UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2018
Linear Systems Fundamentals

FINAL EXAM (WITH SOLUTIONS)

You are allowed two 2-sided sheets of notes.

No books, no other notes, no calculators.

PRINT YOUR NAME ____

Signature ____

Your signature affirms that you have completed this exam on your own and in accordance with the ECE 101 honor code.

Student ID Number ____

Problem	Weight	Score
1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
6	20 pts	
Total	120 pts	

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Tables 3.1, 3.2, 4.1, 4.2, 5.1, 5.2, 9.1, and 9.2 are attached.
Good luck!

Name/Student ID: ________________________________

Problem 1 [Do the Convolution] (20 points, 10pts each part)

Assume you are given a discrete-time LTI system consisting of the cascade shown below:

$$x[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{z[n]} \boxed{h_2[n]} \longrightarrow y[n]$$

The two impulse responses are given by $h_1[n] = 2^n u[n]$ and $h_2[n] = 3^n u[n]$, respectively.

The input signal is given by $x[n] = \delta[n] - 2\delta[n-1]$.

(a) Determine the output $y[n]$. [Hint: determine $z[n]$ first].

$$\begin{aligned}
z[n] &= x[n] * h_1[n] \\
&= (\delta[n] - 2\delta[n-1]) * 2^n u[n] \\
&= \delta[n] * 2^n u[n] - 2\delta[n-1] * 2^n u[n] \\
&= 2^n u[n] - 2(2^{n-1} u[n-1]) \\
&= 2^n u[n] - 2^n u[n-1] \\
&= 2^n (u[n] - u[n-1]) \\
&= 2^n \delta[n] \\
&= \delta[n].
\end{aligned}$$

Now, compute the overall output of the cascade.

$$\begin{aligned}
y[n] &= z[n] * h_2[n] \\
&= \delta[n] * 3^n u[n] \\
&= 3^n u[n].
\end{aligned}$$

Name/Student ID: ______________________________

Problem 1 [Do the Convolution] (cont.)

(b) Consider the same system, shown again below,

$$x[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{z[n]} \boxed{h_2[n]} \longrightarrow y[n]$$

where, again $h_1[n] = 2^n u[n]$ and $h_2[n] = 3^n u[n]$.

Determine the overall impulse response $h[n]$.

$$\begin{aligned}
h[n] &= h_1[n] * h_2[n] \\
&= \sum_{k=-\infty}^{\infty} h_1[k] h_2[n-k] \\
&= \sum_{k=-\infty}^{\infty} 2^k u[k] 3^{n-k} u[n-k] \\
&= \begin{cases} \sum_{k=0}^{n} 2^k 3^{n-k}, & n \geq 0 \\ 0, & n < 0 \end{cases} \\
&= \begin{cases} 3^n \sum_{k=0}^{n} (2/3)^k, & n \geq 0 \\ 0, & n < 0 \end{cases} \\
&= \begin{cases} 3^n \frac{1-(2/3)^{n+1}}{1-(2/3)}, & n \geq 0 \\ 0, & n < 0 \end{cases} \\
&= \begin{cases} 3^{n+1}(1-(2/3)^{n+1}), & n \geq 0 \\ 0, & n < 0 \end{cases} \\
&= \begin{cases} 3^{n+1} - 2^{n+1}, & n \geq 0 \\ 0, & n < 0 \end{cases}
\end{aligned}$$

So, $h[n] = (3^{n+1} - 2^{n+1}) u[n]$.

More generally: $a^n u[n] * b^n u[n] = \frac{a^{n+1} - b^{n+1}}{a-b} u[n]$, for $a \neq b$.

