Problem Set 2 Solutions
ECE 101 Linear Systems Fundamentals

1 Problem 1

1.1 (a)

$y(t) = [\cos(3t)]\, x(t)$

1. Memoryless: $y(t)$ depends only on $x(t)$, therefore the system is memoryless.

2. Time invariant: First, shift the input; let $x_1(t) = x(t - t_0)$, then $y_1(t) = [\cos(3t)]\, x_1(t) = [\cos(3t)]\, x(t - t_0)$. Second, shift the output $y(t - t_0) = [\cos(3\,(t - t_0))]\, x(t - t_0)$. Since $y_1(t) \neq y(t - t_0)$, the system is not time invariant.

3. Linear: if $y_1(t) = [\cos(3t)]\, x_1(t)$ and $y_2(t) = [\cos(3t)]\, x_2(t)$, let $x_3(t) = ax_1(t) + bx_2(t)$, this will give the output

$$\begin{aligned} y_3(t) &= [\cos(3t)]\, x_3(t) \\ &= [\cos(3t)]\, (ax_1(t) + bx_2(t)) \\ &= a\,[\cos(3t)]\, x_1(t) + b\,[\cos(3t)]\, x_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

So the system is linear.

4. Causal: the system is memoryless, so it is causal because surely it does not depend on future values.

5. Stable: Let $|x(t)| \leq B$, then $|y(t)| \leq |[\cos(3t)]\, x(t)| = |\cos(3t)|\, |x(t)|$, since $|\cos(3t)| \leq 1$, so $|y(t)| \leq |x(t)| \leq B$, so the system is stable.

1.1 (b)

$$y(t) = \begin{cases} 0 & x(t) < 0 \\ x(t) + x(t-2) & x(t) \geq 0 \end{cases}$$

1. Memoryless: $y(t)$ depends only on $x(t - 2)$, therefore the system is not memoryless.

2. Time invariant: First, shift the input; let $x_1(t) = x(t - t_0)$, then

$$y_1(t) = \begin{cases} 0 & x_1(t) < 0 \\ x_1(t) + x_1(t-2) & x_1(t) \geq 0 \end{cases}$$

$$y_1(t) = \begin{cases} 0 & x(t-t_0) < 0 \\ x(t-t_0) + x(t-t_0-2) & x(t-t_0) \geq 0 \end{cases}$$

Second, shift the output

$$y(t-t_0) = \begin{cases} 0 & x(t-t_0) < 0 \\ x(t-t_0) + x(t-t_0-2) & x(t-t_0) \geq 0 \end{cases}$$

Since $y_1(t) = y(t-t_0)$, the system is time invariant.

3. Linear: If $x_1(t) = 1$, then $y_1(t) = 1+1 = 2$ and If $x_2(t) = -0.5$. then $y_2(t) = 0$. Let $x_3(t) = x_1(t) + x_2(t) = 1 - 0.5 = 0.5$, so $y_3(t) = 0.5 + 0.5 = 1$, but $y_1(t) + y_2(t) = 2$. This means the system is not additive. Therefore, the system is not linear.

4. Causal: the system depends on $x(t)$ and $x(t-2)$ which are current and previous samples of x, so it is causal because it does not depend on future values.

5. Stable: Let $|x(t)| \leq B$, then $|y(t)| \leq |x(t) + x(t-2)| \leq |x(t)| + |x(t-2)| \leq 2B$, so the system is stable.

1.2 (a)

$y[n] = nx[n]$

1. Memoryless: $y[n]$ depends only on $x[n]$, therefore the system is memoryless.

2. Time invariant: First, shift the input; let $x_1[n] = x[n-n_0]$, then $y_1[n] = nx_1[n] = nx[n-n_0]$. Second, shift the output $y[n-n_0] = (n-n_0)\,x[n-n_0]$. Since $y_1[n] \neq y[n-n_0]$, the system is not time invariant.

3. Linear: if $y_1[n] = nx_1[n]$ and $y_2[n] = nx_2[n]$, let $x_3[n] = ax_1[n] + bx_2[n]$, this will give the output

$$\begin{aligned} y_3[n] =& nx_3[n] \\ =& n\left(ax_1[n] + bx_2[n]\right) \\ =& anx_1[n] + bnx_2[n] \\ =& ay_1[n] + by_2[n] \end{aligned}$$

So the system is linear.

