Lecture 18

Sampling and reconstruction

Preview of today's lecture

- Sampling theorem
  - Establish the fundamental connection between continuous-time bandlimited signals and discrete-time signals
  - Illustrate the impact of sampling in the time and frequency domains

- Reconstruction theorem
  - Define the reconstruction formula
  - Explain the role of the sinc function in reconstruction
  - Illustrate reconstruction in time and frequency domains

- Important example
  - Be able to illustrate spectra with and without aliasing

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Connections back to ECE 45

Lecture 18

graph.
dne.

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 16, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 15 Fourier transform

Fourier

Sampling theorem

Key points

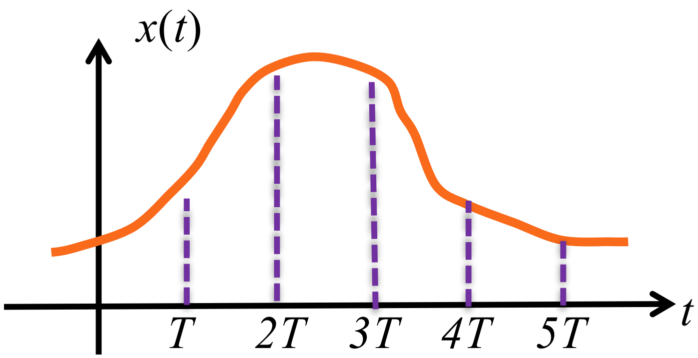
- Establish the fundamental connection between continuous-time bandlimited signals and discrete-time signals
- Illustrate the impact of sampling in the time and frequency domains

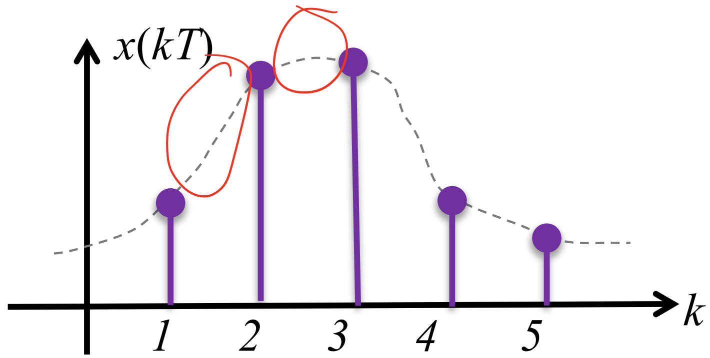
What is sampling?

Gone

- For a given CT signal $x(t)$

Kronecker delta functions



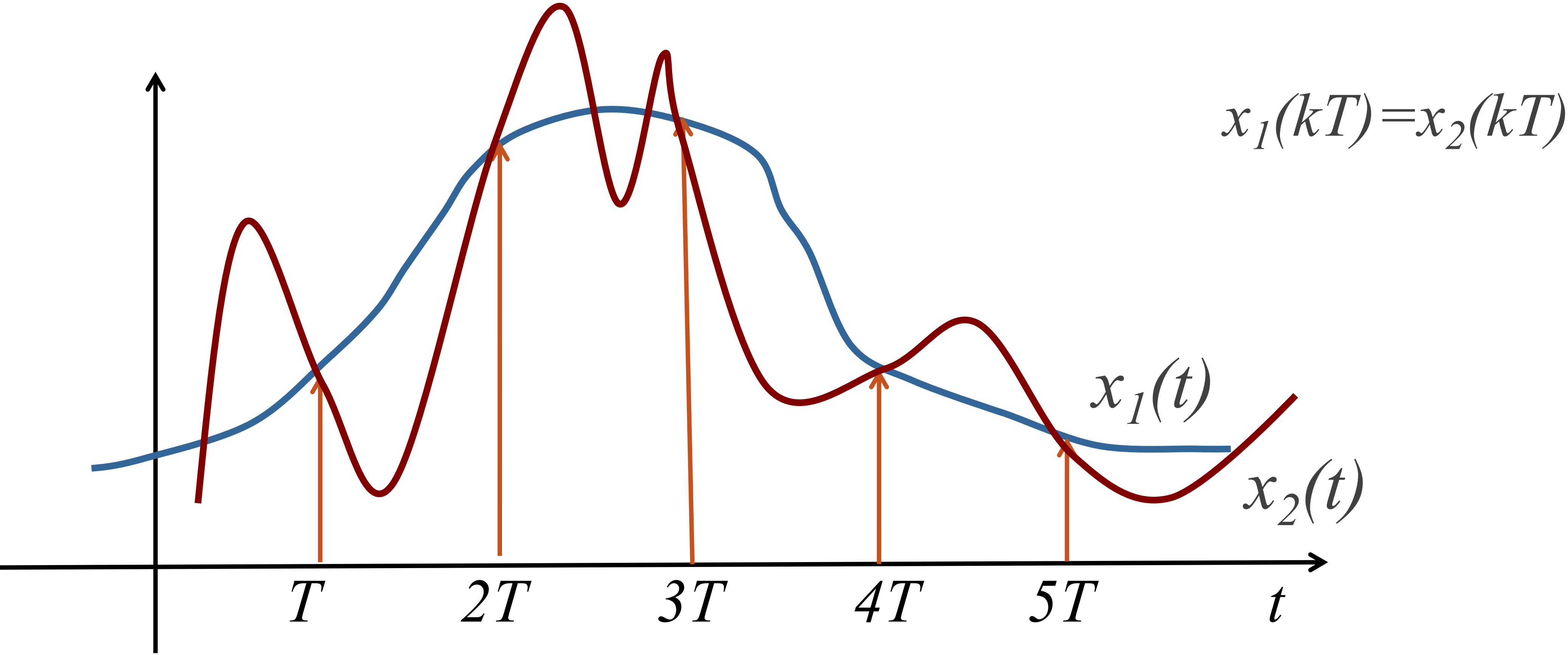


- The signal $x(kT)$ is called a sampled version of $x(t)$
  - The sampled signal is a discrete-time signal, written as *x[k]*

$\{x(kT)\}_{k=-\infty}^{\infty}$

- The critical question related to sampling
  - Is it possible to recover *x(t)* from *x(kT)* ?

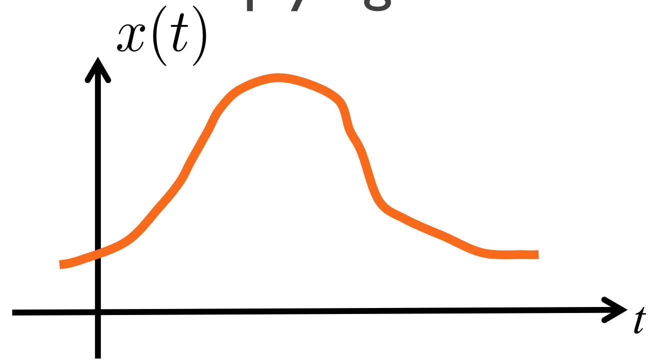
Why is there the potential for ambiguity?

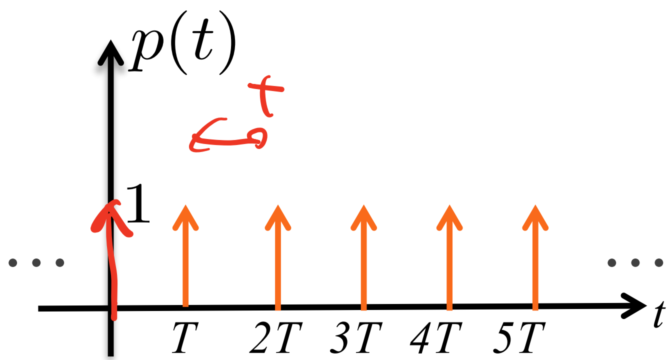


Two different signals can have the same samples

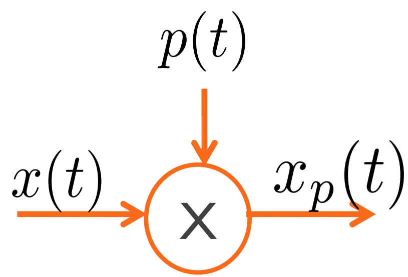
Impulse-train periodic sampling

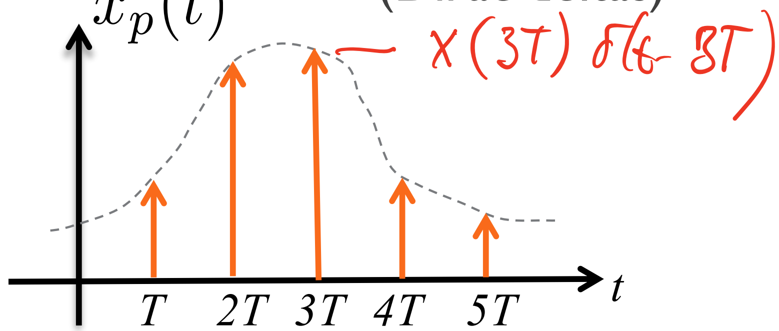
- A convenient way to understand periodic sampling is through multiplying the CT signal by a periodic impulse train





