Lecture 15

Convolution

Preview of today's lecture

- Convolution property
  - Convolution in time is multiplication in frequency
  - Use this fact to compute convolutions with less work!
- Multiplication property
  - Multiplication in time is convolution in frequency
  - Use this fact to explain windowing
- Bandwidth
  - Finite duration signals have infinite bandwidth
  - Different measures of bandwidth are used in practice

Fourier transform properties I $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

	Time domain	Fourier transform
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(j\omega)$

Fourier transform properties 2

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

	Time domain	Fourier transform
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert} x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$	

Fourier transform properties 3

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

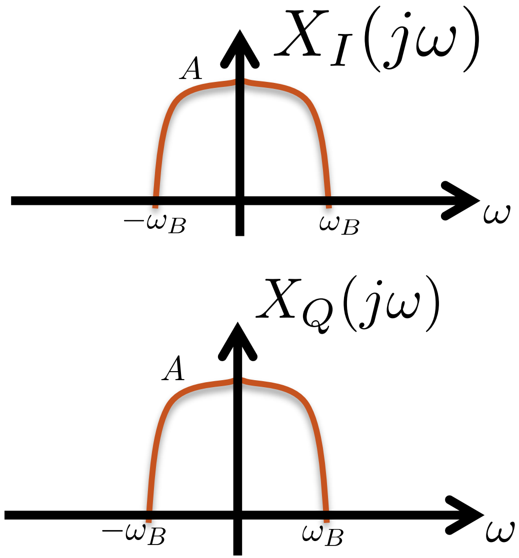
$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$

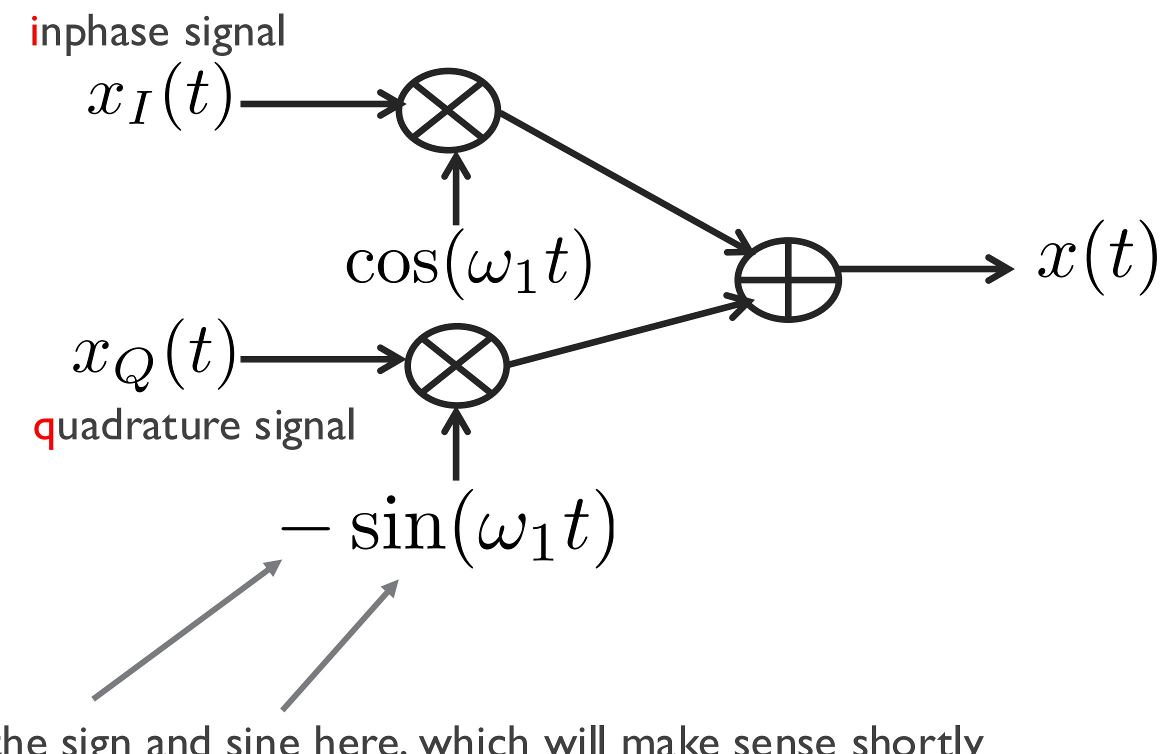
	Time domain	Fourier transform
Convolution in time	$y(t) = h(t) * x(t)$	$Y(j\omega) = H(j\omega)X(j\omega)$
Multiplication in time	$y(t) = h(t)x(t)$	$\underbrace{Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta}$

windowing

Practical application – Inphase and quadrature

- What if two information signals are sent as follows?





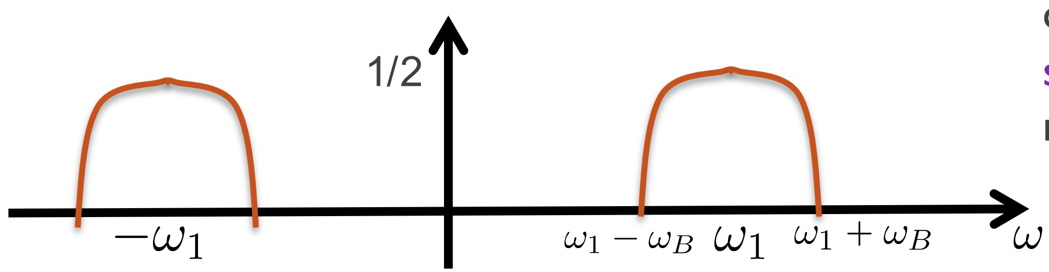
Practical application – Inphase and quadrature (cont.)

- What happens in the frequency domain?
  - Inphase term

$$\mathcal{F}\left\{x_I(t)\cos(\omega_1 t)\right\} = \frac{1}{2}X_I(j(\omega-\omega_1)) + \frac{1}{2}X_I(j(\omega+\omega_1))$$

  - Quadrature term

$$\mathcal{F}\left\{-x_Q(t)\sin(\omega_1 t)\right\} = \frac{j}{2}X_Q(j(\omega-\omega_1)) - \frac{j}{2}X_Q(j(\omega+\omega_1))$$



mixture of inphase and quadrature terms but not the same mixture at positive and negative frequencies

Practical application – Inphase and quadrature (cont.)

- What about demodulation?
  - Trig identities

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u \cos v = \frac{1}{2}[\sin(u-v) + \sin(u+v)]$$

Can recover both inphase and quadrature!

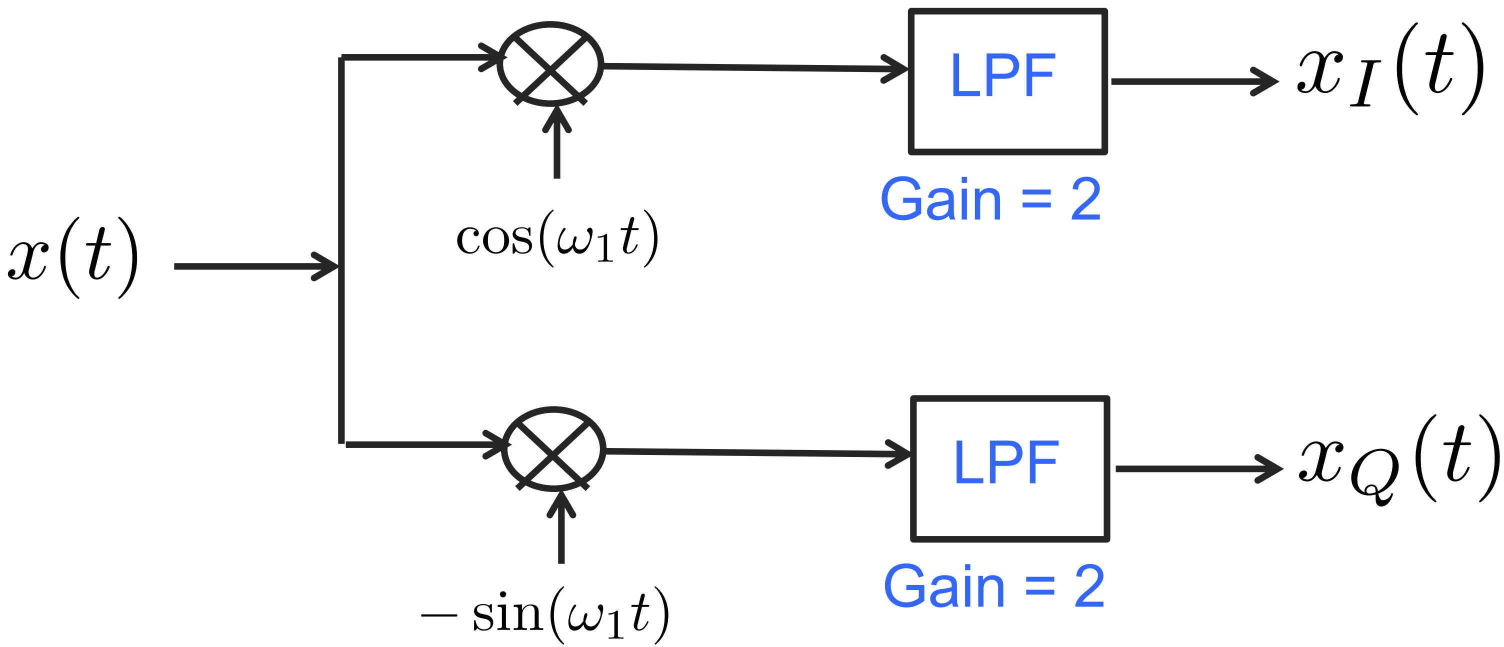
  - Applying the identities

filter out

$$x(t)\cos(\omega_1 t) = \frac{1}{2}x_I(t) + \boxed{\frac{1}{2}x_I(t)\cos(2\omega_1 t) - \frac{1}{2}x_Q(t)\sin(2\omega_1 t)}$$
$$x(t)\sin(\omega_1 t) = -\frac{1}{2}x_Q(t) + \boxed{\frac{1}{2}x_Q(t)\cos(2\omega_1 t) + \frac{1}{2}x_I(t)\sin(2\omega_1 t)}.$$

Practical application – Inphase and quadrature (cont.)

- IQ demodulator



Practical application – Inphase and quadrature (cont.)

- Why do we use complex signals?

