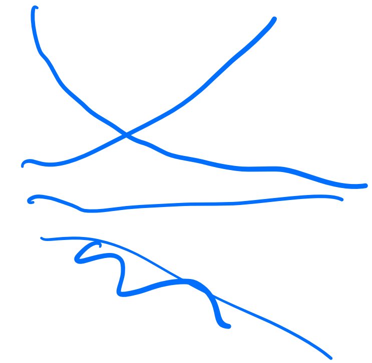
Lecture 4

~~Exponential, sinusoids, complex exponentials, and the delta function~~

Systems, Time invariance, linearity

Complex exponential signals



- General form of complex exponential

$$x(t) = Ce^{at}$$

- Simplifying with $\quad C = |C|e^{j\theta}$ and $a = r + j\omega_0$

$$x(t) = |C|e^{j\theta}e^{rt+j\omega_0 t}$$
$$= |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

not periodic

- Complex sinusoid is the special case $r = 0$

$$x(t) = |C|e^{j(\omega_0 t+\theta)}$$

$\omega_0 = \frac{2\pi}{T}$

Example

- Suppose $C = 0.25\, e^{j0.2\pi}$

  $a = \underbrace{0.5}_{r} + \underbrace{j2\pi1000}_{\omega_0}$

$|x(t)| = 0.25\, e^{t/2}$

$\angle x(t) = 0 + 0 + \underbrace{2\pi1000t + 0.2\pi}_{\text{unwrapped phase}}$

- Determine

$$x(t) = 0.25\, e^{t/2}\, e^{j(2\pi1000t + 0.2\pi)}$$

$$\overline{0.25\, e^{t/2}}\ \{\overline{\cos(2\pi1000t + 0.2\pi)}$$

$$+ j\sin(2\pi1000t + 0.2\pi)\}$$

$$\mathrm{Re}\{x(t)\} = 0.25\, e^{t/2} \cos(2\pi1000t + 0.2\pi)$$

Delta function

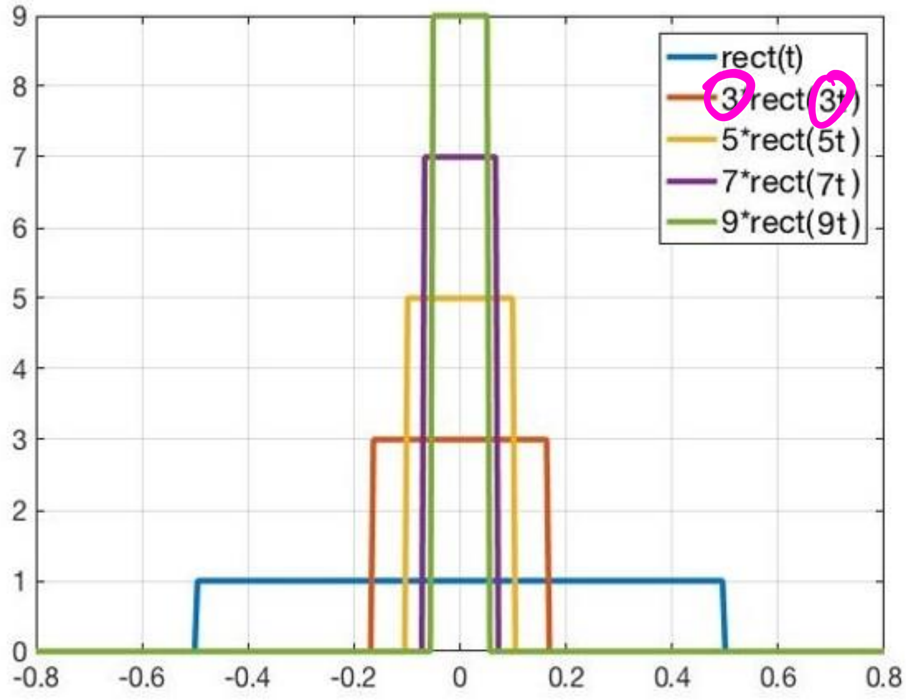
rect(at)