Dirac delta functions



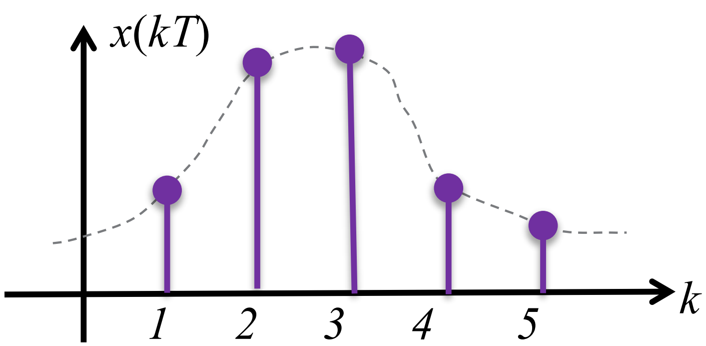


(Dirac deltas)

Contains the same values as the sampled signal

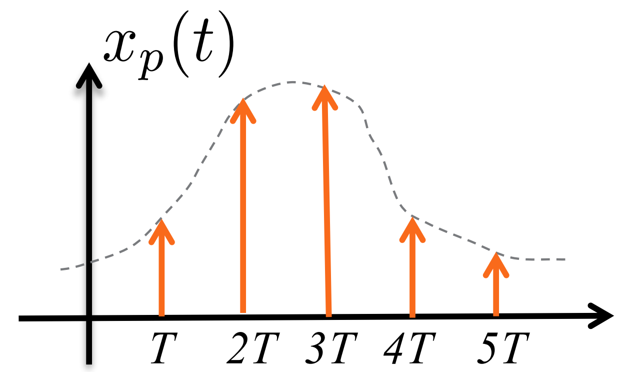
Our approach to sampling in this class

<u>Discrete-time sampled signal</u>



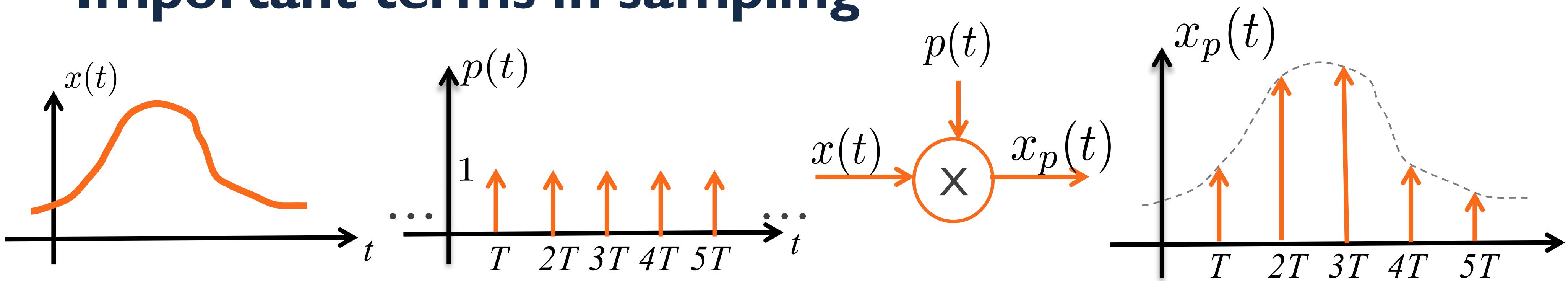
In digital signal processing, as explored in ECE 101, we care about the discrete-time signal

<u>Impulse-train signal</u>



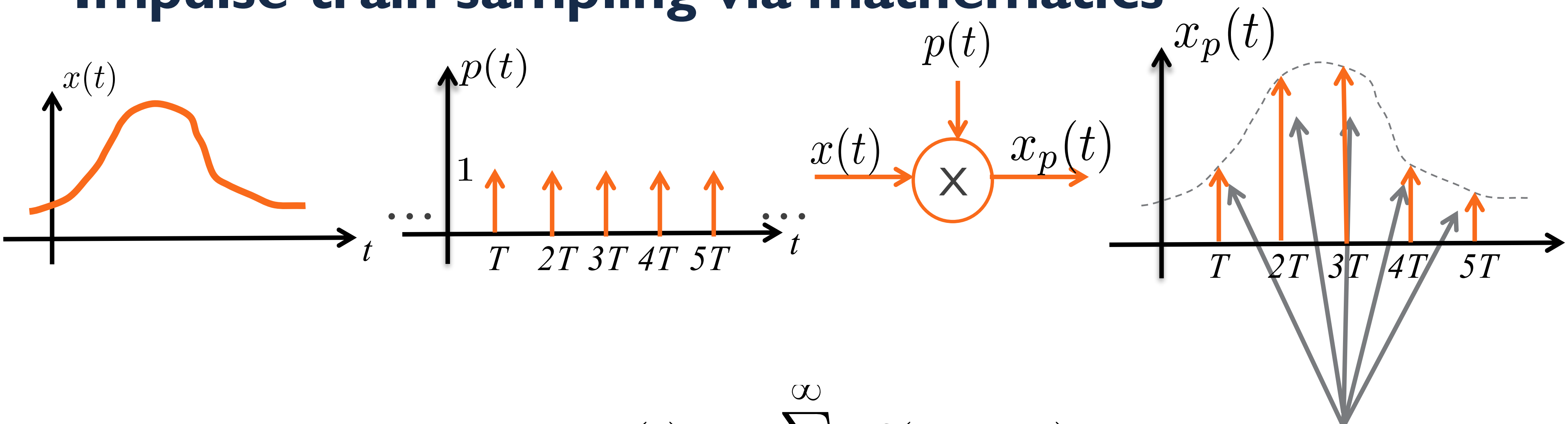
In this class, we will study sampling from the perspective of the impulse-train signal

Important terms in sampling



- The periodic impulse train $p(t)$ is the sampling function
- The period T is the sampling period
- The fundamental frequency of $p(t)$, $\omega_s = \frac{2\pi}{T}$ is the sampling frequency

Impulse-train sampling via mathematics



Sampled values ride the deltas

- The sampling function $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- The output signal $x_p(t) = x(t)p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Sampled signal is here via sifting property

Impulse train of samples in the frequency domain

- Multiplication in time domain → convolution in frequency domain

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

$p(t) \xleftrightarrow{FS} \frac{1}{T}$

where

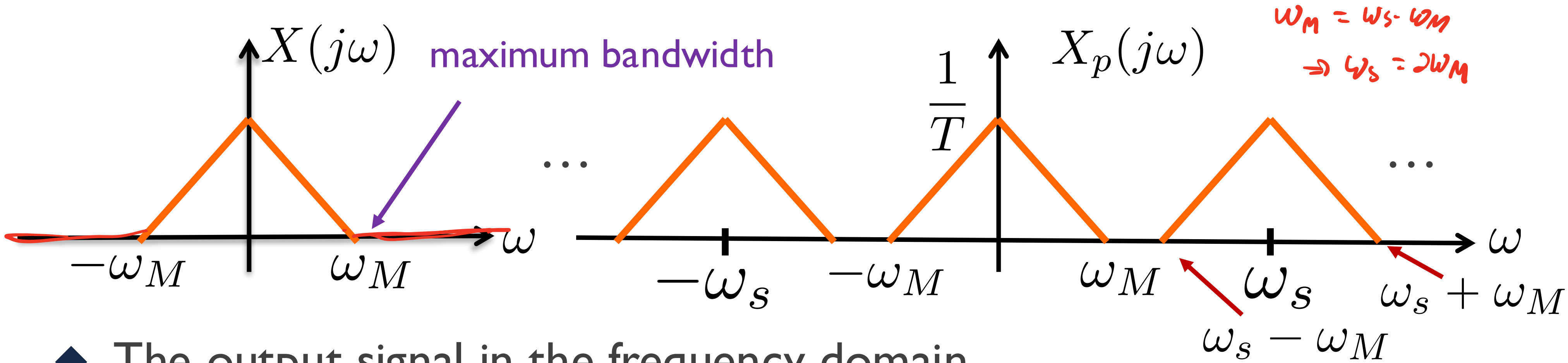
$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

$$\boxed{\omega_s = \frac{2\pi}{T}}$$

- Then

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Understanding the frequency domain effect



- The output signal in the frequency domain

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

- A superposition of shifted versions of $X(j\omega)$ scaled by $\frac{1}{T}$

If $\omega_s - \omega_M > \omega_M$ ➡ $\omega_s > 2\omega_M$ then original spectrum is undistorted

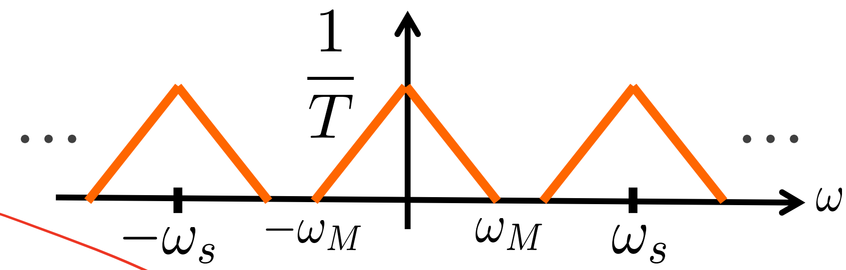
Sampling theorem

- When is $x(t)$ completely determined from $x_p(t)$?