This is called the complex baseband signal

$$x_{bb}(t) = x_I(t) + j x_Q(t)$$

$$\mathrm{Re}\{x_{bb}(t)\} \qquad \mathrm{Im}\{x_{bb}(t)\}$$

Complex signals become a convenient way to work with inphase and quadrature together, avoiding the need for matrix notation

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 16 Fourier transform

Fourier

Convolution property

Key points

- Convolution in time is multiplication in frequency
- Use this fact to compute convolutions

Convolution property

- If $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega) \quad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$
- Then

$$\boxed{y(t) = h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)}$$

Convolution in time is multiplication in frequency

Proof of the convolution property

$$Y(j\omega) = \mathcal{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\}$$

~~analysis~~

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t}dt$$

Apply definition

→ t

$$= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dtd\tau$$

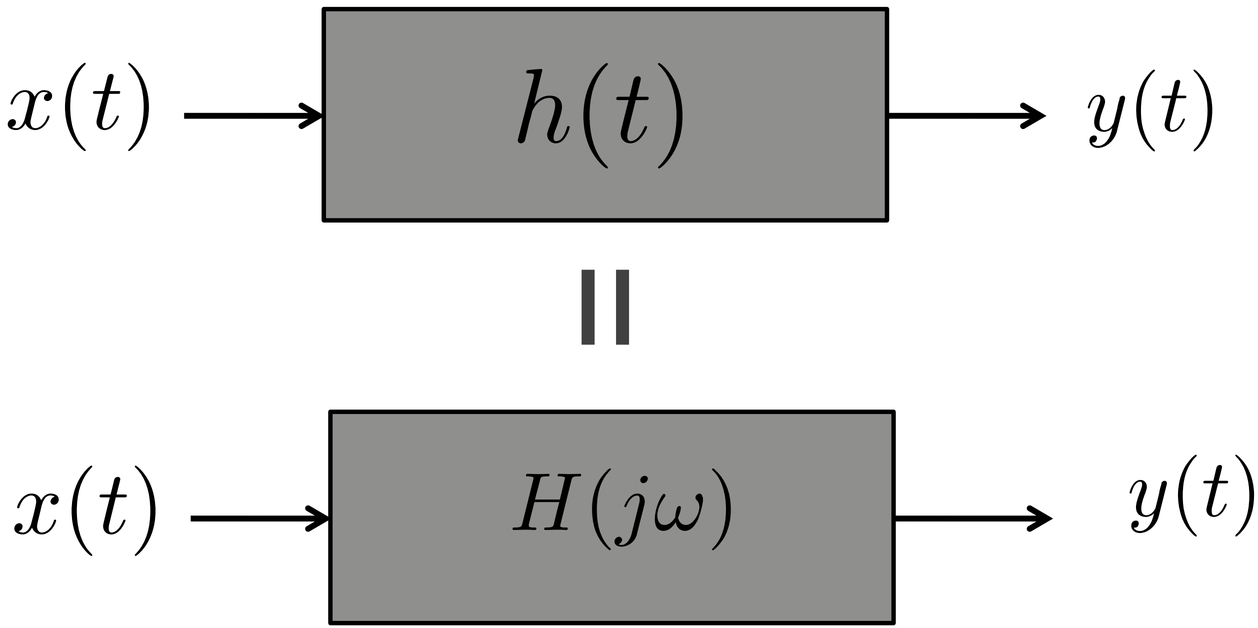
Exchange order of integration

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau$$

Time shift property

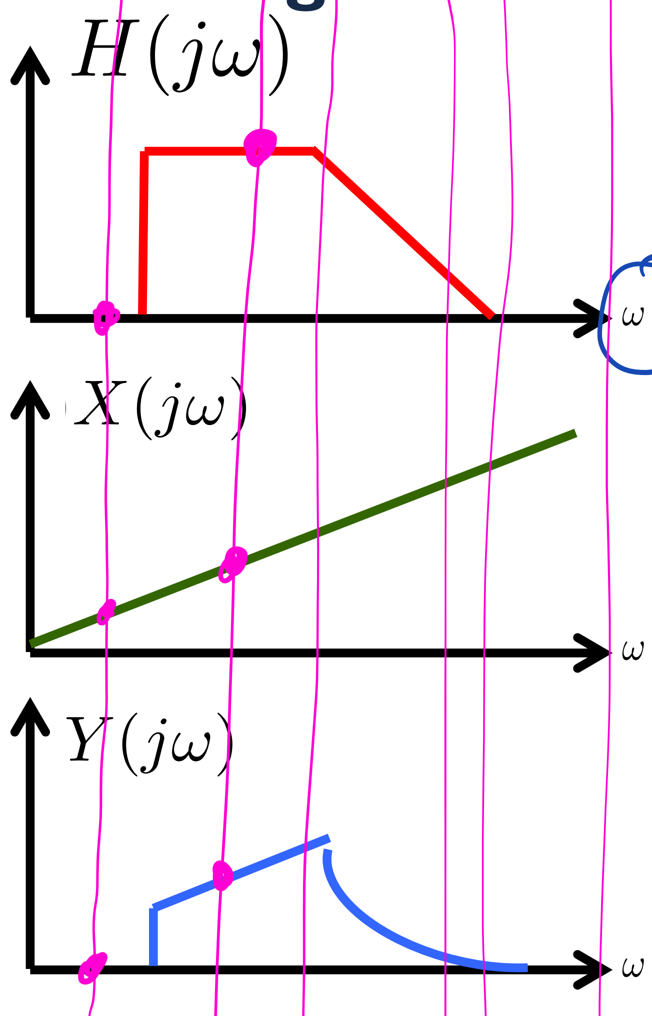
$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \quad = H(j\omega)X(j\omega)$$

Block diagrams



From a notational perspective, an LTI system may be described by the impulse response in the time or frequency domains