Name/Student ID: ______________________________

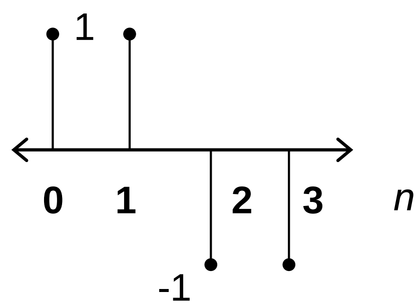
Problem 2 [DT Fourier Series] (20 points, 10pts each part)

Let $x[n]$ be the periodic discrete-time signal with fundamental period $N = 4$ defined by:

$$x[n] = \begin{cases} 1, & n = 0, 1 \\ -1, & n = 2, 3. \end{cases}$$

(a) (i) (2 points)

Sketch one period of $x[n]$ precisely.



(ii) (8 points)

Determine the DT Fourier series coefficients a_k, $k = 0, 1, 2, 3$ of the signal $x[n]$. Write your answers in the box below. **Justify your answers.**

DTFS analysis equation: $a_k = \frac{1}{N}\sum_{k=0}^{N} x[n]e^{-jk\frac{2\pi}{N}n}$.

So:

$$\begin{aligned} a_k &= \frac{1}{4}(x[0] + x[1]e^{-jk\frac{\pi}{2}} + x[2]e^{-jk\pi} + x[3]e^{-jk\frac{3\pi}{2}}) \\ &= \frac{1}{4}(1 + (-j)^k - (-1)^k - j^k). \end{aligned}$$

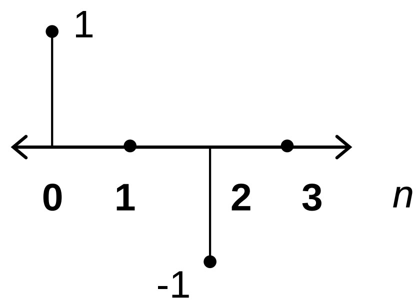
$$\boxed{\begin{aligned} a_0 &= 0 \\ a_1 &= \tfrac{1-j}{2} \\ a_2 &= 0 \\ a_3 &= \tfrac{1+j}{2} \end{aligned}}$$

Name/Student ID: ____________________

Problem 2 [DT Fourier Series] (cont.)

(b) Consider the signal $x_1[n] = \frac{1}{2}(x[n] - x[n-1])$, where $x[n]$ is the signal in part (a).

(i) (2 points)
Sketch one period of $x_1[n]$ precisely.



(ii) (3 points)
Let b_k be the Fourier series coefficients of $x_1[n]$. Express b_k as a function of a_k, the Fourier series coefficients of $x[n]$.

By linearity and time-shifting property 3.2,

$$\begin{aligned} b_k &= \frac{1}{2}(1 - e^{-jk\frac{\pi}{2}})a_k \\ &= \frac{1}{2}(1 - (-j)^k)a_k \end{aligned}$$

(iii) (5 points)
The signal $x_1[n]$ satisfies $x_1[n] = \cos(\omega n)$ for some $\omega \in [0, 2\pi)$. Determine ω. Then determine the values b_k, $k = 0, 1, 2, 3$. Write your answers in the box below. **Justify your answers.**

$x_1[n] = \cos(\frac{\pi}{2}n)$

$$\boxed{\begin{aligned} \omega &= \pi/2 \\ b_0 &= 0 \\ b_1 &= 1/2 \\ b_2 &= 0 \\ b_3 &= 1/2 \end{aligned}}$$

Name/Student ID: ______________________________

Problem 3 [DTFT] (20 points)

Consider a causal and stable LTI system whose input $x[n]$ and output $y[n]$ are related by the first-order difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

(a) (6 pts)

Determine the frequency response $H(e^{j\omega})$ of the system.

Justify your answer.

Taking the DTFT of both sides of the equation gives:

$$Y(e^{j\omega}) - \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

So,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

Name/Student ID: ____

Problem 3 [DTFT] (cont.)

(b) (6 pts)

Determine the impulse response $h[n]$ of the system. Justify your answer.

