Lecture 14

Sinc, rectangle, modulation and convolution

Preview of today's lecture

- Practical implications of the sinc and rect functions
  - Connection between the sinc function and ideal lowpass filters
  - Application of sinc in communication systems

- Communication systems
  - Summarize some key principles of communication systems
  - Explain the role of the modulation property

- Convolution property
  - Convolution in time is multiplication in frequency
  - Use this fact to compute convolutions with less work!

Summarizing the Fourier transform and its inverse

Fourier transform (analysis)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform (synthesis)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\boxed{x(t) \overset{FT}{\longleftrightarrow} X(j\omega)}$$

(more pairs in the book)

Basic Fourier transform pairs 1/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Delta	$\delta(t)$	2π
Constant	$\frac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\frac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega - \omega_0)$
Causal exponential	$e^{-at}u(t)$ $\mathrm{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$

(more pairs in the book)

Basic Fourier transform pairs 2/3

	Time domain *x(t)*	**Frequency domain *X(j ω)***
Cosine	$\cos \omega_0 t$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
Sine	$\sin \omega_0 t$	$\pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

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$\omega_0 = \frac{2\pi}{T}$

(more pairs in the book)

Basic Fourier transform pairs 3/3

	Time domain $x(t)$	Frequency domain $X(j\omega)$
Rectangle	$\mathrm{rect}(t)$	$\dfrac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \mathrm{sinc}\left(\dfrac{\omega}{2\pi}\right)$
Scaled rectangle	$\mathrm{rect}\left(\dfrac{t}{2T_1}\right)$	$2T_1\dfrac{\sin(\omega)}{\omega} = 2T_1\mathrm{sinc}\left(\dfrac{\omega T_1}{\pi}\right)$
Sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\dfrac{\omega}{2\pi}\right)$
Scaled sinc	$\dfrac{B}{2\pi}\mathrm{sinc}\left(\dfrac{Bt}{2\pi}\right)$	$\mathrm{rect}\left(\dfrac{\omega}{B}\right)$

Fourier transform properties I $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

	Time domain	Fourier transform
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(j\omega)$

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Fourier transform properties 2

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

	Time domain	Fourier transform
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert}x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$	

Fourier transform properties 3

$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

$h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(j\omega)$

#14	Time domain	Fourier transform
Convolution in time	$y(t) = h(t) * x(t)$	$Y(j\omega) = H(j\omega)X(j\omega)$
Multiplication in time	$y(t) = h(t)x(t)$	$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$

#15

(more pairs in the book)

Basic Fourier transform pairs 1/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Delta	$\delta(t)$	2π
Constant	$\frac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\frac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega - \omega_0)$
Causal exponential	$e^{-at}u(t)$ $\text{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$

(more pairs in the book)

Basic Fourier transform pairs 2/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Cosine	$\cos \omega_0 t$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
Sine	$\sin \omega_0 t$	$\pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

(more pairs in the book)

Basic Fourier transform pairs 3/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Rectangle	$\mathrm{rect}(t)$	$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
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Sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
Scaled sinc	$\frac{B}{2\pi}\mathrm{sinc}\left(\frac{Bt}{2\pi}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$

Summarizing symmetry

$e(t) + j\,o(t)$

$\frac{x(t) + x^*(t)}{2}$

$$x(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

$$X(j\omega) = \mathrm{Re}\{E(j\omega)\} + j\mathrm{Im}\{E(j\omega)\} + \mathrm{Re}\{O(j\omega)\} + j\mathrm{Im}\{O(j\omega)\}$$

- Key symmetry equation relates real, imaginary, even, and odd in the time and frequency domains
- Can determine signal characteristics in one domain by inspecting the other domain

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 16 Fourier transform

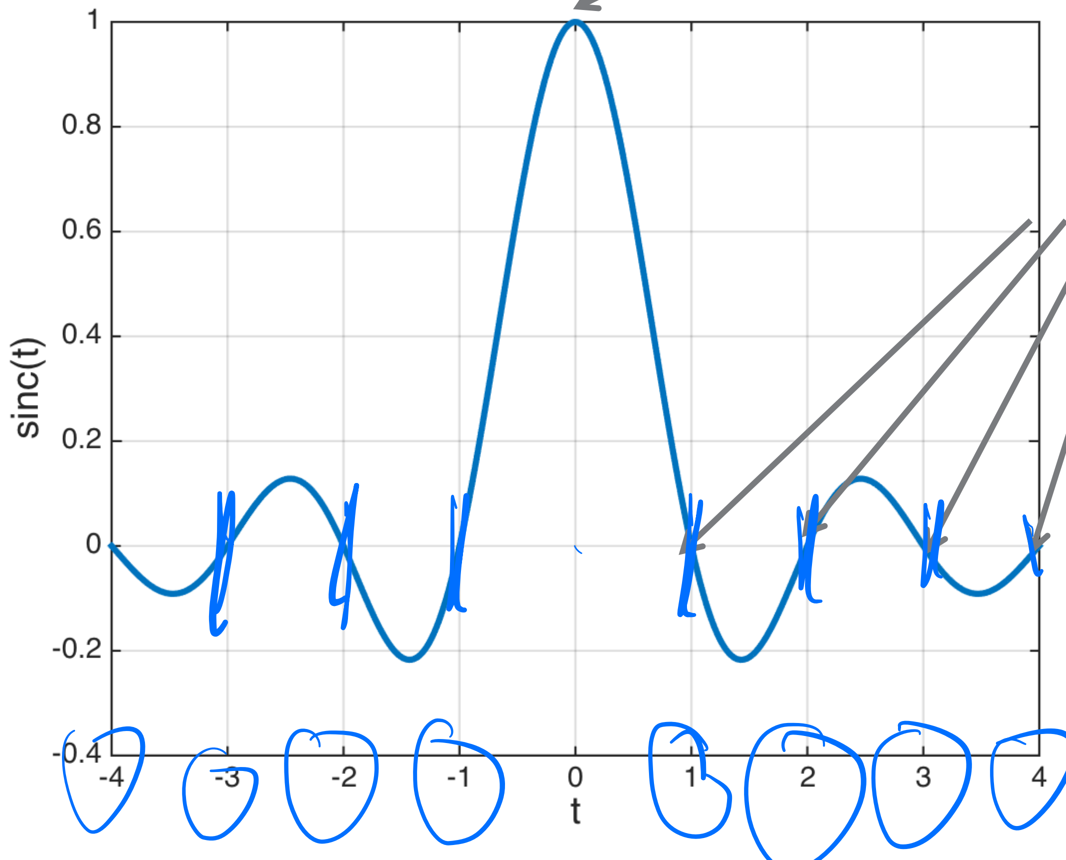
Fourier

Practical implications of the sinc and rect functions

Key points

- Explain the connection between the sinc function and ideal lowpass filters
- Explain the application of sinc in communication systems

Sinc function



Maximum value of 1, i.e. $\text{sinc}(0) = 1$

Zero crossings at +/-1, +/- 2, ….

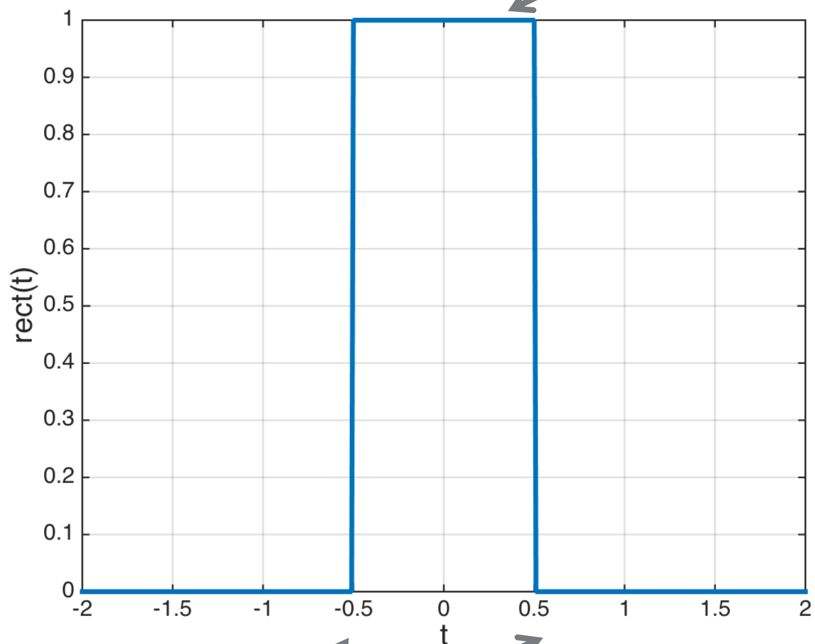
$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Be aware, sometimes sinc is defined like this

$$\text{sinc}(t) = \frac{\sin(t)}{t}$$

Rect function

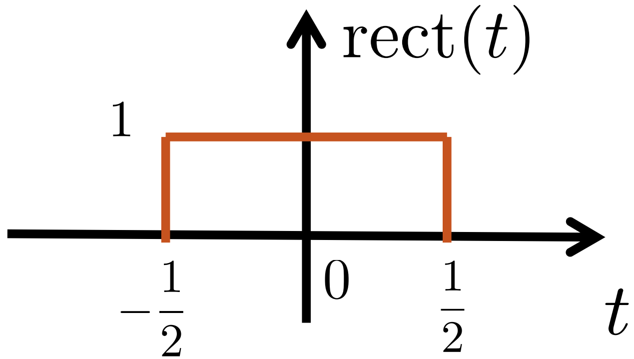
Maximum value of 1, i.e.



