UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2021

Linear Systems Fundamentals

SOLUTIONS TO FINAL EXAM

Preface

- I recognize that final exams are stressful in the best of times, and this is an unusually hard time.
- Please remember that despite the stress, I am counting on you to uphold academic integrity while you complete your final exam.
- Posting or seeking exam questions or answers online, or by consulting unauthorized resources, is a gross violation of our principles of integrity and engineering ethics.
- To be fair to all students, any integrity violations discovered during the final exam will be reported to the Dean of Engineering and to the office of Academic Integrity.
- Please make sure you understand and follow the academic integrity guidelines for the exam. If you are not sure, ask me.
- An honest effort, no matter what the outcome, is something to be proud of, especially in these challenging times.
- I am very proud of the commitment and resilience you have displayed. You should feel proud, too.
- **Good luck on the exam!**

PRINT YOUR NAME Pierre-Simon Laplace

SIGNATURE Laplace

Your signature confirms that you have completed this exam on your own and in accordance with the Academic Integrity Agreement.

Student ID Number LT0000XXXX

Instructions

- No electronics allowed for problem solving.
- Notes and Tables from Chapters 3 ,4, 5, 9 of the textbook are attached.
- Two 2-sided sheets of your own notes are allowed.
- Time allowed: 3 hours (including upload time).
- Write your solutions in the designated spaces in the exam.
- **Justify all of your answers.**
- If extra pages are needed to show all of your work, use the scratch pages at the end of the exam and submit them.
- **Upload your solutions by 6pm via Gradescope.**

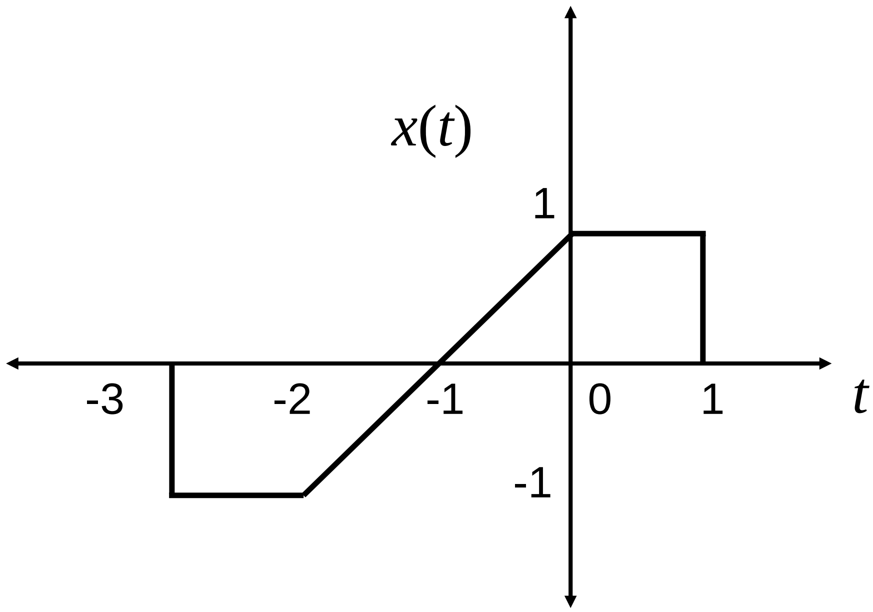
Problem	Weight	Score
1	30 pts	30
2	30 pts	30
3	30 pts	30
4	30 pts	30
5	30 pts	30
6	30 pts	30
Total	180 pts	180

Good luck!

Name/Student ID: ____________________

Problem 1 [CTFT Properties] (30 points, 10 points each part)

Let $x(t)$ be the CT signal below. Let $X(j\omega)$ denote its Fourier transform.



(a) Determine $\int_{-\infty}^{\infty} X(j\omega)d\omega$.

(b) Determine $X(j0)$.

Let $x_1(t) = x(t-1)$, with Fourier transform $X_1(j\omega)$.

(c) Determine $Re\{X_1(j\omega)\}$.

Write answers to parts (a), (b), (c) on the following pages.

Justify your answers.

Name/Student ID: ______________________________

Problem 1 (cont.)

(a) (10 points)

Determine $\int_{-\infty}^{\infty} X(j\omega)d\omega$.

From the synthesis equation,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

we see that, by setting $t = 0$,

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega.$$

Therefore,

$$\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0).$$

For the signal $x(t)$ shown, with $x(0) = 1$, we conclude

$$\boxed{\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi.}$$

Name/Student ID: ______________________________

Problem 1 (cont.)

(b) (10 points)

Determine $X(j0)$.

From the analysis equation,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

we see that, by setting $\omega = 0$,

$$X(j0) = \int_{-\infty}^{\infty} x(t)dt$$

which is the area under the curve.

For the signal $x(t)$ shown, which is a time shift of an odd signal $x(t-1)$, the area under the curve is zero. This can also be seen by direct computation:

$$\int_{-\infty}^{-3} x(t)dt = 0$$
$$\int_{-3}^{-2} x(t)dt = -1$$
$$\int_{-2}^{-1} x(t)dt = -\frac{1}{2}$$
$$\int_{-1}^{0} x(t)dt = \frac{1}{2}$$
$$\int_{0}^{1} x(t)dt = 1$$
$$\int_{1}^{\infty} x(t)dt = 0$$

Therefore,

$$\boxed{X(j0) = 0.}$$

Name/Student ID: ______________________________

Problem 1 (cont.)

