Lecture 11

Frequency response and the Fourier transform

Preview of today's lecture

- Fourier series application examples

- Filtering a periodic signal
  - Impact of filtering on the FS coefficients
  - Types of common filters: lowpass, highpass, bandpass, bandstop

- Introduction to the Fourier transform

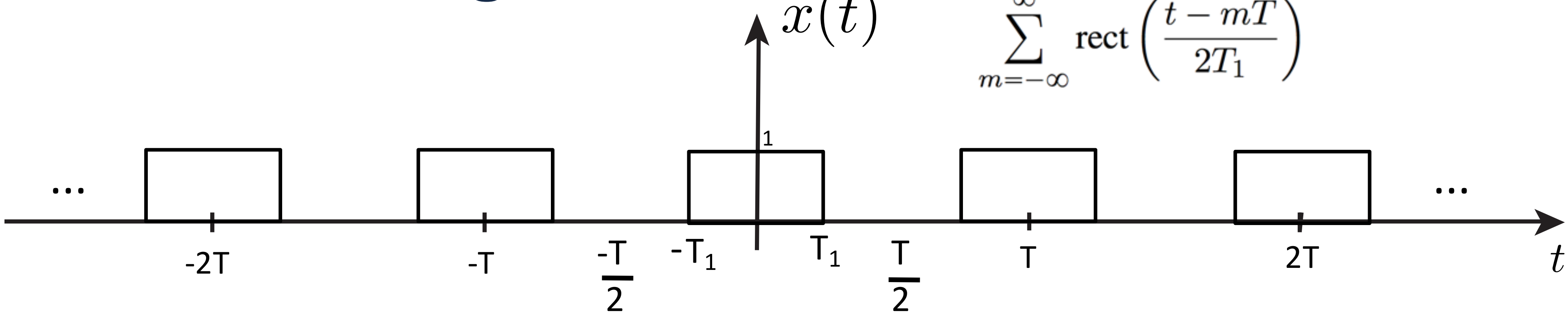
Basic signals

	Time domain $x(t)$	Fourier coefficients a_k
Constant (periodic for any T)	c	$c\delta[k]$
Cosine	$\cos(\omega_0 t)$	$\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$
Sine	$\sin(\omega_0 t)$	$\frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]$
Impulse train	$\sum_{m=-\infty}^{\infty} \delta(t - mT)$	$\frac{1}{T}$

Pulse train in general

From O&W Example 3.5

$$\sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{t-mT}{2T_1}\right)$$



From the book

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \qquad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

$$a_k = \frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

with fundamental frequency

$$a_k = \frac{2T_1}{T} \text{sinc}\left(\frac{k2T_1}{T}\right)$$

simplified

Fourier series properties

- Let $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) \overset{FS}{\longleftrightarrow} a_k \qquad\qquad y(t) \overset{FS}{\longleftrightarrow} b_k$$

	Time domain	FS domain
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shift	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$
Time reversal	$x(-t)$	a_{-k}

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Fourier series properties (continued)

	Time domain	FS domain
Time scaling	$x(\alpha t)$	a_k $\quad T_{\text{new}} = \frac{T}{\alpha}$ period changes
Conjugate	$x^*(t)$	a^*_{-k}
Multiplication	$x(t)y(t)$	$\sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$
Derivative	$\frac{d}{dt}x(t)$	$a_k(jk\omega_0)$
Parseval's Theorem	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	

Fourier series and symmetry

	Time domain	FS domain
Conjugate symmetry	$x(t)$ real	$a_k = a^*_{-k}$
Real and even	$x(t)$ real and even	a_k real and even
Real and Odd	$x(t)$ real and odd	a_k imag. and odd

Application Example 6

(from O&W 3.26)

- Let $x(t)$ be a periodic signal whose FS coefficients are

$$a_k = \begin{cases} 2 & k = 0 \\ j(1/2)^{|k|} & \text{otherwise} \end{cases}$$

- Is $x(t)$ real?
  - Real signals must satisfy $x(t) = x^*(t)$ or $a_k = a^*_{-k}$ not satisfied here
- Is $x(t)$ even?
  - Even signals satisfy $x(t) = x(-t)$ or $a_k = a_{-k}$ yes is satisfied
- Is $\frac{dx(t)}{dt}$ even?
  - The FS coefficients of $\frac{dx(t)}{dt}$ are $(j\omega_0 k)a_k$ for which $(j\omega_0 k)a_k \neq -(j\omega_0 k)a_{-k}$

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

Filtering periodic signals

Key points

- Explain how LTI systems impact signals
- Distinguish between different kinds of frequency filters

Recall the implications of the eigenfunction property

- General case

$$e^{j\omega_0 t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

- Special case for real impulse responses

$$\cos(\omega_0 t) \longrightarrow \boxed{H(j\omega)} \longrightarrow |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$$

How to find the frequency response?

- If the system is described by a differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \Rightarrow \quad H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

- If you have the impulse response, compute the Fourier transform

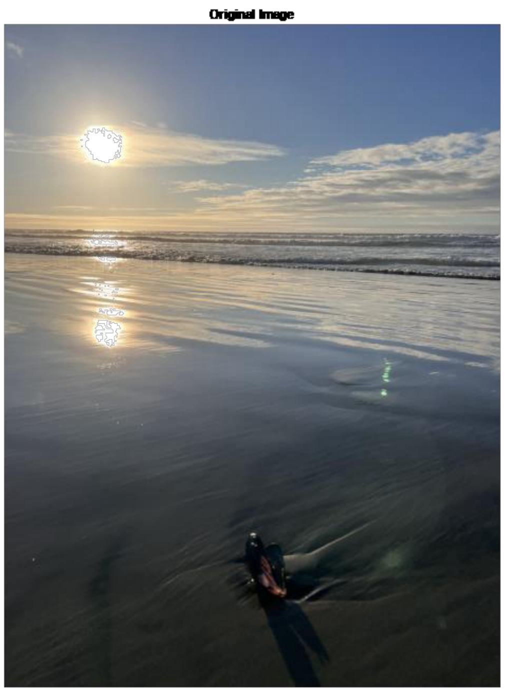
$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

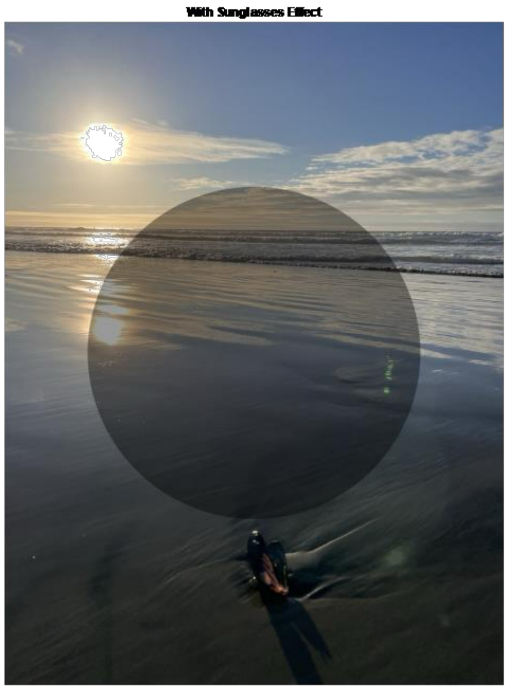
Impact of LTI systems on periodic signals

