UNIVERSITY OF CALIFORNIA SAN DIEGO
Dept. of Electrical and Computer Engineering

Midterm #2

Date: February 25, 2025 Course: ECE 45

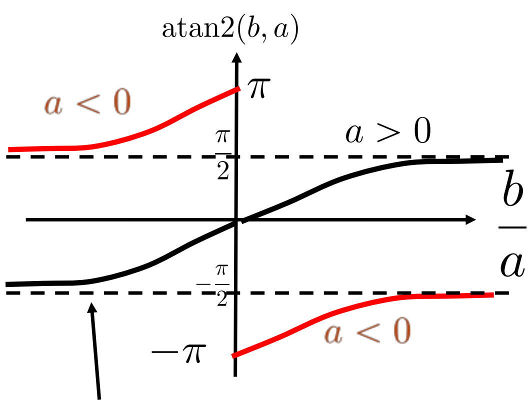
Name: ______________________________
Last, First

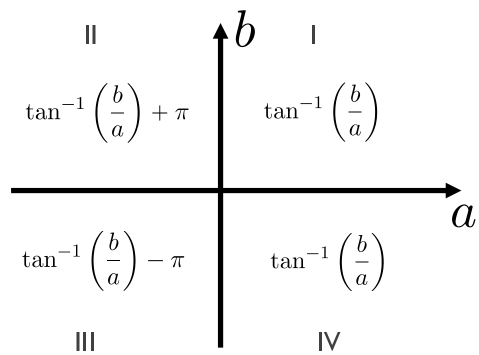
- The exam will last 75 minutes.
- Please highlight your final solution with a box if you want it graded. The solutions must be indicated in the corresponding region for the problem and clearly indicated.
- Point values for a problem are *evenly divided among subproblems* unless otherwise noted.
- The exam must be turned in at the conclusion of class when the instructor says that time is up. Late exams will not be graded.

Problem	Point Value	Your Score	Topic
1	25		Short answers
2	25		Fourier properties
3	25		MATLAB fun
4	25		Fabulous filtering
Total	100		

Table 1: Some identities

$$
\begin{aligned}
e^{ju} &= \cos u + j \sin u \\
\cos u &= \frac{1}{2}\left(e^{ju} + e^{-ju}\right) \\
\sin u &= \frac{1}{2j}\left(e^{ju} - e^{-ju}\right) \\
\sin^2 u + \cos^2 u &= 1 \\
\cos^2 u - \sin^2 u &= \cos 2u \\
2 \sin u \cos u &= \sin 2u \\
\cos^2 u &= \frac{1}{2}(1 + \cos 2u) \\
\sin^2 u &= \frac{1}{2}(1 - \cos 2u) \\
\sin(u \pm v) &= \sin u \cos v \pm \cos u \sin v \\
\cos(u \pm v) &= \cos u \cos v \mp \sin u \sin v \\
\sin u \sin v &= \frac{1}{2}[\cos(u - v) - \cos(u + v)] \\
\cos u \cos v &= \frac{1}{2}[\cos(u - v) + \cos(u + v)] \\
sin u \cos v &= \frac{1}{2}[\sin(u - v) + \sin(u + v)] \\
\mathrm{sinc}(x) &= \frac{\sin(\pi x)}{\pi x}
\end{aligned}
$$





this is the typical arctangent computed from b/a

Table 2: Basic Fourier Series pairs

Time domain signal	Fourier Series coefficients
c	$c\ \delta[k]$
$\sin(\omega_0 t)$	$\frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]$
$\cos(\omega_0 t)$	$\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$
$\sum_{m=-\infty}^{\infty} \delta(t-mT)$	$\frac{1}{T}$
$\sum_{m=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-mT}{2T_1}\right)$	$a_k = \frac{2T_1}{T}\mathrm{sinc}\left(\frac{k2T_1}{T}\right)$
$\sum_{m=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-mT}{2T_1}\right)$	$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \quad k \neq 0$ $a_0 = \frac{2T_1}{T}$

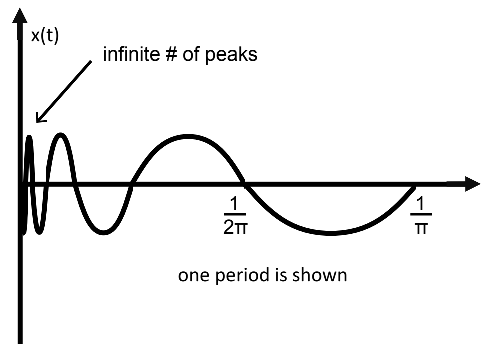
Table 3: Basic Fourier Series relationships

