UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2022
Linear Systems Fundamentals

MIDTERM EXAM WITH SOLUTIONS

You are allowed one 2-sided sheet of notes.

No books, no other notes, no calculators.

PRINT YOUR NAME Marc-Antoine Parseval des Chênes

Signature $\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|a_k\right|^2$

Student ID Number $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$

Problem	Weight	Score
1	32 pts	32
2	32 pts	32
3	36 pts	36
Total	100 pts	100

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Useful facts and Tables 3.1 and 3.2 from the textbook are attached to the exam.

Good luck!

Name/Student ID: ______________________________

Problem 1 [DT Signals] (32 points)

(a) Let $x[n]=u[n+1]-2u[n-1]+u[n-2]$, where $u[n]$ is the unit step signal.

(i) Represent $x[n]$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $x[n]$ precisely.

(ii) Let $y[n] = x[n] * u[n]$. Represent $y[n]$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $y[n]$ precisely.

(b) Let $x[n] = \delta[n] - \delta[n-1]$.

(i) Let $\mathcal{E}v\{x[n]\}$ be the even part of $x[n]$. Represent $\mathcal{E}v\{x[n]\}$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $\mathcal{E}v\{x[n]\}$ precisely.

(ii) Let $\mathcal{O}dd\{x[n]\}$ be the odd part of $x[n]$. Represent $\mathcal{O}dd\{x[n]\}$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $\mathcal{O}dd\{x[n]\}$ precisely.

Write your answers on the following pages.

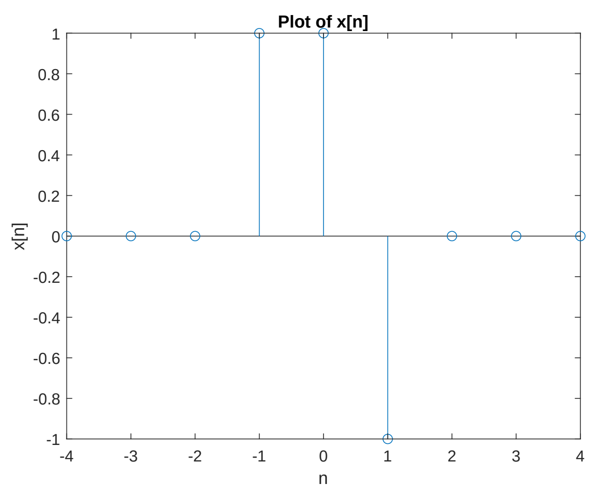
Name/Student ID: ______________________________

Problem 1 [DT Signals](cont.)

(a) Let $x[n]{=}u[n+1]{-}2u[n-1]{+}u[n-2]$, where $u[n]$ is the unit step signal.

(i) (8 points)
Represent $x[n]$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $x[n]$ precisely.

$$\begin{aligned} x[n] &= (u[n+1]-u[n-1])+(-u[n-1]+u[n-2]) \\ &= (\delta[n+1]+\delta[n])-(u[n-1]-u[n-2]) \\ &= \delta[n+1]+\delta[n]-\delta[n-1] \end{aligned}$$



Name/Student ID: ______________________________

Problem 1 [DT Signals](cont.)

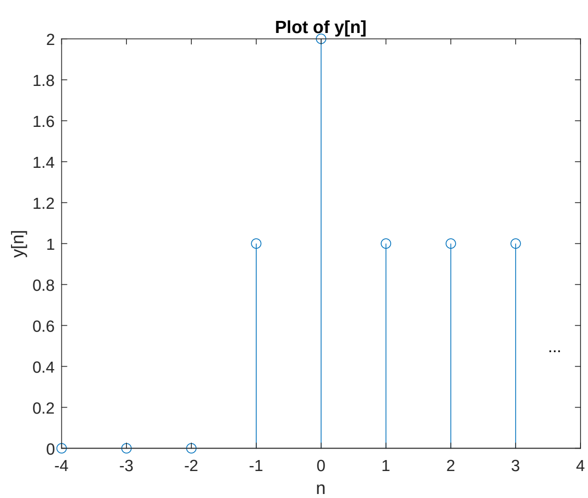
(a) Let $x[n]=u[n+1]-2u[n-1]+u[n-2]$, where $u[n]$ is the unit step signal.