4. Causal: the system is memoryless, so it is causal because surely it does not depend on future values.

5. Stable: Let $|x[n]| \leq B$, then $|y[n]| \leq |nx[n]| = |n|\,|x[n]| \leq |n|\,B$, since $\lim_{n\to\infty} n = \infty$, $y[n]$ is unbounded and the system is not stable.

1.2 (b)

$$y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n] & n \leq -1 \end{cases}$$

1. Memoryless: $y[n]$ depends only on $x[n]$, therefore the system is memoryless.

2. Time invariant: First, shift the input; let $x_1[n] = x[n-n_0]$, then

$$y_1[n] = \begin{cases} x_1[n] = x[n-n_0] & n \geq 1 \\ 0 & n = 0 \\ x_1[n] = x[n-n_0] & n \leq -1 \end{cases}$$

   Second, shift the output

$$y[n-n_0] = \begin{cases} x[n-n_0] & n-n_0 \geq 1 \\ 0 & n-n_0 = 0 \\ x[n-n_0] & n-n_0 \leq -1 \end{cases}$$

   Since $y_1[n] \neq y[n-n_0]$, the system is not time invariant.

3. Linear: if $y_1[n] = nx_1[n]$ and $y_2[n] = nx_2[n]$, let $x_3[n] = ax_1[n] + bx_2[n]$, this will give the output

$$y_3[n] = \begin{cases} x_3[n] = ax_1[n] + bx_2[n] & n \geq 1 \\ 0 & n = 0 \\ x_3[n] = ax_1[n] + bx_2[n]] & n \leq -1 \end{cases} \qquad = ay_1[n] + by_2[n]$$

   So the system is linear.

4. Causal: the system is memoryless, so it is causal because surely it does not depend on future values.

5. Stable: Let $|x[n]| \leq B$, then $|y[n]| \leq |x[n]| = |x[n]| \leq B$, so the system is stable.

2 Problem 2

2.1 (a)

$y(t) = \cos(x(t))$

Let $x_1(t) = 0\ \forall t$, then $y_1(t) = \cos(0) = 1\ \forall t$ and let $x_2(t) = 2\pi\ \forall t$, then $y_2(t) = \cos(2\pi) = 1\ \forall t$.
Since $y_1(t) = y_2(t)$, while $x_1(t) \neq x_2(t)$, the system is not invertible.

2.1 (b)

$y[n] = \begin{cases} x[n/2] & n: even \\ 0 & n: odd \end{cases}$, this expands the input.

so the system is ivertible, and the inverse system is $y[n] = x[2n]$

3 Problem 3

3.1

Let $x[n] = h[n] = \alpha^n u[n]$. The output is

$$\begin{aligned}
y[n] =& x[n] * h[n] \\
=& \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
=& \sum_{m=-\infty}^{\infty} \alpha^m u[m] \alpha^{n-m} u[n-m] \\
=& \sum_{m=-\infty}^{\infty} \alpha^n u[m]u[n-m]
\end{aligned}$$

Note that $u[m] = 1\ m \geq 0$ and $u[n-m] = 1\ m \leq n$, so $u[m]u[n-m] = 1\ 0 \leq m \leq n,\ n > 0$ so

$$y[n] = \sum_{m=0}^{n} \alpha^n = \alpha^n \sum_{m=0}^{n} 1 = \alpha^n (n+1)u[n]$$

3.2

Note $x[n] = u[n] - u[n-5]$ and $h[n] = u[n-2] - u[n-8] + u[n-11] - u[n-17]$.

$$\begin{aligned}
y[n] =& \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
=& \sum_{m=-\infty}^{\infty} \left[u[m] - u[m-5]\right] h[n-m] \\
=& \sum_{m=0}^{4} h[n-m] \\
=& h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]
\end{aligned}$$

In MATLAB, this can be easily computed using the **conv** function. See Figure **??**.

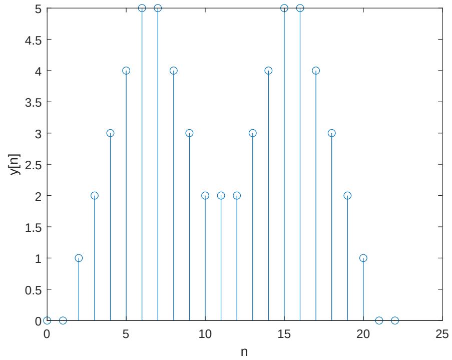

Figure 1: $y[n]$ for problem 2.21 d.