Let $x_1(t) = x(t-1)$, with Fourier transform $X_1(j\omega)$.

(c) (10 points)

Determine $Re\{X_1(j\omega)\}$.

The signal $x_1(t)$ is real and odd.

Therefore, from Table 4.1, Properties of the CTFT, we know that $X_1(j\omega)$ is purely imaginary and odd.

It follows that

$$\boxed{Re\{X_1(j\omega)\} = 0.}$$

Name/Student ID: ______________________________

Problem 2 [DTFT] (30 points; 10 points each part)

Let S be the DT LTI system obtained by concatenating the two DT LTI systems S_1 and S_2 defined by:

$$\begin{aligned} S_1: &\quad y[n] = x[n] - \frac{1}{2}x[n-1] \\ S_2: &\quad y[n] = x[n] + \frac{1}{4}x[n-1] \end{aligned}$$

(a) Determine the frequency response $H(e^{j\omega})$ of S and write it in the form

$$\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

(b) Let S' be the inverse system to S. Determine the frequency response $G(e^{j\omega})$ of S' and write it in the form

$$\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

(c) Let $x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$ be the input signal to the system S'. Determine the output signal $y[n]$.

Write answers to parts (a), (b), (c) on the following pages.

Justify your answers.

Name/Student ID: ______________________________

Problem 2 (cont.)

(a) (10 points)

Determine the frequency response $H(e^{j\omega})$ of S and write it in the form $\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$.

The frequency response of S_1 can be found by using the eigenfunction property of DT LTI systems or by taking the DTFT of both sides, using the Time Shifting property of the DTFT from Table 5.1.

From the latter approach, we get

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega}) - {}^{1}\!/\!{}_{2}e^{-j\omega})X(e^{j\omega}) \\ &= X(e^{j\omega})(1 - \frac{1}{2}e^{-j\omega}) \end{aligned}$$

and

$$H_1(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 - \frac{1}{2}e^{-j\omega}.$$

Similarly,

$$H_2(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + \frac{1}{4}e^{-j\omega}.$$

Since S is the concatenation of S_1 and S_2, the frequency response $H(e^{j\omega})$ satisfies

$$\begin{aligned} H(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\ &= (1 - \frac{1}{2}e^{-j\omega})(1 + \frac{1}{4}e^{-j\omega}) \\ &= 1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega} \end{aligned}$$

So,

$$\boxed{H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}.}$$

Name/Student ID: ______________________________

Problem 2 (cont.)

(b) (10 points)

Let S' be the inverse system to S. Determine the frequency response $G(e^{j\omega})$ of S' and write it in the form $\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$.

Let $h[n]$ be the impulse response of the system S and $g[n]$ be the impulse response of the inverse systems S'.

The concatenation of S' and S is the identity system, so $h[n] * g[n] = \delta[n]$.

Applying the DTFT to both sides, and using the Convolution property from Table 5.1 and the transform pair $\delta[n] \leftrightarrow 1$ from Table 5.2, we find

$$H(e^{j\omega})G(e^{j\omega}) = 1.$$

Therefore,

$$\begin{aligned} G(e^{j\omega}) &= \frac{1}{H(e^{j\omega})} \\ &= \frac{1}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}. \end{aligned}$$

So,

$$\boxed{G(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}.}$$

Name/Student ID: ______________________________

Problem 2 (cont.)

(c) (10 points)

Let $x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$ be the input signal to the system S'. Determine the output signal $y[n]$.

Using the DTFT of $\delta[n]$ and the Time Shifting property, we find the DTFT of $x[n]$

$$X(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}.$$

By the convolution formula for DT LTI systems,

$$y[n] = x[n] * g[n].$$

Applying the Convolution property of DTFT and recalling the factorization of $G(e^{j\omega})$,

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})G(e^{j\omega}) \\ &= (1 - \frac{1}{2}e^{-j\omega})\left(\frac{1}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}\right) \\ &= (1 - \frac{1}{2}e^{-j\omega})\left(\frac{1}{(1 - \frac{1}{2}e^{-j\omega})(1 + \frac{1}{4}e^{-j\omega})}\right) \\ &= (\frac{1}{1 + \frac{1}{4}e^{-j\omega}}). \end{aligned}$$

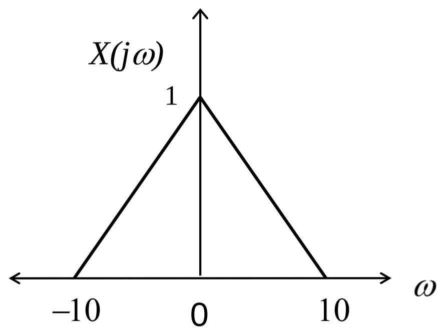
From Table 5.2, DTFT Pairs, we find

$$\boxed{y[n] = \left(-\frac{1}{4}\right)^n u[n].}$$

Name/Student ID: ______________________________

Problem 3 [Modulation] (30 points, 10 points each part)

Let $x(t)$ be the signal with Fourier transform $X(j\omega)$ shown below.