(ii) (8 points)
Let $y[n] = x[n] * u[n]$.
Represent $y[n]$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $y[n]$ precisely.

$$\begin{aligned} y[n] &= x[n] * u[n] \\ &= \sum_{k=-\infty}^{n} x[k] \\ &= \delta[n+1] + 2\delta[n] + \sum_{k=1}^{\infty} \delta[n-k] \end{aligned}$$

Alternatively,

$$\begin{aligned} y[n] &= x[n] * u[n] \\ &= (\delta[n+1] + \delta[n] - \delta[n-1]) * u[n] \\ &= u[n+1] + u[n] - u[n-1] \\ &= \sum_{k=-1}^{\infty} \delta[n-k] + \delta[n] \end{aligned}$$



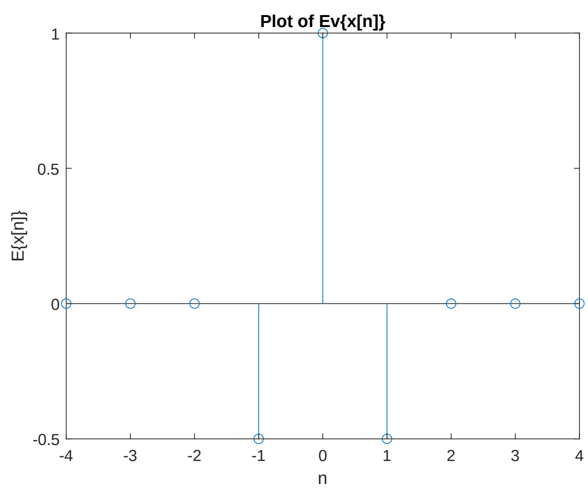
Name/Student ID: ____________________

Problem 1 [DT Signals] (cont.)

(b) Let $x[n] = \delta[n] - \delta[n-1]$.

(i) (8 points)
Let $\mathcal{E}v\{x[n]\}$ be the even part of $x[n]$. Represent $\mathcal{E}v\{x[n]\}$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $\mathcal{E}v\{x[n]\}$ precisely.

$$
\begin{aligned}
\mathcal{E}v\{x[n]\} &= \frac{x[n] + x[-n]}{2} \\
&= \frac{(\delta[n] - \delta[n-1]) + (\delta[-n] - \delta[-n-1])}{2} \\
&= \frac{(\delta[n] - \delta[n-1]) + (\delta[n] - \delta[n+1])}{2} \\
&= \frac{-\delta[n+1] + 2\delta[n] - \delta[n-1]}{2} \\
&= -\frac{1}{2}\delta[n+1]) + \delta[n] - \frac{1}{2}\delta[n-1]
\end{aligned}
$$



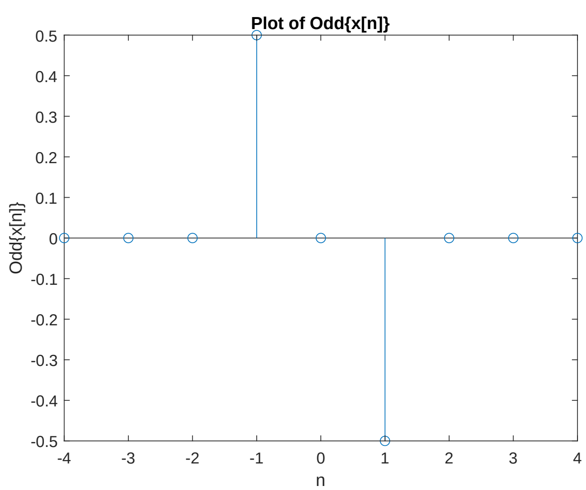
Name/Student ID: ____________________

Problem 1 [DT Signals] (cont.)

(b) Let $x[n] = \delta[n] - \delta[n-1]$.

(ii) (8 points)
Let$\mathcal{O}dd\{x[n]\}$ be the odd part of $x[n]$. Represent $\mathcal{O}dd\{x[n]\}$ mathematically as a linear combination of time-shifted unit impulse signals. Sketch $\mathcal{O}dd\{x[n]\}$ precisely.

$$
\begin{aligned}
\mathcal{O}dd\{x[n] &= \frac{x[n] - x[-n]}{2} \\
&= \frac{(\delta[n] - \delta[n-1]) - (\delta[n] - \delta[n+1])}{2} \\
&= \frac{\delta[n+1] - \delta[n-1]}{2} \\
&= \frac{1}{2}\delta[n+1]) - \frac{1}{2}\delta[n-1]
\end{aligned}
$$



Name/Student ID: ________________________________

Problem 2 [DT Systems] (32 points)

Let S be the DT system with the property that the output signal $y[n]$ produced but the input signal $x[n]$ is given by:

$$y[n] = x[-n+1].$$

(a) Check the appropriate box indicating whether or not the system S satisfies the specified property.

