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Chapter 3
Continuous-time Fourier series

Signals and Systems

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Eigenfunctions of LTI systems

Learning objectives

- Characterize the eigenfunctions of CT and DT LTI systems

LTI systems

- LTI systems are characterized by their impulse responses

- Output is the convolution of the input and the impulse response

$$y(t) = x(t) * h(t)$$

$$y[n] = x[n] * h[n]$$

- Certain special functions called eigenfunctions pass through *almost* untouched by the convolution

Eigenfunctions of a CT LTI system

$$x(t) = e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = H(s)e^{st}$$

Complex exponential with complex "frequency" s

Eigenvalue (a complex scalar)

eigenfunction

- CT complex exponentials are eigenfunctions of LTI systems
  - Eigen comes from the German word "own" or "self"
  - Eigenfunction passes through the LTI system
  - Attenuated and scaled according to $H(s)$ (frequency response)

Eigenfunctions are easy to convolve

Why are complex exponentials so special?

- Consider a CT LTI system with impulse response $h(t)$

$$
\begin{aligned}
y(t) &= h(t) * e^{st} \\
&= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau} d\tau \\
&= e^{st} \boxed{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau} \\
&= e^{st} H(s)
\end{aligned}
$$

Note for the future: $H(s)$ is the Laplace transform of the impulse response

$H(s)$ also called the transfer function

What about an LTI system described by a LCCDE?

- Recall that $\dfrac{d^N}{dt^N}e^{st} = s^N e^{st}$
- Because the system is LTI, it follows that $y(t) = H(s)e^{st}$
- Inserting into the differential equation $\displaystyle\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

$$H(s)e^{st}\sum_{k=0}^{N} a_k s^k = e^{st}\sum_{k=0}^{M} b_k s^k \Rightarrow \boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}$$

Note where the LCCDE coefficients occur

Eigenfunctions of a DT LTI system

$$x[n] = z^n \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = H(z)z^n$$

Complex exponential with base *z*

Eigenvalue
(a complex scalar)

Eigenfunction passes right through

- DT complex exponentials are eigenfunctions of LTI systems
  - Eigenfunction passes through the LTI system
  - Attenuated and scaled according to $H(z)$ (related to impulse response)

Eigenfunctions are easy to convolve

Why are complex exponentials so special?

- Consider a DT LTI system with impulse response $h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$

$$= z^n \boxed{\sum_{k=-\infty}^{\infty} h[k]z^{-k}}$$

$$= z^n H(z)$$

Note for the future: *H(z)* is the Z-transform of the impulse response

H(z) also called the transfer function

What about an LTI system described by a LCCDE?

- Recall that $x[n] = z^n$ and observe that $x[n-1] = z^n z^{-1}$
- Because the system is LTI it follows that $y[n] = H(z)z^n$
- Inserting into the difference equation $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

$$H(z)z^n \sum_{k=0}^{N} a_k z^{-k} = z^n \sum_{k=0}^{M} b_k z^{-k} \quad \Rightarrow \quad \boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}}$$

Cautionary note!!

- Note that

$$e^{\alpha t} \neq e^{\alpha t} u(t)$$
$$\gamma^n \neq \gamma^n u[n]$$

- Only everlasting exponentials are true eigenfunctions

- Causal exponential functions do not have the same nice properties

☹

Eigenfunctions in other contexts

- Linear algebra

eigenvalue

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

matrix eigenvector

- Eigenfaces are eigenvectors used in human face recognition

original faces

eigenfaces

http://www.pages.drexel.edu/~sis26/Eigenface%20Tutorial.htm

Example 1: Constant input

- Consider $x[n] = c$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

- This is just a trivial exponential function

$$x[n] = c1^n$$

- Hence

$$y[n] = H(1)c$$

Example 2: A sinusoid

- Consider $x[n] = \cos(\Omega n)$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- Decomposing using Euler's identity

$$x[n] = \frac{1}{2}\left(e^{j\Omega n} + e^{-j\Omega n}\right)$$

- Gives the output

$$y[n] = \frac{1}{2}H(e^{j\Omega})e^{j\Omega n} + \frac{1}{2}H(e^{-j\Omega})e^{-j\Omega n}$$

- Can be simplified further in some cases

Example 3: Response for a LCCDE

- Consider the LCCDE that describes an LTI system

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = x(t)$$

- Find the response to $x(t) = e^{3t}$

$$H(s) = \frac{1}{\frac{1}{2} + s} = \frac{2}{1 + 2s}$$

$$y(t) = \frac{2}{1 + 2 \cdot 3} e^{3t} = \frac{2}{7} e^{3t}$$

Eigenfunctions in summary

$$x(t) = e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = H(s)e^{st}$$

CT complex exponential with complex s

DT complex exponential with complex z

Eigenvalue
(a complex scalar)

Eigenfunction

$$x[n] = z^{n} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = H(z)z^{n}$$

Convolution is easy with eigenfunctions!

Continuous-time Fourier series

Learning objectives

- Explain the key idea of Fourier series representation of signals
- Specialize the Fourier series to real signals

Fourier series for C T periodic signals

- Consider the periodic signal $x(t)$ with period T: $x(t+T)=x(t)$
- The Fourier series representation of the periodic signal $x(t)$ is

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where $\omega_0=\frac{2\pi}{T}$ is the fundamental frequency

- The Fourier series coefficients of $x(t)$ are $\{a_k\}$ and a_0 is DC
- The k-th harmonic components of $x(t)$ are a_k and a_{-k}

Interpreting the Fourier series

- Can represent (most) periodic signals as

$$x(t) = \underbrace{a_0}_{\text{DC offset}} + \underbrace{a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}}_{1^{st}\text{ harmonic on fundamental term}} + \underbrace{a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t}}_{2^{nd}\text{ harmonic, at } 2\omega_0} + \cdots$$

- Checking periodicity

$$x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t+T)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{jk\omega_0 T}$$

$$= x(t)$$

=1

$$\boxed{\begin{aligned} e^{jk\omega_0 T} &= e^{jk\frac{2\pi}{T}T} \\ &= e^{jk2\pi} \\ &= 1 \end{aligned}}$$

Example of Fourier series addition

- http://www.intmath.com/fourier-series/fourier-graph-applet.php

Special case of real signals

$$x^*(t) = x(t)$$

- Real signals have special symmetry in the Fourier series

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right)^*$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{jk\omega_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$$= \sum_{\ell=-\infty}^{\infty} a_{-\ell}^* e^{j\ell\omega_0 t}$$

$$\equiv \underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}_{x(t)}$$

$$\boxed{a_k = a_{-k}^*}$$ conjugate symmetry

Using the symmetry for real signals

- Suppose that x(t) is real

Decomposition is real, which is expected

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right]$$

z z*

$$= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

$$z + z^* = 2Re\{z\}$$

Writing the coefficients in polar form

- Let $a_k = A_k e^{j\theta_k}$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right)$$

- Write real signals as a sum of phase shifted cosines and DC term

http://en.wikipedia.org/wiki/Polar_bear

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Writing the coefficients in Cartesian form

- Let $a_k = B_k + jC_k$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{(B_k + jC_k)(\cos k\omega_0 t + j\sin k\omega t)\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2(B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t))$$

- Write real signals as a sum of sine, cosine, and DC term

https://guinevereglasfurddotcom.files.wordpress.com/2014/11/rene-descartes_1214990c.jpg

Example 4

- A CT Periodic & real signal $x(t)$ has a fundamental period $T=8$. The non-zero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1} = 2$$

$$a_3 = a^*_{-3} = 4j$$

- Express $x(t)$ in both polar and Cartesian forms

Example 4 - solution

$$
\begin{aligned}
x(t) &= a_1 e^{j\left(\frac{2\pi}{T}\right)t} + a_{-1} e^{-j\left(\frac{2\pi}{T}\right)t} + a_3 e^{j3\left(\frac{2\pi}{T}\right)t} + a_{-3} e^{-j3\left(\frac{2\pi}{T}\right)t} \\
&= 2e^{j\left(\frac{2\pi}{8}\right)t} + 2e^{-j\left(\frac{2\pi}{8}\right)t} + 4je^{j3\left(\frac{2\pi}{8}\right)t} - 4je^{-j3\left(\frac{2\pi}{8}\right)t} \\
&= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{3\pi}{4}t\right) \qquad \text{Cartesian} \\
&= 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right) \qquad \text{Polar}
\end{aligned}
$$

Summary of Fourier series for CT periodic signals

General form of the Fourier series is

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{(jk\omega_0)t}}$$

where $\omega_0 = \dfrac{2\pi}{T}$ is the fundamental frequency

Special forms for when signal is real

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right)$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\left(B_k \cos\left(k\omega_0 t\right) - C_k \sin\left(k\omega_0 t\right)\right)$$

Fourier analysis and orthogonality

Learning objectives

- Connect Fourier series to the inner product of a vector
- Prove orthogonality of complex sinusoids

Fourier analysis and synthesis

time domain signal → **Fourier analysis** → Fourier coefficients (frequency domain)

figuring out what frequencies are used to build the time domain signal

Fourier coefficients (frequency domain) → **Fourier synthesis** → time domain signal

putting the frequencies together to build the time domain signal

Fourier theory involves decomposing signals into their Fourier coefficients and building signals from Fourier coefficients

Key Fourier equations

Finding the coefficients: Use the analysis equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}\mathrm{d}t = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}\mathrm{d}t$$

Integrate over the fundamental period T

Reconstructing the signal: Use the synthesis equations

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

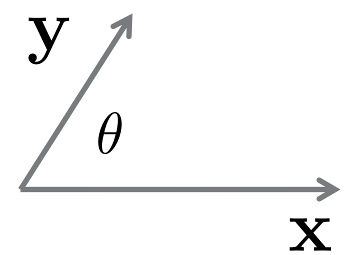
coefficients complex sinusoid

Signal *x(t)* that is periodic with fundamental period *T*

The analysis equation as an inner product

- Inner product of a pair of vectors

$$\mathbf{x}^*\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta)$$



- Inner product between a pair of periodic functions *x(t)* and *y(t)*

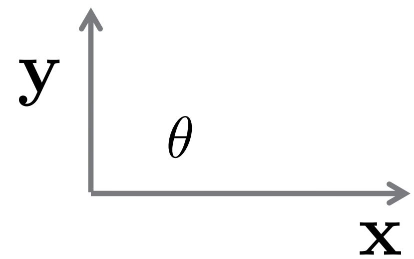
$$\langle x(t), y(t)\rangle = \int_0^T x(t)y^*(t)dt$$

Way to define magnitude and extent of overlap

Orthogonality

- A pair of vectors are orthogonal if

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta) = 0$$



- Two periodic functions *x(t)* and *y(t)* are orthogonal if

$$\langle x(t), y(t) \rangle = \int_0^T x(t) y^*(t) dt = 0$$

Orthogonality of complex sinusoids

- Consider the following periodic signals

$$x(t) = e^{jk\omega_0} \qquad y(t) = e^{jn\omega_0}$$

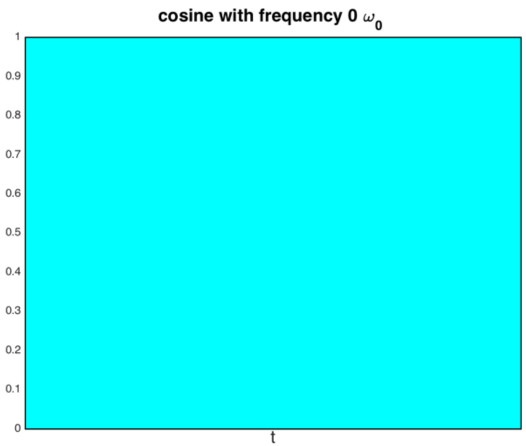
- Compute the inner product between these two signals

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos((k-n)\omega_0 t) + j\sin((k-n)\omega_0 t) dt$$

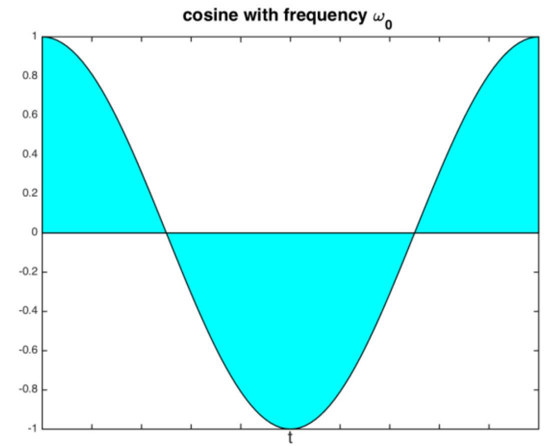
$$= \begin{cases} T & k = n \\ 0 & k \neq n \end{cases} \Rightarrow T\delta[k-n]$$

Complex sinusoids comprised of different harmonics of the fundamental frequency are orthogonal

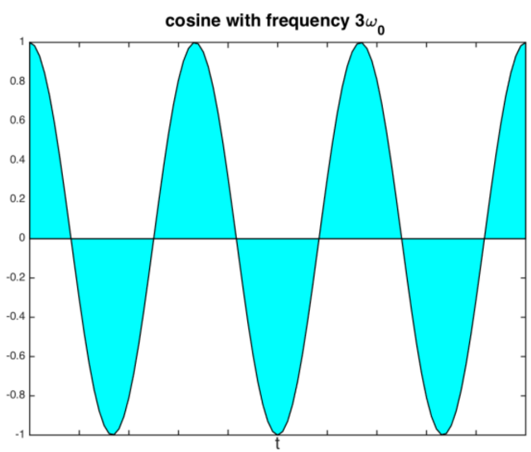
Illustration of integrating the cosine



Area=T



Area=0
("negative area" below x-axis cancels "positive area")



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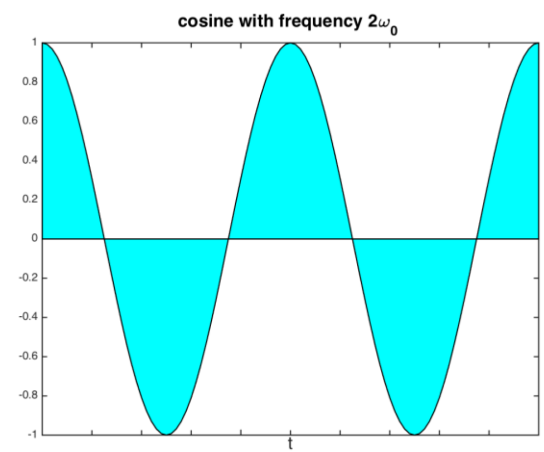


Area=0

Use orthogonality to check analysis & synthesis

- Inserting the synthesis into the analysis equations

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_0^T a_k e^{jk\omega_0 t}e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\frac{1}{T}\int_0^T e^{j(k-n)\omega_0 t}dt}_{\delta[k-n]}$$

- Note: Assumes the order of integration & sum can be exchanged which is not always the case, relates to discussion in next lecture

Summary of finding the Fourier series coefficients

- The Fourier series coefficients are computed from

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk!_0 t}dt$$

- The signal is reconstructed from its coefficients using

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

- The Fourier coefficients give insight into "how much" of the frequency $k\omega_0$ is contained in the signal

Fourier coefficients of a rectangular pulse train

Learning objectives

- Find the Fourier series coefficients of a classic example
- Use the results of this derivation in future lectures

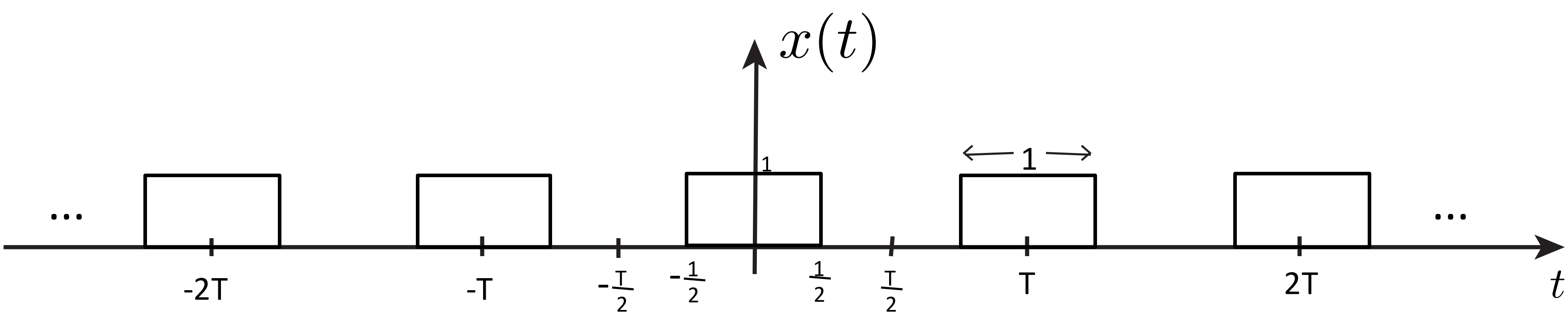
Pulse train I

This is an important reference example. It may not be covered in class. We will use the general result though in other example problems as this is an interesting and relevant signal used in circuits.

