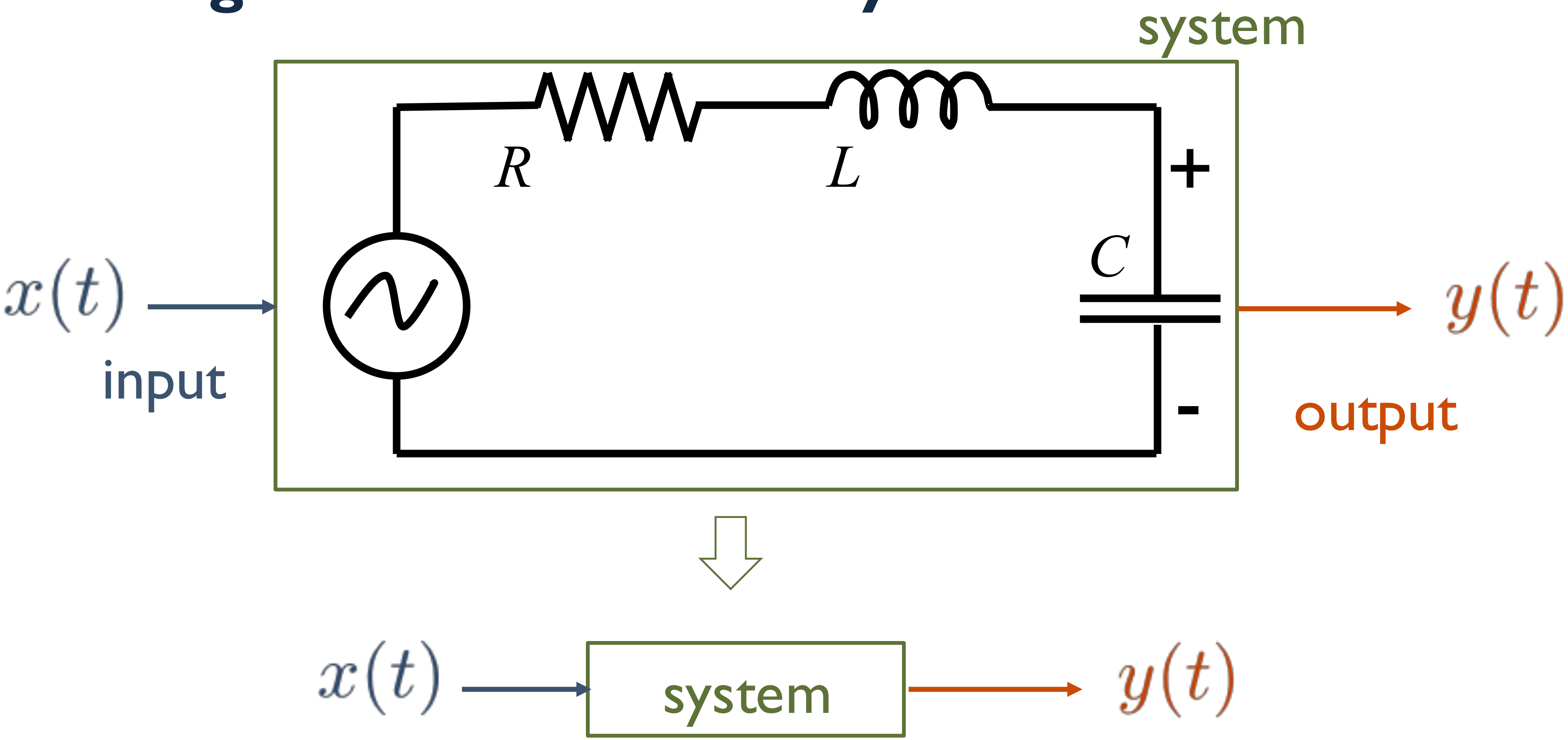
Lecture 2

Rectangle, step functions, signal transformations, periodic, even and odd

Preview of today's lecture

- Signals, unit–step and rectangle functions
  - Describe mathematically the unit step and rectangle functions
- Basic signal transformations
  - Apply different transformations on continuous-time signals
  - Create new signals from these transformations
- Signal characteristics: periodic, even, odd, and conjugate symmetric
  - Distinguish between periodic and aperiodic signals
  - Compute even and odd parts of an arbitrary signal

Thinking about circuits as a system



A mathematical description of how the input is transformed into the output

Connections back to ECE 45

Lectures 2 - 4 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 5 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

UC San Diego

	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, phasors	1.3		
4	1/16	Signals	Dirac delta, Kronecker delta, Sha function	1.4	HW3	HW2
5	1/21	LTI in time	Linear and time-invariant systems	2.1		
6	1/23	LTI in time	Convolution, convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Signals, unit–step and rectangle functions

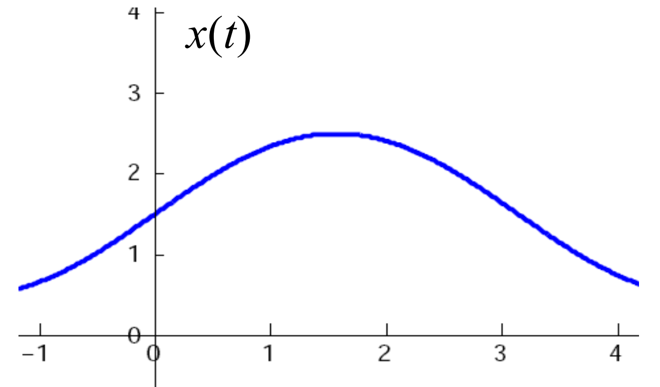
Learning objectives

- Describe mathematically the unit step and rectangle functions

What is a "signal"?

ECE 45 focuses on CT signals

- Representation of a value/info. relative to an independent variable
  - Often a time variable but could be something else



- Continuous-time (CT) signals $x(t)$

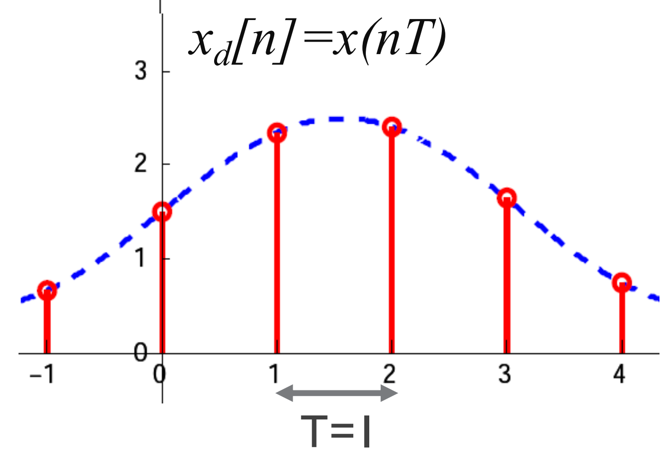
$t \in \mathbb{R}$

Bracket notation used to denote a discrete-time sequence

- Discrete-time (DT) signals $x[n]$
  - Often obtained by *sampling* CT signal
  - Taken at values nT

sampling period

$\cancel{x[n/2]}$



Complex signals

- Signals may take complex values (equivalently they have an amplitude and a phase when in polar form)

$$x(t) = \mathrm{Re}\{x(t)\} + j\mathrm{Im}\{x(t)\} \qquad x[n] = \mathrm{Re}\{x[n]\} + j\mathrm{Im}\{x[n]\}$$

- Complex signals are found in many practical problems
  - Most digital signal processors support complex operations
- Example: "in phase" (real) and "quadrature" (imag) EM signals

"rides" a cosine

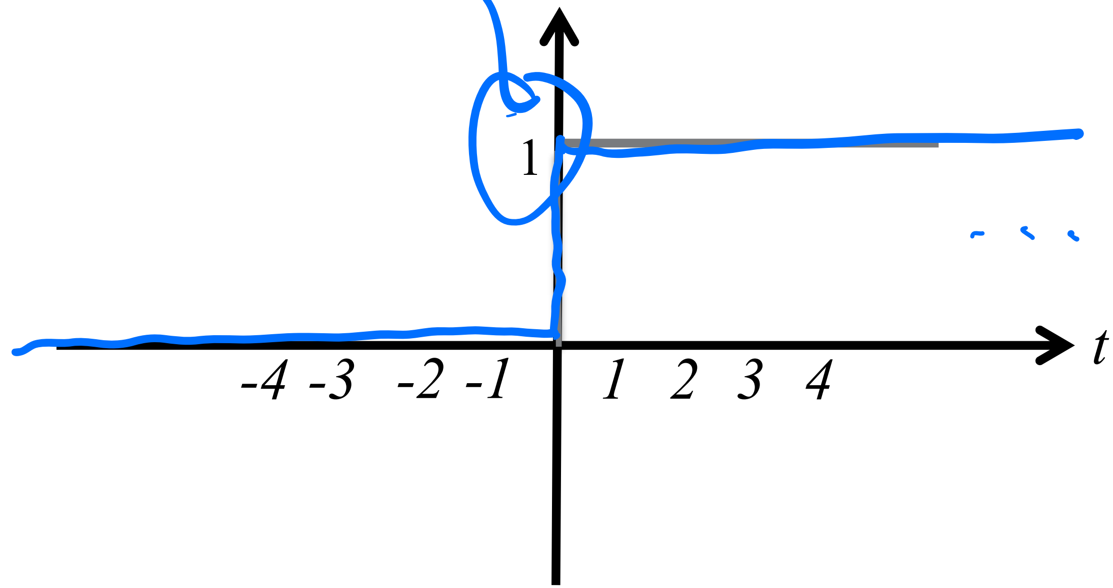
$$x(t) = \mathrm{Re}\{x(t)\} + j\mathrm{Im}\{x(t)\}$$

"rides" a sine

Many connections between complex signals, Maxwell's equations, phasors, etc.