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \sum_{k=-\infty}^{\infty} \overbrace{a_k H(j\omega_0 k)}^{\text{New FS coefficients}} e^{j\omega_0 kt}$$

Fourier series coefficients are modified by the frequency response of the system

General concept of a filter

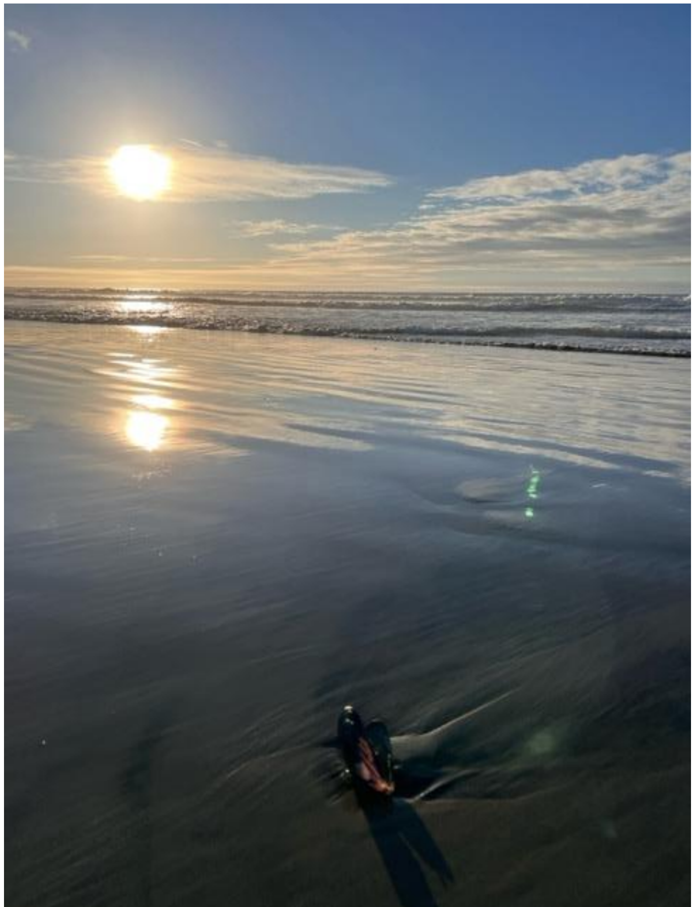


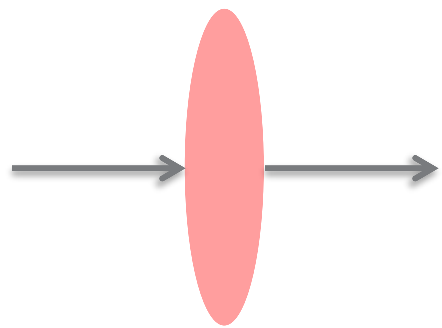


Coffee filter

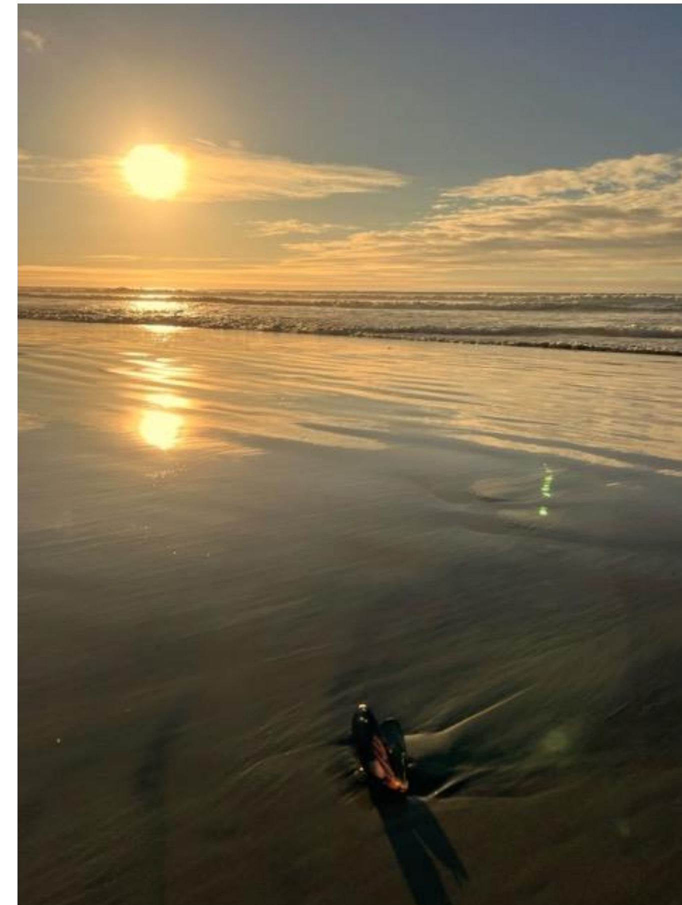
Sunglasses

Filtering an image to enhance the red



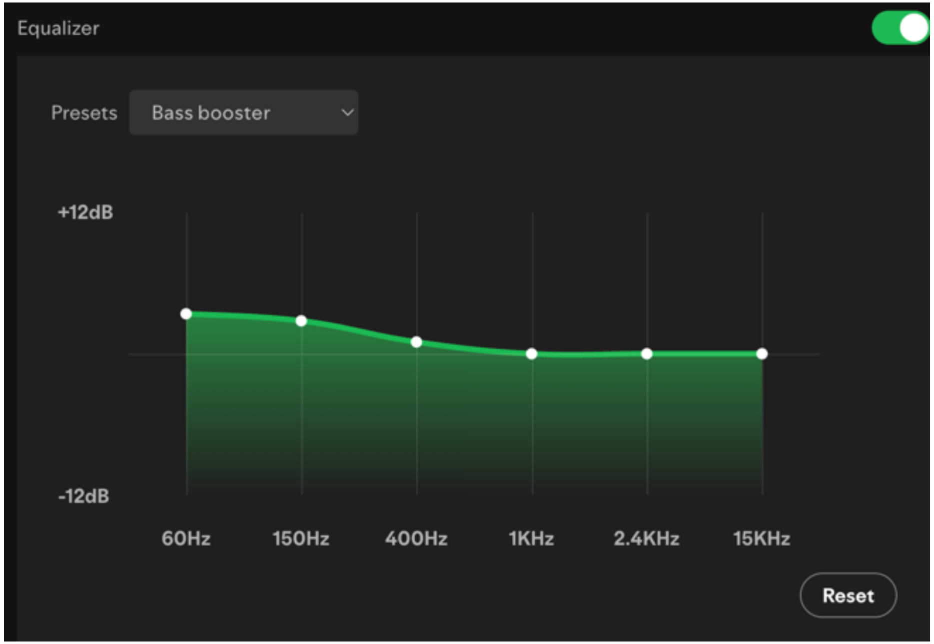


Red filter



Modify the frequency content of the signal to create a sunset effect

Filtering a signal to enhance the bass



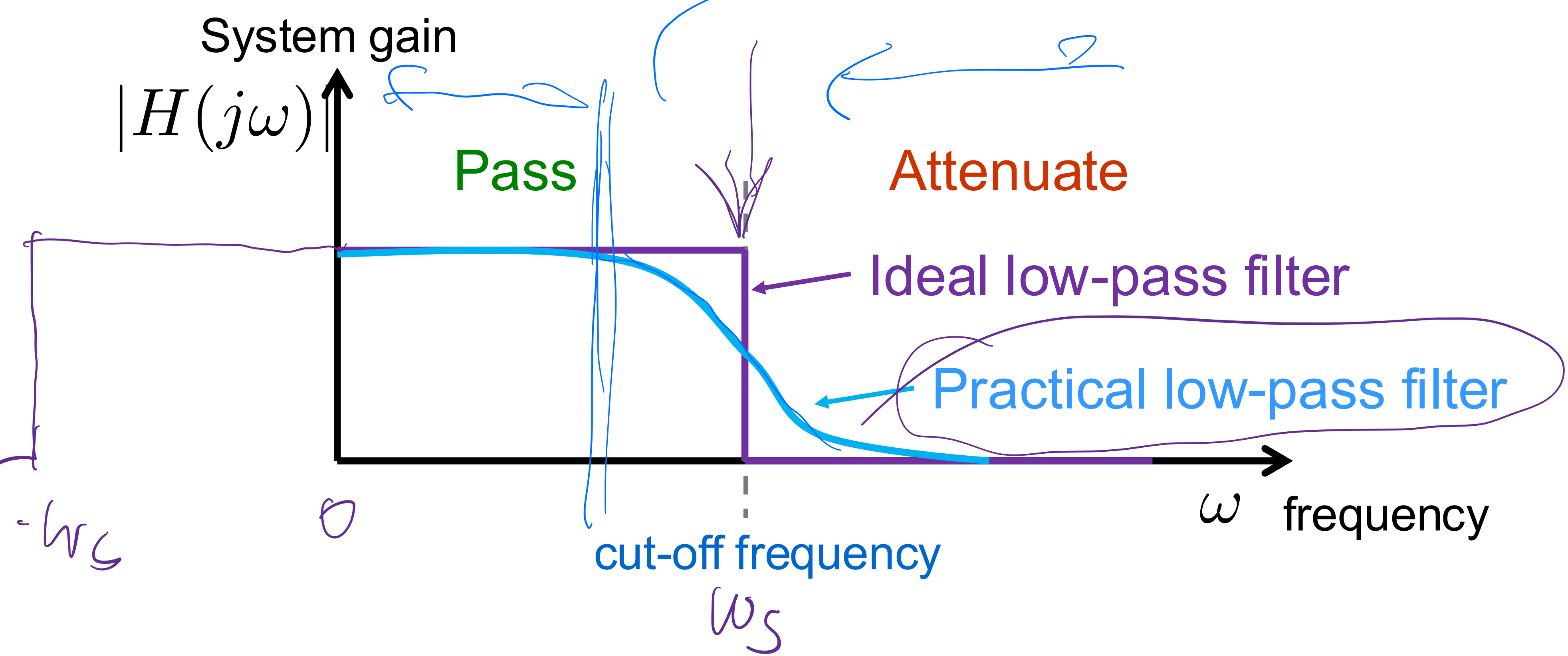
Change the frequency content of audio playback per personal preference

