ECE 101 Linear Systems

Problem Set 1 Solutions

Problem 1 - Signal Power/Energy:

1.3(c)

Using the definition of total energy for continuous signal we obtain

$$E_\infty = \int_{-\infty}^{\infty} |x_3(t)|^2 dt = \int_{-\infty}^{\infty} \cos^2(t) dt = \infty.$$

Using the definition of average power for continuous signal we obtain

$$\begin{aligned} P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x_3(t)|^2 dt &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos^2(t) dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{2}(1+\cos(2t)) dt \\ &= \lim_{T\to\infty} \frac{1}{4T} \left(2T + \sin(2T)\right) \\ &= \frac{1}{2}. \end{aligned}$$

<u>Detailed answer:</u>
Consider

$$\int_{-T}^{T} |x_3(t)|^2 dt = \int_{-T}^{T} \cos^2(t) dt = \int_{-T}^{T} \frac{1}{2}(1+\cos(2t)) dt = \frac{1}{2}\left(2T + \sin(2T)\right).$$

This implies

$$-\frac{1}{2} + T \le \int_{-T}^{T} |x_3(t)|^2 dt \le T + \frac{1}{2},$$

and

$$-\frac{1}{4T} + \frac{1}{2} \le \frac{1}{2T} \int_{-T}^{T} |x_3(t)|^2 dt \le \frac{1}{2} + \frac{1}{4T},$$

Now taking limit we obtain

$$\lim_{T\to\infty} \left(-\frac{1}{2} + T\right) \le \lim_{T\to\infty} \int_{-T}^{T} |x_3(t)|^2 dt \le \lim_{T\to\infty} \left(T + \frac{1}{2}\right),$$

and

$$\lim_{T\to\infty} \left(-\frac{1}{4T} + \frac{1}{2}\right) \le \frac{1}{2T} \int_{-T}^{T} |x_3(t)|^2 dt \le \lim_{T\to\infty} \left(\frac{1}{2} + \frac{1}{4T}\right),$$

Hence, we have

$$\lim_{T\to\infty}\int_{-T}^{T}|x_3(t)|^2dt=\infty \quad \text{and} \quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x_3(t)|^2dt=\frac{1}{2}.$$

Using the definition of total energy and average power for continuous time signal we have $E_\infty=\infty$ and $P_\infty=\frac{1}{2}$.

1.3 (e)

Given $x_2[n]=e^{j\frac{\pi}{2n}+\frac{\pi}{8}}$ we have $|x_2[n]|^2=1$. Using the definition of total energy for discrete signal we obtain

$$E_\infty=\sum_{n=-\infty}^{\infty}|x_2[n]|^2=\sum_{n=-\infty}^{\infty}1=\infty.$$

Using the definition of average power for discrete signal we obtain

$$P_\infty=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x_2[n]|^2=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}1=1.$$

1.13

The signal

$$y(t)=\int_{-\infty}^{t}\delta(\tau+2)-\delta(\tau-2)d\tau=\begin{cases}0, & t\le -2,\\ 1, & -2\le t<2,\\ 0 & t\ge 2.\end{cases}$$

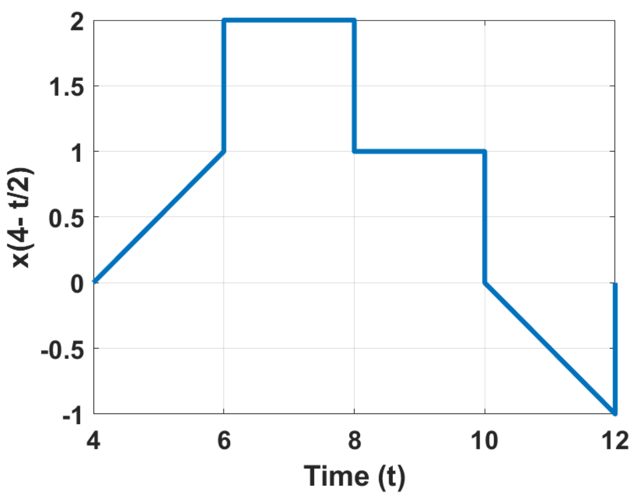
Therefore, the value of E_∞ for $y(t)$ is given by

$$E_\infty=\int_{-\infty}^{\infty}|y(t)|^2dt=\int_{-2}^{2}1dt=4.$$

Problem 2 - Signal Transformations:

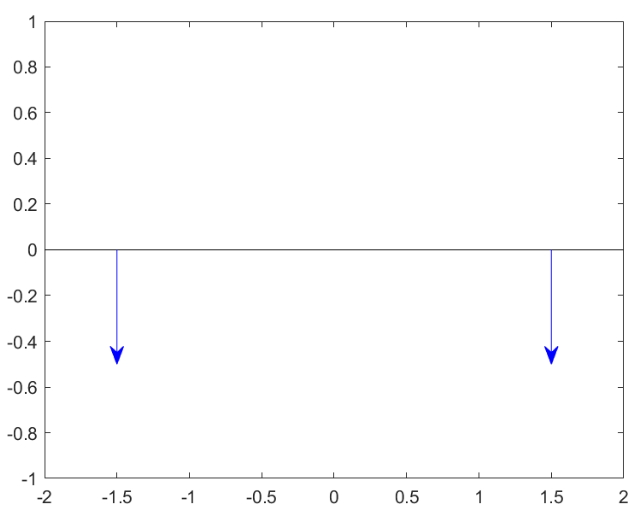
1.21(d)

$x(4 - \frac{t}{2})$



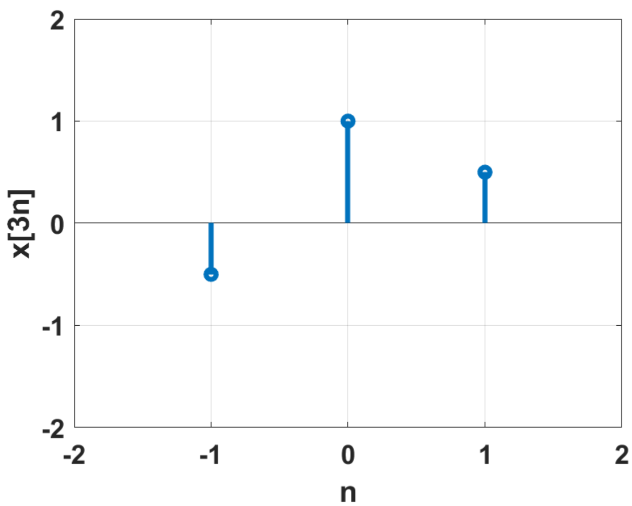
1.21(f)

$x(t)[\delta(t + \frac{3}{2}) - \delta(t - \frac{3}{2})]$



1.22(c)

$x[3n]$



1.22(e)

