UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2020
Linear Systems Fundamentals

FINAL EXAM

Preface

- I recognize that final exams are stressful in the best of times, and this is an unusually hard time.
- Please remember that despite the stress, I am counting on you to uphold academic integrity while you complete your final exam.
- Posting or seeking exam questions or answers online, or by consulting unauthorized resources, is a gross violation of our principles of integrity and engineering ethics.
- To be fair to all students, any integrity violations discovered during the final exam will be reported to the Dean of Engineering and to the office of Academic Integrity.
- Please make sure you understand and follow the academic integrity guidelines for the exam. If you are not sure, ask me.
- An honest effort, no matter what the outcome, is something to be proud of, especially in these challenging times.
- I am very proud of the commitment and resilience you have displayed. You should feel proud, too.
- **Good luck on the exam!**

PRINT YOUR NAME ______________________________

Student ID Number ______________________________

Signature ______________________________

Your signature confirms that you have completed this exam on your own and in accordance with the Academic Integrity Agreement.

Instructions

- Open Canvas website, no electronics allowed for problem solving.
- Tables from Chapters 3 ,4 5, and 9 of the textbook are attached to the exam for your convenience.
- Time allowed: 4 hours (including download and upload time).
- Write your solutions in the designated space in the exam.
- **Justify your answers.**
- If extra pages are needed to show all of your work, use the scratch pages at the end of the exam and submit them.
- **Upload your solutions by 7pm via Gradescope.**
- If you have not submitted the **Academic Integrity Agreement**, or are not sure if you did, **please do so now**.
- **If you do not submit the Academic Integrity Agreement form, you will receive an Incomplete grade for the quarter.**

Problem	Weight	Score
1	30 pts	
2	30 pts	
3	30 pts	
4	30 pts	
5	30 pts	
6	30 pts	
Total	180 pts	

Good luck!

Name/Student ID: ______________________________

Problem 1 [DT Filtering] (30 points)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(a) Determine the frequency response $H(e^{j\omega})$ of the system.

(b) Determine and sketch precisely the magnitude $|H(e^{j\omega})|$ of the frequency response in the range $\omega \in [-\pi, \pi]$.

(c) Determine and sketch precisely the phase $\angle H(e^{j\omega})$ of the frequency response in the range $\omega \in [-\pi, \pi]$. (Add or subtract integer multiples of 2π to keep the phase in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.)

(d) Find a non-zero sinusoidal input signal $x[n]$ to the system that produces the output $y[n] = 2x[n]$.

Write your answers to parts (a), (b), (c), and (d) on the following 4 pages.

Name/Student ID: ____________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(a) (8 points)

Determine the frequency response $H(e^{j\omega})$ of the system.

Name/Student ID: ____________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(b) (8 points)

Determine and sketch precisely the magnitude $|H(e^{j\omega})|$ of the frequency response in the range $\omega \in [-\pi, \pi]$.

Name/Student ID: ____________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(c) (8 points)

Determine and sketch precisely the phase $\angle H(e^{j\omega})$ of the frequency response in the range $\omega \in [-\pi, \pi]$. (Add or subtract integer multiples of 2π to keep the phase in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.)

Name/Student ID: ______________________________

Problem 1 (cont.)

Consider the discrete-time (DT) linear time-invariant (LTI) system defined by the difference equation:

$$y[n] = x[n] - x[n-2].$$

(d) (6 points)

Find a non-zero sinusoidal input signal $x[n]$ to the system that produces the output $y[n] = 2x[n]$.

Name/Student ID: ______________________________

Problem 2 [DTFT] (30 points; 15 points each part)

(a) Find the DT Fourier transform (DTFT) $X(e^{j\omega})$ of the signal

$$x[n] = \left(\frac{1}{3}\right)^n u[n-2].$$

(b) Find the signal $x[n]$ whose DTFT in the period $[-\pi, \pi]$ is given by

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{3} \leq |\omega| \leq \frac{2\pi}{3} \\ 0, & 0 \leq |\omega| < \frac{\pi}{3} \text{ and } \frac{2\pi}{3} < |\omega| \leq \pi. \end{cases}$$

Express your answer in terms of sinusoidals signals, not exponential signals.