(a) Let $x_1(t) = x(t)\cos(30t)$. The signal $x_1(t)$ is received and passed through a bandpass filter with frequency response

$$H_{BP}(j\omega) = \begin{cases} 2, & 20 < |\omega| < 30 \\ 0, & \text{otherwise.} \end{cases}$$

Write the CTFT $X_1(j\omega)$ of $x_1(t)$ in terms of $X(j\omega)$.
Let $y_1(t)$ be the output of the bandpass filter. Sketch precisely the Fourier transform $Y_1(j\omega)$ of the signal $y_1(t)$ **and write it in terms of** $X(j\omega)$.

(b) Using the signal $y_1(t)$ from part (a), define $x_2(t) = y_1(t)\cos(20t)$. The signal $x_2(t)$ is transmitted and then passed through the lowpass filter with frequency response

$$H_{LP}(j\omega) = \begin{cases} 2, & |\omega| < 20 \\ 0, & \text{otherwise.} \end{cases}$$

Let $y_2(t)$ be the output of the lowpass filter. Sketch precisely its Fourier transform $Y_2(j\omega)$ **and write it in terms of** $X(j\omega)$.

(c) Can the original signal $x(t)$ be recovered from the signal $y_2(t)$ of part (c)? If not, explain why not. If so, describe a demodulation scheme to recover $x(t)$. **Write the transform of any intermediate signal you generate from $y_2(t)$ during the recovery process in terms of $X(j\omega)$ and sketch it precisely.**

Write your answers to parts (a), (b), (c) on the following pages.

Name/Student ID: ____________________________________

Problem 3 (cont.)

(a) (10 points)

Let $x_1(t) = x(t)\cos(30t)$. The signal $x_1(t)$ is received and passed through a bandpass filter with frequency response

$$H_{BP}(j\omega) = \begin{cases} 2, & 20 < |\omega| < 30 \\ 0, & \text{otherwise.} \end{cases}$$

Write the CTFT $X_1(j\omega)$ of $x_1(t)$ in terms of $X(j\omega)$.
Let $y_1(t)$ be the output of the bandpass filter. Sketch precisely the Fourier transform $Y_1(j\omega)$ of the signal $y_1(t)$ **and write it in terms of $X(j\omega)$**.

$$X_1(j\omega) = \frac{1}{2}X(j(\omega+30)) + \frac{1}{2}X(j(\omega-30)).$$

$$Y_1(j\omega) = \begin{cases} X(j(\omega+30)), & -30 < \omega < -20 \\ X(j(\omega-30)), & 20 < \omega < 30 \\ 0, & \text{otherwise} \end{cases}$$

$Y_1(\omega)$

Name/Student ID: ____

Problem 3 (cont.)

(b) (10 points)

Using the signal $y_1(t)$ from part (a), define $x_2(t) = y_1(t)\cos(20t)$. The signal $x_2(t)$ is transmitted and then passed through the lowpass filter of with frequency response

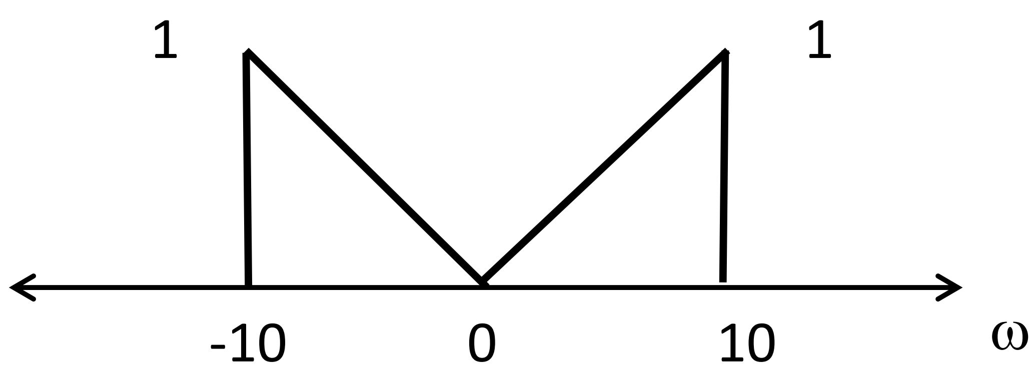
$$H_{LP}(j\omega) = \begin{cases} 2, & |\omega| < 20 \\ 0, & \text{otherwise.} \end{cases}$$

Let $y_2(t)$ be the output of the lowpass filter. Sketch precisely its Fourier transform $Y_2(j\omega)$ **and write it in terms of $X(j\omega)$**.

$$X_2(j\omega) = \frac{1}{2}Y_1(j(\omega+20)) + \frac{1}{2}Y_1(j(\omega-20)).$$

$$Y_2(j\omega) = \begin{cases} X(j(\omega+10)), & -10 < \omega < 0 \\ X(j(\omega-10)), & 0 < \omega < 10 \\ 0, & \text{otherwise} \end{cases}$$

$Y_2(\omega)$



Name/Student ID: ______________________________

Problem 3 (cont.)