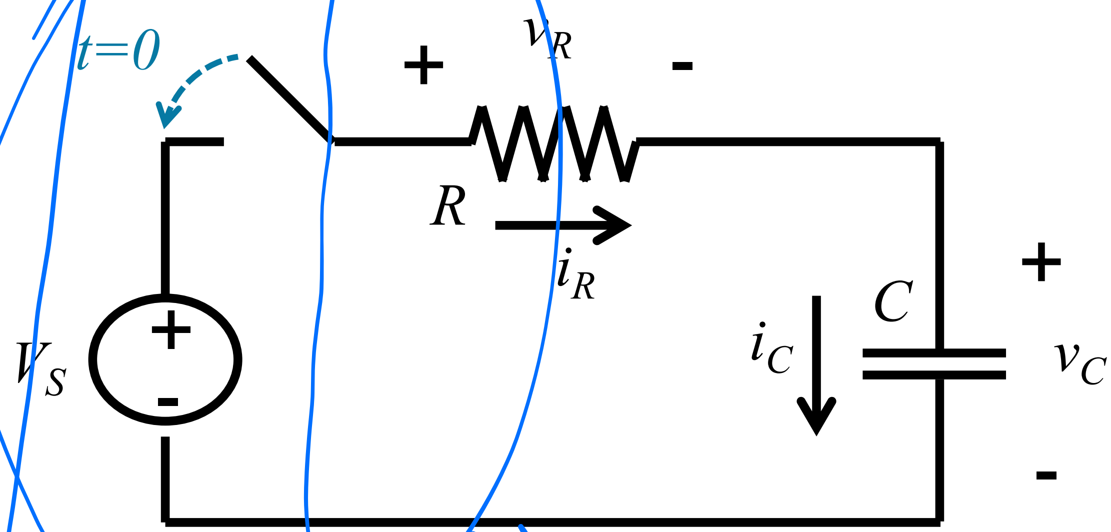
Unit step function

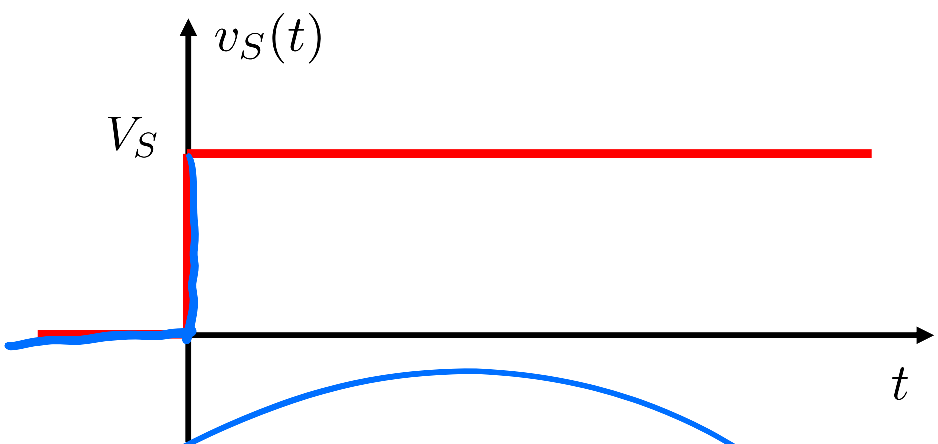
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$



- At t=0, u(t) may be either 0, 1, or ½ depending on the book
  - The specific choice is only important in a mathematical analysis class

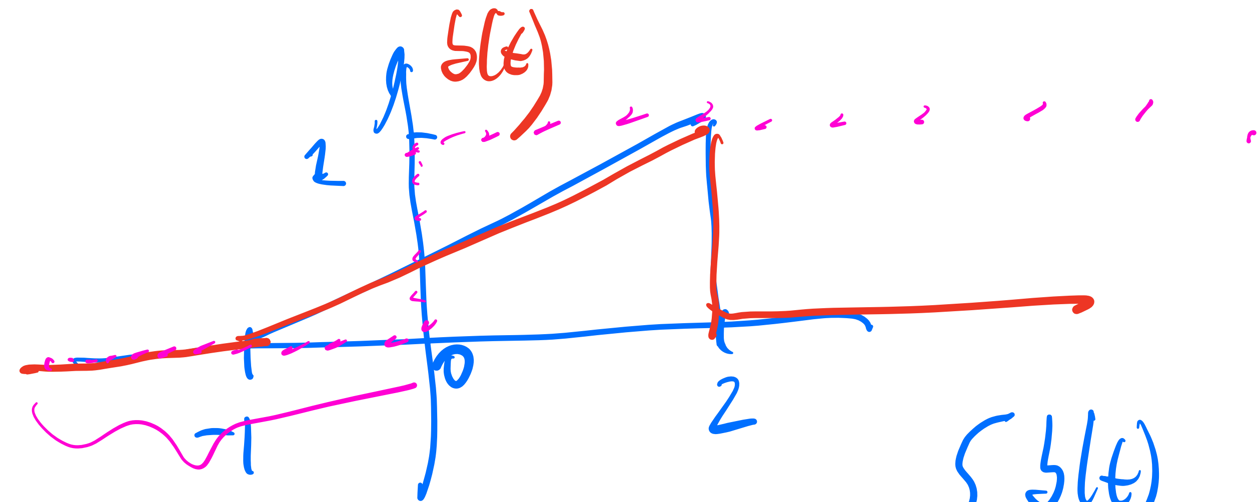
Where have you seen the unit step function?



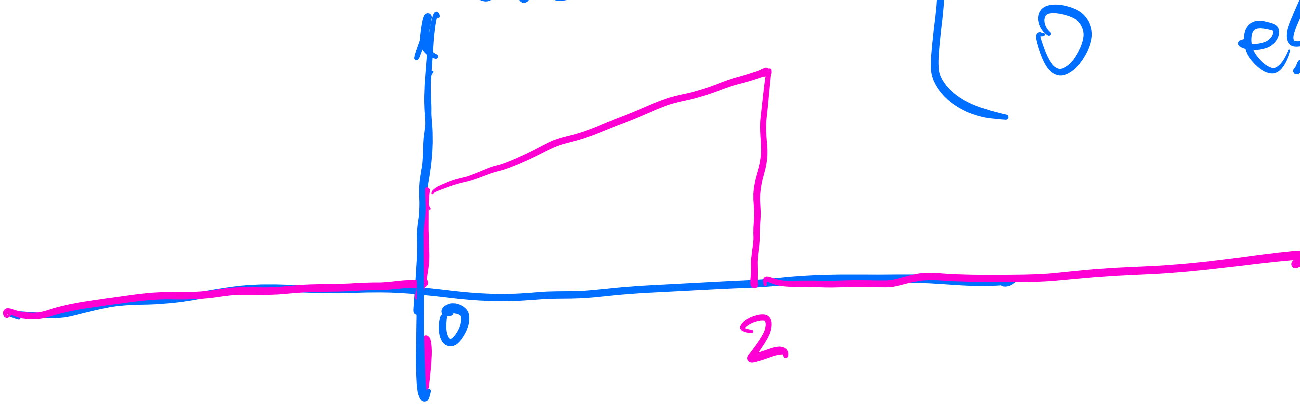


The voltage signal created from switching in a DC source is written using the unit step function

$$\boxed{v_S(t) = V_S\, u(t)}$$

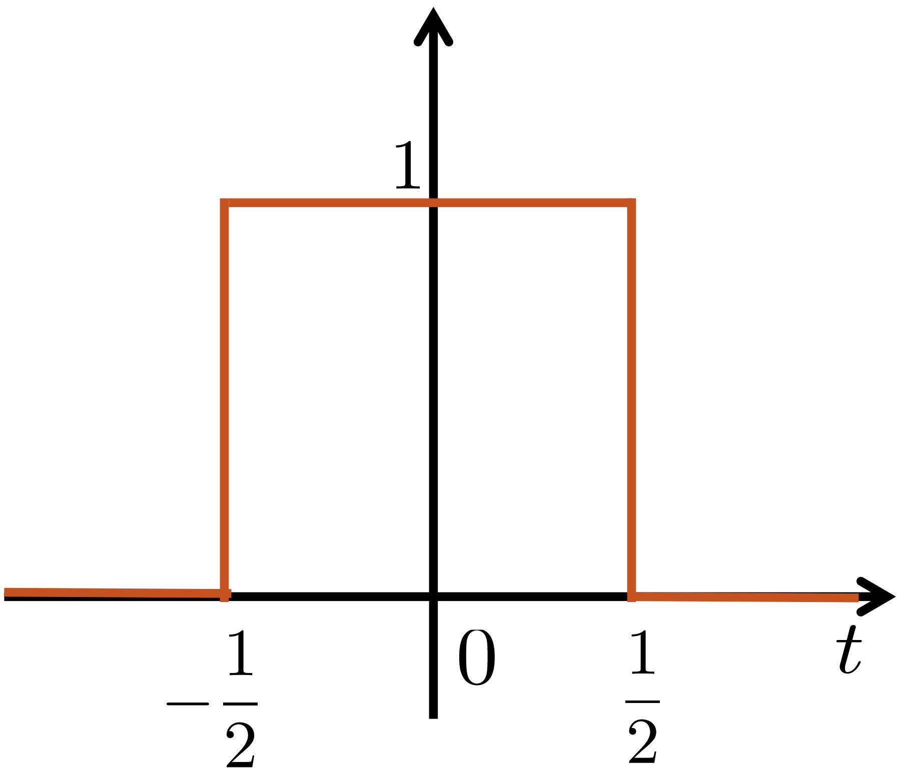


$$g(t)\,u(t) = \begin{cases} g(t) & t \geq 0 \\ 0 & \text{else} \end{cases}$$



Rectangle function

$$\mathrm{rect}(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leqslant \frac{1}{2} \end{cases}$$

