ECE 101 - Spring 2025

Linear Systems Fundamentals

Midterm Exam Review Topics

1 Useful Identities

Complex numbers and trigonometry

- Complex arithmetic, magnitude, phase, triangle inequality
- Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$
- Evaluation of complex exponential $e^{j\theta}$ at standard angles (e.g., $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$)
- Basic trigonometric identities:

$$\begin{aligned}
\cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)\ ;\ \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \\
e^{j(A+B)} &= \cos(A+B) + j\sin(A+B) \\
e^{jA}e^{jB} &= (\cos(A) + j\sin(A))(\cos(B) + j\sin(B)) \\
&= (\cos(A)\cos(B) - \sin(A)\sin(B)) + j(\sin(A)\cos(B) + \cos(A)\sin(B)) \\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)\ ;\ \sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \\
\cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)\ ;\ \sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B) \\
\sin(A)\cos(B) &= \frac{1}{2}(\sin(A+B) + \sin(A-B)) \\
\cos(A)\cos(B) &= \frac{1}{2}(\cos(A+B) + \cos(A-B))\ ;\ \sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))
\end{aligned}$$

- Infinite geometric series:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z},\ \text{for } |z| < 1.$$

$$\sum_{n=0}^{\infty} nz^n = \frac{z}{(1-z)^2},\ \text{for } |z| < 1.$$

- Finite geometric series:

$$\sum_{n=0}^{N-1} z^n = \begin{cases} N & z = 1 \\ \frac{1-z^N}{1-z} & \text{for any complex } z \neq 1. \end{cases}$$

- Useful identities involving complex exponentials:

$$1 + e^{-jM} = e^{-j\frac{M}{2}}\left(e^{j\frac{M}{2}} + e^{-j\frac{M}{2}}\right) = 2e^{-j\frac{M}{2}}\cos\left(\frac{M}{2}\right)$$

$$1 - e^{-jM} = e^{-j\frac{M}{2}}\left(e^{j\frac{M}{2}} - e^{-j\frac{M}{2}}\right) = 2je^{-j\frac{M}{2}}\sin\left(\frac{M}{2}\right)$$

2 Signals - Chapter 1

- Signal energy and power

  Total energy (CT): $E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt$

  Total energy (DT): $E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$

  Average power (CT): $P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

  Average power (DT): $P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

  For periodic signals:

  Average power in one period (CT): $\frac{1}{T}\int_T |x(t)|^2 dt$

  Average power in one period (DT): $\frac{1}{N}\sum_{n=<N>} |x[n]|^2$

- Transformations of independent variable
  - Time shifting, time reversal, time scaling
  - Interpretations of $x(at-b)$
    * Shift by b, then scale by a
    * Scale by a, then shift by b/a
  - Interpretations of $x[an-b]$, $\quad a, b$ integers
    * Shift by b, then scale (decimate) by a

- Periodic signals
  - Periodicity conditions
  - Fundamental period and frequency
  - Finding fund. period/frequency of the sum of periodic signals
  - Periodicity and scaling

- Even and Odd signals
  - Definitions of even and odd signals,
  - Even-odd decomposition theorem: $x(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\}$
    where $\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2}$ and $\mathcal{O}d\{x(t)\} = \frac{x(t)-x(-t)}{2}$

- CT and DT impulse and unit step signals
  - Relationships
    $\sum_{n=-\infty}^{\infty} \delta[n] = 1$
    $\delta[n] = u[n] - u[n-1]$
    $\delta[n] = u[n] - u[n-1]$
    $u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{k=-\infty}^{n} \delta[k]$

$\int_{-\infty}^{\infty} \delta(t)dt = 1$
$\delta(t) = \frac{d}{dt}u(t)$
$u(t) = \int_{0}^{\infty} \delta(t-\tau)d\tau = \int_{-\infty}^{t} \delta(\tau)d\tau$

  - Sampling and sifting properties
    $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
    $\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$
    $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
    $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$
  - Representation property
    $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
    $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

- Complex exponential signals

  CT: $x(t) = ce^{at}, c, a \in \mathbb{C}$

  DT: $x[n] = c\alpha^n, c, \alpha \in \mathbb{C}$

  $x(t) = e^{j\omega_0 t}$ periodic, fund. frequency ω_0, fund. period $T = 2\pi/\omega_0$

  $x[n] = e^{j\Omega_0 n}$ periodic in n if and only if $\Omega_0 = 2\pi m/N$, for $m, N \in \mathbb{Z}, N > 0$

  If $\gcd(m, N) = 1$, fund. period N , fund. frequency $2\pi/N$

  $x[n] = e^{j\Omega n}$ periodic in Ω, period 2π.

3 Systems - Chapters 1 and 2

- **Basic system properties**

  Memoryless: output at time n does not depend on inputs before or after time n

  Invertible: distinct input signals produce distinct output signals

  Causal: output at time n does not depend on inputs after time n

  Stable: bounded input signals produces bounded output signals

  Time-invariant: $x(t)$ produces $y(t) \Rightarrow x(t-t_0)$ produces $y(t-t_0)$

  Linear: additive and scalable

- **System impulse response and step response (for any system)**

  Input $\delta(t)$ produces impulse response $h(t)$

  Input $u(t)$ produces step response $s(t)$

- **LTI systems**

  Linear (additive and scalable) and time-invariant.