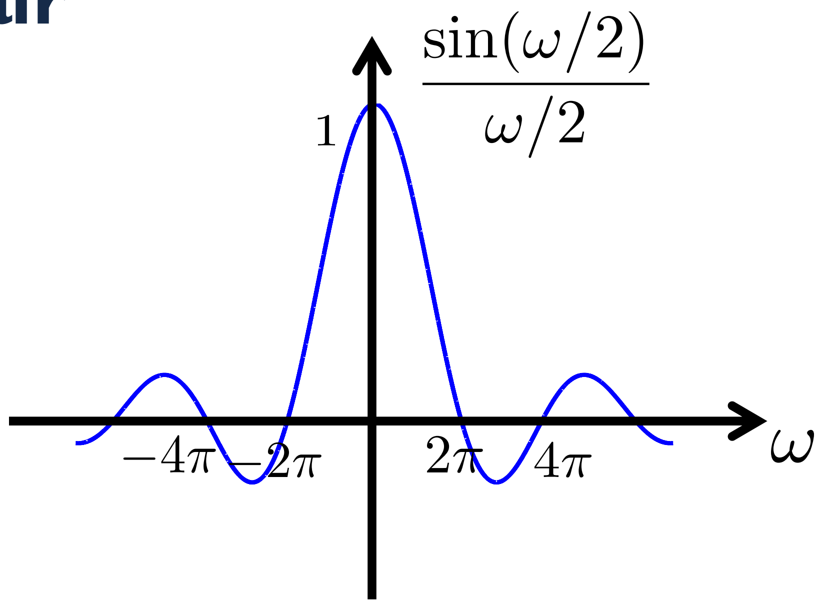
$$\boxed{\mathrm{rect}(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}}$$

Goes to zero at +/- 1/2

Rect – Sinc Fourier transform pair



$\overset{F}{\longleftrightarrow}$



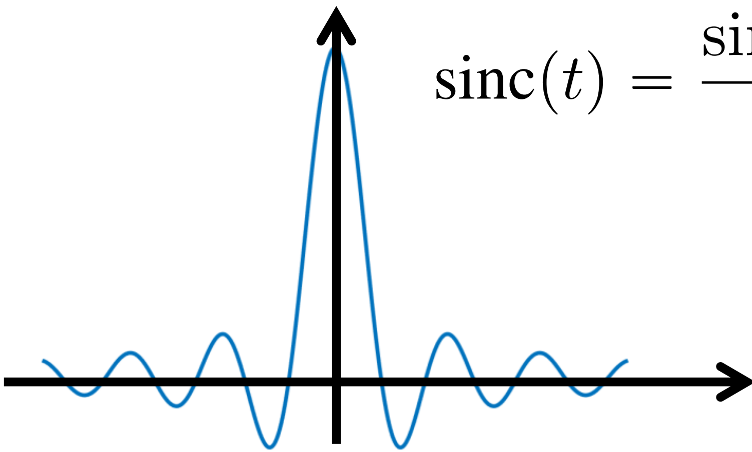
$$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \frac{\sin\frac{1}{2\pi}\pi\omega}{\frac{1}{2\pi}\cdot\pi\omega} = \operatorname{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$\boxed{\operatorname{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2\pi}\right)}$$

Aside: if using Hertz instead of radians/s

$$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \frac{\sin\frac{2\pi f}{2}}{\frac{2\pi f}{2}} = \operatorname{sinc}(f), \quad f \text{ in Hz}$$

Sinc - Rect Fourier transform pair

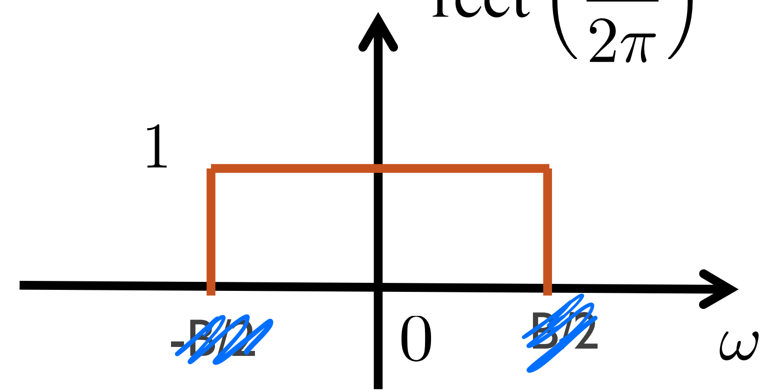


$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Zero crossings at +/-1, +/- 2, ….

$$\xleftrightarrow{F}$$

$$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$



$-\pi$ $\qquad$ π

$$\frac{\omega}{2\pi} \rightarrow \pi \cdot \frac{1}{2\pi}$$

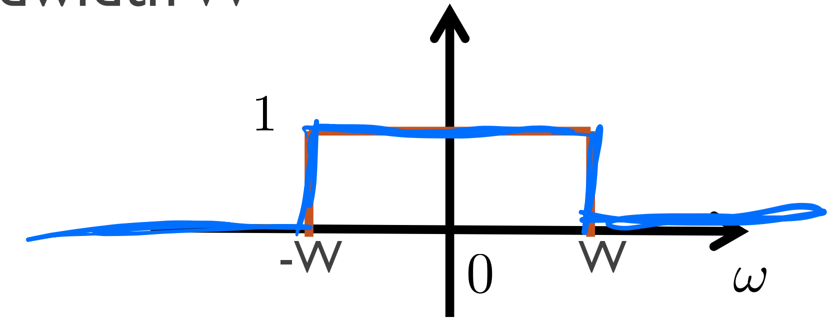
$$\boxed{\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2\pi}\right)}$$

Connection to low pass filter design

- Consider an ideal lowpass filter with bandwidth W

$$X(j\omega) = \begin{cases} 1 & |\omega| \leqslant W \\ 0 & |\omega| > W \end{cases}$$

$X(j\omega) = \text{rect}(\omega/2W)$

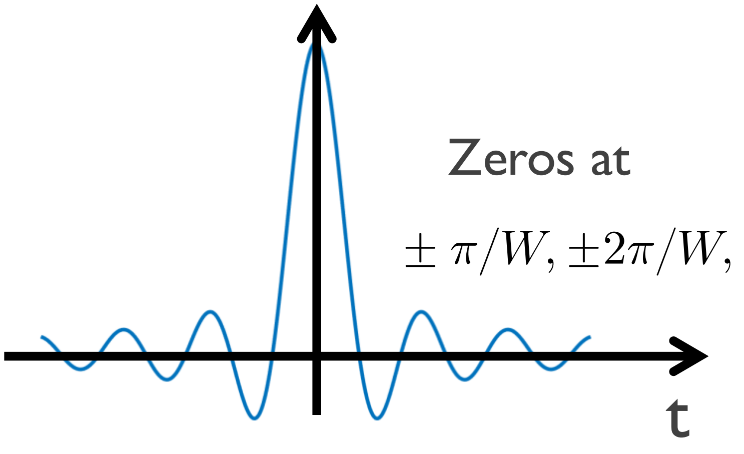


$$\text{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \text{rect}\left(\frac{\omega}{2\pi}\right) \quad \text{pair}$$

$$\frac{1}{|b|} x\left(\frac{t}{b}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jb\omega) \quad \text{scaling}$$

$$\frac{2W}{2\pi} x\left(t\frac{2W}{2\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(j\omega\frac{2\pi}{2W}\right) \quad \text{substitute}$$

$$\frac{W}{\pi} \text{sinc}\left(\frac{tW}{\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} \text{rect}\left(\frac{\omega}{2W}\right) \quad \text{simplify}$$



Implications on filter design

- The ideal lowpass filter is a sinc function with impulse response

$$h(t) = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$

- RLC type circuits though realize a LCCDE with an impulse response that is a linear combination of terms like

