ECE 45 – Circuits and Systems
Winter 2025

Homework #7

Due: February 20 at 11:59pm, submitted via GradeScope.

You can make multiple upload attempts to experiment with the system and the best way to upload. You must correctly mark the answers to the problems in GradeScope, e.g. problem 1, problem 2, problem 3, to get full credit. Note that you must tag your problems when uploading to GradeScope or they will not be graded and you will not receive credit. Any regrade requests must be placed through GradeScope within one week of the return of the homework.

Remember, discussion of homework questions is encouraged. Please be absolutely sure to submit your own independent homework solution.

1. (20 %) Consider two periodic signals with Fourier Series coefficients $x(t) \xleftrightarrow{\text{FS}} a_k$ and $y(t) \xleftrightarrow{\text{FS}} b_k$. As usual, assume that $x(t)$ has period T and thus fundamental frequency $\omega_0 = 2\pi/T$. Use the properties of the Fourier Series to answer the following.

   (a) Find the period and FS coefficients of $y(t) = x(3t)$.

   (b) Find the period and FS coefficients of $y(t) = x(t-5)$.

   (c) Find the period and FS coefficients of $y(t) = x(3t-5)$.

   (d) Express $y(t)$ in terms of $x(t)$ if $b_k = 1 + (a_k - a^*_{-k})$.

   (e) Find the period and FS coefficients of $y(t) = x(1-t) + x(t)$.

   (f) Compute $\frac{1}{T}\int_T |e^{j\omega_0 t}x(3t-5)|^2 dt$.

   Solution:

   (a)

$$y(t) = x(3t) \xleftrightarrow{\text{FS}} b_k = a_k$$

   Time period : T/3

   (b)

$$y(t) = x(t-5) \xleftrightarrow{\text{FS}} b_k = a_k e^{-jk\omega_0 5}$$

   Time period : T

   (c) $y(t) = x(3(t-5/3))$ Consider a new function $z(t) = x(3t)$, and has the FS coefficients c_k. Then,

$$z(t) = x(3t) \xleftrightarrow{\text{FS}} c_k = a_k$$

   and the fundamental frequency of $z(t)$ is $\omega_{z0} = 3\omega_0$.

Now, using $z(t)$, we can write $y(t)$ as follows:

$$y(t) = z(t - 5/3)$$

By using the time shift property, we can write:

$$\begin{aligned} y(t) = z(t - 5/3) &\xleftrightarrow{\text{FS}} c_k e^{-jk\omega_{z0}\frac{5}{3}} \\ y(t) &\xleftrightarrow{\text{FS}} a_k e^{-jk(3\omega_0)\frac{5}{3}} \\ y(t) &\xleftrightarrow{\text{FS}} a_k e^{-jk5\omega_0}. \end{aligned}$$

Time period : T/3

(d) We know from the lectures that,

$$T \sum_{k=-\infty}^{\infty} \delta(t - kT) \xleftrightarrow{\text{FS}} 1$$

$$x^*(t) \xleftrightarrow{\text{FS}} a^*_{-k}$$

Using the linearity property of fourier transform, if

$$b_k = 1 + a_k - a^*_{-k}$$

Then,

$$y(t) = T \sum_{k=-\infty}^{\infty} \delta(t - kT) + x(t) - x^*(t)$$

Time period = T.

(e) We know,

$$\begin{aligned} x(t) &\xleftrightarrow{\text{FS}} a_k \\ x(1-t) &\xleftrightarrow{\text{FS}} a_{-k} e^{-j\omega_0 k} \end{aligned}$$

Now $y(t) = x(t) + x(1 - t)$, this gives us,

$$y(t) = x(1 - t) + x(t) \xleftrightarrow{\text{FS}} a_k + a_{-k} e^{-j\omega_0 k}$$

Time period = T

(f) We know,

$$\begin{aligned} x(t) &\xleftrightarrow{\text{FS}} a_k \\ x(3t - 5) &\xleftrightarrow{\text{FS}} e^{-5j\omega_0 k}\ a_k \end{aligned}$$