example of Spotify pre-set equalizer settings

Low-pass filter

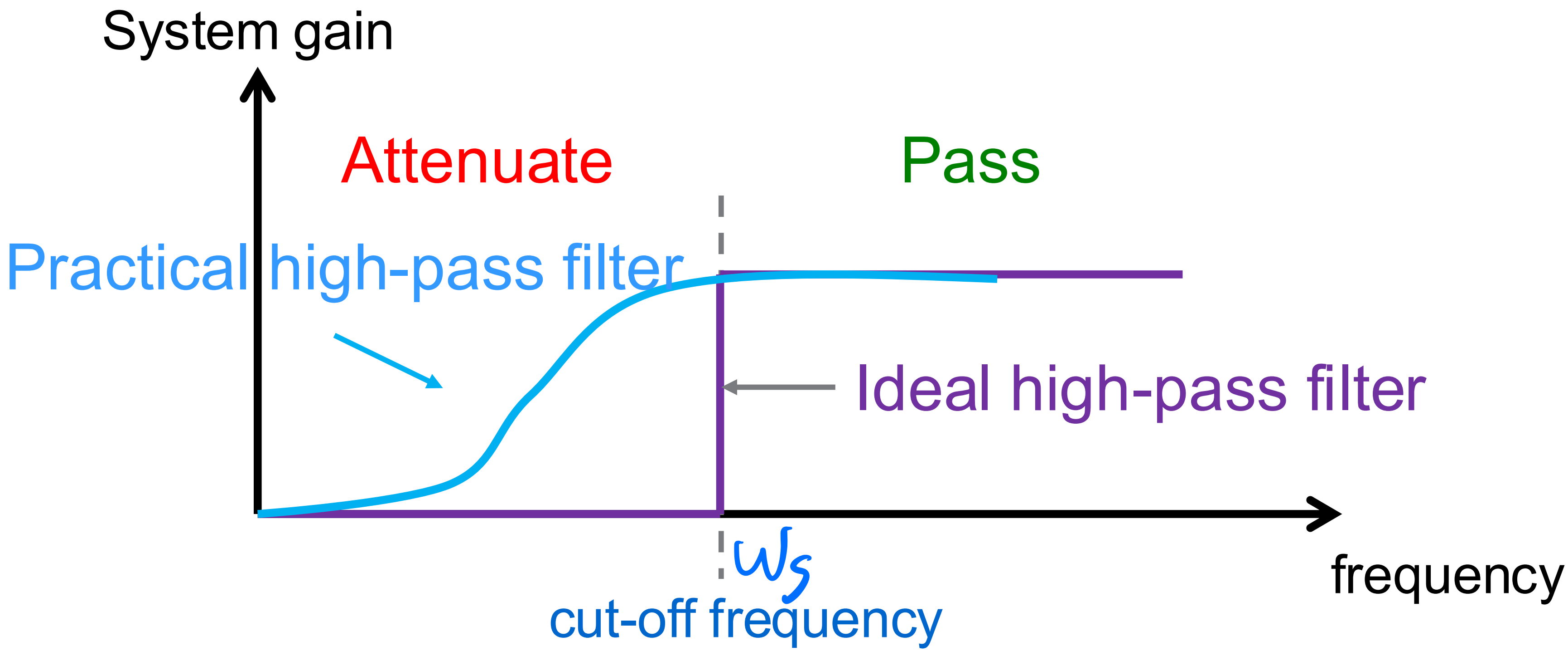
Transition

- Systems that pass low frequencies, attenuate high frequencies



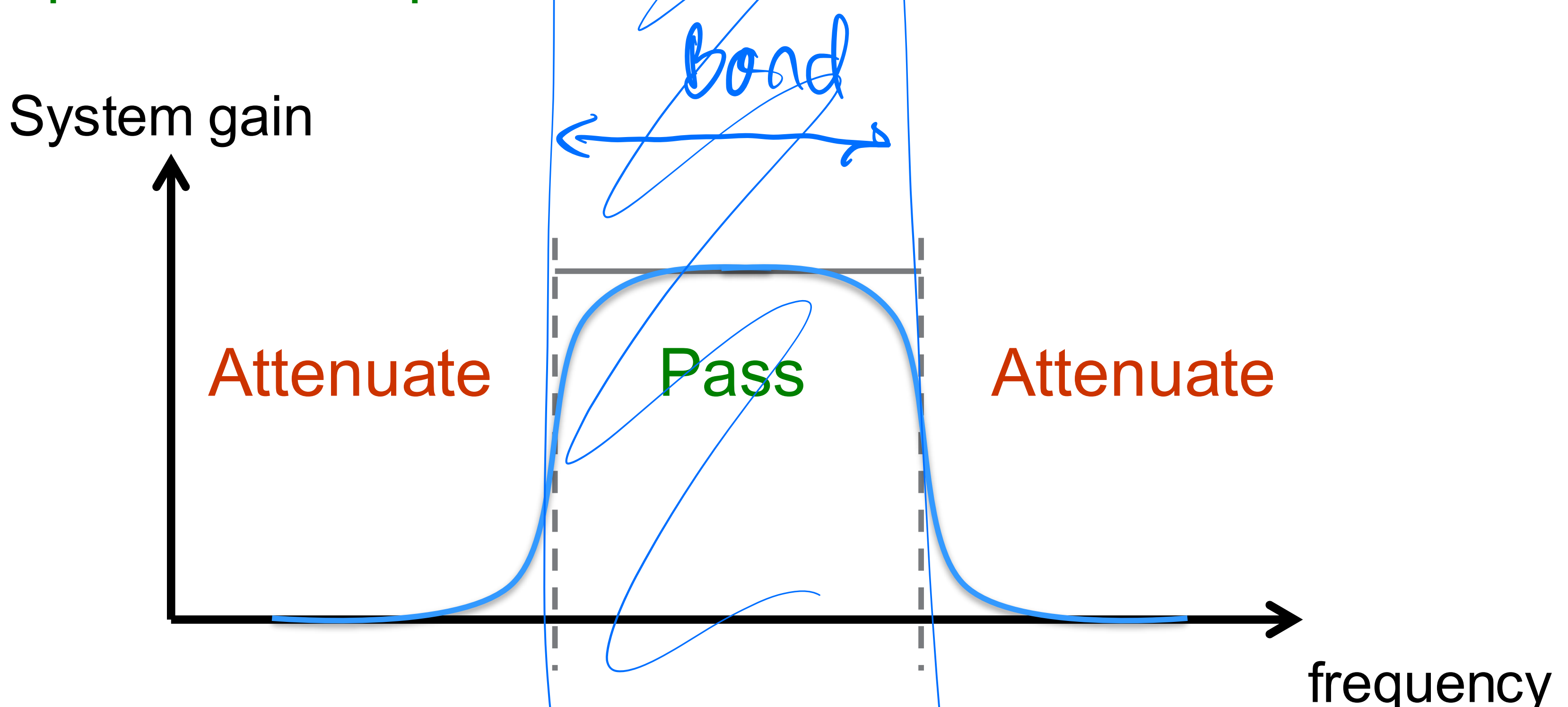
High-pass filters (HPFs)