$x[n]u[3-n]$

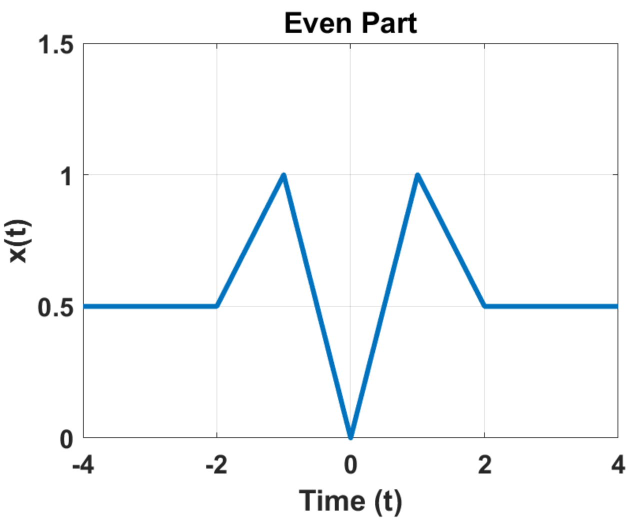
Note that

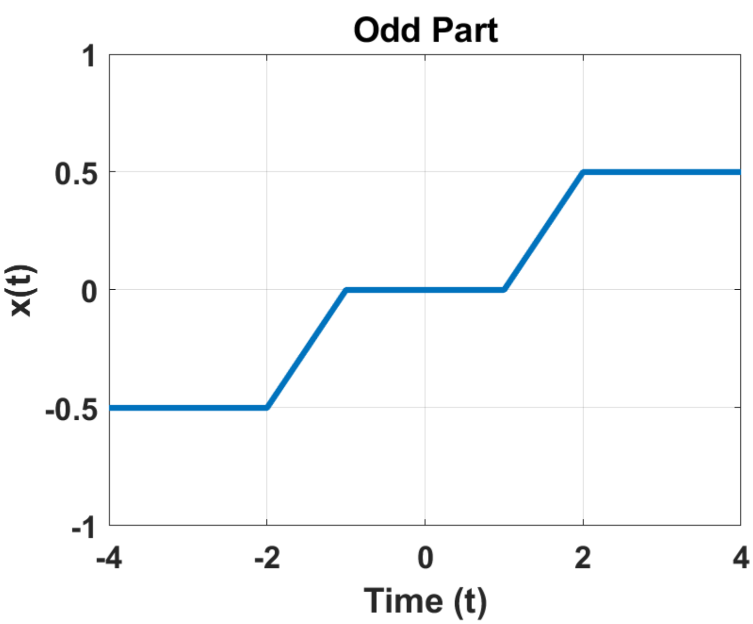
$$u[3-n] = \begin{cases} 0 & n > 3, \\ 1 & n \leq 3. \end{cases} \tag{1}$$

Hence $x[n]u[3-n] = x[n]$.

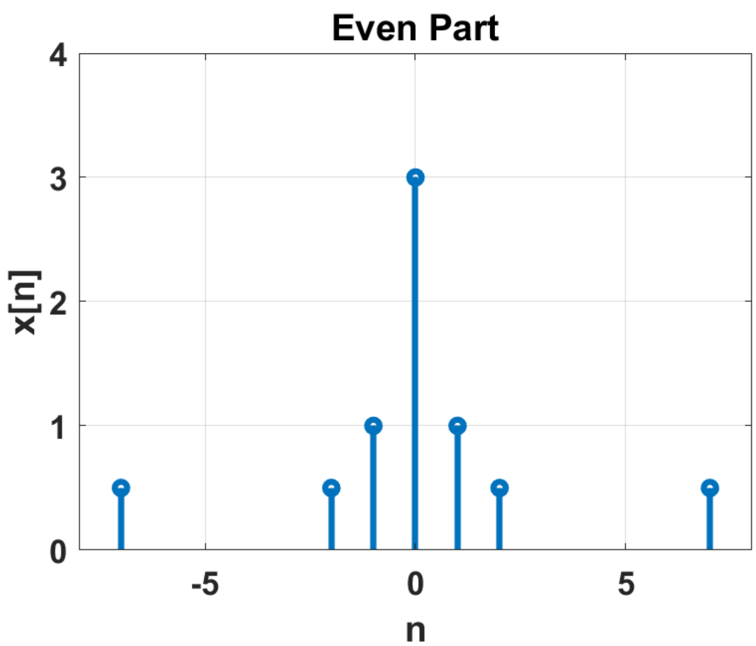
Problem 3 - Even/Odd :