- Sampling theorem
  - Let $x(t)$ be a band-limited signal with $X(j\omega) = 0 \text{ for } |\omega| > \omega_M$.
  - Then, $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \ldots$

If $\boxed{\omega_s > 2\omega_M}$ $\qquad \omega_s = \dfrac{2\pi}{T}$

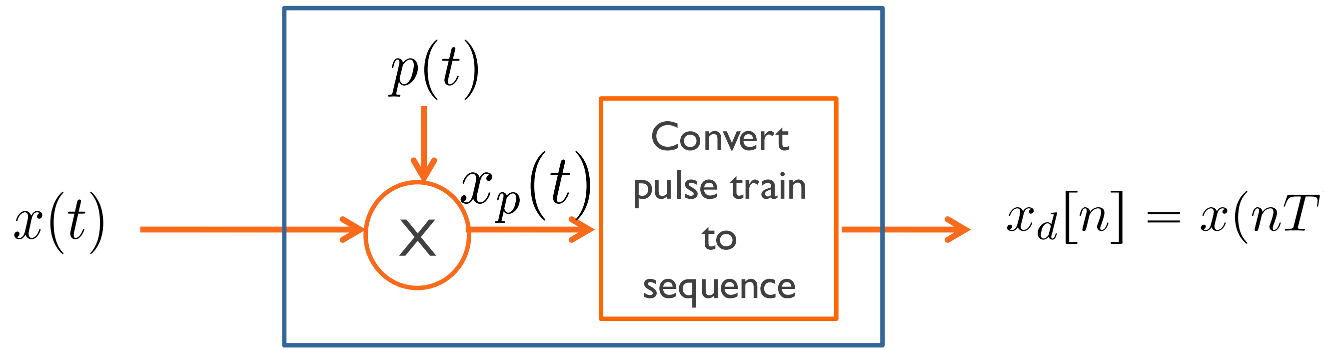
Nyquist frequency

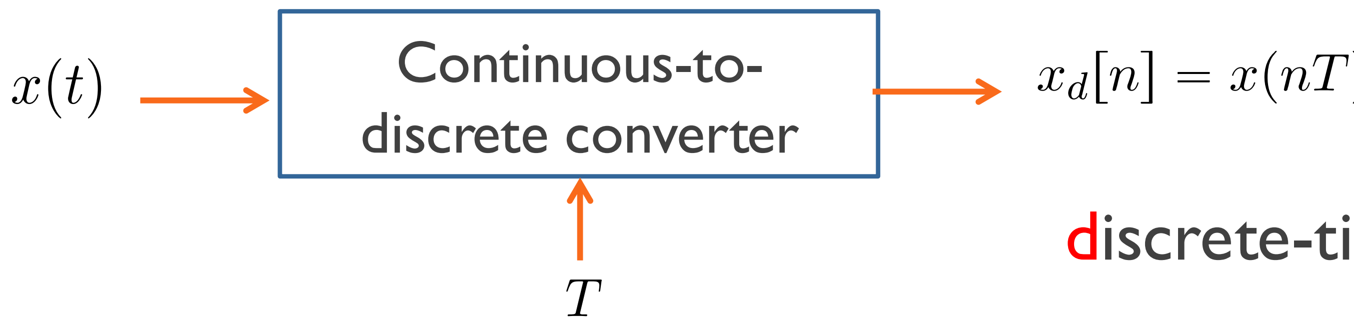


- The product $2\omega_M$ is called the "Nyquist rate"

Ideal continuous-to-discrete converter

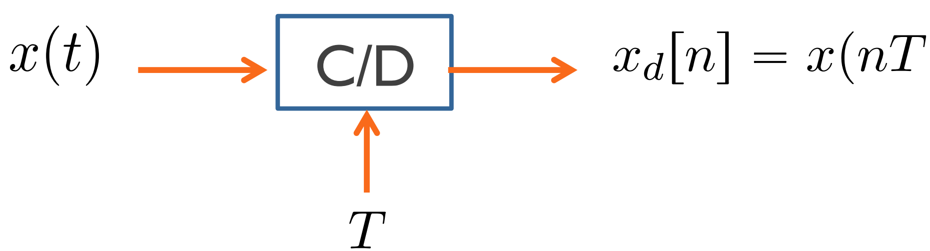
mathematical description





discrete-time

shorthand notation

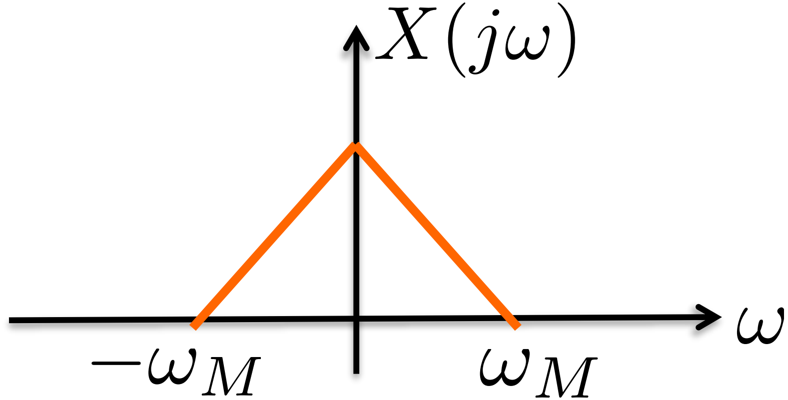


Connecting the domains

	time domain	frequency domain
CT signal	$x(t)$	$X(j\omega)$
impulse train of samples	$x_p(t) = \sum_n x(nT)\delta(t - nT)$	$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