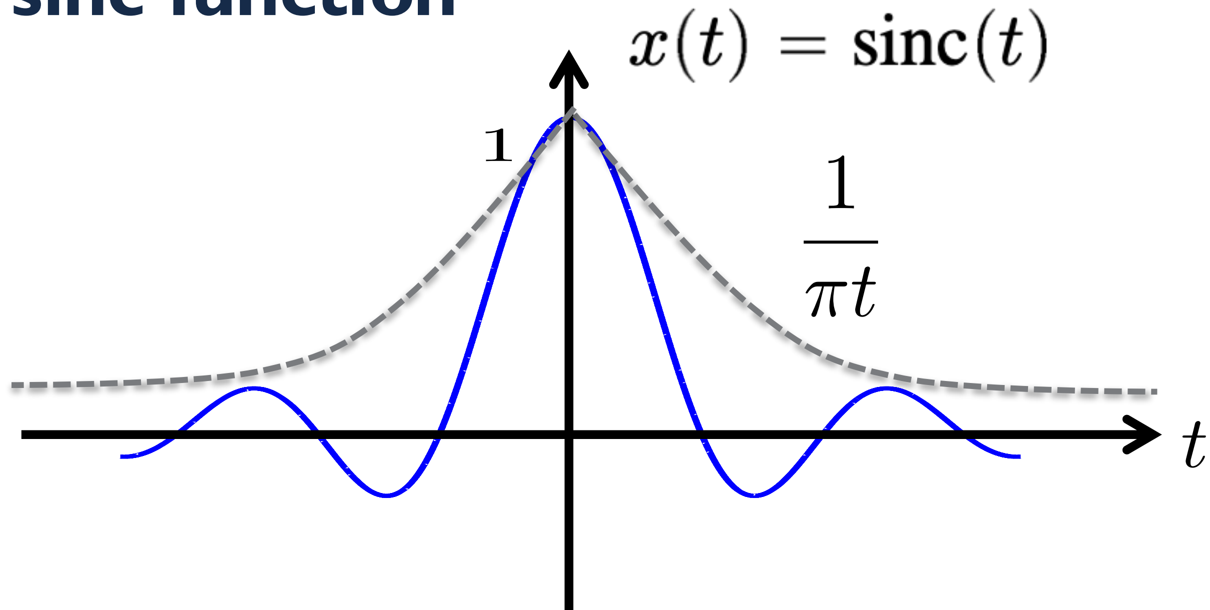
$$\delta(t) \qquad e^{-\lambda t}u(t) \qquad t^k e^{-\lambda t}u(t)$$

$h(t)$

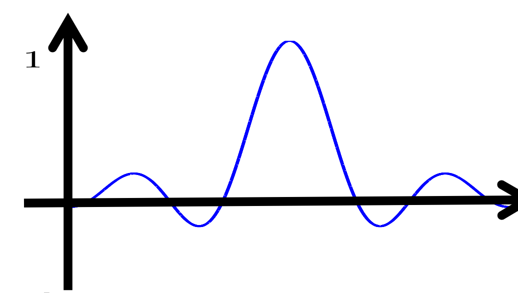
Circuits are design to only approximate ideal filters

Working with the sinc function

$$\boxed{\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}}$$

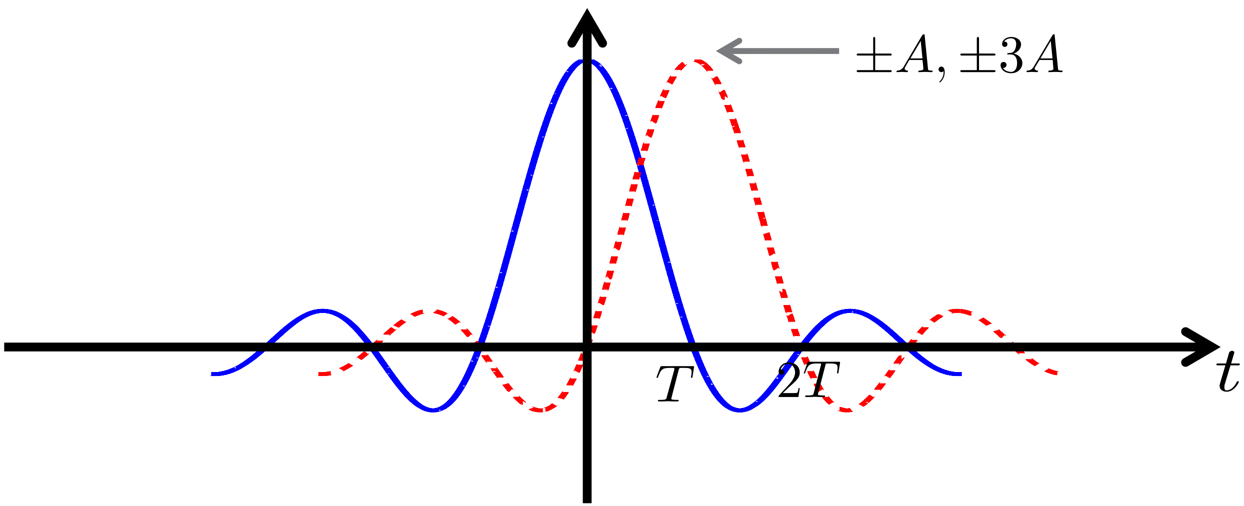


- Everlasting, non-causal time domain signal
- Truncating the sinc function
  - After 20 crossings, less than 5% of peak value
- Making the sinc causal
  - Shift to make approximately causal with delay based on truncation

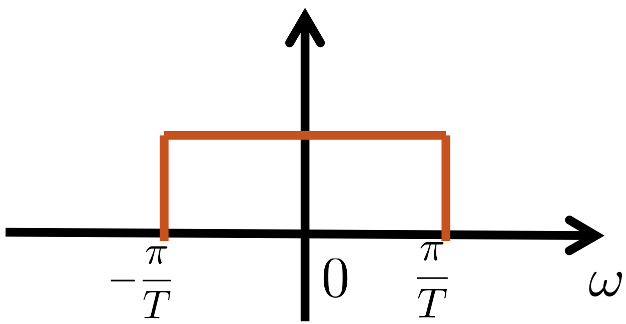


Connection to communications 1/2

- If we used the sinc to send a pulse $\pm 1, \pm 3$ every T seconds



Spectrum has the form



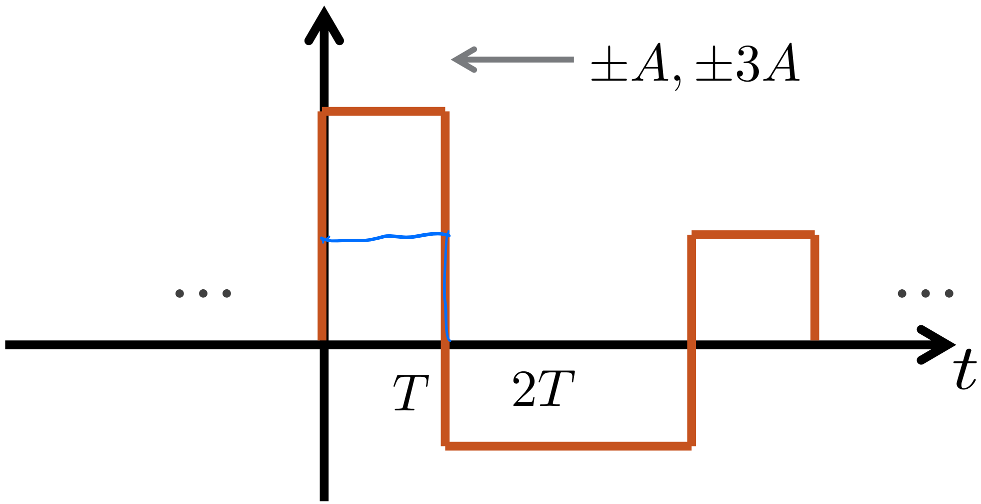
$$\underbrace{\text{sinc}\left(\frac{t}{T}\right)}_{u(t)} \perp \underbrace{\text{sinc}\left(\frac{t \pm kT}{T}\right)}_{v(t)} \quad \forall k$$

$$\int u(t)v(t)dt = 0$$

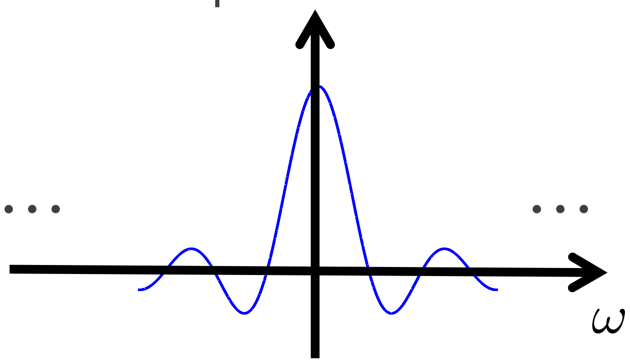
Sincs are good choices to carry data because shifted sinc functions are orthogonal

