Lecture 3

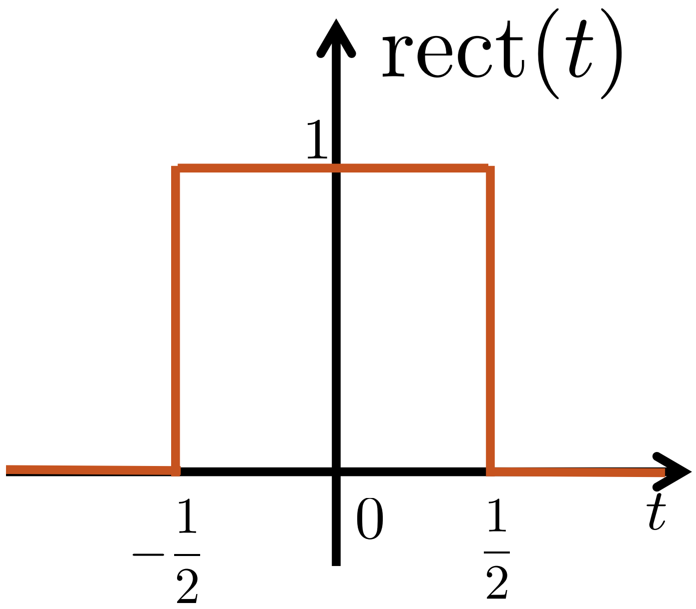
Exponential, sinusoids, complex exponentials, and the delta function

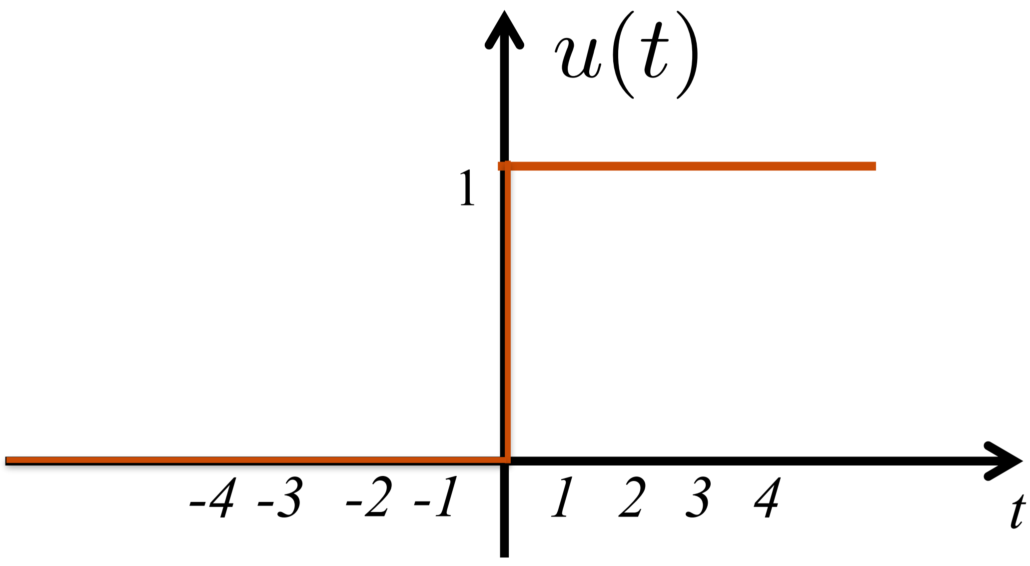
Preview of today's lecture

- CT sinusoids and exponentials
  - Determine the key parameters of a complex sinusoid and exponential
  - Sketch a complex exponential based on its form

- Unit-impulse function also known as the Dirac delta function
  - Explain the properties of delta unit impulse function
  - Exploit the sifting property to simplify expressions with deltas
  - Exploit the integration property to simplify expressions with deltas

From Lecture #2: two common functions





Examples: Shifting, inverting and scaling

- Let

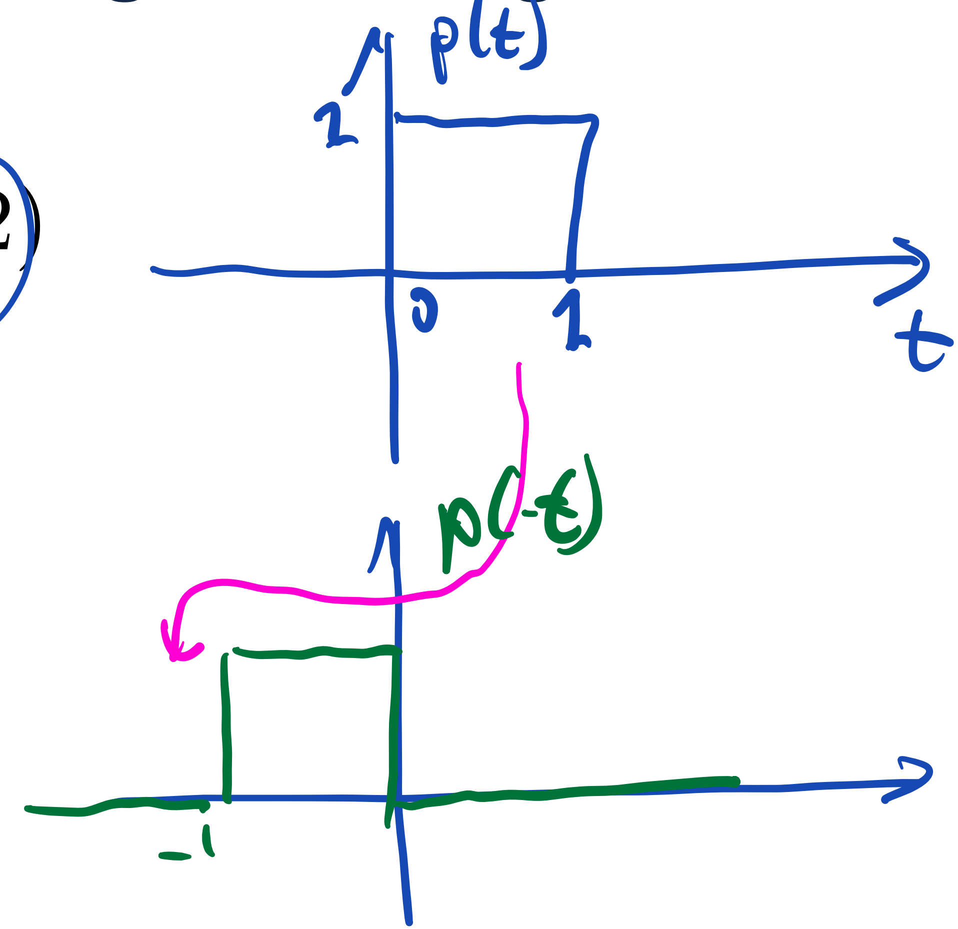
$$p(t) = \text{rect}(t - 1/2)$$

- Find and plot

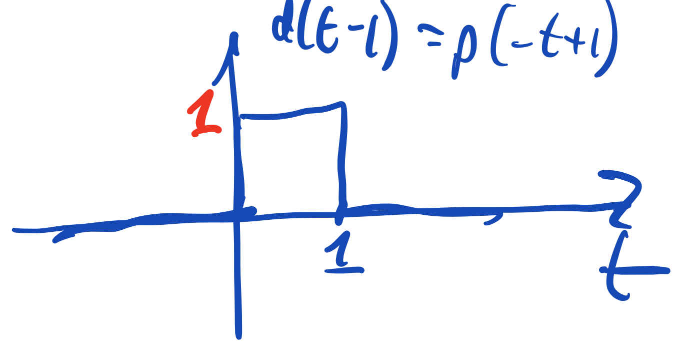
$$p(-t) = d(t)$$

$$p(-t+1) = b(t)$$

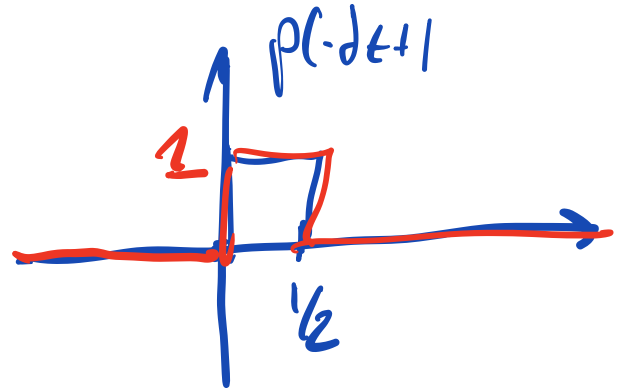
$$p(-2t+1)$$



$$p(-t+1) = p(-(t-1)) = d(t-1)$$



$$p(-2t+1) = d(2t) = p(-2\cdot t + 1)$$



$d(t)$ $\quad t=1 \quad 2t=1$

$t = 1/2$

Connections back to ECE 45

Lectures 2 - 4 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 5 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

