ECE 45 – Circuits and Systems
Winter 2025

Homework #9
Due: March 6 at 11:59pm, submitted via GradeScope.

You can make multiple upload attempts to experiment with the system and the best way to upload. You must correctly mark the answers to the problems in GradeScope, e.g. problem 1, problem 2, problem 3, to get full credit. Note that you must tag your problems when uploading to GradeScope or they will not be graded and you will not receive credit. Any regrade requests must be placed through GradeScope within one week of the return of the homework.

Remember, discussion of homework questions is encouraged. Please be absolutely sure to submit your own independent homework solution.

1. (20 %) Compute the following Fourier transforms using the tables and properties. Let $x(t) = 2\text{sinc}(2t)$.
   (a) $X(j\omega) = \mathcal{F}\{x(t)\}$
   (b) $Y(j\omega) = \mathcal{F}\{x(t+2)\}$
   (c) $Y(j\omega) = \mathcal{F}\{x(3t+2)\}$
   (d) $Y(j\omega) = \mathcal{F}\{e^{-j2\pi 100t}x(3t+2)\}$
   (e) $Y(j\omega) = \mathcal{F}\{\frac{d}{dt}e^{-j2\pi 100t}x(3t+2)\}$
   (f) $Y(j\omega) = \mathcal{F}\{\left(\frac{d}{dt}e^{-j2\pi 100t}x(3t+2)\right) * e^{-3t}u(t)\}$
   (g) $Y(j\omega) = \mathcal{F}\{\text{rect}(t/100)\left(\frac{d}{dt}e^{-j2\pi 100t}x(3t+2) * e^{-3t}u(t)\right)\}$ (you do not have to solve the convolution)

   <u>Solution:</u>

   Given: $x(t) = 2\text{sinc}(2t)$

   (a) $\mathcal{F}\{x(t)\}$:

$$\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}\left(\frac{\omega}{2\pi}\right)$$
$$\text{sinc}(2t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi}\right) \quad \text{[Time Scaling]}$$
$$2\text{sinc}(2t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}\left(\frac{\omega}{4\pi}\right).$$

   (b) $\mathcal{F}\{x(t+2)\}$: Time shifting property.

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}\left(\frac{\omega}{2\pi}\right)$$
$$x(t+2) \overset{\mathcal{F}}{\longleftrightarrow} e^{j\omega 2}\text{rect}\left(\frac{\omega}{4\pi}\right)$$

(c) $\mathcal{F}\{x(3t+2)\}$: Scale, then shift.

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \text{rect}\left(\frac{\omega}{4\pi}\right)$$
$$x(3t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{3}\text{rect}\left(\frac{\omega}{12\pi}\right) \quad \text{[Time Scaling]}$$

Writing $x(3t+2) = x(3(t+2/3))$:

$$x(3(t+2/3)) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{3}e^{j\omega\frac{2}{3}}\text{rect}\left(\frac{\omega}{12\pi}\right) \quad \text{[Time Shifting]}$$

(d) $\mathcal{F}\{e^{-j2\pi 100t}x(3t+2)\}$
We will use the results from our previous question and apply frequency shifting operation to it.

$$x(3t+2) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{3}e^{j\omega\frac{2}{3}}\text{rect}\left(\frac{\omega}{12\pi}\right)$$
$$e^{-j2\pi 100t}x(3t+2) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{3}e^{j\frac{2}{3}(\omega+2\pi 100)}\text{rect}\left(\frac{\omega+2\pi 100}{12\pi}\right)$$

(e) $\mathcal{F}\{\frac{d}{dt}e^{-j2\pi 100t}x(3t+2)\}$
Let us consider $y(t) = e^{-j2\pi 100t}x(3t+2)$ and correspondingly, $Y(j\omega) = X(j\omega)$.
Then, we have:

$$y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

Now, applying the differentiation property on $y(t)$:

$$\frac{d}{dt}y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega\; Y(j\omega)$$
$$\frac{d}{dt}e^{-j2\pi 100t}x(3t+2) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{j\omega}{3}e^{j\frac{2}{3}(\omega+200\pi)}\text{rect}\left(\frac{\omega+200\pi}{12\pi}\right)$$

(f) $Y(j\omega) = \mathcal{F}\{\left(\frac{d}{dt}e^{-j2\pi 100t}x(3t+2)\right) * e^{-3t}u(t)\}$

$$\mathcal{F}\{\frac{d}{dt}e^{-j2\pi 100t}x(3t+2) * e^{-3t}u(t)\} = \mathcal{F}\{\frac{d}{dt}e^{-j2\pi 100t}x(3t+2)\} \times \mathcal{F}\{e^{-3t}u(t)\}$$

We need to calculate the second term in the multiplication:

$$\mathcal{F}\{e^{-3t}u(t)\} = \frac{1}{3+j\omega} \quad \text{[from the FT tables]}$$

