]$$

- Find the particular solution assuming $x[n] = \beta^n u[n] \qquad \beta \neq \gamma$

Example 1 (continued)

$$
\begin{aligned}
y_p[n] &= x[n] * h[n] \\
&= \sum_{k=0}^{\infty} h[k]x[n-k] \\
&= \sum_{k=0}^{\infty} \gamma^k u[k] \beta^{n-k} u[n-k] \\
&= \sum_{k=0}^{\infty} \gamma^k \beta^{n-k} u[n-k]
\end{aligned}
$$

Example 1 (continued)

nonzero for $n - k \geqslant 0 \Rightarrow n \geqslant k$

$$y_p[n] = \sum_{k=0}^{\infty} \gamma^k \beta^{n-k} u[n-k]$$

from the sum $k \geqslant 0$

sum is non-zero only for $n \geqslant 0$

$$= u[n]\beta^n \sum_{k=0}^{n} \left(\frac{\gamma}{\beta}\right)^k$$

$$= u[n]\beta^n \frac{1 - \left(\frac{\gamma}{\beta}\right)^{n+1}}{1 - \frac{\gamma}{\beta}}$$

$$= \frac{\beta^{n+1} - \gamma^{n+1}}{\beta - \gamma} u[n]$$

Step 4: Computing the total solution

- The total solution is the sum of the particular solution and the homogenous solution

$$y[n] = y_p[n] + y_h[n]$$

Example I (concluded)

- Consider again the following difference equation

$$y[n] - \gamma y[n-1] = x[n]$$

- Now find the total solution given *y[-1]* and $x[n] = \beta^n u[n]$

$$\beta \neq \gamma$$

Already found in previous parts

$$\begin{aligned} y[n] &= y_p[n] + y_c[n] \\ &= y_c[-1]\gamma^{n+1} + \frac{\beta^{n+1} - \gamma^{n+1}}{\beta - \gamma} u[n] \end{aligned}$$

When do LCCDE describe LTI systems?

IMPORTANT

- A LCCDE described by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- is **also** an LTI system if the **initial conditions** are **zero**

$$y[n] = \cancel{y_h[n]} + y_p[n]$$

  - System must be **at rest** to be LTI
  - Makes sense because $y_p[n] = x[n] * h[n]$

Reference example with two roots.

Two root example

- Solve the following difference equation

$$y[n] - 0.6y[n-1] - 0.16y[n-2] = 5x[n]$$

$$y[-1] = 0 \quad y[-2] = 25/4$$

$$x[n] = 4^{-n}u[n]$$

Two root example (continued)

- Characteristic polynomial

$$r^2 - 0.6r - 0.16 = (r + 0.2)(r - 0.8)$$

- Characteristic equation

$$(r + 0.2)(r - 0.8) = 0$$

- Characteristic roots

$$r_1 = -0.2 \;\; r_2 = 0.8$$

- Characteristic modes

$$(-0.2)^n \text{ and } (0.8)^n$$

Two root example (continued)

- Homogeneous response is

$$y_h[n] = c_1(-0.2)^n + c_2(0.8)^n$$

- Solve for the unknown coefficients using the initial conditions

$$\begin{aligned} y_h[-1] &= c_1(-0.2)^{-1} + c_2(0.8)^{-1} \\ &= 0 \\ y_h[-2] &= c_1(-0.2)^{-2} + c_2(0.8)^{-2} \\ &= \frac{25}{4} \end{aligned}$$

$$\Rightarrow \quad c_1 = \frac{1}{5} \qquad c_2 = \frac{4}{5}$$

$$\boxed{y_h[n] = \frac{1}{5}(-0.2)^n + \frac{4}{5}(0.8)^n}$$

Two root example (continued)

- Find the impulse response

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

- In this case $y_c[n] = c_1(-0.2)^n + c_2(0.8)^n$ and $b_N = 0$

$$\Rightarrow \quad h[n] = [c_1(-0.2)^n + c_2(0.8)^n]u[n]$$

- Need to solve for unknown constants using *h[-1]=0, h[-2]=0*
- From the definition of impulse response

$$h[n] - 0.6h[n-1] - 0.16h[n-2] = 5\delta[n]$$

Two root example (continued)

- Rearranging terms

$$h[n] = 5\delta[n] + 0.6h[n-1] + 0.16h[n-2]$$

- Using initial conditions

$$\begin{aligned} h[0] &= 5\delta[0] + 0.6h[-1] + 0.16h[-2] \\ &= 5 \\ h[1] &= 5\delta[1] + 0.6h[0] + 0.16h[-1] \\ &= 0.6 \cdot 5 = 3 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} 5 &= c_1 + c_2 \\ 3 &= c_1(-0.2) + c_2(0.8) \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 4 \end{aligned}$$

Two root example (continued)

- Impulse response is

$$\boxed{h[n] = [(-0.2)^n + 4(0.8)^n]u[n]}$$

- The particular response is found by solving

$$\begin{aligned} y_p[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \end{aligned}$$

$$= \sum_{k=-\infty}^{\infty} (-0.2)^k u[k] 4^{-(n-k)} u[n-k] + 4 \sum_{k=-\infty}^{\infty} (0.8)^k u[k] 4^{-(n-k)} u[n-k]$$

Two root example (continued)

- Solve the convolution in a similar way as the past example

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} (-0.2)^k u[k] 4^{-(n-k)} u[n-k] + 4 \sum_{k=-\infty}^{\infty} (0.8)^k u[k] 4^{-(n-k)} u[n-k]$$

$$= \left[\frac{(0.25)^{n+1} - (-0.2)^{n+1}}{0.25 - (-0.2)} + 4 \frac{(0.25)^{n+1} - (0.8)^{n+1}}{0.25 - 0.8} \right] u[n]$$

$$= \boxed{[-05.05(0.25)^{n+1} - 0.22(-0.2)^{n+1} + 7.27(0.8)^{n+1}]u[n]}$$

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Two root example (continued)

- The total solution is

$$y[n] = y_h[n] + y_p[n]$$

$$= \boxed{\frac{1}{5}(-0.2)^n + \frac{4}{5}(0.8)^n + [-05.05(0.25)^{n+1} - 0.22(-0.2)^{n+1} + 7.27(0.8)^{n+1}]u[n]}$$