UC San Diego

	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, phasors	1.3		
4	1/16	Signals	Dirac delta, Kronecker delta, Sha function	1.4	HW3	HW2
5	1/21	LTI in time	Linear and time-invariant systems	2.1		
6	1/23	LTI in time	Convolution, convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

CT sinusoids and exponentials

Learning objectives

- Determine the key parameters of a complex sinusoid and exponential
- Sketch a complex exponential based on its form

CT real sinusoid

$\mathbb{R}$

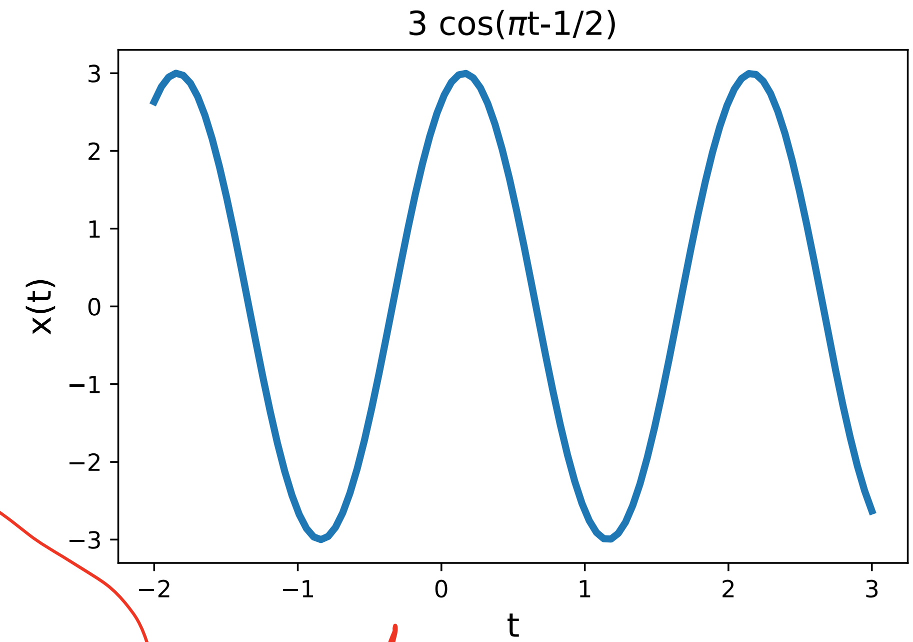
amplitude > 0 phase

$$x(t) = C\cos(\omega_0 t + \theta)$$

frequency in radians/s

Periodic with period $T = \dfrac{2\pi}{\omega_0}$

$\cos(\omega_0 t + \pi/3)$



$\pi/3$

$C = 3$

$\omega_0 = \pi$

$\theta = -1/2$

$$\cos\left(\omega_0\left(t+\underbrace{\frac{2\pi}{\omega_0}}_{T}\right)\right)$$

$$= \cos\left(\omega_0 t + \frac{\cancel{\omega_0}\cdot 2\pi}{\cancel{\omega_0}}\right)$$

$$= \cos(\omega_0 t + 2\pi) = \cos(\omega_0 t)$$

Complex numbers

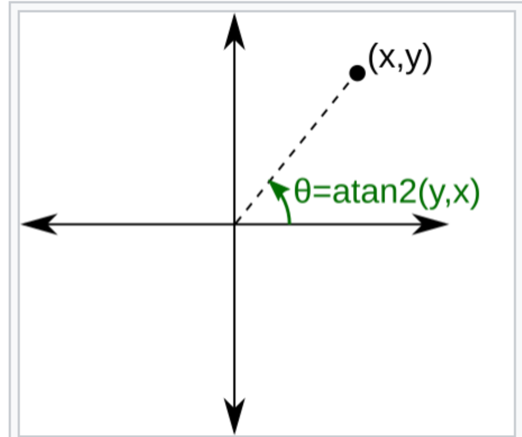
$$C = x + jy \qquad \text{Cartesian}$$

$$= |C|\, e^{j\theta} \qquad \text{Polar}$$

Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\boxed{C = \underbrace{|C|\cos(\theta)}_{\mathrm{Re}\,\{C\}} + j\,\underbrace{|C|\sin(\theta)}_{\mathrm{Im}\,\{C\}}}$$



atan2(y, x) returns the angle θ between the ray to the point (x, y) and the positive x-axis, confined to $(-\pi, \pi]$.

https://en.wikipedia.org/wiki/Atan2

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

CT complex sinusoid

$C \in \mathbb{R}$, $C > 0$

complex frequency in radians/s

$$x(t) = Ce^{j\omega_0 t} \qquad \text{period } T = \frac{2\pi}{\omega_0}$$

$$= |C|e^{j\theta}e^{j\omega_0 t}$$

$$= \boxed{|C|e^{j(\omega_0 t+\theta)}}$$

amplitude phase

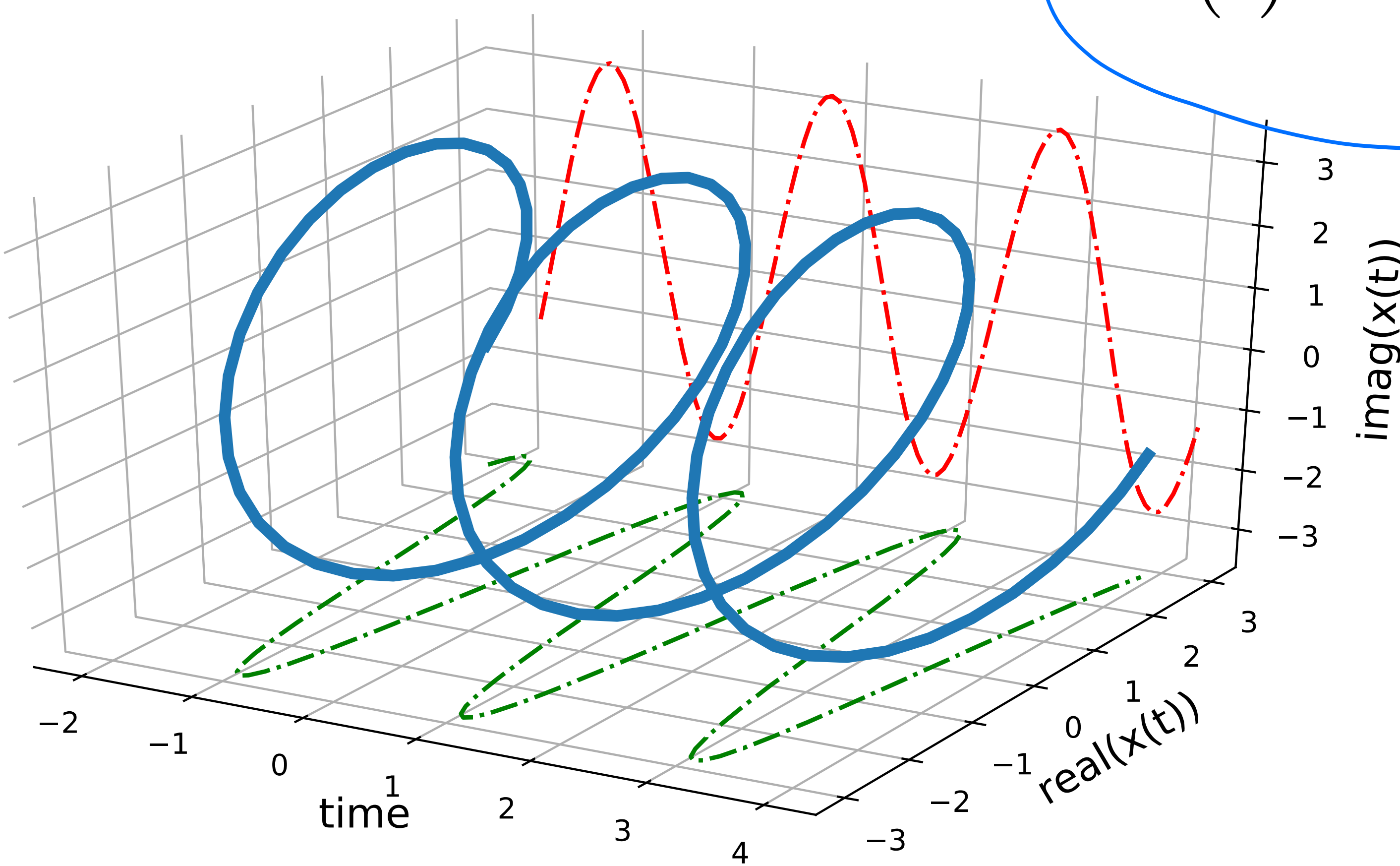
$$= |C|\cos(\omega_0 t + \theta) + j|C|\sin(\omega_0 t + \theta)$$