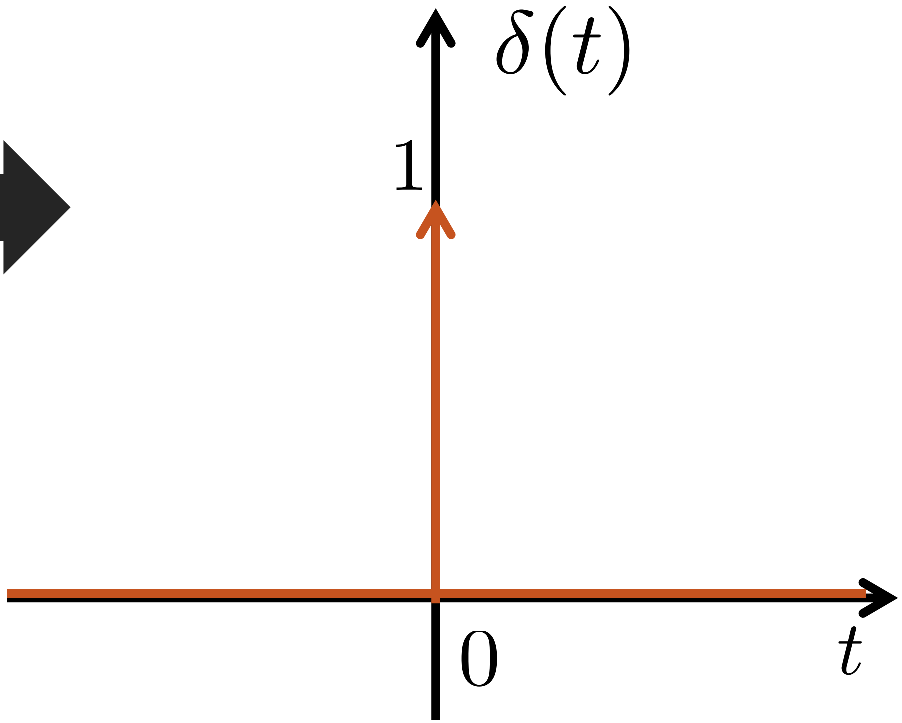
at=1/2

$1/a$

$-1/2a$ $1/2a$ t

Dirac delta or unit-impulse function





Delta function as the limit of a sequence of every narrowing unit energy rectangles

Summary of the delta function

- Sifting with deltas pulls out the signal value but leaves the delta

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

1

- Integrating with deltas eliminates the delta and gives a value

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

$$\int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\cancel{\int_{-\infty}^{\infty}\delta(t-t_0)\,dt}$$

- Other properties

$$\int_{-\infty}^{\infty}\delta(t)dt = 1 \qquad \boxed{\delta(at) = \frac{1}{|a|}\delta(t)}$$

Examples

- Simplify the following

1. $(3r^2 + 2r + 1)\delta(r - 1)$ $= (3 \cdot 1^2 + 2 \cdot 1 + 1)\delta(r-1)$ $= \boxed{6\delta(r-1)}$ 1

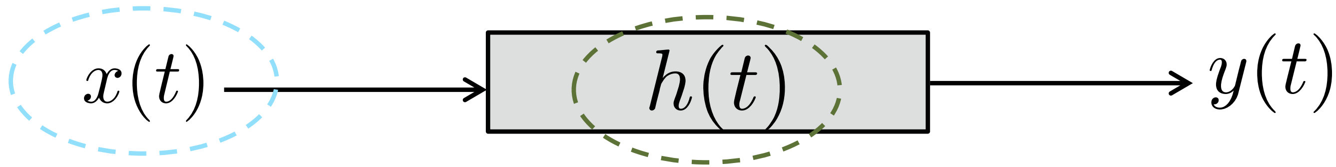
2. $\int_{-\infty}^{\infty} (3r^2 + 2r + 1)\delta(r - 1)dr$ $= 6$ 2

3. $3r^2\delta(2r)$ $= 3r^2 \cdot \frac{1}{2} \cdot \delta(r)$ $= \frac{3r^2}{2} \cdot \delta(r - 0)$

