Discussion Session 5:

Fourier Series Tutorial:

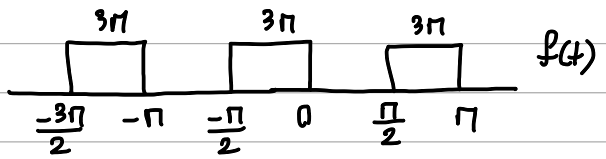
Fourier Series is a mathematical tool used to represent periodic function as a sum of sines and cosines. A periodic signal $f(t)$ with period T_0 can be expressed as:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_n t} = \sum_{n=-\infty}^{\infty} \underbrace{F_n}_{\text{Fourier Coefficient}} e^{jn\omega_0 t}$$

$$\omega_n = n\frac{2\pi}{T_0} = n\omega_0$$

F_n are given by :
$$F_n = \frac{1}{T_0}\int_{T_0} F(t)\, e^{-j\omega_n t}\, dt$$

Example # 1:



a) What is the input's fundamental frequency (ω_0) ?

b) What is the Fourier Series coefficient (F_n) ?

Answer:

$$T_0 = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2$$

b) $$F_n = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\, e^{-jn2t}\, dt = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{0} 3\pi\, e^{-jn2t}\, dt =$$

$$= \frac{3}{-jn2}\left[1 - e^{-jn\pi}\right] = \frac{3}{2jn}\left[e^{-jn\pi} - 1\right]$$

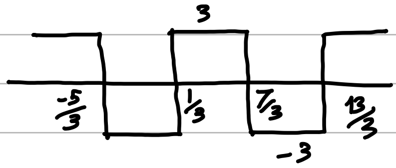
Because of n in denominator, we need to split up into cases:

For $n \neq 0$ and n = even, $F_n = \frac{3}{2jn}\left[1-1\right] = 0$

For $n \neq 0$, and n = odd, $F_n = \frac{3}{2jn}\left[-1-1\right] = \frac{-3}{jn}$

For $n = 0$, $F_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{0} 3\pi\, e^{-j(0)2t}\, dt = 3 \cdot \frac{\pi}{2} = \frac{3\pi}{2}$

Example #2:



a) What is the input's fundamental frequency (w_0)?

b) What is this input's Fourier Series coefficient (F_n)?

Answer:

a) Finding $T_0 = \frac{13}{3} - \frac{1}{3} = 4 \longrightarrow W_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

b) $$F_n = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jnw_0 t}\, dt = \frac{1}{4}\int_{1/3}^{13/3} f(t)\, e^{-jn\frac{\pi}{2}t}\, dt = \underbrace{\frac{1}{4}\int_{1/3}^{7/3} 3\, e^{-jn\frac{\pi}{2}t}\, dt}_{A}$$

$$+ \underbrace{\frac{1}{4}\int_{7/3}^{13/3} (-3)\, e^{-jn\frac{\pi}{2}t}\, dt}_{B}$$

$$A = \frac{3}{4\left(-jn\frac{\pi}{2}\right)}\left[e^{-jn\frac{\pi}{2}\left(\frac{7}{3}\right)} - e^{-jn\frac{\pi}{2}\left(\frac{1}{3}\right)}\right] = \frac{-3}{j2\pi n}\left[e^{-j\frac{7\pi}{6}n} - e^{-j\frac{\pi}{6}n}\right]$$

$$B = \frac{-3}{4\left(-jn\frac{\pi}{2}\right)}\left[e^{-jn\frac{\pi}{2}\left(\frac{13}{3}\right)} - e^{-jn\frac{\pi}{2}\left(\frac{7}{3}\right)}\right] = \frac{3}{j2\pi n}\left[e^{-j\frac{13\pi}{6}n} - e^{-j\frac{7\pi}{6}n}\right]$$

$$F_n = A + B = \frac{3}{j2\pi n}\left[e^{-j\frac{\pi}{6}n} - e^{-j\frac{7\pi}{6}n} + e^{-j\frac{13\pi}{6}n} - e^{-j\frac{7\pi}{6}n}\right]$$