- Systems that pass high frequencies, attenuate low frequencies



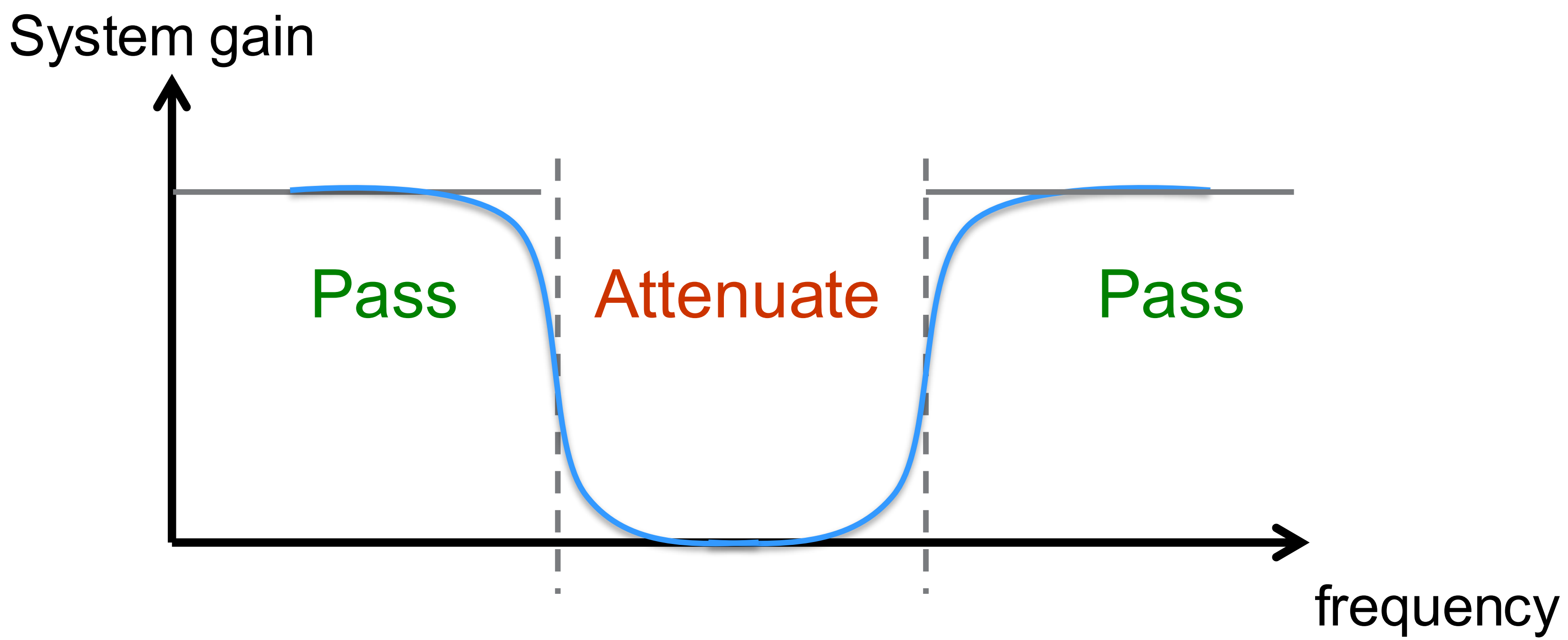
Band-pass filters

- A specific band is passed, and outside this band is attenuated



Band-stop (notch) filters

- Stop (attenuates) a certain band, and passes the other frequencies

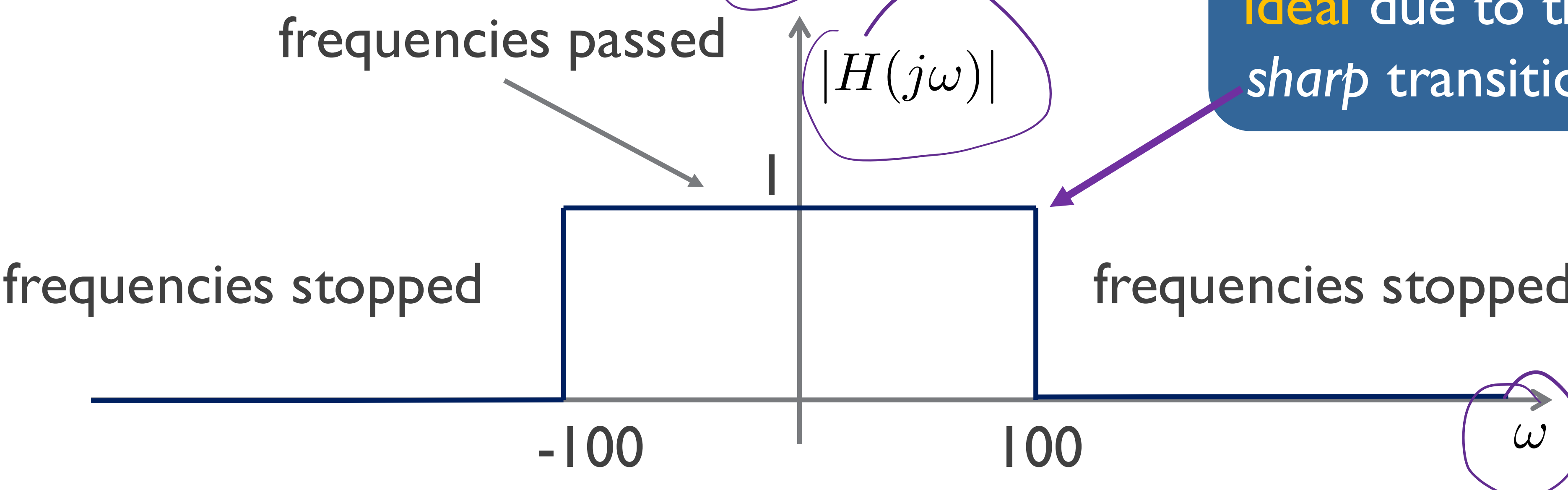


Example: Low-pass filtering a periodic signal I

- Consider an ideal low-pass filter whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leqslant 100 \\ 0, & |\omega| > 100 \end{cases}$$