real and imaginary portions related through Eulers

Note: if the amplitude C is negative, then we could simply compensate for the negative by shifting the phase by pi

Visualizing a complex sinusoid

$$x(t) = 3e^{j(\pi t - \frac{1}{2})}$$



$|x(t)|$

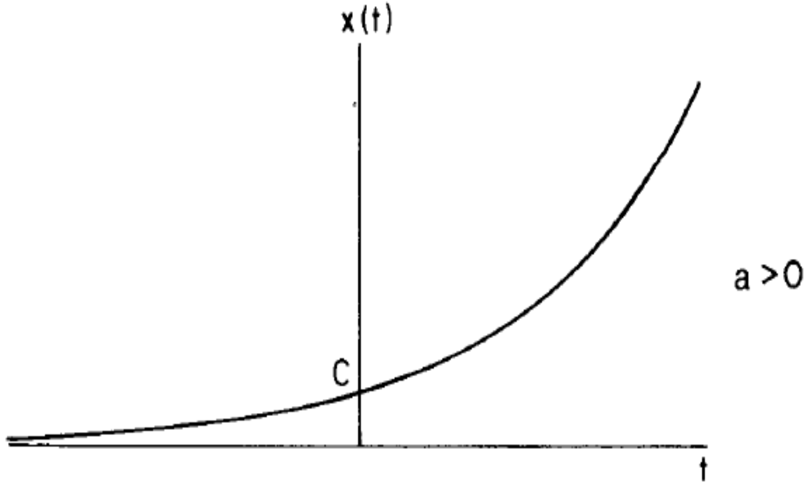
$|3e^{j(\pi t - 1/2)}|$

$= 3$

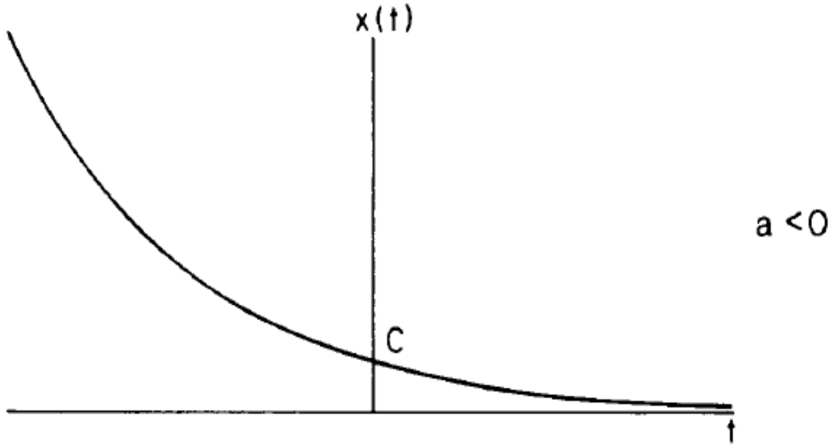
CT real exponential signal

$$x(t) = Ce^{at}$$

real real



Increasing exponential
(a > 0, C > 0)



Decaying exponential
(a < 0, C > 0)

CT complex exponential: general case

$$x(t) = Ce^{at}$$

complex complex

polar of C

car of a

$C = c_\sigma + jc_\omega;$ cartesian

$= |C|e^{j\theta},$ polar

$a = r + j\omega_0,$ cartesian

$$x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t}$$

$e^{rt}e^{j\omega_0 t}$

$$\boxed{x(t) = |C|e^{rt}e^{j(\omega_0 t+\theta)}}$$

General case includes real exponential, real sinusoid, and complex sinusoid as special cases

Visualizing CT complex exponentials

$$x(t) = \overbrace{|C|e^{rt}}^{\text{real}} e^{j(\omega_0 t + \theta)}$$

Increasing or decaying exponential "envelope"

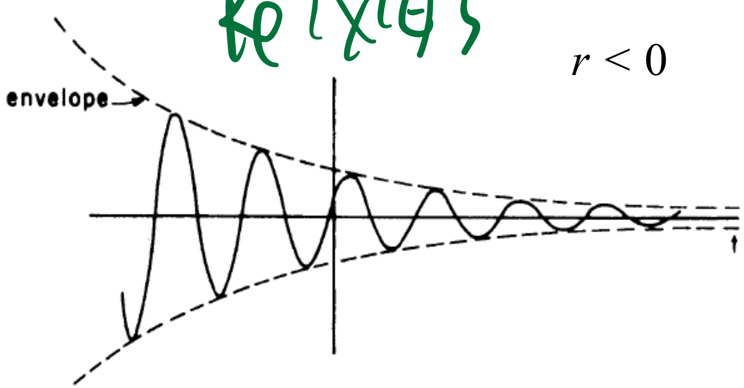
Complex sinusoid

$$Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t + \theta)$$
$$Im\{x(t)\} = |C|e^{rt}\sin(\omega_0 t + \theta)$$

$$e^{rt} \text{ is the "damping" term} \longrightarrow \begin{cases} r > 0 \rightarrow \text{ blows up} \\ r < 0 \rightarrow \text{ decays to zero} \end{cases}$$