Write your answers to parts (a) and (b) on the following 2 pages.

Name/Student ID: ____

Problem 2 (cont.)

(a) (15 points)

Find the DT Fourier transform (DTFT) $X(e^{j\omega})$ of the signal

$$x[n] = \left(\frac{1}{3}\right)^n u[n-2].$$

Name/Student ID: ______________________________

Problem 2 (cont.)

(b) (15 points)

Find the signal $x[n]$ whose DTFT in the period $[-\pi, \pi]$ is given by

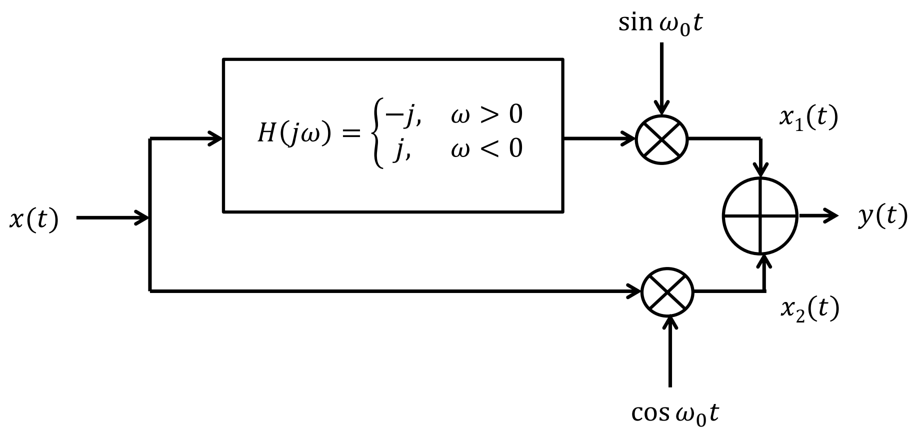
$$X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{3} \leq |\omega| \leq \frac{2\pi}{3} \\ 0, & 0 \leq |\omega| < \frac{\pi}{3} \text{ and } \frac{2\pi}{3} < |\omega| \leq \pi. \end{cases}$$

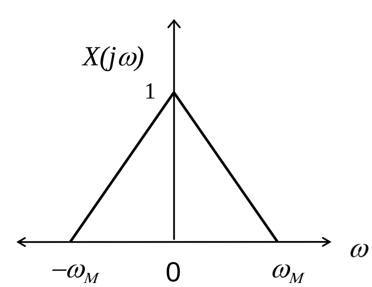
Express your answer in terms of sinusoidals signals, not exponential signals.

Name/Student ID: ______________________________

Problem 3 [Amplitude Modulation] (30 points)

Consider the modulation system shown below. The input signal $x(t)$ has a Fourier transform $X(j\omega)$ that is zero for $|\omega| > \omega_M$, also shown below. Assume that $\omega_0 > \omega_M$





(a) Detemine and sketch precisely the Fourier transform $X_1(j\omega)$ of $x_1(t)$, the Fourier transform $X_2(j\omega)$ of $x_2(t)$, and the Fourier transform $Y(j\omega)$ of $y(t)$.

(b) Is there a demodulation scheme that recovers the signal $x(t)$ from $y(t)$? If so, describe one. If not, why not?

Write your answers to parts (a) and (b) on the following 2 pages.

Name/Student ID: ______________________________

Problem 3 (cont.)

(a) (20 points)

Detemine and sketch the Fourier transform $X_1(j\omega)$ of $x_1(t)$, the Fourier transform $X_2(j\omega)$ of $x_2(t)$, and the Fourier transform $Y(j\omega)$ of $y(t)$.

Name/Student ID: ____

Problem 3 (cont.)

(b) (10 points)

Is there a demodulation scheme that recovers the signal $x(t)$ from $y(t)$? If so, describe one. If not, why not?

Name/Student ID: ____________________

Problem 4 [Sampling] (30 points; 10 points each part)

Consider the sinusoidal signal $x(t) = \sin(100\pi t)$. Apply continuous-time impulse train sampling to this signal, with sampling frequency ω_s.