$= 0^2 \cdot \delta(r)$ $= 0$ 3

Connections back to ECE 45

Lectures 2 - 3 working with signals



Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

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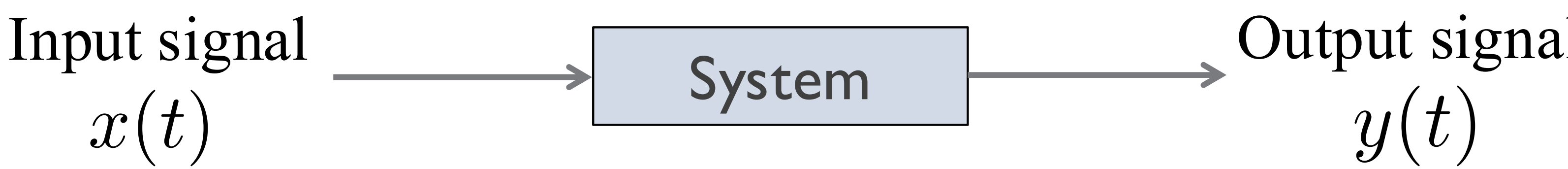
	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, Dirac Delta	1.3 - 1.4		
4	1/16	LTI in time	Systems, linearity, time invariance	1.6.5, 1.6.6	HW3	HW2
5	1/21	LTI in time	Impulse response and convolution	2.2		
6	1/23	LTI in time	Convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Introduction to Systems

Learning objectives

- Describe the output of the systems in terms of their inputs
- Give examples of continuous-time systems

Continuous-time systems



Amplifier (or all-pass amplifier)

$\cos(24\pi t)$ $\quad\quad$ $A\cos(\underbrace{24\pi t})$

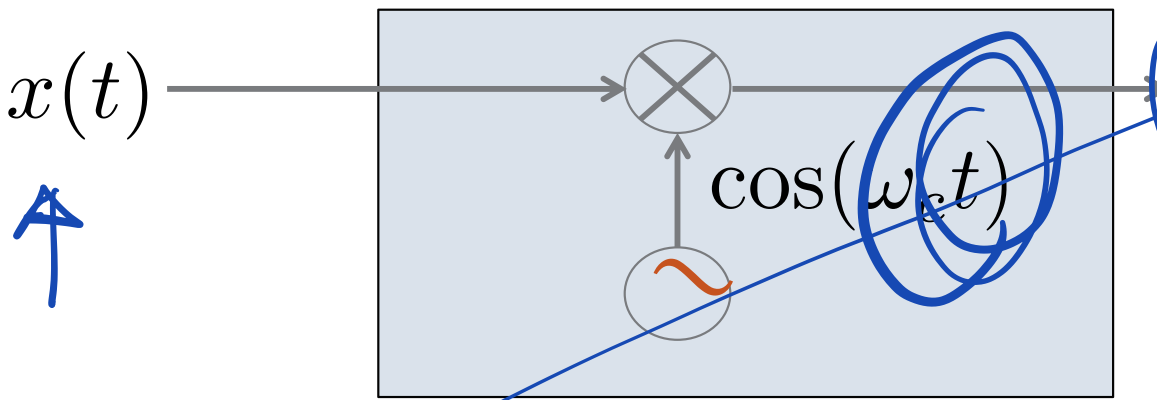
$$x(t) \longrightarrow \boxed{A} \longrightarrow y(t) = Ax(t)$$

$A > 0$ "good"

- A is the gain $\quad A > 0$

- Passes all input frequencies equally (makes more sense w/ Fourier)

Amplitude modulation



$$y(t) = x(t)\cos(\omega_c t)$$

Information "rides" the carrier

$\omega_c = 2\pi \cdot 800{,}000$

Amplitude Modulation transmitter (AM radio)

- At the receiver:

$$(x(t)\cos(\omega_c t))\cos(\omega_c t) = \cos^2(\omega_c t)x(t)$$

$$= \frac{1}{2}(1 + \cancel{\cos(2\omega_c t)})x(t)$$

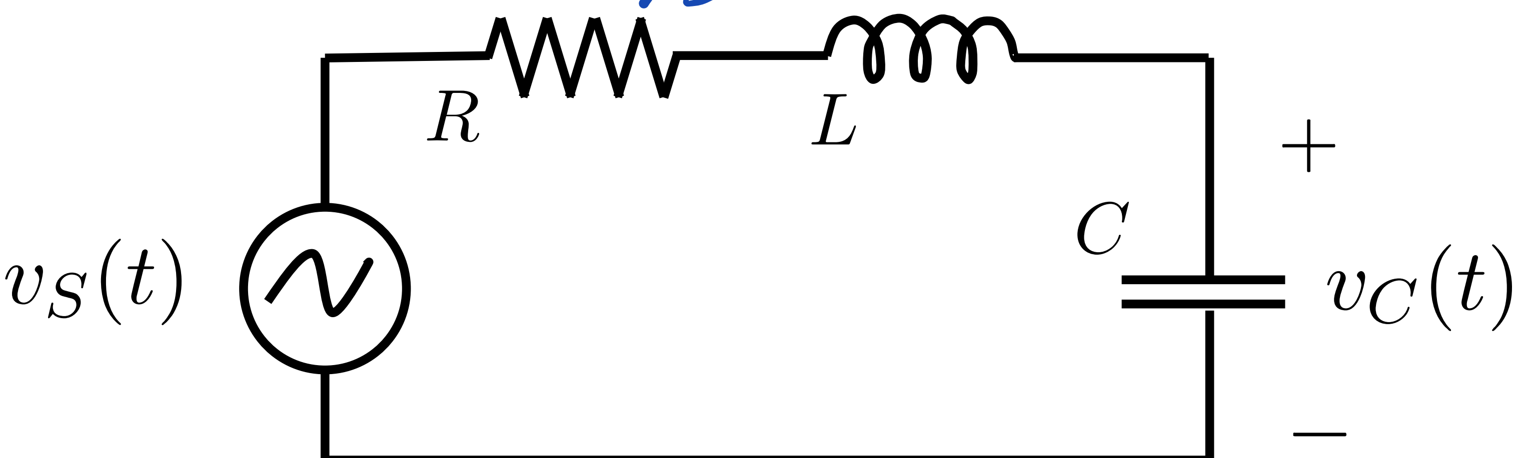
$$\longrightarrow x(t)$$

Filter out
high frequencies

Note: The filter operation only works for certain bandlimited signals x(t), not true in general (we will cover concepts of filtering and bandlimited in subsequent lectures)