We call this ideal due to the *sharp* transition



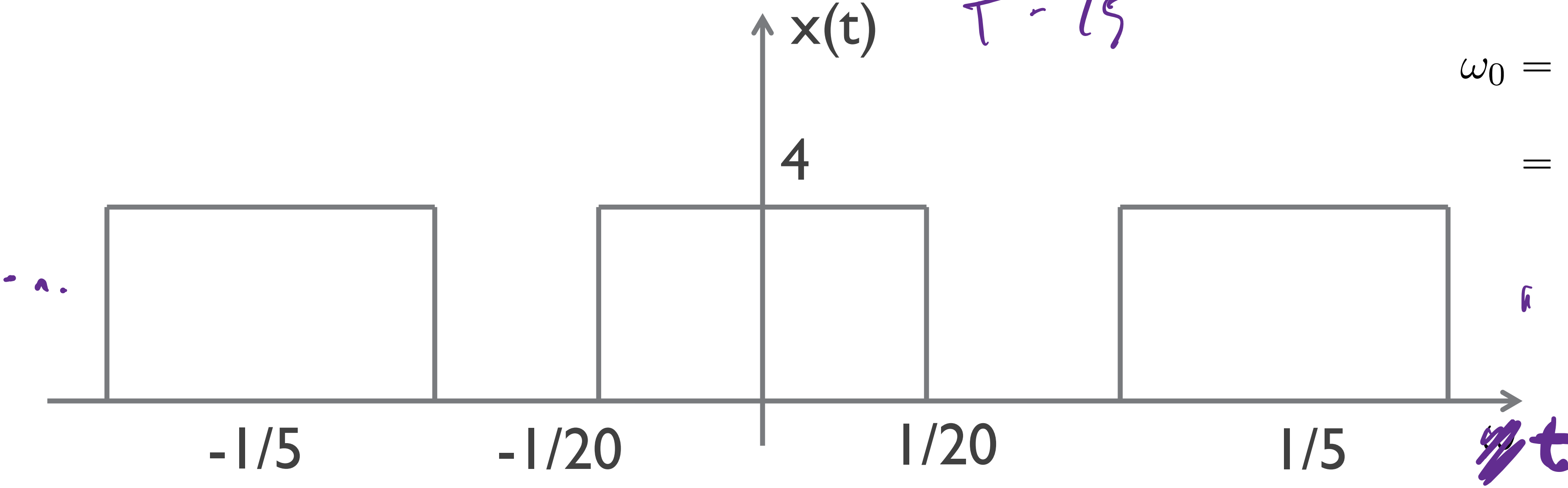
Example: Low-pass filtering a periodic signal 2

- Find the output if the input signal is

$T_1 = 1/20$

$T = 1/5$

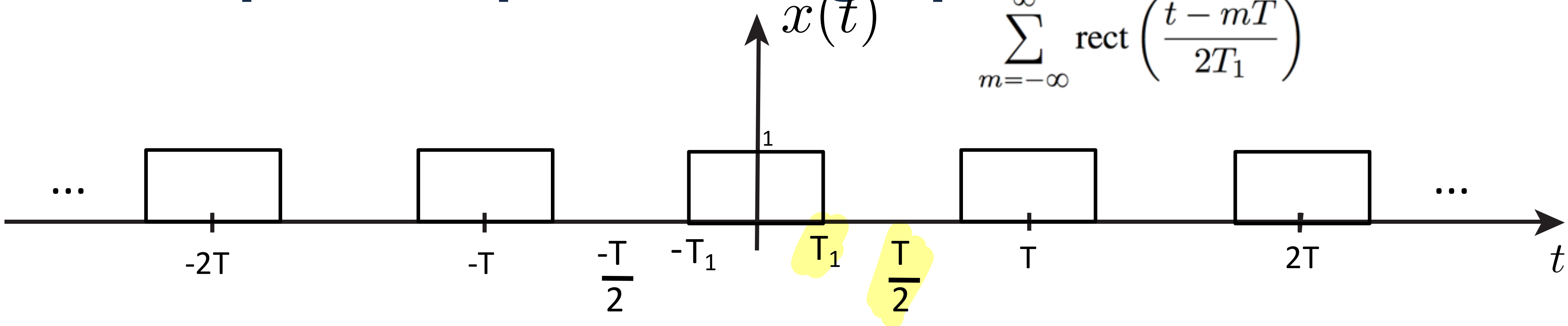
$$T = \frac{1}{5}$$
$$T_1 = \frac{1}{20}$$
$$\omega_0 = \frac{2\pi}{1/5} = 10\pi$$



Square wave with period T=1/5

From O&W Example 3.5

Example: Low-pass filtering a periodic signal 3



$$\sum_{m=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-mT}{2T_1}\right)$$