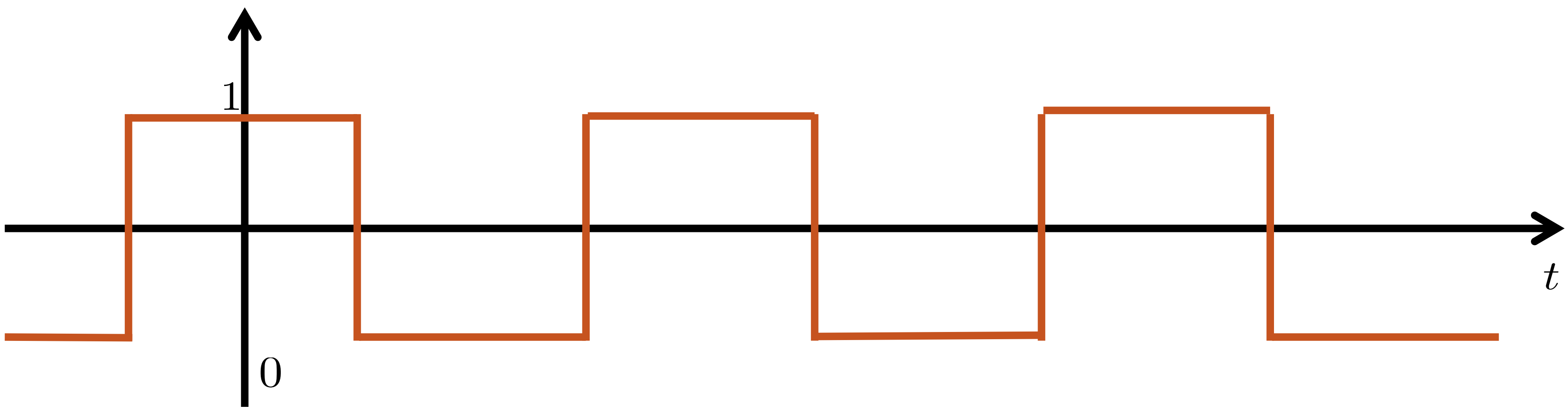


Also called the box function

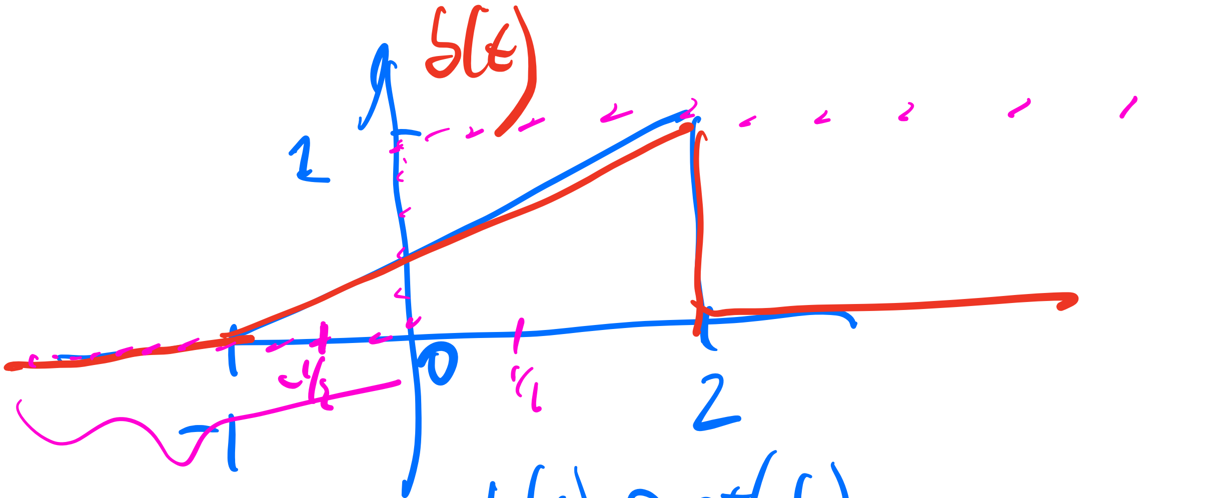
May have different notation in other books like $\Pi(t)$

Where will you see the rectangle function?

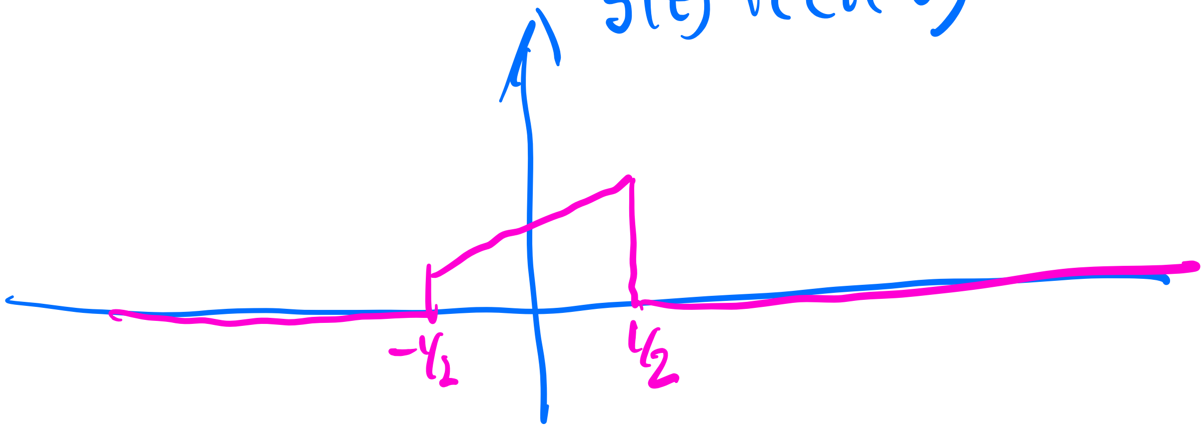
- Part of many common convolution examples
- Used to build a square waves, which is analyzed via Fourier Series



- Shows up as part of "windowing," e.g. $x(t)\,\mathrm{rect}(t)$



$g(t) \cdot rect(t)$



Why spend time on these basic signals?

- Unit step functions
  - An important building block signals for other parts of the course
  - Shows up often when signals ”start at zero”

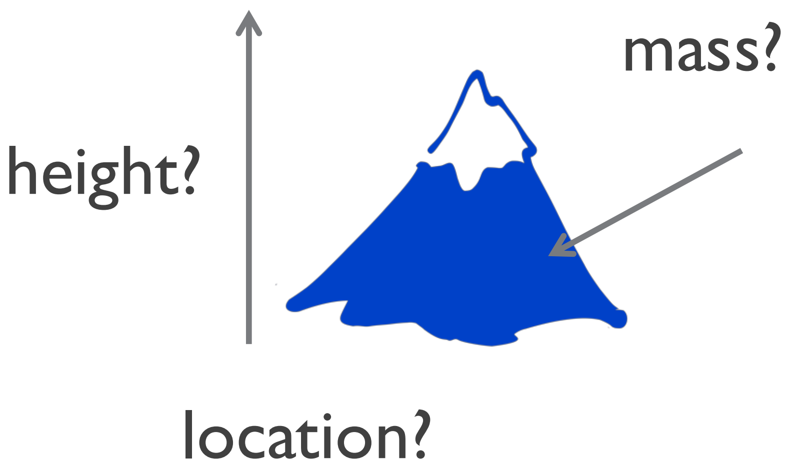
- Rectangle function
  - Another important building block signal
  - Used for many examples in convolution and Fourier

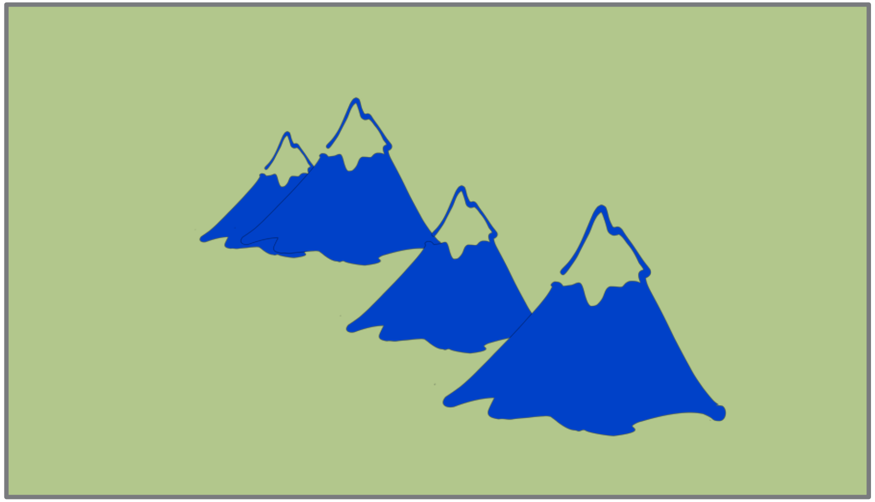
Basic signal transformations

Learning objectives

- Apply different transformations on continuous-time signals
- Create new signals from these transformations

Moving mountains (function is over space here)





- Use mountain description to build a map
  - Where are the mountains located?
  - What are their size?