4 Problem 4

4.1

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau \\
&= \int_0^t e^{(\beta-\alpha)\tau}e^{-\beta t}d\tau,\ t \geq 0 \\
&= e^{-\beta t}\left.\frac{e^{(\beta-\alpha)\tau}}{(\beta-\alpha)}\right|_0^t,\ t \geq 0 \\
&= e^{-\beta t}\frac{e^{(\beta-\alpha)t}-1}{(\beta-\alpha)},\ t \geq 0 \\
&= \frac{e^{-\alpha t}-e^{-\beta t}}{(\beta-\alpha)}u(t)
\end{aligned}$$

If $\alpha = \beta$, then

$$\begin{aligned}
y(t) &= \int_0^t e^0 e^{-\beta t}d\tau,\ t \geq 0 \\
&= e^{-\beta t}\int_0^t 1d\tau,\ t \geq 0 \\
&= te^{-\beta t}u(t)
\end{aligned}$$

Some examples of $y(t)$ for representative values of α and β are shown in Figures 2 through 5.

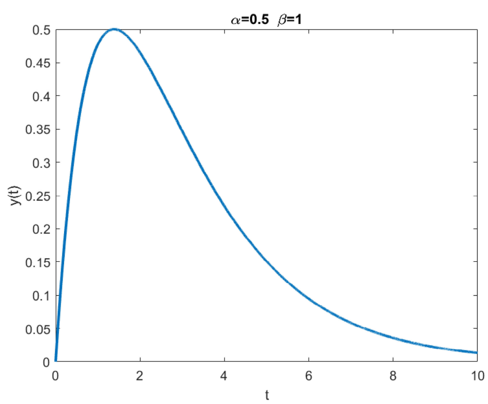

Figure 2: $\alpha = 0.5, \beta = 1$

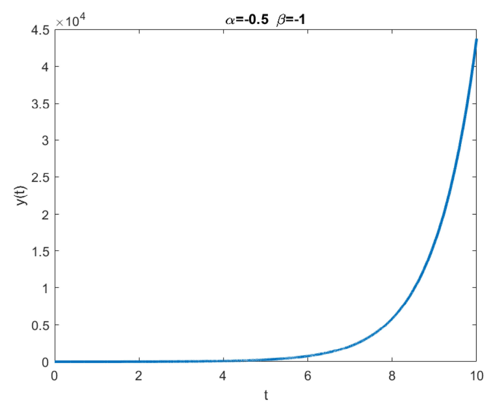

Figure 3: $\alpha = -0.5, \beta = -1$

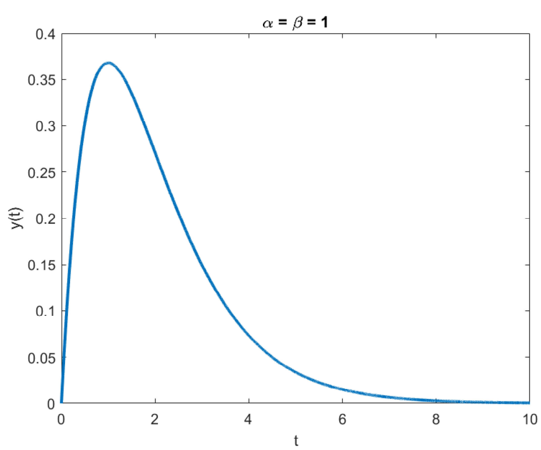

Figure 4: $\alpha = 1, \beta = 1$

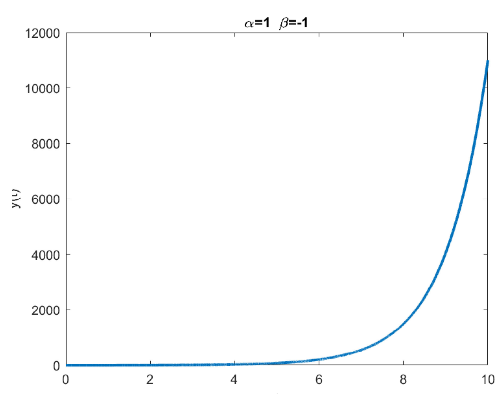

Figure 5: $\alpha = 1, \beta = -1$

4.2

$x(t)$: one period of $\sin(\pi t)$ and $h(t) = 2u(t-1) - 2u(t-3)$

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_0^2 \sin(\pi\tau)\left(2u(t-\tau-1) - 2u(t-\tau-3)\right)d\tau \\
&= \int_0^2 2\sin(\pi\tau)u(t-\tau-1) - \int_0^2 2\sin(\pi\tau)u(t-\tau-3)d\tau \\
&= A + B
\end{aligned}$$