$$= \frac{3}{j2\pi n}\left[e^{-j\frac{\pi}{6}n} + e^{-j\frac{13\pi}{6}n} - 2e^{-j\frac{7\pi}{6}n}\right]$$

$$= \frac{3}{j2\pi n} e^{-jn\frac{\pi}{6}}\left[1 + \underbrace{\underbrace{e^{-j\frac{12\pi}{6}n}}_{e^{-j(2\pi)n}}}_{=1} - 2\underbrace{e^{-j\frac{6\pi}{6}n}}_{e^{-jn\pi}}\right]$$

$$= \frac{3}{j2\pi n} e^{-jn\frac{\pi}{6}}\left[2 - 2e^{-jn\pi}\right]$$

This is the point we can split into cases since there is an n in denominator

For $n \neq 0$, n: even : $e^{-jn\pi} = 1 \longrightarrow F_n = \frac{3}{j2\pi n} e^{-j\frac{\pi}{6}n} \underbrace{[2-2]}_{0} = 0$

For $n \neq 0$, $n =$ odd : $e^{-jn\pi} = -1 \longrightarrow F_n = \frac{3}{j2\pi n} e^{-j\frac{\pi}{6}n} [2+2] = \frac{6}{jn\pi} e^{-j\frac{\pi}{6}n}$

For $n = 0$, $F_n = \frac{1}{4}\int_{1/3}^{7/3} (3)\, dt + \frac{1}{4}\int_{7/3}^{13/3} (-3)\, dt = \frac{3}{4}\left(\frac{7}{3}-\frac{1}{3}\right) - \frac{3}{4}\left(\frac{13}{3}-\frac{7}{3}\right) = \frac{6}{4} - \frac{6}{4} = 0$

If $f(x)$ is a periodic function with period T_0, it can be expressed as:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi}{T_0} nx\right) + b_n \sin\left(\frac{2\pi}{T_0} nx\right)\right]$$

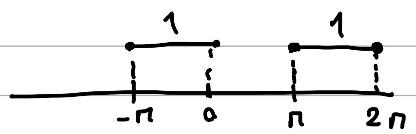
with $a_0 = \frac{1}{T_0}\int_{T_0} f(x)\, dx$

for $n \neq 0$

$$a_n = \frac{2}{T_0}\int_{T_0} f(x) \cos\left(\frac{2\pi}{T_0} nx\right) dx$$

$$b_n = \frac{2}{T_0}\int_{T_0} f(x) \sin\left(\frac{2\pi}{T_0} nx\right) dx$$

Example #3:



Find a_n and b_n.

Answer:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{0} 1\, dx = 0.5$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos\left(\frac{2\pi}{2\pi} nx\right) dx = \frac{1}{\pi}\int_{-\pi}^{0} 1 \cos(nx)\, dx = \frac{1}{n\pi} \sin(nx)\Big|_{x=-\pi}^{x=0}$$

$$= \frac{1}{n\pi}\left(0 - \underbrace{\sin(-n\pi)}_{0}\right) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin\left(\frac{2\pi}{2\pi} nx\right) dx = \frac{1}{\pi}\int_{-\pi}^{0} 1 \sin(nx)\, dx = \frac{-1}{nx}\left(\cos(nx)\right)\Big|_{x=-\pi}^{x=0}$$

$$= \frac{-1}{n\pi}\left(1 - \cos(n\pi)\right) = \frac{-1}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & n = \text{even} \\ \frac{-2}{n\pi} & n = \text{odd} \end{cases}$$

Pick an appropiate value of x, to show that: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Answer: we showed that $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin\left(\frac{2\pi}{2\pi} nx\right) = \frac{1}{2} - \frac{2}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \cdots\right)$

Comparing with the formula we should have $\sin(x) = 1$, $\sin(3x) = -1$, $\sin(5x) = 1, \ldots$

So we put $x = \frac{\pi}{2}$, which gives us

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{2} - \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \cdots\right)$$

$$\rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$