RLC circuit

Example RLC circuit



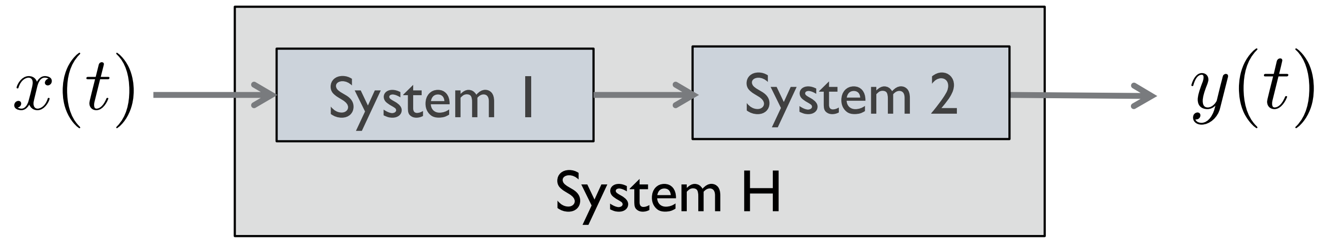
output

in

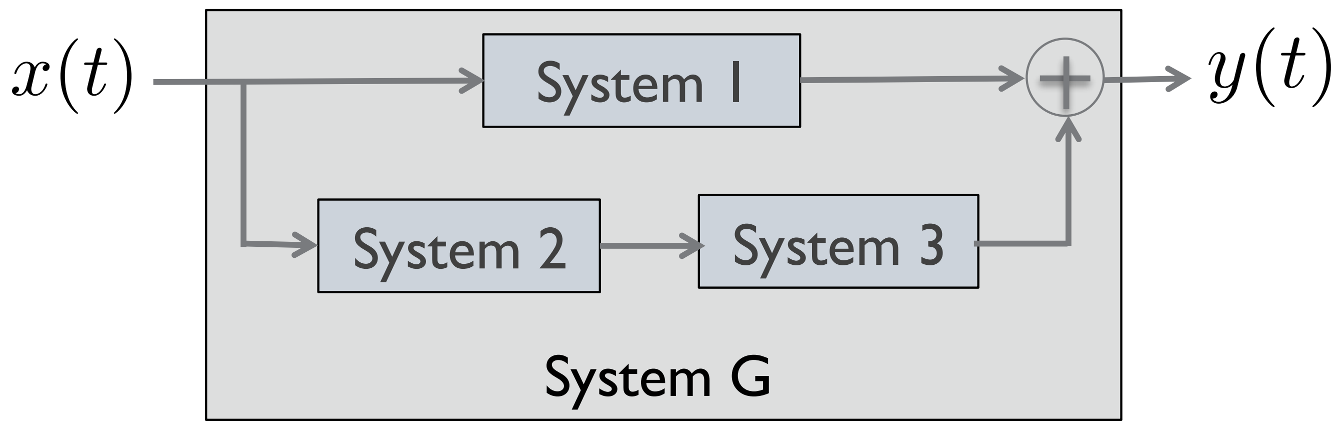
$$LC\frac{d^2 v_C(t)}{dt^2} + RC\frac{d v_C(t)}{dt} + v_C(t) = v_S(t)$$

Inter-connected systems – "systems-of-systems"

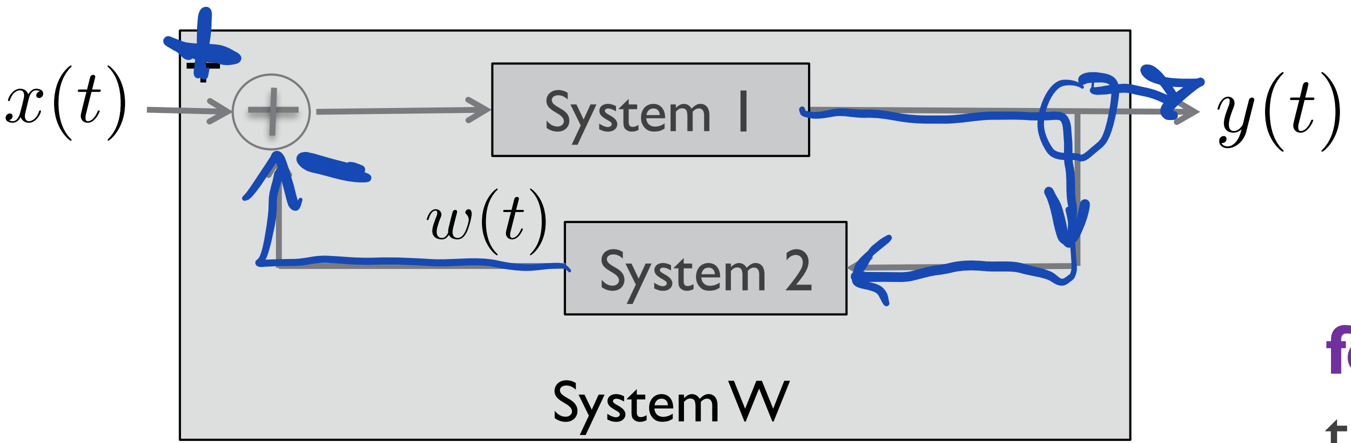
Serial



Parallel



Interconnected systems with feedback



feedback is when the output goes back into the input

Feedback is used in control systems

System introduction in summary

- A system is a functional unit that relates an input signal to an output signal
- Systems can be described in many ways including mathematically or via a block diagram
- Continuous-time systems have continuous inputs and outputs

Time invariance

Learning objectives

- Determine if a system is time invariant or time varying

Time invariant (TI)

A system is time invariant if behaves in the same way regardless of the current time

Formally: consider the system

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If

$$x(t - t_o) \longrightarrow \boxed{\text{System}} \longrightarrow y(t - t_o)$$

For all t_0 then the system is time invariant otherwise it is time varying.

How to check if the system is time-invariant?

- Method #1 (direct approach)
  - Consider the system

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

  - Apply the shifted input $x_2(t) = x_1(t - t_0)$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

  - Does the $\boxed{y_2(t) = y_1(t - t_0)?}$ If yes, the system is TI

$\forall t_0$

How to check if the system is time-invariant?