- Find the Fourier series coefficients of the unit pulse train

$$x(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \frac{1}{2} < |t| < \frac{T}{2} \end{cases}$$

and is repeated every T



Pulse train 2

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt, \quad \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-jk\omega_0 t}dt$$

$$= \frac{-1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{jk\omega_0 T}\left(e^{\frac{-jk\omega_0}{2}} - e^{\frac{jk\omega_0}{2}}\right)$$

$$= \frac{2}{k\omega_0 T}\frac{1}{2j}\left(e^{\frac{jk\omega_0}{2}} - e^{-\frac{jk\omega_0}{2}}\right)$$

Pulse train 3

$$a_k = \frac{1}{T}\frac{\sin\left(\frac{k\omega_0}{2}\right)}{\frac{k\omega_0}{2}}$$

$$= \frac{1}{T}\frac{\sin\left(\pi\frac{k\omega_0}{2\pi}\right)}{\pi\frac{k\omega_0}{2\pi}}$$

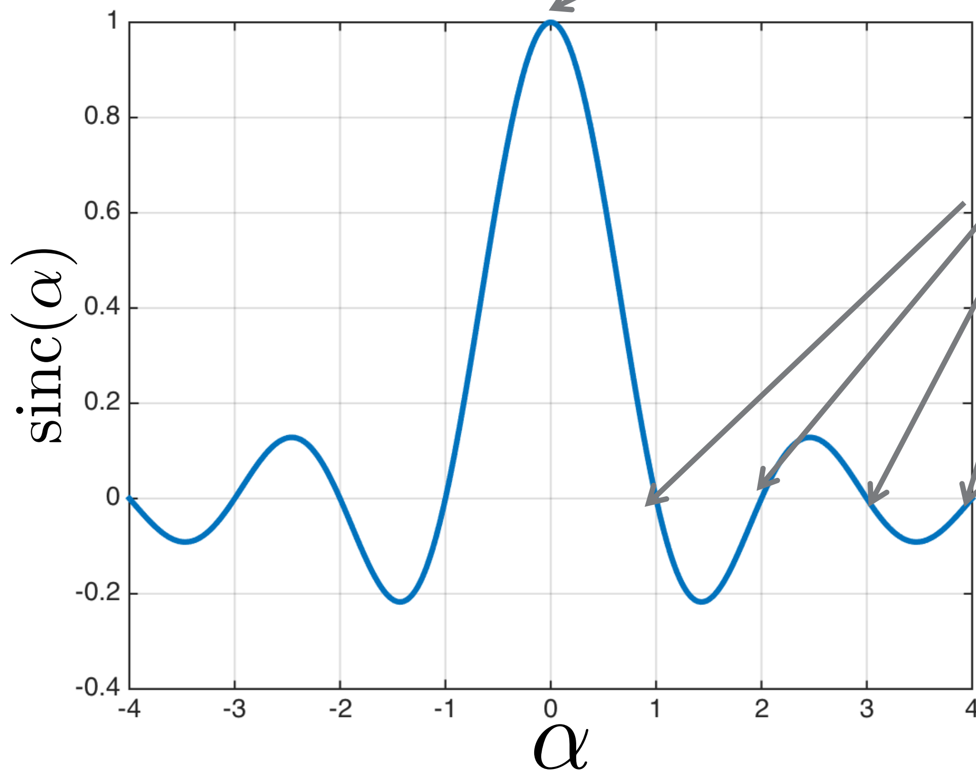
$$= \frac{1}{T}\text{sinc}\left(\frac{k\omega_0}{2\pi}\right)$$

We define the sinc as

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Pulse train 4

Maximum value of 1, i.e. $\mathrm{sinc}(0) = 1$



Zero crossings at +/-1, +/- 2, ...

$$\mathrm{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

Be aware, sometimes sinc is defined like this

$$\mathrm{sinc}(\alpha) = \frac{\sin(\alpha)}{\alpha}$$

Pulse train 5

- What about k=0?

$$a_0 = \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{x(t)}_{1}\, dt = \frac{1}{T}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{T}$$

- As an side, for the sinc function

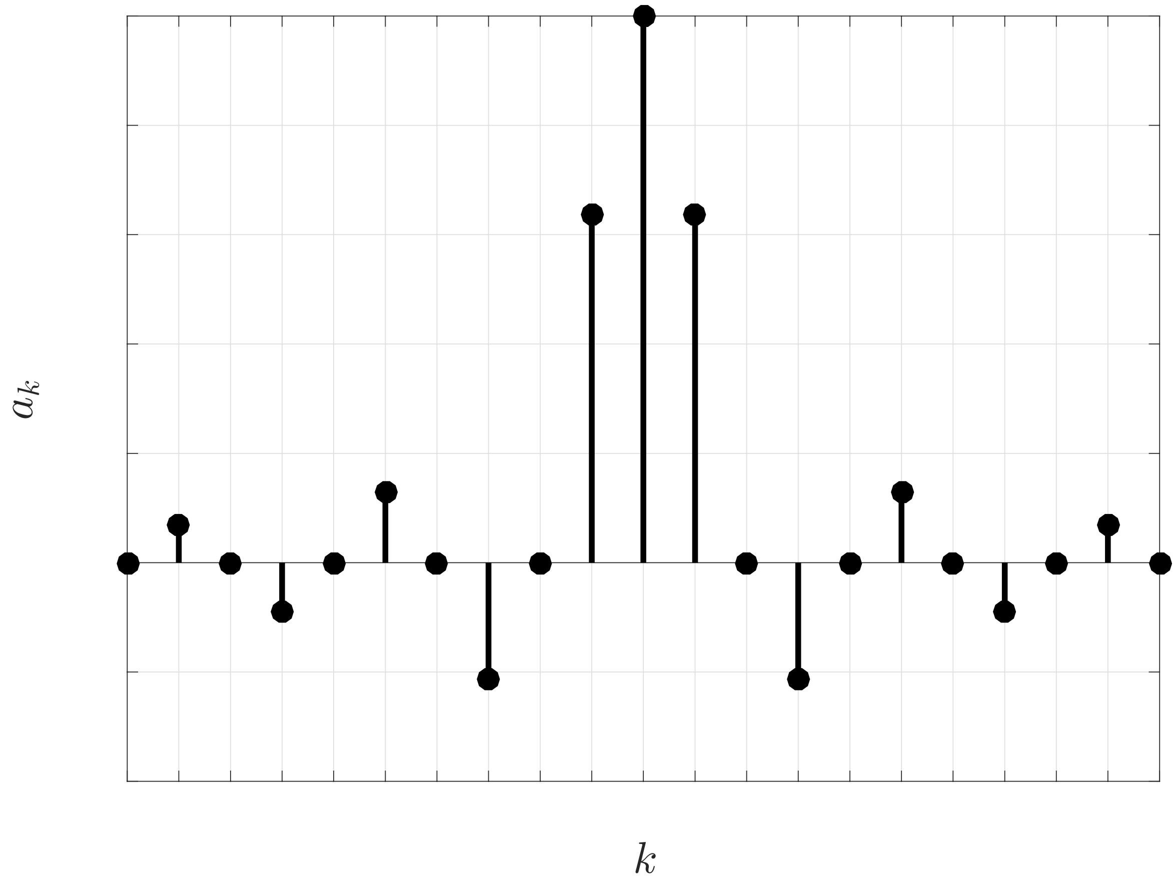
$$\lim_{t\to\infty} \frac{\sin(\pi t)}{\pi t} = \lim_{t\to\infty} \frac{\pi\cos(\pi t)}{\pi} = 1$$

- Therefore the following holds for all values of k

$$a_k = \frac{1}{T}\text{sinc}\left(\frac{k\omega_0}{2\pi}\right) = \frac{\omega_0}{2\pi}\text{sinc}\left(\frac{k\omega_0}{2\pi}\right)$$

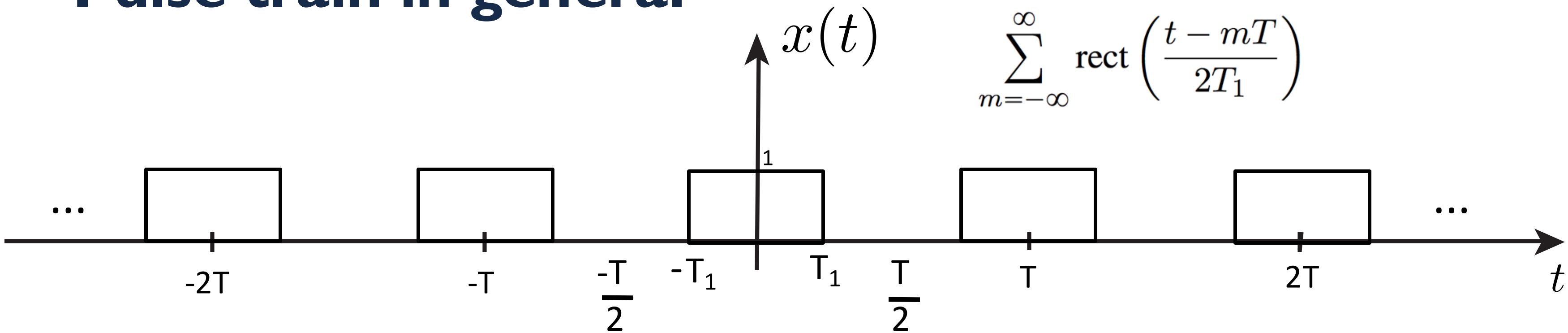
Pulse train 6

Example with T=2



Pulse train in general

From O&W Example 3.5



$$\sum_{m=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-mT}{2T_1}\right)$$

From the book

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \qquad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

$$a_k = \frac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

with fundamental frequency

$$a_k = \frac{2T_1}{T}\mathrm{sinc}\left(\frac{k2T_1}{T}\right)$$

simplified

Sufficient conditions for a periodic signal to have a Fourier series representation

Learning objectives

- Understand the Gibbs phenomena
- Determine whether a periodic signal satisfy Dirichlet conditions

The issues in a nutshell

- If we approximate the Fourier Series with a finite number of terms, is that a good approximation of the original signals?

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \xrightarrow{?} x(t)$$

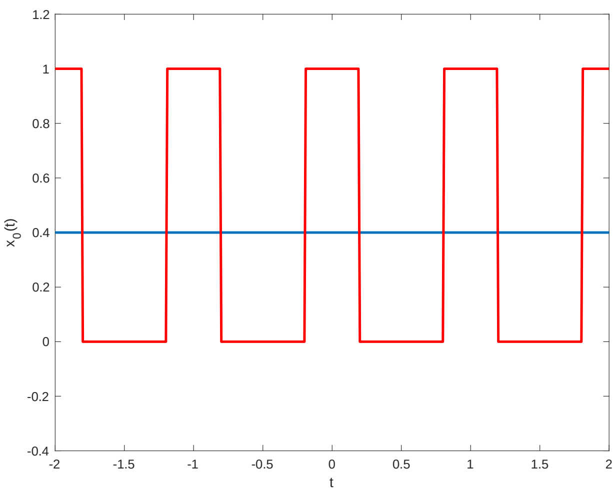
- What does it mean for a signal to have a Fourier Series representation?

Synthesizing a square wave

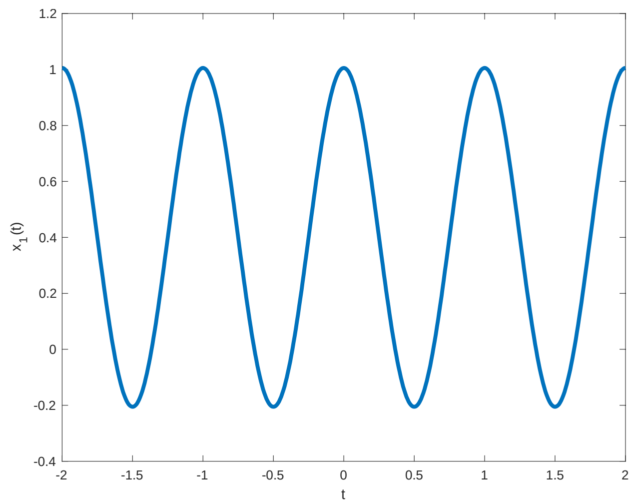
$$x(t) = \frac{T_1}{T} + 2\sum_{k=0}^{\infty} \frac{\sin(k\omega_0 T_1/2)}{k\pi} \cos(k\omega_0 t)$$

Suppose we synthesize a square wave (T_1=0.4T) with a finite number of terms

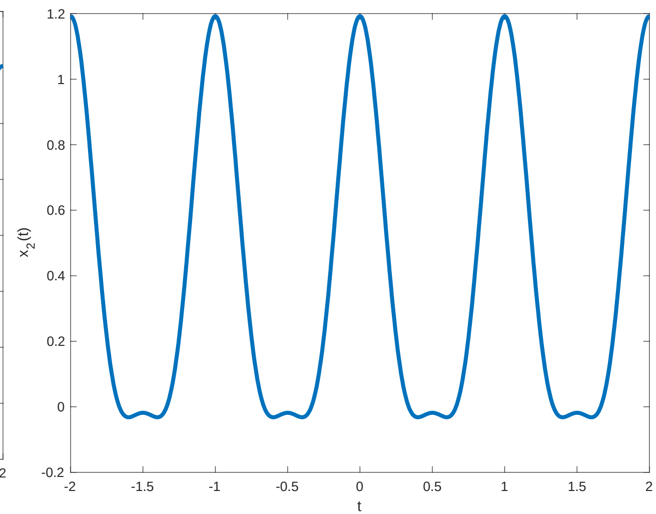
$$\boxed{x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}}$$