Justify your answers by either proving that the property holds for the system, or giving a specific counterexample to prove that it does not.

True	**False**	
☐	☐	Causal
☐	☐	Invertible
☐	☐	Stable
☐	☐	Linear
☐	☐	Time-invariant

(b) Determine and sketch precisely the impulse response $h[n]$ of system S. Describe in words the action of the system S on the input signal $x[n]$.

(c) Let S_1 be the **LTI system** that has the same impulse response as S.

Check the appropriate box indicating whether or not the system S_1 satisfies the specified property.

Justify your answers by reference to the impulse response.

True	**False**	
☐	☐	Causal
☐	☐	Invertible
☐	☐	Stable

Describe in words the action of the system S_1 on the input signal $x[n]$.

Write your answers on the following pages.

Name/Student ID: ______________________________

Problem 2 [DT Systems] cont.)

Let S be the DT system with the property that the output signal $y[n]$ produced but the input signal $x[n]$ is given by: $y[n] = x[-n+1]$.

(a) (15 points) Check the appropriate boxes for the system S. **Justify your answers.**

True **False**

☐ ☒ Causal
$y[0] = x[1]$, so the output at time $n = 0$ depends on the input at the future time $n = 1$.

☒ ☐ Invertible
The system action can be described as "shift by -1, then flip" or "flip, then shift by 1".
The inverse system operation is "flip, then shift by 1
or "shift by -1, then flip".
The inverse system is given by $y[n] = x[1-n] = x[-n+1]$.
The system S is its own inverse.

☒ ☐ Stable
If $|x[n]| \leq B\ \forall n$, then $|y[n]| = |x[-n+1]| \leq B\ \forall n$.
Bounded input produces bounded output (BIBO), so the system is stable.

☒ ☐ Linear
Suppose $x_1[n] \to y_1[n] = x_1[-n+1]$ and $x_2[n] \to y_2[n] = x_2[-n+1]$.
Let $x_3[n] = ax_1[n] + bx_2[n]$, where $a, b \in \mathbb{C}$.
Then $x_3[n] \to y_3[n] = x_3[-n+1] = ax_1[-n+1] + bx_2[-n+1]$
So $y_3[n] = ay_1[n] + by_2[n]$.
The system satisfies linear superposition, so it is linear.

☐ ☒ Time-invariant
Let $x_1[n] = x[n-1]$.
Then $x_1[n] \to y_1[n] = x_1[-n+1] = x[-n+1-1] = x[-n]$.
Now $y[n-1] = x[-(n-1)+1] = x[-n+2] \neq y_1[n]$.
So the system is not time-invariant.

Name/Student ID: ______________________________

Problem 2 [DT Systems] (cont.)

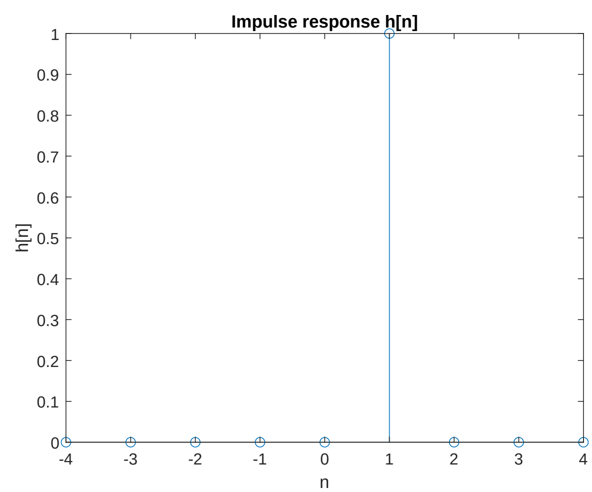
Let S be the DT system with the property that the output signal $y[n]$ produced but the input signal $x[n]$ is given by: $y[n] = x[-n+1]$.

(b) (5 points)

Determine and sketch precisely the impulse response $h[n]$ of system S. Describe in words the action of the system S on the input signal $x[n]$.