- Method #2 (counter example)

  If we suspect the system is time variant, find an example where time invariance fails "a counterexample"

  - This method is often quicker

- Note:
  - If you can not find a counter example, then you have to use Method 1
  - The counter example is just a simple way to **disprove** TI

- Hint: Usually, but not always, if the output includes any function of time other than $x(t)$, it is time-varying

Time invariance example I

System described by $y(t) = 3tx(t-3)$

$$y_1(t) = 3tx_1(t-3)$$ Output for generic input $x_1(t)$

$$y_2(t) = 3tx_2(t-3)$$ Output for generic input $x_2(t)$

Now, let: $x_2(t) = x_1(t-t_0)$

$$y_2(t) = 3tx_1(t-t_0-3)$$

$$y_1(t-t_0) = 3(t-t_0)x_1(t-t_0-3)$$

As $y_2(t)$ does not match $y_1(t-t_0) \longrightarrow$ **time-variant system**

Time invariance example 2

System described by $y(t) = x(t)x(t-1)$

$$y_1(t) = x_1(t)x_1(t-1)$$

$$y_2(t) = x_2(t)x_2(t-1)$$

$y_1(t-t_0)$
$= x_1(t-t_0)$
$\times x_1(t-t_0-1)$

Let $x_2(t) = x_1(t-t_0)$

$$y_2(t) \overset{?}{=} x_1(t-t_0)x_1(t-t_0-1)$$

$$= y_1(t-t_0)$$

The system is **time invariant (TI)**

Time invariance example 3

System with AM modulation $y(t) = x(t)\cos(\omega_c t)$

Counterexample, let $x_1(t) = \delta(t), x_2(t) = \delta(t - \frac{\pi}{2\omega_c})$

$$y_1(t) = \delta(t)$$

$$y_2(t) = \delta\left(t - \frac{\pi}{2\omega_c}\right)\cos(\omega_c t) = 0$$

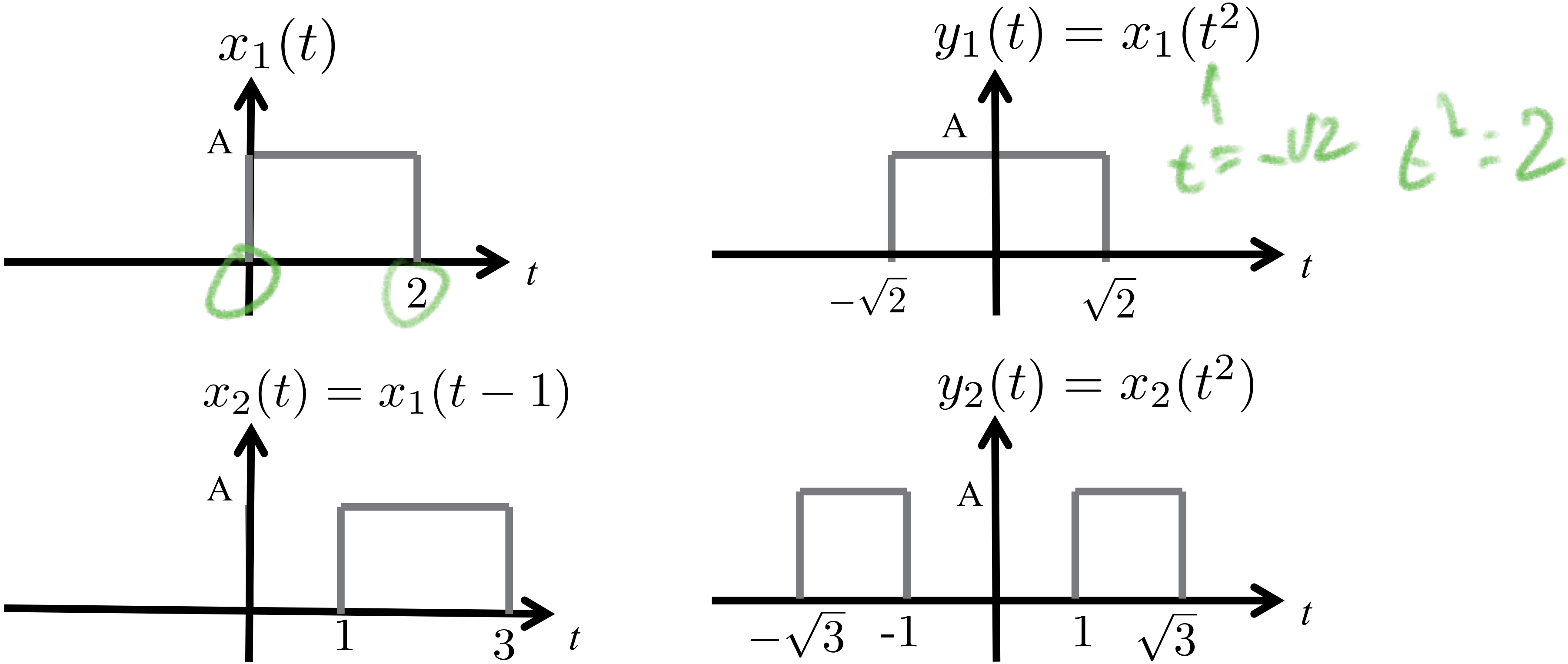
$$y_2(t) \neq y_1\left(t - \frac{\pi}{2\omega_c}\right)$$