Signal transformations are ways to describe and manipulate signals

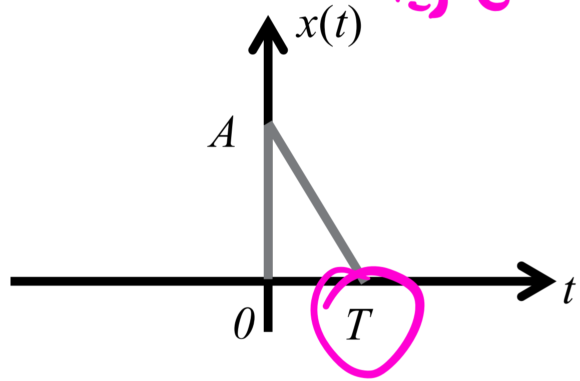
from clipartpanda.com

Example: time shift

$t_0 > 0$ in this example

$t-2 = T$?

$\Rightarrow t = T+2$



2

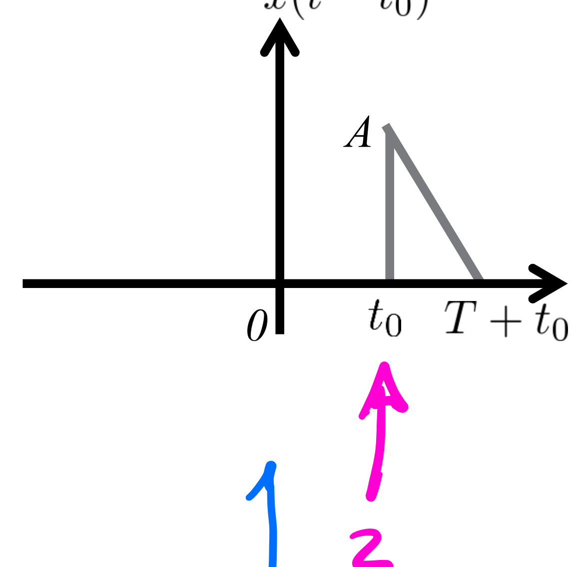
$x(t - t_0)$

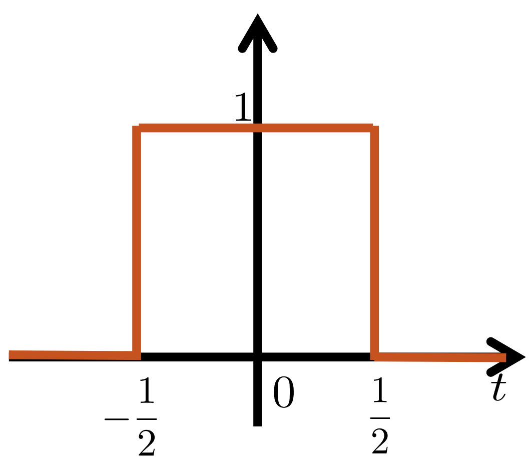
right shift

$t_0 = 2$

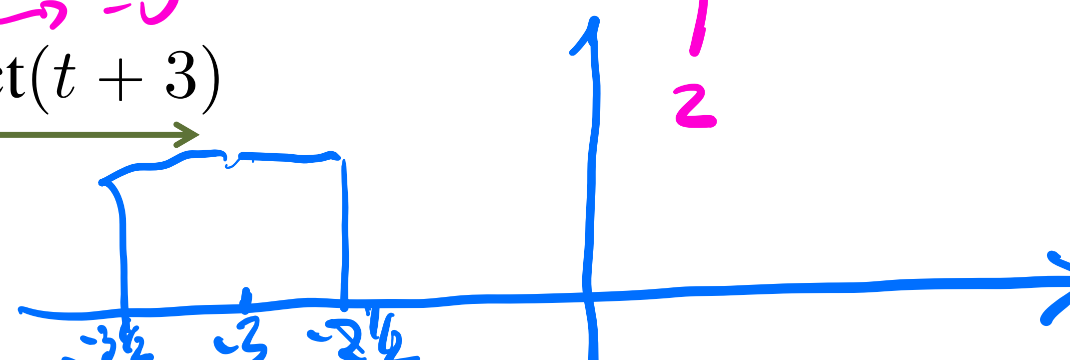
$x(t-2)$ $t = ?$

$= 0$





$\text{rect}(t+3)$



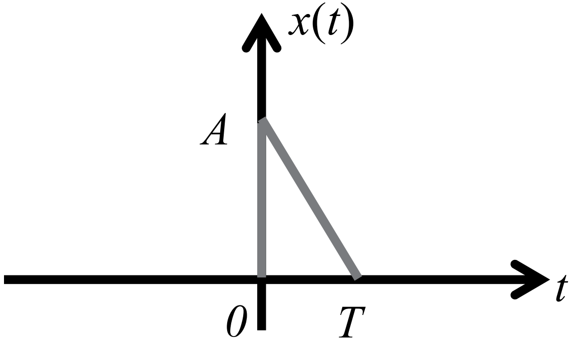
$t+3 = 0$? $\Rightarrow t = -3$

$$x(t-(-3))$$

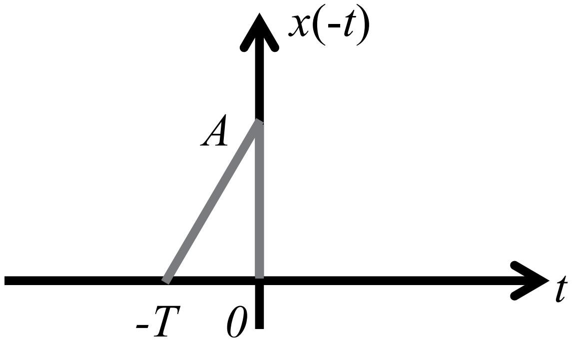
$$t_0 = -3 \; ?$$

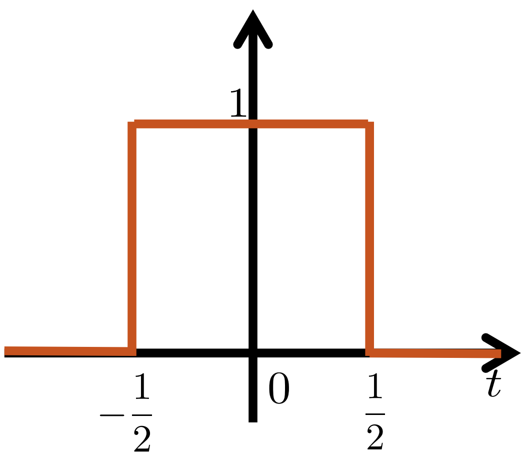
$$x(t+3)$$

Example: reflection or time reversal



$x(-t)$



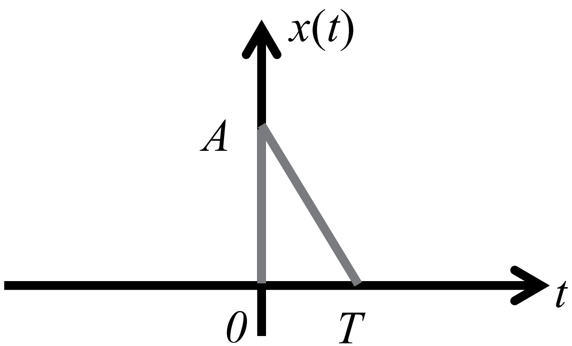


$\mathrm{rect}(-t)$

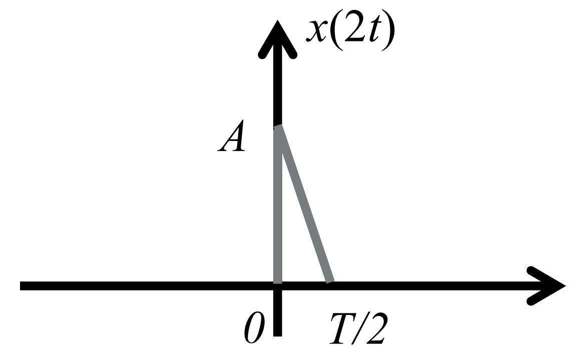
$x(-(-t))$
$= x(t)$

Same!!

Example: time compression / downsampling

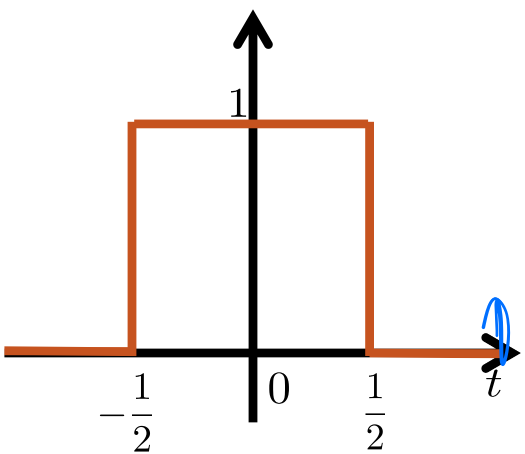


$\xrightarrow{x(2t)}$



2t = T ?