since $u(t-\tau-1) = 1$, $\tau \leq t-1$ and zero otherwise, while $u(t-\tau-3) = 1$, $\tau \leq t-3$ and zero otherwise. The first integration A has three cases:

1. $t-1 \leq 0 \Rightarrow t \leq 1$: $A = 0$
2. $0 \leq t-1 \leq 2 \Rightarrow 1 \leq t \leq 3$: $A = \int_0^{t-1} 2\sin(\pi\tau)\,d\tau = \left.\frac{-2\cos(\pi\tau)}{\pi}\right|_0^{t-1} = \frac{2-2\cos(\pi(t-1))}{\pi}$
3. $t-1 \geq 2 \Rightarrow t \geq 3$: $A = \int_0^2 2\sin(\pi\tau)\,d\tau = \left.\frac{-2\cos(\pi\tau)}{\pi}\right|_0^2 = 0$

The second integration B has three cases:

1. $t-3 \leq 0 \Rightarrow t \leq 3$: $B = 0$
2. $0 \leq t-3 \leq 2 \Rightarrow 3 \leq t \leq 5$: $B = \int_0^{t-3} 2\sin(\pi\tau)\,d\tau = \left.\frac{-2\cos(\pi\tau)}{\pi}\right|_0^{t-3} = \frac{-2+2\cos(\pi(t-3))}{\pi}$
3. $t-3 \geq 2 \Rightarrow t \geq 5$: $B = \int_0^2 2\sin(\pi\tau)\,d\tau = 0$

$$y(t) = \begin{cases} 0 & t \leq 1 \\ \frac{2-2\cos(\pi(t-1))}{\pi} & 1 \leq t \leq 3 \\ \frac{-2+2\cos(\pi(t-3))}{\pi} & 3 \leq t \leq 5 \\ 0 & t \geq 5 \end{cases}$$

A plot of the signal $y(t)$ is shown in Figure 6.

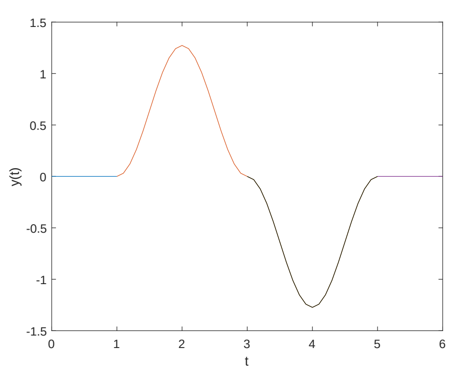

Figure 6: $y(t)$ for problem 2.22 c.

5 Problem 5

5.1

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$ and since $T=1$, so $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$ and $y(t) = x(t) * h(t)$, such that

$$\begin{aligned} \delta(t) * h(t) &= h(t) \\ \delta(t-1) * h(t) &= h(t-1) \\ &\vdots \\ \delta(t-k) * h(t) &= h(t-k) \end{aligned}$$

so $y(t) = \sum_{k=-\infty}^{\infty} \delta(t-k) * h(t) = \sum_{k=-\infty}^{\infty} h(t-k)$ and from Figure 7, we see that $y(t) = 1$.

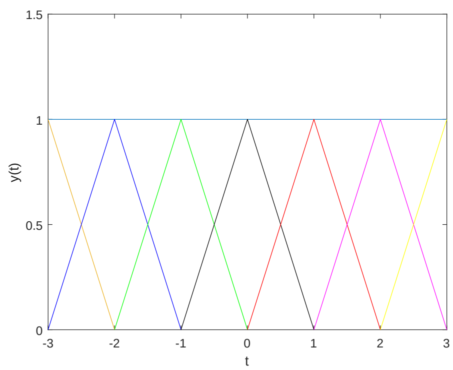

Figure 7: $y(t)$ for problem 2.23 d.