DC



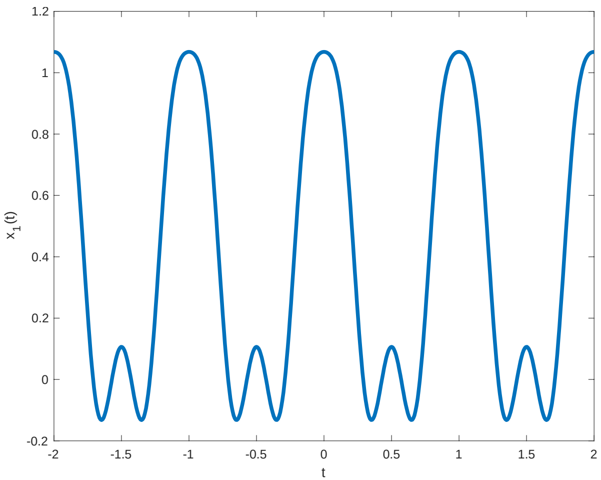
N=1 (k is -1,0,1)



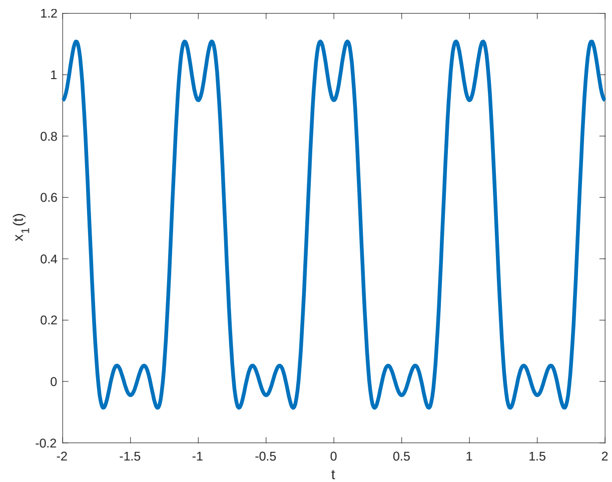
N=2 (k is -2,…,2)

Adding more terms

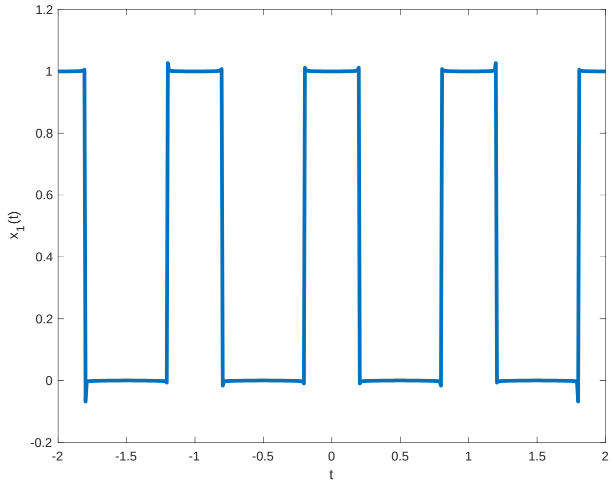
$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$



N= 3 (k is -3,…,3)

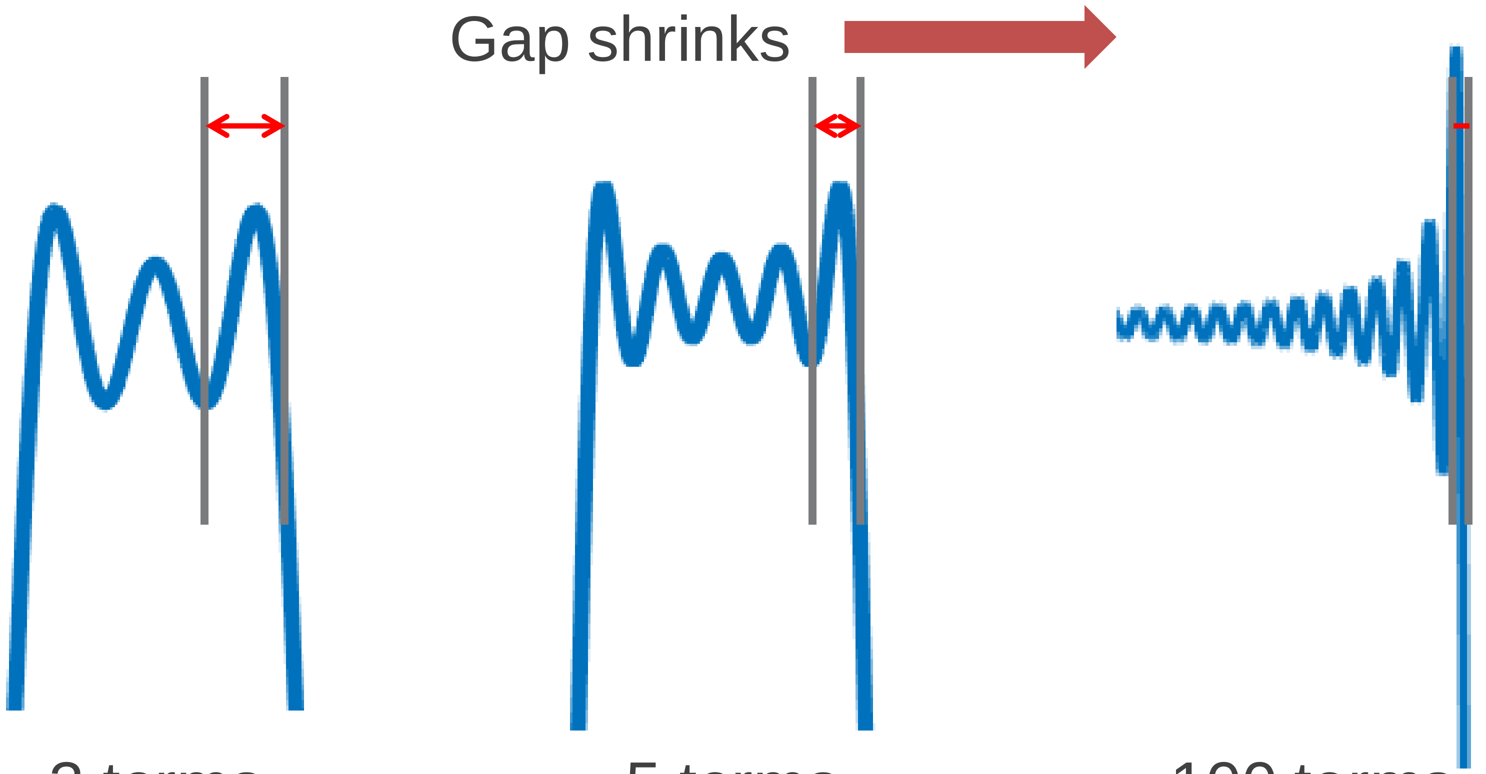


N=4 (k is -4,…,4)



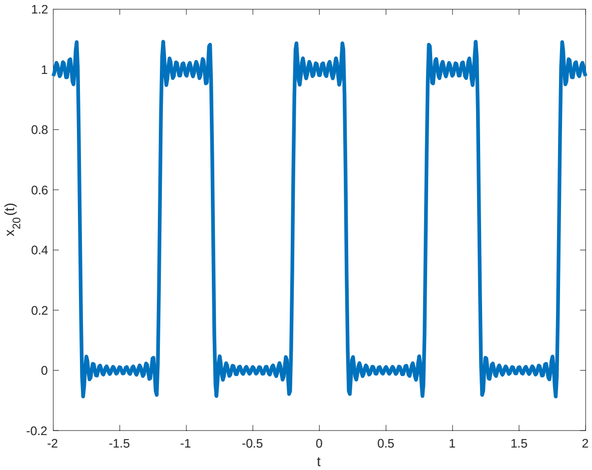
N=1000
(k is -1000,…,1000)

Zooming in on the ripple

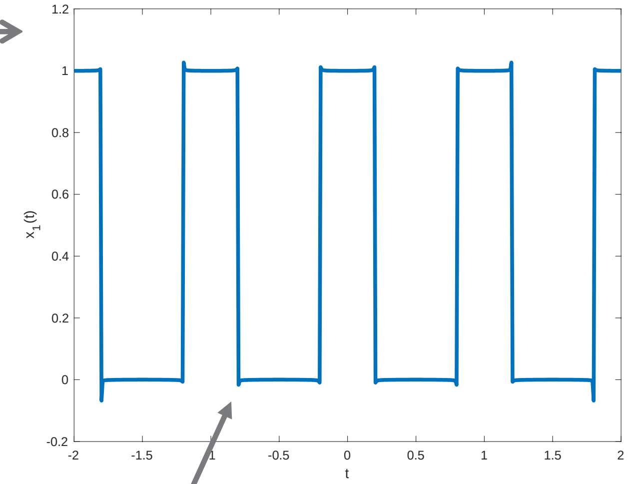


Gap goes to zero for very large number of terms

Gibbs phenomena



Ripple becomes smaller



The overshoot at the point of discontinuity is known as Gibbs phenomena

Sufficient condition to have a Fourier series

A periodic signal x(t) that satisfies the Dirichlet ("Diri-klay") conditions

(1) Absolute integrability

(2) Finite number of minima and maxima for a given time period

(3) Finite number of discontinuities for a period T

has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where equality holds for all t except possibly at the points of discontinuity

Gibbs phenomena occurs at those points

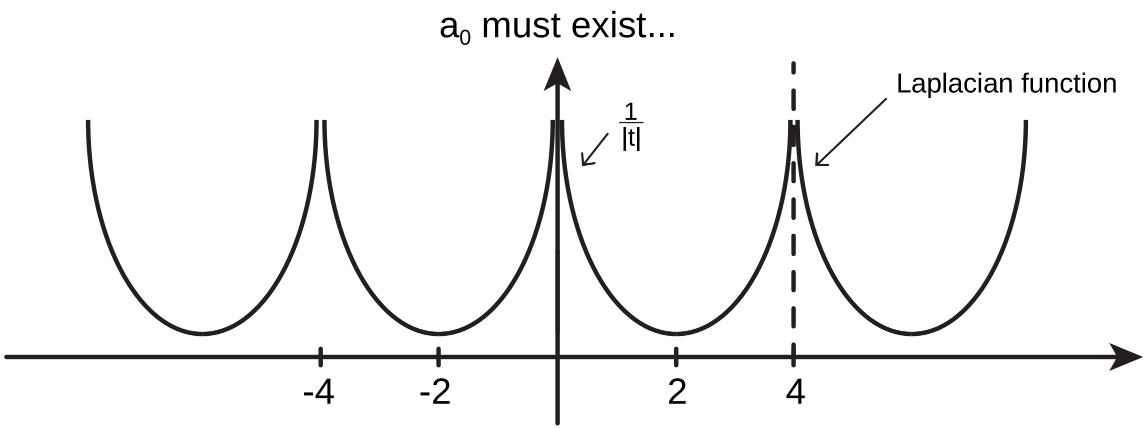
http://www.dictionary.com/browse/dirichlet

#1: Absolute integrability

$$\boxed{\int_T |x(t)|dt < \infty}$$

- Example of violation

$$x(t) = \begin{cases} \frac{1}{|t|}, & t \in (-2, 2) \\ \text{repeat for all T} \end{cases}$$

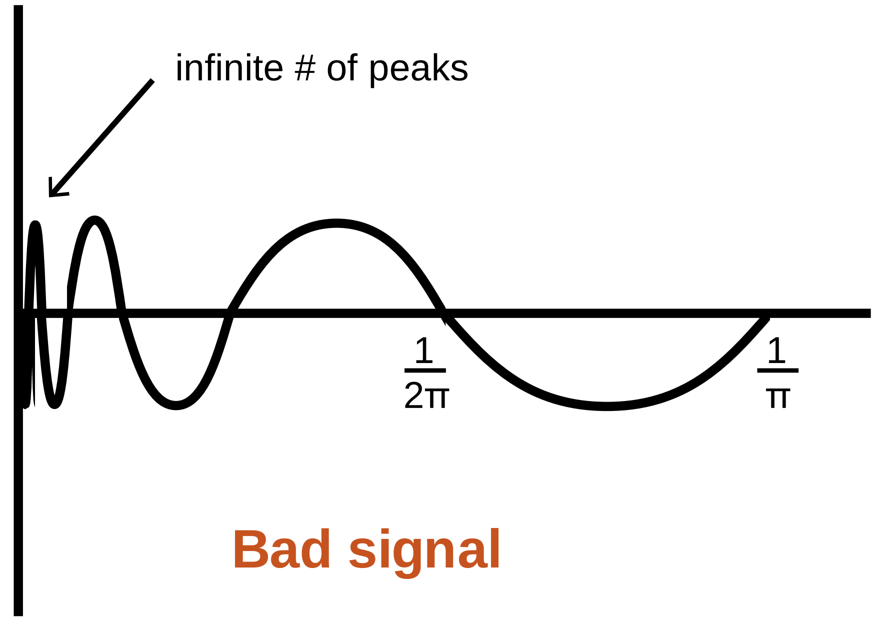


Bad signal

#2: Finite number of min and max for a given period

- Example of violation (just one period shown)

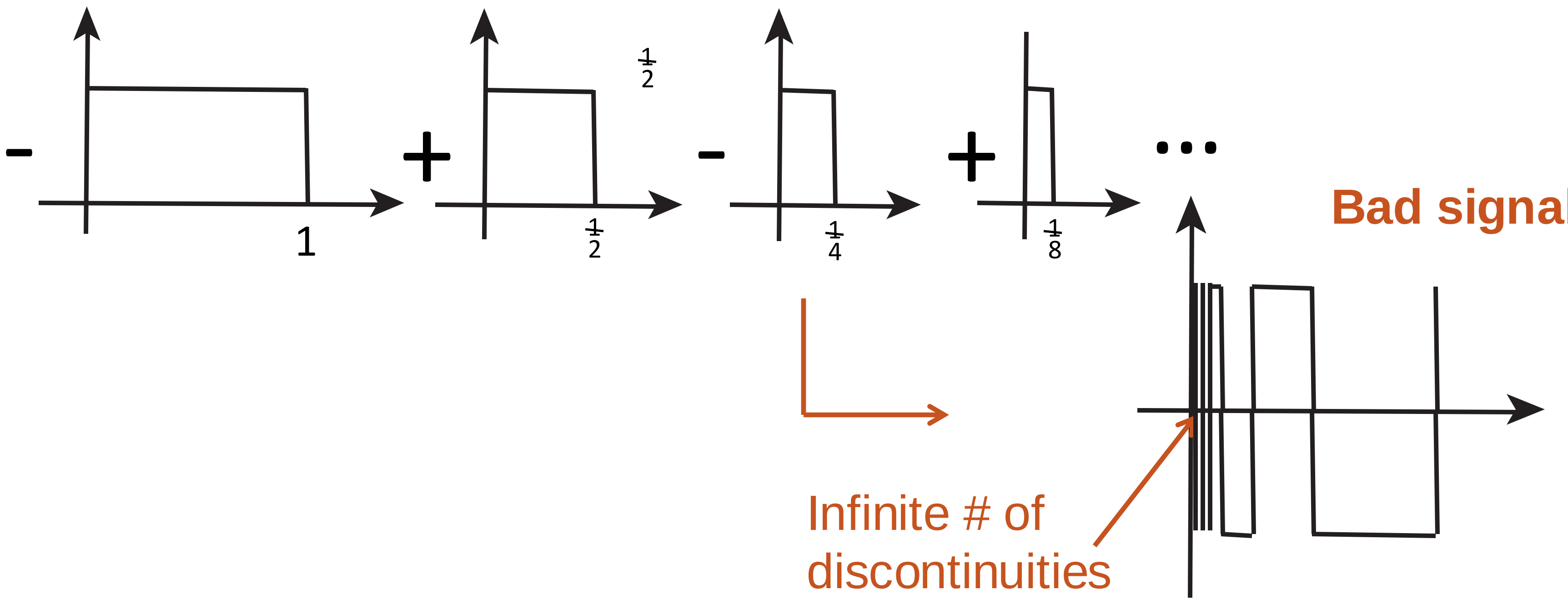
$$x(t) = \begin{cases} \sin\left(\frac{1}{t}\right), & t \in \left(0, \frac{1}{\square}\right) \\ \text{repeat every } \frac{1}{\square} \text{ secs} \end{cases}$$