The impulse response is the output produced by the input $x[n] = \delta[n]$.

$$\begin{aligned} h[n] &= x[-n+1] \\ &= \delta[-n+1] \\ &= \delta[n-1] \end{aligned}$$



As mentioned in the analysis of invertibility in part (a), the action of the system can be described as follows:

Description 1: Shift $x[n]$ by -1, then flip the result about $n = 0$.

Description 2: Flip $x[n]$, then shift the result by 1.

Name/Student ID: ______________________________

Problem 2 [DT Systems] (cont.)

Let S be the DT system with the property that the output signal $y[n]$ produced but the input signal $x[n]$ is given by: $y[n] = x[-n+1]$.

(c) (12 points)

Let S_1 be the **LTI system** S_1 that has the same impulse response as S.

Check the appropriate boxes for the system S_1.

Justify your answers by reference to the impulse response.

The system S_1 is defined by: $y[n] = x[n] * h[n] = x[n] * \delta[n-1]$.

True	False	
☒	☐	Causal Since $h[n] = 0 \ \forall n < 0$, the system is causal.
☒	☐	Invertible Let $g[n] = \delta[n+1]$. Then $g[n] * h[n] = \delta[n+1] * \delta[n-1] = \delta[n]$. So the system is invertible. (The inverse system is "shift by -1".)
☒	☐	Stable $\sum_{n=-\infty}^{\infty} \lvert h[n]\rvert = \sum_{n=-\infty}^{\infty} \lvert\delta[n-1]\rvert = 1 < \infty$. So the system is stable.

Describe in words the action of the system S_1 on the input signal $x[n]$.

The system produces the output $y[n] = x[n] * \delta[n-1] = x[n-1]$, so it shifts $x[n]$ by 1.

Note that the system S_1 is different from the system S.

Name/Student ID: ____________________________________

Problem 3 [DTFS and DT LTI Systems] (36 points)

Let $x[n]$ be a periodic DT signal with fundamental period $N = 4$. You are given the following information about the DTFS coefficients of $x[n]$:

$$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 0.$$

(a) **Referring to Table 3.2** (Properties of DTFS), answer the following questions. **Justify your answers.**

- (i) Is $x[n]$ an even signal?
- (ii) Is $x[n]$ a real signal?
- (iii) What is the average power $\frac{1}{4}\sum_{n=0}^{3} |x[n]|^2$ in one period of $x[n]$?

(b) Write one period of the signal $x[n]$: $x[0],\ x[1],\ x[2],\ x[3]$.
Express the values as complex numbers in rectangular form.

(c) A DT LTI system S has impulse response $h[n] = \delta[n] + \delta[n-1]$.

- (i) Determine the transfer function $H(z)$ and the frequency response $H(e^{j\omega})$ of S.
- (ii) Determine and sketch precisely the phase $\angle H(e^{j\omega})$ and gain $|H(e^{j\omega})|$ over the frequency interval $[0, 2\pi]$. Express both of them in the simplest mathematical form that you can, using only linear and sinusoidal functions of ω.
- (iii) When the input to system S is the signal $x[n]$ above, determine the DTFS of the output signal $y[n]$.
- (iv) Determine one period of $y[n]$: $y[0],\ y[1],\ y[2],\ y[3]$. Express the values as complex numbers in rectangular form.

Write your answers on the following pages.

Name/Student ID: ______________________________

Problem 3 [DTFS and DT LTI Systems] (cont.)

Let $x[n]$ be a periodic DT signal with fundamental period $N = 4$. You are given the following information about the DTFS coefficients of $x[n]$:
$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 0.$

(a) **Referring to Table 3.2** (Properties of DTFS), answer the following questions. **Justify your answers.**

   (i) (4 points)
   Is $x[n]$ an even signal?

   The Time Reversal property says that the DTFS coefficients of $x[-n]$ are $b_k = a_{-k}$. If $x[n]$ is even, then $x[n] = x[-n]$, so this would imply $a_k = b_k = a_{-k}$.
   By the periodicity of DTFS, we know $a_{-1} = a_3 = 0$. But $a_1 = 1$.
   So $a_{-1} \neq a_1$. This means $x[n] \neq x[-n]$, so $x[n]$ **not** an even signal.

   (ii) (4 points)
   Is $x[n]$ a real signal?

   The Conjugate Symmetry property says that if $x[n]$ is real, then $a_k = a^*_{-k}$. But $a_1 = 1$ and $a_{-1} = 0$, so $a_1 \neq a^*_{-1}$. So, $x[n]$ is **not** real.