Sampling in the frequency domain – Nyquist OK

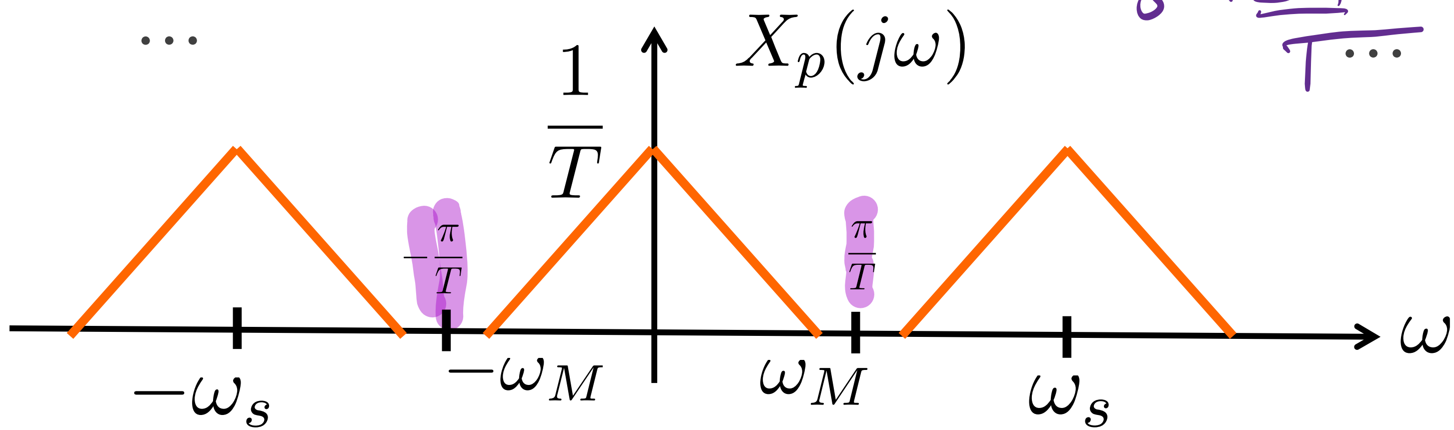
CT signal



$\omega_s > 2\omega_M$

$\omega_s = \frac{2\pi}{T} \cdots$

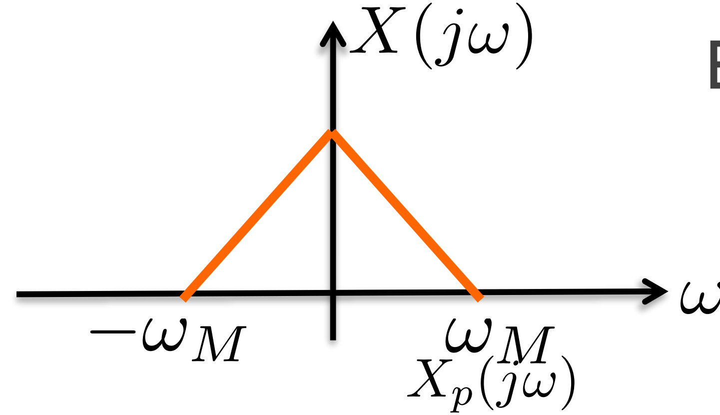
impulse train of samples



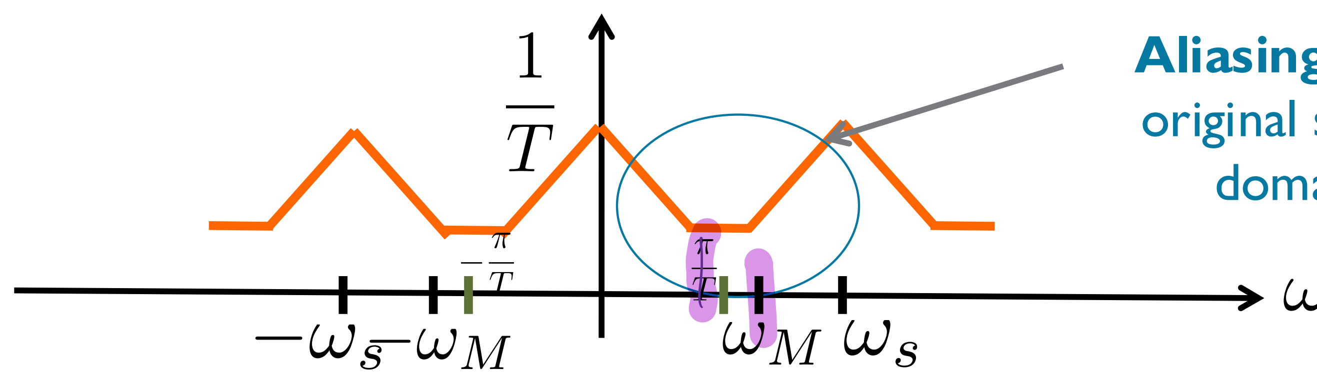
Sampling in the frequency domain – Nyquist Not OK

CT signal

E.g.
$\omega_M < \omega_s < 2\omega_M$



impulse train of samples



Aliasing! The shape original signal in freq domain is distort

Sampling summary

- The sampling theorem tells us when a continuous time signal may be periodically sampled with no loss

- The signal must be perfectly bandlimited and the sampling period must be small enough

- If the sampling theorem is not satisfied, it is still possible to sample the signal but aliasing will result

Classical example – the wagon wheel

Key points

- Explain the wagon wheel effect

The wagon wheel effect

- Classic demo
  - https://www.youtube.com/watch?v=VNftf5qLpiA
- Another demo that shows the effect of the number of spokes
  - https://www.youtube.com/watch?v=9MN5MF72PHs&t=34s
- Typical explanation on wikipedia
  - https://en.wikipedia.org/wiki/Wagon-wheel_effect

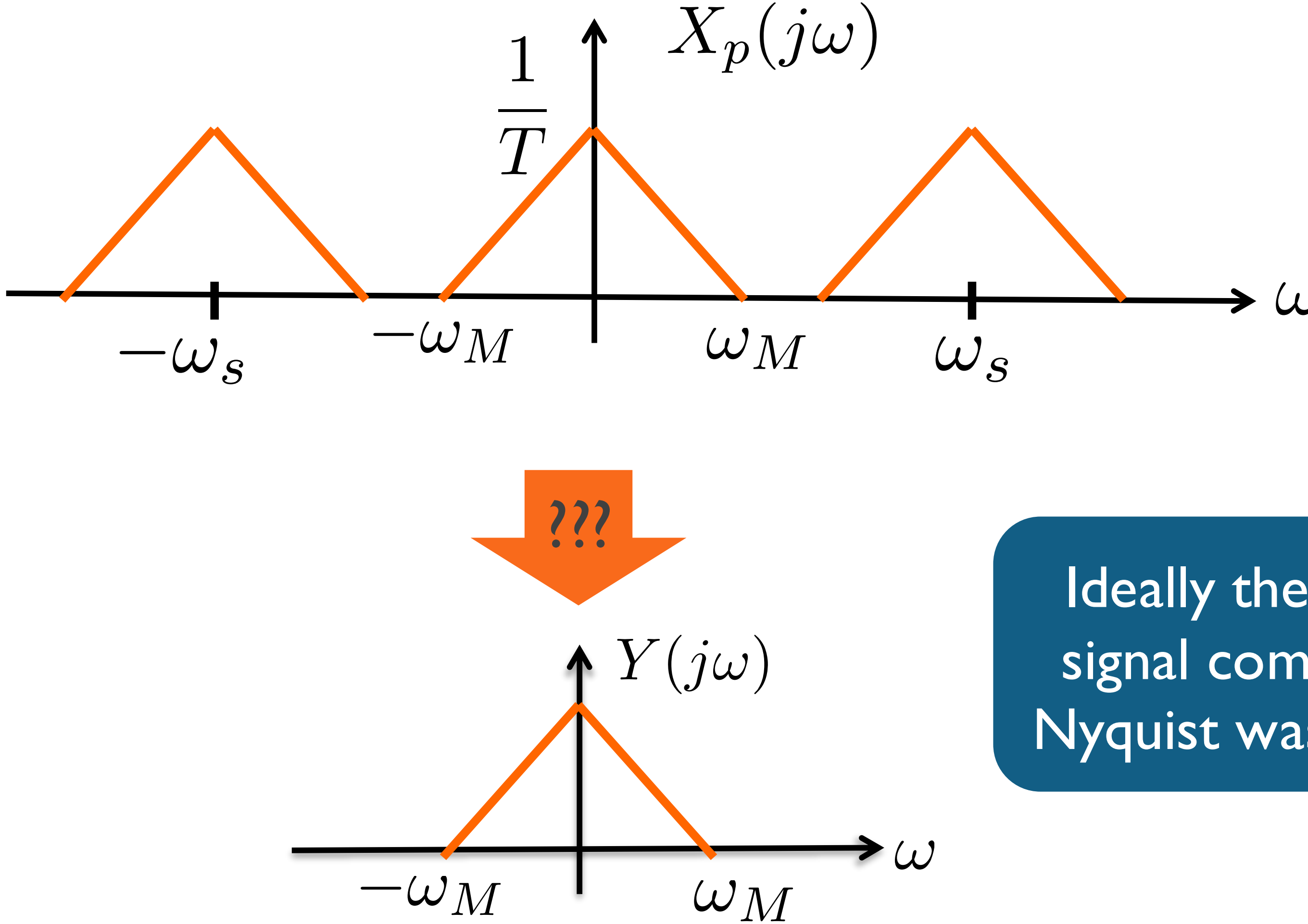
The reason that spoked wheels seem to go backwards is explained by the Nyquist sampling theorem

Reconstruction of a signal from its samples

Key points

- Define the reconstruction formula
- Explain the role of the sinc function in reconstruction
- Illustrate reconstruction in time and frequency domains

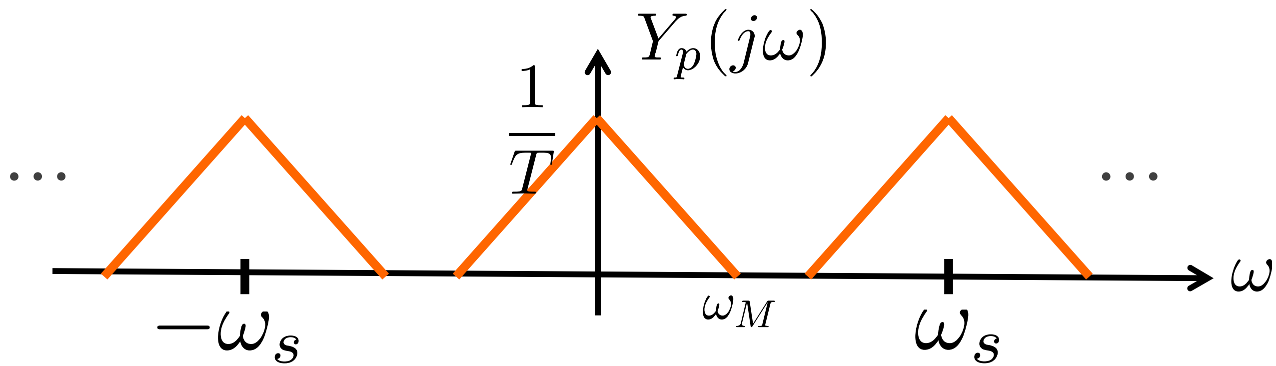
How to recover the original signal?