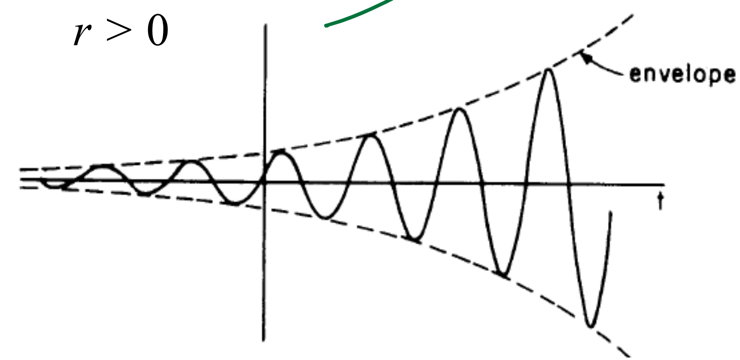
Re{x(t)}

$r < 0$



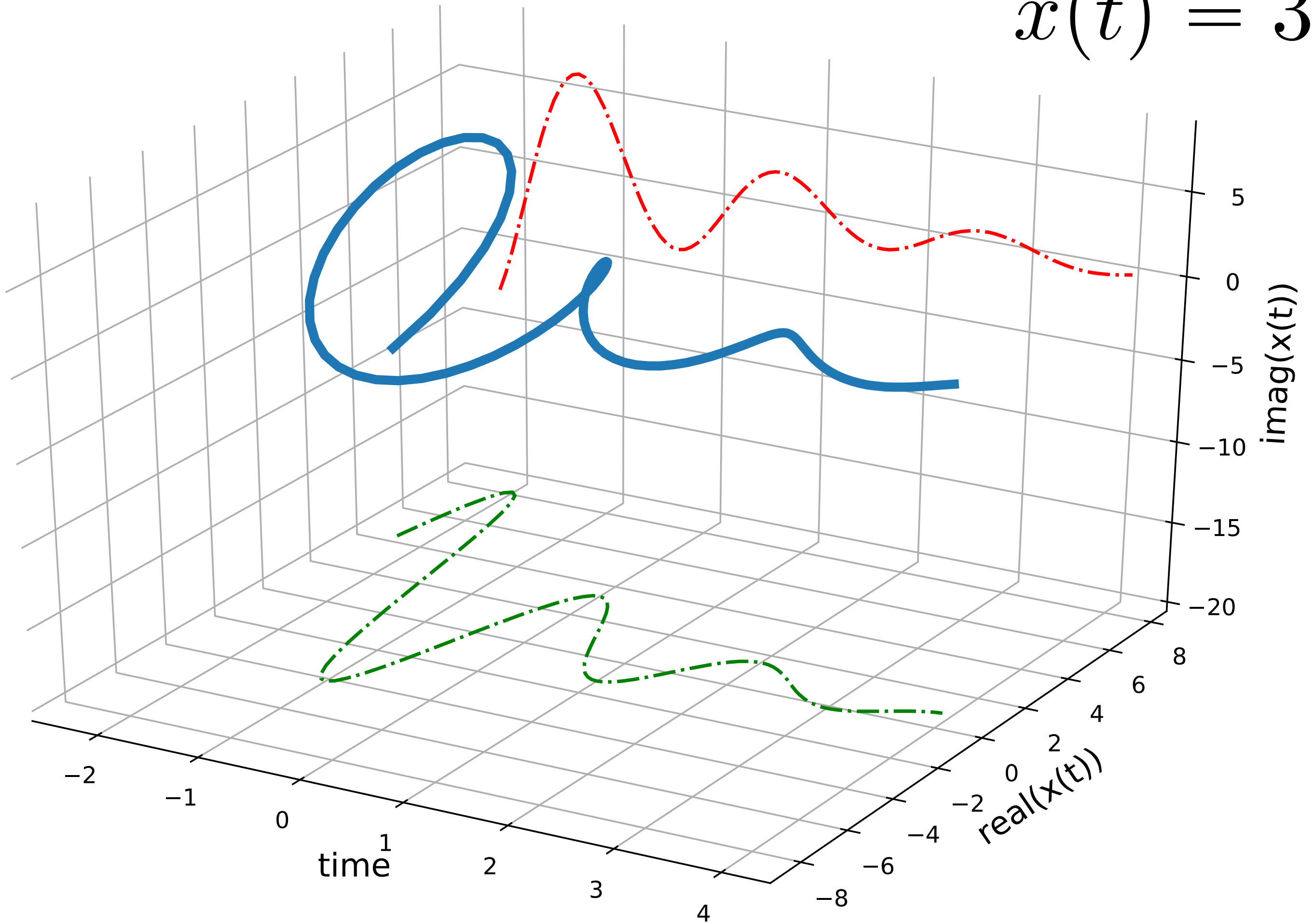
Re{x(t)}

$r > 0$



Visualizing CT complex exponentials in 3D

$$x(t) = 3e^{-t/2}e^{j(\pi t - 1/2)}$$



Example: Sketching a complex exponential

- Consider a complex exponential with the following values

$$x(t) = Ce^{at}$$

$$C = 3 - j, \quad a = 1 + 10j$$

- The problem:
  - Express x(t) in terms of its envelope and complex sinusoidal parts
  - Express the real and imaginary parts of x(t)
  - Express the magnitude of x(t)
  - Plot all of these from time t = 0 to t = 3, showing the envelope

Solution I

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

- Convert $C = 3 - j$ to polar form

$$C = \sqrt{3^1 + 1}e^{j\operatorname{atan2}(-1,3)}$$
$$= \sqrt{10}e^{-j0.3218}$$

- Substitute to get the general form (recall $a = 1 + 10j$)

$$x(t) = Ce^{at}$$
$$= \sqrt{10}e^{-j0.3218}e^{t(1+10j)}$$
$$= \sqrt{10}e^{t}e^{j(10t-0.3218)}$$

See more about the two argument arc tangent here https://en.wikipedia.org/wiki/Atan2

Solution 2

- Apply Euler's to get real and imaginary parts:

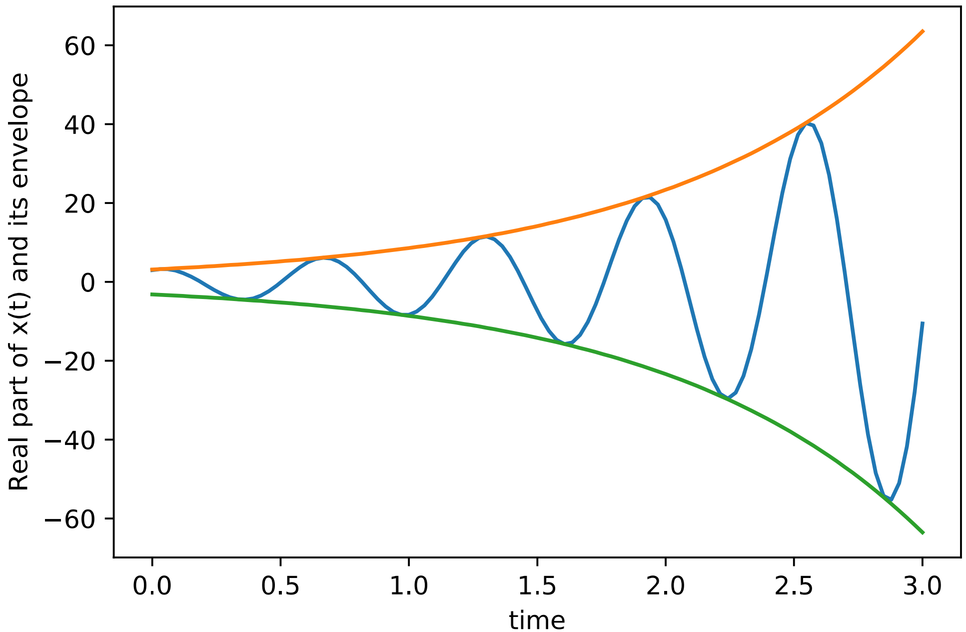
$$Re\{x(t)\} = \sqrt{10}e^t \cos(10t - 0.3218)$$
$$Im\{x(t)\} = \sqrt{10}e^t \sin(10t - 0.3218)$$