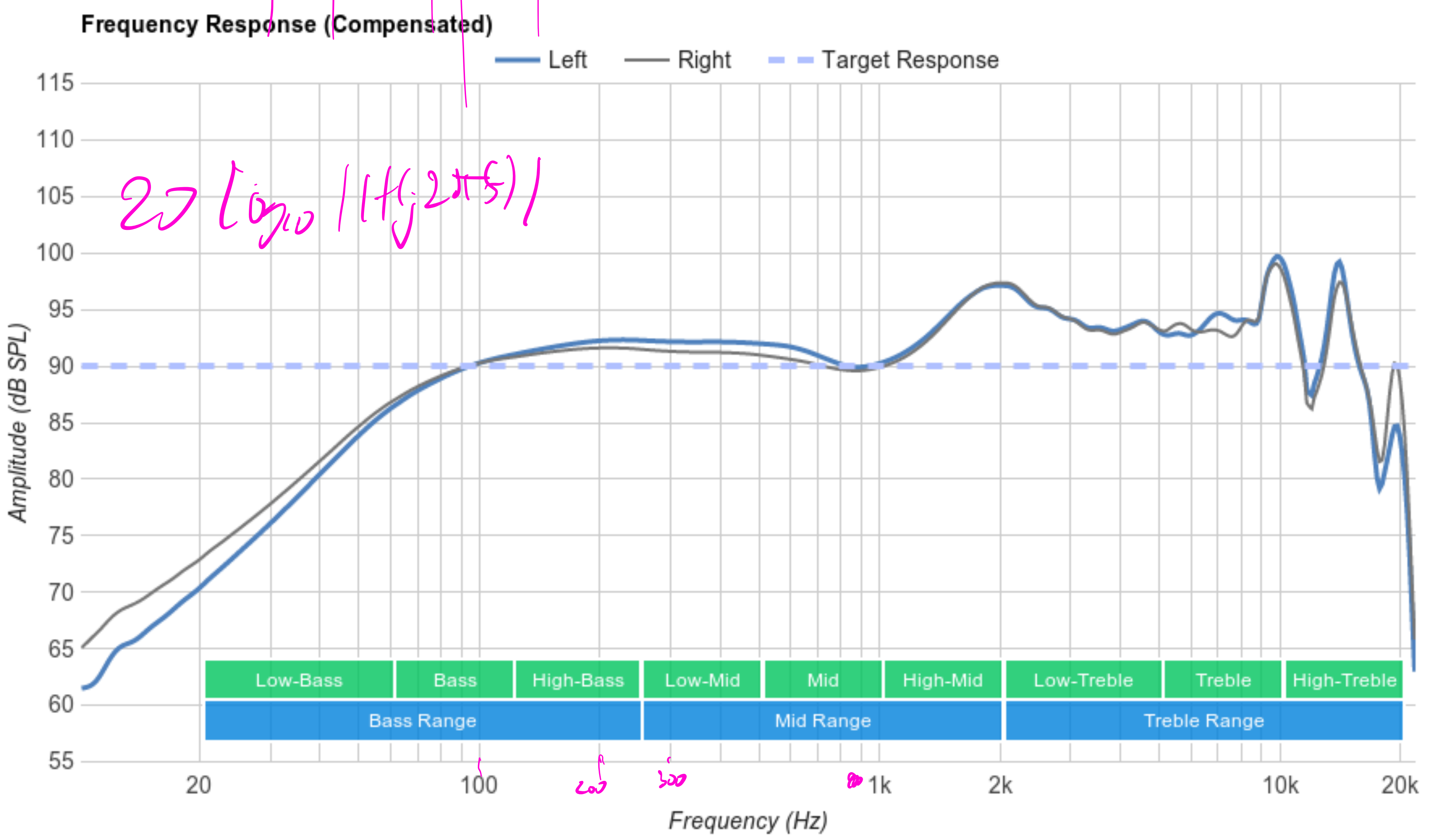
Visualizing the convolution property

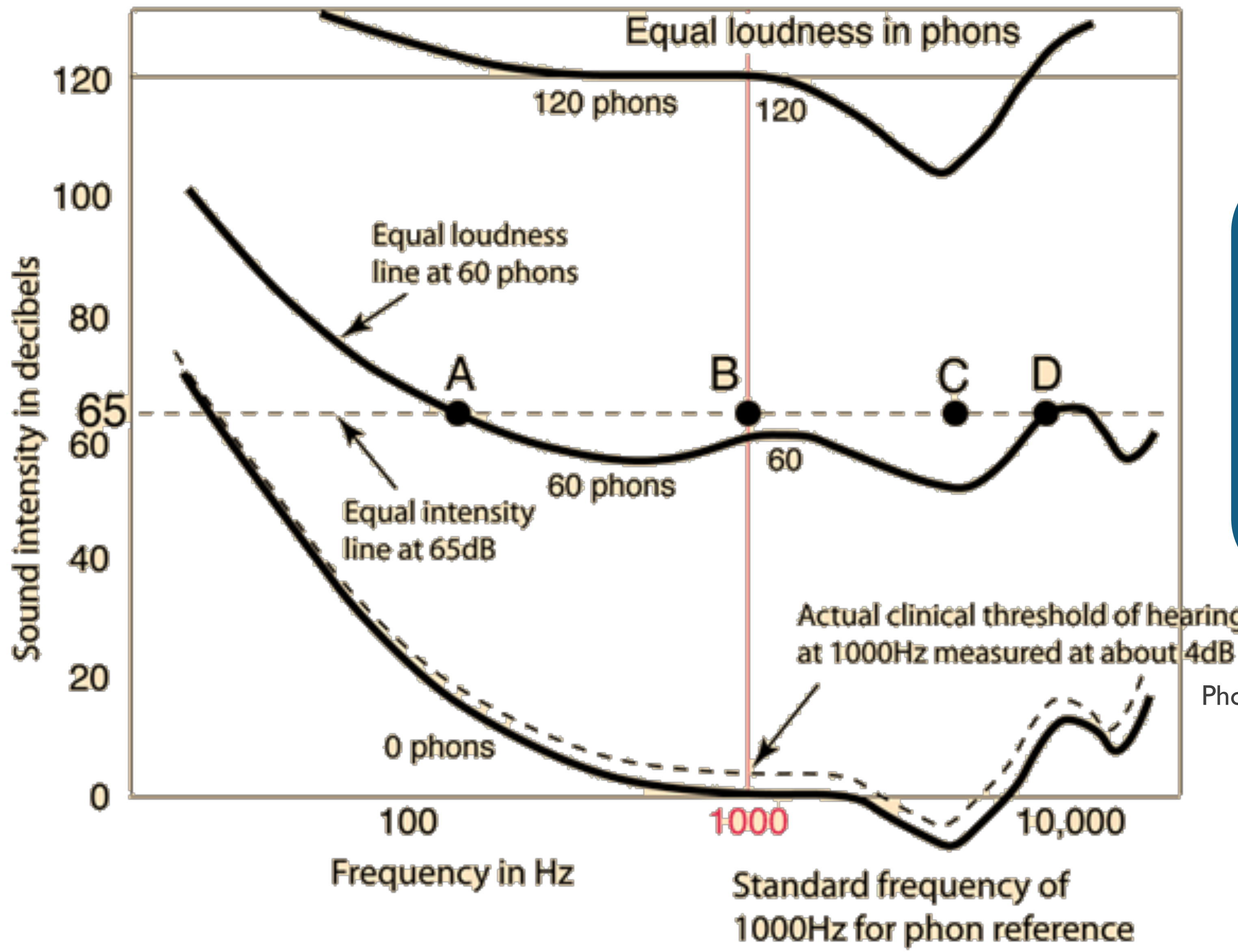


$$Y(j\omega) = H(j\omega)X(j\omega)$$

frequency

Direct multiplication at each frequency


Frequency Response (Compensated)
Left
Right
Target Response
115
110
105
100
95
90
85
80
75
70
65
60
55
Amplitude (dB SPL)
20
100
1k
2k
10k
20k
Frequency (Hz)
Low-Bass
Bass
High-Bass
Low-Mid
Mid
High-Mid
Low-Treble
Treble
High-Treble
Bass Range
Mid Range
Treble Range
$20\log_{10}|H(j2\pi f)|$
200
300



Perceptual response to sound intensity is not uniform in frequency

Phon is sound referenced to 1kHz

http://hyperphysics.phy-astr.gsu.edu/hbase/Sound/earcrv.html

Using the convolution property to do convolutions

- Compute the following convolution

$$y(t) = h(t) * x(t)$$

- Convert the two signals into the frequency domain

$$H(j\omega) = \mathcal{F}\{h(t)\}$$
$$X(j\omega) = \mathcal{F}\{x(t)\}$$

- Compute the product

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Go from frequency domain back into the time domain

$$y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$$

Double sinc example

- Given where $\omega_i > 0$ and $\omega_c > 0$

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \qquad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

- Find

$$y(t) = h(t) * x(t)$$

Double sinc example (continued)

- Solve by going into the frequency domain
- First find

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Need to compute

$$\mathcal{F}\left\{\frac{\sin(\omega_i t)}{\pi t}\right\}\mathcal{F}\left\{\frac{\sin(\omega_c t)}{\pi t}\right\}$$

- But note that

$$\text{sinc}\left(\frac{t}{2\pi}\right) = \frac{\sin\left(t/2\right)}{t/2} \quad \text{and} \quad \text{sinc}\left(\frac{t}{2\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\text{rect}(\omega)$$

Double sinc example (continued)

- Using the scaling property

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- Write

LPF w/ ω_i

$$\frac{\sin(\omega_i t)}{\pi t} = \frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}$$

- It follows that

$\mathrm{rect}(\omega/2\omega_i)$

$$\mathcal{F}\left\{\frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}\right\} = 2\pi \frac{\omega_i}{\pi} \frac{1}{|2\omega_i|} \mathrm{rect}(\omega/2\omega_i)$$

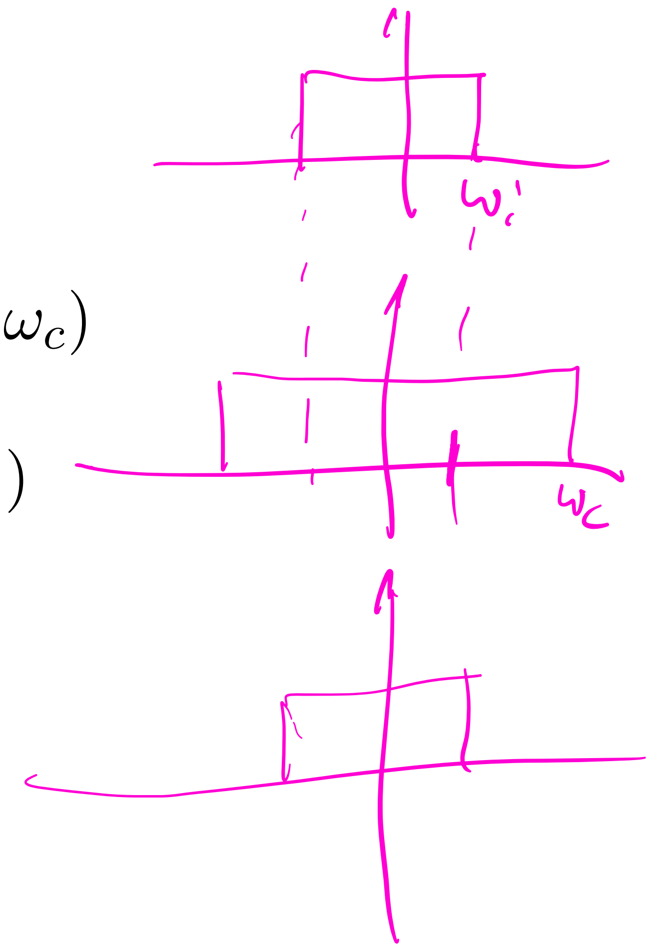
$$= \mathrm{rect}(\omega/2\omega_i)$$

Double sinc example (continued)

- The convolution is then

$$Y(j\omega) = \mathrm{rect}(\omega/2\omega_i)\mathrm{rect}(\omega/2\omega_c)$$

$$= \mathrm{rect}(\omega/2\min(\omega_c, \omega_i))$$



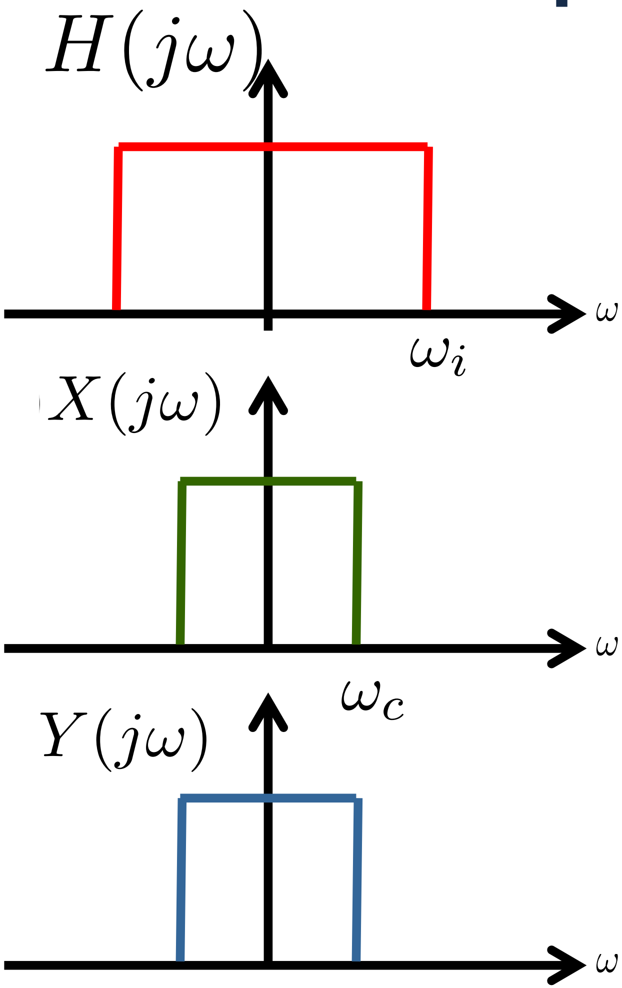
- Back in the time domain

$$y(t) = \frac{\sin(\min(\omega_i, \omega_c)t)}{\pi t}$$

This is a general result that sinc convolved with sinc gives sinc

Double sinc example (concluded)

Visualizing the effect in the frequency domain



$$Y(j\omega) = H(j\omega)X(j\omega)$$

Example where ω_i is bigger than ω_c

Summarizing the convolution property

- Convolution between two signals in time becomes the product of the Fourier transforms of those signals in the frequency domain

- Convolutions are easy to do in the frequency domain as they involve a simple point-wise multiplication

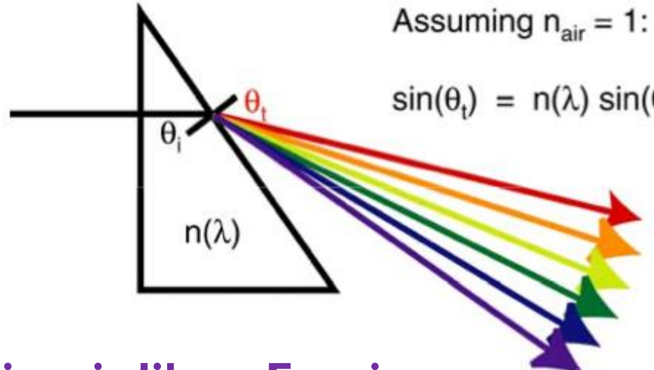
- The convolution property explains how the frequency response of a system directly effects the frequencies of the input signal to create the output signal