Ideally the original signal comes out if Nyquist was satisfied

Filter the reconstructed the signal

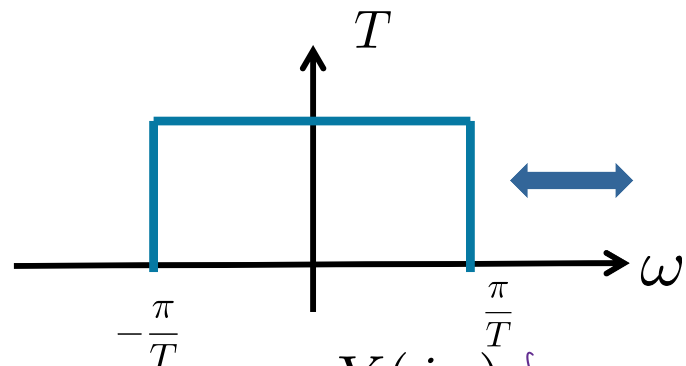
frequency domain



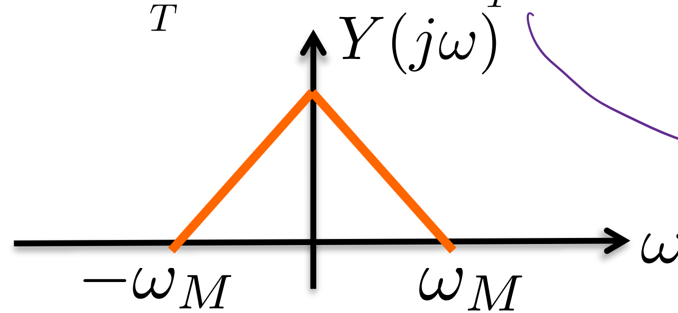
Ideal low pass filter

Cutoff $\frac{\pi}{T}$

Gain T



$$T\text{rect}\left(\frac{\omega}{2\pi/T}\right) \leftrightarrow \text{sinc}\left(\frac{t}{T}\right)$$



$\frac{\omega_s}{2}$

Filtering to reconstruct the signal

time domain

$x_d[n]$ → Convert to pulse train of samples → Lowpass filter cutoff π/T, gain T → $y(t)$

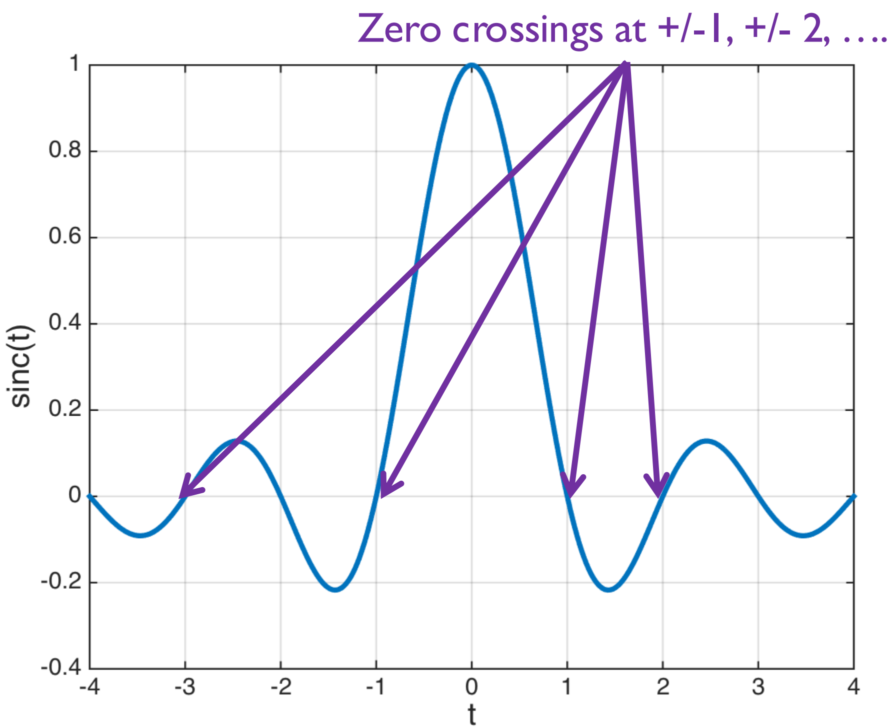
$$\begin{aligned}
y(t) &= y_p(t) * h(t) \\
&= h(t) * \sum_{n=-\infty}^{\infty} x_d[n]\delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_d[n]h(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_d[n]\mathrm{sinc}\left(\frac{t - nT}{T}\right)
\end{aligned}$$

$x(nT)$

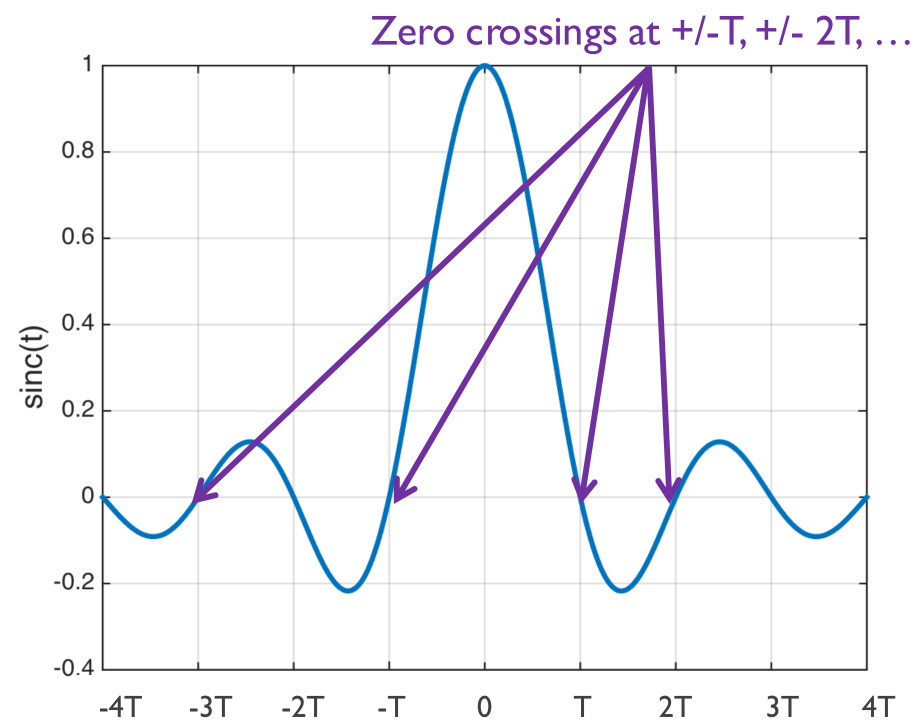
sinc interpolation

Reconstruction formula!