Connection to communications 2/2

- If we used the rect to send a pulse $\pm 1, \pm 3$ every T seconds



Spectrum has the form

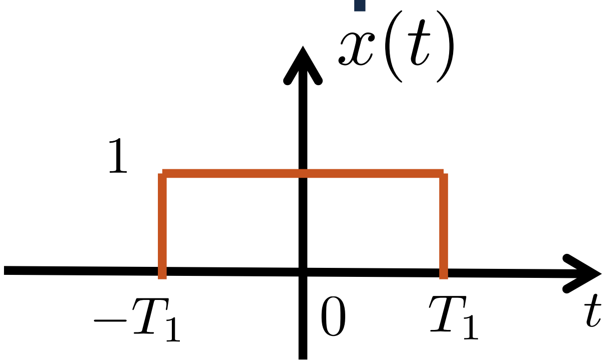


Zero crossings at

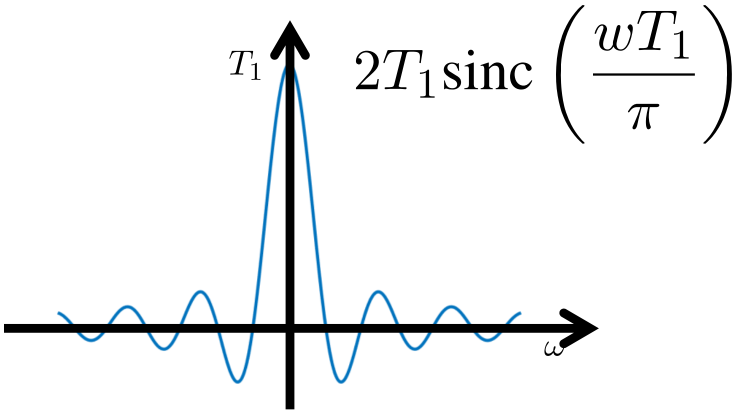
$\pm 2\pi/T, \pm 4\pi/T, \ldots$

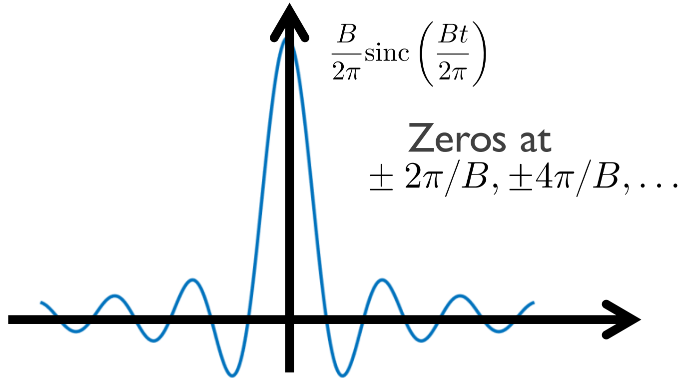
- Rectangle pulse uses infinite bandwidth!
- Sinc pulses or variations are used extensively in communications (have fixed, minimum bandwidth – very efficient)
  - Spectrum is expensive, more than $1000/Hz in the US for prime broadband spectrum (so a 10 MHz channel is about $10b)

Sinc – rect pair **duality**



$\overset{F}{\longleftrightarrow}$





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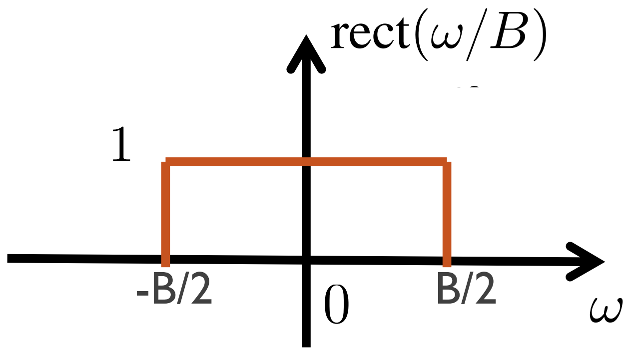


It is possible to figure out one set of transforms from the other

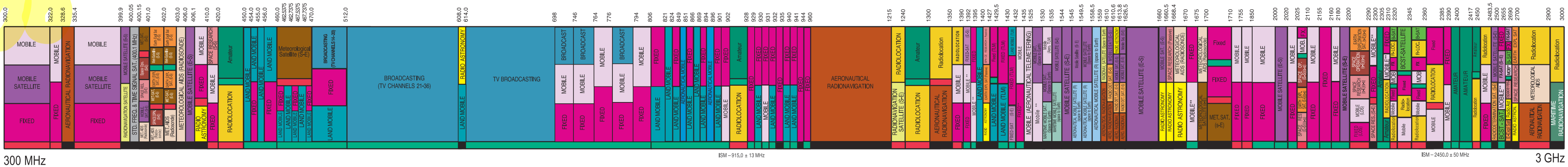
Application to communication systems

Key points

- Summarize some key principles of communication systems
- Explain the connection between communication and the frequency domain

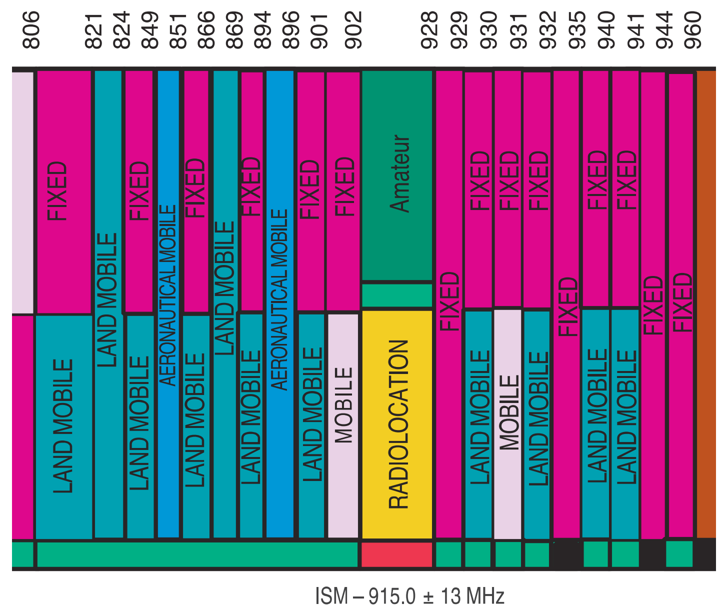
Spectrum allocation

Numbers are the frequency given in MHz, multiply by 2π to get Mrad/s



UHF band

Cellular is called "Land mobile"



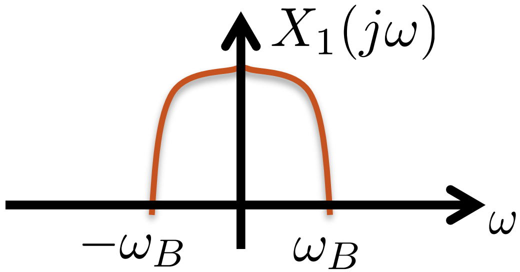
ISM band here is used for cordless phones, sensors

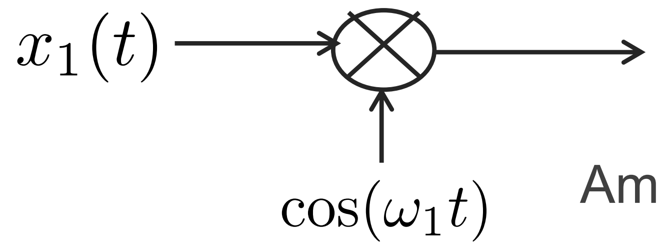
Practical application – Modulation

- Modulation is a concept widely used in communication systems
- While the "modulation" can mean different things, in communications it usually refers using a sinusoid to carry information
- Many types of modulation
  - Amplitude modulation (AM)
  - Frequency modulation (FM)
  - Phase modulation (PM)
- In this lecture: focus on an example with amplitude modulation

Practical application – Amplitude modulation

- One classically type of modulation is AM





$$x(t) = x_1(t)\cos(\omega_1 t)$$

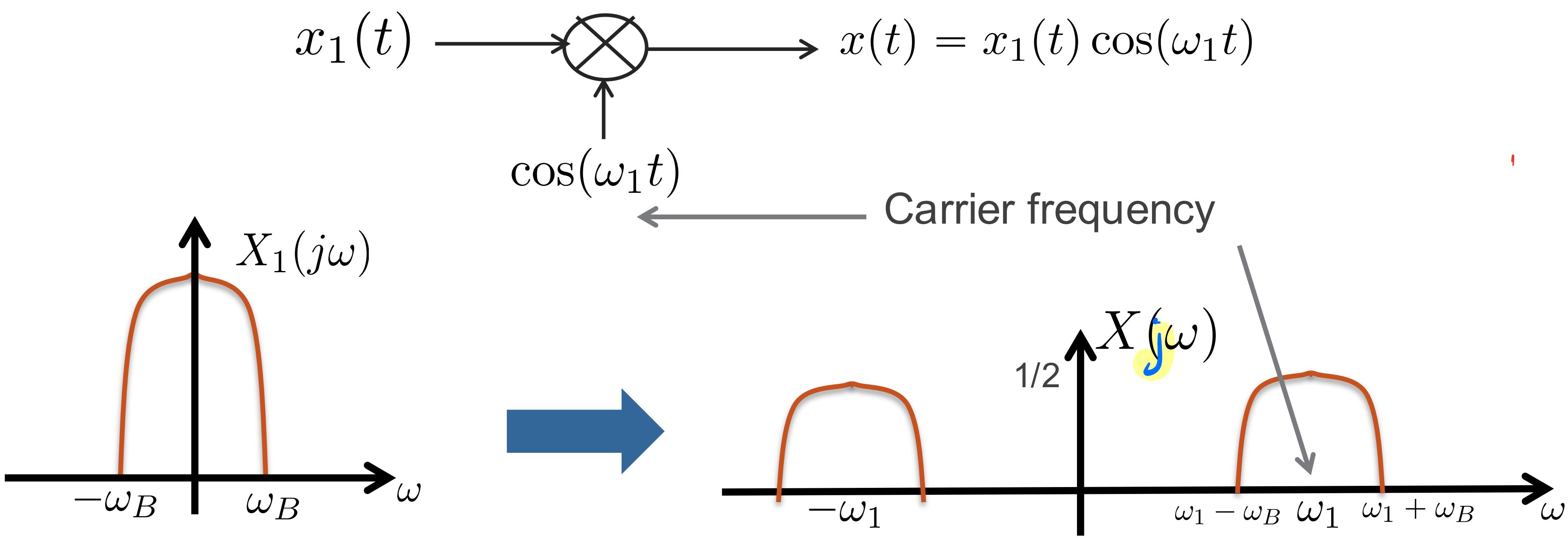
Amplitude of cosine is modulated

- What happens in the frequency domain?