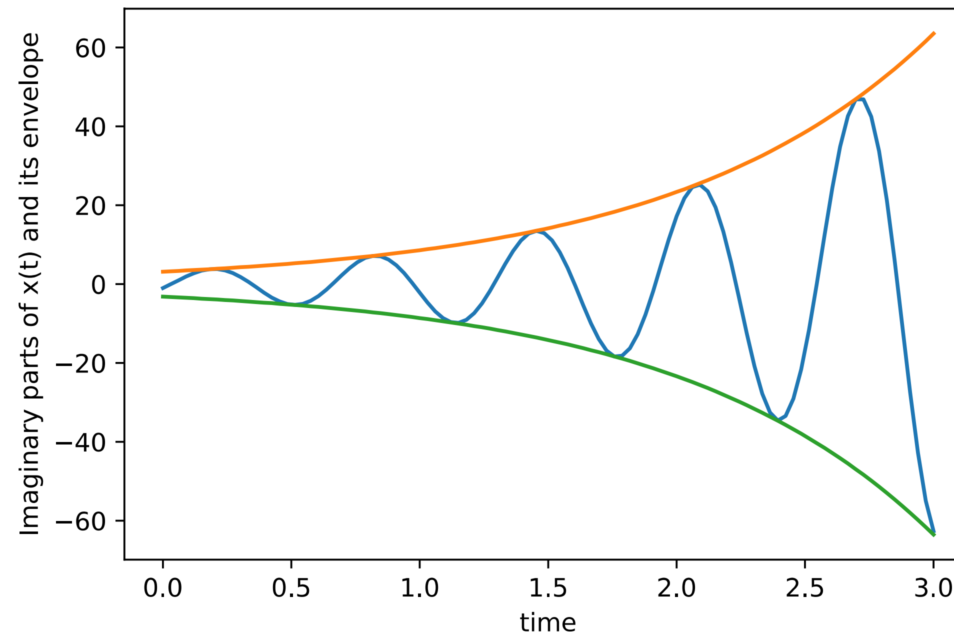
- Use the general form to find the magnitude

$$|x(t)| = \sqrt{10}e^t$$

- Sketch the signal behavior

Real and imaginary parts are just π/2 shifts (magnitude is just the positive envelope)





$$Re\{x(t)\} = \sqrt{10}e^{t}\cos(10t - 0.3218)$$

$$Im\{x(t)\} = \sqrt{10}e^{t}\sin(10t - 0.3218)$$

MATLAB to create these plots

```
C = 3-j;
a = 1+10j;
t = 0:.01:3; % equivalent to linspace (0,3,101)
x = C*exp(a*t);
env = abs(C)*exp(real(a) *t);

figure(1);
plot(t,real(x),t,env,t,-env);
xlabel('time');
ylabel('Real part of x(t) and its envelope');

figure(2);
plot(t,imag(x),t,env,t,-env);
xlabel('time');
ylabel('Imaginary parts of x(t) and its envelope');
```

CT exponential and sinusoidal signals in summary

- Complex exponentials and sinusoids
  - Important building blocks for future lectures
  - Can be understood through Euler's identity
  - Related to real exponentials and real sinusoids
  - Will become close friends by the end of the course ☺

- You should be able to
  - Identify complex exponentials and complex sinusoids
  - Determine the period of a complex sinusoid
  - Plot a complex exponential sketch and in MATLAB

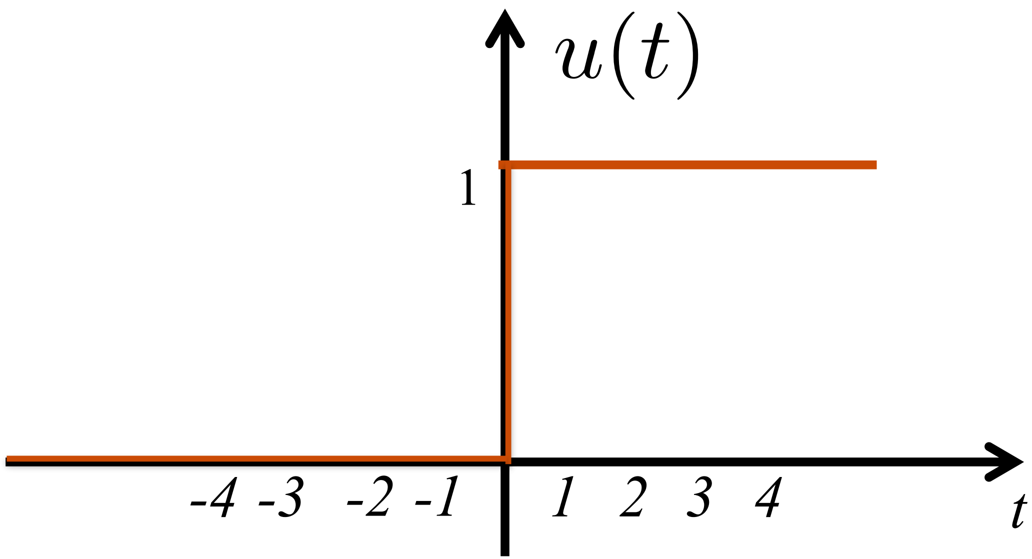
Unit-impulse function
also known as the Dirac delta function

Learning objectives

- Explain the properties of delta unit impulse function
- Exploit the sifting property to simplify expressions with deltas
- Exploit the integration property to simplify expressions with deltas

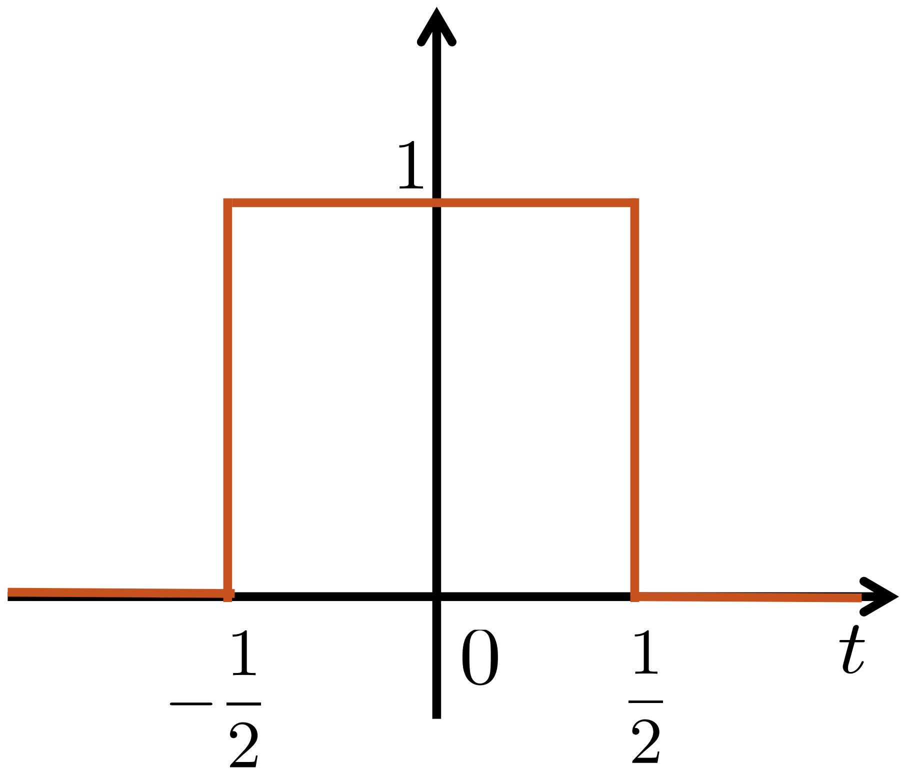
Unit step function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