#3: Finite number of discontinuities for a period T

- Example of violating signal (shown and defined over one period)

$$x(t) = \sum_{k=0}^{\infty} \left[u\left(t - \frac{1}{2^k}\right) - u(t) \right] (-1)^k$$



Summary of Fourier series conditions

- Not every periodic signal has a Fourier series representation but…
- A large class of signals do have such representations if they satisfy the Dirichlet conditions
  - Fortunately this includes all practical signals (e.g. can not create signals with an infinite number of points of discontinuity)
- The Fourier series expansion of a signal may have a few points where equality is not satisfied, known as Gibbs phenomena

Working with the Fourier series

Learning objectives

- Compute the output of an LTI system to a periodic input
- Use the Fourier series coefficients of these signals in other problems

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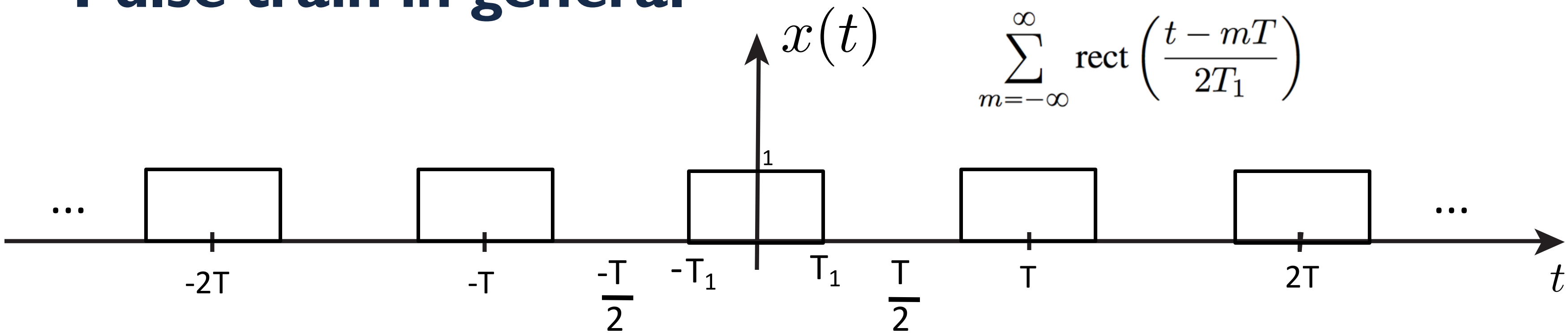
Basic signals

	Time domain $x(t)$	Fourier coefficients a_k
Constant (periodic for any T)	c	$c\delta[k]$
Cosine	$\cos(\omega_0 t)$	$\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$
Sine	$\sin(\omega_0 t)$	$\frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]$
Impulse train	$\sum_{m=-\infty}^{\infty} \delta(t-mT)$	$\frac{1}{T}$

Pulse train in general

From O&W Example 3.5

$$\sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{t-mT}{2T_1}\right)$$



From the book

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \qquad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

$$a_k = \frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

with fundamental frequency

$$a_k = \frac{2T_1}{T} \text{sinc}\left(\frac{k2T_1}{T}\right)$$

simplified

Output of an LTI system

$$\int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}$$

$$e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \overbrace{H(s)}e^{st}$$

$$\alpha e^{s_1 t} + \beta e^{s_2 t} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \alpha H(s_1)e^{s_1 t} + \beta H(s_2)e^{s_2 t}$$

New FS coefficients

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 k t}$$

Fourier series coefficients are modified by the frequency response of the system

Properties of Fourier series

Learning objectives

- Use Fourier series properties to simplify calculation & build intuition
- Analyze problems that include FS properties

Properties of the Fourier series

- The following notation is used to denote a signal and its FS coefficients

$$x(t) \xleftrightarrow{FS} a_k$$

- Properties are used to figure out how transformations of the input signal lead to transformations of the FS coefficients, helps to avoid direct computation!

Fourier series properties

- Let $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) \overset{FS}{\longleftrightarrow} a_k \qquad\qquad y(t) \overset{FS}{\longleftrightarrow} b_k$$

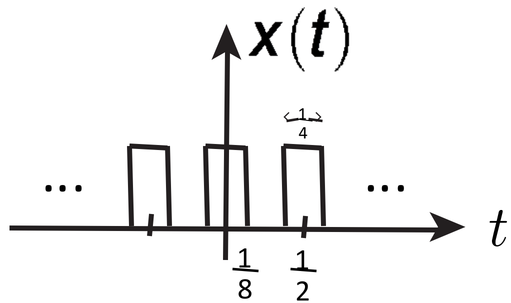
	Time domain	Frequency domain
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shift	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$
Time reversal	$x(-t)$	a_{-k}

Fourier series properties (continued)

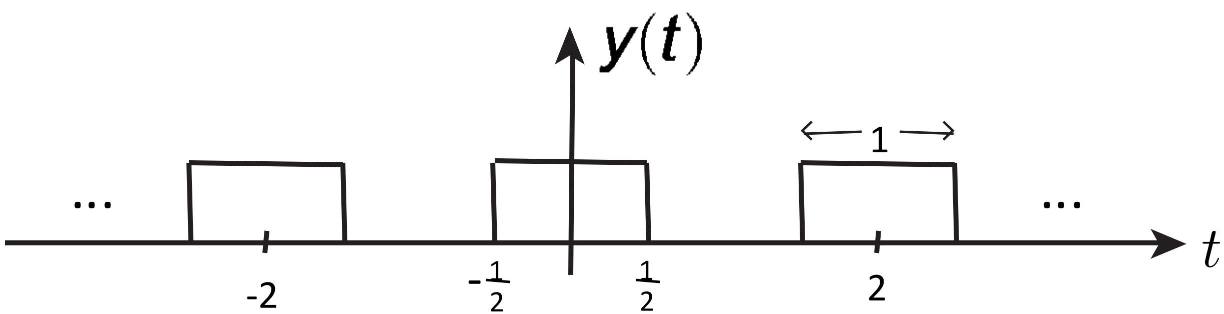
	Time domain	Frequency domain
Time scaling	$x(\alpha t)$	a_k $\quad T_{\text{new}} = \frac{T}{\alpha}$ period changes
Conjugate	$x^*(t)$	a^*_{-k}
Multiplication	$x(t)y(t)$	$\sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$
Derivative	$\frac{d}{dt}x(t)$	$a_k(jk\omega_0)$
Parseval's Theorem	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	

Visualizing time scaling

- Example $y(t) = x(\alpha t)$ $\alpha = \dfrac{1}{4}$



Stretched signal has same structure



Reinforcing the time-scaling property

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$
- If $y(t) = x(\alpha t)$, $\alpha > 0$

$$\boxed{\begin{array}{l} \alpha < 1 \rightarrow \text{stretching} \\ \alpha > 1 \rightarrow \text{compression} \end{array}}$$

- Then $y(t) = x(\alpha t)$ is periodic with period $T_{\text{new}} = \frac{T_{\text{old}}}{\alpha}$

$$x(\alpha t) \overset{FS}{\longleftrightarrow} a_k$$

Scale in time does not change the FS coefficients

Example – Making use of the table

- Let $x(t)$ be a periodic signal with a fundamental period T, and FS coefficients a_k . Derive the FS coefficients of the following signal

$$x(t-t_0)+x(t+t_0)$$

- Solution

$$x(t) \xleftrightarrow{FS} a_k$$

$$x(t-t_0) \xleftrightarrow{FS} a_k e^{-jk\omega_0 t_0}$$

$$x(t+t_0) \xleftrightarrow{FS} a_k e^{jk\omega_0 t_0}$$

$$x(t-t_0)+x(t+t_0) \xleftrightarrow{FS} a_k e^{-jk\omega_0 t_0} + a_k e^{jk\omega_0 t_0}$$

$$= 2\cos(k\omega_0 t_0) a_k$$

Example – An implication of time scaling

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$
- If $x(t)$ is even then $x(t) = x(-t)$ and it follows that

$$a_k = a_{-k}$$

- If $x(t)$ is odd then $x(-t) = -x(t)$ and it follows that

$$a_k = -a_{-k}$$

Symmetry in the signal leads to structure in FS coefficients

Example - Using Parseval's theorem

- Consider the signal $x(t) = \cos(\omega_0 t)$

$$= \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

- The FS coefficients: $a_0 = 0, \quad a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0 \text{ else}$
  - Find the power using Parseval's theorem

$$\frac{1}{T}\int_T |\cos(\omega_0 t)|^2 dt = \sum |a_k|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

  - Find the power directly in the time domain

$$\cos^2 \omega_0 t = \frac{1}{2}(1 + \cos 2\omega_0 t) \qquad \frac{1}{T}\int_0^T \frac{1}{2} dt + \frac{1}{T}\int_T \cos 2\omega_0 t\, dt = \frac{1}{2}$$

Summary of Fourier series properties

- Fourier series properties relate transformations of signals in the time domain and transformations of Fourier series coefficients
- Understanding the properties is valuable for developing intuition on how signals behave in the time and frequency domains
- Exploiting the properties has the practical advantage of avoiding tedious Fourier Series or inverse Fourier Series calculations
- While you can refer to the table for solving homework and exam problems, you must internalize the properties in your brain to use in the real world

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Reference material on properties

Learning objectives

- Prove the relations given in the property table

Property #1: Linearity

- If $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

$$z(t) = Ax(t) + By(t)$$

- Then $$z(t) \overset{FS}{\longleftrightarrow} Aa_k + Bb_k$$

FS of a sum of signals is the sum of their FS coefficients

Property #2: Time shifting

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$
- If $y(t) = x(t - t_0)$, $y(t)$ is periodic with the same period

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

- Then $b_k = a_k e^{-jk\omega_0 t_0}$

Note $|b_k| = |a_k|$ since $|e^{jk}| = 1$

Shift in time results in a phase shift in frequency

Example 1

- Let $x(t)$ be a periodic signal with a fundamental period T, and FS coefficients a_k . Derive the FS coefficients of the following signal

$$x(t - t_0) + x(t + t_0)$$

- Solution

$$x(t) \stackrel{FS}{\longleftrightarrow} a_k$$

$$x(t - t_0) \stackrel{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0}$$

$$x(t + t_0) \stackrel{FS}{\longleftrightarrow} a_k e^{jk\omega_0 t_0}$$

$$x(t - t_0) + x(t + t_0) \stackrel{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0} + a_k e^{jk\omega_0 t_0}$$

$$= 2\cos(k\omega_0 t_0) a_k$$

Property #3: Time reversal

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \overset{FS}{\longleftrightarrow} a_k$
- Then $y(t) = x(-t)$, $y(t)$ is periodic with the same period
- and

$$y(t) \overset{FS}{\longleftrightarrow} a_{-k}$$

Reverse in time results in reverse in frequency

Time reversal proof

- Suppose that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Then

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$$

- Changing variables

$$y(t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t} \quad \Rightarrow \quad y(t) \overset{FS}{\longleftrightarrow} a_{-k}$$

Implications of time reversal on even and odd

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \stackrel{FS}{\longleftrightarrow} a_k$

- If $x(t)$ is even then $x(t) = x(-t)$ and it follows that

$$a_k = a_{-k}$$

- If $x(t)$ is odd then $x(-t) = -x(t)$ and it follows that

$$a_k = -a_{-k}$$

Symmetry in the signal leads to structure in FS coefficients

Property #4: Time scaling

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \xleftrightarrow{FS} a_k$

- If $y(t) = x(\alpha t)$, $\alpha > 0$

$$\alpha < 1 \rightarrow \text{stretching}$$
$$\alpha > 1 \rightarrow \text{compression}$$

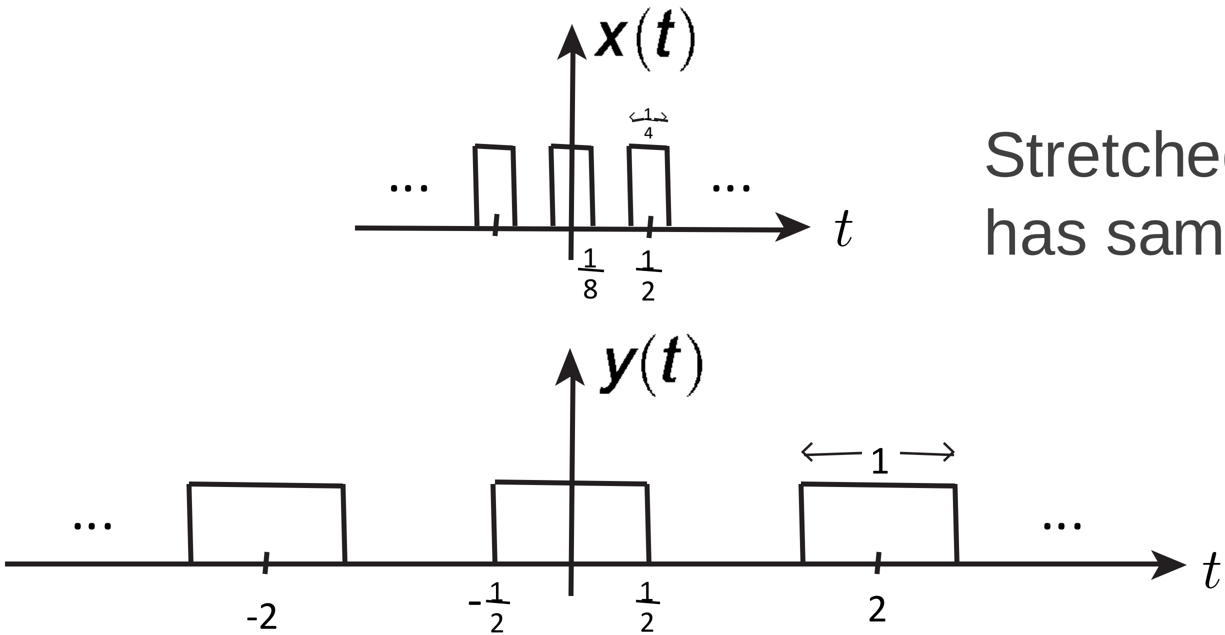
- Then $y(t) = x(\alpha t)$ is periodic with period $T_{\text{new}} = \frac{T_{\text{old}}}{\alpha}$

$$x(\alpha t) \xleftrightarrow{FS} a_k$$

Scale in time does not change the FS coefficients

Visualizing time scaling

- Example $y(t) = x(\alpha t)$ $\alpha = \dfrac{1}{4}$



Stretched signal has same structure

Time scaling proof

- Since $x(t) \overset{FS}{\longleftrightarrow} a_k$ it follows that $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

- Then

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0) t}$$

$$x(\alpha t) \overset{FS}{\longleftrightarrow} a_k$$

Property #5: Multiplication

- If $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

- Then for $z(t) = x(t)y(t)$

Product in time leads to convolution in frequency

$$z(t) = x(t)y(t) \overset{FS}{\longleftrightarrow} h_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