We know Parseval's theorem,

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=\infty}^{\infty} |a_k|^2$$

Now, exploiting the fact that $|e^{j\tau}| = 1$, for any $\tau \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{T}\int_T |e^{j\omega_0 t}x(3t-5)|^2 dt &= \frac{1}{T}\int_T |x(3t-5)|^2 dt \\
&= \sum_{k=-\infty}^{\infty} |e^{-5j\omega_0 k}a_k|^2 \quad \text{[using time shift and parseval theorem]} \\
&= \sum_{k=-\infty}^{\infty} |e^{-5j\omega_0 k}|^2 |a_k|^2 \\
&= \sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{T}\int_T |x(t)|^2 dt
\end{aligned}$$

2. (20 %) Using the table of common transform pairs and the properties, find the FS series coefficients of the following signals.

   (a) Let $x(t)$ be periodic with $T = 10$ signal and first period $t \in [-5, 5)$ given by

   $$x(t) = \text{rect}\,(t-1) - \text{rect}\,(t+1)\,.$$

   (b) Let $x(t)$ be periodic with $T = 5$ signal given by

   $$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-5k) - \delta(t-2-5k).$$

   (c) Let $x(t)$ be periodic with $T = 10$ signal and first period $t \in [-5, 5)$ given by

   $$x(t) = 2\ \text{rect}(t) + \delta(t-1).$$

Solution:

(a) From the lecture, the Fourier series (FS) coefficients of $\text{rect}(t)$ with period $T = 10$ and width $T_1 = 1$ are

$$\text{rect}(t) \overset{\text{FS}}{\longleftrightarrow} \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right).$$

Using the time-shifting property, for $x(t) = \text{rect}(t-1)$, the FS coefficients are given by

$$\text{rect}(t-1) \overset{\text{FS}}{\longleftrightarrow} \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right) e^{-jk\omega_0},$$

where $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{5}$. Similarly, for $\text{rect}(t+1)$, the FS coefficients are

$$\text{rect}(t+1) \overset{\text{FS}}{\longleftrightarrow} \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right) e^{jk\omega_0}.$$

Since $x(t) = \text{rect}(t-1) - \text{rect}(t+1)$, using linearity of the Fourier series,

$$x(t) \overset{\text{FS}}{\longleftrightarrow} \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right) e^{-jk\omega_0} - \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right) e^{jk\omega_0}.$$

Factoring out the common terms:

$$c_k = \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right)\left(e^{-jk\omega_0} - e^{jk\omega_0}\right).$$

Using the identity $e^{-j\theta} - e^{j\theta} = -2j\sin(\theta)$, we obtain

$$c_k = -\frac{2j}{10}\text{sinc}\left(\frac{k}{10}\right)\sin(k\omega_0).$$

Thus, the Fourier series coefficients of $x(t)$ are

$$c_k = -\frac{j}{5}\text{sinc}\left(\frac{k}{10}\right)\sin\left(\frac{k\pi}{5}\right).$$

(b)

$$\sum_{k=-\infty}^{\infty}\delta(t-5k) \xleftrightarrow{FS} \frac{1}{T} = \frac{1}{5}$$

Hence,

$$\sum_{k=-\infty}^{\infty}\delta(t-5k-2) \xleftrightarrow{FS} \frac{1}{T} = \frac{1}{5}e^{-jk\omega_0 * 2}$$

Thus,

$$x(t) \xleftrightarrow{FS} \frac{1}{5}(1 - e^{-j2k\omega_0})$$

(c) From the lecture, for $\text{rect}(t)$ with period $T = 10$

$$\text{rect}(t) \xleftrightarrow{\text{FS}} \frac{2T_1}{T}\text{sinc}(\frac{2kT_1}{T}) = \frac{2 * 0.5}{10}\text{sinc}(\frac{2k * 0.5}{10}) = \frac{1}{10}\text{sinc}(\frac{k}{10})$$

Likewise, for periodic $\delta(t)$ with period $T = 10$,

$$\delta(t) \xleftrightarrow{\text{FS}} \frac{1}{T} = \frac{1}{10}$$

Hence,

$$\delta(t-1) \xleftrightarrow{\text{FS}} \frac{1}{10}e^{-jk\omega_0}$$

Thus,

$$x(t) = 2\ \text{rect}(t) + \delta(t-1) \xleftrightarrow{\text{FS}} \frac{1}{5}\text{sinc}(\frac{k}{10}) + \frac{1}{10}e^{-jk\omega_0},$$

where $\omega_0 = \frac{\pi}{5}$.