Fourier in practice

Key points

- Fourier concepts show up everywhere

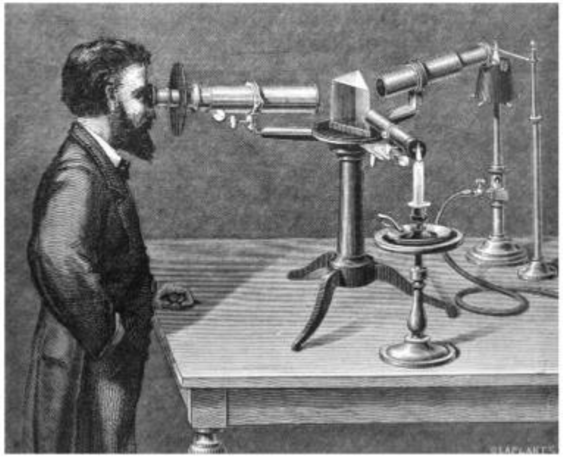
Spectrometer



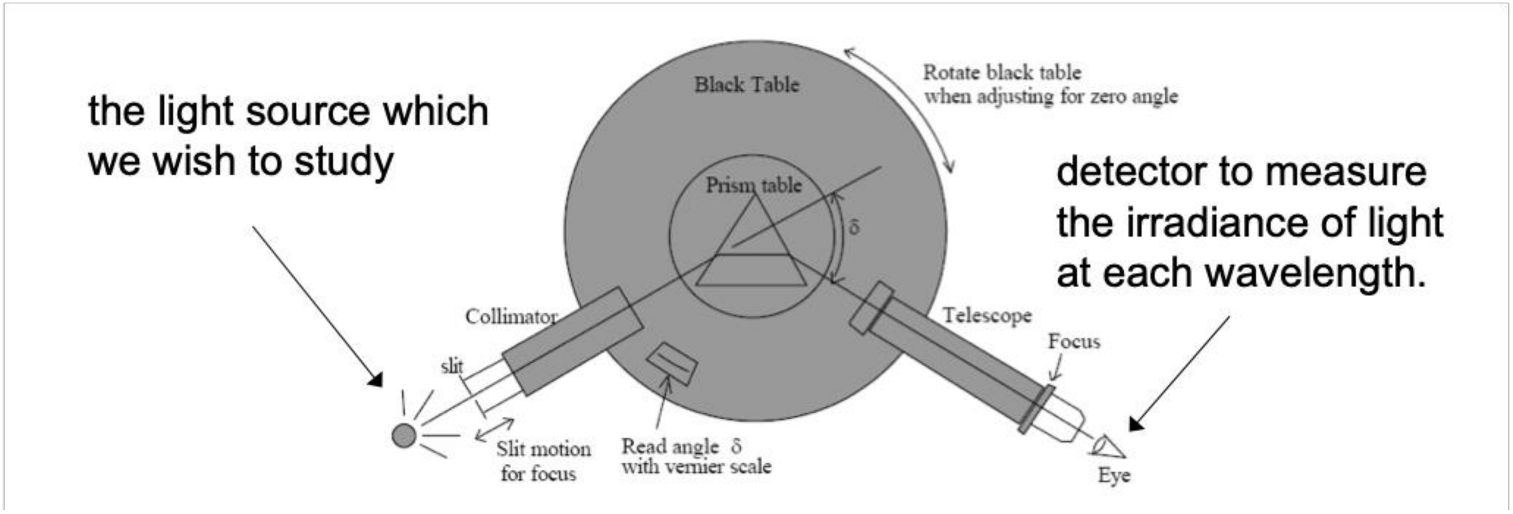
Assuming $n_{air} = 1$:

$$\sin(\theta_t) = n(\lambda) \sin(\theta_i)$$

Prism is like a Fourier operator

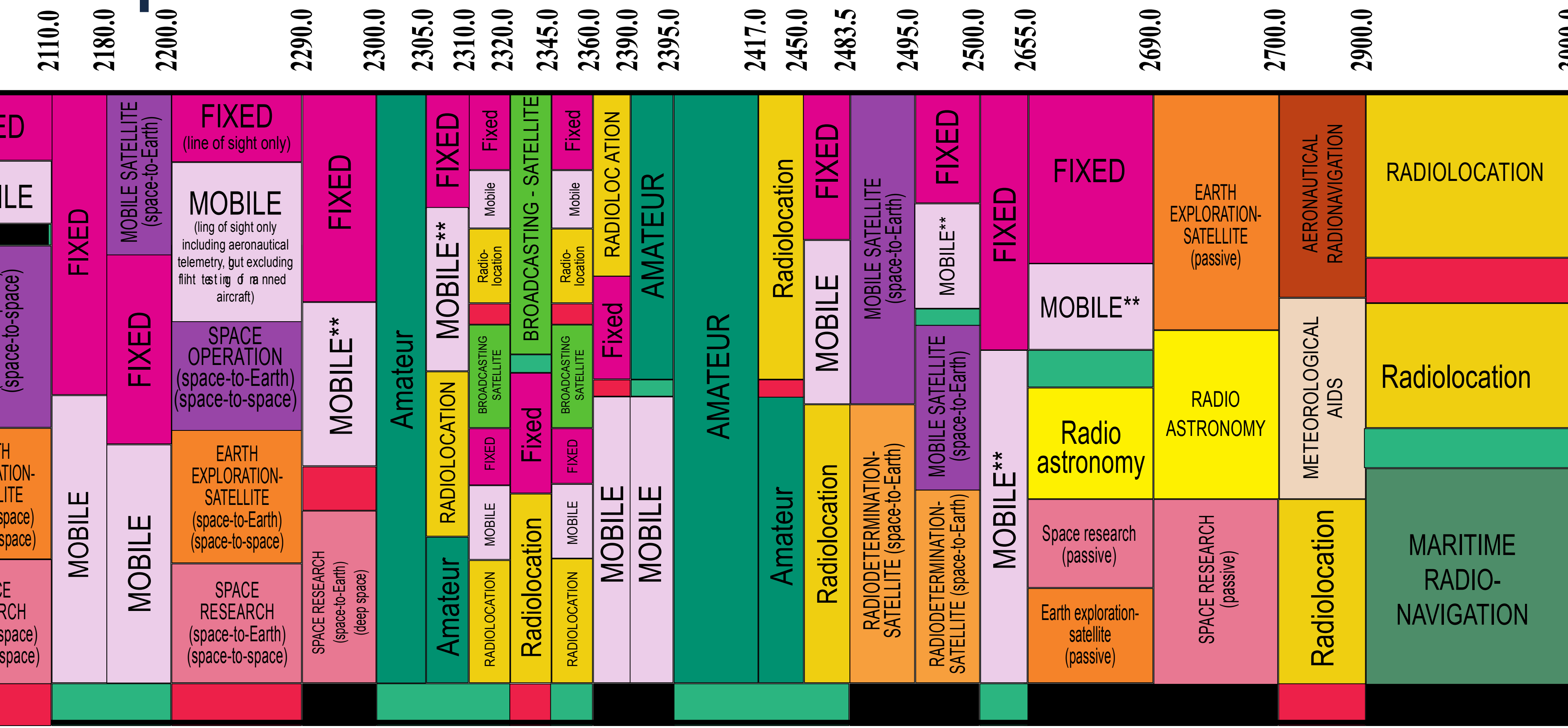


Robert Bunsen, 1859



Figures from https://www.brown.edu/research/labs/mittleman/sites/brown.edu.research.labs.mittleman/files/uploads/lecture19_0.pdf

Spectrum allocation



ISM – 2450.0± .50 MHz

3 GHz

https://www.ntia.doc.gov/files/ntia/publications/january_2016_spectrum_wall_chart.pdf

Communications

Spectrum for in-band signal

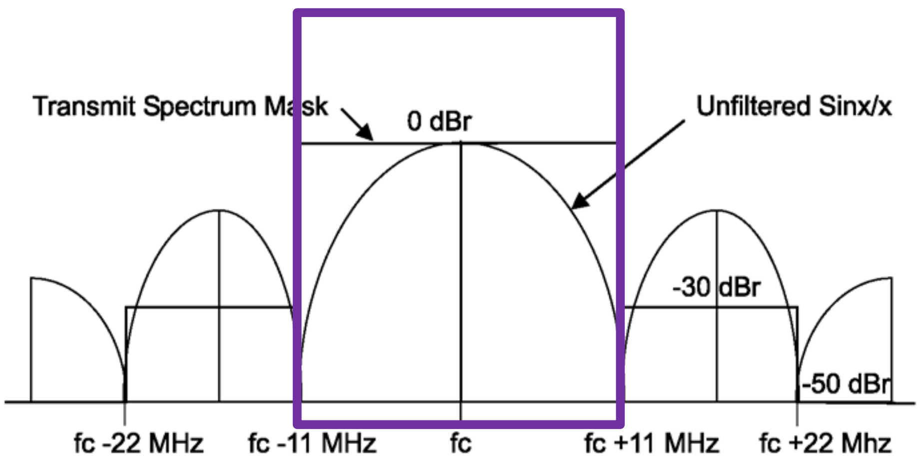
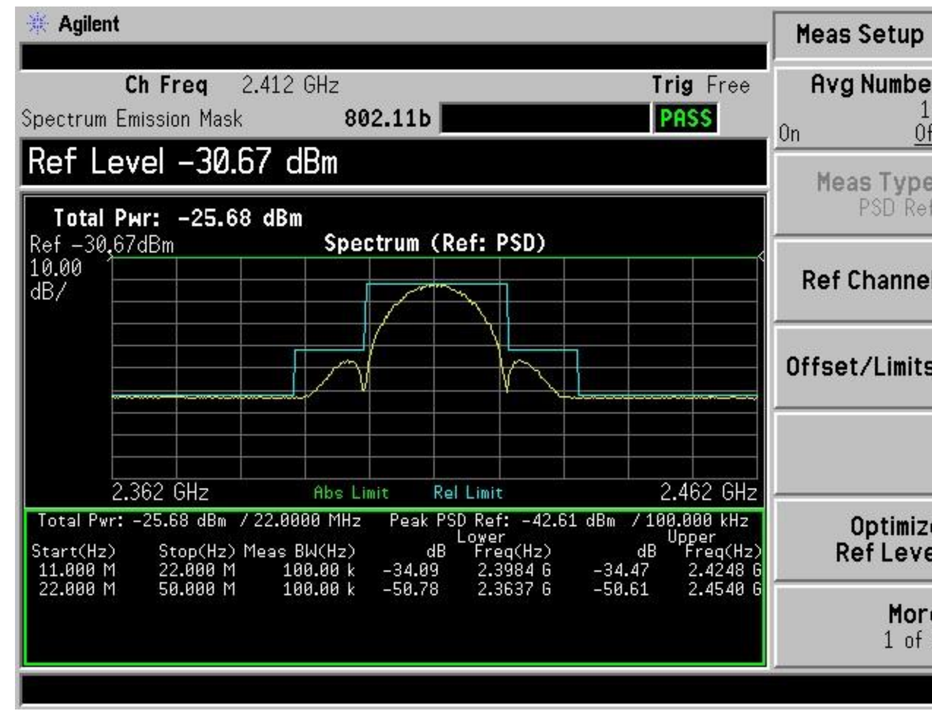


Figure 15-10—Transmit spectrum mask



Allowed out-of-band leakage

Transmit spectrum mask from IEEE 802.11-2016, 15.4.5.5 WiFi!
https://www.keysight.com/us/en/lib/resources/user-manuals/transmit-spectrum-mask-332766.html

