ECE 45 – Circuits and Systems
Winter 2025

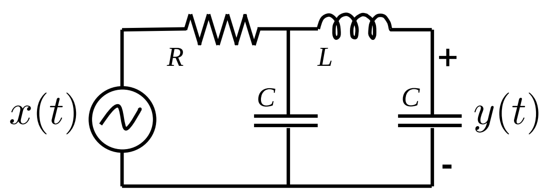
Homework #6

Due: February 13 at 11:59pm, submitted via GradeScope.

You can make multiple upload attempts to experiment with the system and the best way to upload. You must correctly mark the answers to the problems in GradeScope, e.g. problem 1, problem 2, problem 3, to get full credit. Note that you must tag your problems when uploading to GradeScope or they will not be graded and you will not receive credit. Any regrade requests must be placed through GradeScope within one week of the return of the homework.

Remember, discussion of homework questions is encouraged. Please be absolutely sure to submit your own independent homework solution.

1. (15 %) Consider the following circuit with input as voltage $x(t)$ and output voltage $y(t)$.



(a) Find the differential equation relating the input and output.

Solution:

With the current for the right capacitor (denoted as C_2): $i_{C_2}(t) = C\frac{dy(t)}{dt}$, and the voltage of the inductor: $v_L(t) = L\frac{di_{C_2}(t)}{dt}$, we have the voltage for the left capacitor (denoted as C_1): $v_{C_1}(t) = y(t) + L\frac{di_{C_2}(t)}{dt}$, and its current

$$\begin{aligned} i_{C_1}(t) &= C\frac{dv_{C_1}(t)}{dt} \\ &= C\frac{d(y(t) + L\frac{di_{C_2}(t)}{dt})}{dt} \\ &= Cy'(t) + C^2Ly'''(t). \end{aligned}$$

The current of R is $i_{C_1}(t) + i_{C_2}(t)$, and the total voltage can be expressed as

$$\begin{aligned} x(t) &= (i_{C_1}(t) + i_{C_2}(t))R + y(t) + v_L(t) \\ &= (Cy'(t) + C^2Ly'''(t) + Cy'(t))R + y(t) + LCy''(t) \\ &= y(t) + 2CRy'(t) + LCy''(t) + C^2LRy'''(t) \end{aligned}$$

(b) Find the frequency response $H(j\omega)$.
Solution:

$$\begin{aligned} H(j\omega) &= \frac{1}{1 + 2CRj\omega + LC(j\omega)^2 + C^2LR(j\omega)^3} \\ &= \frac{1}{1 - LC\omega^2 + j2CR\omega - jC^2LR\omega^3} \end{aligned}$$

(c) Prove that $H^*(j\omega) = H(-j\omega)$.
Solution:
Left-hand side:

$$\begin{aligned} H^*(j\omega) &= \frac{1}{(1 - LC\omega^2 + j2CR\omega - jC^2LR\omega^3)^*} \\ &= \frac{1}{1 - LC\omega^2 - j2CR\omega + jC^2LR\omega^3} \end{aligned}$$

Right-hand side:

$$\begin{aligned} H(-j\omega) &= \frac{1}{1 - LC\omega^2 - j2CR\omega + jC^2LR\omega^3} \\ &= H^*(j\omega) \end{aligned}$$

(d) Find $y(t)$ if $x(t) = e^{j\omega t}$.
Solution:

$$\begin{aligned} y(t) &= H(j\omega)e^{j\omega t} \\ &= \frac{e^{j\omega t}}{1 - LC\omega^2 + j2CR\omega - jC^2LR\omega^3} \end{aligned}$$

(e) Find $y(t)$ if $x(t) = e^{-j\omega t}$.
Solution:

$$\begin{aligned} y(t) &= H(-j\omega)e^{-j\omega t} \\ &= \frac{e^{-j\omega t}}{1 - LC\omega^2 - j2CR\omega + jC^2LR\omega^3} \end{aligned}$$