   (iii) (4 points)
   What is the average power $\frac{1}{4}\sum_{n=0}^{3}|x[n]|^2$ in one period of $x[n]$?

   Parseval's Relation for Periodic Signals states:
   $\frac{1}{N}\sum_{n=<N>}|x[n]|^2 = \sum_{k=<N>}|a_k|^2$
   So, $\frac{1}{4}\sum_{n=0}^{3}|x[n]|^2 = \sum_{k=0}^{3}|a_k|^2 = 2.$

Name/Student ID: ________________________________

Problem 3 [DTFS and DT LTI Systems] (cont.)

Let $x[n]$ be a periodic DT signal with fundamental period $N = 4$. You are given the following information about the DTFS coefficients of $x[n]$:
$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 0.$

(b) (4 points)

Write one period of the signal $x[n]$: $x[0],\ x[1],\ x[2],\ x[3]$.
Express the values as complex numbers in rectangular form.

The Synthesis Equation states:

$$\begin{aligned} x[n] &= \sum_{k=0}^{3} a_k e^{jk\frac{2\pi}{N}n}. \\ &= e^{j\frac{2\pi}{4}n} + e^{j2\frac{2\pi}{4}n} \\ &= e^{j\frac{\pi}{2}n} + e^{j\pi n} \\ &= j^n + (-1)^n \end{aligned}$$

Evaluating this for $n = 0, 1, 2, 3$ yields:

$$\begin{aligned} x[0] &= 2 \\ x[1] &= -1 + j \\ x[2] &= 0 \\ x[3] &= -1 - j \end{aligned}$$

Name/Student ID: ______________________________

Problem 3 [DTFS and DT LTI Systems] (cont.)

(c) A DT LTI system S has impulse response $h[n] = \delta[n] + \delta[n-1]$.

(i) (4 points)

Determine the transfer function $H(z)$ and the frequency response $H(e^{j\omega})$ of S.

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ &= 1 + z^{-1} \end{aligned}$$

Substituting $z = e^{j\omega}$ into the transfer function yields the frequency response:

$$H(e^{j\omega}) = 1 + e^{-j\omega}$$

Name/Student ID: ______________________________

Problem 3 [DTFS and DT LTI Systems] (cont.)

(c) A DT LTI system S has impulse response $h[n] = \delta[n] + \delta[n-1]$.

(ii) (8 points)

Determine and sketch precisely the phase $\angle H(e^{j\omega})$ and gain $|H(e^{j\omega})|$ over the frequency interval $[0, 2\pi]$.

Using the identity (from the Midterm Exam Review Topics)

$1 + e^{-jM} = e^{-j\frac{M}{2}}\left(e^{j\frac{M}{2}} + e^{-j\frac{M}{2}}\right) = 2e^{-j\frac{M}{2}}\cos\left(\frac{M}{2}\right)$

we get

$H(e^{j\omega}) = 1 + e^{-j\omega} = 2e^{-j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right)$

So,

$$\begin{aligned}
\angle H(e^{j\omega}) &= \angle 2 + \angle e^{-j\frac{\omega}{2}} + \angle\cos\left(\frac{\omega}{2}\right) \\
&= 0 + -\frac{\omega}{2} + 0\ (\text{if } \omega \in [0,\pi]) + \pi(\text{if } \omega \in (\pi, 2\pi]) \\
&= \begin{cases} -\frac{\omega}{2}, & \omega \in [0,\pi] \\ -\frac{\omega}{2} + \pi, & \omega \in (\pi, 2\pi] \end{cases}
\end{aligned}$$

$|H(e^{j\omega})| = |2e^{-j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right)| = 2|\cos\left(\frac{\omega}{2}\right)|$

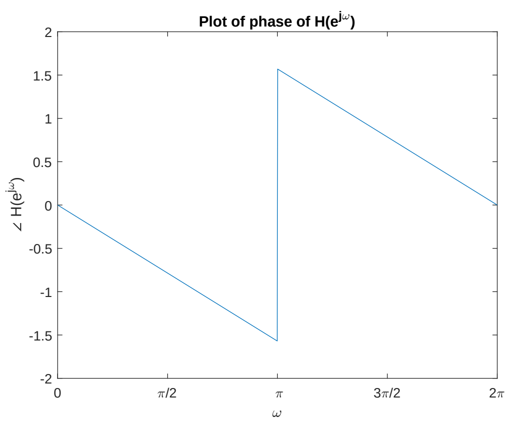


Figure 1: Phase of H(e$^{j\omega}$)