Multiplication property

Key points

- Multiplication in time is convolution in frequency
- Use this fact to explain windowing

Multiplication property

- If $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega) \quad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

- Then

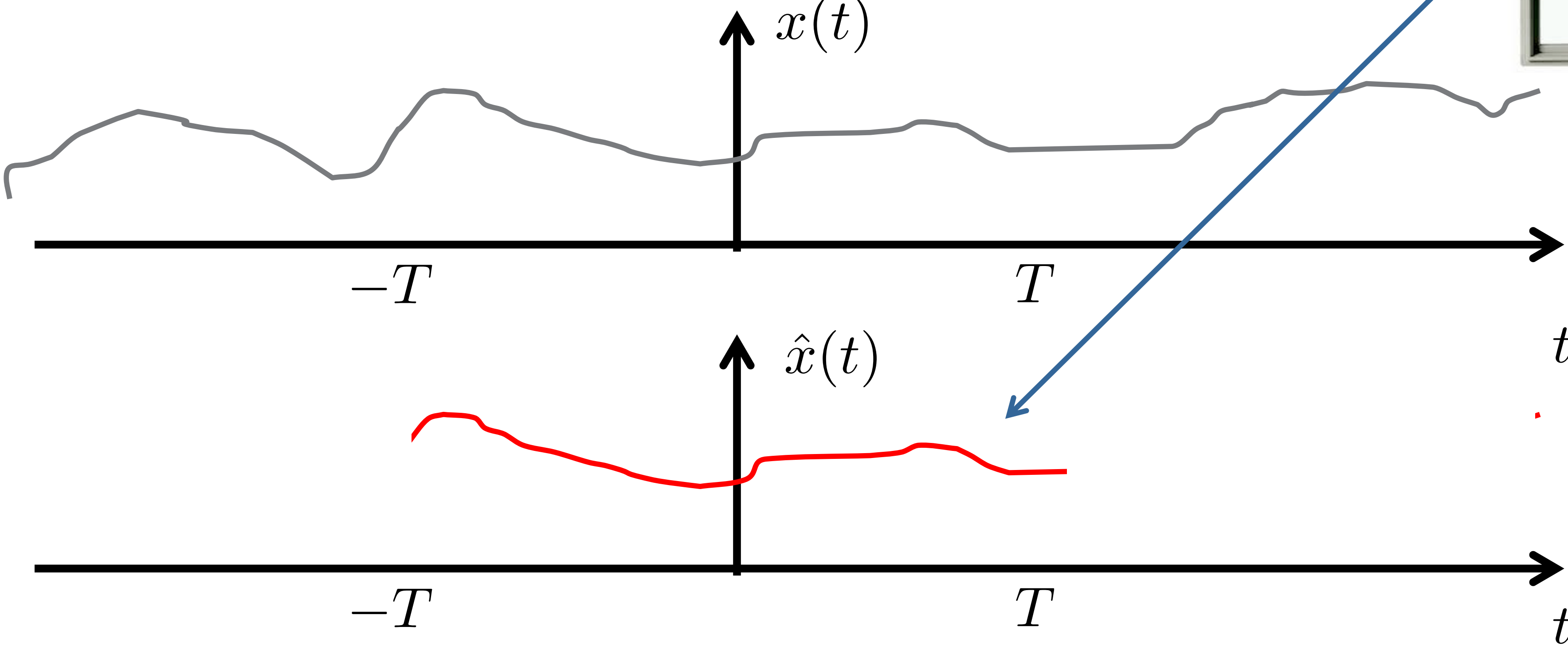
$$\boxed{y(t) = h(t)x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta}$$

Product in time is convolution in frequency

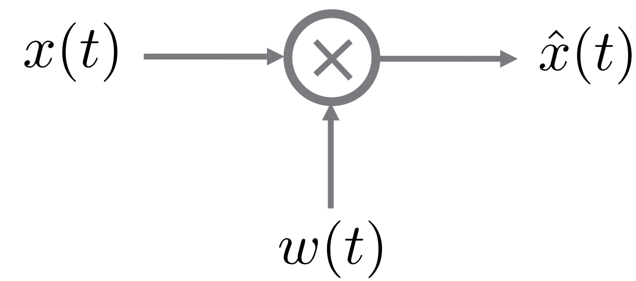
Implication of product property

- Suppose that you have a signal x(t)
  - But you only measure x(t) from –T….T
  - What happens?

Have only a window of the data



Windowing the spectrum



- The observed signal can be written as

hat

$$\hat{x}(t) = \underbrace{\mathrm{rect}(t/(2T))}_{w(t)} x(t)$$

- In the frequency domain

$$\hat{X}(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2T\mathrm{sinc}\left(\frac{2T\theta}{2\pi}\right) X(j(\omega-\theta))d\theta$$

Spectrum is filtered by the sinc function

Example – windowing a cosine

$$x(t) = \cos(\omega_c t) \longrightarrow \bigotimes \longrightarrow \hat{x}(t)$$

$$w(t) = \mathrm{rect}(t/(2T))$$

$$X(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

$$W(j\omega) = 2T\mathrm{sinc}\left(\frac{2T\omega}{2\pi}\right)$$

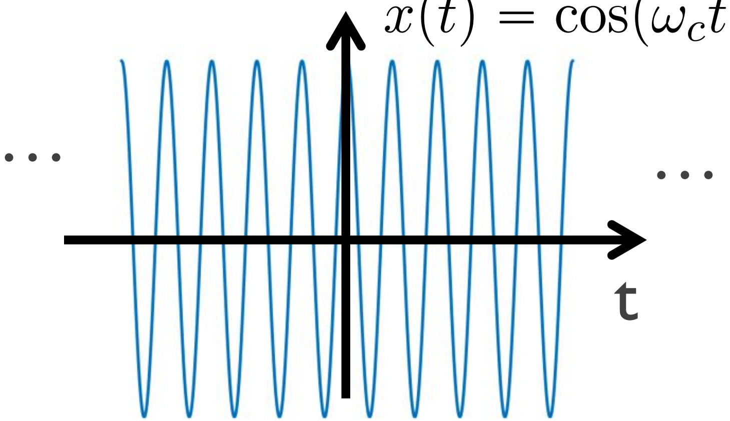
$$\hat{X}(j\omega) = \frac{T}{\pi}\mathrm{sinc}\left(\frac{2T(\omega - \omega_c)}{2\pi}\right) + \frac{T}{\pi}\mathrm{sinc}\left(\frac{2T(\omega + \omega_c)}{2\pi}\right)$$

