Lecture 13

Properties of the Fourier transform

Preview of today's lecture

- Fourier transform properties
  - Understand important connections between signal operations in the time and frequency domains
  - Use the table of common transform pairs and the table of common properties to compute transforms without doing integration

- Fourier symmetry properties
  - Connect signal properties like symmetric, even and odd in both time and frequency domains

Summarizing the Fourier transform and its inverse

Fourier transform (analysis)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform (synthesis)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\boxed{x(t) \leftrightarrow X(j\omega)}$$

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 16 Fourier transform

Fourier

Essential Fourier transforms

Key points

- Know these important FT pairs

(more pairs in the book)

Basic Fourier transform pairs 1/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Delta	$\delta(t)$	1
Constant	$\frac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\frac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega - \omega_0)$
Causal exponential	$e^{-at}u(t)$ $\mathrm{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$

(more pairs in the book)

Basic Fourier transform pairs 2/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Cosine	$\cos \omega_0 t$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
Sine	$\sin \omega_0 t$	$\pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

(more pairs in the book)

Basic Fourier transform pairs 3/3

	Time domain *x(t)*	Frequency domain *X(j ω)*
Rectangle	$\mathrm{rect}(t)$	$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
Scaled rectangle	$\mathrm{rect}\left(\frac{t}{2T_1}\right)$	$2\frac{\sin\left(\omega T_1\right)}{\omega} = 2T_1\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$
Sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
Scaled sinc	$\frac{B}{2\pi}\mathrm{sinc}\left(\frac{Bt}{2\pi}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$

Fourier transform properties

Key points

- Use FT properties to simplify calculation & build intuition
- Analyze problems that include FT properties

Fourier transform properties I $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

	Time domain	Fourier transform
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(j\omega)$

Fourier transform properties 2 $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

	Time domain	Fourier transform
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert}x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2 d\omega$	

Fourier transform properties 3

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$$

	Time domain	Fourier transform
Convolution in time	$y(t) = h(t) * x(t)$	$Y(j\omega) = H(j\omega)X(j\omega)$
Multiplication in time	$y(t) = h(t)x(t)$	$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$

These last two properties are important and feature in future lectures

Linearity

- If $$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega), \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$
- Then $$ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$$

Sums in time lead to sums in frequency

Linearity example

- Consider

$$\cos t \overset{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega - 1) + \delta(\omega + 1)]$$

$$\sin t \overset{\mathcal{F}}{\longleftrightarrow} \pi j[\delta(\omega + 1) - \delta(\omega - 1)]$$

- By linearity

$$\begin{aligned}\cos t + j \sin t &\overset{\mathcal{F}}{\longleftrightarrow} \pi\delta(\omega - 1) + \underbrace{\pi\delta(\omega + 1) - \pi\delta(\omega + 1)}_{0} + \pi\delta(\omega - 1) \\ &\overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - 1) \\ &\overset{\mathcal{F}}{\longleftrightarrow} \mathcal{F}\{e^{jt}\}\end{aligned}$$

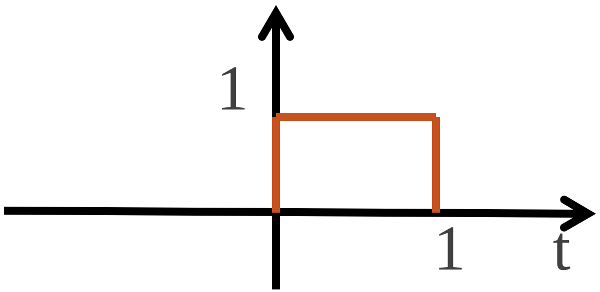
Time shifting

- If
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
- Then
$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$
- Time shifting does not change the magnitude in the freq. domain
$$|X(j\omega)e^{-j\omega t_0}| = |X(j\omega)||e^{-j\omega t_0}|$$
- Phase changes are linear with frequency ω and shift t_0
$$\angle(X(j\omega)e^{-j\omega t_0}) = \angle X(j\omega) - \omega t_0$$