Step 2: Sinc is critical to reconstruction



$$\operatorname{sinc}(t)$$

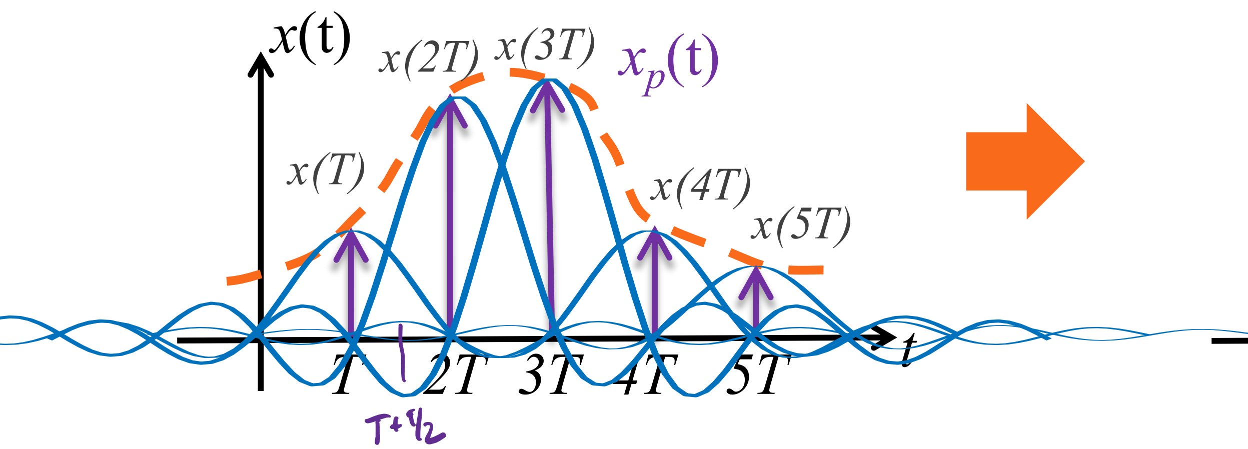


$$\operatorname{sinc}\left(\frac{t}{T}\right)$$

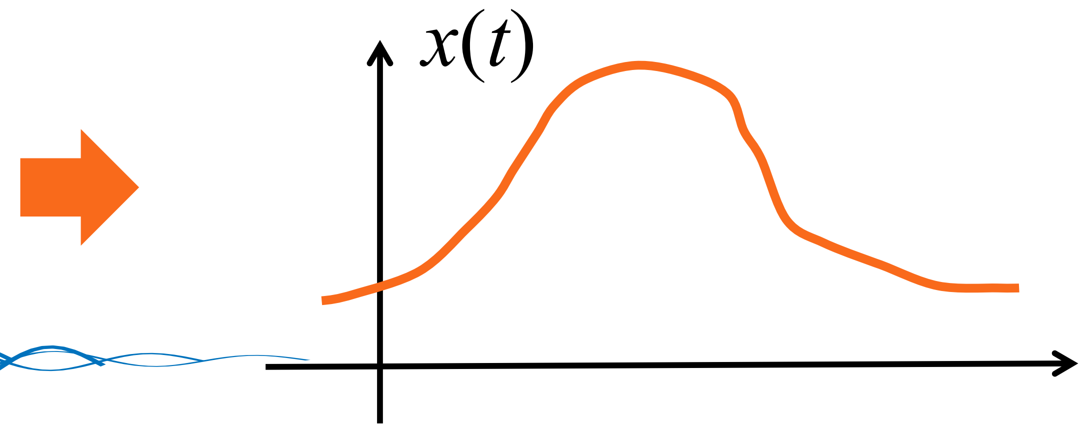
Zero crossings occur exactly at the sampling intervals

Reconstructing using the sinc functions

time domain

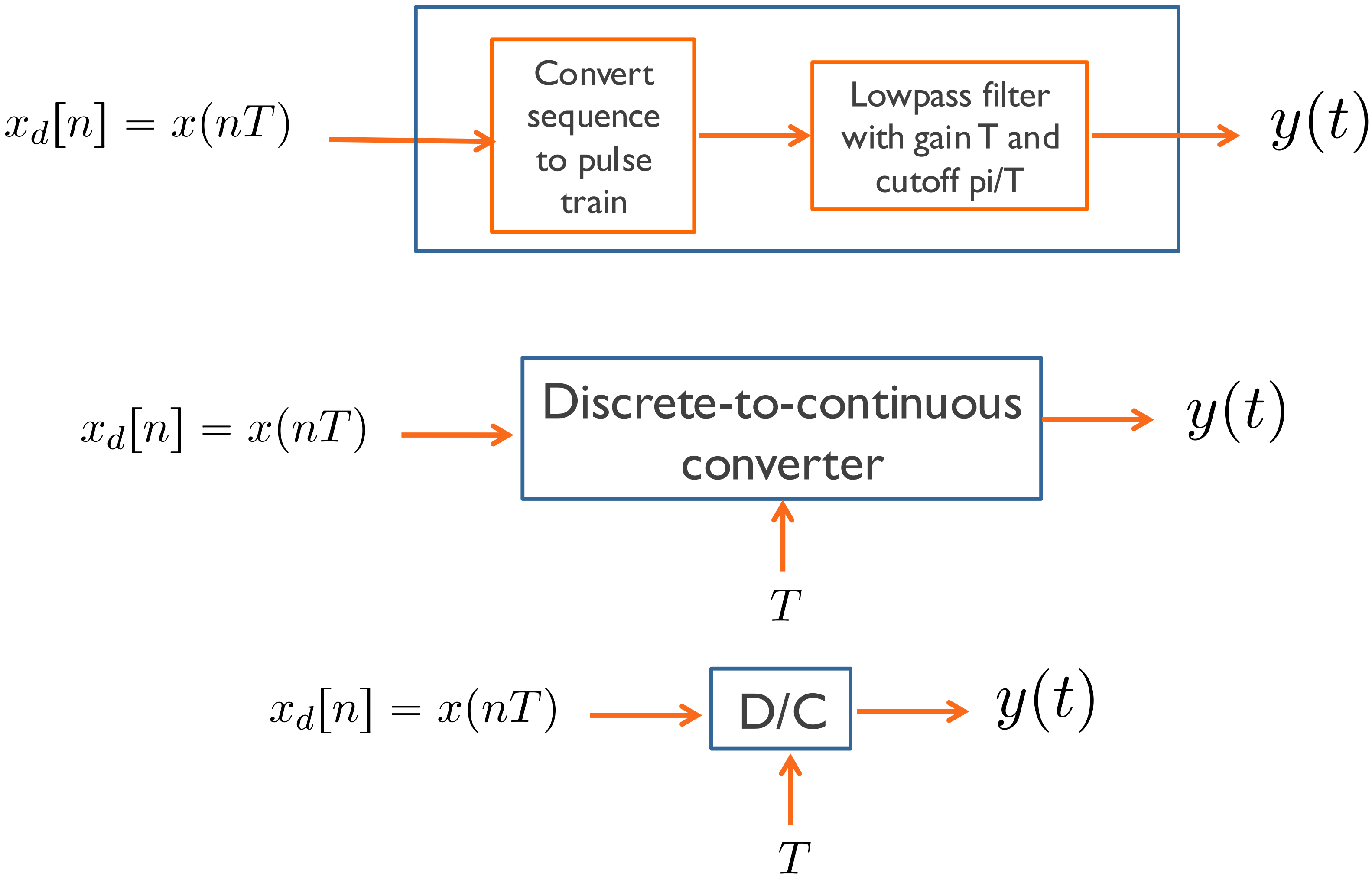


$$\sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$



Reconstructed signal results from a superposition of sinc functions

Ideal discrete-to-continuous converter



Reconstruction in the time and frequency domains

	time domain	frequency domain
impulse train of samples	$y_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$	$Y_p(j\omega) = X_p(j\omega)$
CT signal	$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\frac{t-nT}{T}\right)$	$Y(j\omega) = T\text{rect}\left(\frac{\omega}{2\pi/T}\right)Y_p(j\omega)$ $= T\text{rect}\left(\frac{\omega}{2\pi/T}\right)X_p(j\omega)$

Output is always bandlimited no matter the input

If Nyquist is satisfied then

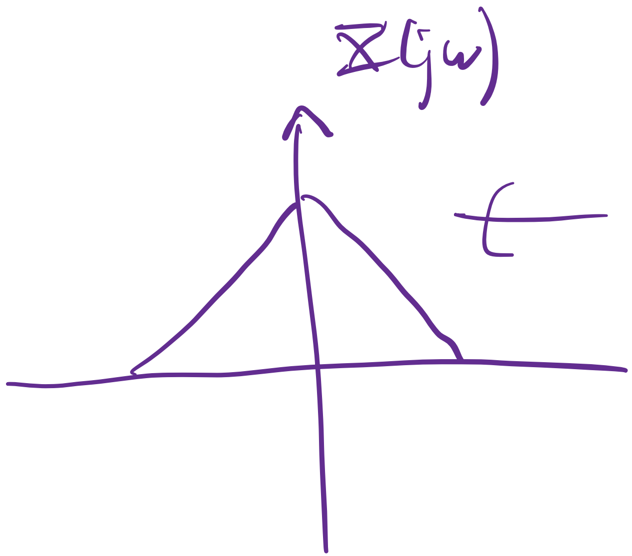
- In the frequency domain

$$Y(j\omega) = X(j\omega)$$

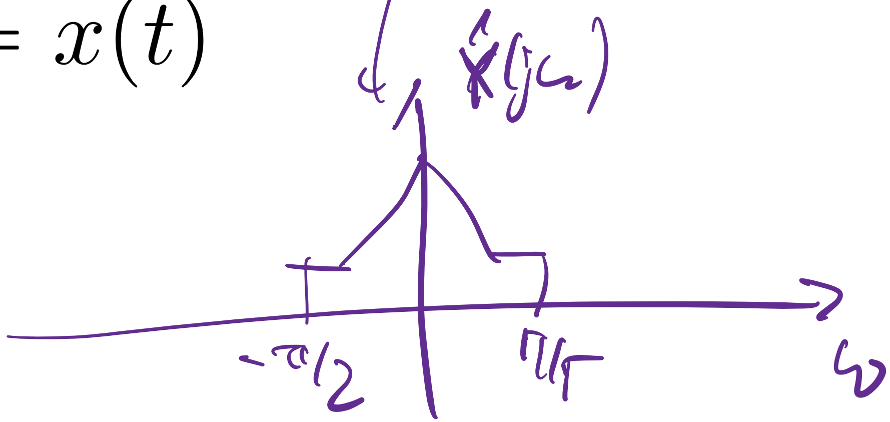
w/ alias

- In the time domain

$$y(t) = x(t)$$



w/ no alias



Thoughts on reconstruction

- Optimal reconstruction involves interpolation of the samples with a sinc function, with the bandwidth determined by the reconstruction frequency
- The output of the discrete-to-continuous converter is always bandlimited
- Aliasing is created by sampling a signal with a sampling frequency less than the Nyquist rate and thus is the "fault" of the continuous-to-discrete conversion, reconstruction just operates on the samples already given

