ECE 45 – Circuits and Systems
Winter 2025

Homework #5

Due: February 6 at 11:59pm, submitted via GradeScope.

You can make multiple upload attempts to experiment with the system and the best way to upload. You must correctly mark the answers to the problems in GradeScope, e.g. problem 1, problem 2, problem 3, to get full credit. Note that you must tag your problems when uploading to GradeScope or they will not be graded and you will not receive credit. Any regrade requests must be placed through GradeScope within one week of the return of the homework.

Remember, discussion of homework questions is encouraged. Please be absolutely sure to submit your own independent homework solution.

1. (50 %) Consider an LTI system with the specified impulse response $h(t) = e^{-2t}u(t)$.

   (a) Find the transfer function $H(s)$ for this system and also the range of values of s for which the integral converges. This is known as the region of convergence. NOTE: The region of convergence and Laplace transforms are dealt with in other courses, this aspect of the problem is just provided here to make the problem complete. You will not need to be able to solve for the region of convergence for this coures in general.

   Solution: Here, we need to compute the eigenvalue for the LTI system with impulse response $h(t)$ in response to the complex exponential e^{st}. Computing $H(s)$ from $h(t)$:

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-2t}e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+2)t}dt \\ &= \frac{1}{s+2}. \end{aligned}$$

   An important point to note is that the last integral only converges for values where $\mathrm{Re}(s) > -2$. This notion of region of convergence is something that is associated with the Laplace transform and is beyond the scope of this course. In this course, we only compute $H(s)$ to get $H(j\omega)$. In other problems going forward, you will not need to deal with this convergence issue.

   (b) Find the frequency response $H(j\omega)$ for this system.

   Solution: Substituting $s = j\omega$:

$$H(j\omega) = \frac{1}{j\omega + 2}.$$

(c) Determine the output if the input is e^{-3t}.

Solution: Since e^{-3t} is an eigenfunction of an LTI system, the output is given by:

$$\begin{aligned} y(t) &= H(-3)e^{-3t} \\ &= \frac{1}{-3+2}e^{-3t} \\ &= -e^{-3t}. \end{aligned}$$

(d) Determine the output if the input is 3.

Solution: Since 3 is a constant input, using the property of LTI systems:

$$\begin{aligned} y(t) &= 3H(0) \\ &= 3\frac{1}{0+2} \\ &= \frac{3}{2}. \end{aligned}$$

(e) Determine the output if the input is $3\delta(t-1)$.

Solution: Eigenfunction property is not applied for $\delta(t-1)$, but according to the property of $\delta(t-1)$, we have

$$\begin{aligned} y(t) &= 3h(t-1) \\ &= 3e^{-2t+2}u(t-1). \end{aligned}$$

(f) Determine the output if the input is $\cos(100\pi t)$.

Solution: Decompose using Euler's identity:

$$\cos(100\pi t) = \frac{1}{2}e^{j100\pi t} + \frac{1}{2}e^{-j100\pi t}$$

The output is

$$\begin{aligned} y(t) &= \frac{1}{2}H(j100\pi)e^{j100\pi t} + \frac{1}{2}H(-j100\pi)e^{-j100\pi t} \\ &= \frac{e^{j100\pi t}}{2(j100\pi+2)} + \frac{e^{-j100\pi t}}{2(-j100\pi+2)} \\ &= \frac{e^{j100\pi t}(-j100\pi+2) + e^{-j100\pi t}(j100\pi+2)}{2(100^2\pi^2+4)} \\ &= \frac{-j100\pi(e^{j100\pi t} - e^{-j100\pi t}) + 2(e^{j100\pi t} + e^{-j100\pi t})}{2(100^2\pi^2+4)} \\ &= \frac{100\pi\sin(100\pi t) + 2\cos(100\pi t)}{100^2\pi^2+4} \end{aligned}$$

(g) Determine the output if the input is $2e^{j2\pi 10t}$.

Solution: Substitute $s = j2\pi 10$ to $H(s)$

$$\begin{aligned} y(t) &= H(j2\pi 10) \cdot 2e^{j2\pi 10t} \\ &= \frac{2e^{j2\pi 10t}}{j2\pi 10 + 2} \\ &= \frac{e^{j2\pi 10t}}{j10\pi + 1} \end{aligned}$$

(h) Determine the output if the input is $e^{-3t}u(t)$.

Solution:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{-3\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)u(t-\tau)d\tau \\ &= \int_{0}^{t} e^{-3\tau}e^{-2t+2\tau}d\tau \\ &= e^{-2t}(1-e^{-t}) \\ &= e^{-2t} - e^{-3t} \end{aligned}$$

2. (30 %) Write the frequency response for the following systems described by differential equations related input $x(t)$ to output $y(t)$.

(a) $y(t) + \frac{1}{2}\frac{d}{dt}y(t) = x(t)$

Solution: In this problem, we use the following observation that for a differential equation

$$\begin{aligned} & a_N\frac{d^N y}{dt^N} + a_{N-1}\frac{d^{N-1}y}{dt^{N-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) \\ = \; & b_M\frac{d^M x}{dt^M} + b_{M-1}\frac{d^{M-1}x}{dt^{M-1}} + \cdots + b_1\frac{dx}{dt} + b_0 x(t) \end{aligned}$$

the transfer function and frequency response can be computed as

$$\begin{aligned} H(s) &= \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \\ H(j\omega) &= \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \end{aligned}$$

With the equation given

$$\begin{aligned} H(s) &= \frac{1}{1+\frac{s}{2}} \\ H(j\omega) &= \frac{1}{1+\frac{j\omega}{2}} \end{aligned}$$

(b) $y(t) = x(t) - \frac{1}{2}\frac{d}{dt}y(t)$

Solution:

$$H(s) = \frac{1}{1+\frac{s}{2}}$$
$$H(j\omega) = \frac{1}{1+\frac{j\omega}{2}}$$

(c) $y(t) = x(t) - \frac{1}{2}\frac{d}{dt}x(t)$

Solution:

$$H(s) = 1 - \frac{s}{2}$$
$$H(j\omega) = 1 - \frac{j\omega}{2}$$

(d) $\frac{d^2}{dt^2}y(t) + y(t) + \frac{1}{2}\frac{d}{dt}y(t) = \frac{d}{dt}x(t)$

Solution:

$$H(s) = \frac{s}{s^2+1+\frac{s}{2}}$$
$$H(j\omega) = \frac{j\omega}{1-\omega^2+\frac{j\omega}{2}}$$

3. (20 %) Based on the solutions, correct your previous week's homework using a colored pen (or annotation) so it's obvious what you've corrected. If you got a problem exactly right, just use a red check mark to indicate as such.