$$x_1(t)\cos(\omega_1 t) = x_1(t)\frac{1}{2}e^{j\omega_1 t} + x_1(t)\frac{1}{2}e^{-j\omega_1 t}$$

$$\Rightarrow \quad \mathcal{F}\left\{x_1(t)\cos(\omega_1 t)\right\} = \frac{1}{2}X_1(j(\omega-\omega_1)) + \frac{1}{2}X_1(j(\omega+\omega_1))$$

Practical application – Amplitude modulation (cont.)



Spectrum of the information signal (aka baseband signal)

Spectrum of the modulated signal (aka the passband signal)

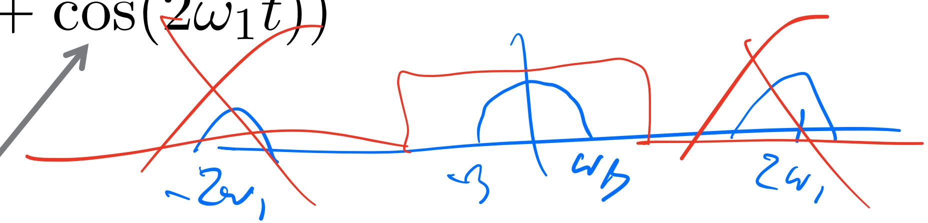
Practical application – Demodulation

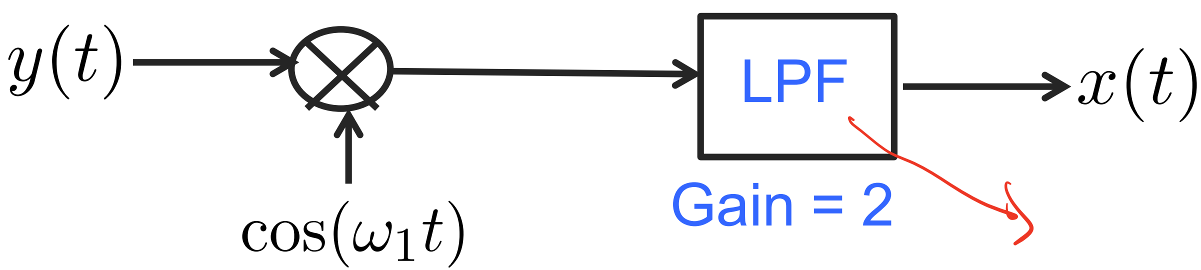
- The reverse of **modulation** is called **demodulation**
  - Exploit the fact that

$\omega_B \leq \omega_1$

$$y(t) = x(t)\cos^2(\omega_1 t) = x(t)\frac{1}{2}(1 + \cos(2\omega_1 t))$$

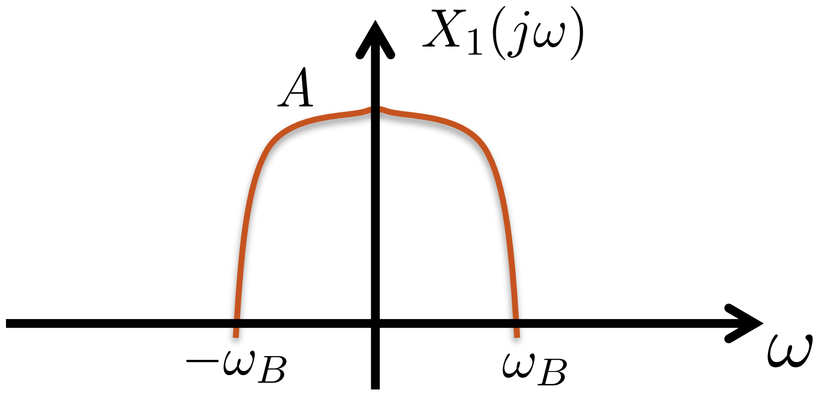
Can eliminate this with a lowpass filter!

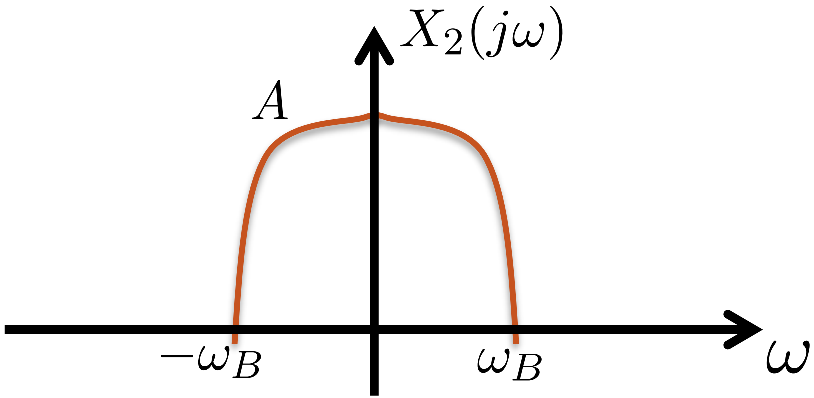




Practical application – Frequency division multiplexing

- Consider a communication system with two users
- Suppose that the signal for each user has frequency response

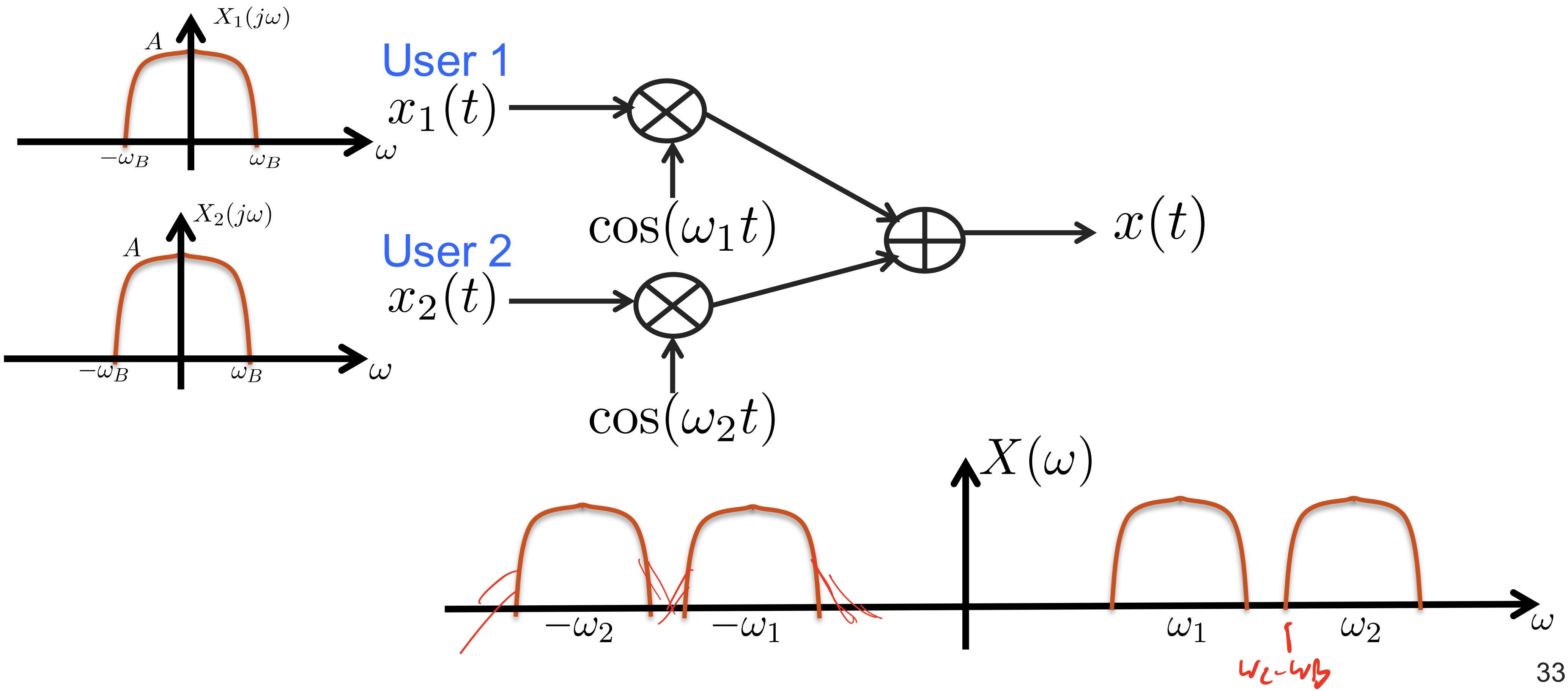




- Objective: Create a communication signal that carries the information contained in each users' signal