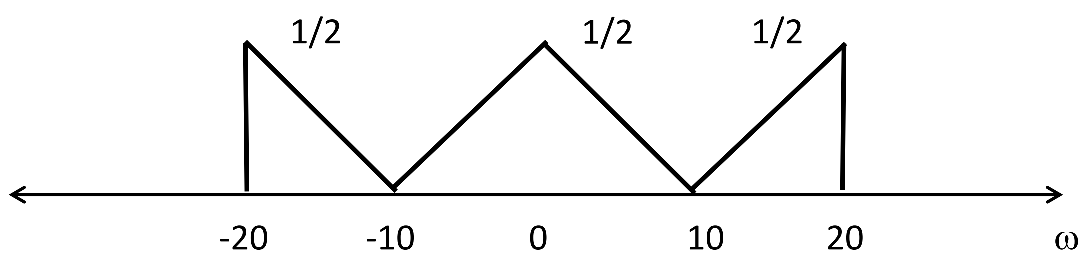
(c) (10 points)

Can the original signal $x(t)$ be recovered from the signal $y_2(t)$ of part (c)? If not, explain why not. If so, describe a demodulation scheme to recover $x(t)$. **Write the transform of any intermediate signal you generate from $y_2(t)$ during the recovery process in terms of $X(j\omega)$ and sketch it precisely.**

Yes. Define $x_3(t) = y_2(t)\cos(10t)$. The resulting signal has Fourier transform

$$X_3(j\omega) = \begin{cases} \frac{1}{2}X(j(\omega + 20)), & -20 < \omega < 10 \\ \frac{1}{2}X(j(\omega)), & -10 < \omega < 10 \\ \frac{1}{2}X(j(\omega - 20)), & 10 < \omega < 20 \\ 0, & \text{otherwise} \end{cases}$$

$X_3(\omega)$



Pass $x_3(t)$ the result through a lowpass filter with frequency response

$$H_3(j\omega) = \begin{cases} 2, & |\omega| < 10 \\ 0, & \text{otherwise.} \end{cases}$$

The resulting signal $y_3(t)$ has transform $Y_3(j\omega) = X(j\omega)$, so $y_3(t) = x(t)$.

Name/Student ID: ______________________________

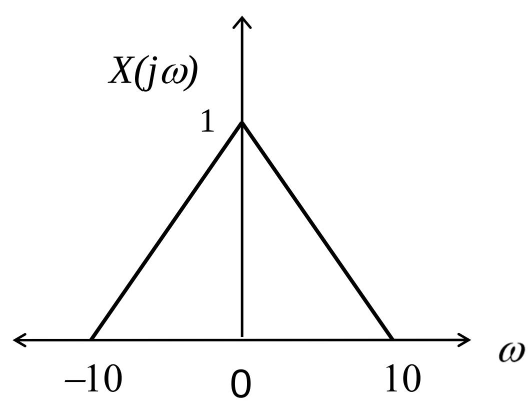
Problem 4 [Sampling Theory] (30 points)

(a) For each of the following signals $x(t)$, determine if a sampling period of $T = \frac{1}{4}$ sec is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples. **Justify your answers by determining the Fourier Transforms $X(j\omega)$ of the signals.**

(i) $x(t) = \frac{\sin(3\pi t)}{\pi t}$

(ii) $x(t) = \begin{cases} 1, & |t| < 3\pi \\ 0, & |t| > 3\pi \end{cases}$

(b) Let $x(t)$ be the signal with Fourier transform $X(j\omega)$ shown below.



Let $x_p(t)$ be obtained from $x(t)$ by impulse train sampling with sampling period $T = \frac{\pi}{20}$, with Fourier transform by $X_p(j\omega)$. Let $x_d[n] = x(nT)$ be the DT sampled signal, with Fourier transform $X_d(e^{j\Omega})$.

(i) Write $X_p(j\omega)$ in terms of $X(j\omega)$. Write the formula relating $X_d(e^{j\Omega})$ to $X_p(j\omega)$. Determine ω_0 such that $X_d(e^{j2\pi}) = X_p(j\omega_0)$.

(ii) Sketch precisely $X_p(j\omega)$ in the range $|\omega| \leq \omega_0$ and $X_d(e^{j\Omega})$ in the range $|\Omega| \leq 2\pi$.

Justify your answers by referring to the formula for $X_p(j\omega)$ and its relation to $X_d(e^{j\Omega})$

Write your answers to parts (a), (b) on the following pages.

Name/Student ID: ______________________________

Problem 4 (cont.)

(a) (16 points)

For each of the following signals $x(t)$, determine if a sampling period of $T = \frac{1}{4}$ sec is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples.

(i) $x(t) = \frac{\sin(3\pi t)}{\pi t}$

From Table 4.2, CTFT Pairs,

$$X(j\omega) = \begin{cases} 1, & |\omega| < 3\pi \\ 0, & |\omega| > 3\pi \end{cases}$$

So, the maximum frequency is $\omega_M = 3\pi$.

With sampling period $T = \frac{1}{4}$, the sampling frequency is $\omega_s \frac{2\pi}{T} = 8\pi$.