Time domain signal	Fourier Series coefficients
$x(t)$	a_k
$y(t)$	b_k
$Ax(t) + By(t)$	$Aa_k + Bb_k$
$x(t-t_0)$	$a_k e^{-jk\omega_0 t_0}$
$x(-t)$	a_{-k}
$x(\alpha t)$	a_k note $T_{\text{new}} = T/\alpha$
$x^*(t)$	a_{-k}^*
$x(t)y(t)$	$\sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$
$\frac{\mathrm{d}}{\mathrm{d}t}x(t)$	$a_k(jk\omega_0)$
$x(t)$ real	$a_k = a_{-k}^*$
$x(t)$ real and even	a_k real and even
$x(t)$ real and odd	a_k imaginary and odd
Parseval's Theorem	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 \mathrm{d}t = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$

Problem 2.1 Short answers (25 points)

(a) Does the following periodic signal have a Fourier Series? Circle one.

Yes No



Solution: No as it does not satisfy the Dirichlet conditions (has an infinite number of maxima and minima).

(b) Consider these two complex sinusoids $e^{j\pi t}$ and $e^{-j\pi t}$. Do these two signals satisfy the orthogonality property over one period?

Yes No

Solution: Yes as these correspond to different frequencies of π and $-\pi$ radians.

(c) A signal has Fourier Series coefficients that satisfy (a) $a_k = 0$ for all k even and (b) the remaining a_k are conjugate symmetric. Is the signal real?

Yes No

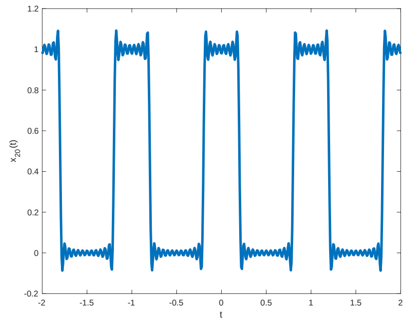
Solution: Yes because the coefficients are conjugate symmetric.

(d) The signal $x(t) = 2\cos(4\pi t)$ is input into a system. The result is $y(t) = e^{j2\pi t}$. Is the system linear?

Yes No

Solution: No because the input was two complex sinusoids with frequency 4π and -4π but the output was a complex sinusoid with frequency 2π. It cannot be an LTI system.

(e) The extra ripple overshoot at the point of discontinuity, in the following reconstruction of a pulse train, is known figure is known as the ______________________________



Solution: Gibb's phenomena

Problem 2.2 Fourier properties

Find the period and the FS series coefficients of the following signals.

(a) $x(t) = \cos\left(3\pi t + \frac{\pi}{6}\right)$

<u>Solution:</u>

Rewrite $x(t) = \cos(3\pi(t + 1/18))$, we have $\omega_0 = 3\pi$, and observe that $x(t)$ can be acquired by shifting the signal $\cos(3\pi t)$ to the left by $t_0 = -\frac{1}{18}$. Therefore,

$$\begin{aligned} T &= \frac{2\pi}{\omega_0} \\ &= \frac{2}{3} \end{aligned}$$

and

$$\begin{aligned} a_k &= \left(\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right) e^{jk\omega_0 \frac{1}{18}} \\ &= \frac{1}{2}\delta[k-1]e^{j\frac{\pi}{6}k} + \frac{1}{2}\delta[k+1]e^{j\frac{\pi}{6}k}. \end{aligned}$$

a_k has non-zero values when $k = \pm 1$:

$$\begin{aligned} a_1 &= \frac{\sqrt{3}}{4} + \frac{1}{4}j \\ a_{-1} &= \frac{\sqrt{3}}{4} - \frac{1}{4}j \\ a_k &= 0, k \neq \pm 1 \end{aligned}$$

(b) $q(t) = \frac{d}{dt}\left(\cos\left(3\pi t + \frac{\pi}{6}\right)\right)$

Solution:

We have two ways to solve the problem. The first way is to use the property: $\frac{d}{dt}x(t) \leftrightarrow a_k(jk\omega_0)$. From 2.2.(a) we know $x(t) \xleftrightarrow{\text{FS}} a_k$. The period of $q(t)$ is $T = \frac{2}{3}$. Let $q(t) \xleftrightarrow{\text{FS}} b_k$, we have

$$\begin{aligned} b_k &= j3\pi k\left(\frac{1}{2}\delta[k-1]e^{j\frac{\pi}{6}k} + \frac{1}{2}\delta[k+1]e^{j\frac{\pi}{6}k}\right) \\ &= \frac{3\pi k}{2}\delta[k-1]e^{j(\frac{\pi}{6}k+\frac{\pi}{2})} + \frac{3\pi k}{2}\delta[k+1]e^{j(\frac{\pi}{6}k+\frac{\pi}{2})} \\ &= \frac{3\pi}{2}\delta[k-1]e^{j(\frac{\pi}{6}+\frac{\pi}{2})} - \frac{3\pi}{2}\delta[k+1]e^{j(-\frac{\pi}{6}+\frac{\pi}{2})} \end{aligned}$$