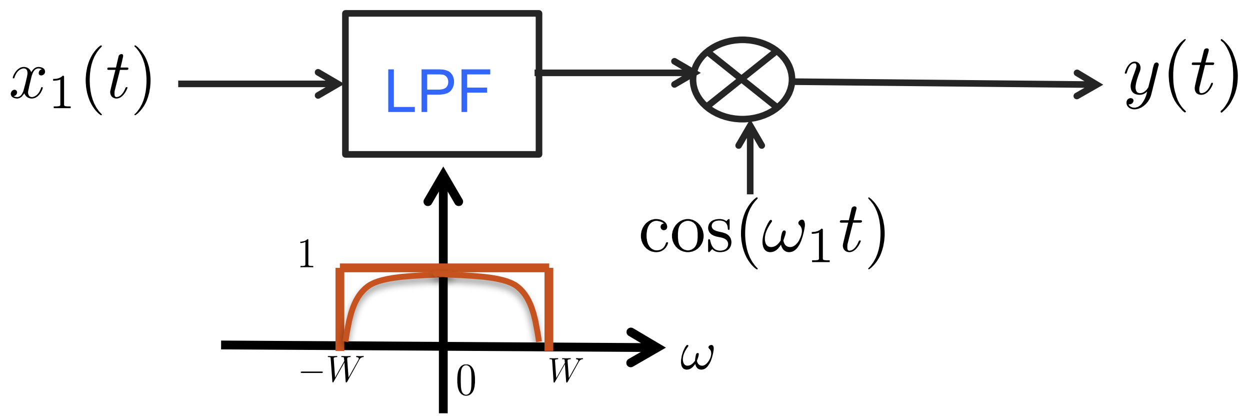
Practical application – Frequency division multiplexing (cont.)

- One possible solution is AM with different carrier frequencies

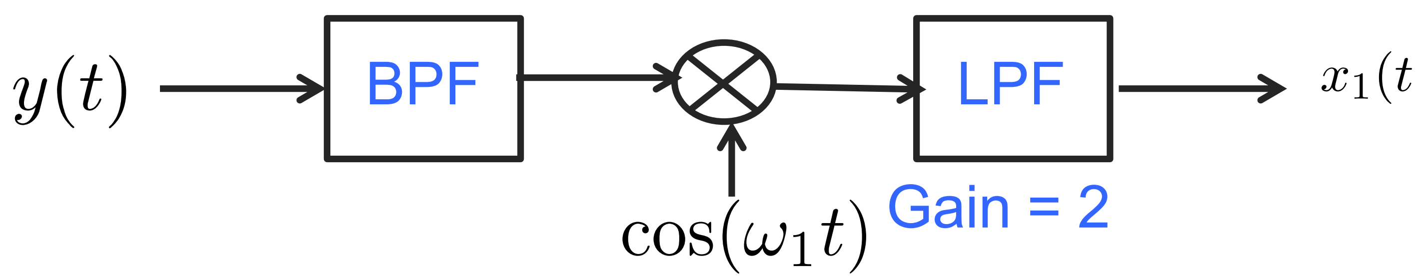


Practical application – Frequency division multiplexing (cont.)

- How to prevent possible overlap in the signals?
  - At the transmitter

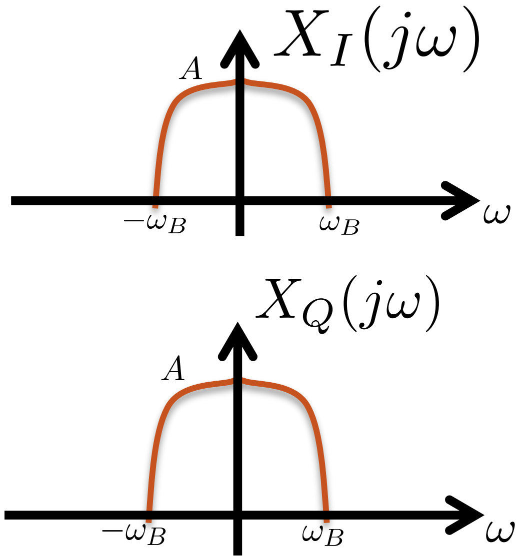


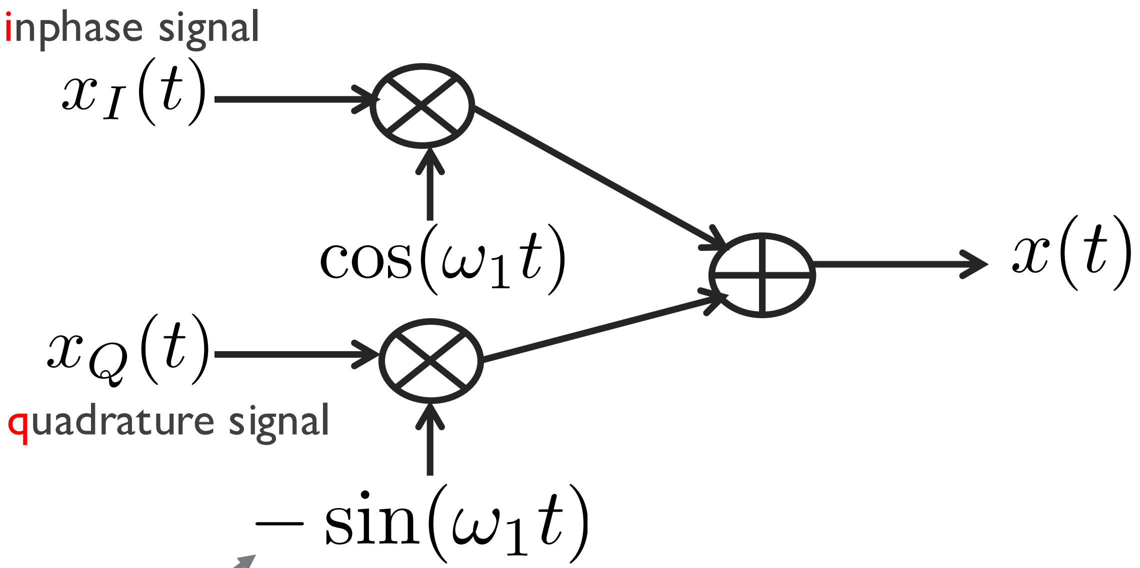
  - At the receiver



Practical application – Inphase and quadrature

- What if two information signals are sent as follows?





note the sign and sine here, which will make sense shortly

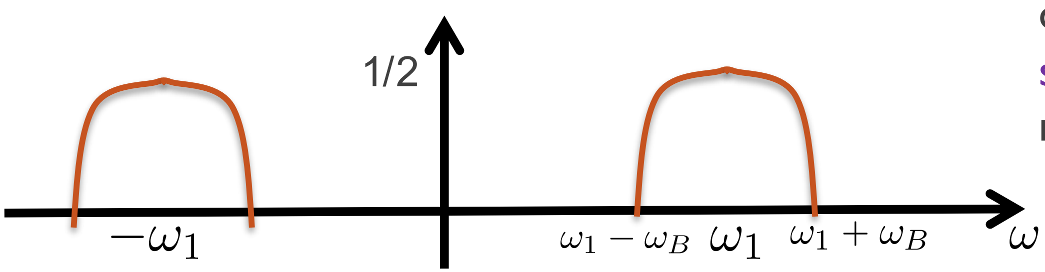
Practical application – Inphase and quadrature (cont.)

- What happens in the frequency domain?
  - Inphase term

$$\mathcal{F}\left\{x_I(t)\cos(\omega_1 t)\right\} = \frac{1}{2}X_I(j(\omega-\omega_1)) + \frac{1}{2}X_I(j(\omega+\omega_1))$$

  - Quadrature term

$$\mathcal{F}\left\{-x_Q(t)\sin(\omega_1 t)\right\} = \frac{j}{2}X_Q(j(\omega-\omega_1)) - \frac{j}{2}X_Q(j(\omega+\omega_1))$$



mixture of inphase and quadrature terms but not the same mixture at positive and negative frequencies

Practical application – Inphase and quadrature (cont.)

- What about demodulation?
  - Trig identities

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u \cos v = \frac{1}{2}[\sin(u-v) + \sin(u+v)]$$

Can recover both inphase and quadrature!