From Table 5.1, or by direct inverse transform calculation, we find

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

(c) (4 pts)

Evaluate the following integral.

$$\frac{1}{2\pi}\int_0^{2\pi} H(e^{j\omega})e^{j2\omega}d\omega.$$

The Synthesis Equation states:

$$h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(e^{j\omega})e^{j\omega n}d\omega.$$

So the integral equals $h[2] = \frac{1}{4}$.

Name/Student ID: ______________________________

Problem 3 [DTFT] (cont.)

(d) (4 pts)

Evaluate the following integral.

$$\frac{1}{2\pi}\int_0^{2\pi}\left|H(e^{j\omega})\right|^2 d\omega.$$

Parseval's Relation (Table 5.1) states:

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|X(e^{j\omega})\right|^2 d\omega.$$

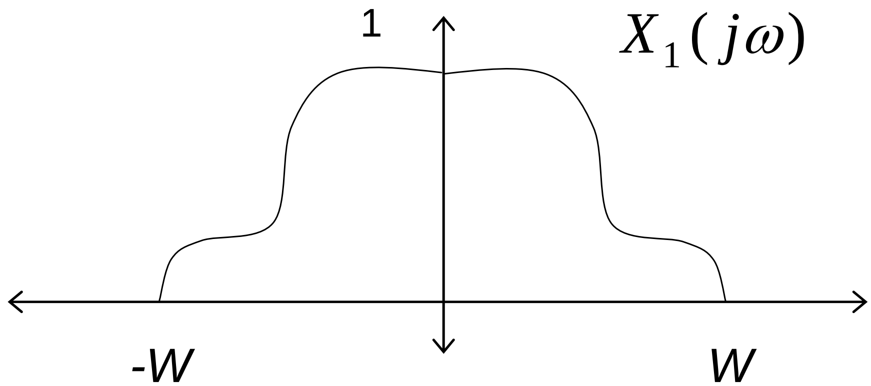
So, the integral equals

$$\sum_{n=-\infty}^{\infty}|h[n]|^2 = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n} = \frac{1}{1-(1/4)} = 4/3.$$

Name/Student ID: ______________________________

Problem 4 [Amplitude Modulation]

Let $x_1(t)$ be a continuous-time signal bandlimited to $[-W, W]$, with Fourier transform $X_1(j\omega)$ shown below.



You wish to transmit the signal using amplitude modulation, i.e., by modulating the signal with a carrier signal $c(t) = \cos(\frac{3W}{2}t)$. Just as you are about the transmit the modulated signal, a saboteur who wishes to prevent a successful transmission secretly adds a scrambling signal $x_2(t)$, also band-limited to $[-W, W]$, but mistakenly modulated with the carrier signal $s(t) = \sin(\frac{3W}{2}t)$. The signal actually transmitted is therefore:

$$y(t) = x_1(t)c(t) + x_2(t)s(t).$$

Note: The following identities may be helpful to you in solving this problem:

$$\begin{aligned}
\cos^2(\theta) &= \frac{1+\cos(2\theta)}{2} \\
\sin^2(\theta) &= \frac{1-\cos(2\theta)}{2} \\
\cos(\theta)\sin(\theta) &= \frac{\sin(2\theta)}{2}
\end{aligned}$$

Name/Student ID: ________________________________

Problem 4 [Amplitude Modulation](cont.)

(a) (4 pts)

Express the Fourier transform $Y(j\omega)$ of the output signal in terms of the transforms $X_1(j\omega)$ and $X_2(j\omega)$ of the signals $x_1(t)$ and $x_2(t)$.

$$\begin{aligned} Y(j\omega) &= \frac{1}{2}\left(X_1(j(\omega - \frac{3W}{2})) + X_1(j(\omega + \frac{3W}{2}))\right) \\ &+ \frac{1}{2j}\left(X_2(j(\omega - \frac{3W}{2})) - X_2(j(\omega + \frac{3W}{2}))\right). \end{aligned}$$

Name/Student ID: ______________________________

Problem 4 [Amplitude Modulation](cont.)