(f) Find $y(t)$ if $x(t) = \cos(\omega t)$. Simplify as much as possible
Solution:

$$\begin{aligned} y(t) &= \frac{1}{2}H(j\omega)e^{j\omega t} + \frac{1}{2}H(-j\omega)e^{-j\omega t} \\ &= \frac{1}{2}H(j\omega)e^{j\omega t} + \frac{1}{2}H^*(j\omega)e^{-j\omega t} \end{aligned}$$

Because of conjugate symmetry

$$\begin{aligned} y(t) &= 2\mathrm{Re}\left[\frac{1}{2}H(j\omega)e^{j\omega t}\right] \\ &= |H(j\omega)|\cos(\omega t + \angle H(j\omega)), \end{aligned}$$

where

$$|H(j\omega)| = \frac{1}{\sqrt{(1 - LC\omega^2)^2 + (2CR\omega - C^2LR\omega^3)^2}}$$

and

$$\angle H(j\omega) = \tan^{-1}\left(-\frac{2CR\omega - C^2LR\omega^3}{1 - LC\omega^2}\right)$$

(g) Find $y(t)$ if $x(t) = 2\sin(24\pi t) + \cos(45\pi t - \pi/3)$.
Solution:

$$\begin{aligned} x(t) &= 2\sin(24\pi t) + \cos(45\pi t - \pi/3) \\ &= 2\cos(24\pi t - \pi/2) + \cos(45\pi t - \pi/3) \end{aligned}$$

$$\begin{aligned} y(t) &= \mathrm{Re}\left[2e^{-j\frac{\pi}{2}} H(j24\pi) e^{j24\pi t})\right] + \mathrm{Re}\left[e^{-j\frac{\pi}{3}} H(j45\pi) e^{j45\pi t}\right] \\ &= 2|H(j24\pi)|\cos\left(24\pi t - \pi/2 + \angle H(j24\pi)\right) + |H(j45\pi)|\cos\left(45\pi t - \pi/3 + \angle H(j45\pi)\right), \end{aligned}$$

where $|H(j\omega)|$ and $\angle H(j\omega)$ are derived in 1.(f).

2. (10 %) Solve OW problem 3.21 but take $T = 6$. Also present the Cartesian form.

Solution: The period is $T = 6$, so the fundamental frequency is:

$$w_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

The Fourier series representation in its complex exponential form is:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

Substituting the given coefficients into the Fourier series expression, we get:

$$\begin{aligned} x(t) &= a_{-5}e^{-j5\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_1 e^{j\omega_0 t} + a_5 e^{j5\omega_0 t} \\ &= 2e^{-j5(\pi/3)t} - je^{-j(\pi/3)t} + je^{j(\pi/3)t} + 2e^{j5(\pi/3)t} \end{aligned}$$

Combining terms with positive and negative indices:

$$x(t) = 2\left[e^{j5(\pi/3)t} + e^{-j5(\pi/3)t}\right] + j\left[e^{j(\pi/3)t} - e^{-j(\pi/3)t}\right]$$

Using Euler's formulas, $\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$ and $\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j}$, we have:

$$\begin{aligned} x(t) &= 2\left[2\cos\left(\frac{5\pi}{3}t\right)\right] + j\left[2j\sin\left(\frac{\pi}{3}t\right)\right] \\ &= 4\cos\left(\frac{5\pi}{3}t\right) - 2\sin\left(\frac{\pi}{3}t\right) \end{aligned}$$

We can also express the sine term as a cosine with a phase shift:

$$x(t) = 4\cos\left(\frac{5\pi}{3}t\right) + 2\cos\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$$

3. (50 %) Compute the Fourier series coefficients of the following signals.

(a) Periodic function with period 2π given by

$$x(t) = t \qquad \forall t \in [-\pi, \pi].$$

Solution:

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T} \\ &= 1 \end{aligned}$$

Therefore

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-jkt}dt \\ &= \frac{1}{2\pi}\cdot\frac{1}{-jk}\int_{-\pi}^{\pi} tde^{-jkt} \\ &= \frac{1}{-j2\pi k}\left(te^{-jkt}\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-jkt}dt\right) \\ &= \frac{j}{2\pi k}\left(\pi e^{-jk\pi} + \pi e^{jk\pi} - \frac{e^{-jkt}}{-jk}\Big|_{-\pi}^{\pi}\right) \\ &= \frac{j}{\pi k}\left(\pi\cos(k\pi) - \frac{\sin(k\pi)}{k}\right) \end{aligned}$$