Shift in time leads to linear phase shift in frequency

Time shifting example

- Find the Fourier transform of



- This signal is just a shifted rectangle function $\mathrm{rect}(t - 1/2)$

- Using the time shifting property

$$\mathrm{rect}(t - 1/2) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega/2}\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

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Differentiation

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$\frac{dx}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

- Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\underbrace{\frac{dx}{dt}}_{\text{new func}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}(e^{j\omega t})d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega X(j\omega)}_{\text{new FT}} e^{j\omega t}d\omega$$

Differentiation example

- What is the FT of the system characterized by

$$\frac{dy}{dt} + ay(t) = x(t)$$

- Solution:
  - Take FT of both sides

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$
$$(j\omega + a)Y(j\omega) = X(j\omega)$$

  - Therefore $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{1}{j\omega + a}$

Integration

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

DC component

Time scaling

- If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Time expansion |a|<1 leads to frequency compression

Time compression |a|>1 leads to frequency expansion

Frequency scaling

- If
$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

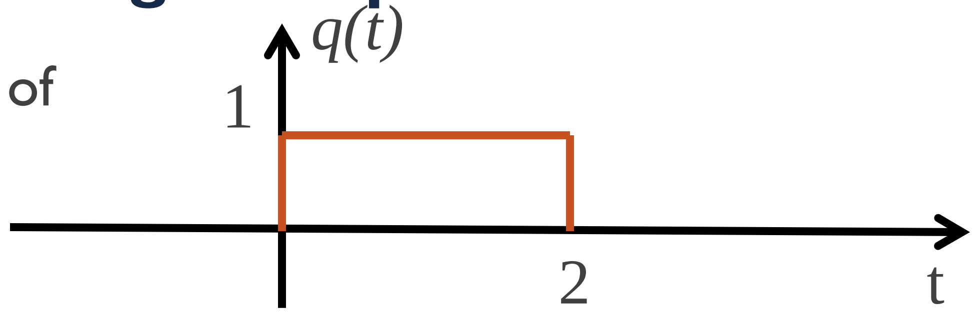
- Then
$$\frac{1}{|b|} x\left(\frac{t}{b}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} X(jb\omega)$$

Frequency expansion |b|<1 leads to time compression

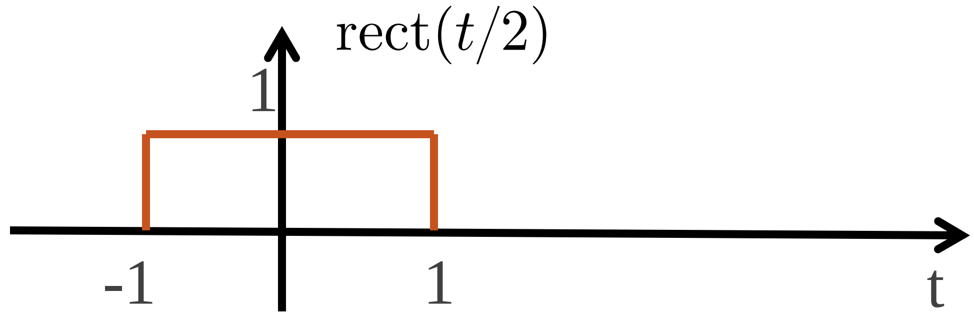
Frequency compression |b|>1 leads to time expansion

Shift and scaling example

- Find the FT of



- To start, notice that



- Shifting gives our function $q(t) = \text{rect}\left(\frac{t-1}{2}\right) = \text{rect}\left(\frac{t}{2} - \frac{1}{2}\right)$

Shift and scaling example (continued)

- From the scaling property

$$\mathrm{rect}\left(\frac{t}{2}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\mathrm{sinc}\left(\frac{2\omega}{2\pi}\right) = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

- From the shift property

$$\mathrm{rect}\left(\frac{t-1}{2}\right) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega} 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