Therefore:

$$\frac{d}{dt}e^{-j2\pi 100t}x(3t+2) * e^{-3t}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{j\omega}{3(3+j\omega)}e^{j\frac{2}{3}(\omega+200\pi)}\text{rect}\left(\frac{\omega+200\pi}{12\pi}\right).$$

(g) $Y(j\omega) = \mathcal{F}\{\text{rect}(t/100)\left(\frac{d}{dt}e^{-j2\pi 100t}x(3t+2) * e^{-3t}u(t)\right)\}$ (you do not have to solve the convolution)

Writing

$$y(t) = \frac{d}{dt} e^{-j2\pi 100t} x(3t+2) * e^{-3t}u(t)$$

and its corresponding Fourier transform is $Y(j\omega)$.
Also, let us take $\text{rect}(t/100) = z(t)$.
Then, the question can be written as: $\mathcal{F}\{z(t)\ y(t)\}$, which can be solved using multiplication property:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\theta)Y(j(\omega-\theta))d\theta.$$

Calculating $Z(j\omega)$:

$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\frac{\omega}{2\pi}\right)$$
$$\text{rect}\left(\frac{t}{100}\right) \overset{\mathcal{F}}{\longleftrightarrow} 100\ \text{sinc}\left(\frac{50\omega}{2\pi}\right).$$

$$\begin{aligned}\mathcal{F}\{z(t)\ y(t)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\theta)Y(j(\omega-\theta))d\theta \\ &= \frac{100}{2\pi}\int_{-\infty}^{\infty} \text{sinc}\left(\frac{50\theta}{\pi}\right)\frac{j(\omega-\theta)}{3(3+j(\omega-\theta))}e^{j\frac{2}{3}((\omega-\theta)+200\pi)}\text{rect}\left(\frac{\omega-\theta+200\pi}{12\pi}\right)d\theta\end{aligned}$$

2. (10 %) Let $x(t)$ be any signal with Fourier transform $X(j\omega)$. The frequency-shift property of the Fourier transform may be stated as

$$e^{j\omega_0 t}x(t) \overset{FT}{\leftrightarrow} X(j(\omega-\omega_0)).$$

(a) Prove the frequency-shift property by applying the frequency shift to the analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\text{dt}.$$

(b) Prove the frequency-shift property by utilizing the Fourier transform of $e^{j\omega_0 t}$ in conjunction with the multiplication property of the Fourier transform.

Solution:

(a) We are tasked with proving the frequency-shift property of the Fourier transform:

$$e^{j\omega_0 t}x(t) \overset{FT}{\leftrightarrow} X(j(\omega-\omega_0)).$$

Start with the Fourier transform of $y(t) = e^{j\omega_0 t}x(t)$:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}\,dt$$

Substitute $y(t) = e^{j\omega_0 t}x(t)$:

$$Y(j\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}\,dt$$

Simplify the exponent:

$$Y(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\,dt$$

This is the Fourier transform of $x(t)$ evaluated at $\omega - \omega_0$, so:

$$Y(j\omega) = X(j(\omega - \omega_0))$$

Thus, we have shown:

$$e^{j\omega_0 t}x(t) \overset{FT}{\leftrightarrow} X(j(\omega - \omega_0)).$$

(b) We are tasked with proving the frequency-shift property using the Fourier transform of $e^{j\omega_0 t}$ and the multiplication property of the Fourier transform.
The Fourier transform of $e^{j\omega_0 t}$ is:

$$e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

The multiplication property of the Fourier transform states that if $x(t)$ has Fourier transform $X(j\omega)$, then:

$$x(t)e^{j\omega_0 t} \overset{FT}{\leftrightarrow} X(j(\omega - \omega_0))$$

Thus, the Fourier transform of $y(t) = e^{j\omega_0 t}x(t)$ is the convolution of the Fourier transforms of $e^{j\omega_0 t}$ and $x(t)$:

$$Y(j\omega) = (2\pi\delta(\omega - \omega_0)) * X(j\omega)$$

The convolution of $2\pi\delta(\omega - \omega_0)$ and $X(j\omega)$ is:

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega' - \omega_0)X(j(\omega - \omega'))\,d\omega'$$

Using the sifting property of the Dirac delta function, we obtain:

$$Y(j\omega) = X(j(\omega - \omega_0))$$

Therefore, we have shown that:

$$e^{j\omega_0 t}x(t) \overset{FT}{\leftrightarrow} X(j(\omega - \omega_0))$$

This completes the proof of the frequency-shift property.

3. (20 %) Consider a communication system.Suppose that we use amplitude modulation to send the bit sequence '110' using three pulses. Let the symbol period $T = 0.5\mu s$.

   (a) Suppose that the transmitted signal is $a(t) = \text{rect}(t/T) + \text{rect}((t-T)/T) - \text{rect}((t-2T)/T)$. Plot this signal in Matlab and explain what is happening. We have seen a variation of this signal before in a previous homework.