Property #6: Conjugation and symmetry

- If $x(t)$ is periodic with period $T = \frac{2\pi}{\omega_0}$ and $x(t) \xleftrightarrow{FS} a_k$
- Then $x^*(t) \xleftrightarrow{FS} a^*_{-k}$
- Impliciations
  - If $x(t)$ is real, then the FS coefficients are conjugate symmetric

$$a^*_{-k} = a_k$$

  - If $x(t)$ is real and even, then the FS coefficients are real and even

$$a_k = a^*_k$$

  - If $x(t)$ is real and odd, then the FS coefficients are imaginary and odd

Property #7: Parseval's theorem

- Consider a periodic signals with FS representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- The power in the signal is

$$\boxed{\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$

Power is the same whether in the time or frequency domain

Proof of Parseval's theorem

$$\frac{1}{T}\int |x(t)|^2 dt = \frac{1}{T}\int_T \left|\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right|^2 dt$$

for a complex number

$$|x|^2 = xx^*$$

$$= \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_k e^{jk\omega_0 t} a_\ell^* e^{-j\ell\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \frac{1}{T}\int_T a_k e^{jk\omega_0 t} a_\ell^* e^{-j\ell\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \frac{1}{T}\int_T e^{j(k-\ell)\omega_0 t} dt$$

Use orthogonality property

$\Rightarrow \delta[k-\ell]$

Proof of Parseval's theorem (cont.)

$$\frac{1}{T}\int |x(t)|^2 dt = \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \delta[k-\ell]$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

Orthogonality is key to the proof

Parseval's theorem – Example

- Consider the signal $x(t) = \cos(\omega_0 t)$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

- The FS coefficients: $a_0 = 0, \quad a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0 \text{ else}$
  - Find the power using Parseval's theorem

$$\frac{1}{T}\int_T |\cos(\omega_0 t)|^2 dt = \sum |a_k|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

  - Find the power directly in the time domain

$$\cos^2 \omega_0 t = \frac{1}{2}(1 + \cos 2\omega_0 t) \qquad \frac{1}{T}\int_0^T \frac{1}{2} dt + \frac{1}{T}\int_T \cos 2\omega_0 t\, dt = \frac{1}{2}$$

Property #8: Derivative

- Consider a periodic signal $x(t)$ with $T = \frac{2\pi}{\omega_0}$ and

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

- Then

$$\frac{dx(t)}{dt} \leftrightarrow a_k(jk\omega_0)$$

Each FS coefficient scaled as a function of the frequency

Proof of the derivative property

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\frac{d}{dt}x(t) = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k (jk\omega_0) e^{jk\omega_0 t}$$

Application of Fourier series properties

Learning objectives

- Use the Fourier series properties to infer information about signals

Several important examples are provided here, though note that they may not be covered in the lecture

Application Example 1

(from O&W 3.25)

- Consider the following three CT signals with a fundamental period of T=1/2

$$x(t) = \cos(4\pi t)$$

$$y(t) = \sin(4\pi t)$$

$$z(t) = x(t)y(t)$$

- Determine the FS coefficients of the three signals

The purpose of this example is to infer the FS coefficients from the fact that these are all sinusoidal signals. Using this fact means that integration will not be required to find the coefficients.

Application Example I

$$x(t) = \cos(4\pi t)$$
$$= \underbrace{\frac{1}{2}}_{a_1} e^{j4\pi t} + \underbrace{\frac{1}{2}}_{a_{-1}} e^{-j4\pi t}$$

$$y(t) = \sin(4\pi t)$$
$$= \underbrace{\frac{1}{2j}}_{b_1} e^{j4\pi t} + \underbrace{\frac{-1}{2j}}_{b_{-1}} e^{-j4\pi t}$$

Application Example I

Alternative derivation, exploit

$$\sin x \cdot \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$$

$$x(t)y(t) = \cos(4\pi t)\sin(4\pi t)$$

$$\stackrel{FS}{\longleftrightarrow} c_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

$$= \sum_{\ell=-\infty}^{\infty} \left(\frac{1}{4j}\delta[\ell-1]\delta[k-\ell-1] - \frac{1}{4j}\delta[\ell-1]\delta[k-\ell+1] \right.$$

$$\left. + \frac{1}{4j}\delta[\ell+1]\delta[k-\ell-1] - \frac{1}{4j}\delta[\ell+1]\delta[k-\ell+1] \right)$$

$$= \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k] + \frac{1}{4j}\delta[k] - \frac{1}{4j}\delta[k+2]$$

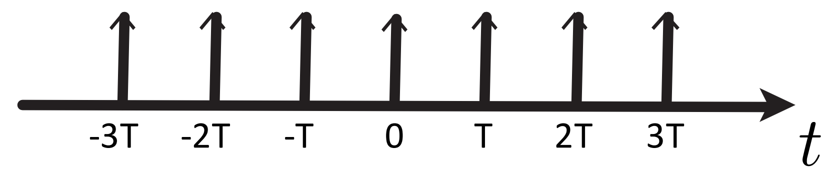
$$= \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2]$$

Application Example 2

This impulse train signal shows up later in the course in sampling. This kind of signal can be used to build waveforms that are good for radar for example.

◆ Consider the impulse train signal

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \text{period } T$$

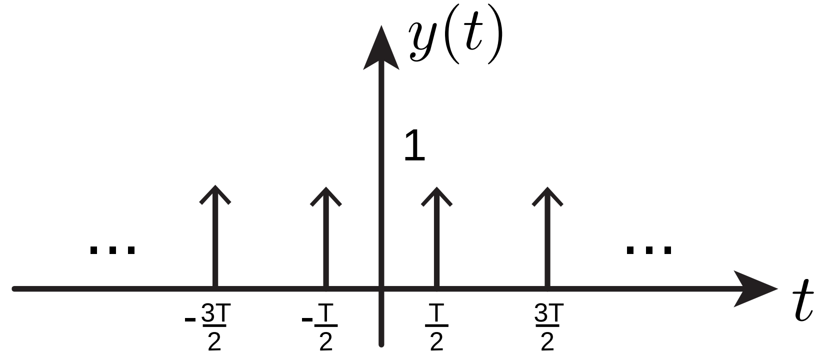


◆ Calculate the FS coefficients

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\frac{2\pi}{T}t}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{0}dt \\ &= \frac{1}{T} \quad \forall k \end{aligned}$$

Application Example 3

- Consider the pulse train



t

- Calculate the FS coefficients
- Because $y(t) = x\left(t - \dfrac{T}{2}\right)$

$$
\begin{aligned}
b_k &= a_k e^{-jk\omega_0 t_0} \\
&= a_k e^{-jk\frac{2\pi}{T}\frac{T}{2}} \\
&= a_k e^{-jk\pi} \\
&= \frac{1}{T}\cos k\pi \\
&= \frac{(-1)^k}{T}
\end{aligned}
$$

Application Example 4

- Let $x(t)$ be a periodic signal with a fundamental period T, and FS coefficients a_k . Derive the FS coefficients of the following signal

$$\frac{d^2x(t)}{dt^2}$$

Application Example 4

- From the definition of the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}$$

- Differentiating

$$\frac{d^2x(t)}{dt^2} = \frac{d^2}{dt^2} \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}$$

$$= \sum_{k=-\infty}^{\infty} \frac{d^2}{dt^2} a_k e^{j(2\pi/T)kt}$$

$$= \sum_{k=-\infty}^{\infty} \boxed{-k^2 \frac{4\pi^2}{T^2} a_k} e^{j(2\pi/T)kt}$$

Application Example 5

(from O&W 3.23 (a))

- Consider the FS coefficients of a CT signal that is periodic with period 4. Determine the signal $x(t)$

$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \dfrac{\sin(k\pi/4)}{k\pi}, & \text{otherwise} \end{cases}$$

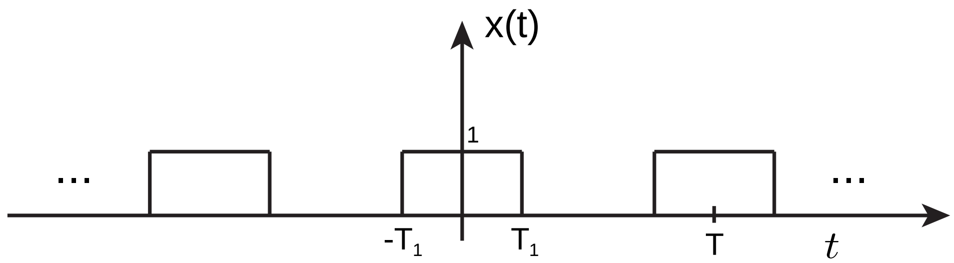
- Approach
  - Start with a known FS
  - Make transformations to reach the required signal

This is a detailed example that involves working backwards from the FS coefficients to find the underlying signal. It requires look at the expression and thinking about the properties differently than other problems as we are doing some detective work here.

Application Example 5 (continued)

- Use the known FS and the FS properties to recover signals from their FS coefficients

- Consider this function from an earlier lecture



$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi}$$

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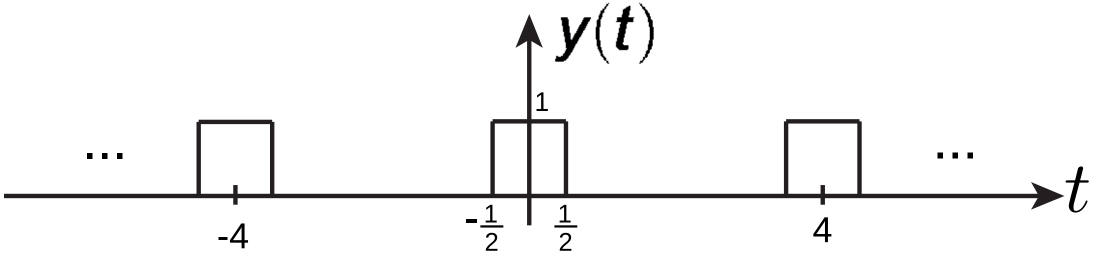
Application Example 5 (continued)

- Solution:
  - Consider the signal $y(t) \leftrightarrow b_k$ with FS coefficients $$b_k = \frac{\sin\frac{k\pi}{4}}{k\pi}$$
  - As T=4

$$b_k = \frac{\sin\frac{k\pi}{4}}{k\pi} = \frac{\sin\left(k\cdot\frac{2\pi}{T}\cdot T_1\right)}{k\pi}$$
$$= \frac{\sin\left(k\frac{\pi}{2}\cdot T_1\right)}{k\pi} \rightarrow T_1 = \frac{1}{2}$$



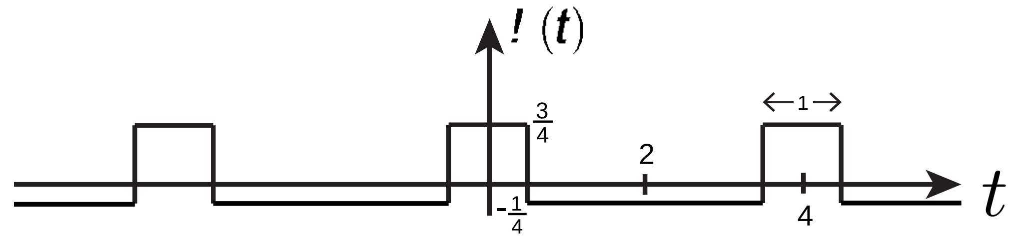
Application Example 5 (continued)

- The dc component of the signal $y(t)$ is

$$b_0 = \frac{1}{T}\int_T y(t)dt$$
$$= \frac{1}{4}\cdot 1 = \frac{1}{4}$$

- But the DC component of $y(t)$ is 0, so subtract it
  - Define the signal $w(t) \leftrightarrow c_k$ as $\omega(t) = y(t) - \frac{1}{4}$
  - Then $c_0 = 0$

$$c_k = \frac{\sin\frac{\pi k}{4}}{\pi k}$$



Application Example 5 (continued)

- Now, what is remaining is to add j^k
  - We know that $j^k = (e^{j\frac{\pi}{2}})^k = e^{j\frac{\pi}{2}k}$
  - So, now consider $x(t) = w(t - t_0)$
  - Using the FS properties

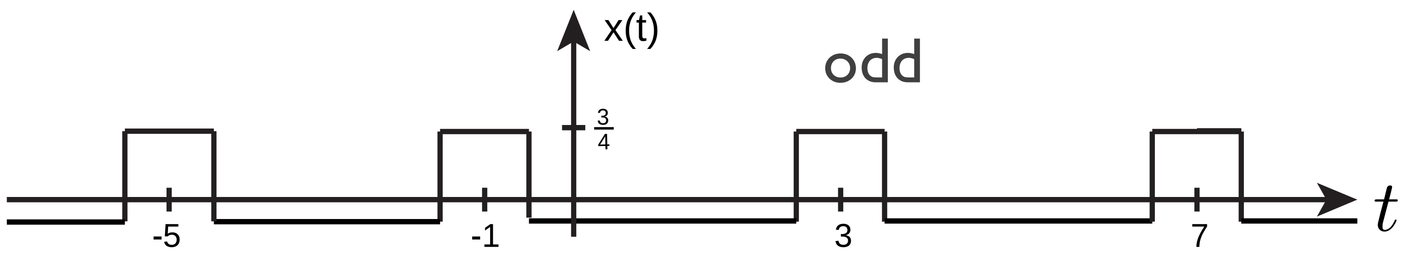
If $w(t) \leftrightarrow c_k$ then $w(t - t_0) \leftrightarrow c_k \cdot e^{j\frac{\pi}{2}k} = a_k$

$$e^{j\frac{\pi}{2}k} = e^{-jk\omega_0 t_0}, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$= e^{-jk\frac{\pi}{2}t_0} \longrightarrow t_0 = -1$$

Application Example 5 (concluded)

- Hence, $x(t) = w(t - t_0) = w(t + 1)$



$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \dfrac{\sin(k\pi/4)}{k\pi}, & \text{otherwise} \end{cases}$$

Application Example 6

(similar to O&W 3.23 (d))

- Consider the FS coefficients of a CT signal that is periodic with period 4. Determine the signal $x(t)$

$$a_k = \begin{cases} 1, & k \text{ odd} \\ 2, & k \text{ even} \end{cases}$$

- Solution: Use the fact that

$$\sum_{k=-\infty}^{\infty} \delta(t-k) \xleftrightarrow{FS} a_k = 1 \ \forall k$$

Application Example 6 (continued)

Step 1

- Consider a train of deltas with period 4

$$\sum_{k=-\infty}^{\infty} \delta(t-4k) = \sum_{k=-\infty}^{\infty} \delta(4(t/4-k))$$

- Using the scaling property of the delta function $\delta(t/a) = |a|\delta(t)$

$$\sum_{k=-\infty}^{\infty} \delta(t-4k) = \frac{1}{4}\sum_{k=-\infty}^{\infty} \delta(t/4-k)$$

- Now from the time scaling property and linearity property

$$\frac{1}{4}\sum_{k=-\infty}^{\infty} \delta(t/4-k) \xleftrightarrow{FS} b_k = \frac{1}{4} \; \forall k$$

Application Example 6 (continued)

- As a result we can conclude from

$$\sum_{k=-\infty}^{\infty} \delta(t-k) \overset{FS}{\longleftrightarrow} a_k = 1 \; \forall k$$

- That

$$\sum_{k=-\infty}^{\infty} \delta(t-4k) \overset{FS}{\longleftrightarrow} b_k = \frac{1}{4} \; \forall k$$

- And

$$4 \sum_{k=-\infty}^{\infty} \delta(t-4k) \overset{FS}{\longleftrightarrow} b_k = 1 \; \forall k$$

Application Example 6 (continued)

Step 2

- Consider a signal with

$$c_k = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

- Inserting into the synthesis equation

$$x_2(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{4}kt}$$

$$= \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{4}2kt}$$

$$= \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{2}kt}$$

But this is just the synthesis of a signal with period 2 and FS coefficients {1}

$$\Rightarrow \quad 2\sum_{k=-\infty}^{\infty} \delta(t-2k)$$

Application Example 6 (concluded)