b_k has non-zero values when $k = \pm 1$:

$$\begin{aligned} b_1 &= -\frac{3\pi}{4} + \frac{3\sqrt{3}\pi}{4}j \\ b_{-1} &= -\frac{3\pi}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}j\right) \\ &= -\frac{3\pi}{4} - \frac{3\sqrt{3}\pi}{4}j \\ b_k &= 0, k \neq \pm 1 \end{aligned}$$

The second way is to derive

$$\begin{aligned} q(t) &= \frac{d}{dt}\left(\cos\left(3\pi t + \frac{\pi}{6}\right)\right) \\ &= -3\pi\sin(3\pi t + \frac{\pi}{6}) \end{aligned}$$

Use the property: $\sin(\omega_0 t) \xleftrightarrow{\text{FS}} \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]$, we have

$$\begin{aligned} b_k &= \frac{-3\pi}{2j}\delta[k-1]e^{j\frac{\pi}{6}k} + \frac{3\pi}{2j}\delta[k+1]e^{j\frac{\pi}{6}} \\ &= \frac{3\pi}{2}\delta[k-1]e^{j(\frac{\pi}{6}+\frac{\pi}{2})} - \frac{3\pi}{2}\delta[k+1]e^{j(-\frac{\pi}{6}+\frac{\pi}{2})}, \end{aligned}$$

which gives the same result as above.

(c) $y(t) = \sum_{m=-\infty}^{\infty} \text{rect}(t/2 - 4m)$

Solution:

$$\begin{aligned} y(t) &= \sum_{m=-\infty}^{\infty} \text{rect}(t/2 - 4m) \\ &= \sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{t-8m}{2}\right) \end{aligned}$$

From the lecture, we have $T = 8$, $T_1 = 1$, and

$$\begin{aligned} a_k &= \frac{\sin(\pi k \frac{2T_1}{T})}{k\pi} \\ &= \frac{2T_1}{T}\text{sinc}(\frac{k2T_1}{T}) \\ &= \frac{1}{4}\text{sinc}\left(\frac{k}{4}\right) \end{aligned}$$

Problem 2.3 MATLAB Fun

As you saw in homework, the approximation of $x(t)$ obtained from using N harmonics is

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

where a_k refers to the Fourier series coefficients of $x(t)$. Consider the function $x(t)$ with period $T = 2\pi$, and $f(x) = x^2$ for $-\pi \leqslant x \leqslant \pi$. Its Fourier series coefficients are

$$a_k = \begin{cases} \frac{\pi^2}{3} & k = 0 \\ \frac{2(-1)^k}{k^2} & k \neq 0 \end{cases}$$

Write a Matlab code to plot $x_{10}(t)$, the approximation of $x(t)$ with $N = 10$ harmonics from $-\pi$ to π.

Some reminders:

- `y = linspace(x1, x2, n)` generates n points, evenly spaced between x1 and x2.
- `plot(X, Y)` creates a 2-D line plot of the data Y versus the corresponding values in X.

Solution:

```
clc; clear; close all;

% Define parameters
T = 2 * pi; % Period
omega0 = 2 * pi / T; % Fundamental frequency
t = linspace(-2*T, 2*T, 1000); % Time vector

% Define the original function
xt = (mod(t + pi, 2*pi) - pi).^2; % Periodic extension of x(t) = t^2 for
    [-pi, pi]

% Create figure for Fourier approximations
figure();
N = 10;
x_approx = (pi^2) / 3 * ones(size(t)); % Initialize approximation
for k = -N:N
    if k ~= 0
        ak = 2 * (-1)^k / (k^2); % Fourier coefficients
        x_approx = x_approx + ak * exp(1j * k * omega0 * t);
    end
end

plot(t, real(x_approx), 'r', 'LineWidth', 2); hold on;
plot(t, xt, 'b', 'LineWidth', 2);
xlabel('t'); ylabel('x(t)');
title(['N = ', num2str(N)]);
legend('Fourier Approximation', 'Original Function');
```

```
grid on;
```

Listing 1: MATLAB Code for Fourier Series Approximation

(extra page for problem 3 if needed)