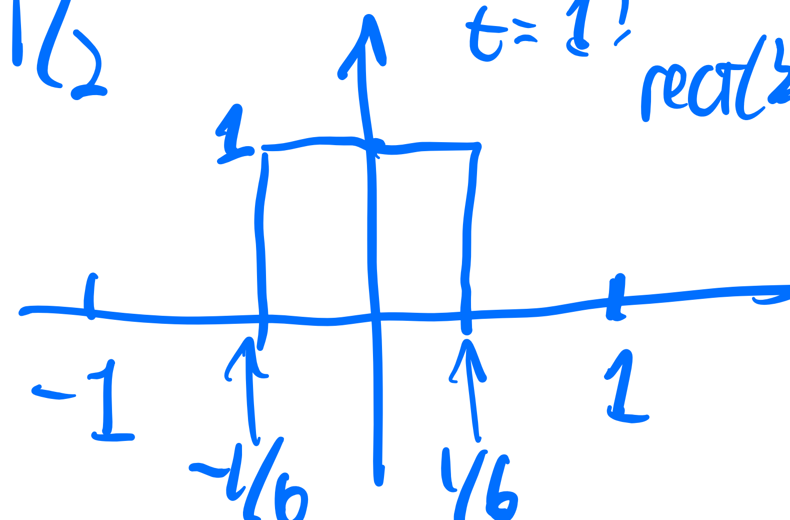
t = T/2



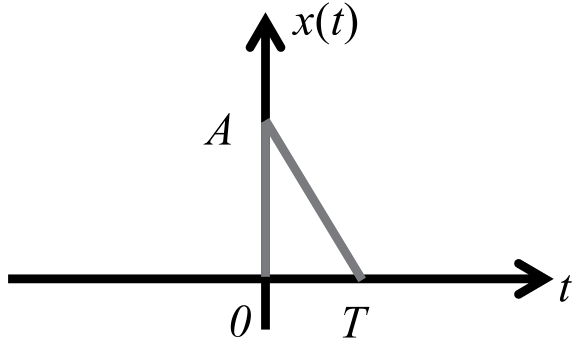
$\xrightarrow{\mathrm{rect}(3t)}$

t = 1 ?

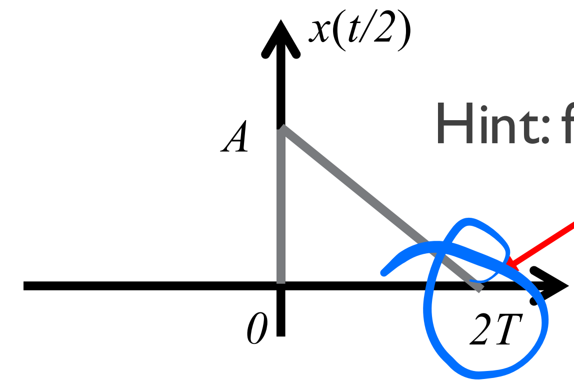
rect(3t)



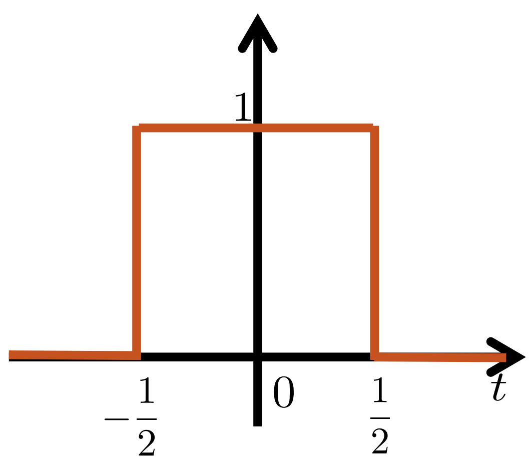
Example: time expansion (upsampling)



$x(t/2)$ $\longrightarrow$

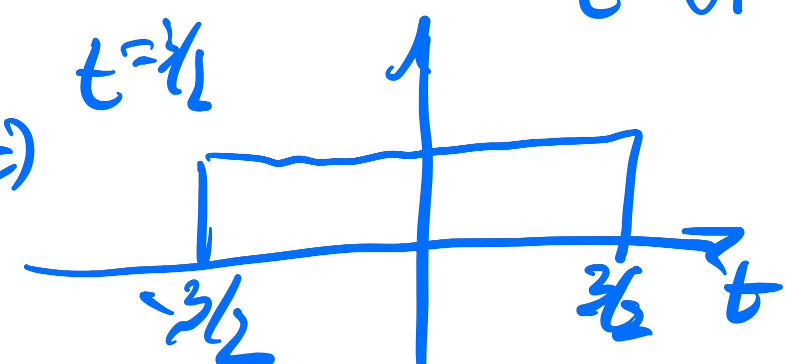


$t/2 = T \Rightarrow t = 2T$



$\mathrm{rect}(t/3)$ $\longrightarrow$

$t/3 = 1/2 \Rightarrow t = 3/2$



Example

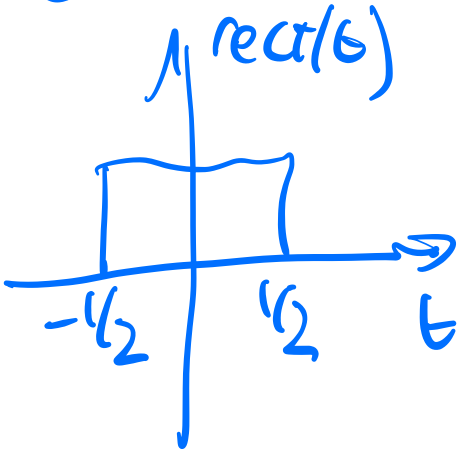
- Plot the signal $\text{rect}(2t-3)$

① Let $a(t) = \text{rect}(t-3)$

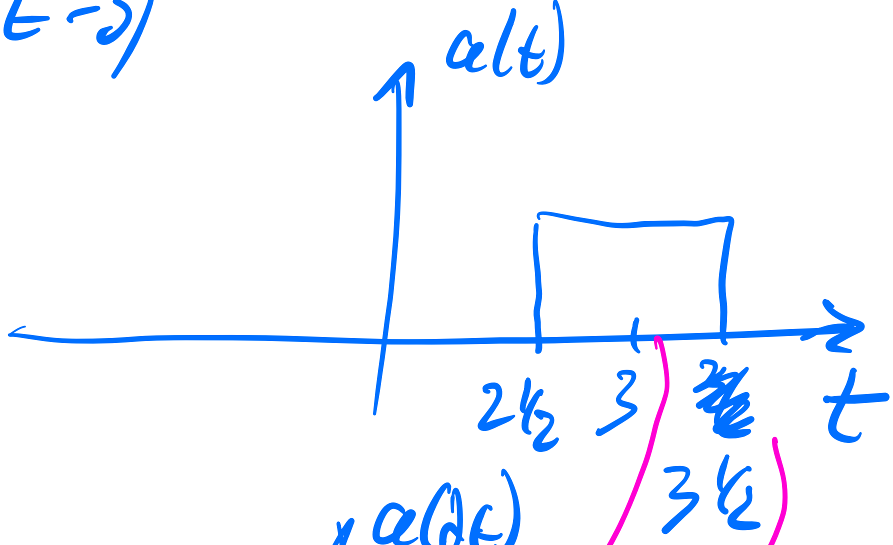
② $a(2t) = \text{rect}(2t-3)$

- You may shift then scale, or scale then shift (carefully)