(b) (8 pts)

At the receiving end of the transmission, you have a demodulation circuit that first produces the signal $z(t) = y(t)c(t)$.

(i) Write an expression for the demodulated signal $z(t) = y(t)c(t)$.

Using the trigonometric identities, we find:

$$\begin{aligned} z(t) &= y(t)c(t) \\ &= x_1(t)c(t)c(t) + x_2(t)s(t)c(t) \\ &= \frac{1}{2}x_1(t)(1+\cos(3Wt)) + \frac{1}{2}x_2(t)\sin(3Wt) \\ &= \frac{1}{2}x_1(t) + \frac{1}{2}x_1(t)\cos(3Wt)) + \frac{1}{2}x_2(t)\sin(3Wt). \end{aligned}$$

(ii) Write an expression for its Fourier transform $Z(j\omega)$.

The Fourier Transform $Z(j\omega)$ is:

$$\begin{aligned} Z(j\omega) &= \frac{1}{2}X_1(j\omega) + \frac{1}{4}\left(X_1(j(\omega-3W)) + X_1(j(\omega+3W))\right) \\ &\quad + \frac{1}{4j}\left(X_2(j(\omega-3W)) - X_2(j(\omega+3W))\right). \end{aligned}$$

Name/Student ID: ______________________________

Problem 4 [Amplitude Modulation](cont.)

(c) (4 pts)

The signal $z(t)$ is then input to a low-pass filter with high frequency cutoff at $\frac{3W}{2}$ and gain 2.

Show that the output $y_1(t)$ of the low-pass filter is equal to $x_1(t)$, the desired signal.

The low-pass filter removes all but the first term, which when scaled by the gain factor 2, produces the output $y_1(t)$ whose Fourier Transform is

$$Y_1(j\omega) = X_1(j\omega).$$

Therefore, $y_1(t) = x_1(t)$, as claimed.

(d) (4 pts)

How would you recover $x_2(t)$ from $y(t)$? (You'll want to show it to the saboteur, of course.)

Multiply the received signal by $s(t) = \sin(\frac{3W}{2}t)$, rather than $c(t) = \cos(\frac{3W}{2}t)$, and then apply the resulting signal to the input of the same low-pass filter used in part (c).

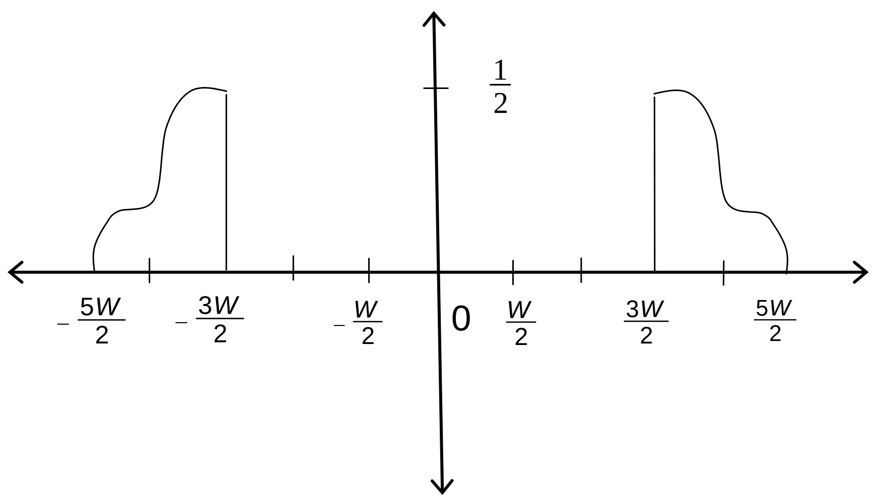
Name/Student ID: ______________________

Problem 4 [Amplitude Modulation](cont.)