Since $\omega_s = 8\pi > 6\pi = 2\omega_M$, the required condition of the Sampling Theorem is satisfied, so

$$\boxed{\text{x(t) can be reconstructed from its samples.}}$$

(ii) $x(t) = \begin{cases} 1, & |t| < 3\pi \\ 0, & |t| > 3\pi \end{cases}$

Referring to Table 4.2,

$$X(j\omega) = 2\frac{\sin(\omega 3\pi)}{\omega}.$$

This can be rewritten as

$$X(j\omega) = 6\pi\,\mathrm{sinc}(3\omega).$$

As a sinc function, this transform does not have finite bandwidth.

Therefore, the conditions of the Sampling Theorem cannot be satisfied no matter how small the sampling period T is.

In particular,

$$\boxed{\text{x(t) can not be reconstructed from its samples.}}$$

Name/Student ID: ____

Problem 4 (cont.)

(b) (14 points) Let $x_p(t)$ be obtained from $x(t)$ by impulse train sampling with sampling period $T = \frac{\pi}{20}$, with Fourier transform by $X_p(j\omega)$. Let $x_d[n] = x(nT)$ be the DT sampled signal, with Fourier transform $X_d(e^{j\Omega})$.

(i) (4 points)
Write $X_p(j\omega)$ in terms of $X(j\omega)$. Write the formula relating $X_d(e^{j\Omega})$ to $X_p(j\omega)$. Determine ω_0 such that $X_d(e^{j2\pi}) = X_p(j\omega_0)$.

With sampling period $T = \frac{\pi}{20}$, the sampling frequency is $\omega_s = \frac{2\pi}{T} = 40$.

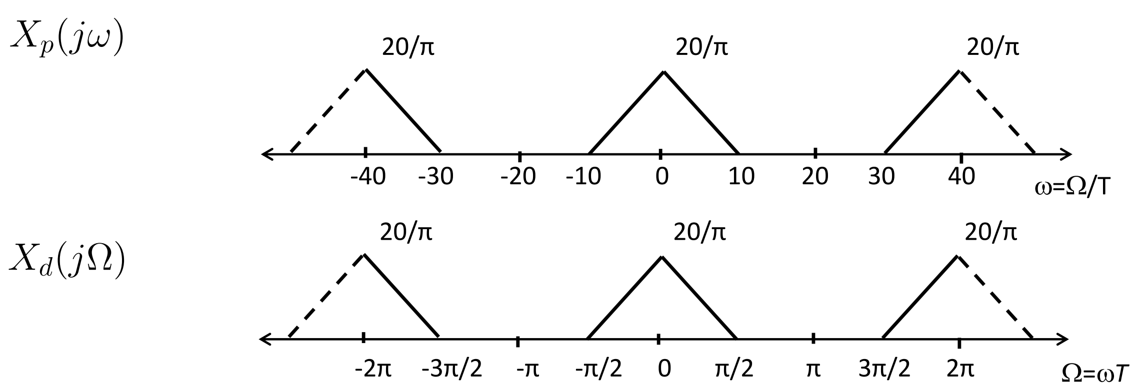
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)).$$

$$X_d(e^{j\Omega}) = X_p(j\Omega/T) \text{ or } X_d(e^{j\omega T}) = X_p(j\omega).$$

So, with $\Omega = 2\pi$, we find

$$\boxed{\omega_0 = \Omega/T = 40.}$$

(ii) (10 points)
Sketch precisely $X_p(j\omega)$ in the range $|\omega| \leq \omega_0$ and $X_d(e^{j\Omega})$ in the range $|\Omega| \leq 2\pi$.



Name/Student ID: ____

Problem 5 [Laplace Transform and LTI Systems] (30 points)

Consider an LTI system S with transfer function

$$H(s) = \frac{s-1}{(s^2-2s+2)(s+1)}.$$

(a) Sketch precisely the pole-zero plot associated with $H(s)$. Indicate the order of all poles and zeros.

(b) Suppose S is stable.

  (i) Determine $|H(j0)|$ and $\angle H(j0)$.

  (ii) Determine $|H(j\omega)|$ and $\angle H(j\omega)$ in the limit of as $\omega \to \pm\infty$.

  (iii) Is the system ~~have~~ lowpass, highpass, bandpass, or allpass?

(c) Suppose S is causal. If the impulse response is $h(t)$, does $h(t)e^{-2t}$ have a Fourier transform?

Write your answers to parts (a), (b), (c) on the following pages.

Name/Student ID: ________________________________

Problem 5 (cont.)