Step 3

- Write signal with period 4 and FS coefficients

$$a_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$$

- As the sum of signals with FS coefficients

$$b_k = 1 \qquad \text{and} \qquad c_k = \begin{cases} 1 & k \ \text{even} \\ 0 & k \ \text{odd} \end{cases}$$

- Time domain signal is then

$$\boxed{x(t) = 4 \sum_{k=-\infty}^{\infty} \delta(t - 4k) + 2 \sum_{k=-\infty}^{\infty} \delta(t - 2k)}$$

Application Example 7

(from O&W 3.23 (c))

- In the following, we specify the FS coefficients of a CT signal that is periodic with period 4. Determine the signal $x(t)$

$$a_k = \begin{cases} jk & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$$

- Solution:

$$\begin{aligned} x(t) &= a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t} \\ &= j e^{j\omega_0 t} - j e^{-j\omega_0 t} + 2j e^{j2\omega_0 t} - 2j e^{-j2\omega_0 t} \\ &= -2\sin(\omega_0 t) - 4\sin(2\omega_0 t) \end{aligned}$$

Application Example 8

(from O&W 3.26)

- Let $x(t)$ be a periodic signal whose FS coefficients are

$$a_k = \begin{cases} 2 & k = 0 \\ j(1/2)^{|k|} & \text{otherwise} \end{cases}$$

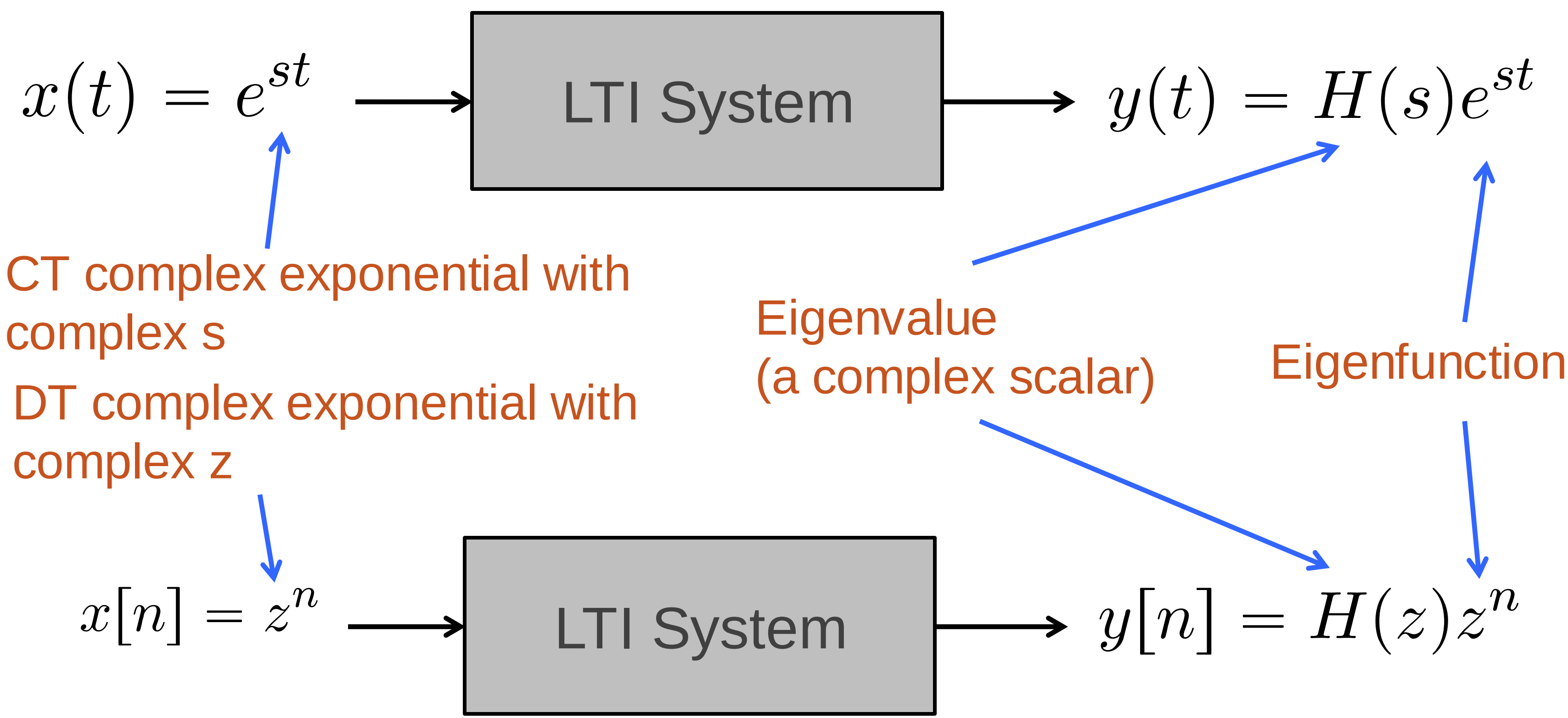
- Is $x(t)$ real?
  - Real signals must satisfy $x(t) = x^*(t)$ or $a_k = a^*_{-k}$ not satisfied here
- Is $x(t)$ even?
  - Even signals satisfy $x(t) = x(-t)$ or $a_k = a_{-k}$ yes is satisfied
- Is $\frac{dx(t)}{dt}$ even?
  - The FS coefficients of $\frac{dx(t)}{dt}$ are $(j\omega_0 k)a_k$ for which $(j\omega_0 k)a_k \neq -(j\omega_0 k)a_{-k}$

Building intuition on the frequency response

Key points

- Explain how some systems represent frequency filters
- Distinguish between different kinds of frequency filters

Recall the eigenfunction property



Convolution is easy with eigenfunctions!

Frequency response from eigenfunctions with s = j ω_0

frequency response performs filtering

$$e^{j\omega_0 t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

$$\cos(\omega_0 t) \longrightarrow \boxed{H(j\omega)} \longrightarrow |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$$

$H(j\omega_0)$ is the gain of the system at frequency ω_0

How to find the frequency response?

- If you already know the system response then

$$H(j\omega) = H(s)|_{s=j\omega}$$

- If you have the impulse response, compute the Fourier transform (more soon)

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

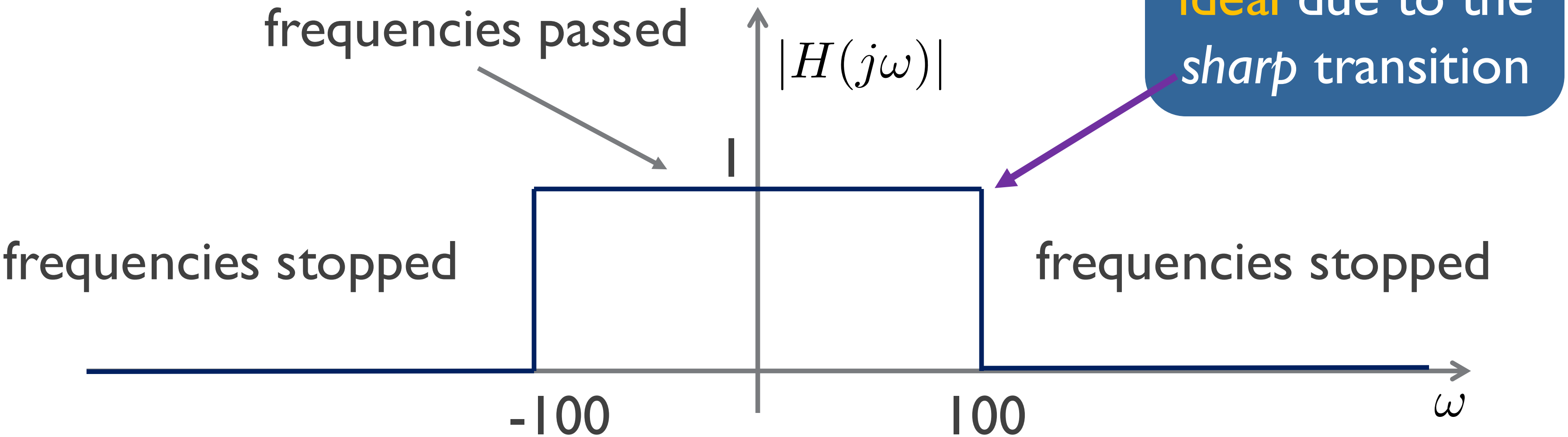
- If the system is described by a differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Rightarrow H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Example: Low-pass filtering a periodic signal I

- Consider an ideal low-pass filter whose frequency response is

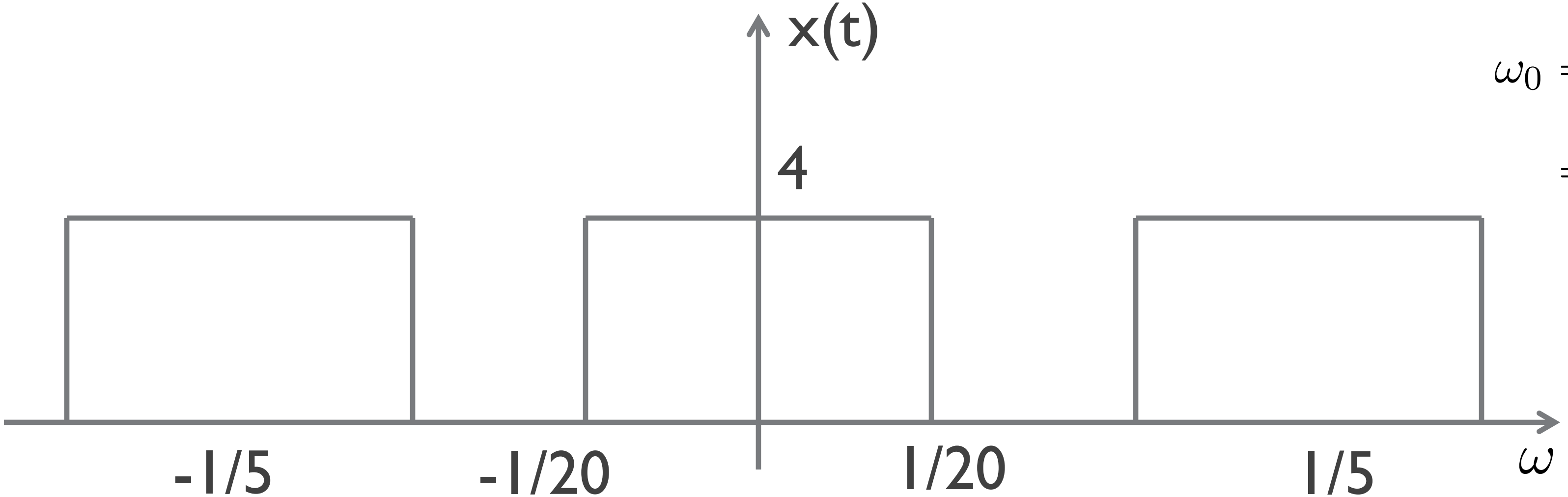
$$H(j\omega) = \begin{cases} 1, & |\omega| \leqslant 100 \\ 0, & |\omega| > 100 \end{cases}$$



Example: Low-pass filtering a periodic signal 2

$$T = \frac{1}{5}$$
$$T_1 = \frac{1}{20}$$
$$\omega_0 = \frac{2\pi}{1/5} = 10\pi$$

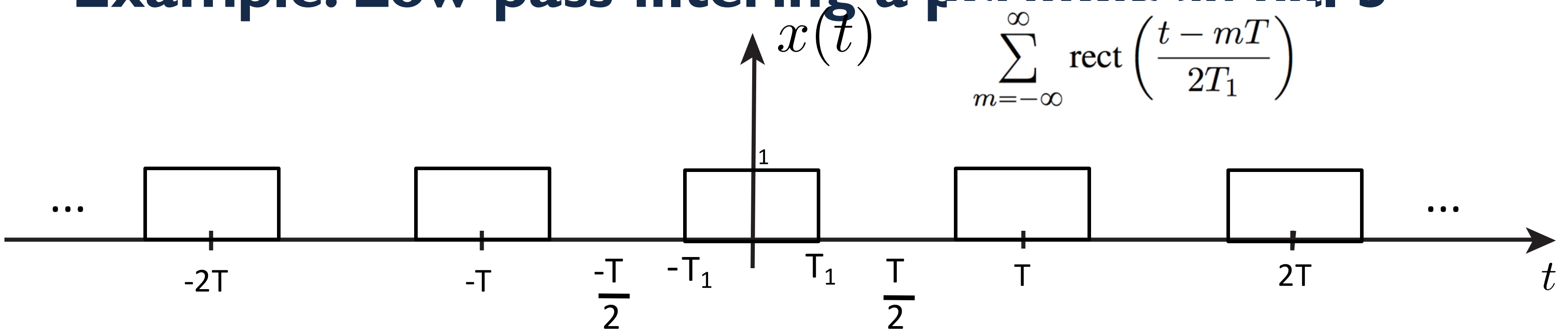
- Find the output if the input signal is



Square wave with period T=1/5

From O&W Example 3.5

Example: Low-pass filtering a periodic signal 3



$$\sum_{m=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-mT}{2T_1}\right)$$

From the book

Rewritten using the sinc function $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

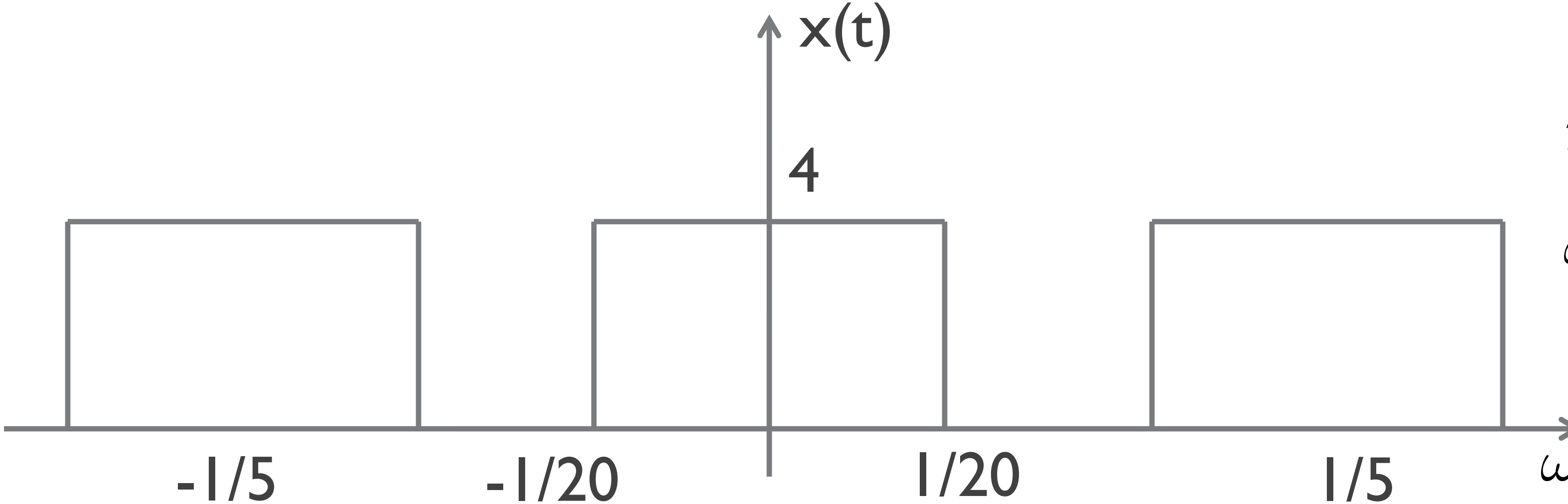
(for connecting to results in later lectures)