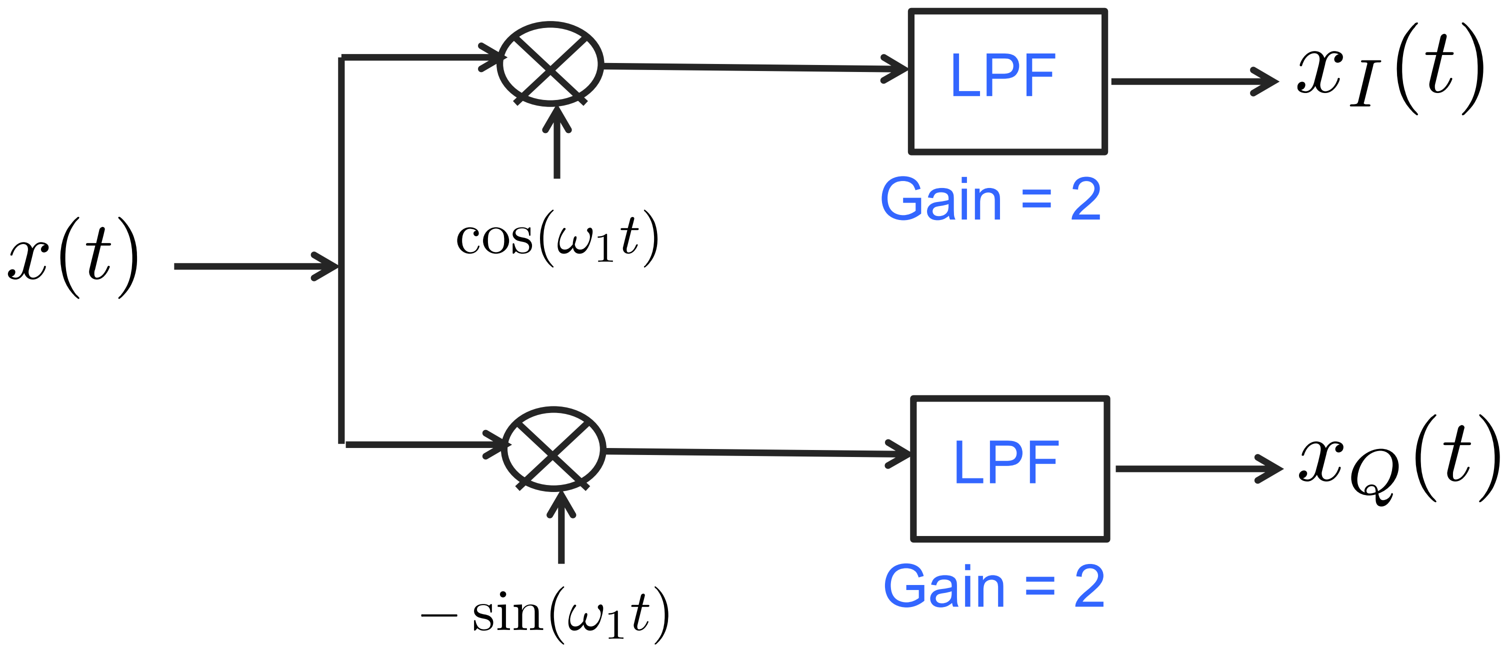
  - Applying the identities

filter out

$$x(t)\cos(\omega_1 t) = \frac{1}{2}x_I(t) + \boxed{\frac{1}{2}x_I(t)\cos(2\omega_1 t) - \frac{1}{2}x_Q(t)\sin(2\omega_1 t)}$$
$$x(t)\sin(\omega_1 t) = -\frac{1}{2}x_Q(t) + \boxed{\frac{1}{2}x_Q(t)\cos(2\omega_1 t) + \frac{1}{2}x_I(t)\sin(2\omega_1 t)}$$

Practical application – Inphase and quadrature (cont.)

- IQ demodulator



Practical application – Inphase and quadrature (cont.)

- Why do we use complex signals?

This is called the complex baseband signal

$$x_{bb}(t) = x_I(t) + jx_Q(t)$$

$\text{Re}\{x_{bb}(t)\}$ $\qquad$ $\text{Im}\{x_{bb}(t)\}$

Complex signals become a convenient way to work with inphase and quadrature together, avoiding the need for matrix notation

Convolution property

Key points

- Convolution in time is multiplication in frequency
- Use this fact to compute convolutions

Convolution property

- If $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$ $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

- Then

$$\boxed{y(t) = h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)}$$

Convolution in time is multiplication in frequency

Proof of the convolution property

$$Y(j\omega) = \mathcal{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\}$$

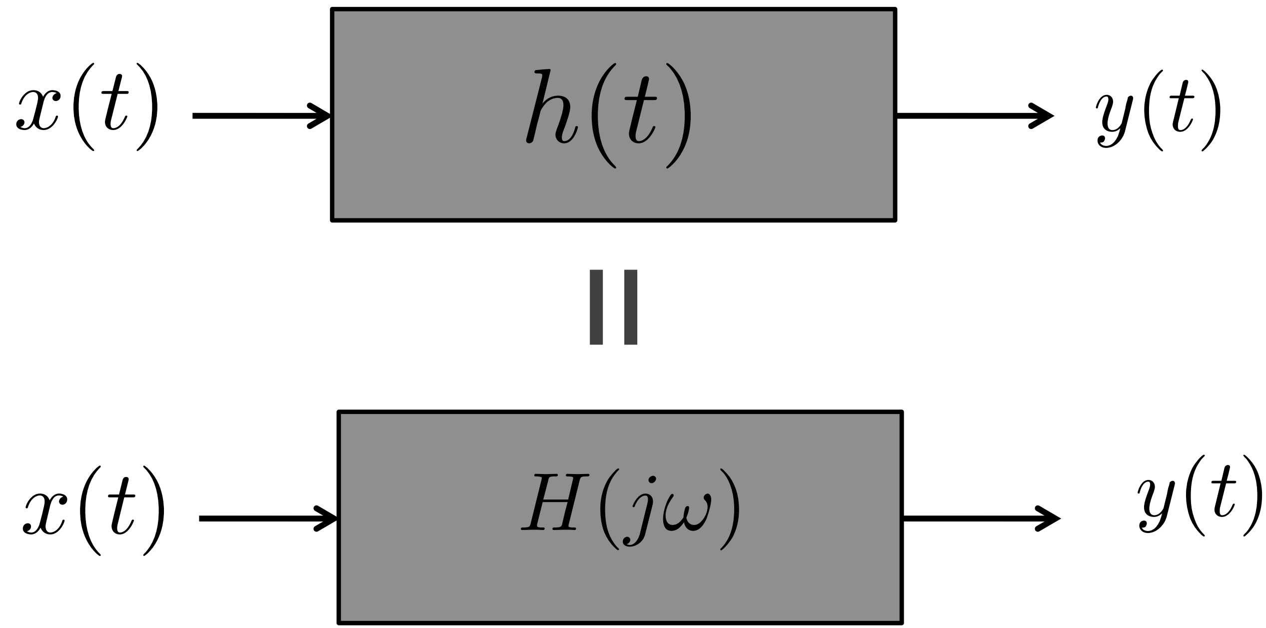
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t}dt$$ Apply definition

$$= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dtd\tau$$ Exchange order of integration

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau$$ Time shift property

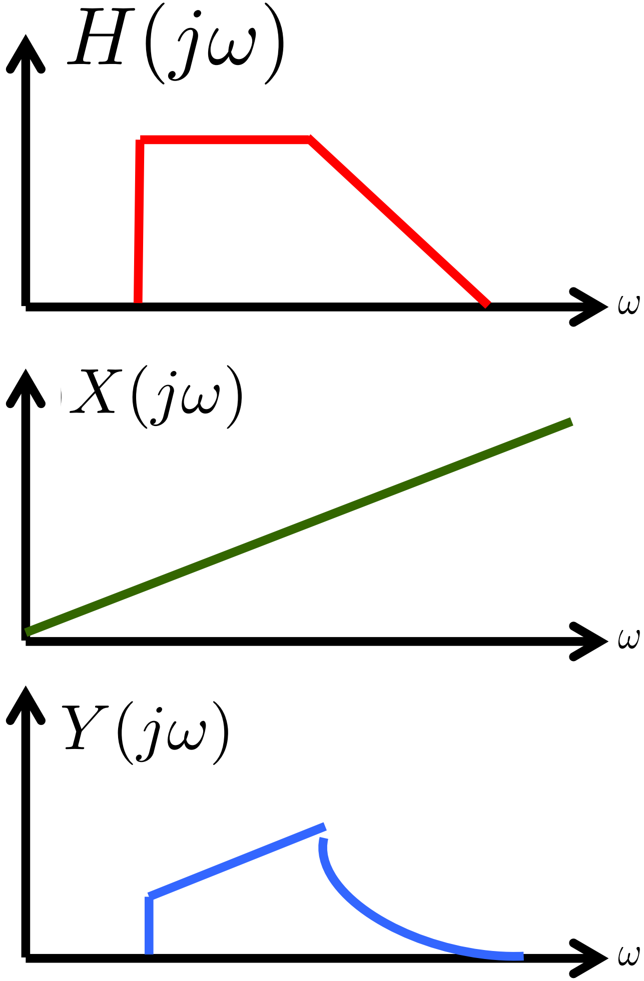
$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \quad = H(j\omega)X(j\omega)$$