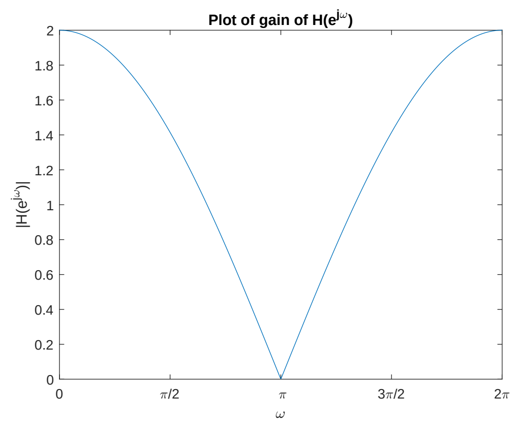


Figure 2: Gain of H(e$^{j\omega}$)

Name/Student ID: ____

Problem 3 [DTFS and DT LTI Systems] (cont.)

Let $x[n]$ be a periodic DT signal with fundamental period $N = 4$. You are given the following information about the DTFS coefficients of $x[n]$:
$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 0.$

(c) A DT LTI system S has impulse response $h[n] = \delta[n] + \delta[n-1]$.

(iii) (4 points)
When the input to system S is the signal $x[n]$ above, determine the DTFS of the output signal $y[n]$.

$y[n] = \sum_{k=0}^{3} b_k e^{jk\frac{2\pi}{4}n} = \sum_{k=0}^{3} b_k e^{jk\frac{\pi}{2}n}$ where $b_k = H(e^{jk\frac{2\pi}{4}})a_k = H(e^{jk\frac{\pi}{2}})a_k$.

$b_0 = 0$ because $a_0 = 0$

$b_1 = H(e^{j\frac{\pi}{2}})a_1 = (1 + e^{-j\frac{\pi}{2}}) \cdot a_1 = (1-j)\cdot 1 = 1-j$

$b_2 = H(e^{j\pi})a_2 = (1 + e^{-j\pi}) \cdot a_2 = 0 \cdot 1 = 0$

$b_3 = 0$

This shows that $y[n]$ has fundamental frequency $\frac{\pi}{2} = \frac{2\pi}{4}$ and fundamental period $N = 4$.

So, one period of the DTFS coefficients of $y[n]$ is

$$b_0 = 0, b_1 = 1-j, b_2 = 0, b_3 = 0.$$

The DTFS representation of y is $y[n] = (1-j)e^{j\frac{\pi}{2}n}$.

Alternatively, note that

$$y[n] = x[n] * h[n] = x[n] * \delta[n] + \delta[n-1] = x[n] - x[n-1].$$

The DFTS representation of $y[n]$ can be determined from that of $x[n]$ using this expression.

Name/Student ID: ________________________________

Problem 3 [DTFS and DT LTI Systems] (cont.)

Let $x[n]$ be a periodic DT signal with fundamental period $N = 4$. You are given the following information about the DTFS coefficients of $x[n]$:
$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 0.$

(c) A DT LTI system S has impulse response $h[n] = \delta[n] + \delta[n-1]$.

(iv) (4 points)
Determine one period of $y[n]$: $y[0],\ y[1],\ y[2],\ y[3]$.
Express the values as complex numbers in rectangular form.

From the previous part, $y[n]$ has fundamental period $N = 4$ with one period of the DTFS given by $b_0 = 0, b_1 = 1-j, b_2 = 0, b_3 = 0$.

The Synthesis Equation states:

$$\begin{aligned} yn] &= \sum_{k=0}^{3} b_k e^{jk\frac{2\pi}{N}n}. \\ &= (1-j)e^{j\frac{2\pi}{4}n} \\ &= (1-j)e^{j\frac{\pi}{2}n} \\ &= (1-j)j^n \end{aligned}$$

Evaluating this for $n = 0, 1, 2, 3$ yields:

$$\begin{aligned} y[0] &= 1-j \\ y[1] &= (1-j)j = 1+j \\ y[2] &= (1-j)j^2 = -1+j \\ y[3] &= (1-j)j^3 = -1-j \end{aligned}$$

Scratch page **Name/Student ID:** ____________________

Scratch page **Name/Student ID:** ______________________

Scratch page **Name/Student ID:** ______________________________