From $\frac{\sin(k\pi)}{k} = 0, \forall k \in \mathbb{Z}, k \neq 0$, and $\cos(k\pi) = (-1)^k, \forall k \in \mathbb{Z}$, we have

$$a_k = \frac{j(-1)^k}{k}, k \neq 0$$

For $k = 0$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} tdt \\ &= 0 \end{aligned}$$

(b) Periodic function with period $T = 6$ with first period $t \in [0, 6)$ given by

$$x(t) = \text{rect}\left(\frac{t-2}{2}\right) + \text{rect}\left(\frac{t-1}{2}\right).$$

Solution:

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T} \\ &= \pi/3 \end{aligned}$$

Therefore

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{6}\int_0^6 x(t)e^{-jkt\pi/3}dt \\ &= \frac{1}{6}\int_0^1 e^{-jkt\pi/3}dt + \frac{1}{3}\int_1^2 e^{-jkt\pi/3}dt + \frac{1}{6}\int_2^3 e^{-jkt\pi/3}dt \\ &= \frac{-e^{-jk\pi/3} - 1 + e^{-jk2\pi/3} + e^{-jk\pi}}{-jk2\pi}, k \neq 0 \end{aligned}$$

For $k = 0$

$$\begin{aligned} a_0 &= \frac{1}{6}\int_{-\pi}^{\pi} x(t)dt \\ &= 2/3 \end{aligned}$$

(c) Periodic function with $T = 8$ with first period $t \in [-4, 4)$ given by

$$x(t) = \text{rect}\,(t - 5/2) - \text{rect}\,(t + 5/2)$$

<u>Solution:</u>

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T} \\ &= \pi/4 \end{aligned}$$

Therefore

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{8}\int_{-4}^{4} x(t)e^{-jkt\pi/4}dt \\ &= -\frac{1}{8}\int_{-3}^{-2} e^{-jkt\pi/4}dt + \frac{1}{8}\int_2^3 e^{-jkt\pi/4}dt \\ &= \frac{-e^{jk\pi/2} + e^{3jk\pi/4} + e^{-jk3\pi/4} - e^{-jk\pi/2}}{-jk2\pi} \\ &= \frac{2\cos(3k\pi/4) - 2\cos(k\pi/2)}{-jk2\pi}, k \neq 0 \end{aligned}$$

For $k = 0$

$$\begin{aligned} a_0 &= \frac{1}{8}\int_{-\pi}^{\pi} x(t)dt \\ &= 0 \end{aligned}$$

(d) Periodic function with period $T = 4$ given by

$$x(t) = \sum_{m=-\infty}^{\infty} \delta(t - 4m) - \delta(t - 2 - 4m).$$

Solution:

$$\begin{aligned}\omega_0 &= \frac{2\pi}{T}\\ &= \pi/2\end{aligned}$$

Therefore

$$\begin{aligned}a_k =& \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\\ =& \frac{1}{4}\int_0^4 x(t)e^{-jkt\pi/2}dt\\ =& \frac{1}{4}\left(e^{-jk\pi/2\cdot 0} - e^{-jk\pi/2\cdot 2}\right)\\ =& \frac{1}{4}(1 - e^{-jk\pi})\end{aligned}$$

4. (30 %) Consider the Fourier Series representation of a periodic signal

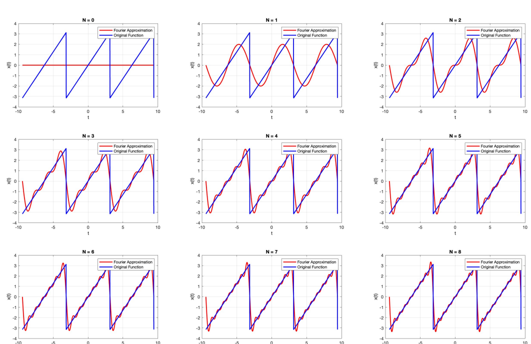
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