(a) Assume $\omega_s = 80\pi$, producing the sampled signal $x_1(t)$. Determine $X_1(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_1(t)$?

(b) Assume $\omega_s = 120\pi$, producing the sampled signal $x_2(t)$. Determine $X_2(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_2(t)$?

(c) Suppose you applied a reconstruction filter with frequency response

$$H(j\omega) = \begin{cases} \frac{1}{40}, & |\omega| < 80\pi \\ 0, & |\omega| > 80\pi \end{cases}$$

to the signals $x_1(t)$ and $x_2(t)$ of parts (a) and (b). Determine the corresponding outputs $z_1(t)$ and $z_2(t)$.

Write your answers to parts (a), (b), and (c) on the following 3 pages.

Name/Student ID: ______________________________

Problem 4 (cont.)

Consider the sinusoidal signal $x(t) = \sin(100\pi t)$. Apply continuous-time impulse train sampling to this signal, with sampling frequency ω_s.

(a) (10 points)

Assume $\omega_s = 80\pi$, producing the sampled signal $x_1(t)$. Determine $X_1(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_1(t)$?

Name/Student ID: ______________________________

Problem 4 (cont.)

Consider the sinusoidal signal $x(t) = \sin(100\pi t)$. Apply continuous-time impulse train sampling to this signal, with sampling frequency ω_s.

(b) (10 points)

Assume $\omega_s = 120\pi$, producing the sampled signal $x_2(t)$. Determine $X_2(j\omega)$ and sketch it precisely in the frequency range $[-100\pi, 100\pi]$. How would you generate the output $y(t) = \sin(20\pi t)$ from $x_2(t)$?

Name/Student ID: ____________________

Problem 4 (cont.)

(c) (10 points)

Suppose you applied a reconstruction filter with frequency response

$$H(j\omega) = \begin{cases} \frac{1}{40}, & |\omega| < 80\pi \\ 0, & |\omega| > 80\pi \end{cases}$$

to the signals $x_1(t)$ and $x_2(t)$ of parts (a) and (b). Determine the corresponding outputs $z_1(t)$ and $z_2(t)$.

Name/Student ID: ______________________________

Problem 5 [Laplace Transform and ROC] (30 points)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(a) Sketch precisely the pole-zero plot associated with $H(s)$. Indicate the order of all poles and zeros.

(b) Identify all possible regions of convergence (ROC) associated with $H(s)$ and indicate for each ROC whether the corresponding system is causal and/or stable.

(c) For each possible ROC in part (b), indicate whether the corresponing impulse response $h(t)$ is left-sided, right-sided, two-sided, or finite-duration, and whether or not it has a Fourier transform $H(j\omega)$.

(d) Determine a linear constant-coefficient differential equation that characterizes the LTI systems with system function $H(s)$.

Write your answers to parts (a), (b), (c), and (d) on the following 4 pages.

Name/Student ID: ______________________________

Problem 5 (cont.)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(a) (8 points)

Sketch precisely the pole-zero plot associated with $H(s)$. Indicate the order of all poles and zeros.

Name/Student ID: ______________________________

Problem 5 (cont.)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(b) (8 points)

Identify all possible regions of convergence (ROC) associated with $H(s)$ and indicate for each ROC whether the corresponding system is causal and/or stable.

Name/Student ID: ______________________________

Problem 5 (cont.)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(c) (8 points)

For each possible ROC in part (b), indicate whether the corresponing impulse response $h(t)$ is left-sided, right-sided, two-sided, or finite-duration, and whether or not it has a Fourier transform $H(j\omega)$.

Name/Student ID: ____

Problem 5 (cont.)

Consider the LTI system function

$$H(s) = \frac{s-1}{(s^2+2s+2)(s^2-4)}.$$

(d) (6 points)

Determine a linear constant-coefficient differential equation that characterizes the LTI systems with system function $H(s)$.