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \quad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

$$a_k = \frac{2T_1}{T} \mathrm{sinc}\left(\frac{k2T_1}{T}\right)$$

Example: Low-pass filtering a periodic signal 4



$$T = \frac{1}{5}$$
$$T_1 = \frac{1}{20}$$
$$\omega_0 = \frac{2\pi}{1/5} = 10\pi$$

Period $T = \frac{1}{5}$

Rectangle size $T_1 = \frac{1}{20}$

Fundamental frequency $\omega_0 = \frac{2\pi}{1/5} = 10\pi$

Fourier series coefficients

$$k \neq 0 \quad a_k = \frac{4}{\pi k} \sin\left(\pi k \tfrac{1}{2}\right)$$
$$k = 0 \quad a_0 = 2$$

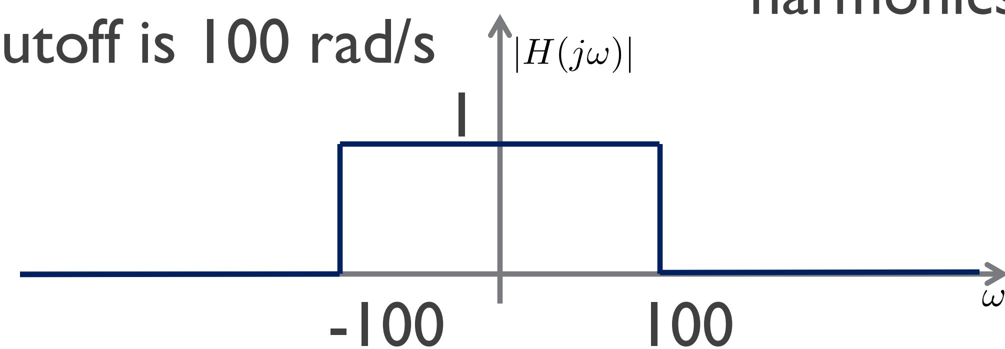
(note effect of scaling by 4)

Example: Low-pass filtering a periodic signal 5

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt}$$

harmonics are at frequencies $k\omega_0$

cutoff is 100 rad/s



$$\omega_0 = 10\pi \approx 31.4$$

$$2\omega_0 = 2 \cdot 10\pi \approx 62.8$$

$$3\omega_0 = 3 \cdot 10\pi \approx 94.2$$

$$\xcancel{4\omega_0 = 4 \cdot 10\pi \approx 125.6}$$

Fourier series coefficients are modified by the frequency response of the system

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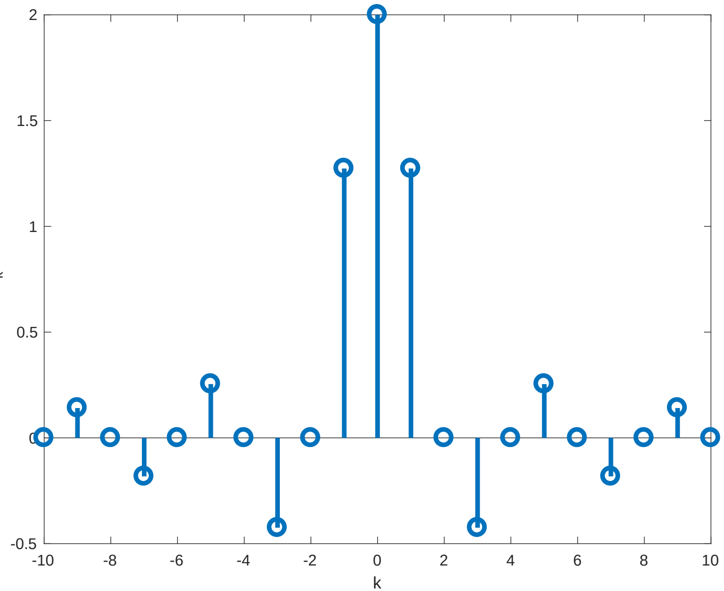
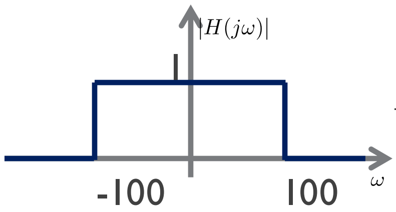
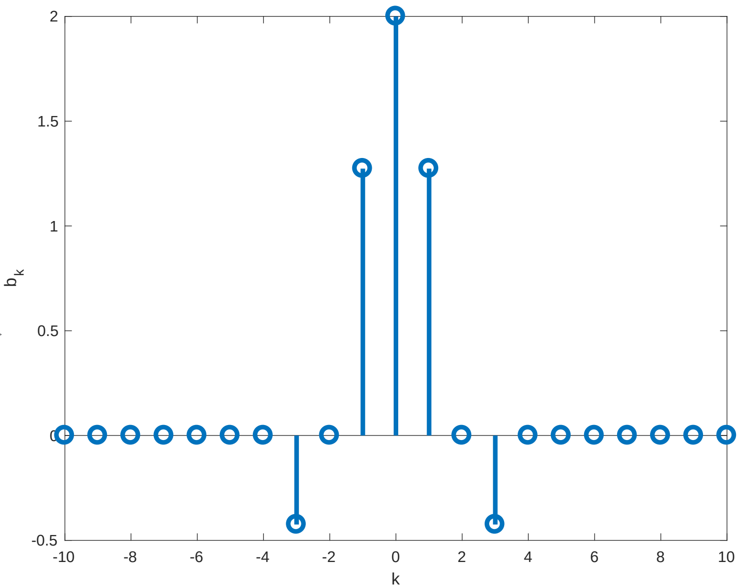
Example: Low-pass filtering a periodic signal 6



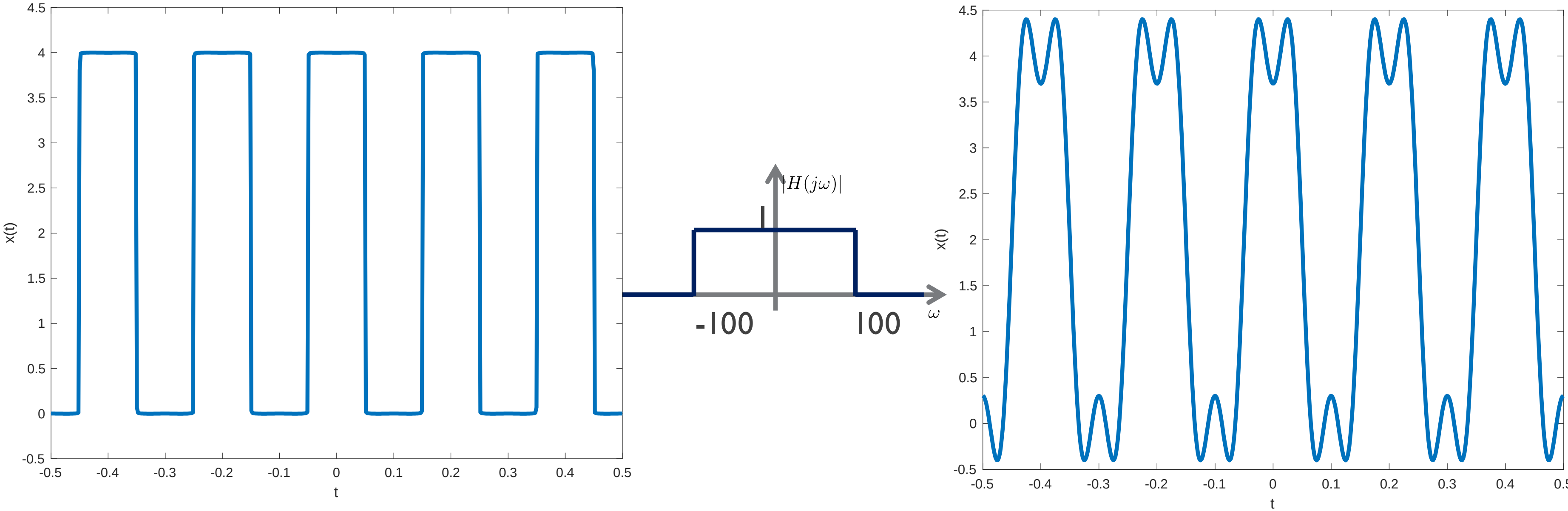




Fourier coefficients before the LTI system

Fourier coefficients after the LTI system

Example: Low-pass filtering a periodic signal 7



Time domain signal before the LTI system

Time domain signal after the LTI system

Frequency response summary

- If the input to an LTI system is periodic, then the output is also periodic with the same period

- LTI systems impact the amplitude and phase of the Fourier series coefficients as determined by the frequency response of the system

- To determine the effect of an LTI system on a periodic signal, compute the Fourier transform of the impulse response and evaluate it at multiples of the fundamental frequency $k\omega_0$

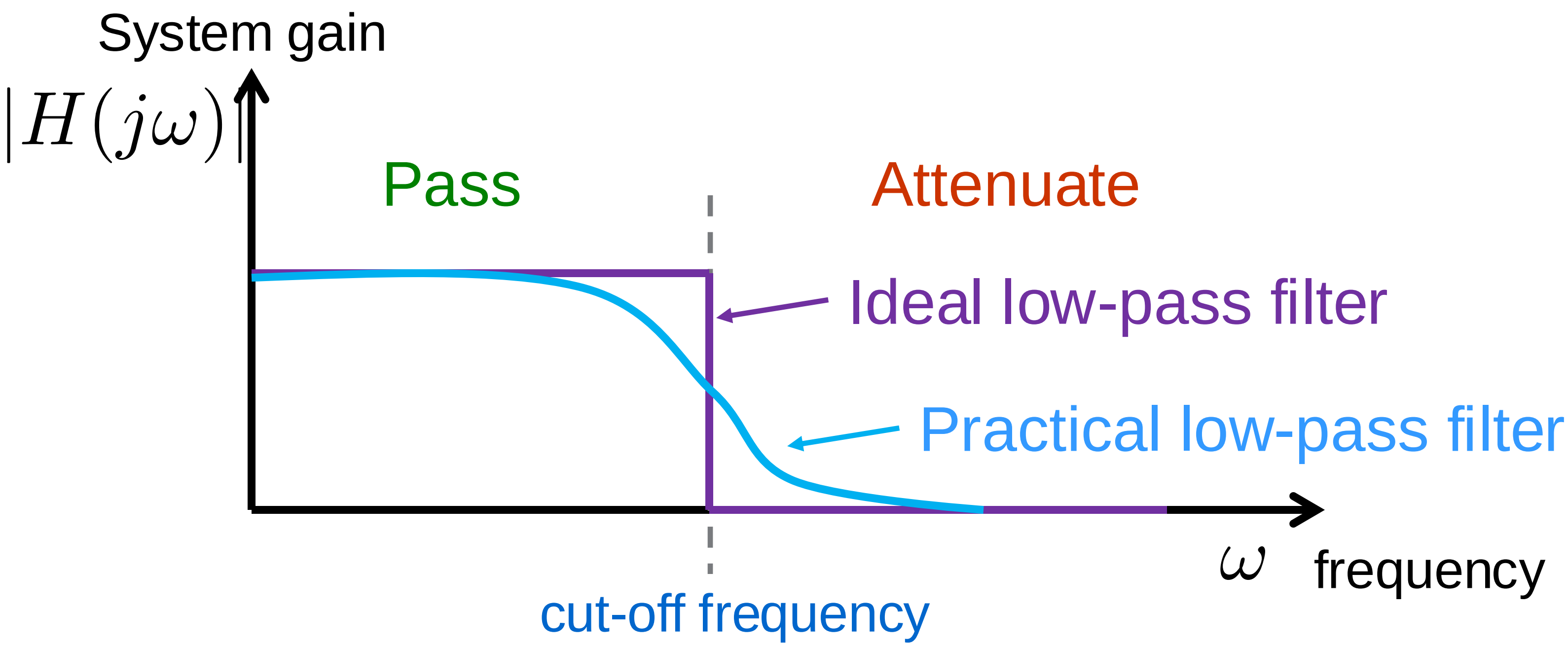
Types of filters

Key points

- Distinguish between different kinds of frequency filters
- Explain low-pass, high-pass, and bandpass filter concepts

Low-pass filter

- Systems that pass low frequencies, attenuate high frequencies



1st order low-pass filters

- Consider the 1st order differential equation

$$\frac{dy}{dt} + Ay(t) = x(t)$$

$$\underbrace{(s + A)}_{Q(s)}Y(s) = \underbrace{1}_{P(s)}X(s)$$

$$\boxed{H(j!_0) = \frac{P(j!_0)}{Q(j!_0)} = \frac{1}{j!_0 + A}}$$

1st order low-pass filters

- What is the system gain $|H(j!)|$?

$$|H(j!)| = \frac{1}{\sqrt{!^2 + A^2}}$$

- In dB

$$|H(j!)|(dB) = 20\log_{10}|H(j!)|$$

$$= 20\log_{10}\left((!^2 + A^2)^{-\frac{1}{2}}\right)$$

$$= -10\log_{10}(!^2 + A^2)$$

1st order low-pass filters

- What does this gain mean in terms of the frequency response?