   (b) What is the Fourier transform of $a(t)$? Plot $|A(j\omega)|$ in Matlab and explain the shape and the connection to $T = 0.5\mu s$. Does this signal use a little spectrum or a lot of spectrum?

(c) Suppose that the transmitted signal is $b(t) = \text{sinc}(t/T) + \text{sinc}((t-T)/T) - \text{sinc}((t-2T)/T)$. Plot this signal in Matlab and explain what is happening. How is it different than using square pulses?

(d) What is the Fourier transform of $b(t)$? Plot $|A(j\omega)|$ in Matlab and explain the shape and the connection to $T = 0.5\mu s$. Does this signal use a little spectrum or a lot of spectrum?

(e) Now suppose that we use wireless communications to instead send the bits. Suppose that $x(t) = b(t)\cos(2\pi 2.4 \times 10^9 t)$. The frequency of this sinusoid is 2.4 GHz, which is known as the *carrier frequency* as it "carries" the pulses. Plot the spectrum of the resulting signal $X(j\omega)$ in Matlab. You may plot in this case for values of $\omega > 0$. Explain what has happened. Interpret the results in terms of the FCC spectrum chart. Is 2.4 GHz meaningful in a practical system?

(f) The *sinc* pulses are infinitely long in the time domain. That is not very practical. What if we just truncated the transmitted signal and instead send $\hat{x}(t) = x(t)$ for $t = (-3T, 3T)$? Compare the spectrum of $X(j\omega)$ and $\hat{X}(j\omega)$ in Matlab. Explain what has happened.

<u>Solution:</u> Solution to this question is uploaded in piazza.

4. (20 %) Let us reconsider the building of a square wave for the Fourier series coefficients. Consider the following example:

```
openExample('matlab/xfourier')
```

You may have to type this in to avoid font issues in the copy and paste. This examples opens an `.mlx` file. This is a type of computational notebook, similar to those used extensively in machine learning. Save the example with a new file name in your own workspace.

In this problem, we see the importance of the phase, which we often do not discuss in class. Modify this line in the "finale" of the example

```
%x = x + sin(k*t)/k;
theta = 2*pi*rand([1,1]); % new line
x = x + sin(k*t+theta)/k; % new line
```

Run the example a few times and try to understand the effect. You might play with different values of `theta`.

Now create a MATLAB live script to teach what you have observed. Essentially, modify this file and the explanation as if you were teaching a future student in the course. For example, you might start by updating the title of the document and explanation and adding some sections at the end to explain further.

<u>Solution:</u>

Important Observations to be noted:

(i) When the phase of a signal changes, its amplitude and frequency do not change. So, even after randomly shifts in phase, all harmonics still have the same amplitude and frequency response. There are no changes in Fourier Series coefficients here. Frequency remains the same, and hence the harmonic contribution to the total signal; does not change. What changes is the time at which this harmonic starts with respect to other signals, since all harmonics are out of phase.

(ii) When there is a phase difference between different signals, the original overlapping points change. Due to this, during the reconstruction of the original signal from these Fourier Series Components (harmonics), signals start overlapping randomly, resulting in a distorted square wave.

Other Observations:

(i) Even multiples of the harmonics together give a saw-tooth wave.

(ii) Odd multiples contribute to building a square wave.

(iii) The final output of randomly shifted sine signals is a distorted square signal, but it is periodic.

5. (10 %) Consider the sign function

$$\mathrm{sgn}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases}$$

(a) Does $\mathrm{sgn}(t)$ satisfy the finite energy sufficient condition for a Fourier transform to exist? Recall that the energy of a signal $x(t)$ is defined as $\int_{-\infty}^{\infty} |x(t)|^2 dt$.

(b) Does $\mathrm{sgn}(t)$ satisfy the Dirichlet conditions for a Fourier transform to exist?

(c) Using the Fourier transform of the unit step function $u(t)$, suggest a Fourier transform for $\mathrm{sgn}(t)$.

Solution:

(a) Check the energy of $\mathrm{sgn}(t)$:

$$\int_{-\infty}^{\infty} |\mathrm{sgn}(t)|^2 dt = \int_{-\infty}^{0} |-1|^2 dt + \int_{0}^{\infty} |1|^2 dt$$

so, no it does not satisfy the conditions.

(b) No because $\int_{-\infty}^{\infty} |\mathrm{sgn}(t)| dt = \infty$ and therefore absolute integrability is not satisfied.

(c) From the table:

$$u(t) \Longleftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$
$$\mathrm{sgn}(t) = 2u(t) - 1$$

Therefore:

$$\begin{aligned} \mathcal{F}\{\mathrm{sgn}(t)\} &= 2\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) - (2\pi\delta(\omega)) \\ &= \frac{2}{j\omega} \end{aligned}$$