3. (20 %) Let $x(t)$ be a periodic signal with period $T = 8$. Find $x(t)$ given that its Fourier series coefficients are given by the following. Ideally, you should think about how to build signals with these coefficients from the fundamental signals that we know from class with suitable manipulations.

(a)

$$a_k = \begin{cases} e^{j2\pi k/3}\frac{\sin(3k\pi)}{k\pi}, & k \neq 0 \\ 3, & k = 0 \end{cases}$$

(b)

$$a_k = \begin{cases} 0 & k = 0 \\ 1 & k > 0 \\ -1 & k < 0 \end{cases}$$

<u>Solution:</u>

(a) We can solve the problem in two different ways. The more direct one is to rewrite it using the sinc function.

Let $w(t)$ be a pulse train, with c_k as its Fourier coefficients: $w(t) \xleftrightarrow{\text{FS}} c_k$. We know from the lecture that $T_1 = 12$, and

$$c_k = \frac{\sin(3k\pi)}{k\pi}, k \neq 0$$
$$c_0 = 3$$

Consider $y(t) = w(t - t_0)$, with b_k as its Fourier coefficients. Then we have to make an observation that a delay in time is equivalent to a shift in phase (AKA a multiplication by a complex exponential) in the frequency domain. So we have: $w(t - t_0) \iff e^{-j\omega t_0}c_k$. Substitute with $e^{j2\pi k/3}$, i.e., $e^{-j\frac{\pi}{4}kt_0} = e^{j2\pi/3}$, we have $t_0 = -\frac{8}{3}$. Therefore,

$$b_k = e^{j2\pi k/3}\frac{\sin(3k\pi)}{k\pi}, k \neq 0$$
$$b_0 = 3$$

As $a_k = b_k$,

$$\begin{aligned} x(t) &= w\left(t + \frac{8}{3}\right) \\ &= \sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{t + 8/3 - 8m}{24}\right) \\ &= 3 \end{aligned}$$

So the result is only the DC part.

The less direct approach is to recognize that this special case of a DC. We observe that $\frac{\sin(3k\pi)}{k\pi} = 0,\ k \neq 0$, and $a_k = 3\delta(k\omega_0)$. Therefore, $x(t) = 3$ as $3 \xleftrightarrow{\text{FS}} 3\delta[k]$.

Note that, if you plot the rect pulse train you will see that it becomes just a constant.

(b) Writing out a few terms of the Fourier synthesis equation helps use tackle this one. Recognize that $w_0 = \pi/4$

$$\begin{aligned}
x(t) &= \sum_{m=-\infty}^{\infty} a_k e^{jk\pi t/4} \\
&= 0 + e^{j\pi t/4} - e^{-j\pi t/4} + e^{j(2)\pi t/4} - e^{-j(2)\pi t/4} + e^{j(3)\pi t/4} - e^{-j(3)\pi t/4} \ldots \\
&= 2j(\sin(\pi t/4) + \sin((2)\pi t/4) + \sin((3)\pi t/4)\ldots) \\
&= 2j \sum_{m=1}^{\infty} \sin(m\pi t/4)
\end{aligned}$$

4. (20 %) Consider a periodic signal $x(t)$ with period $T = 0.035$ and Fourier series coefficients

$$a_k = \begin{cases} 1 & k \text{ even} \\ j^k & k \text{ odd} \end{cases}.$$

Suppose signal $x(t)$ is input into an LTI system with frequency response

$$H(j\omega) = \begin{cases} 1 - \frac{|\omega|}{1000\pi} & |\omega| < 1000\pi \\ 0 & |\omega| \geqslant 1000\pi \end{cases}$$

to generate the output $y(t)$. Answer the following questions.

(a) Find the Fourier Series coefficients of $y(t)$.

(b) Find an expression for $y(t)$ in terms of the Fourier Series coefficients.

(c) We say that $H(j\omega)$ acts as a filter on $x(t)$. Based on how it affects $x(t)$, how would you describe the behavior of this filter?