Impulses get smeared due to windowing

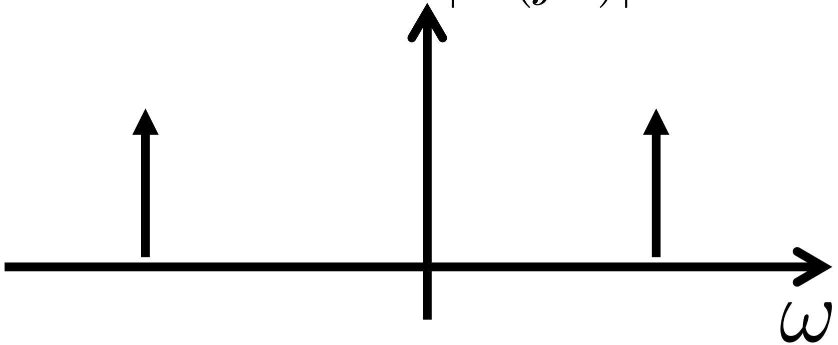
Intuition on windowing

$$x(t) = \cos(\omega_c t)$$

cosine

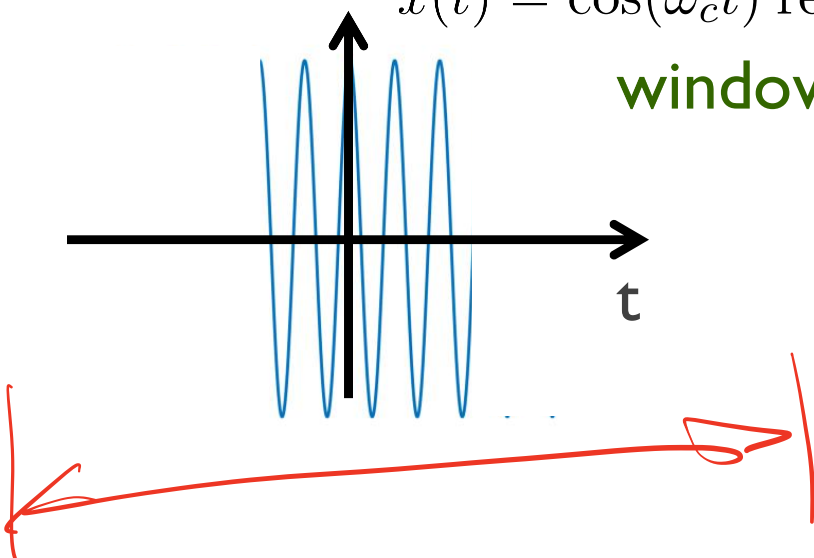


$$|X(j\omega)|^2$$

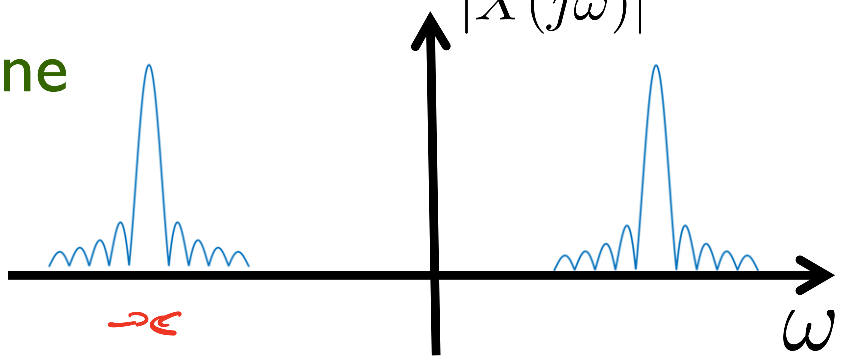


$$\hat{x}(t) = \cos(\omega_c t)\,\mathrm{rect}(t/2T)$$

windowed cosine



$$|\hat{X}(j\omega)|^2$$



Impact of windowing on resolution

- Suppose that we window a sum of two cosines

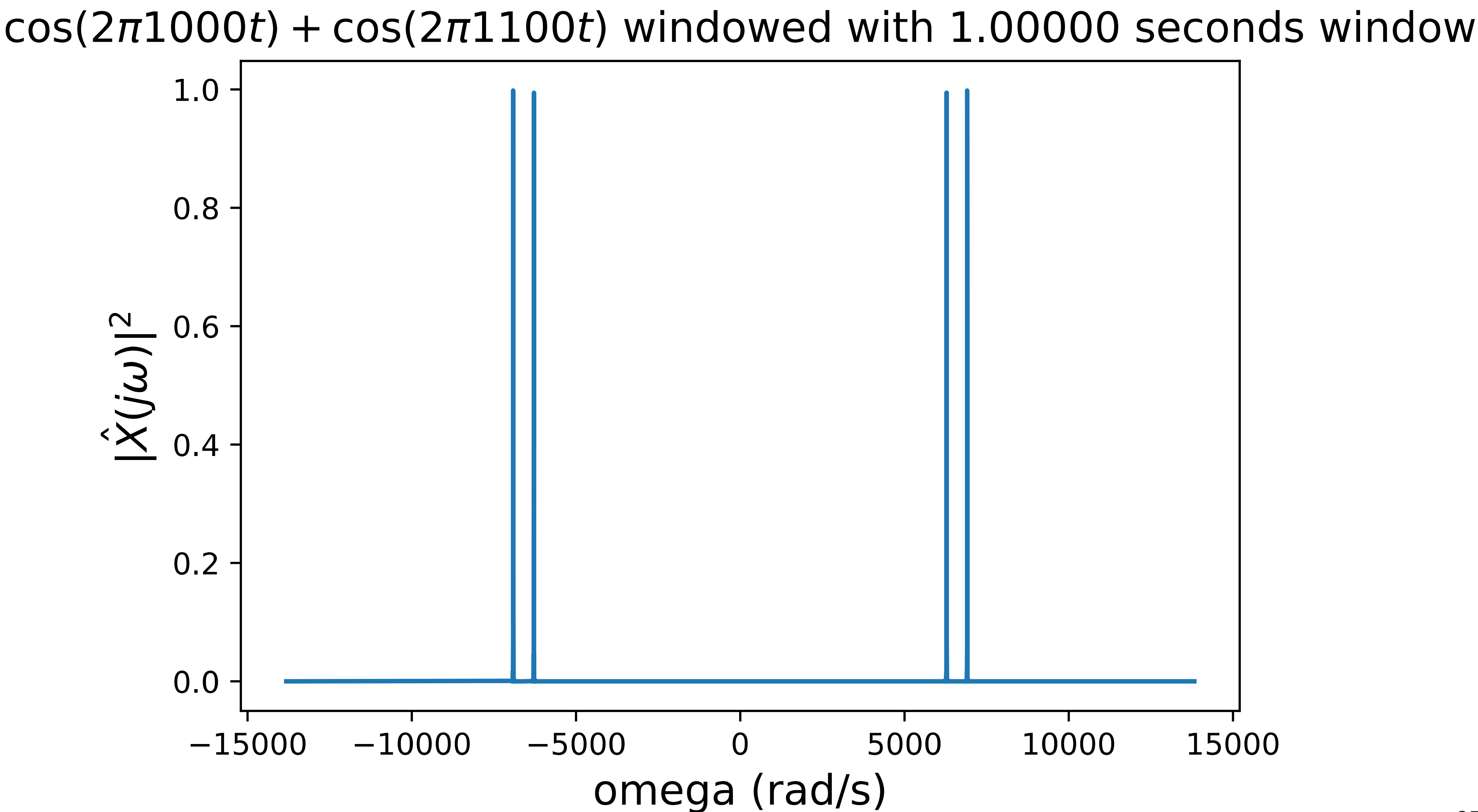
$$\cos(2\pi 1000 t) + \frac{1}{2}\cos(2\pi 1100 t) \longrightarrow \otimes \longrightarrow \hat{x}(t)$$

$$\mathrm{rect}(t/(2T))$$

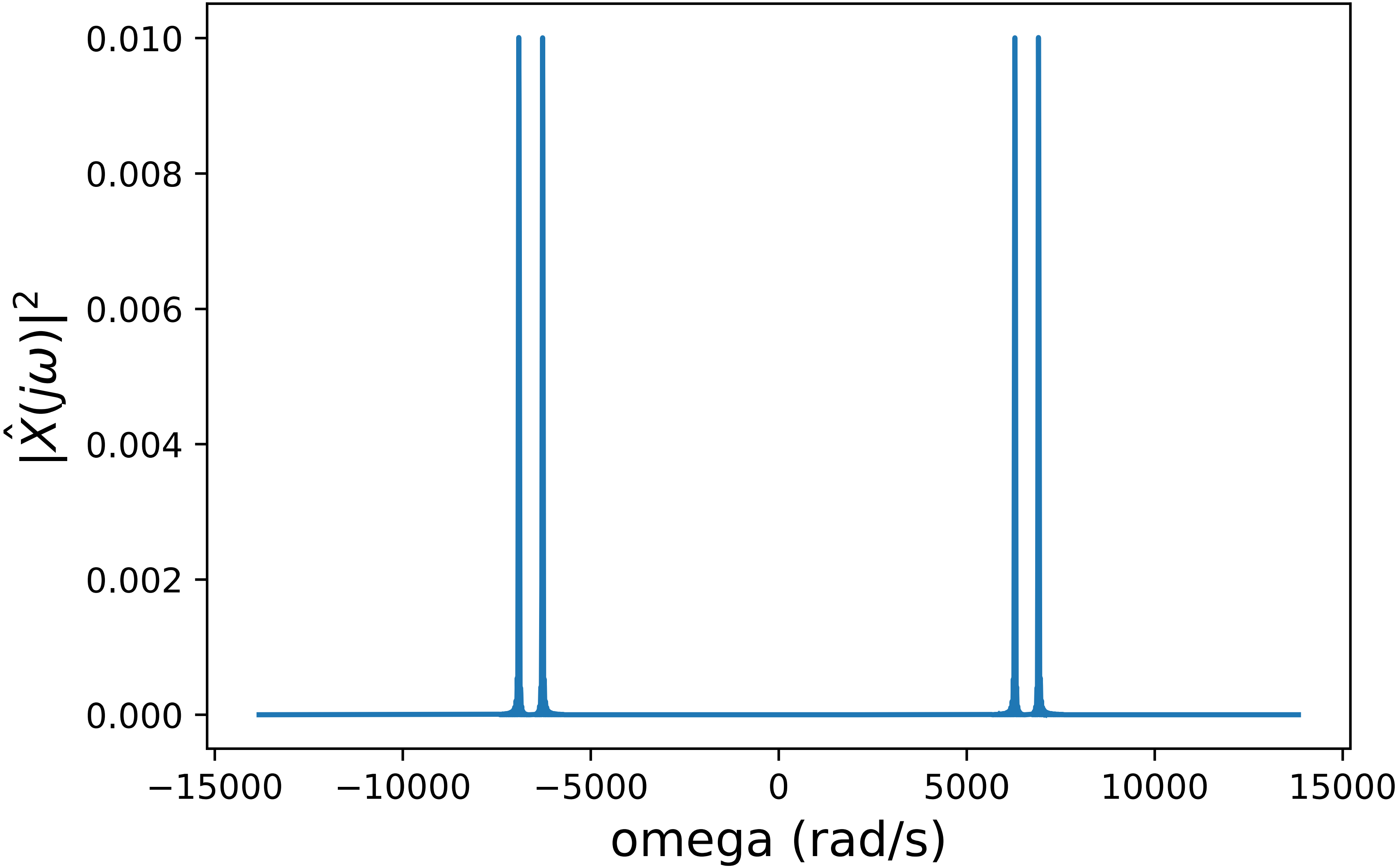
1000 Hz or 6,280 rad/s

1100 Hz or 6,911 rad/s

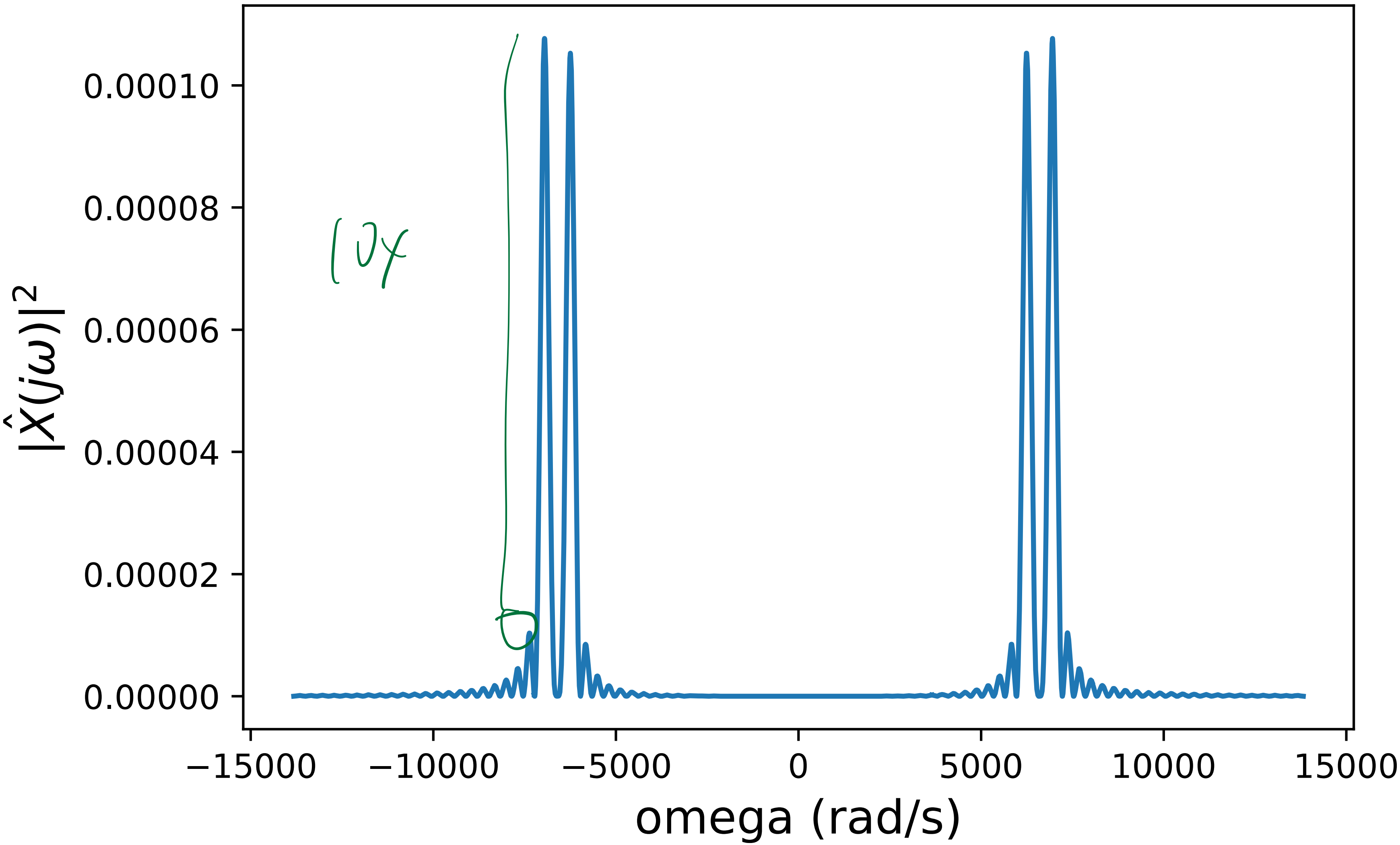
What is the impact of the window size T on the ability to resolve the sinusoids?