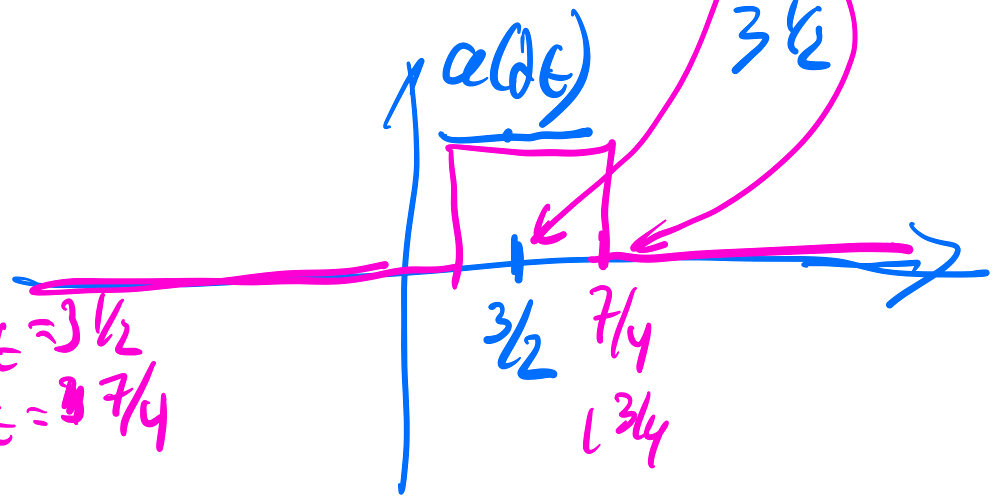
1) $a(t) = \mathrm{rect}(t-3)$



$\Rightarrow$



$a(t) \longrightarrow$



$2t = 3$
$t = 3/2$

$2t = 3\frac{1}{2}$
$t = 7/4$

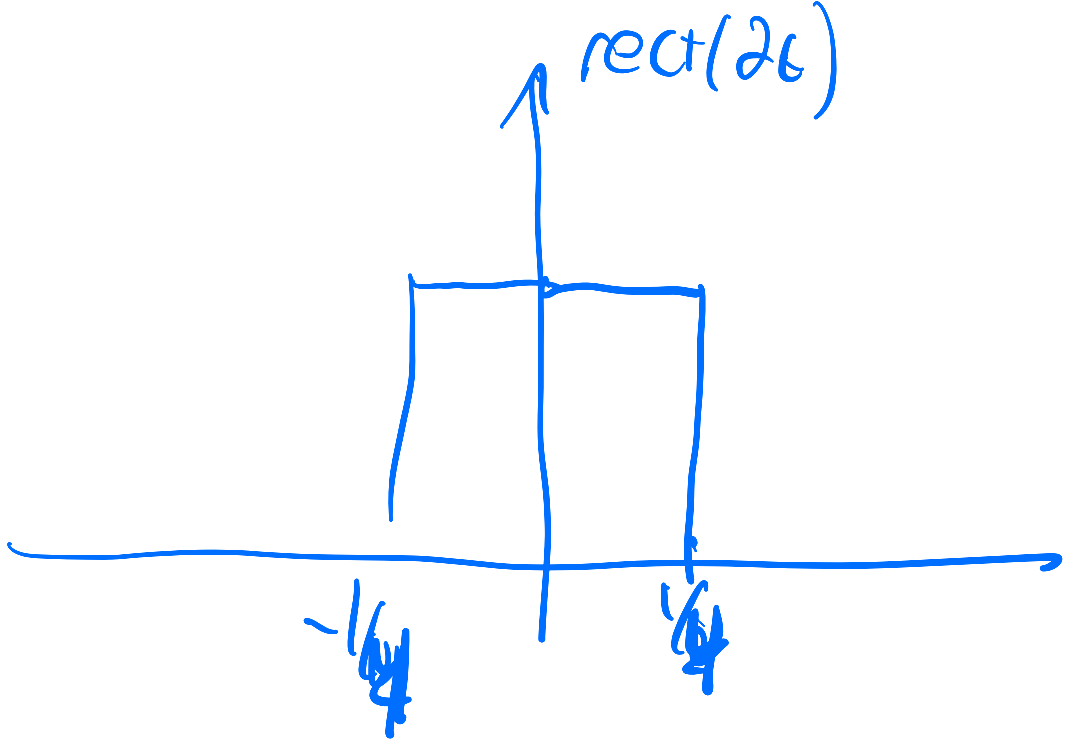
Example

- Plot the signal $\mathrm{rect}(2t-3)$ $= \mathrm{rect}(2(t-3/2))$

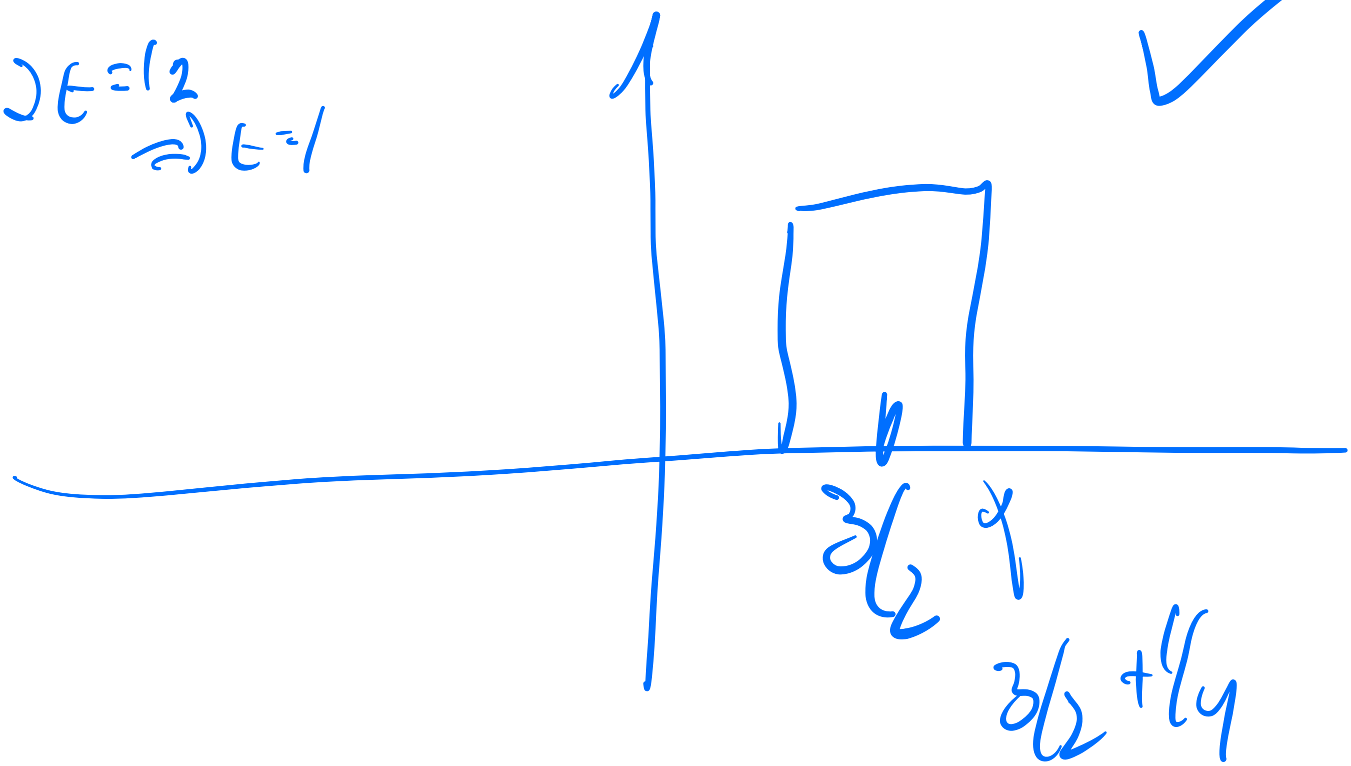
$b(t) = \mathrm{rect}(2t)$

$c(t) = b(t-3/2) = \mathrm{rect}(2t-3)$

- You may shift then scale, or scale then shift (carefully)



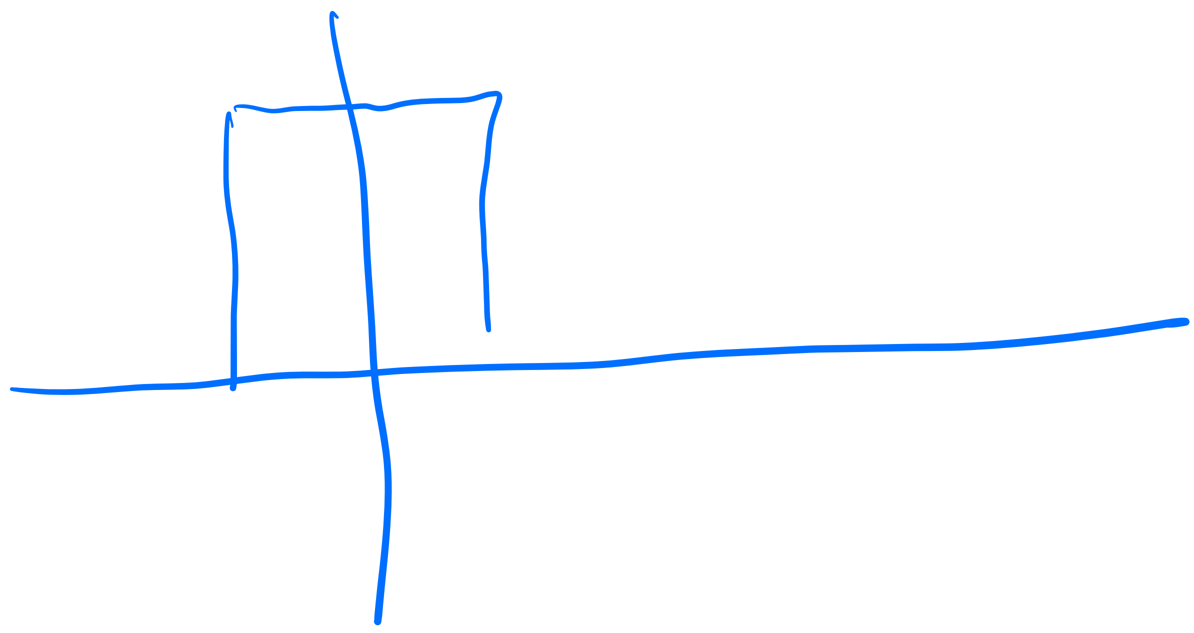
$2t = 1/2$
$\Rightarrow t = 1$



Signal transformation summary

- There are different ways to transform a signal
  - This section focused on transformations of the independent variable

- Transformations
  - Time shifting changes the starting point of a signal
  - Time scaling changes how fast the signal is "played"
  - Time reversal flips a signal
  - Compression / expansion change the "speed" of a signal



Signal characteristics: periodic, even, and odd

Learning objectives

- Distinguish between periodic and aperiodic signals
- Compute the period of a periodic signal
- Compute even and odd parts of an arbitrary signal

Periodic signals

$T > 0$

- Periodic signals satisfy for some finite non-zero T ~~or t~~

$$x(t) = x(t + T)$$ periodic with period T for all t

$\forall t$

- Period is the smallest non-zero solution
  - This is called the fundamental period
  - Normally period means fundamental period

$x(t+2T) = x((t+T)+T) = x(t+T) = x(t)$

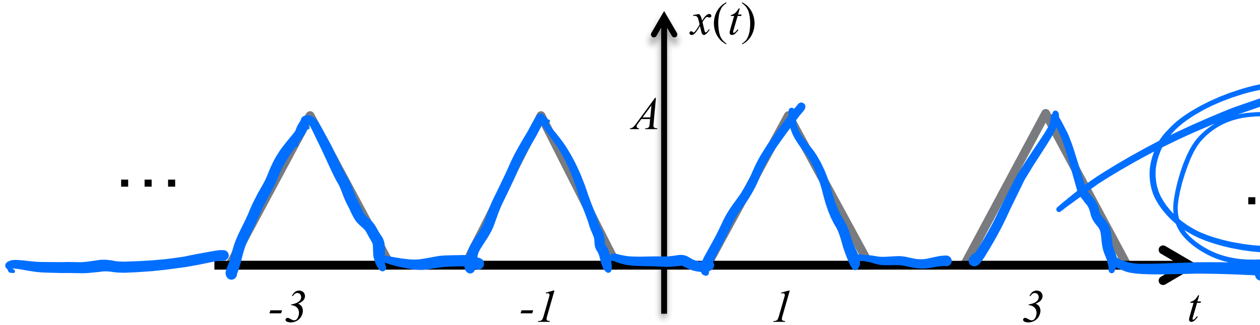
$x(t+T) = x(t)$

Periodic signals can be treated with special mathematical tools

Examples of periodic signals

pulse train of triangles

Dots indicate that the signal continues



period?