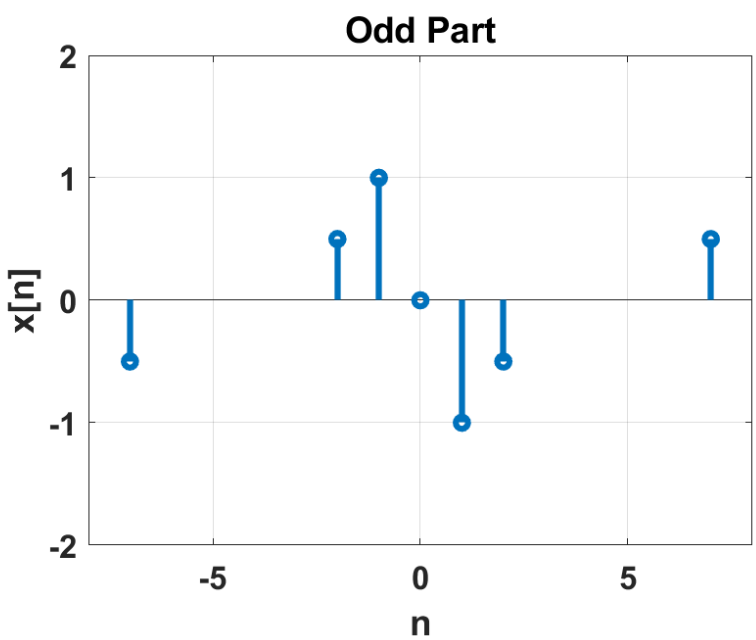
1.23(b)





1.24(b)





Problem 4 - Periodicity:

1.25(e)

Given

$$x(t) = \frac{\sin(4\pi t)u(t) + \sin(-4\pi t)u(-t)}{2} = \sin(4\pi t)\frac{u(t) - u(-t)}{2} = \begin{cases} \frac{\sin(4\pi t)}{2} & t \geq 0, \\ \frac{\sin(4\pi(-t))}{2} & t < 0, \end{cases}$$

implies

$$x(t) = \frac{\sin(4\pi|t|)}{2}.$$

This is not periodic. To see this, note that any period of $x(t)$ would have to be a multiple of $1/2$, which is the period of $\sin(4\pi t)$. However, given $T = k/2$, $k \geq 1$, the value of $x(t+T)$ on the interval $[-T, 0]$ equals $\sin(4\pi t)/2$, whereas the value of $x(t)$ on that interval would be $\sin(-4\pi t)/2 = -\sin(4\pi t)/2$, implying $x(t+T) \neq x(t)$ on this interval.

1.26(d)

Recall that

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right).$$

Therefore,

$$x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n) = \frac{1}{2}\left(\cos(\frac{3\pi}{4}n) + \cos(\frac{\pi}{4}n)\right). \tag{2}$$

The period of each term on the right is 8, so the period of the sum is: $\mathrm{lcm}(8,8) = 8$.

1.32

Given $x(t)$, let $y_1(t) = x(2t)$ and $y_2(t) = x(t/2)$.

(1) Given $x(t)$ is periodic implies there exists $T > 0$ such that $x(t) = x(t+T)$ for every t. Since $y_1(t) = x(2t) = x(2t+T) = y_1(t + \frac{T}{2})$, $y_1(t)$ is periodic with period $\frac{T}{2}$. This implies that the fundamental period T_x of $x(t)$ is greater than or equal to two times the fundamental period T_1 of $y_1(t)$, i.e., $T_x \geq 2T_1$. (Otherwise, if $T_x < 2T_1$, then $T_x/2 < T_1$ would satisfy the periodicity condition for $y_1(t)$, which is a contradiction.)

(2) Given $y_1(t)$ is periodic implies there exists $T > 0$ such that $y_1(t) = y_1(t+T)$ for every t. Since $x(t) = y_1(\frac{t}{2}) = y_1(\frac{t}{2} + T) = x(t + 2T)$, $x(t)$ is periodic with period $2T$. This implies that the fundamental period T_1 of $y_1(t)$ is greater than or equal to one half the fundamental period T_x of $x(t)$, i.e., $T_1 \geq T_x/2$. (Otherwise, if $T_1 < T_x/2$, then $2T_1 < T_x$ would satisfy the periodicity condition for $x(t)$, which is a contradiction.)

    Together, the inequalities relating the fundamental periods from parts (1) and (2) imply $T_x \geq 2T_1 \geq 2(T_x/2) = T_x$, which means that that $T_x = 2T_1$.