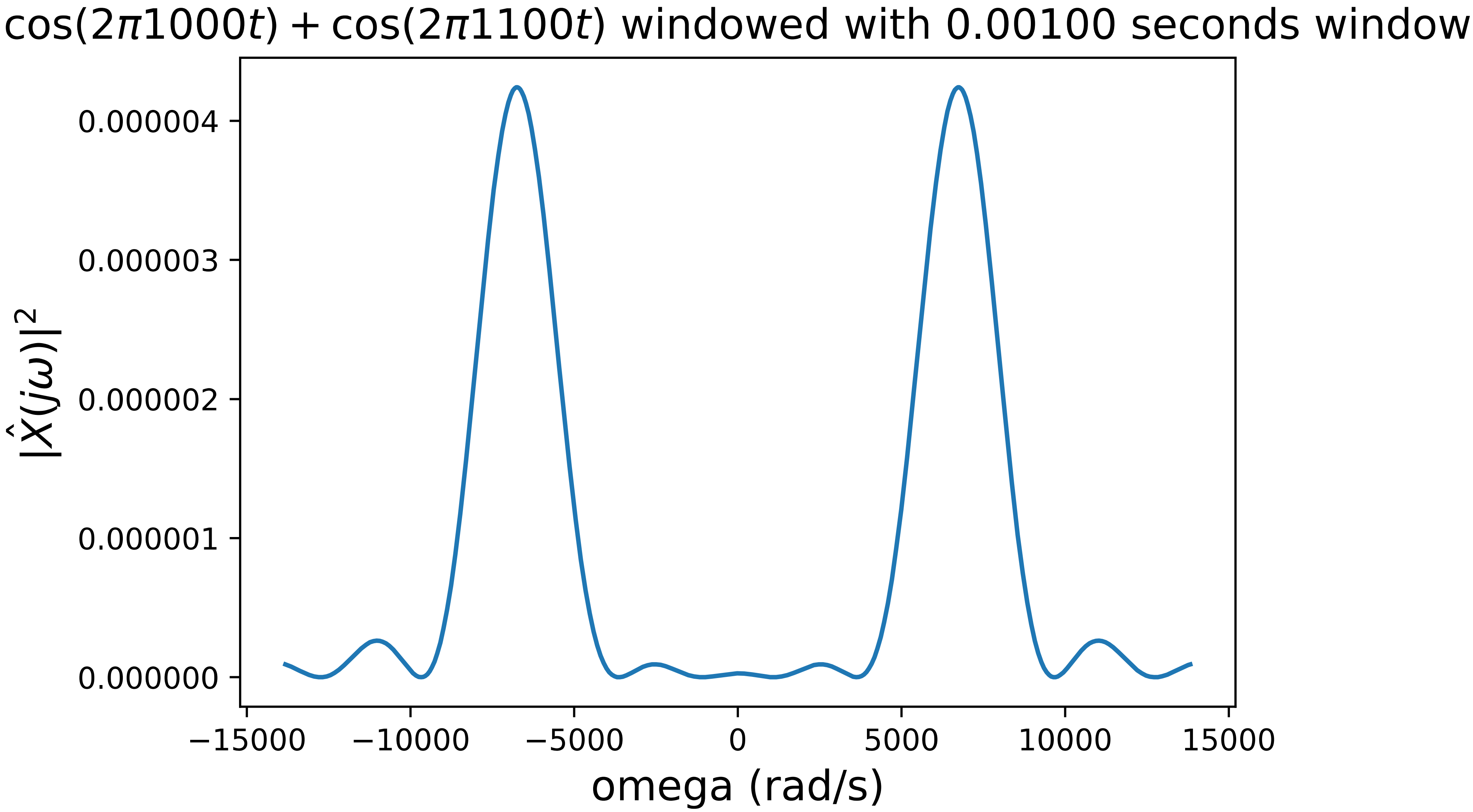

cos(2π1000t) + cos(2π1100t) windowed with 1.00000 seconds window
1.0
0.8
0.6
0.4
0.2
0.0
|X̂(jω)|²
−15000
−10000
−5000
0
5000
10000
15000
omega (rad/s)

$\cos(2\pi 1000 t) + \cos(2\pi 1100 t)$ windowed with 0.10000 seconds window

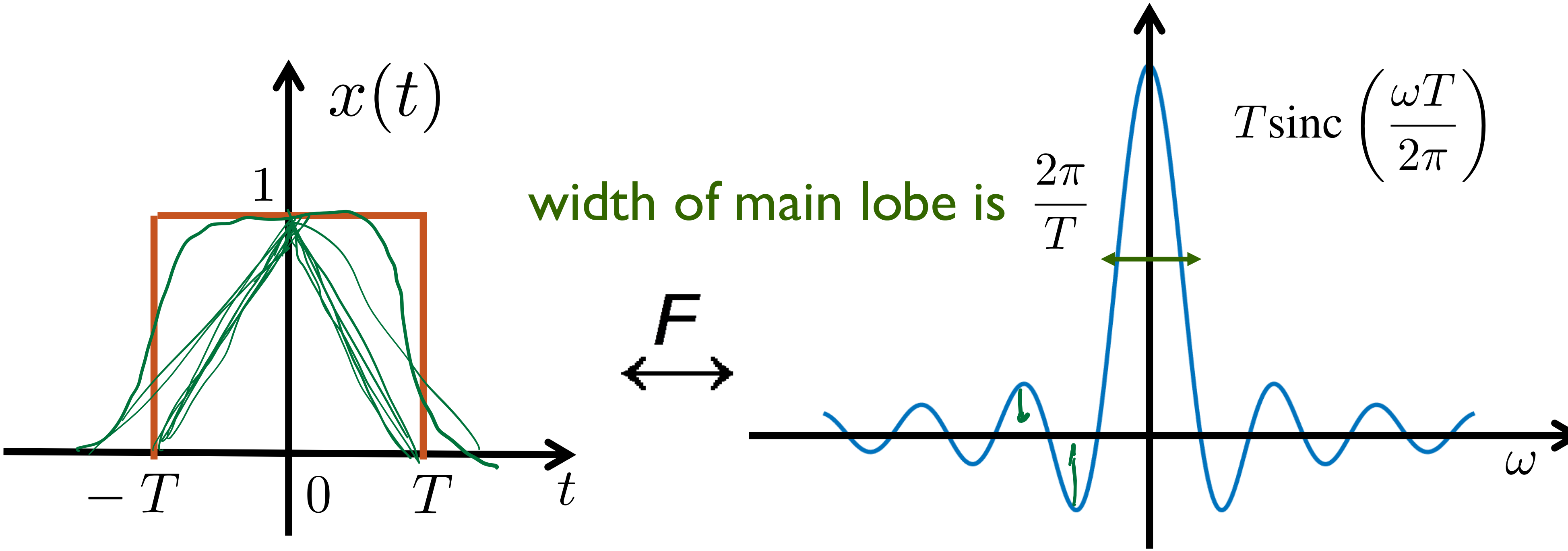


$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 0.01000 seconds window




$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 0.00100 seconds window
$|\hat{X}(j\omega)|^2$
0.000004
0.000003
0.000002
0.000001
0.000000
−15000
−10000
−5000
0
5000
10000
15000
omega (rad/s)

How much time is needed to resolve these cosines?



Zeros at $\pm\,\pi/T, \pm\pi/T, \ldots$

One main lobe of separation

$$\frac{2\pi}{T} = |\omega_1 - \omega_2| \quad \Rightarrow \quad T = \frac{2\pi}{|\omega_1 - \omega_2|} = \frac{2\pi}{2\pi 100} = 0.01\,\mathrm{s}$$

Summarizing the multiplication property

- Product between two signals in time becomes the (scaled) convolution of the Fourier transforms of those signals in the frequency domain

- Truncating a real signal for analysis, called windowing, leads to a distortion of the original signal's Fourier transform

- The ability to resolve different frequencies in a signal improves as the observation window grows longer

Bandwidth

Key points

- Finite duration signals have infinite bandwidth
- Different measures of bandwidth are used in practice

Isolation in time and frequency

- From the windowing theorem

sinc has infinite duration

$$x(t)\text{rect}(t/2T) \quad \Rightarrow \quad \frac{2T}{2\pi}\left[X(j\omega) * \text{sinc}(2T\omega/2\pi)\right]$$

Finite duration in time

Infinite duration in frequency

- From the convolution theorem

sinc has infinite duration

$$X(j\omega)\text{rect}(\omega/2B) \quad \Rightarrow \quad x(t) * 2B\text{sinc}(2Bt/2\pi)$$

Finite duration in frequency

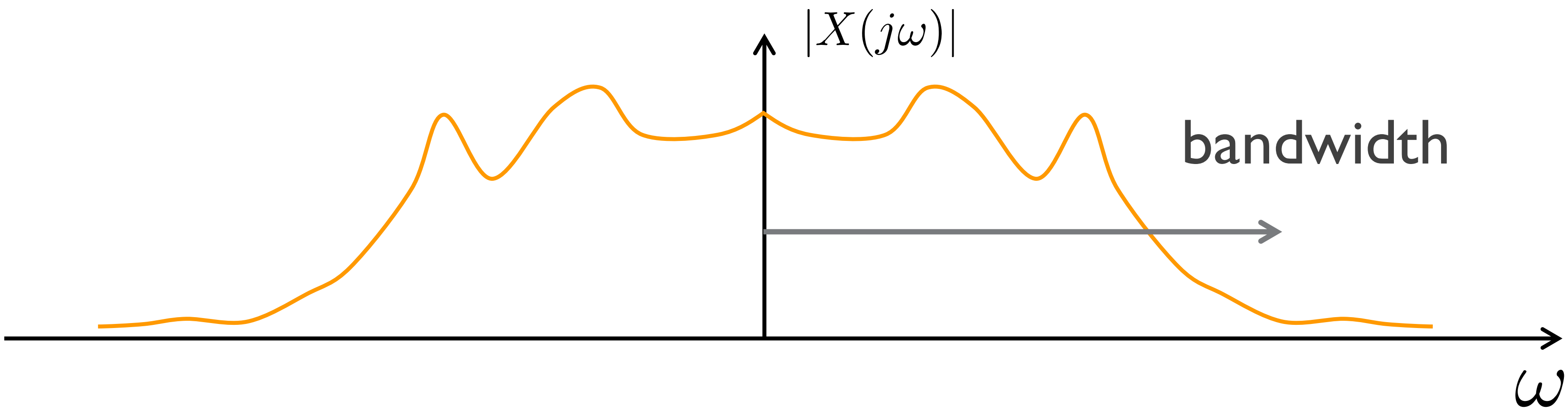
Infinite duration in time

Windowing and convolution have impact on the spectrum of practical signals (infinite) and the impulse response of an ideal low-pass filter (infinite)