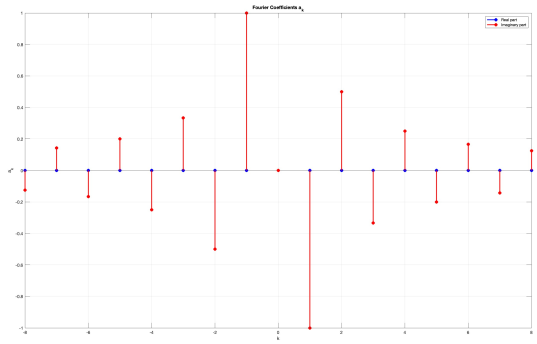
Let us denote the approximation of $x(t)$ obtained from using N harmonics as

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

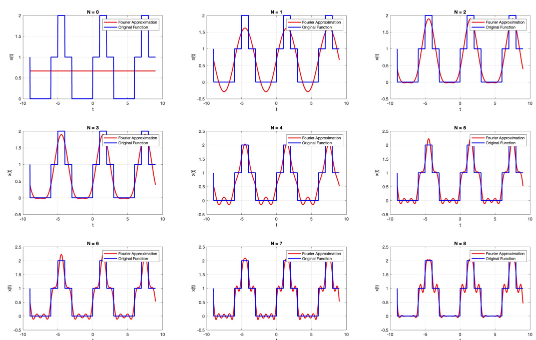
For each the previous signals in (a)-(c), plot $x(t)$ from $-3T/2$ to $3T/2$. On each plot, including $x(t)$, $x_0(t)$, $x_1(t)$, $\ldots$, $x_8(t)$. On a separate plot, plot the real and imaginary Fourier Series coefficient $\{a_k\}_{k=-8}^{8}$. Be sure to label the results and use markers or and/or different line types.
Solution:

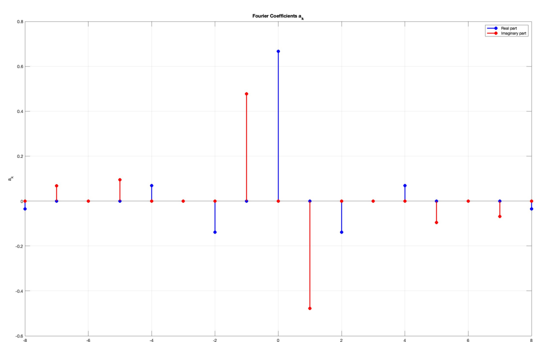
(a) Part a



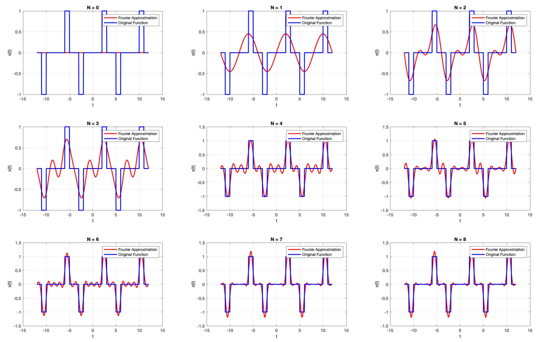


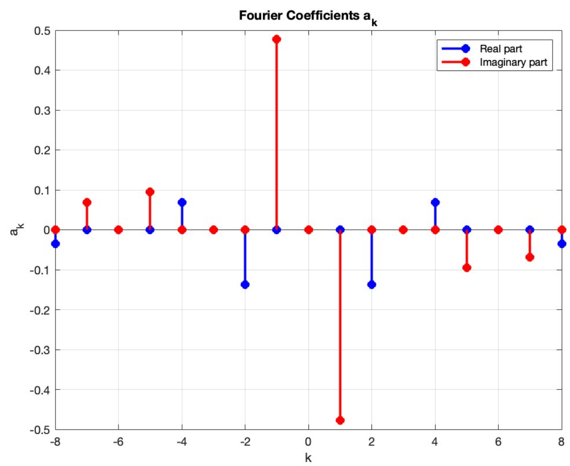
(b) Part b





(c) Part c





5. (20 %) Based on the solutions, correct your previous week's homework using a colored pen (or annotation) so it's obvious what you've corrected. If you got a problem exactly right, just use a red check mark to indicate as such.