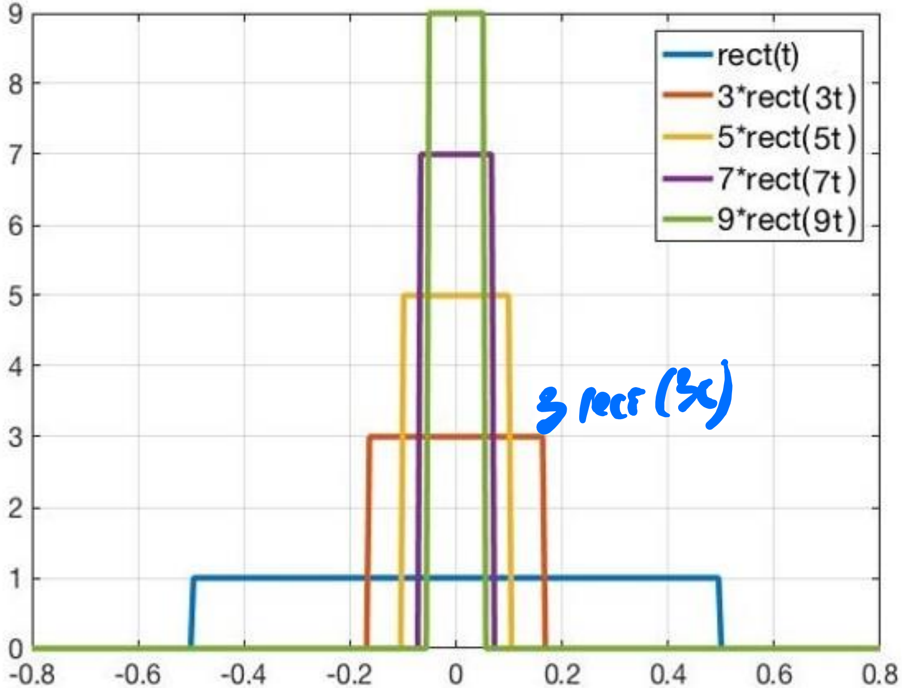


Rectangle function

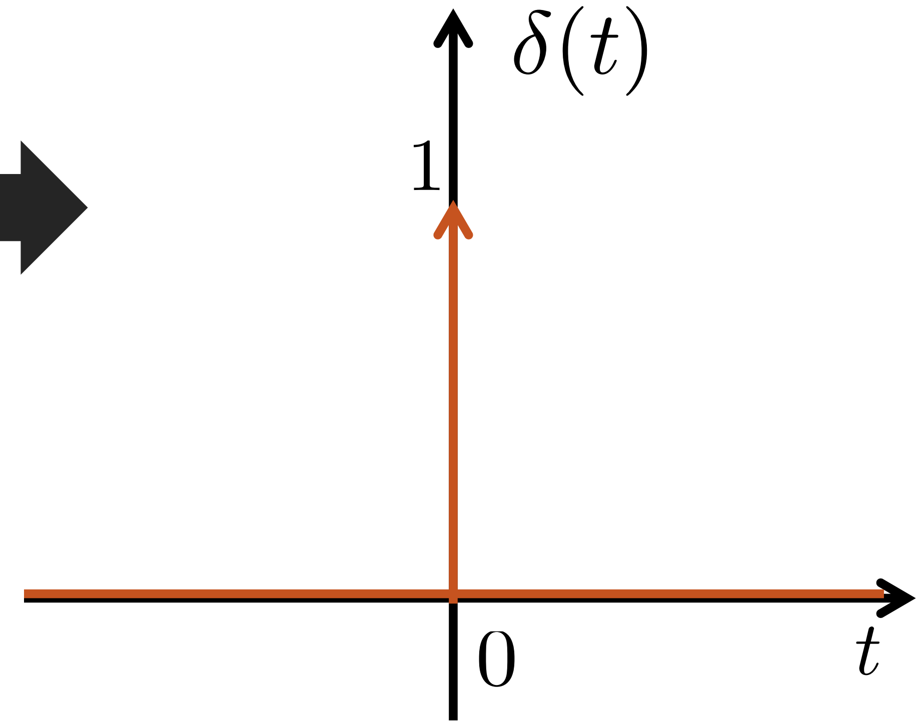
$$\mathrm{rect}(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leqslant \frac{1}{2} \end{cases}$$