Problem 2.4 Fabulous filtering 25 points

Consider the following signal $r(t)$ given by

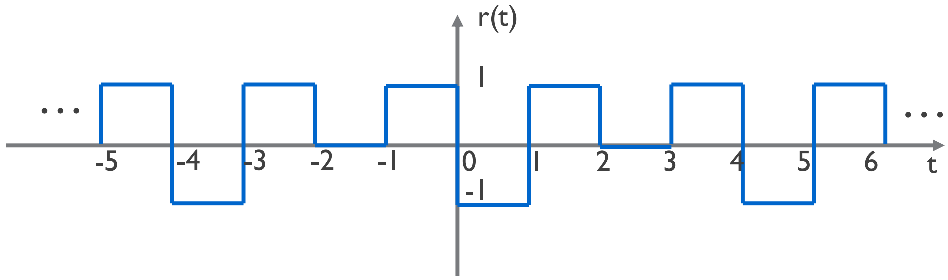


Figure 1: Signal of interest

(a) Determine the Fourier Series coefficients of this signal and of course the period.

<u>Solution:</u> This signal is periodic with period $T = 4$. We can build this signal from pulse trains. Let $p(t)$ denote a rectangular pulse train with period $T = 4$ and width $T_1 = 1/2$. Then from the table

$$\sum_{m=-\infty}^{\infty} \text{rect}\,(t - 4m) \overset{\mathcal{FS}}{\longleftrightarrow} a_k = \frac{1}{4}\text{sinc}\left(\frac{k}{4}\right)$$

We can write this signal using $p(t)$ as

$$r(t) = p\left(t + \frac{1}{2}\right) - p\left(t - \frac{1}{2}\right) + p\left(t - \frac{3}{2}\right). \tag{1}$$

Now note using the Fourier Series properties that

$$\begin{aligned}
p\left(t - \frac{1}{2}\right) &\overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{4}\text{sinc}\left(\frac{k}{4}\right) e^{-j\pi\frac{k}{4}} \\
p\left(t - \frac{3}{2}\right) &\overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{4}\text{sinc}\left(\frac{k}{4}\right) e^{-j\pi 3\frac{k}{4}} \\
p\left(t + \frac{1}{2}\right) &\overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{4}\text{sinc}\left(\frac{k}{4}\right) e^{j\pi\frac{k}{4}} \\
p\left(t + \frac{1}{2}\right) - p\left(t - \frac{1}{2}\right) + p\left(t - \frac{3}{2}\right) &\overset{\mathcal{FS}}{\longleftrightarrow} a_k = \frac{1}{4}\text{sinc}\left(\frac{k}{4}\right)\left[e^{+j\pi\frac{k}{4}} - e^{-j\pi\frac{k}{4}} + e^{-j\pi 3\frac{k}{4}}\right].
\end{aligned}$$

(b) Now suppose that $\omega_0 = \pi/2$. Suppose that $r(t)$ is input into an LTI system with frequency response $H_1(j\omega)$ to produce the output $g(t)$. Find an expression for $g(t)$ in the time domain.

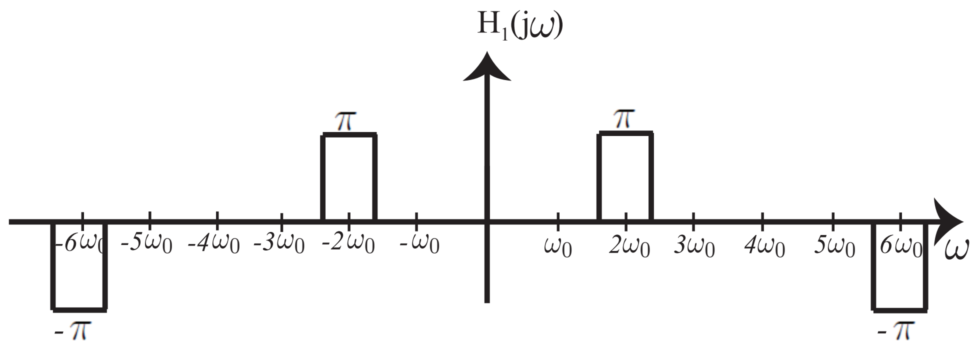


Figure 2: The frequency response $H_1(j\omega)$, where $\omega_0 = \frac{\pi}{2}$.

Solution: Note that since $r(t)$ is periodic it can be written as

$$r(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

where $\omega_0 = 2\pi/4 = \pi/2$.

The signal after the filtering operation is also periodic and can be expressed as

$$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} H_1\left(j\omega_0 k\right).$$

Based on the structure of $H_1(j\omega)$

$$\begin{aligned} g(t) &= \pi a_2 e^{j\omega_0 2t} + \pi a_{-2} e^{-j\omega_0 2t} - \pi a_6 e^{j\omega_0 6t} + \pi a_{-2} e^{-j\omega_0 6t} \\ &= 2\pi \cos(\omega_0 2t) a_2 - 2\pi \cos(\omega_0 6t) a_6 \end{aligned}$$