Establishing periodicity

- To prove a CT signal is periodic
  - Direct: find a $T>0$ such that $x(t)=x(t+T)$ for all t
  - Indirect: show that $x(t)$ is in a known class of periodic signals
- To prove a CT signal is aperiodic
  - Direct: Show that there is no $T>0$ such that $x(t)=x(t+T)$ for all t
- Remember the fundamental period is the smallest non-zero T

Example of direct method

- Determine whether or not the following signal is periodic? If it is periodic, determine its fundamental period.

$\omega = \frac{2\pi}{T} = 2$ $\quad T = \pi$

$$x(t) = \sin(2t) + \cos(4t + \pi/2)$$

$\sin(2\pi f t) = \frac{2\pi}{T}$

- Want to find T such that $x(t) = x(t+T)$

$$x(t+T) = \sin(2(t+T)) + \cos(4(t+T) + \pi/2)$$ shifted signal

$$\sin(2(t+T)) = \sin(2t) \text{ for } T = k\pi \text{ where } k \in \mathbb{Z}$$ multiple solutions

$$\cos(4(t+T) + \pi/2) = \cos(4t + \pi/2) \text{ for } T = k\pi/2 \text{ where } k \in \mathbb{Z}$$ multiple solutions

$$\boxed{T = \pi}$$ fundamental period

Example of indirect method

- Determine whether or not the following signal is periodic? If it is periodic, determine its fundamental period.

$$x(t) = [\cos(2t - \pi/3)]^2$$

- Recall that $\cos^2(x) = ½ (1 + \cos 2x)$

$x(t) = [1 + \cos(4t - 2\pi/3)]/2$. Periodic, period $= 2\pi/(4) = \pi/2$.

- By showing this simplifies to a known periodic function, we can also conclude it is period and find the frequency flipped

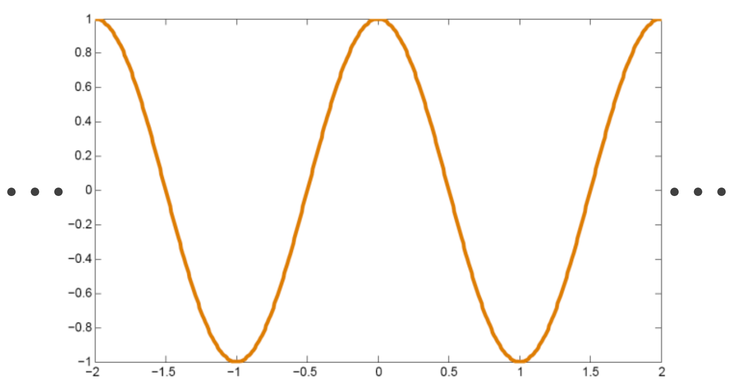
Even signals (real)

- Even signals satisfy

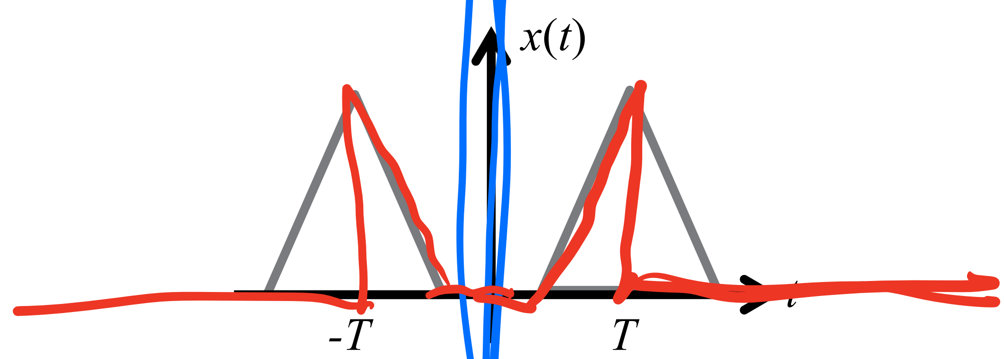
$$x(-t) = x(t)$$

same

- Examples (need not be periodic)



$\cos(2\pi f t)$



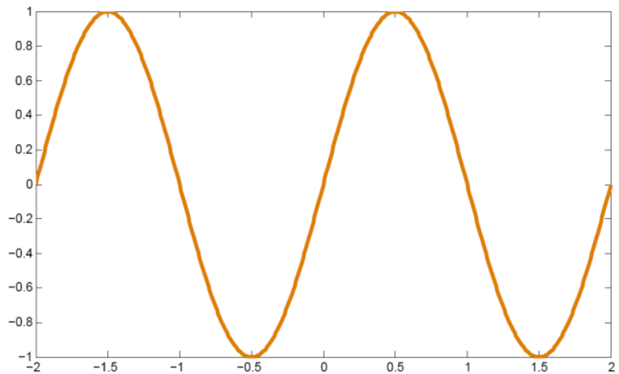
Odd signals (real)