(e) (4 pts)

Annoyed at being outsmarted, the saboteur decides to ruin your next transmission by applying to your modulated signal $x_1(t)c(t)$ a high-pass filter with low frequency cutoff $\frac{3W}{2}$ and unity gain. The resulting transmitted signal is denoted $v(t)$.

Sketch precisely the Fourier transform $V(j\omega)$ of the transmitted signal $v(t)$. Express $V(j\omega)$ in terms of $X_1(j\omega)$.



$$V(j\omega) = \begin{cases} \frac{1}{2}X_1(j(\omega + \frac{3W}{2})), & \omega \leq -\frac{3W}{2} \\ 0, & |\omega| \leq \frac{3W}{2} \\ \frac{1}{2}X_1(j(\omega - \frac{3W}{2})), & \omega \geq \frac{3W}{2} \end{cases}$$

(f) (2 pts)

How would you recover $x_1(t)$ from $v(t)$?

Demodulate using $c(t) = \cos(\frac{3W}{2}t)$, then pass through a low-pass filter with high frequency cutoff at $3W/2$ and gain 4.

Name/Student ID: ____________________

Problem 5 [Sampling Theory] (20 pts)

(a) (10 points)

Let $x(t)$ be a continuous-time band-limited signal with

$$X(j\omega) = 0 \text{ for } |\omega| > W$$

For each of the signals below, indicate the condition that, according to the sampling theorem, the sampling frequency ω_s should satisfy so that the signal can be reconstructed from its samples, and also indicate the corresponding condition on the sampling period T.

Justify your answers by explicit reference to the Fourier Transforms $Y(j\omega)$ of the signals.

(i) $y(t) = x(t/3)$.

By the Time and Frequency Scaling property,

$$Y(j\omega) = 3X(j3\omega).$$

So $Y(j\omega) = 0$, for $|\omega| > W/3$.
The sampling theorem conditions are:

$$\omega_s > 2W/3$$
$$T < \frac{3\pi}{W}$$

(ii) $y(t) = x(t)x(-t)$.

By the Time Reversal property, $x(-t) \leftrightarrow X(-j\omega)$.
By the multiplication property,

$$y(t) = x(t)x(-t) \leftrightarrow Y(j\omega) = \frac{1}{2\pi}X(j\omega) * X(-j\omega).$$

So, $Y(j\omega) = 0$, for $|\omega| > 2W$.
The sampling theorem conditions are:

$$\omega_s > 4W$$
$$T < \frac{\pi}{2W}$$

Name/Student ID: ____________________

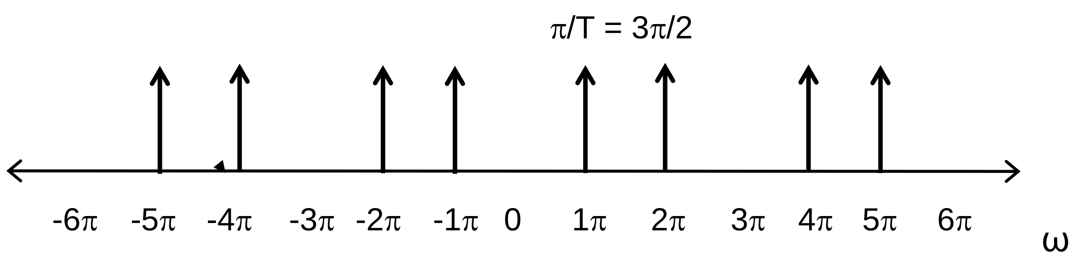
Problem 5 (cont.)

(b) (10 points)

Consider the signal $x(t) = \cos(2\pi t)$. Let $x_p(t)$ be obtained by impulse-train sampling $x(t)$ with sampling frequency $\omega_s = 3\pi$.

(i) Let $X_p(j\omega)$ be the Fourier transform of the signal $x_p(t)$.
Determine $X_p(j\omega)$ and sketch it precisely in the range $[-6\pi, 6\pi]$.