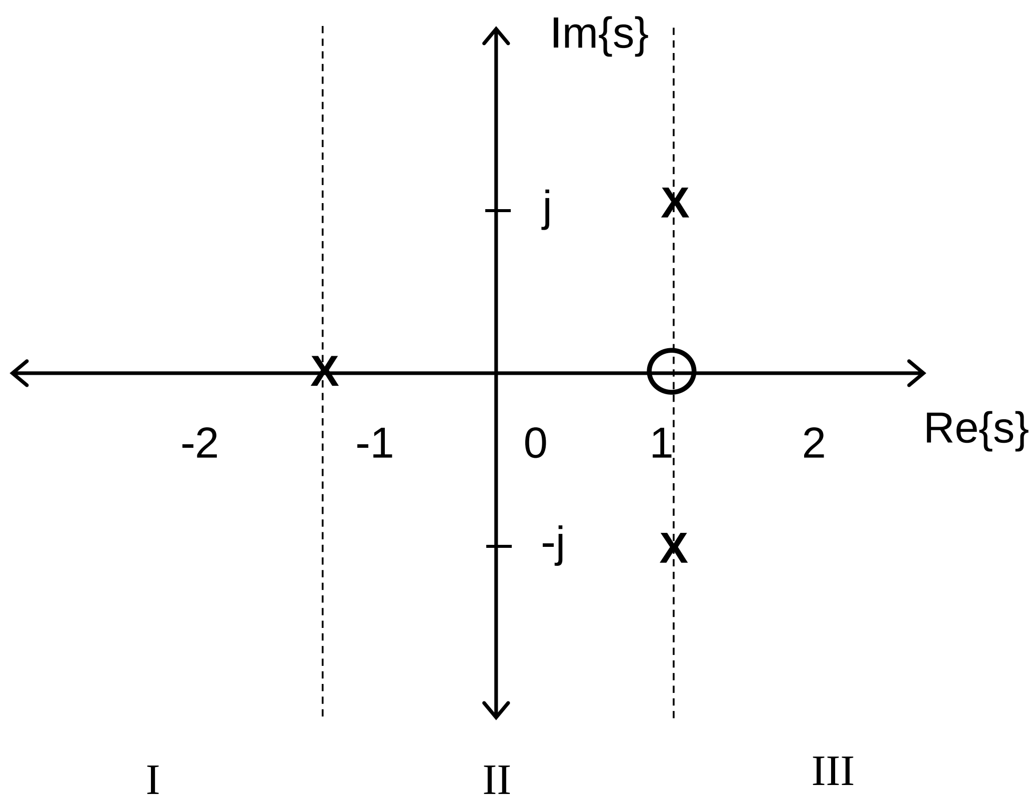
Consider the LTI system function $H(s) = \frac{s-1}{(s^2-2s+2)(s+1)}$.

(a) (10 points)

Sketch precisely the pole-zero plot associated with $H(s)$. Indicate the order of all poles and zeros.

$s^2 - 2s + 2 = (s - (1 + j))(s(-(1 - j))$

Zeros: $s = 1$, order 1.
Poles: $s = -1$, $s = 1 - j$, $s = 1 + j$, all order 1.



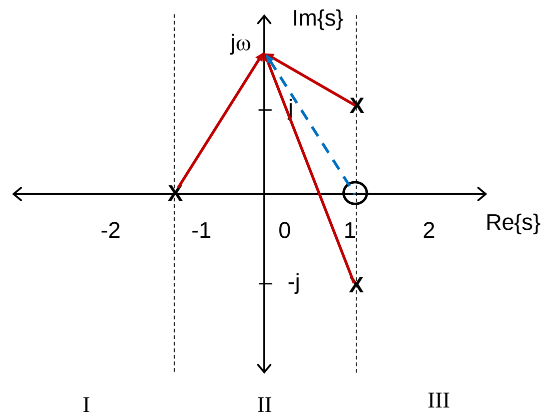
Name/Student ID: ______________________________

Problem 5 (cont.)

Consider the LTI system function $H(s) = \frac{s-1}{(s^2-2s+2)(s+1)}$.

(b) (12 points)

Suppose S is stable. The ROC is region II in the pole-zero plot, including the $j\omega$-axis.



(i) Determine $|H(j0)|$ and $\angle H(j0)$.

$|H(j\omega)| = \frac{|j\omega-1|}{|j\omega-(1+j)|\cdot|j\omega-(1-j)|\cdot|j\omega+1|} = \frac{1}{|j\omega-(1+j)|\cdot|j\omega-(1-j)|}$.
For $\omega = 0$, this yields

$$|H(j0)| = \frac{|0-1|}{0-(1+j)|\cdot|0-(1-j)|\cdot|0+1|} = \frac{1}{\sqrt{2})\cdot\sqrt{2}\cdot 1} = \frac{1}{2}.$$

$$\boxed{|H(j0)| = \frac{1}{2}.}$$

$\angle H(j\omega) = \angle(j\omega - 1) - (\angle(j\omega-(1+j)) + \angle(j\omega-(1-j)) + \angle(j\omega+1)$.
For $\omega = 0$, this yields

$$\angle H(j0) = \pi - (\frac{5\pi}{4} + \frac{3\pi}{4} + 0) = \pi - 2\pi = -\pi = \pi \mod 2\pi.$$

$$\boxed{\angle H(j0) = \pi.}$$

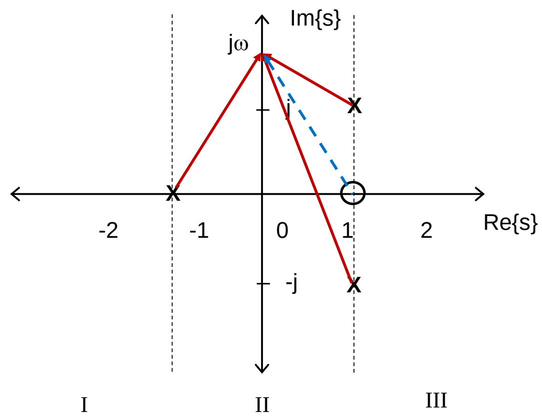
Name/Student ID: ______________________

Problem 5 (cont.)

Consider the LTI system function $H(s) = \frac{s-1}{(s^2-2s+2)(s+1)}$.