Name/Student ID: ______________________________

Problem 6 [Laplace Transform and LTI Systems] (30 points)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(a) Determine and sketch precisely the magnitude $|H(j\omega)|$ of the frequency response $H(j\omega)$. How would you describe the filter characteristics of the system - lowpass, highpass, bandpass, allpass?

(b) Determine (but do not sketch) the phase $\angle H(j\omega)$ of the frequency response $H(j\omega)$.

(c) Determine the impulse response $h(t)$ of the system.

Hint: Write $(s-1)^2$ in the form $(s+1)^2 + f(s)$.

(d) Suppose the input to the system is $x(t) = te^t u(t)$. Determine the corresponding output $y(t)$.

Write your answers to parts (a), (b), (c), and (d) on the following 4 pages.

Name/Student ID: ____________________

Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(a) (8 points) Determine and sketch precisely the magnitude $|H(j\omega)|$ of the frequency response $H(j\omega)$.

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Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(b) (8 points)

Determine (but do not sketch) the phase $\angle H(j\omega)$ of the frequency response $H(j\omega)$. How would you describe the filter characteristics of the system - lowpass, highpass, bandpass, allpass?

Name/Student ID: ____

Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(c) (8 points)

Determine the impulse response $h(t)$ of the system.

Hint: Write $(s-1)^2$ in the form $(s+1)^2 + f(s)$.

Name/Student ID: ______________________________

Problem 6 (cont.)

Consider the causal LTI system with Laplace transform and region of convergence

$$H(s) = \frac{(s-1)^2}{(s+1)^2}, \quad \mathcal{R}e(s) > -1.$$

(d) (6 points)

Suppose the input to the system is $x(t) = te^t u(t)$. Determine the correreponding output $y(t)$.

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Chapter 2: LTI systems

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k] \qquad y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

If $h(t)$ is the impulse response to the CT system S, and $g(t)$ is the impulse response to the CT system S', then it follows that if $g(t)*h(t)=\delta(t)$, then S' is the inverse of S. The same property holds true for DT systems.

An LTI system is causal iff $h[n]=0$ for $n<0$ ($h(t)=0$ for $t<0$).

If an impulse response is absolutely summable ($\sum_{k=-\infty}^{\infty}|h[k]|<\infty$), then the LTI system is stable.

If an impulse response is absolutely integrable ($\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$), then the LTI system is stable.

Chapter 3: Fourier series and Fourier coefficients

synthesis equation — analysis equation

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \qquad a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad \omega_0=2\pi/T$$

$$x[n]=\sum_{k=<N>}a_k e^{jk\omega_0 n} \qquad a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk\omega_0 n} \qquad \omega_0=2\pi/N$$

Response of LTI system to complex exponential

e^{st} (z^n) is called the eigenfunction and $H(s)$ ($H(z)$) the eigenvalue of the LTI system.

$$e^{st}\to H(s)e^{st} \qquad H(s)=\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau$$

$$z^n\to H(z)z^n \qquad H(z)=\sum_{k=-\infty}^{\infty}h[k]z^{-k}$$

Let $x(t)\to y(t)$ ($x[n]\to y[n]$) in an LTI system. Then

$$\text{if} \quad x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \quad \text{then} \quad y(t)=\sum_{k=-\infty}^{\infty}a_k H(jk\omega_0)e^{jk\omega_0 t} \qquad \omega_0=2\pi/T$$

$$\text{if} \quad x[n]=\sum_{k=<N>}a_k e^{jk\omega_0 n} \quad \text{then} \quad y[n]=\sum_{k=<N>}a_k H(e^{jk\omega_0})e^{jk\omega_0 n} \qquad \omega_0=2\pi/N$$

Chapters 4 and 5

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega \qquad X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

Triangle Inequality and summation formula:

$$|X+Y|\le|X|+|Y| \qquad \sum_{k=0}^{\infty}z^k=\frac{1}{1-z}\quad \text{if} \quad |z|<1 \qquad \sum_{k=0}^{N-1}z^k=\frac{1-z^N}{1-z}$$

Chapter 7

Impulse Train Sampling

Let $x(t)$ be a continuous function, T be the sample period, and $x_p(t)$ the sampled function.