6 Problem 6

6.1

$h[n] = 0.8^n u[n+2]$

- Causal: $h[-1] \neq 0$, so the system is not causal.
- Stable: Compute $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |0.8^n u[n+2]| = \sum_{n=-2}^{\infty} 0.8^n = \frac{0.8^{-2}}{1-0.8} = \frac{125}{16} < \infty$, so the system is stable.

6.2

$h(t) = te^{-t}u(t)$

- Causal: $h(t)$ is right handed signal where $h(t) = 0$ when $t < 0$, so the system is causal.
- Stable: $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left|te^{-t}u(t)\right| dt = \int_{0}^{\infty} \left|te^{-t}\right| dt = -te^{-t}\Big|_0^{\infty} + \int_0^{\infty} e^{-t} dt = -te^{-t} - e^{-t}\Big|_0^{\infty} = 1$, so the system is stable.

7 Problem 7

7.1

$y[n] + 2y[n-1] = x[n]$,

Initial rest: $x[n]=\delta[n]$, so $x[n]=y[n]=0,\ n<0$
In order to find the impulse response, we find $y[n]$ for $x[n]=\delta[n]$
$y[n]=-2y[n-1]+x[n],\ y[-1]=0$
$y[0]=-2y[-1]+x[0]=0+1=1=(-2)^0$
$y[1]=-2y[0]+x[1]=-2+0=-2$
$y[2]=-2y[1]+x[2]=(-2)^2+0=(-2)^2$
$\vdots$
$y[k]=-2y[k-1]+x[k]=(-2)^k+0=(-2)^k$
so $h[n]=(-2)^n,\ n\geq 0$, i.e. $h[n]=(-2)^n\,u[n]$

8 More on LTI systems

(a): True

$h(t)$ periodic and non-zero, so $\int_{-\infty}^{\infty}|h(t)|\,dt=\infty$, so the system is unstable.

(b): False

counter example: $h[n]=\delta[n-1]$ is causal, but $h_{inv}[n]=\delta[n+1]$ is not causal.

(c): False

counterexample: $h[n]=u[n]$, where $|h[n]|=|u[n]|\leq 1$,
but $\sum_{n=-\infty}^{\infty}|h[n]|=\sum_{n=-\infty}^{\infty}u[n]=\sum_{n=0}^{\infty}1=\infty$, so it is not stable

(d): True

for $h[n]$: finite duration, $\sum_{n=-\infty}^{\infty}|h[n]|=\sum_{n:finite}|h[n]|<\infty$, so the system is stable.

(e): False

counterexample: $h(t)=e^t u(t)$ is causal, but $\int_{-\infty}^{\infty}|h(t)|\,dt=\int_0^{\infty}e^t dt=\infty$, so the system is not absolutely integrable and therefore it is not stable.

(f): False

counterexample: If $h_1[n]=\delta[n-1]$ and $h_2[n]=\delta[n+1]$, then the first system is causal and the second is non-causal.
But $h[n]=h_1[n]*h_2[n]=\delta[n]$, so the cascade is causal.

(g): False

counterexample: $h(t)=e^{-t}u(t)$, has a step response of $s(t)=\left(1-e^{-t}\right)u(t)$.
Then $\int_{-\infty}^{\infty}|s(t)|\,dt=\int_0^{\infty}1-e^{-t}dt=t-e^{-t}\Big|_0^{\infty}=\infty$
and $\int_{-\infty}^{\infty}|h(t)|\,dt=\int_0^{\infty}e^{-t}dt=-e^{-t}\Big|_0^{\infty}=1$.
So the system is stable, even though $\int_{-\infty}^{\infty}|s(t)|\,dt=\infty$

(h): True

Recall that $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$, so $s[n] = \sum_{k=0}^{\infty} h[n-k] = \sum_{k=-\infty}^{n} h[k]$.

If $s[n] = 0,\ \forall n < 0$, then $h[n] = s[n] - s[n-1] = 0,\ \forall n < 0$. Therefore, the system is causal.

Conversely, if the system is causal, then $h[n] = 0,\ \forall n < 0$, so $s[n] = \sum_{k=-\infty}^{n} h[k] = 0,\ \forall n < 0$.