- Odd signals satisfy $x(-t) = -x(t)$

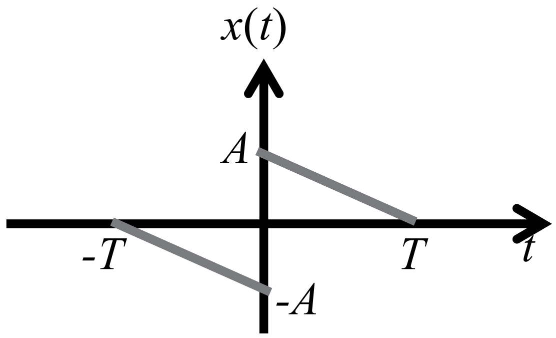
flipped

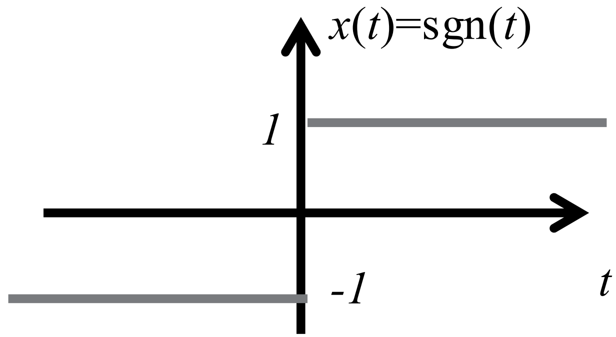
negated

- Examples



$\sin(2\pi f t)$





Generalization to complex signals

- Even becomes conjugate symmetric

$$x^*(-t) = x(t)$$

- Odd becomes conjugate antisymmetric

$$x^*(-t) = -x(t)$$

Interesting facts about even and odd functions

- Any real function can be written in terms of its even and odd parts

$$e(t) = \frac{1}{2}[f(t) + f(-t)]$$

$$e(-t) = \frac{1}{2}\{f(-t) + f(t)\} = e(t)$$

$$o(t) = \frac{1}{2}[f(t) - f(-t)]$$

$$e(-t) = \frac{1}{2}[f(-t) + f(t)] = e(t)$$

$$o(-t) = \frac{1}{2}[f(-t) - f(t)] = -o(t)$$

$$\boxed{f(t) = e(t) + o(t)}$$

Interesting facts about even and odd functions

- Integration properties

$$\int_{-\infty}^{\infty} e(t)dt = 2\int_{0}^{\infty} e(t)dt$$

also true

$$\int_{-\infty}^{\infty} o(t)dt = 0 = \int_{-T_0}^{T_0} o(t)dt$$

- Multiplications of two odd or two even functions → even function

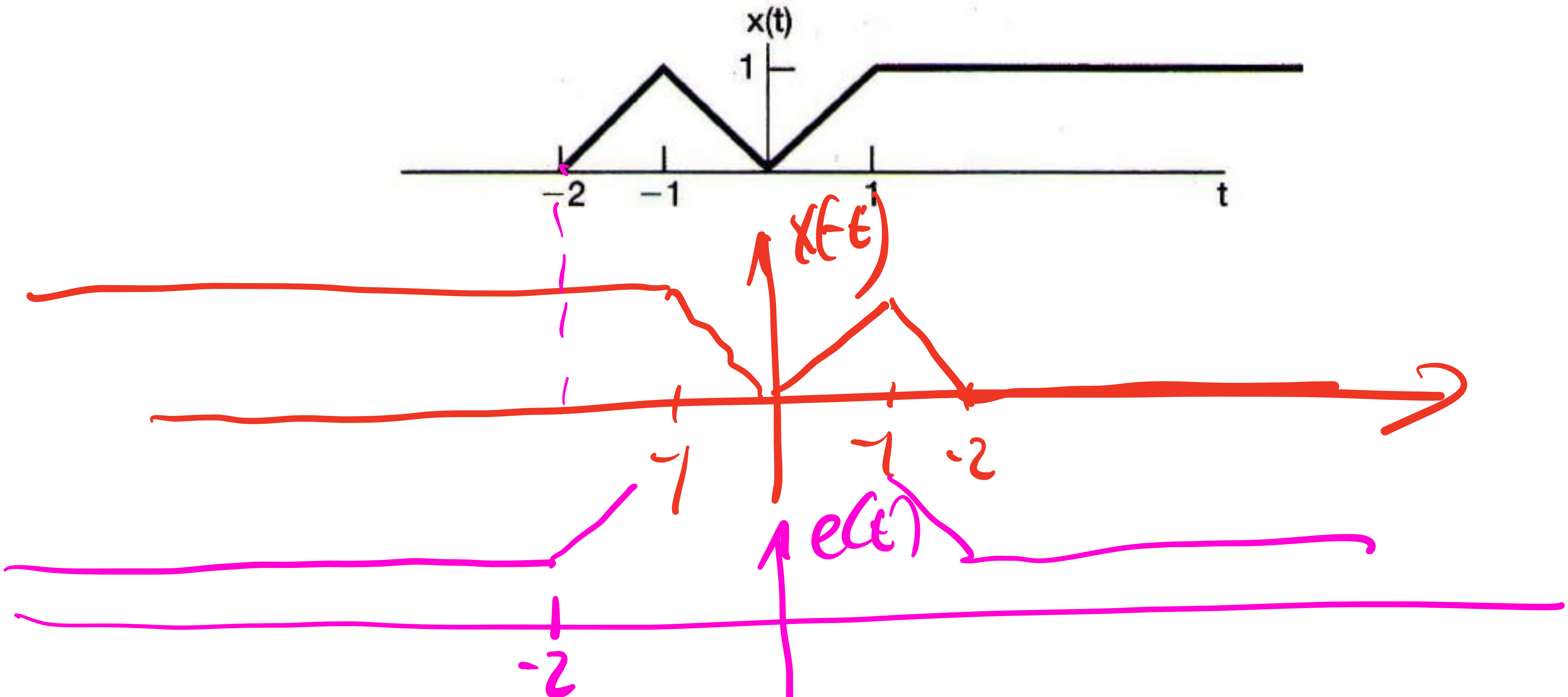
$$y(t) = o_1(t)o_2(t) = e(t)$$

$$y(t) = e_1(t)e_2(t) = \text{even function}$$

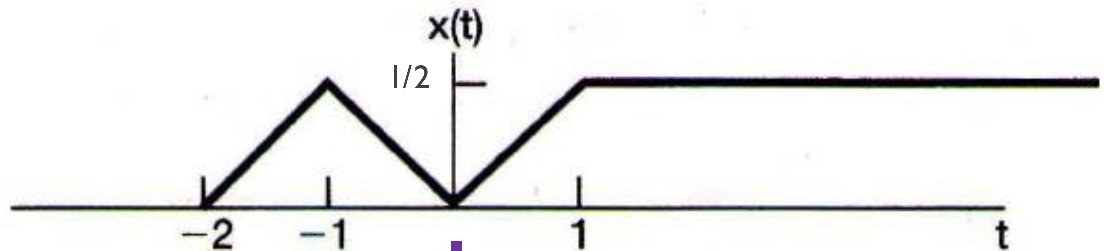
$$y(t) = o(t)e(t) = \text{odd function}$$

Example

- Determine and sketch the even and odd parts of the signal

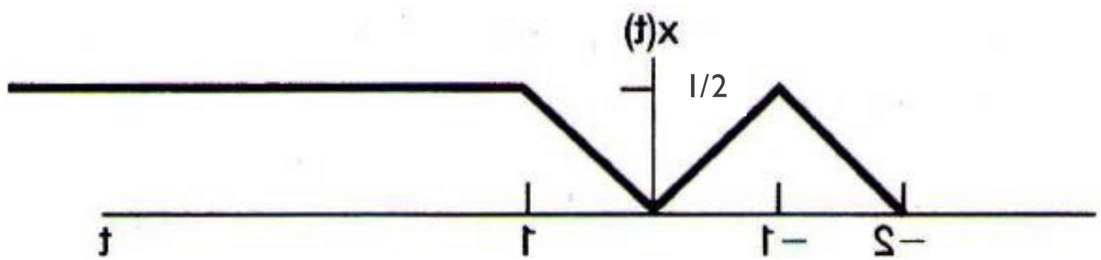


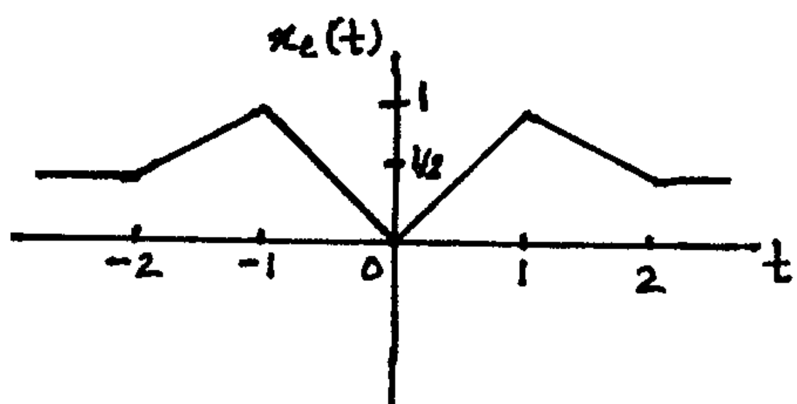
Sketching the even solution



plus

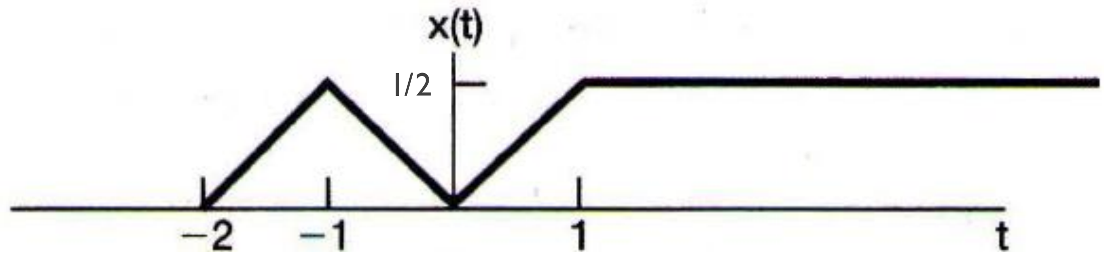
$$e(t) = \tfrac{1}{2}\,(x(t) + x(-t))$$



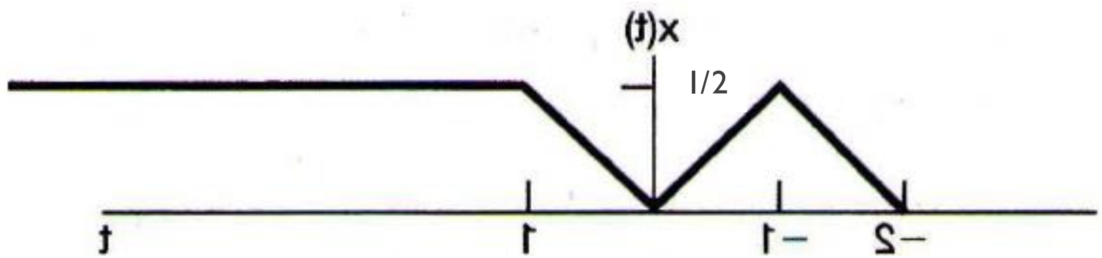


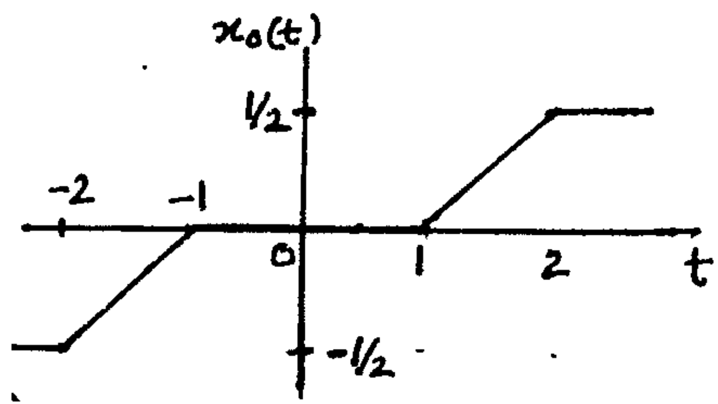
Sketching the odd solution

$o(t) = ½ (x(t) - x(-t))$



minus





Similar properties for complex signals

$$E(t) = \frac{1}{2}[f(t) + f^*(-t)]$$ Conjugate symmetric

$$O(t) = \frac{1}{2}[f(t) - f^*(-t)]$$ Conjugate antisymmetric

$$E^*(-t) = \frac{1}{2}[f^*(-t) + f(t)] = E(t)$$

$$O^*(-t) = \frac{1}{2}[f^*(-t) - f(t)] = -O(t)$$

$$f(t) = E(t) + O(t)$$

Note: Conjugate symmetric & real is even (why?)

Signal characteristics summary

- Periodic signals
  - Special type of signals that repeat
  - Need to determine if a signal is periodic and its period

- Even, odd, conjugate symmetric, conjugate antisymmetric
  - Signals with symmetry about the y axis
  - Fact used to simplify computations and derive intuition

- These special signal structures will be used in Fourier analysis