Consider the following sequence

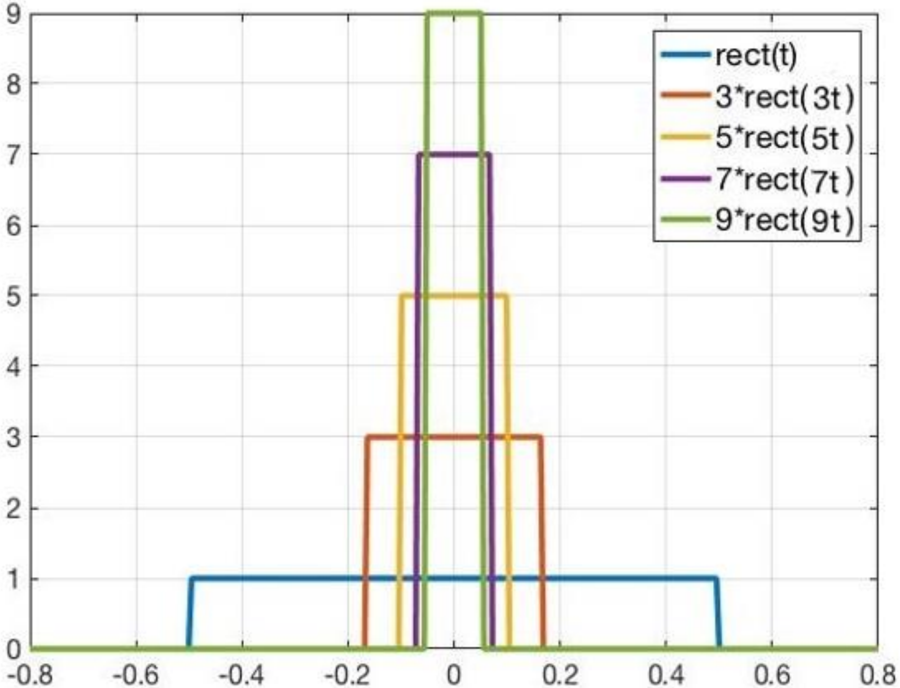


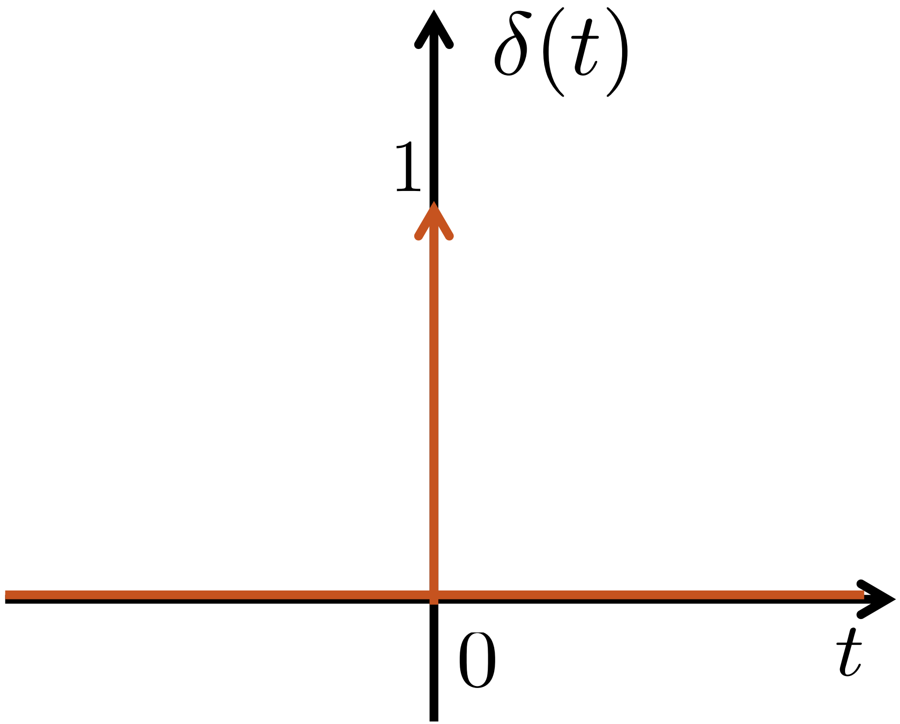
Dirac delta or unit-impulse function



Delta function is a generalized function that requires some care

Unit area property





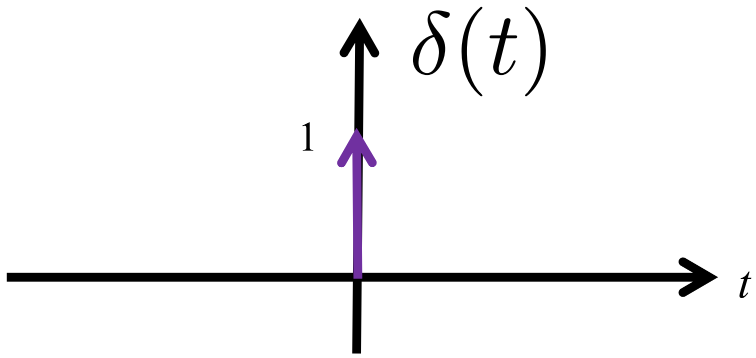
Bt

$$\int_{-\infty}^{\infty} B \,\mathrm{rect}\left(\frac{t}{B}\right) dt = 1$$

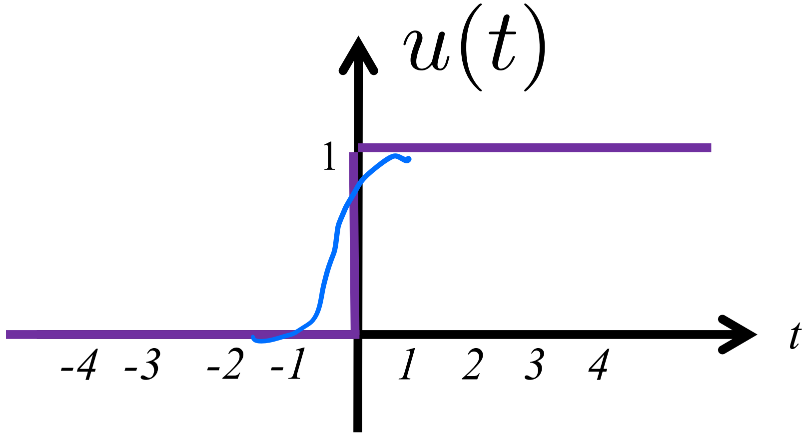
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Connection to the unit step function

$$\frac{du(t)}{dt} = \delta(t)$$



$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$



Sifting property at zero

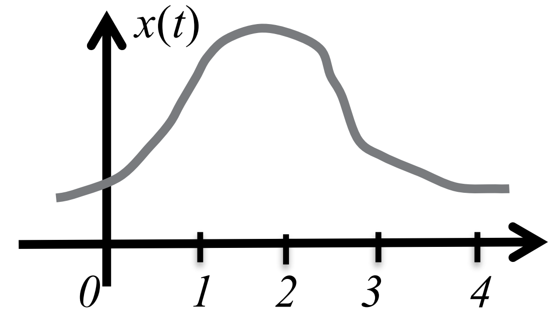
any function

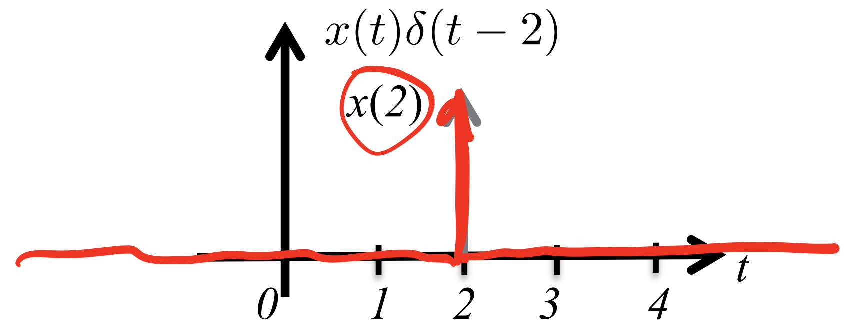
$$\boxed{x(t)}\delta(t) = x(0)\underline{\delta(t)}$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

Avoid common error: be sure to leave in delta function!

Example





From https://www.epicurious.com/ingredients/do-you-really-need-to-sift-flour-article

Sifting property example

- Consider the following signal

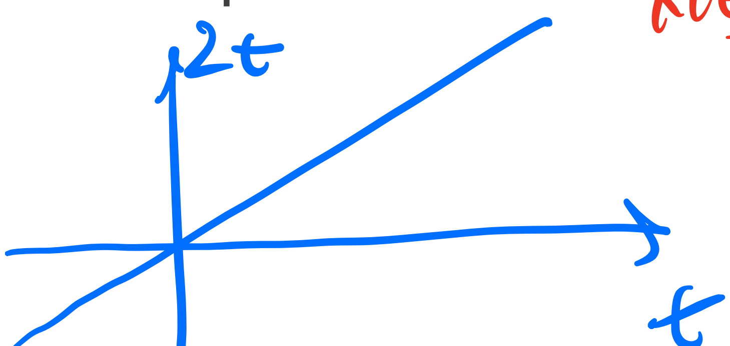
$$x(t) = 2t$$

- Find a simplified expression for

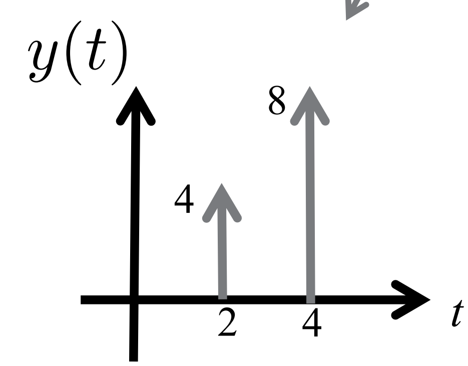
$$y(t) = x(t)(\delta(t-2) + \delta(t-4))$$

and plot the result

$$x(t)\delta(t-2) + x(t)\delta(t-4) = x(2)\delta(t-2) + x(4)\delta(t-4)$$



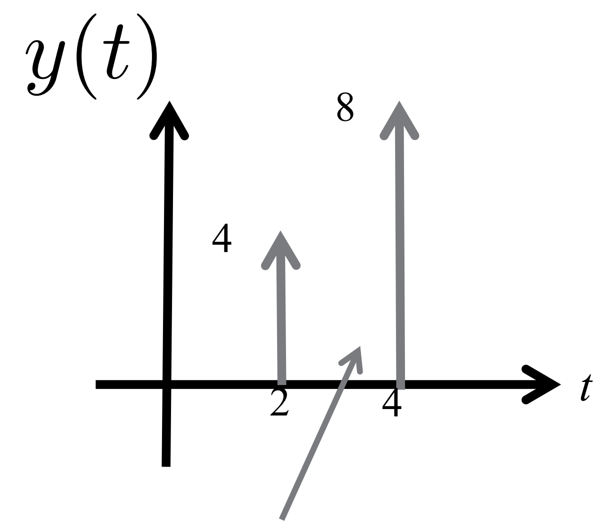
We draw deltas like this



Sifting property example

$$x(t) = 2t$$

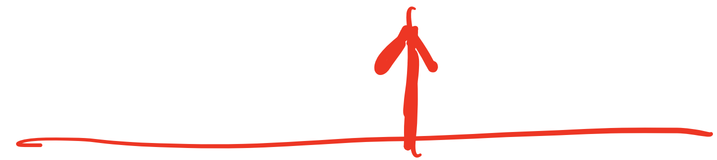
$$\begin{aligned} y(t) &= x(t)(\delta(t-2) + \delta(t-4)) \\ &= 2t(\delta(t-2) + \delta(t-4)) \\ &= 2t\delta(t-2) + 2t\delta(t-4) \\ &= 4\delta(t-2) + 8\delta(t-4) \end{aligned}$$



We label deltas like this

$$\int_{-\infty}^{\infty} y(t)\,dt = \int_{-\infty}^{\infty} 4\delta(t-2)\,dt + \int_{-\infty}^{\infty} 8\delta(t-4)\,dt = 12$$

Integration property



$$\int_{-\infty}^{\infty} x(\tau)\delta(\tau)d\tau = x(0)$$

$$\int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau = x(t)$$

There are <u>no delta functions</u> in the final expression, they were integrated out