From the book

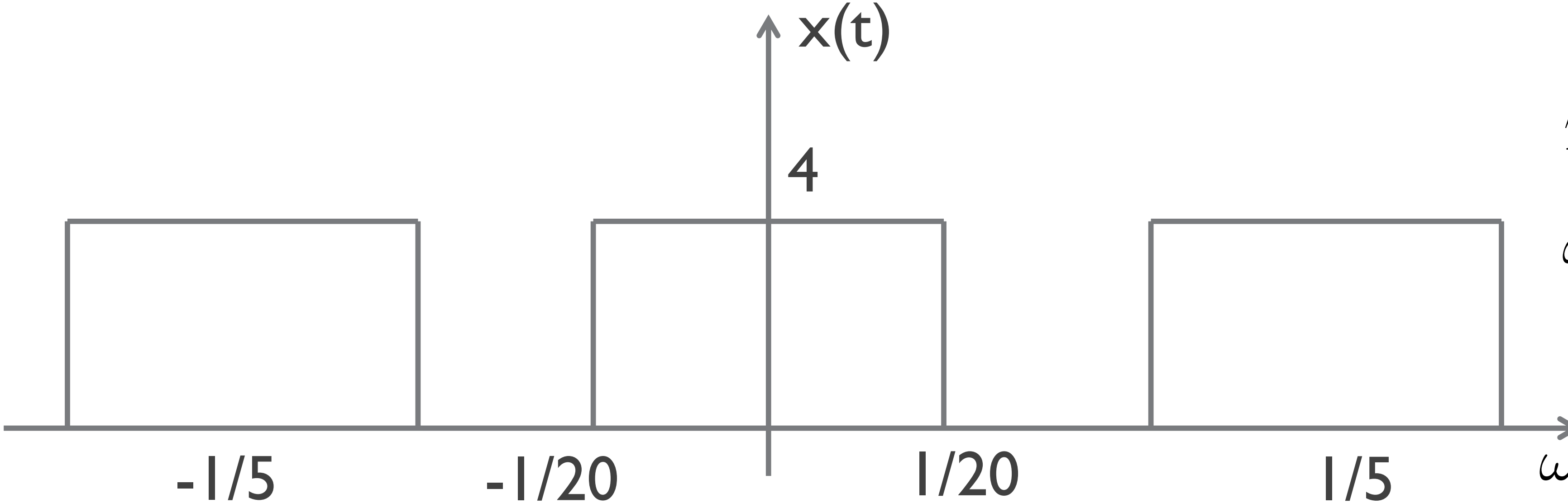
$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \quad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$
(for connecting to results in later lectures)

$$a_k = \frac{2T_1}{T}\mathrm{sinc}\left(\frac{k2T_1}{T}\right)$$

Example: Low-pass filtering a periodic signal 4



$$T = \frac{1}{5}$$
$$T_1 = \frac{1}{20}$$
$$\omega_0 = \frac{2\pi}{1/5} = 10\pi$$

Period $T = \frac{1}{5}$

Rectangle size $T_1 = \frac{1}{20}$

Fundamental frequency $\omega_0 = \frac{2\pi}{1/5} = 10\pi$

Fourier series coefficients

$$k \neq 0 \quad a_k = \frac{4}{\pi k} \sin\left(\pi k \tfrac{1}{2}\right)$$
$$k = 0 \quad a_0 = 2$$

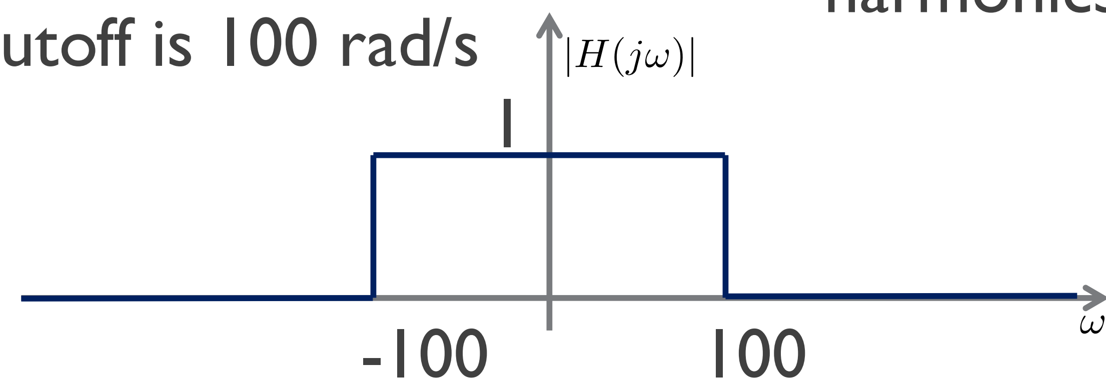
(note effect of scaling by 4)

Example: Low-pass filtering a periodic signal 5

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt}$$

harmonics are at frequencies $k\omega_0$

cutoff is 100 rad/s



$$\omega_0 = 10\pi \approx 31.4$$

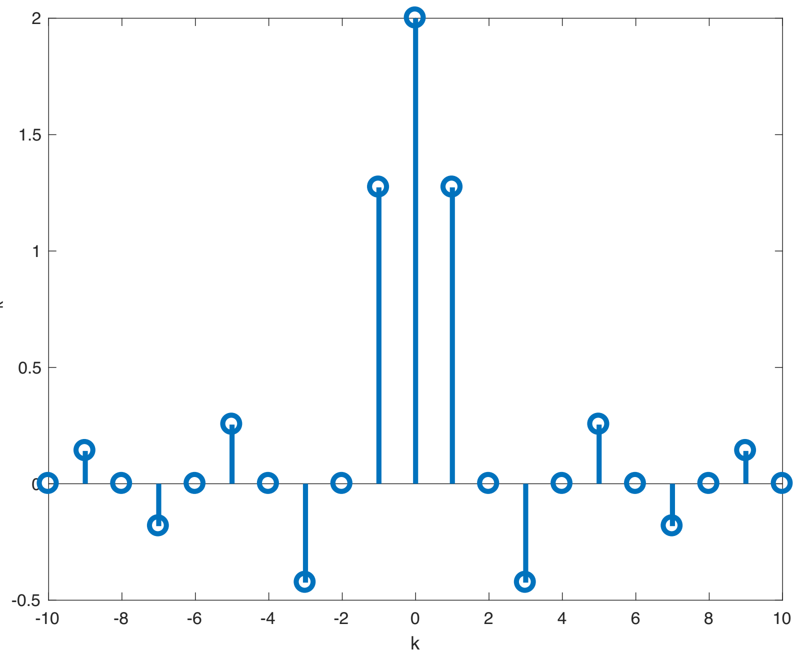
$$2\omega_0 = 2 \cdot 10\pi \approx 62.8$$

$$3\omega_0 = 3 \cdot 10\pi \approx 94.2$$

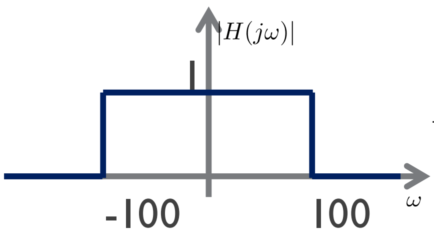
$$\cancel{4\omega_0 = 4 \cdot 10\pi \approx 125.6}$$