$$= \delta\left(t - \frac{\pi}{2\omega_c}\right)$$

Time-variant

Time invariance example 4

$$y(t) = x(t^2)$$



Time-variant

(hint didn't work, Method 1 also tricky to see)

Time invariance summary

- A system is time invariant if behaves in the same way regardless of the current time
- Time invariant systems are much easier to design and analyze compared to time varying systems
- The main way to check time invariance is to shift the input and see if the output is always shifted
- A counter example is sufficient to show a system is not time invariant

Linearity

Learning objectives

- Determine if a system is linear or nonlinear
- Understand sub-properties of superposition and scaling

Scaling the input

If a system obeys the scaling property, then scaled inputs lead to scaled outputs

Consider the following system:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

If the scaling property is satisfied then for any scalar value A

$$Ax(t) \longrightarrow \boxed{\text{System}} \longrightarrow Ay(t)$$

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Superposition property

If a system obeys the superposition property, then system acts in the same way onto each system

If for two different inputs:

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

then superpositon holds if for input $x(t) = x_1(t) + x_2(t)$ then:

$$x_1(t) + x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t) + y_2(t)$$

Linear systems

A system is linear if scaling is preserved and superposition holds

Consider two different inputs and outputs

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

If the following holds for any scalar A and B then the system is linear

$$Ax_1(t) + Bx_2(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow Ay_1(t) + By_2(t)$$

How to check if the system linear?

- Direction approach
  - Check that scaling holds $Ax(t) \Rightarrow Ay(t)$
  - Check the superposition holds $x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)$
- Slightly faster direct approach
  - Check that $Ax_1(t) + Bx_2(t) \Rightarrow Ay_1(t) + By_2(t)$
- Find a counter example
  - One that may be useful (from the scaling property)

$$0 \longrightarrow \boxed{\text{System}} \longrightarrow 0$$

  - If a system generates a non-zero output to a signal that is zero for all time then it is non-linear

Linearity example I

$$y(t) = x(t)x(t-1)$$

Let us check if the scaling property holds $x_1(t) = Ax(t)$

$$\begin{aligned} y_1(t) &= x_1(t)Ax_1(t-1) \\ &= Ax(t)Ax(t-1) \\ &= A^2x(t)x(t-1) \\ &\neq Ay(t) \end{aligned}$$