Important example involving the sampling and reconstruction of a sinusoid

Key points

- You should be able to determine the frequency of an undersampled sinusoid after reconstruction
- When Nyquist is not satisfied, aliasing is created

Sampling a sinusoid

- Consider the following signal

$$x(t) = \cos(37\pi t + \pi/4)$$

- Determine the following
  - Nyquist frequency $\omega_M = 37\pi$
  - Nyquist rate $2\omega_M = 74\pi$
  - Maximum sampling period $T < \dfrac{2\pi}{2\omega_M} = \dfrac{2\pi}{74\pi} = \dfrac{1}{37}$ seconds

- General form of sampled signal

$$x(nT) = \cos(37\pi nT + \pi/4)$$

Suppose Nyquist is satisfied

$$T < \frac{2\pi}{2\omega_M} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

$$\omega_s = \frac{2\pi}{T} = 2\pi 74 = 148\pi$$

- Suppose that $T = 1/74$ seconds
- Find the impulse train signal

$$\begin{aligned} x_p(t) &= \sum_n x(nT)\delta(t - n/74) \\ &= \sum_n \cos((37\pi/74)n + \pi/4)\delta(t - n/74) \\ &= \sum_n \cos((\pi/2)n + \pi/4)\delta(t - n/74) \end{aligned}$$

Find the CT transforms

$$T < \frac{2\pi}{2\omega_M} = \frac{2\pi}{74\pi} = \frac{1}{37}$$

$$\omega_s = \frac{2\pi}{T} = 2\pi 74 = 148\pi$$

- Suppose that $T = 1/74$

- Find the CTFT

$$X(j\omega) = \pi e^{j\pi/4}\delta(\omega - 37\pi) + \pi e^{-j\pi/4}\delta(\omega + 37\pi)$$

- Find the CTFT of the impulse train signal

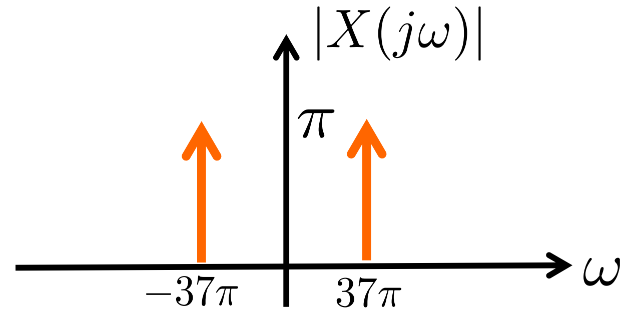
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$= 74 \sum_{k=-\infty}^{\infty} \left(\pi e^{j\pi/4}\delta(\omega - k148\pi - 37\pi) + \pi e^{-j\pi/4}\delta(\omega - k148\pi + 37\pi)\right)$$

Oversampling (using a rate greater than Nyquist)

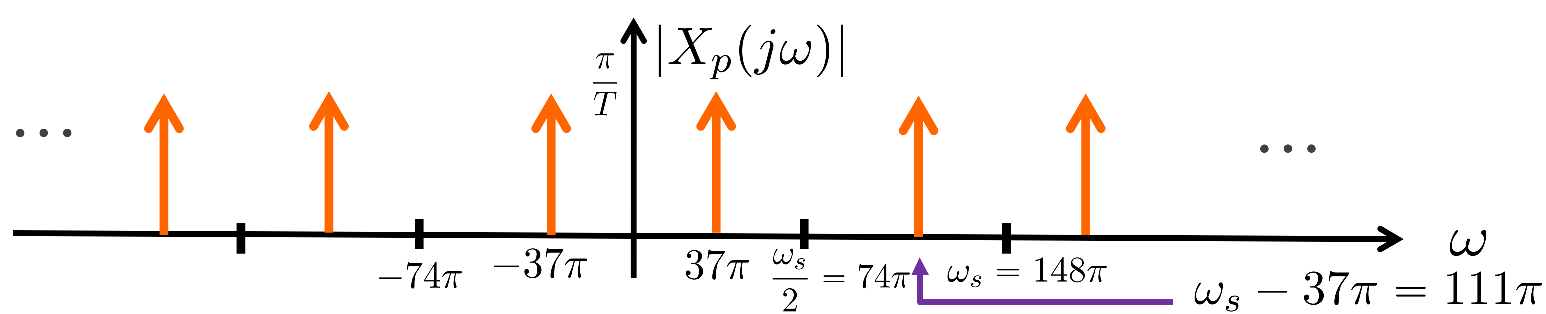
T=1/74

CT signal



$$\omega_s = \frac{2\pi}{T} = 148\pi$$

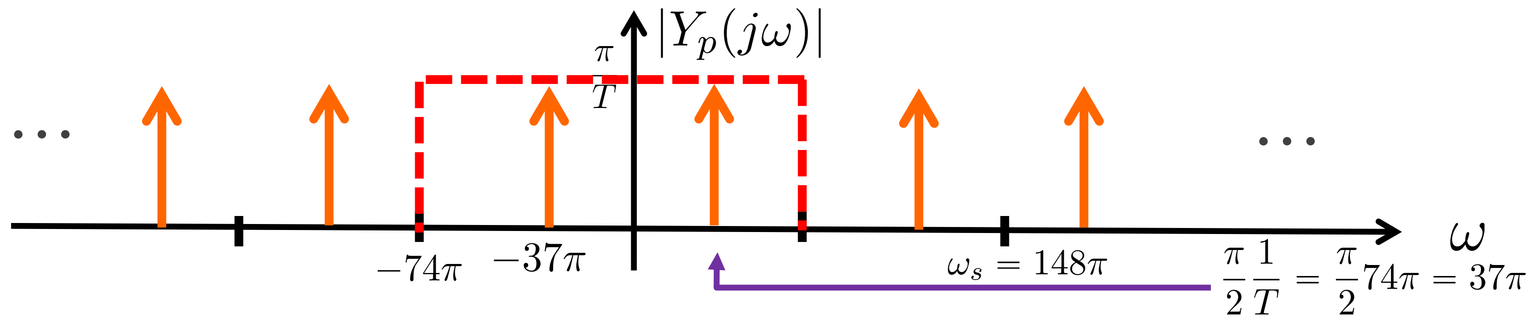
impulse train of samples



Reconstruction (using a rate greater than Nyquist)

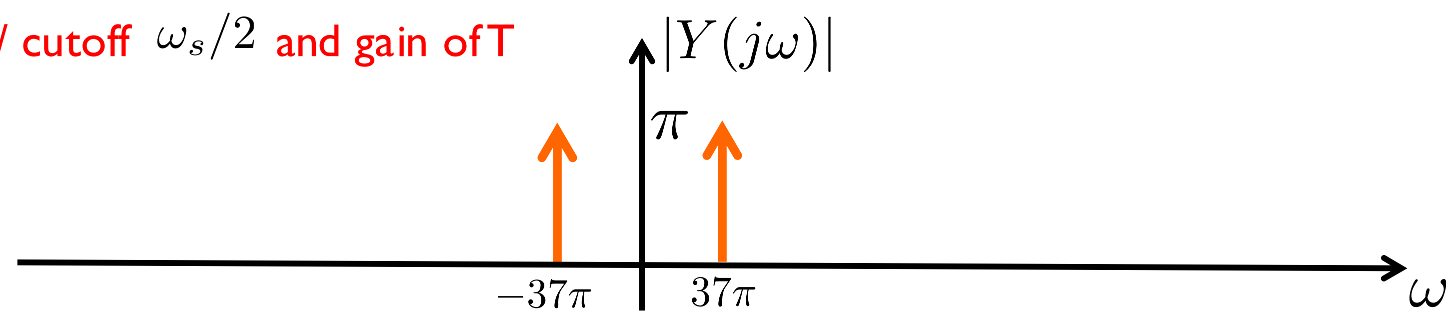
T=1/74

impulse train of samples



Ideal LPF w/ cutoff $\omega_s/2$ and gain of T

CT signal



What if Nyquist is not satisfied?