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad\qquad x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Defining $\omega_s = 2\pi/T$, the transforms are

$$P(j\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \qquad\qquad X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$$

yielding

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Sampling Theorem

Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, ...$ if

$$\omega_s > 2\omega_M.$$

Discrete-Time Processing of Continuous-Time Signals

Let $x_c(t)$ be a continuous function, T be the sample period, and $x_p(t)$ the sampled function. Define the discrete function $x[n]$ as $x[n] = x_c(nT)$.

$$x_p(t) = x_c(t)p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \qquad\qquad X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT}$$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n} \qquad \text{and thus} \qquad X_d(e^{j\Omega}) = X_p(j\Omega/T)$$

Chapter 9

$$\text{Laplace Transform:} \qquad X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Table 3.1 PROPERTIES OF THE CONTINUOUS-TIME FOURIER SERIES
$x(t)$ and $y(t)$ are periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$

Section	Property	Periodic Signal	Fourier Series Coefficients
	both $x(t)$ and $y(t)$ periodic with period T and frequency $\omega_0 = 2\pi/T$	$x(t)$ $y(t)$	a_k b_k
3.5.1	Linearity	$A\,x(t) + B\,y(t)$	$A\,a_k + B\,b_k$
3.5.2	Time Shifting	$x(t - t_0)$	$a_k\,e^{-jk\omega_0 t_0}$
	Frequency Shifting	$\exp[jM\omega_0 t]\,x(t)$	a_{k-M}
3.5.6	Conjugation	$x^*(t)$	a^*_{-k}
3.5.3	Time Reversal	$x(-t)$	a_{-k}
3.5.4	Time Scaling	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
	Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$T\,a_k\,b_k$
3.5.5	Multiplication	$x(t)\,y(t)$	$\sum_{\ell=-\infty}^{\infty} a_\ell\,b_{k-\ell}$
	Differentiation	$dx(t)/dt$	$j\,k\,\omega_0\,a_k$
	Integration (requires $a_0 = 0$)	$\int_{-\infty}^{t} x(t')\,dt'$	$a_k/(j\,k\,\omega_0)$
3.5.6	Conjugate Symmetry	$x(t)$ is real	$\begin{cases} \lvert a_k\rvert = \lvert a_{-k}\rvert \\ a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \angle a_k = -\angle a_{-k} \end{cases}$
3.5.6	real and even signals	$x(t)$ real and even	a_k purely real and even
3.5.6	real and odd signals	$x(t)$ real and odd	a_k purely imaginary and odd
3.5.6	Even Decomposition ($x(t)$ is real)	$x_e(t) = \text{Ev}\,\{x(t)\}$	$\mathfrak{Re}\,\{a_k\}$
3.5.6	Odd Decomposition ($x(t)$ is real)	$x_o(t) = \text{Od}\,\{x(t)\}$	$j\mathfrak{Im}\,\{a_k\}$

Parseval's Relation for Periodic Signals $\quad (1/T)\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Table 3.2 PROPERTIES OF THE DISCRETE-TIME FOURIER SERIES

$x[n]$ and $y[n]$ are periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$

Property	Periodic Signal	Fourier Series Coefficients
both $x[n]$ and $y[n]$ periodic with period N and frequency $\omega_0 = 2\pi/N$	$x[n]$ $y[n]$	a_k (periodic, period N) b_k (periodic, period N)
Linearity	$A\,x[n] + B\,y[n]$	$A\,a_k + B\,b_k$
Time Shifting	$x[n-n_0]$	$a_k\,e^{-jk\omega_0 n_0}$
Frequency Shifting	$\exp[jM\omega_0 n]\,x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & n \text{ Mod } m = 0 \\ 0, & n \text{ Mod } m \neq 0 \end{cases}$	$\dfrac{a_k}{m}$ (periodic, period mN)
Periodic Convolution	$\sum_{r=<N>} x[r]y[n-r]$	$N\,a_k\,b_k$
Multiplication	$x[n]\,y[n]$	$\sum_{\ell=<N>} a_\ell\,b_{k-\ell}$
First Difference	$x[n] - x[n-1]$	$(1-e^{-j\,k\,\omega_0})a_k$
Running Sum (requires $a_0 = 0$)	$\sum_{k=-\infty}^{n} x[k]$	$\left(\dfrac{1}{1-e^{-j\,k\,\omega_0}}\right) a_k$
Conjugate Symmetry	$x[n]$ is real	$\begin{cases} \|a_k\| = \|a_{-k}\| \\ a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \angle a_k = -\angle a_{-k} \end{cases}$
real and even signals	$x[n]$ real and even	a_k purely real and even
real and odd signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even Decomposition ($x[n]$ is real)	$x_e[n] = \text{Ev}\,\{x[n]\}$	$\mathfrak{Re}\,\{a_k\}$
Odd Decomposition ($x[n]$ is real)	$x_o[n] = \text{Od}\,\{x[n]\}$	$j\mathfrak{Im}\,\{a_k\}$