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

Integration property examples

- Simplify the following expressions

$$\int_{-\infty}^{\infty} \cos\left(\frac{\pi t^2}{2}\right) \delta(t+2)dt$$

$$= \int_{-\infty}^{\infty} \cos\left(\frac{\pi (-2)^2}{2}\right) \delta(t+2)\, dt$$

$$= \cos(2\pi) \cdot \underbrace{\int_{-\infty}^{\infty} \delta(t+2)\, dt}_{1}$$

$$= 1$$

$$\int_{-\infty}^{\infty} \delta(t-2)\delta(t+2)dt = 0$$

$g(t)$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \int_{-\infty}^{\infty} x(t)\, \delta(t-\tau)\, d\tau$$

$$x(t) \int_{-\infty}^{\infty} \delta(t-\tau)\, d\tau = x(t)$$

$$\int_{-\infty}^{\infty} x(\eta)\,\delta(t-\eta)\,d\eta = x(t)$$

Integration property examples

- Simplify the following expressions

$\int_{-\infty}^{\infty} \cos\left(\frac{\pi t^2}{2}\right) \delta(t+2) dt$	$\int_{-\infty}^{\infty} \cos\left(\pi \frac{(-2)^2}{2}\right) \delta(t+2) dt = \cos(2\pi) = 1$
$\int_{-\infty}^{\infty} \delta(t-2)\delta(t+2) dt$	0
$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau$	$x(t)$

Time scaling the delta function (1/3)

- What is $\int_{-\infty}^{\infty} \delta(a\tau)d\tau$?
- Suppose that $a > 0$

$$\int_{-\infty}^{\infty} \delta(a\tau)d\tau = \int_{-\infty}^{\infty} \frac{1}{a}\delta(t)dt$$
$$= \frac{1}{a}$$

real number

$a > 0$

$$\int_{-\infty}^{\infty} \delta(a\tau)\, d\tau$$

$t = a\tau \qquad dt = a\, d\tau$

$d\tau = \frac{1}{a}\, dt$

$$\int_{-\infty}^{\infty} \frac{1}{a}\, \delta(t)\, d\tau = \frac{1}{a}$$

$a < 0$

$$\int_{-\infty}^{\infty} \delta(at)\, dt$$

$t = a\tau \qquad dt = a\, d\tau$

$$\int_{+\infty}^{-\infty} \frac{1}{a}\, \delta(t)\, dt$$

$$= -\int_{-\infty}^{\infty} \frac{1}{a}\, \delta(t)\, dt = \frac{1}{-a}$$

Time scaling the delta function (2/3)

- Suppose that $a < 0$

$$\int_{-\infty}^{\infty} \delta(a\tau)d\tau = \int_{-\infty}^{\infty} \frac{1}{-a}\delta(t)dt$$
$$= \frac{1}{-a}$$

Time scaling the delta function (3/3)

- Conclude that for any real value a

$$\int_{-\infty}^{\infty} \delta(a\tau)d\tau = \frac{1}{|a|}$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\int_{-\infty}^{\infty} \cos\left((\pi t)^2/2\right) \delta(2\cdot t)\, dt$$

$$\frac{1}{2}\delta(t)$$

$$\int_{-\infty}^{\infty} \frac{1}{2}\cdot \cos\left((\pi t)^2/2\right) \delta(t)\, dt$$

$$\frac{1}{2}\cdot \cos\left((\pi\cdot 0)^2/2\right) = \frac{1}{2}$$

Cautionary notes on the delta function

- The unit-impulse function, also called the Direct delta function, is really a *generalized function*
  - It does not really behave like a normal function
  - It is either zero or undefined
- We should technically only be using $\delta(t)$ under the integral sign
  - It is well defined in the integral sign
  - Some Professors will complain if not in the integral sign (but not me)
  - Take real analysis in the math department for further enlightenment
- Despite these quirks, <u>the delta function is extremely useful</u> for modeling and understanding signals and systems

"All models are wrong, but some are useful" – George Box (statistician)

Summary of the delta function

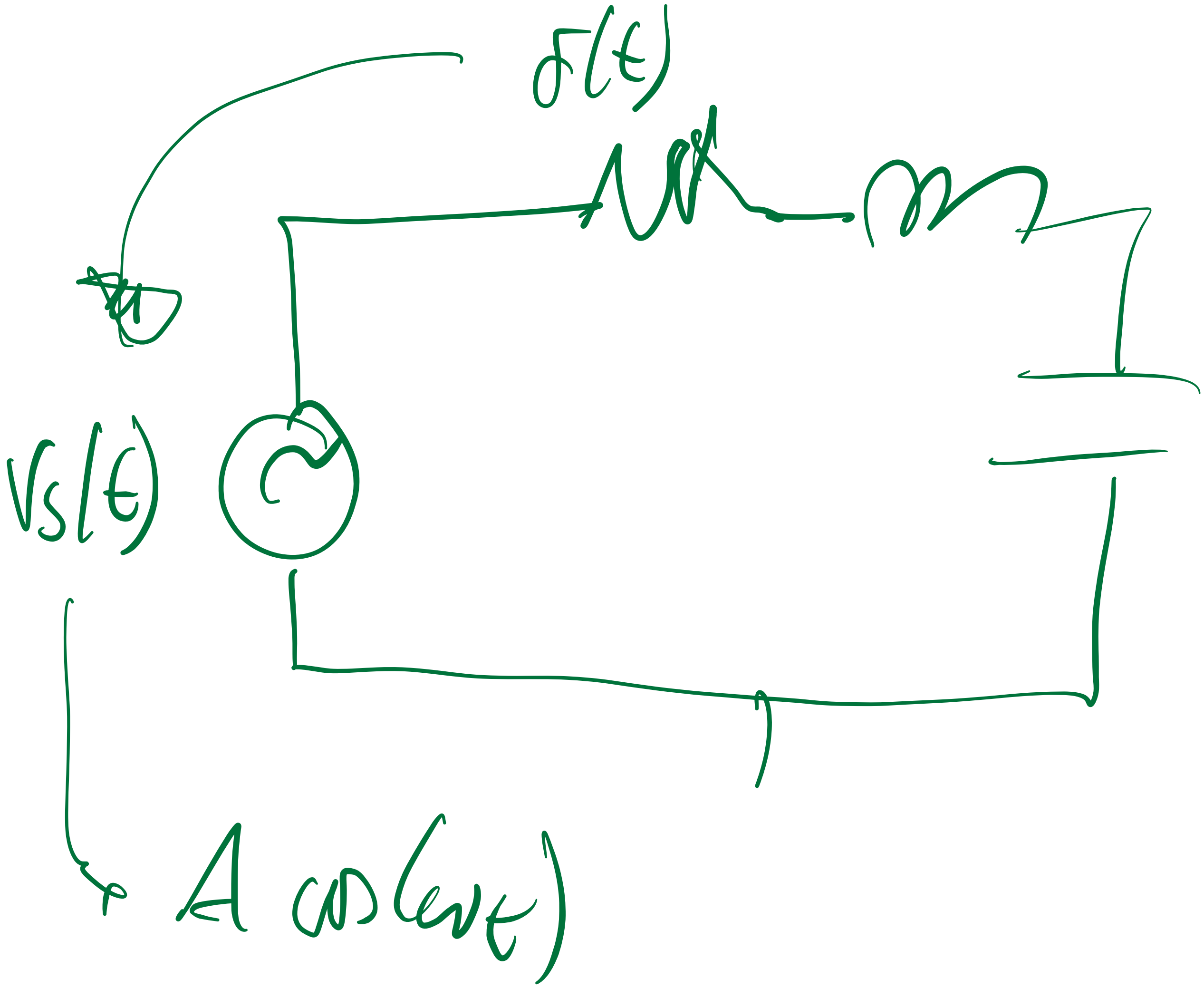
- Unit-impulse or Dirac delta in important basic signal
  - Generalized function of time

- Sifting with deltas pulls out the signal value but leaves the delta

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

- Integrating with deltas eliminates the delta and gives a value

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$


δ(t)
$v_s(t)$
$A\cos(\omega t)$