Block diagrams



From a notational perspective, an LTI system may be described by the impulse response in the time or frequency domains

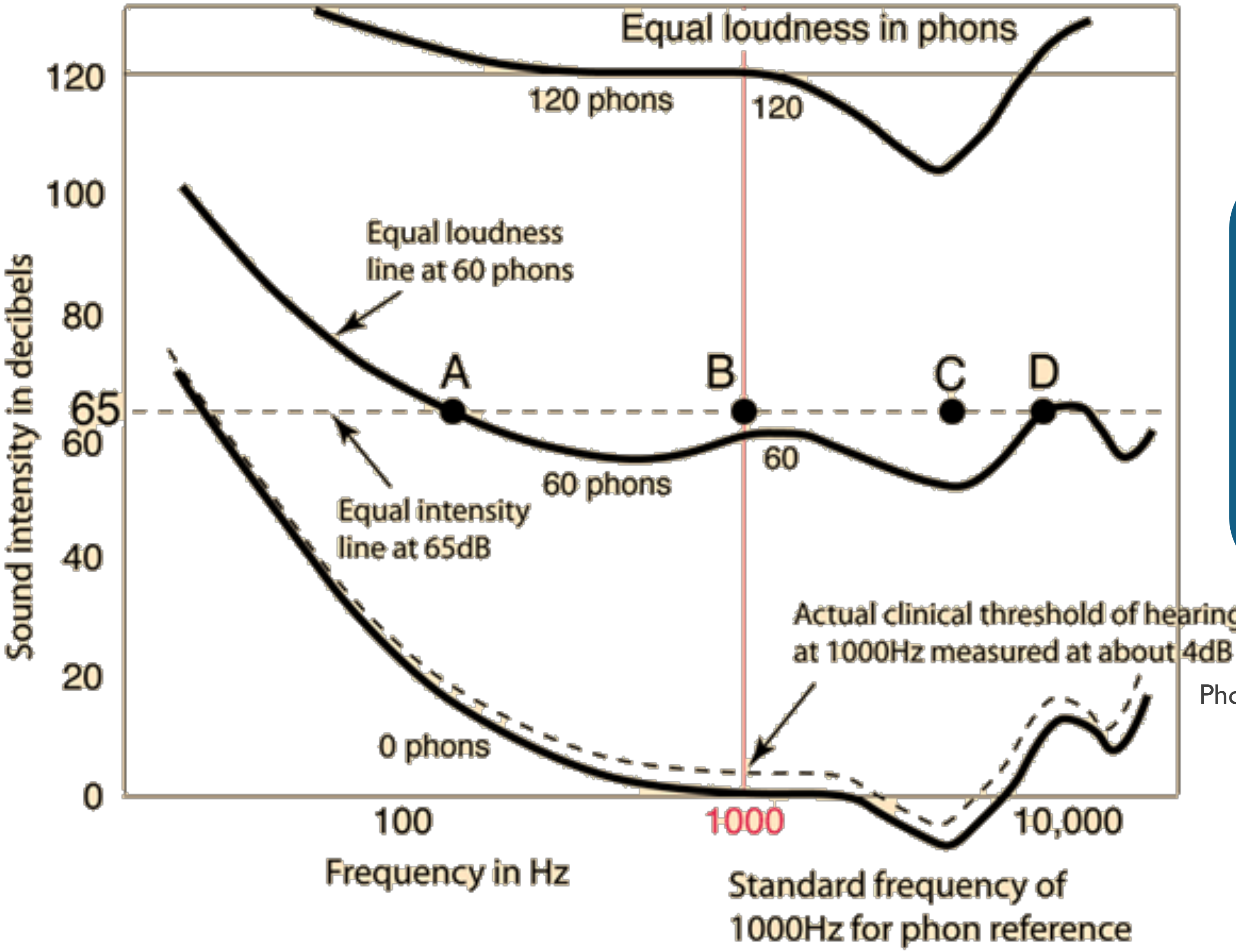
Visualizing the convolution property



$$Y(j\omega) = H(j\omega)X(j\omega)$$

Direct multiplication at each frequency

Contributed by EE 313 student Erte Bablu from http://www.rtings.com/headphones/reviews/apple/wireless-airpods



Perceptual response to sound intensity is not uniform in frequency

Phon is sound referenced to 1kHz

http://hyperphysics.phy-astr.gsu.edu/hbase/Sound/earcrv.html

Using the convolution property to do convolutions

- Compute the following convolution

$$y(t) = h(t) * x(t)$$

- Convert the two signals into the frequency domain

$$H(j\omega) = \mathcal{F}\{h(t)\}$$
$$X(j\omega) = \mathcal{F}\{x(t)\}$$

- Compute the product

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Go from frequency domain back into the time domain

$$y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$$

Double sinc example

- Given where $\omega_i > 0$ and $\omega_c > 0$

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \qquad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

- Find

$$y(t) = h(t) * x(t)$$

Double sinc example (continued)

- Solve by going into the frequency domain
- First find

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Need to compute

$$\mathcal{F}\left\{\frac{\sin(\omega_i t)}{\pi t}\right\}\mathcal{F}\left\{\frac{\sin(\omega_c t)}{\pi t}\right\}$$

- But note that

$$\operatorname{sinc}\left(\frac{t}{2\pi}\right) = \frac{\sin(t/2)}{t/2} \quad \text{and} \quad \operatorname{sinc}\left(\frac{t}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\operatorname{rect}(\omega)$$

Double sinc example (continued)

- Using the scaling property

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- Write

$$\frac{\sin(\omega_i t)}{\pi t} = \frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}$$

- It follows that

$$\begin{aligned}\mathcal{F}\left\{\frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}\right\} &= 2\pi \frac{\omega_i}{\pi} \frac{1}{|2\omega_i|} \mathrm{rect}(\omega/2\omega_i) \\ &= \mathrm{rect}(\omega/2\omega_i)\end{aligned}$$

Double sinc example (continued)

- The convolution is then

$$Y(j\omega) = \mathrm{rect}(\omega/2\omega_i)\mathrm{rect}(\omega/2\omega_c)$$

$$= \mathrm{rect}(\omega/2\min(\omega_c, \omega_i))$$

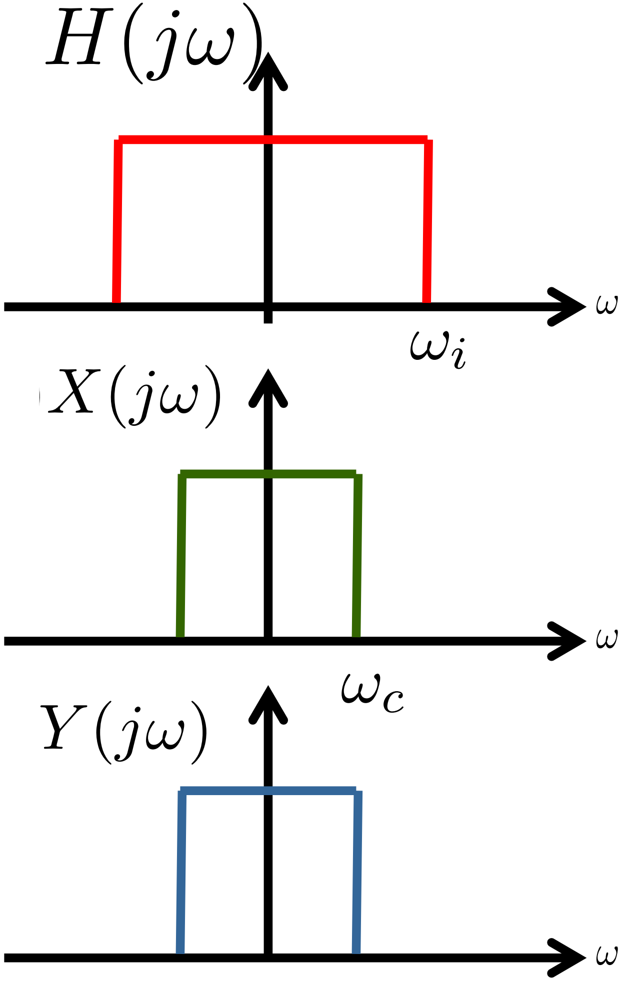
- Back in the time domain

$$y(t) = \frac{\sin(\min(\omega_i, \omega_c)t)}{\pi t}$$

This is a general result that sinc convolved with sinc gives sinc

Double sinc example (concluded)

Visualizing the effect in the frequency domain



$$Y(j\omega) = H(j\omega)X(j\omega)$$

Example where ω_i is bigger than ω_c

Summarizing the convolution property

- Convolution between two signals in time becomes the product of the Fourier transforms of those signals in the frequency domain
- Convolutions are easy to do in the frequency domain as they involve a simple point-wise multiplication
- The convolution property explains how the frequency response of a system directly effects the frequencies of the input signal to create the output signal