Parseval's Relation for Periodic Signals $\quad \dfrac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$

Table 4.1 PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

Property	Aperiodic Signal	Fourier Transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$a\,x(t) + b\,y(t)$	$a\,X(j\omega) + b\,Y(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Frequency Shifting	$\exp(j\omega_0 t)\,x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time Scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)\,y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
Differentiation (time)	$dx(t)/dt$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t')\,dt'$	$\frac{1}{j\omega} X(j\omega) + \pi\,X(0)\delta(\omega)$
Differentiation (frequency)	$t\,x(t)$	$j\frac{d}{d\omega} X(j\omega)$
Conjugate Symmetry	$x(t)$ is real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathfrak{Re}\{X(j\omega)\} = \mathfrak{Re}\{X(-j\omega)\} \\ \mathfrak{Im}\{X(j\omega)\} = -\mathfrak{Im}\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \angle X(j\omega) = -\angle X(-j\omega) \end{cases}$
real and even signals	$x(t)$ real and even	$X(j\omega)$ purely real and even
real and odd signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even Decomposition ($x(t)$ is real)	$x_e(t) = \mathrm{Ev}\,\{x(t)\}$	$\mathfrak{Re}\,\{X(j\omega)\}$
Odd Decomposition ($x(t)$ is real)	$x_o(t) = \mathrm{Od}\,\{x(t)\}$	$j\mathfrak{Im}\,\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega$

Table 4.2 CONTINUOUS-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$ otherwise
$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = 1/2$ $a_k = 0$ otherwise
$\sin(\omega_0 t)$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = -j/2$ $a_k = 0$ otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \quad k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq T/2 \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	aperiodic
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	aperiodic
$\delta(t)$	1	aperiodic
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	aperiodic
$\delta(t - t_0)$	$e^{-j\omega t_0}$	aperiodic
$e^{-at}u(t), \ \mathfrak{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	aperiodic
$t\, e^{-at}u(t), \ \mathfrak{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	aperiodic
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathfrak{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	aperiodic

Table 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Discrete Fourier Transform always has a period of 2π.

Property	Aperiodic Signal	Fourier Transform
	$x[n]$ $y[n]$	$X(e^{j\omega})$ $Y(e^{j\omega})$
Linearity	$a\,x[n] + b\,y[n]$	$a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$
Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
Frequency Shifting	$\exp[j\omega_0 n]\,x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & n \text{ Mod } k = 0 \\ 0, & n \text{ Mod } k \neq 0 \end{cases}$	$X(e^{j\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})\,Y(e^{j\omega})$
Multiplication	$x[n]\,y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\,Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1-e^{-j\omega})\,X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}} X(e^{j\omega})$ $+\pi\,X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
Differentiation (frequency)	$n\,x[n]$	$j\frac{d}{d\omega} X(e^{j\omega})$
Conjugate Symmetry	$x[n]$ is real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathfrak{Re}\{X(e^{j\omega})\} = \mathfrak{Re}\{X(e^{-j\omega})\} \\ \mathfrak{Im}\{X(e^{-j\omega})\} = -\mathfrak{Im}\{X(e^{j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}) \end{cases}$
real and even signals	$x[n]$ real and even	$X(e^{j\omega})$ purely real and even
real and odd signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even Decomposition ($x[n]$ is real)	$x_e[n] = \text{Ev}\,\{x[n]\}$	$\mathfrak{Re}\left\{X(e^{j\omega})\right\}$
Odd Decomposition ($x[n]$ is real)	$x_o[n] = \text{Od}\,\{x[n]\}$	$j\mathfrak{Im}\left\{X(e^{j\omega})\right\}$

Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2\,d\omega$

Table 5.2 DISCRETE-TIME FOURIER TRANSFORM PAIRS

In the pairs given below, if the signal is periodic, then $\omega_0 = 2\pi m/N$, where $m, N \in \mathbb{Z}$.
If $\omega_0/(2\pi)$ is irrational, then the signal is aperiodic.
Discrete Fourier Transform always has a period of 2π.

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=<N>} a_k e^{j(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi\ell)$	$a_k = \begin{cases} 1, & k = m + qN \text{ where } q \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$\cos\omega_0 n$	$\pi \sum_{\ell=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi\ell) + \delta(\omega + \omega_0 - 2\pi\ell)\}$	$a_k = \begin{cases} 1/2, & k = \pm m + qN \text{ where } q \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$\sin\omega_0 n$	$-j\pi \sum_{\ell=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi\ell) - \delta(\omega + \omega_0 - 2\pi\ell)\}$	$a_k = \begin{cases} -j/2, & k = m + qN \text{ where } q \in \mathbb{Z} \\ j/2, & k = -m + qN \\ 0, & \text{otherwise} \end{cases}$
$x[n] = 1$	$2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - 2\pi\ell)$	$a_k = \begin{cases} 1, & k = qN \text{ where } q \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + 1/2)]}{N \sin[2\pi k/2N]}$, k Mod $N \neq 0$; $a_k = (2N_1 + 1)/N$, k Mod $N = 0$
$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	aperiodic
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$	aperiodic
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\text{sinc}\left(\frac{Wn}{\pi}\right)$, $0 < W < \pi$	$X(e^{j\omega}) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$	aperiodic
$\delta[n]$	1	aperiodic
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$	aperiodic
$\delta[n - n_0]$	$e^{-j\omega n_0}$	aperiodic
$(n+1)a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{(1 - a\,e^{-j\omega})^2}$	aperiodic
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{(1 - a\,e^{-j\omega})^r}$	aperiodic

Table 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a\,x_1(t) + b\,x_2(t)$	$a\,X_1(s) + b\,X_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-s\,t_0} X(s)$	R
Shifting in s	$e^{s_0\,t}\,x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
Time scaling	$x(a\,t)$	$\frac{1}{\lvert a\rvert} X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)\,X_2(s)$	At least $R_1 \cap R_2$
Differentiation	$\frac{d}{dt}x(t)$	$s\,X(s)$	At least R
Differentiation in s	$-t\,x(t)$	$\frac{d}{ds}X(s)$	R
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathfrak{Re}\{s\} > 0\}$

Initial Value Theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then $x(0^+) = \lim_{s\to\infty} s\,X(s)$.

Final Value Theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then $\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s)$.

Table 9.2 LAPLACE TRANSFORM PAIRS

Signal	Laplace Transform	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathfrak{Re}\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathfrak{Re}\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	s^{-n}	$\mathfrak{Re}\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	s^{-n}	$\mathfrak{Re}\{s\} < 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathfrak{Re}\{s\} > -\alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathfrak{Re}\{s\} < -\alpha$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathfrak{Re}\{s\} > -\alpha$
$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathfrak{Re}\{s\} < -\alpha$
$\delta(t-T)$	e^{-sT}	All s
$[\cos\omega_0 t]\,u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > 0$
$[\sin\omega_0 t]\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > 0$
$[e^{-\alpha t}\cos\omega_0 t]\,u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > -\alpha$
$[e^{-\alpha t}\sin\omega_0 t]\,u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > -\alpha$
$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	all s
$u_{-n}(t) = u(t) * \ldots * u(t)$ n times	s^{-n}	$\mathfrak{Re}\{s\} > 0$