(3) Given $x(t)$ is periodic implies there exists $T > 0$ such that $x(t) = x(t+T)$ for every t. Since $y_2(t) = x(\frac{t}{2}) = x(\frac{t}{2} + T) = y_2(t + 2T)$, $y_2(t)$ is periodic with period $2T$. Therefore, the fundamental period T_x of $x(t)$ is greater than or equal to one half the fundamental period T_2 of $y_2(t)$, i.e., $T_x \geq T_2/2$.

(4) Given $y_2(t)$ is periodic implies there exists $T > 0$ such that $y_2(t) = y_2(t+T)$ for every t. Since $x(t) = y_2(2t) = y_1(2t+T) = x(t+\frac{T}{2})$, $x(t)$ is periodic with period $\frac{T}{2}$. Therefore, the fundamental period T_2 of $y_2(t)$ is greater than or equal to two times the fundamental period T_x of $x(t)$, i.e., $T_2 \geq 2T_x$.

Together, the inequalities relating the fundamental periods from parts (3) and (4) imply $T_x \geq T_2/2 \geq (2T_x)/2) = T_x$, which means that that $T_x = T_2/2$.

Problem 5 - Exponentials and Periodicity:

1.9(b)

The signal $x_2(t) = e^{(-1+j)t}$ can be written as $x_2(t) = e^{-t}e^{jt}$. The magnitude is $|x_2(t)| = e^{-t}$, which is a real-valued decaying exponential and is not periodic. Since the magnitude of the signal $x_2(t)$ is not periodic, the signal $x_2(t)$ can not be periodic.

1.9(d)

Given $x_4[n] = e^{j\frac{3\pi}{5}(n+\frac{1}{2})}$. If $x[n]$ is a periodic signal, then there exists $N > 0$ such that $x[n] = x[n+N]$. Therefore,

$$\begin{aligned}
x[n] &= x[n+N] \\
e^{j\frac{3\pi}{5}(n+\frac{1}{2})} &= e^{j\frac{3\pi}{5}(n+N+\frac{1}{2})} \\
1 &= e^{j\frac{3\pi}{5}N} \\
e^{j2k\pi} &= e^{j\frac{3\pi}{5}N} \\
N &= \frac{10k}{3}.
\end{aligned}$$

The smallest positive value $N = 10$ occurs when $k = 3$, so the fundamental period is $N_0 = 10$.

Problem 6 - Impulse and Step Functions:

1.14

Note that the signal $x(t)$ for $t \in [0, 2)$ can be written as $u(t) - 3u(t-1) + 2u(t-2)$ because

$$\begin{aligned}
u(t) - 3u(t-1) + 2u(t-2) &= [u(t) - u(t-1)] - 2\,[u(t-1) - u(t-2)] \\
&= \begin{cases} 1 & 0 < t \leq 1, \\ -2 & 1 < t < 2, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

This implies

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} \left(u(t-2k) - 3u(t-1-2k) + 2u(t-2-2k)\right) \\
&= \sum_{k=-\infty}^{\infty} \left(u(t-2k) - 3u(t-1-2k)\right) + \sum_{k=-\infty}^{\infty} 2u(t-2-2k) \\
&= \sum_{k=-\infty}^{\infty} \left(u(t-2k) - 3u(t-1-2k)\right) + \sum_{k'=-\infty}^{\infty} 2u(t-2k') \\
&= \sum_{k=-\infty}^{\infty} \left(3u(t-2k) - 3u(t-1-2k)\right).
\end{aligned}
$$

Hence, the derivative is

$$
\begin{aligned}
\frac{dx(t)}{dt} &= 3\sum_{k=-\infty}^{\infty} \delta(t-2k) - 3\sum_{k=-\infty}^{\infty} \delta(t-1-2k) \\
&= 3g(t) - 3g(t-1).
\end{aligned}
$$

Therefore, we have $A_1 = 3$, $A_2 = -3$, $t_1 = 0$ and $t_2 = 1$.