  The impulse response $h(t)$ or $h[n]$ characterizes the system

  CT: $y(t) = x(t) * h(t)$ DT: $y[n] = x[n] * h[n]$

- **Relationship between impulse response and step response**

  $s(t) = \int_{-\infty}^{t} h(\tau) d\tau$ $h(t) = \frac{ds(t)}{dt}$

  $s[n] = \sum_{k=-\infty}^{n} h[k]$ $h[n] = s[n] - s[n-1]$

- **Convolution formulas (DT and CT)**

  Convolution sum formula: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$

  Convolution integral formula $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$

- **Properties of convolution**

  Commutativity, associativity, distributivity over addition

  Convolution with shifted impulse:

  CT: $x(t) * \delta(t - t_0) = x(t - t_0)$; DT: $x[n] * \delta[n - n_0] = x[n - n_0]$)

  Convolution with unit step:

  CT: $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$; DT: $x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$

- **Impulse response of serial and parallel concatenations of LTI systems.**

  (shown here for CT systems; similar for DT systems)

  Serial: $h(t) = h_1(t) * h_2(t)$

  Parallel: $h(t) = h_1(t) + h_2(t)$

- **Impulse response and properties of LTI systems**

  (shown here for CT systems; similar for DT systems)

  Memoryless: $h(t) = a\delta(t)$

  Invertible: There exists $g(t)$ such that $h(t) * g(t) = \delta(t)$

  (necessary condition, also sufficient for inputs $x(t)$ with $x(t) * h(t)$ and $x(t) * g(t)$ both well defined and finite)

  Causal: $h(t) = 0$ for $t < 0$

  Stable: $h(t)$ is absolutely integrable

- **Differentiation property of CT LTI systems**

  $x(t) \to y(t) \Rightarrow \frac{dx(t)}{dt} \to \frac{dy(t)}{dt}$

- **LTI systems defined by differential/difference equations**

  $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$ $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

- **Non-recursive filters**

  Non-recursive filters: no feedback of the output

  $y[n] = \sum_{k=0}^{M} b_k x[n-k]$

  DT blur filters with rectangular impulse response

  $h[n] = \frac{1}{N} \sum_{k=0}^{N-1} \delta[n-k] = \frac{1}{N}(u[n] - u[n-N])$

- **Recursive filters**

  Recursive filters: feedback of output

  $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

  DT first order filters: $y[n] - ay[n-1] = x[n]$, a real, $|a| < 1$

  a positive: low-pass filter; $h[n] = a^n u[n]$ right-sided decaying exponential

  a negative: high-pass filter; $h[n] = a^n u[n]$ right-sided alternating-polarity decaying exponential

4 CT and DT Fourier Series - Chapter 3

- **Key equations**

  CTFS equations for periodic signal, fund. period T, fund. frequency $\omega_0 = 2\pi/T$

  Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

  Analysis: $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$

  DTFS equations for periodic signal, fund. period N, fund. frequency $\omega_0 = 2\pi/N$

  Synthesis: $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$

  Analysis: $a_k = \frac{1}{N} \sum_{n=<N} x[n] e^{-jk\omega_0 n}$

- **Response of LTI system to complex exponential**

  System Functions $H(s)$ and $H(z)$

  e^{st} (resp. z^n) is an *eigenfunction*

  $H(s)$ (resp. $H(z)$) is the corresponding *eigenvalue*. It is called the *system function*

  $H(j\omega)$ (resp. $H(e^{j\omega})$) is called the *system frequency response*

  $e^{st} \rightarrow H(s)e^{st}$ $\quad\quad H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$

  $z^n \rightarrow H(z) z^n$ $\quad\quad H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$

- **How to determine if system may be LTI by action on complex exponential signals**

  Check if eigenfunction property is satisfied. If violated, system is not LTI.

- **Filtering of periodic signal through an LTI system**

  Assume frequency response $H(j\omega)$ (resp. $H(e^{j\omega})$)

  $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$

  $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \rightarrow y[n] = \sum_{k=<N>} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$

- **Key examples**

  Periodic complex exponentials, sinusoidal signals, rectangular waves, impulse train

- **Properties of CTFS/DTFS (Tables 3.1 and 3.2 - provided with exam)**

  Periodicity of DTFS: $x[n]$ fund. period $N \Rightarrow a_k$ period N

  Linearity

  Time shifting, frequency shifting

  Time reversal, time scaling

  Periodic convolution, multiplication

  Differentiation, integration (CT) / First difference, running sum (DT)

  Parseval's relation

  FS and signal properties: real (conjugate symmetry), real & even, imaginary & odd

- **Finding system function and frequency response of causal LTI system from differential/difference equations**

  $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$ $\quad H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$ $\quad H(j\omega)$ - plug in $s = j\omega$

  $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$ $\quad H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$ $\quad H(e^{j\omega})$ - plug in $z = e^{j\omega}$

- **Effect of LTI system with *real* impulse response on sinusoids**

  $\cos(\omega_0 t) \rightarrow |H(j\omega_0)| \cdot \cos(\omega_0 t + \angle H(j\omega_0))$

  $\cos(\omega_0 n) \rightarrow |H(e^{j\omega_0})| \cdot \cos(\omega_0 n + \angle H(e^{j\omega_0}))$