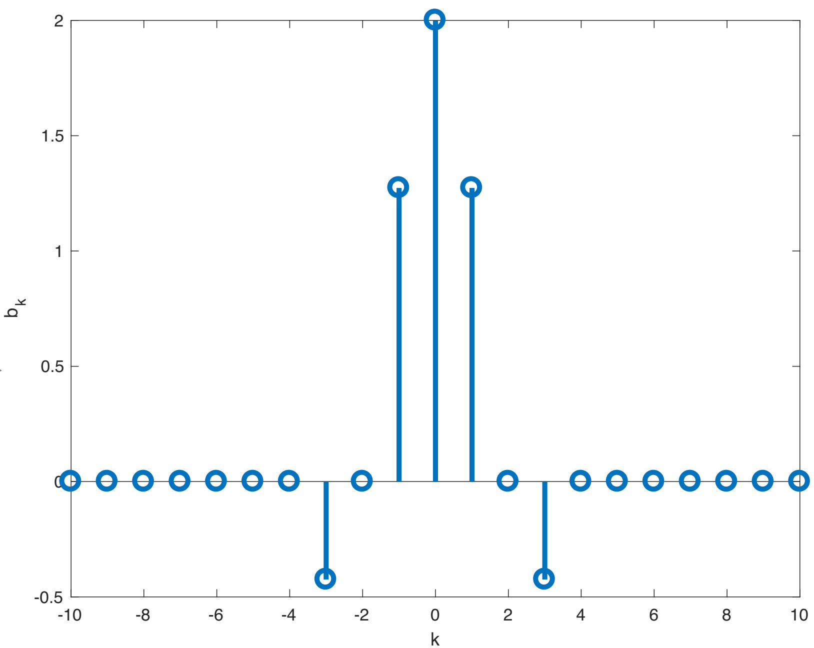
Fourier series coefficients are modified by the frequency response of the system

Example: Low-pass filtering a periodic signal 6



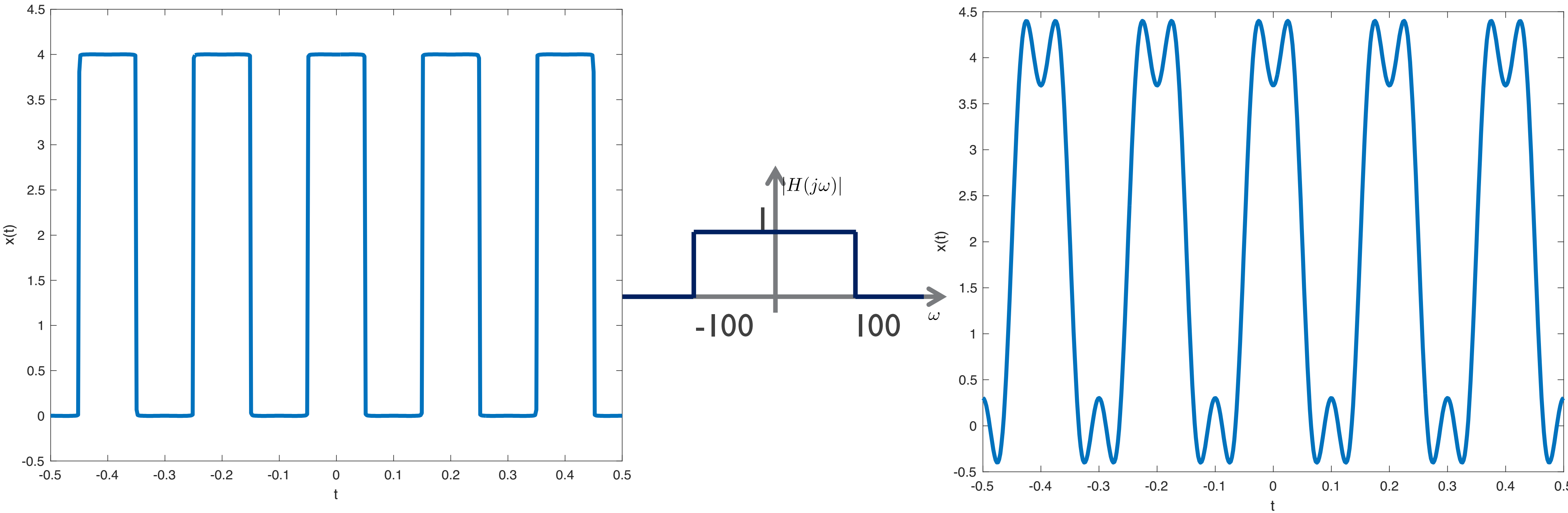
Fourier coefficients before the LTI system





Fourier coefficients after the LTI system

Example: Low-pass filtering a periodic signal 7



Time domain signal before the LTI system

Time domain signal after the LTI system

Frequency response summary

- If the input to an LTI system is periodic, then the output is also periodic with the same period
- LTI systems impact the amplitude and phase of the Fourier series coefficients as determined by the frequency response of the system
- To determine the effect of an LTI system on a periodic signal, compute the Fourier transform of the impulse response and evaluate it at multiples of the fundamental frequency $k\omega_0$

Fourier transform

Key points

- Define Fourier transform
- Determine the Fourier transforms of ~~CT and DT signals~~

Fourier transform for all signals

$j\omega$

- For a signal $x(t)$, the Fourier transform (FT) $X(\cancel{j\omega})$ is

$$X(j\omega) = \mathcal{F}\{x(t)\}$$
$$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- The notation $\mathcal{F}\{\cdot\}$ means to take the Fourier transform of the function inside the brackets

- In some books, $X(\omega)$ is used instead of $X(j\omega)$

No periodicity assumption in the signal

Fourier series for periodic signals

- Consider the periodic signal $x(t)$ with period T: $x(t+T)=x(t)$
- The Fourier series representation of the periodic signal $x(t)$ is

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$k\omega_0$

where $\omega_0=\dfrac{2\pi}{T}$ is the fundamental frequency

- The Fourier series coefficients of $x(t)$ are $\{a_k\}$ and a_0 is DC
- The k-th harmonic components of $x(t)$ are a_k and a_{-k}

Interpreting the Fourier transform

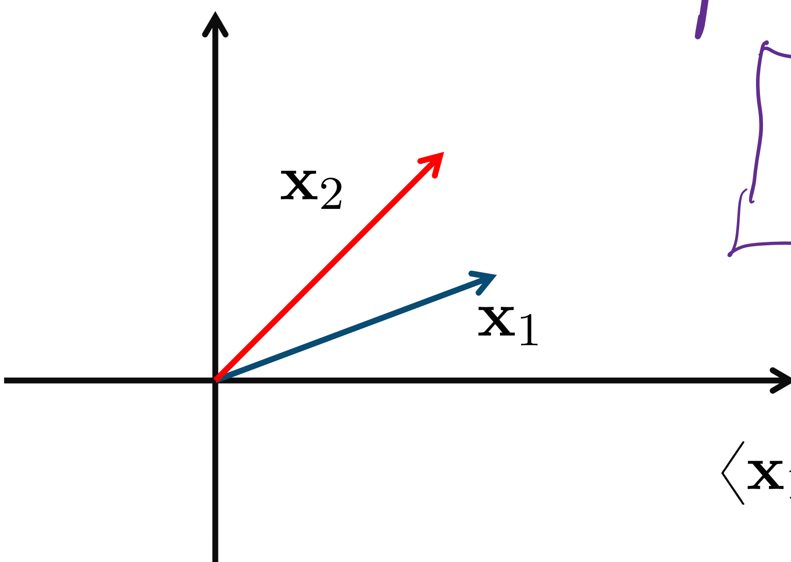
- The Fourier transform can be written

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \langle x(t), e^{j\omega t} \rangle$$

Inner product

How much does $x(t)$ look like complex sinusoid $e^{j\omega t}$ with frequency ω?



$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ Measures how much the vectors "line up" with each other

Why do we care about complex sinusoids anyways?