$X(j\omega) = \pi\delta(\omega - 2\pi) + \pi\delta(\omega + 2\pi)$
$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k3\pi))$, where $T = \frac{2\pi}{\omega_s} = \frac{2}{3}$.



(ii) Suppose $x_p(t)$ is passed through a low-pass filter with gain $T = 2\pi/\omega_s = 2/3$ and cutoff frequency $\omega_c = \omega_s/2 = 3\pi/2$.
Determine the output $y(t)$ of the low-pass filter and determine whether aliasing occurred.
From part (i), we see that the transform of the filter output is

$$Y(j\omega) = \pi\delta(\omega - \pi) + \pi\delta(\omega + \pi)$$

So,

$$y(t) = \cos(\pi t).$$

Therefore, aliasing has occurred.

Name/Student ID: ____________________

Problem 6

(a) (10 points)

Let S be a stable LTI system defined by the differential equation

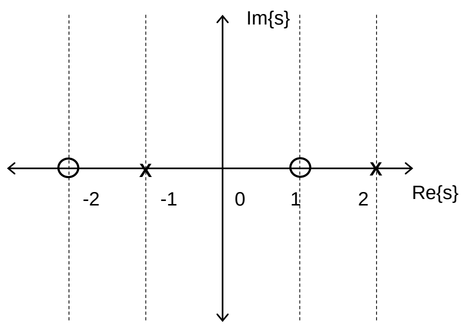
$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = \frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} - 2x(t)$$

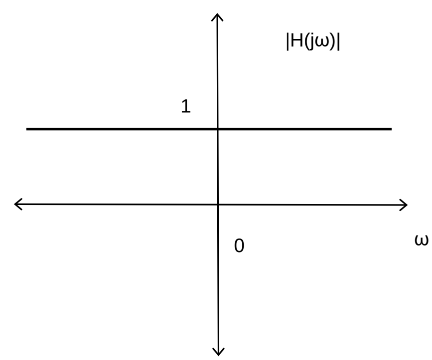
Let $H(j\omega)$ be the frequency response of the system.

(i) Draw the pole-zero plot and sketch precisely the magnitude of the frequency response $\left|H(j\omega)\right|$. The system response is give by

$$H(s) = \frac{\sum_k b_k s^k}{\sum_k a_k s^k} = \frac{(s-1)(s+2)}{(s+1)(s-2)}.$$

So there are first order poles at $s = -1, 2$ and first-order zeros at $s = 1, -2$.





The frequency response is therefore

$$H(j\omega) = \frac{(j\omega - 1)(j\omega + 2)}{(j\omega + 1)(j\omega - 2)}.$$

Using the geometric evaluation technique, we see that

$$|H(j\omega)| = \frac{|j\omega - 1||j\omega + 2|}{|j\omega + 1||j\omega - 2|} = 1, \quad \forall\omega.$$

(ii) Is the frequency response lowpass, highpass, bandpass, or all-pass? Indicate your answer below by checking the appropriate box.

Low-pass	High-pass	Band-pass	All-pass
☐	☐	☐	X

Name/Student ID: ____________________________________

Problem 6 (cont.)

(b) (10 points) Let S be a **causal** LTI system with system function

$$H(s) = \frac{s-2}{s^2-2s-8}.$$

Determine the ROC and the impulse response $h(t)$.

$$H(s) = \frac{s-2}{s^2-2s-8} = \frac{s-2}{(s+2)(s-4)}.$$

Since S is causal, the ROC is $\{s|\mathcal{R}e\{s\} > 4\}$.

Note that the system is not stable.

Using partial fraction expansion,

$$\begin{aligned} H(s) &= \frac{A}{s+2} + \frac{B}{s-4} \\ &= \frac{2/3}{s+2} + \frac{1/3}{s-4} \end{aligned}$$

The impulse response is

$$h(t) = \frac{2}{3}e^{-2t}u(t) + \frac{1}{3}e^{4t}u(t).$$