Solution:

(a) Let $y(t) \overset{\text{FS}}{\longleftrightarrow} b_k$. Since $H(j\omega)$ is linear, we have

$$\begin{aligned}
b_k &= a_k H(jk\omega_0) \\
&= \begin{cases} 0 & |k\omega_0| \geqslant 1000\pi \\ a_k \left(1 - \frac{|k\omega_0|}{1000\pi}\right) & |k\omega_0| < 1000\pi \end{cases}
\end{aligned}$$

Substitute

$$\begin{aligned}
\omega_0 &= \frac{2\pi}{T} \\
&= \frac{2\pi}{0.035} \\
&= \frac{400}{7}\pi
\end{aligned}$$

We have

$$b_k = \begin{cases} 0 & |k| \geqslant 17.5 \\ a_k \left(1 - \frac{2|k|}{35}\right) & |k| < 17.5 \end{cases}$$

(b)

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \\
&= \sum_{k=-17}^{17} a_k \left(1 - \frac{2|k|}{35}\right) e^{jk\frac{400\pi}{7}t}
\end{aligned}
$$

(c) The given system drops all the frequencies beyond $\omega = 1000\pi$ as $b_k = 0$ for $|k\omega_0| \geqslant 0$, hence it's a low-pass filter.

5. (10 %) Consider the signal $x(t) = \cos(2\pi t)$. Since $x(t)$ is periodic with fundamental period of 1, it is also periodic with period N where N is any positive integer. What are the Fourier series coefficients of $x(t)$ if we regard it as a periodic signal with period of 4?

<u>Solution:</u> The given signal is

$$x(t) = \cos(2\pi t).$$

Since $x(t)$ has a fundamental period of $T_0 = 1$, it is also periodic with period $T = 4$. The Fourier series representation of a periodic signal with period T is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kt}.$$

where the Fourier coefficients are computed as

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt.$$

Using Euler's formula, we write

$$\cos(2\pi t) = \frac{e^{j2\pi t} + e^{-j2\pi t}}{2}.$$

Substituting this into the integral expression for c_k:

$$c_k = \frac{1}{4}\int_0^4 \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} e^{-j\frac{2\pi}{4}kt}dt.$$

Rearranging,

$$c_k = \frac{1}{8}\int_0^4 e^{j(2\pi - \frac{2\pi}{4}k)t}dt + \frac{1}{8}\int_0^4 e^{-j(2\pi + \frac{2\pi}{4}k)t}dt.$$

We use the standard integral result:

$$\int e^{j\alpha t}dt = \frac{e^{j\alpha t}}{j\alpha}.$$

Applying this, we compute each term separately:

For the first term,

$$\int_0^4 e^{j(2\pi - \frac{2\pi}{4}k)t}dt = \frac{e^{j(2\pi - \frac{2\pi}{4}k)4} - 1}{j(2\pi - \frac{2\pi}{4}k)}.$$

Since $e^{j2\pi} = 1$, we simplify:

$$= \frac{1 - 1}{j(2\pi - \frac{2\pi}{4}k)} = 0, \quad \text{if } k \neq \pm 4.$$

The same argument applies to the second term, unless the denominator becomes zero. The denominator is zero when

$$2\pi - \frac{2\pi}{4}k = 0 \quad \Rightarrow \quad k = 4,$$

or

$$-(2\pi + \frac{2\pi}{4}k) = 0 \quad \Rightarrow \quad k = -4.$$

For $k = 4$, the integral simplifies to

$$\int_0^4 e^{j(2\pi - \frac{2\pi}{4}\cdot 2)t} dt = \int_0^4 e^0 dt = 4.$$

Thus,

$$c_4 = \frac{1}{8} \times 4 = \frac{1}{2}.$$

Similarly, for $k = -4$,

$$c_{-4} = \frac{1}{2}.$$

For all other k, $c_k = 0$.

The Fourier series coefficients are

$$c_k = \begin{cases} \frac{1}{2}, & k = \pm 4, \\ 0, & \text{otherwise.} \end{cases}$$

6. (20 %) Based on the solutions, correct your previous week's homework using a colored pen (or annotation) so it's obvious what you've corrected. If you got a problem exactly right, just use a red check mark to indicate as such.