- For an LTI system, with input $e^{j\omega t}$ it is easy to compute the output

$$e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega)e^{j\omega t}$$

- If a signal can be represented as a sum of sinusoids, then it is possible to compute the output of an LTI system

$\boxed{\text{without convolution}}$

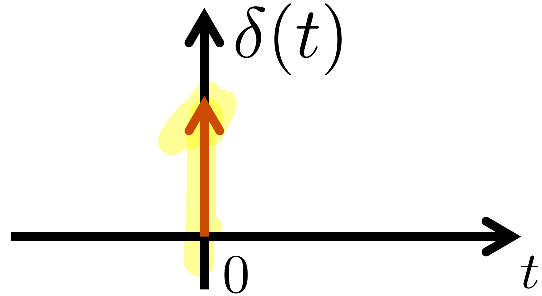
Fourier transform!!

- Further notice that

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

Example – Unit impulse

- Consider the signal $\boxed{x(t) = \delta(t)}$



① Plug in

- Its FT is given by

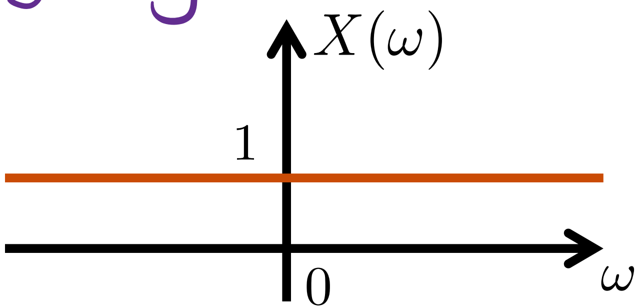
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

sifting

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega 0} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) dt$$

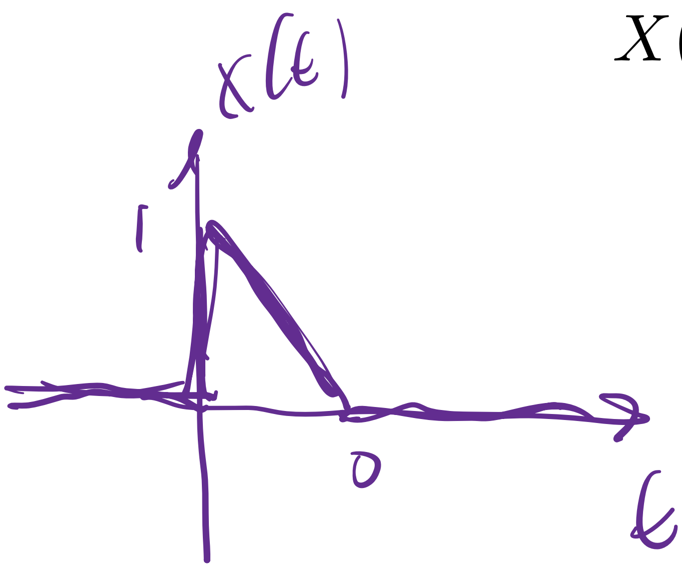
$$= 1$$



Example - Triangle

- Compute the Fourier transform of $$x(t) = \begin{cases} 1 - t & t \in [0,1] \\ 0 & \text{else} \end{cases}$$

- Solution:



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_0^1 (1-t) e^{-j\omega t} dt$$

$$= \int_0^1 e^{-j\omega t} dt - \int_0^1 t e^{-j\omega t} dt$$

plug in + simplify

Example - Triangle (continued)

- Compute the Fourier transform of $x(t) = \begin{cases} 1-t & t \in [0,1] \\ 0 & \text{else} \end{cases}$

integration by parts

- Solution:

$$X(j\omega) = \int_0^1 e^{-j\omega t} dt - t\frac{1}{-j\omega}e^{-j\omega t}|_0^1 + \int_0^1 \frac{1}{-j\omega} e^{-j\omega t} dt$$

$$= \int_0^1 e^{-j\omega t} dt - t\frac{1}{-j\omega}e^{-j\omega t}\Big|_0^1 + \left(\frac{1}{j\omega}\right)^2 e^{-j\omega t}\Big|_0^1$$

$$= \frac{1}{j\omega} - \frac{1}{j\omega}e^{-j\omega} + \frac{1}{j\omega}e^{-j\omega} - \frac{1}{\omega^2}e^{-j\omega} + \frac{1}{\omega^2}$$

$$= \frac{1}{j\omega} - \frac{1}{\omega^2}e^{-j\omega} + \frac{1}{\omega^2}$$

Inverse Fourier transform

$j\omega$

- Given the frequency response $X(j\omega)$, $x(t)$ is given by the

inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

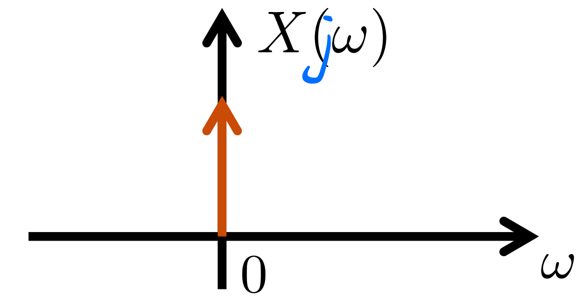
$$= \mathcal{F}^{-1}\{X(j\omega)\}$$

- Transform looks similar to direct transform except
  - Sign of the exponential is different
  - Scaling factor in front (results from using radians and not Hertz)

Similarity will lead to the concept of duality

Example – Unit impulse in frequency

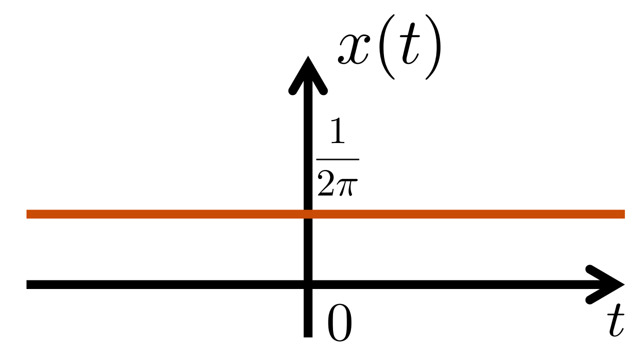
- Consider the signal $\boxed{X(j\omega) = \delta(\omega)}$



- Its FT is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}$$



Notice the duality here in each domain

Example – Shifted delta in frequency

- Consider an impulse in the frequency domain $X(j\omega) = \delta(\omega - \omega_0)$

- Its inverse FT is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega$$

$$= \frac{e^{j\omega_0 t}}{2\pi} \underbrace{\int_{-\infty}^{\infty} \delta(\omega - \omega_0) d\omega}_{1}$$

$$= \frac{e^{j\omega_0 t}}{2\pi}$$

Example – A few deltas in frequency

- Use the Fourier transform synthesis equation to determine the inverse Fourier transform of

$$X(\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$$

- Solution

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)\right) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\left(2\pi e^{j0t} + \pi e^{j4\pi t} + \pi e^{-j4\pi t}\right)$$

$$= 1 + \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t}$$

$$= 1 + \cos(4\pi t)$$

a_1

a_{-1}

Summarizing the Fourier transform and its inverse

Fourier transform (analysis)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform (synthesis)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\boxed{x(t) \leftrightarrow X(j\omega)}$$

Note: An alternative formulation that is common uses Hertz rather than radians/sec

Fourier transform in Hertz

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad f \text{ in Hz}$$

Inverse Fourier transform from X(f) (note lack of 1/2π)

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

In EE 45 we will use $X(j\omega)$