Scaling fails → Nonlinear

Linearity example 1 – alternative solution

$$y(t) = x(t)x(t-1)$$

Consider inputs

$$x_1(t) \to y_1(t) = x_1(t)x_1(t-1)$$

$$x_2(t) \to y_2(t) = x_2(t)x_2(t-1)$$

$$x_3(t) = Ax_1(t) + Bx_2(t)$$

Note that

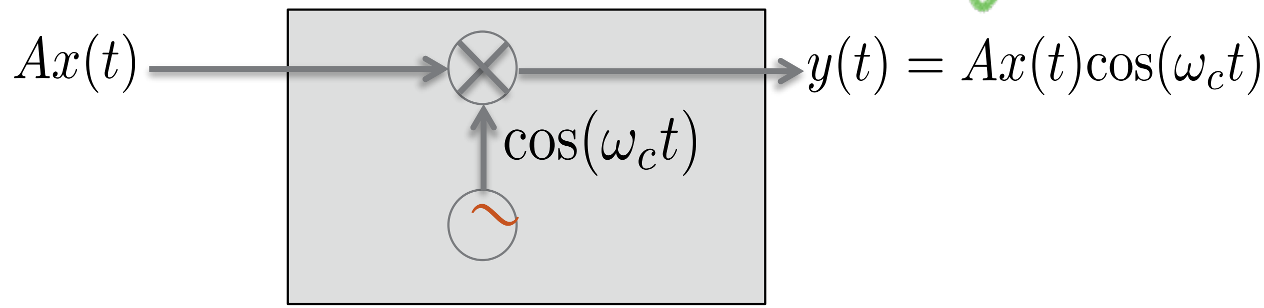
Nonlinear

$$\begin{aligned} y_3(t) &= x_3(t)x_3(t-1) \\ &= \left(Ax_1(t) + Bx_2(t)\right)\left(Ax_1(t-1) + Bx_2(t-1)\right) \\ &= A^2x_1(t) + B^2x_2(t) + ABx_1(t)x_2(t-1) + ABx_1(t-1)x_2(t) \\ &\neq Ay_1(t) + By_t(t) = Ax_1(t)x_1(t-1) + Bx_2(t)x_2(t-1) \end{aligned}$$

Linearity example 2

System with AM modulation

Scaling is preserved



What about superposition?

$$y_1(t) = x_1(t)\cos\omega_c t$$

$$y_2(t) = x_2(t)\cos\omega_c t$$

$$\begin{aligned} x(t) = x_1(t) + x_2(t) \longrightarrow y(t) &= (x_1(t) + x_2(t))\cos\omega_c t \\ &= x_1(t)\cos\omega_c t + x_2(t)\cos\omega_c t \\ &= y_1(t) + y_2(t) \end{aligned}$$

Linear system

Linearity example 3

- Consider the affine system

$$y[n] = 2x[n] + 1$$

- Suppose that

$$x_1[n] \rightarrow y_1[n]$$

$$x_2[n] \rightarrow y_2[n] \text{ where } x_2[n] = Ax_1[n]$$

- Now observe that

$$\begin{aligned} y_2[n] &= 2x_2[n] + 1 \\ &= 2Ax_1[n] + 1 \\ &\neq Ay_1[n] \end{aligned}$$

Scaling does not hold therefore is a **nonlinear system**

Linearity summary

- A system linear if *scaled inputs* lead to *scaled outputs* and the *sum of inputs* leads to a *sum of outputs* if the inputs were applied separately
- Linear systems are easier to design and analyze
- Many systems in practice are nonlinear but are designed to be as linear as possible, or are only used with inputs where they behave in a linear fashion
- Need to check both the scaling and superposition properties to prove that a system is linear

Reference example with details

- Is the following system linear? Time-invariant?

$$y(t) = t^2 x(t-1)$$

Reference example – checking time invariance

Consider the output to input $x_1(t)$

$$y_1(t) = t^2 x_1(t-1)$$

Define a new input

$$x_2(t) = x_1(t - t_0)$$

Compute the output

$$\begin{aligned} y_2(t) &= t^2 x_2(t-1) \\ &= t^2 x_1(t - 1 - t_0) \end{aligned}$$

Not time invariant!

Compare with a shifted version of the first output

$$y_1(t - t_0) = (t - t_0)^2 x_1(t - 1 - t_0) \neq y_2(t)$$

Reference example – checking linearity

Consider the inputs and outputs

$$x_1(t) \rightarrow y_1(t) = t^2 x_1(t-1)$$

$$x_2(t) \rightarrow y_2(t) = t^2 x_2(t-1)$$

Define a new input

$$x_3(t) = a x_1(t) + b x_2(t)$$

Compute the output

Linear!

$$\begin{aligned} y_3(t) &= t^2 x_3(t-1) \\ &= t^2 (a x_1(t-1) + b x_2(t-1)) \\ &= a y_1(t) + b y_2(t) \end{aligned}$$