- Three regimes

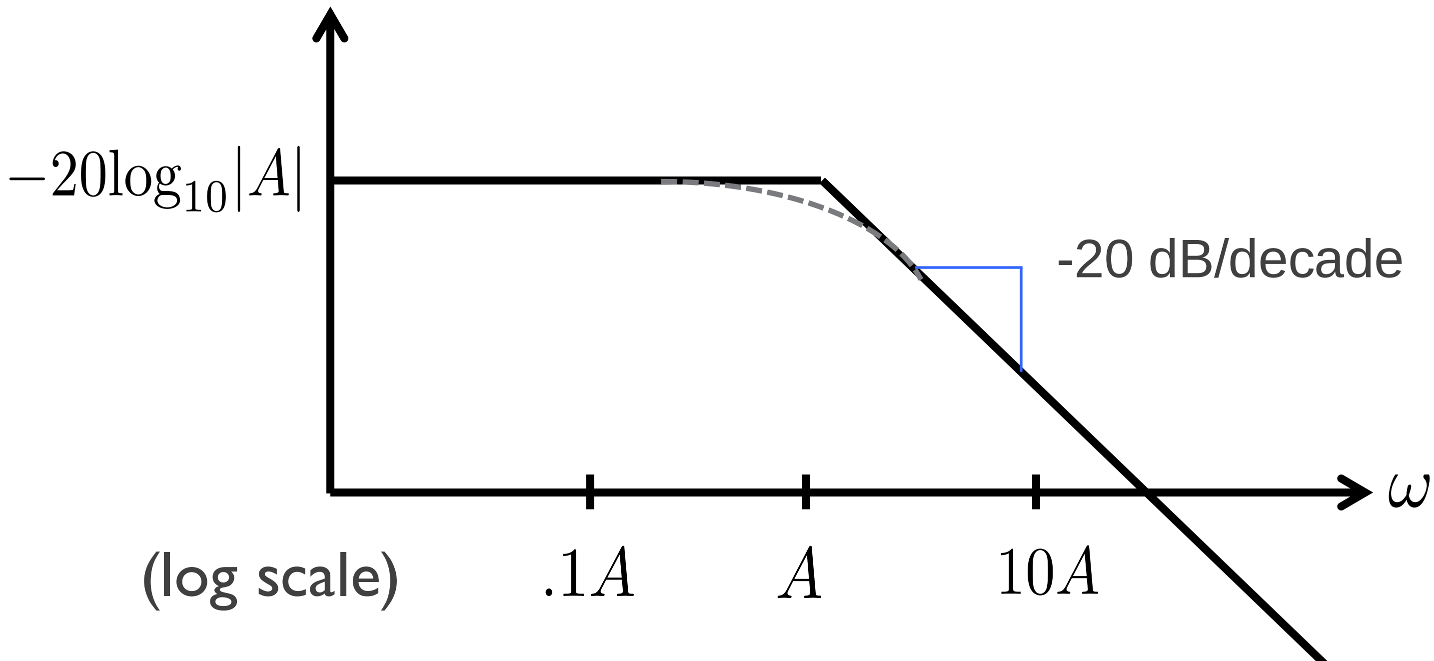
$$\boxed{|H(j\omega)|(dB) = -10\log_{10}\left(\omega^2 + A^2\right)}$$

  - $\omega \ll A, \quad |H(j\omega)| = -20\log_{10}|A|$
  - $\omega \gg A, \quad |H(j\omega)| = -20\log_{10}\omega, \quad \omega > 0$
  - $\omega \approx A,$ Transition region, usually can be ignored

1st order low-pass filters

- What does this gain mean in terms of the frequency response?

$$|H(j\omega)|(dB) = -10\log_{10}(\omega^2 + A^2)$$



Example– Designing a simple audio filter

- Design a LPF
  - 1st order filter
  - Low pass cutoff of 1.6 KHz or 10,000 radians/sec
  - Passband amplification of 40 dB (power increase of 10,000x)

- General 1st order filter has the form

$$H(j\omega) = \frac{G}{j\omega + A}$$

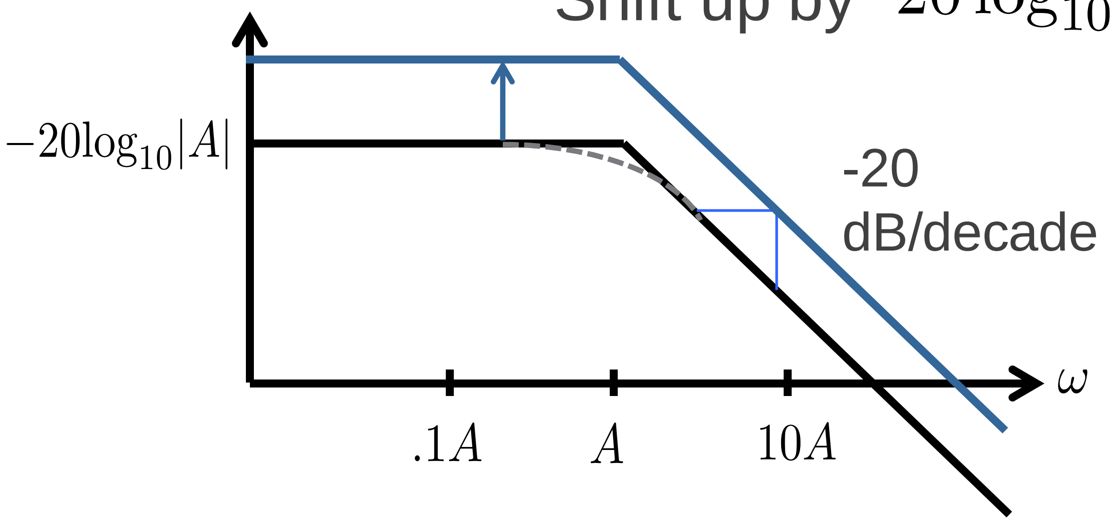
  - We want to determine the values of A and G

Example – Designing a simple audio filter (cont.)

- Gain in terms of dB

$$|H(j\omega)|(dB) = 20\log_{10} G - 10\log_{10}(\omega^2 + A^2)$$

Shift up by $20\log_{10} G$.



Example – Designing a simple audio filter (cont.)

- Cutoff frequency is still determined by A=10Krad/s
- For G, we need amplification of 40 dB

$$|H(j\omega)|(dB) = 20\log_{10} G - 10\log_{10}(\omega^2 + A^2)$$

$$\Rightarrow \quad 40 = 20\log_{10} G - 10\log_{10}(10^8)$$

$$40 = 20\log_{10} G - 80$$

$$120 = 20\log_{10} G$$

$$10^6 = G$$

Example – Designing a simple audio filter (cont.)

- Resulting frequency response

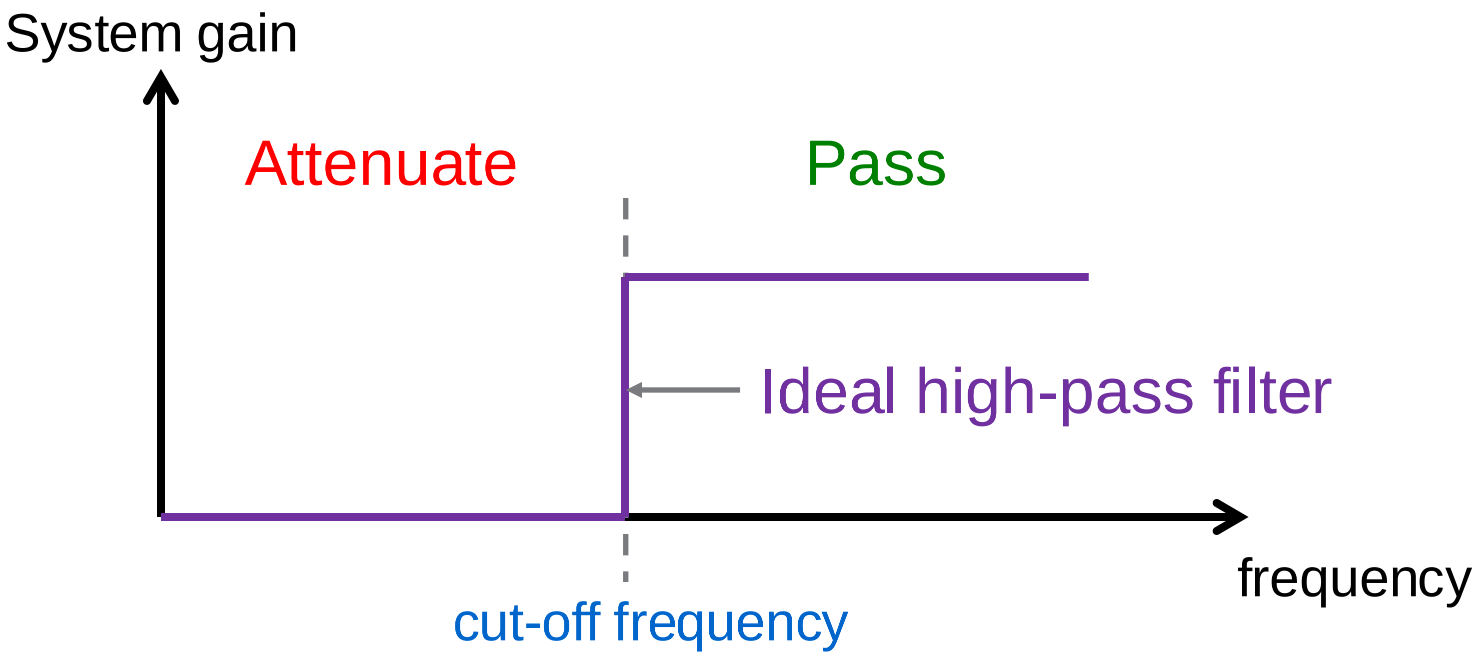
$$H(j\omega) = \frac{10^6}{j\omega + 10^4} \longrightarrow \begin{cases} P(j\omega) = 10^6 \\ Q(j\omega) = j\omega + 10^4 \\ Q(D) = D + 10^4 \end{cases}$$

- Resulting differential equation

$$\boxed{\frac{dy}{dt} + 10^4 y(t) = 10^6 x(t)}$$

High-pass filters (HPFs)

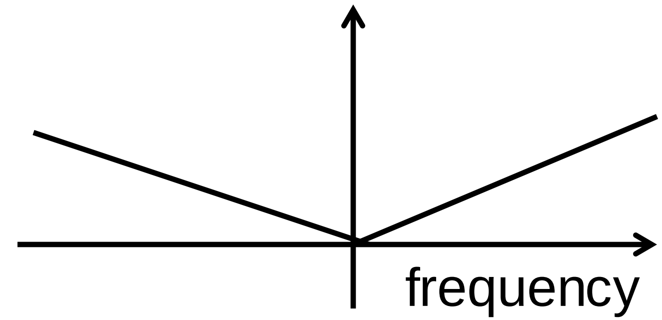
- Systems that pass high frequencies, attenuate low frequencies



High-pass filters (HPFs)

- Example: a differentiator

$$y(t) = \frac{dx}{dt}$$



✦ For $x(t) = e^{j\omega_0 t}$ → $y(t) = j\omega_0 e^{j\omega_0 t}$

$$y(t) = H(j\omega_0)e^{j\omega_0 t}$$

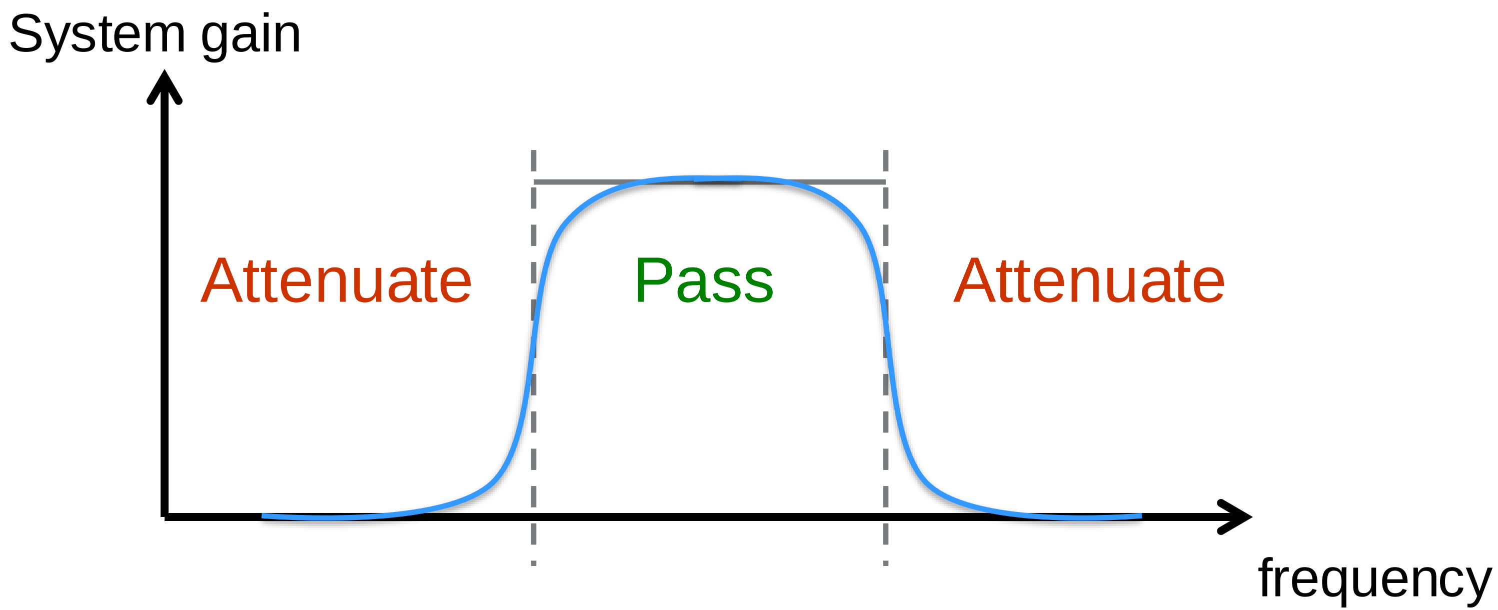
$$H(j\omega_0) = j\omega_0$$

$$\boxed{|H(\omega)| = \omega_0}$$

High gain with high frequencies

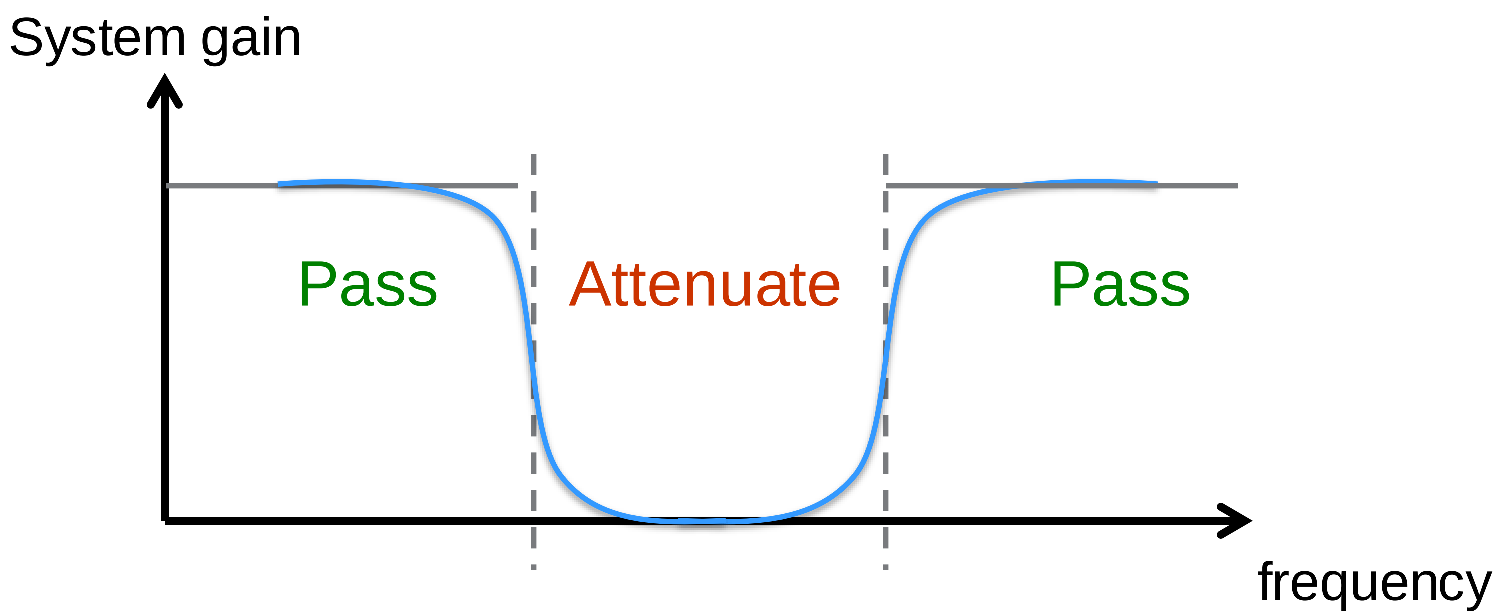
Band-pass filters

- A specific band is passed, and outside this band is attenuated



Band-stop (notch) filters

- Stop (attenuates) a certain band, and passes the other frequencies



Types of filters summary

- LTI systems change the phase and amplitude of the Fourier series coefficients in what is commonly known as filtering (convolution)

- There are many classical types of filters
  - Lowpass filters attenuate unwanted high frequencies
  - Highpass filters attenuate unwanted low frequencies
  - Band-pass filters accentuate frequencies in a target band
  - Band-stop or notch filters get rid of frequencies around a target band

- There are well known designs for filters, e.g. Butterworth or Chebyshev, and algorithms and software packages for designing filters