Bandwidth of a practical signal

- If time duration is finite → bandwidth is infinite
  - For any practical signal, the absolute bandwidth is infinite

- Define a "bandwidth" to measure the extent of frequency content



Common definitions of bandwidth

- Fractional containment bandwidth
  - Bandwidth such that a fraction of energy is contained
  - Solve for ω_B such that

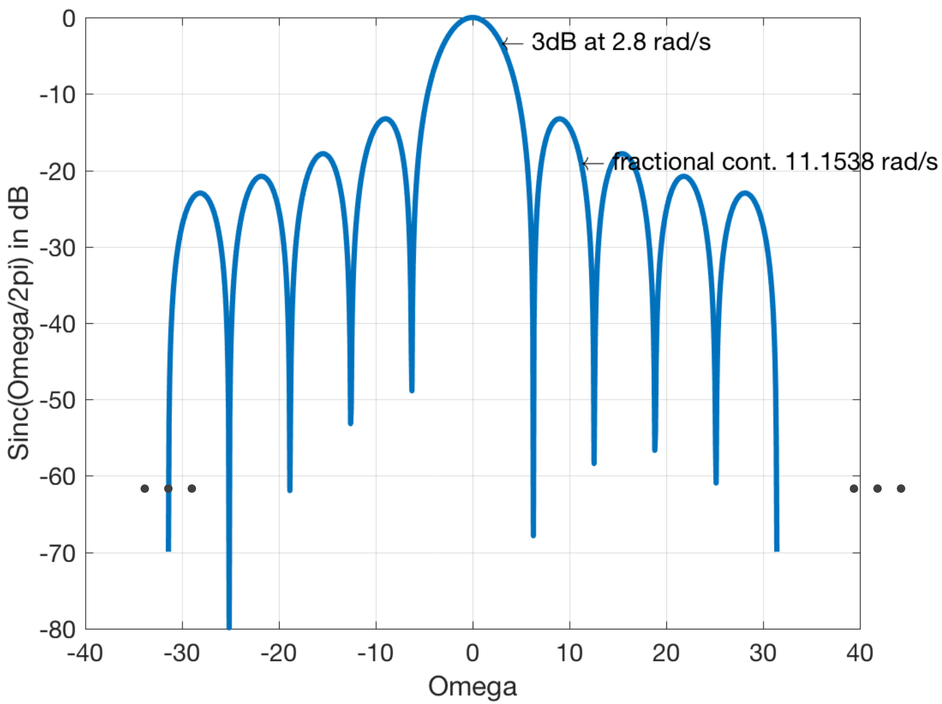
$$\int_{-\omega_B}^{\omega_B} |X(j\omega)|^2 d\omega \geqslant (1-\epsilon) \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- 3dB bandwidth (or half-power bandwidth)
  - Bandwidth where the signal achieves half the peak value
  - Makes the most sense with simple filters

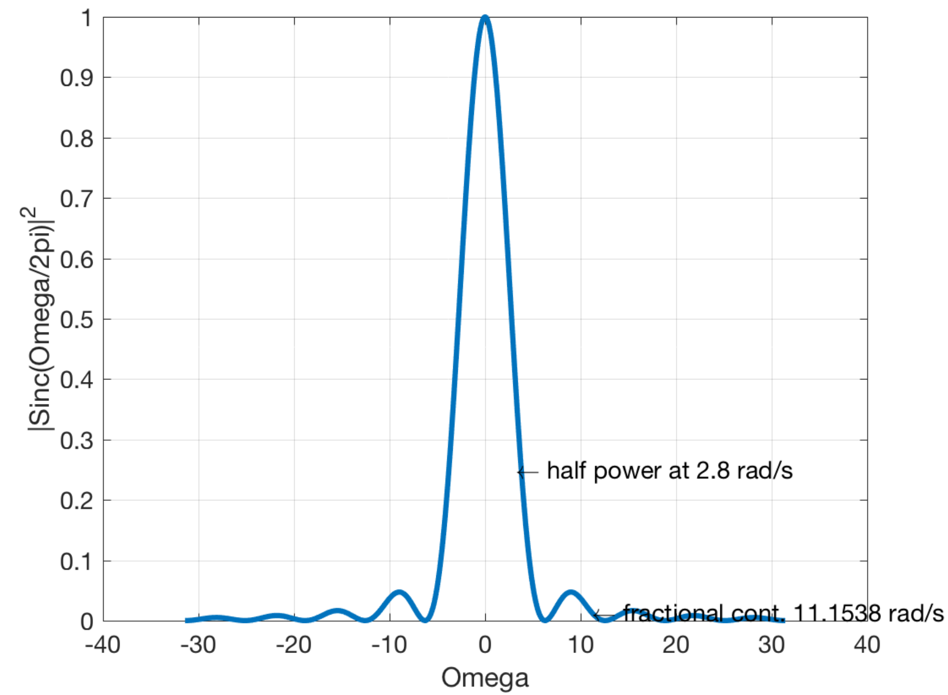
$$|X(j\omega_B)|^2 = \frac{1}{2} \max_{\omega} |X(j\omega)|^2$$

Examples of bandwidth

In dB



In linear scale (magnitude squared)



- Bandwidth of $\operatorname{sinc}(\omega/2\pi)$ with ½ power or 95% containment

Example fractional containment calculation

- Consider the following facts about Gaussian signals (proof of these facts is beyond the scope of this course)

1) Gaussian is its own Fourier transform $$e^{-t^2} \overset{\mathcal{F}}{\longleftrightarrow} \sqrt{\pi}e^{-\frac{\omega^2}{4}}$$

2) Integral of tail is "known" $$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt = Q(x) = \frac{1}{2}\mathrm{erfc}(x/\sqrt{2})$$

3) Unit area $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-t^2/2}dt = 1$$

Gaussian distribution is a big part of probability and statistics

Example fractional containment calculation (cont.)

- Consider signal $x(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$
- Find an expression for the fractional containment bandwidth

$$\int_{-\omega_B}^{\omega_B} |X(j\omega)|^2 d\omega \geqslant (1-\epsilon) \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- Note that

$$|X(j\omega)| = e^{-\frac{\omega^2}{4}}$$

$$|X(j\omega)|^2 = e^{-\frac{\omega^2}{2}}$$

Example fractional containment calculation (cont.)

$$\int_{-\omega_B}^{\omega_B} e^{-\frac{\omega^2}{2}} d\omega = (1-\epsilon) \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega$$

- For the RHS note that

$$\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega = 1$$

- For the LHS

$$\begin{aligned}
\int_{-\omega_B}^{\omega_B} e^{-\frac{\omega^2}{2}} d\omega &= \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \int_{\omega_B}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \int_{-\infty}^{-\omega_B} e^{-\frac{\omega^2}{2}} d\omega \\
&= \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - 2\int_{\omega_B}^{\infty} e^{-\frac{\omega^2}{2}} d\omega \\
&= 1 - 2Q(\omega_B)
\end{aligned}$$

Example fractional containment calculation (cont.)

- Simplifying, we need to solve

$$1 - 2Q(\omega_B) = 1 - \epsilon$$

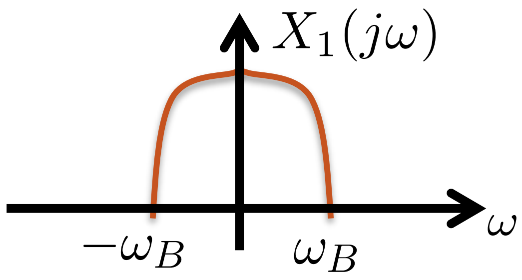
- Rearranging terms

$$Q(\omega_B) = \epsilon/2$$

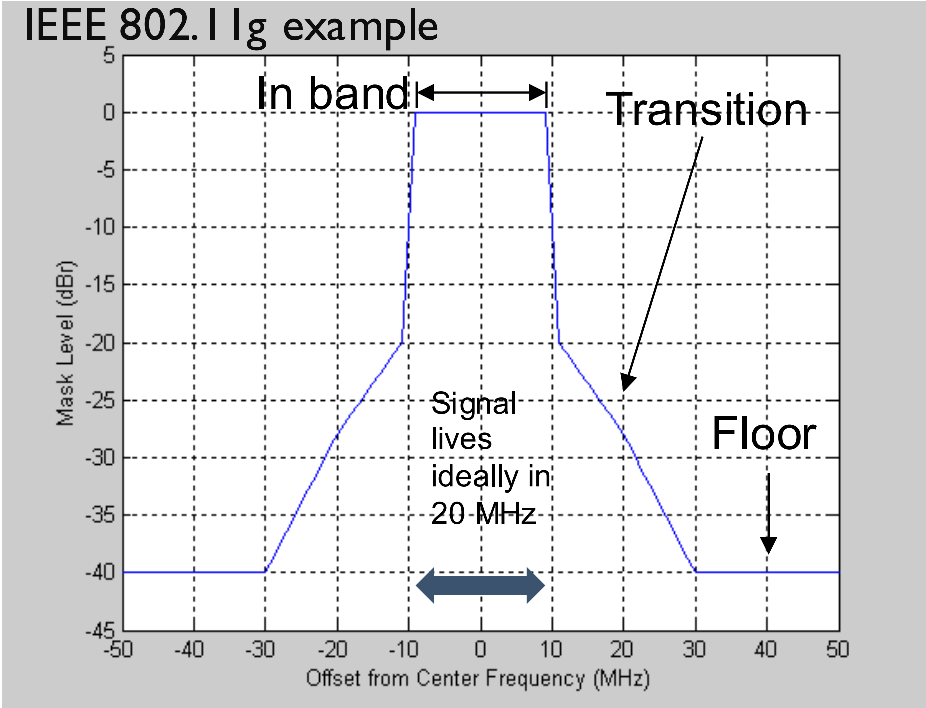
$$\omega_B = Q^{-1}(\omega_B)$$

inverse Q function, available in some form (maybe with a different name) in Excel, Python, MATLAB, etc.

Spectrum masks



Since communication spectrum is not exactly band limited, the allowed profile is called a spectrum mask



In Band: encompasses the desired signal
Transition: bounds adjacent channel interference
Floor: bounds other channel interference

Material copied from Christopher Hansen, Broadcom IEEE 802.11-04/0060r0

Summarizing bandwidth

- Bandwidth is a measure of the extent of the non-zero frequency components present in a signal

- Practical signals always have infinite bandwidth due to be generated in a finite amount of time, a result of the windowing property

- There are different ways to define the bandwidth of a practical signal based on determining when the frequencies are sufficiently small