$$\boxed{T < \frac{2\pi}{\omega_s} = \frac{2\pi}{74\pi} = \frac{1}{37}}$$

- Suppose that $T = \left(\frac{4}{3}\right)\frac{1}{37}$ which does not satisfy Nyquist

$$\omega_s = \frac{2\pi}{T} = \frac{111\pi}{2} = 55.5\pi$$

- Find the CTFT

$$X(j\omega) = \pi e^{j\pi/4}\delta(\omega - 37\pi) + \pi e^{-j\pi/4}\delta(\omega + 37\pi)$$

- Find the CTFT of the impulse train signal

$$\frac{1}{T} = 37\frac{3}{4} = 27.75$$

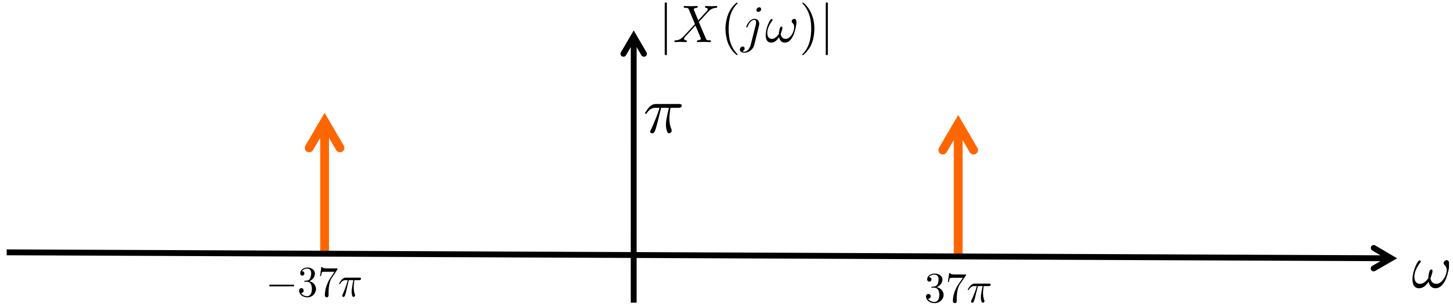
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$= 27.75 \sum_{k=-\infty}^{\infty} \left(\pi e^{j\pi/4}\delta(\omega - k55.5\pi - 37\pi) + \pi e^{-j\pi/4}\delta(\omega - k55.5\pi + 37\pi)\right)$$

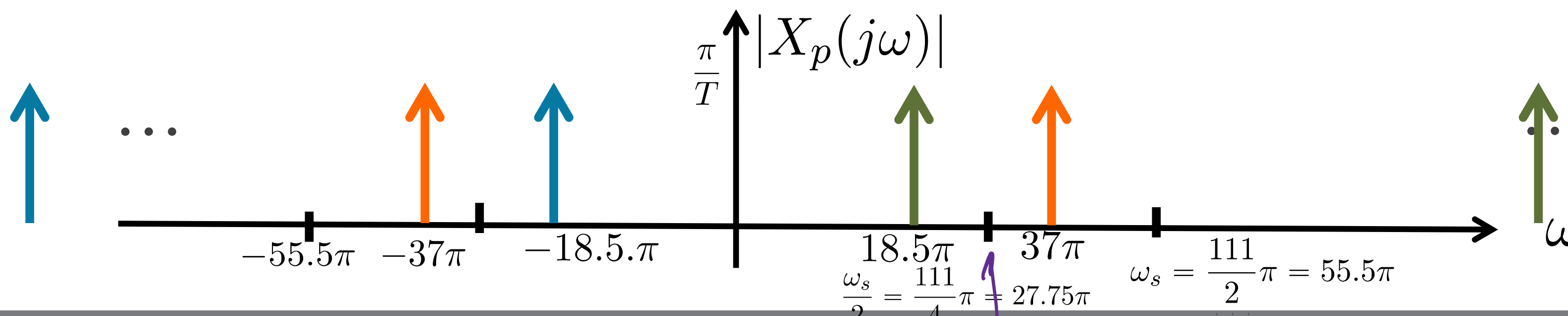
Undersampling (using a rate less than Nyquist)

$$T = \left(\frac{4}{3}\right)\frac{1}{37}$$

CT signal



impulse train of samples



Replica at $-\omega_s$ Original signal Replica at ω_s

$\pi/T = \frac{55\pi}{2}$

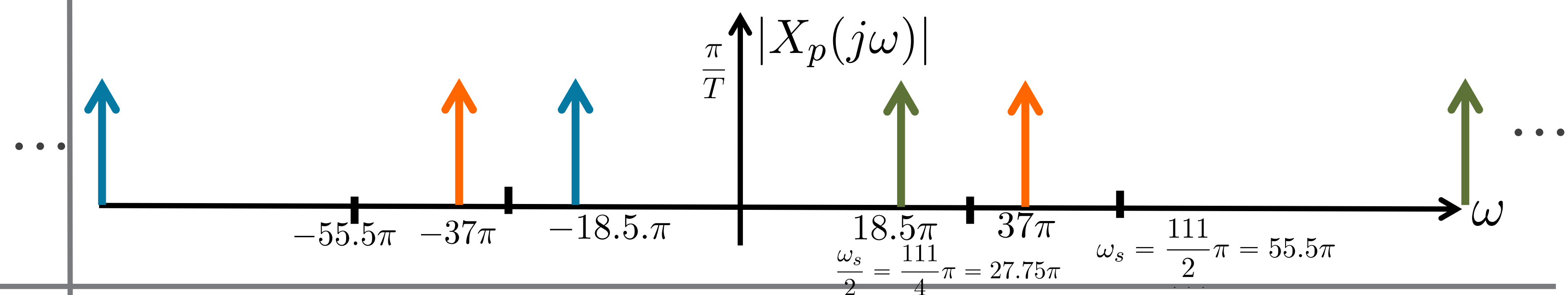
$$\omega_s - 37\pi = \frac{111 - 74}{2}\pi = 18.5\pi$$

Reconstruction (using a rate less than Nyquist)

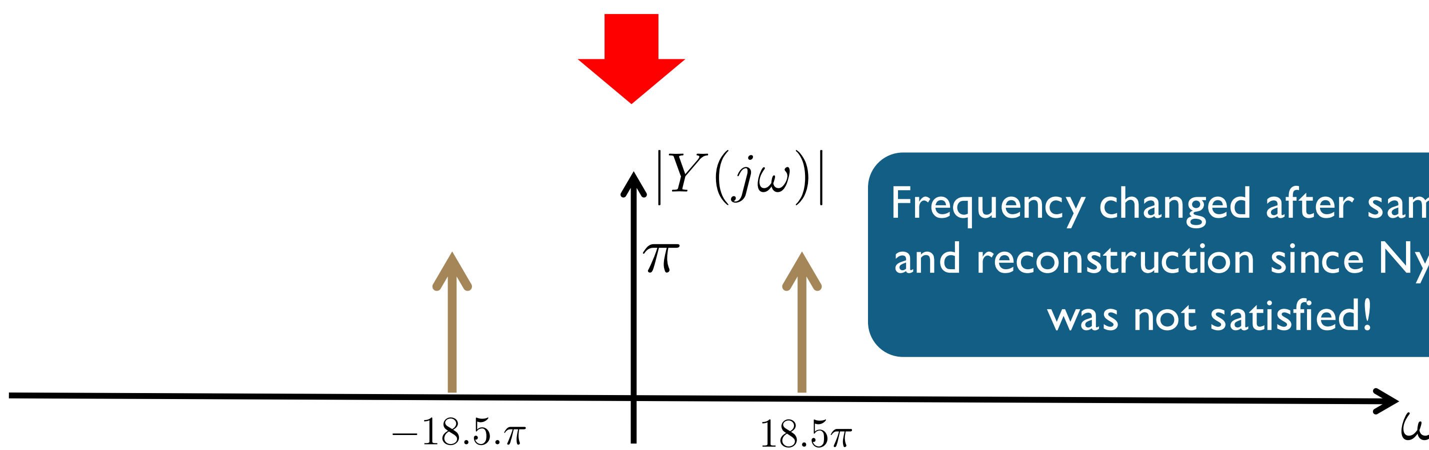
$$T = \left(\frac{4}{3}\right)\frac{1}{37}$$

ω

impulse train of samples



CT signal



Frequency changed after sampling and reconstruction since Nyquist was not satisfied!