(b) (12 points)

Suppose S is stable. The ROC is region II in the pole-zero plot, including the $j\omega$-axis.



(ii) Determine $|H(j\omega)|$ and $\angle H(j\omega)$ in the limit of as $\omega \to \pm\infty$.

$$\lim_{\omega\to\pm\infty} |H(j\omega)| = \lim_{\omega\to\pm\infty} \frac{1}{|j\omega-(1+j)|\cdot|j\omega-(1-j)|} = \frac{1}{\infty\cdot\infty} = 0.$$

$$\boxed{\lim_{\omega\to\pm\infty} |H(j\omega)| = 0.}$$

$$\lim_{\omega\to\pm\infty} \angle H(j\omega) = \frac{\pm\pi}{2} - (\frac{\pm\pi}{2} + \frac{\pm\pi}{2} + \frac{\pm\pi}{2}) = -\pi = \pi \mod 2\pi.$$

$$\boxed{\lim_{\omega\to\pm\infty} \angle H(j\omega) = \pi.}$$

(iii) Is the system ~~have~~ lowpass, highpass, bandpass, or allpass?

$\boxed{\text{Lowpass.}}$

Name/Student ID: ______________________________

Problem 5 (cont.)

Consider the LTI system function $H(s) = \frac{s-1}{(s^2-2s+2)(s+1)}$.

(c) (8 points)

Suppose S is causal. If the impulse response is $h(t)$, does $h(t)e^{-2t}$ have a Fourier transform?

If S is causal, the ROC is $\{s \in \mathbb{C} | Re\{s\} > 1\}$, region III in the pole-zero plot.

If we write $s = \sigma + j\omega$, then the line $\{s \in \mathbb{C} | Re\{s\} = \sigma = 2\}$ lies in the ROC.

This means that, for $s = 2 + j\omega$ the following integral exists:

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} h(t)e^{-(2+j\omega)t}dt \\ &= \int_{-\infty}^{\infty} h(t)e^{-2t}e^{-j\omega t}dt \end{aligned}$$

This means that the signal $h(t)e^{-\sigma t}$ has a Fourier transform.

$\boxed{\text{The answer is : Yes.}}$

Alternatively, apply the Shifting in s property in Table 9.1, Laplace Transform Properties, and note that the $j\omega$-axis lies in the ROC of $H(s+2)$, which is the original ROC shifted to the left by 2.

Name/Student ID: ______________________________

Problem 6 [LT and Differential Equations] (30 points, 10 points each)

Consider the causal LTI system whose input $x(t)$ and output $y(t)$ satisfy the differential equation

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = \frac{dx(t)}{dt} - x(t).$$

(a) Determine the transfer function $H(s)$ and sketch precisely the pole-zero plot.

(b) Determine the region of convergence (ROC). and determine the impulse response $h(t)$.

(c) Draw a block diagram for the direct form (canonical) realization of the system using a cascade of integrators with feedforward and feedback components.

Write your answers to parts (a), (b), (c), and (d) on the following pages.

Name/Student ID: ____________________

Problem 6 (cont.)

Causal LTI system defined by $\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = \frac{dx(t)}{dt} - x(t)$.

(a) (10 points)

Determine the transfer function $H(s)$ and sketch precisely the pole-zero plot.

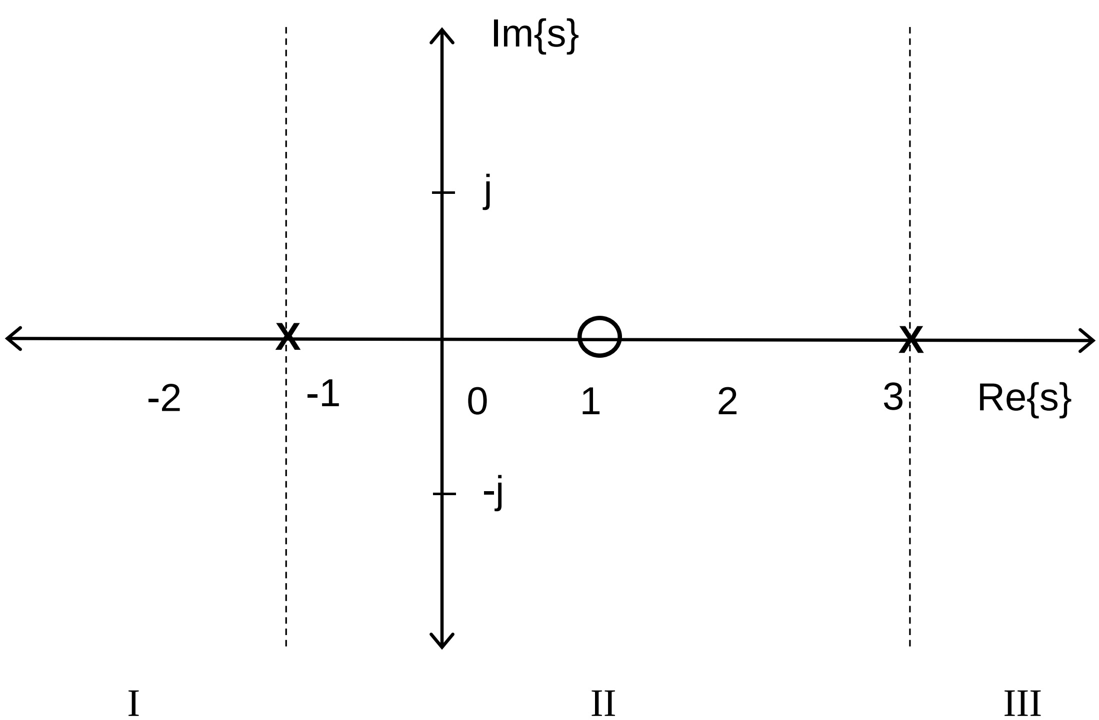
W apply the Laplace Transform to both sides. Using Linearity and the Differentiation in Time property, we get:

$$s^2Y(s) - 2sY(s) - 3Y(s) = sX(s) - X(s).$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-1}{s^2 - 2s - 3}.$$

Factoring the denominator gives

$$H(s) = \frac{s-1}{(s+1)(s-3)}.$$



Name/Student ID: ________________________________

Problem 6 (cont.)

Causal LTI system defined by $\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = \frac{dx(t)}{dt} - x(t)$.

(b) (10 points)

Determine the region of convergence (ROC). and determine the impulse response $h(t)$.

The Laplace Transform is rational and the system is causal, so the ROC is region III, the half-plane to the right of the rightmost pole, namely $s = 3$.

To find $h(t)$ use Partial Fraction Expansion:

$$H(s) = \frac{A}{s+1} + \frac{B}{s-3}$$

$$\begin{aligned} H(s)(s+1)\Big|_{s=-1} &= A + B\frac{s-1}{s+3}\Big|_{s=-1} \\ \frac{-2}{-4} &= A + 0 \Rightarrow A = \frac{1}{2}. \\ H(s)(s-3)\Big|_{s=3} &= A\frac{s-3}{s+1}\Big|_{s=-1} + B \\ \frac{2}{4} &= 0 + b \Rightarrow B = \frac{1}{2}. \end{aligned}$$

So $H(s) = \frac{1/2}{s_1} + \frac{1/2}{s-3}$.

Referring to Table 9.2, Laplace Transform Pairs

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{3t}u(t).$$

Note that the system is not stable because the $j\omega$-axis is not in the ROC.

Name/Student ID: ______________________

Problem 6 (cont.)

Causal LTI system defined by $\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = \frac{dx(t)}{dt} - x(t)$.

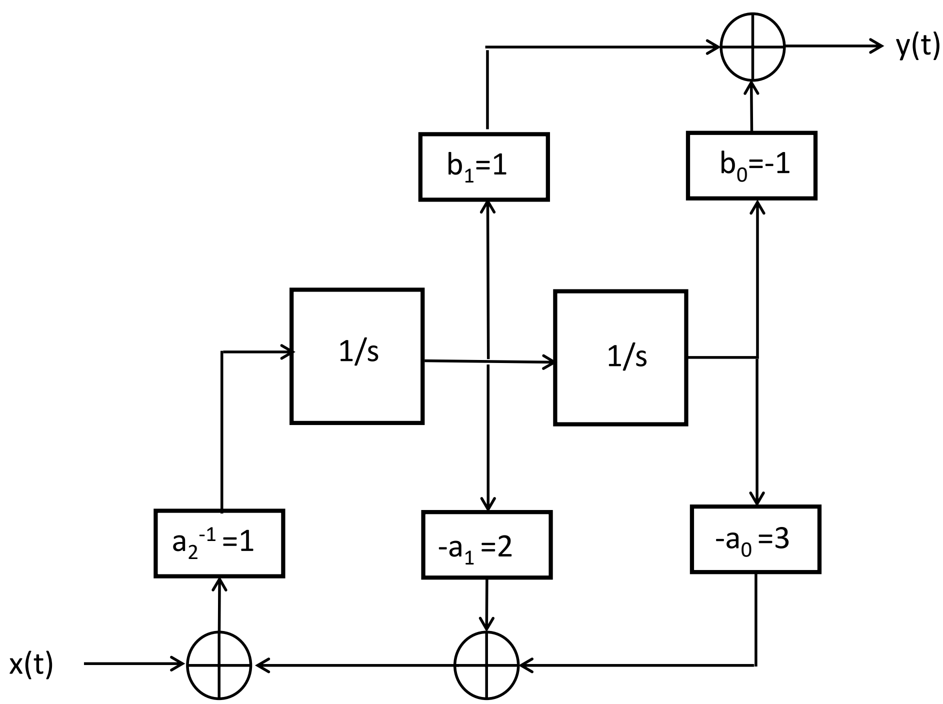
(c) (10 points)

Draw a block diagram for the direct form (canonical) realization of the system using a cascade of integrators with feedforward and feedback components.

From the differential equation, we have

$$a_2 = 1,\ a_1 = -2,\ a_0 = -3$$

$$b_2 = 0,\ b_1 = 1,\ b_0 = -1$$



Scratch page **Name/Student ID:** ____________________

Scratch page **Name/Student ID:** ____________________

Scratch page **Name/Student ID:** ____________________________

Scratch page **Name/Student ID:** ______________________________

Scratch page **Name/Student ID:** ______________________________