Another scaling example

- What is the inverse Fourier transform of $\mathrm{sinc}(\omega)$?
- We know that
  - From the rect-sinc Fourier pair $$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$
  - From the scaling law $$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$
- Using the scaling property $$\mathrm{rect}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \mathrm{sinc}\left(\omega\right)$$
- Therefore using linearity $$\frac{1}{2\pi}\mathrm{rect}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\omega\right)$$

Inversion

- Find the FT of $x(-t)$

- This is just a special case of time and frequency scaling

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- With $a = -1$

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$$

Example using scaling and time shift

- Determine the Fourier transform of $\text{sinc}(1-2t)$

$$\begin{aligned} x(t) &= \text{sinc}(1-2t) \\ &= y(2t) \\ y(t) &= \text{sinc}(1-t) \\ &= \text{sinc}(-(t-1)) \\ &= z(t-1) \\ z(t) &= \text{sinc}(-t) \\ &= \text{sinc}(t) \end{aligned}$$

$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}(\omega/2\pi)$$

$$\text{sinc}(t/2\pi) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\text{rect}(-\omega) = 2\pi\text{rect}(\omega)$$

$$\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(\omega/(2\pi))$$

$$\begin{aligned} Z(j\omega) &= \text{rect}(\omega/(2\pi)) \\ Y(j\omega) &= e^{-j\omega} Z(j\omega) \\ X(j\omega) &= \frac{1}{2} Y\left(j\frac{\omega}{2}\right) \\ &= \frac{1}{2} e^{-j\omega/2} \text{rect}(\omega/(4\pi)) \end{aligned}$$

Frequency shifting

- If $$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
- Then $$x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

this is called modulation

- Corollary $$x(t)\cos\omega_0 t \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$$

Modulate in time leads to shift in frequency

Example combining shift and scaling

- Determine the inverse Fourier transform of

$$X(j\omega) = \frac{2\sin\left(3(\omega - 2\pi)\right)}{(\omega - 2\pi)}$$

Example (continued)

- Given
$$X(j\omega) = \frac{2\sin\left(3(\omega - 2\pi)\right)}{(\omega - 2\pi)}$$

- Use the following fact
$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$$

- For convenience let $R(j\omega) = \dfrac{\sin(\omega/2)}{\omega/2}$

- Rewrite as another shifted function
$$X(j\omega) = Y(j(\omega - 2\pi))$$
$$Y(j\omega) = \frac{2\sin(3\omega)}{\omega}$$

Example (continued)

- Rewrite again as

$$Y(j\omega) = 3\frac{2\sin(6\omega/2)}{6\omega/2}$$
$$= 6R(j6\omega)$$

- Using the scaling property

$$Y(j\omega) = 6R(j6\omega) \Rightarrow y(t) = \mathrm{rect}(t/6)$$

- Using the shift property

$$X(j\omega) = Y(j(\omega - 2\pi)) \Rightarrow x(t) = e^{j2\pi t}\,\mathrm{rect}(t/6)$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Energy of the signal in the time domain

Energy of the signal in the frequency domain

- This is a result of conservation of energy
- Scaling factor is because of radians

Example using Parseval's theorem

- If the signal $x(t)$ has the FT below $X(j\omega)$

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2 \qquad X(j\omega) = \begin{cases} \frac{j}{2\pi}, & -2 \leqslant \omega < 0 \\ -\frac{j}{2\pi}, & 0 \leqslant \omega \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$$

- Calculate

$$A = \int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

Example with Parseval's theorem (cont.)

$$\int_{-\infty}^{\infty} t^2 \left(\frac{\sin(t)}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi^3}$$

Duality in the Fourier transform

- If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

- Then

$$X(jt) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

If you know one Fourier pair then you know the other Fourier pair

Applications of duality

- Reproving frequency shift
  - Consider
    $$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$
  - Then
    $$x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))$$
- Impulse in time and frequency
  - Consider
    $$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$
  - Then
    $$1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega)$$

Duality example

- Consider

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

- Then

$$\mathrm{sinc}\left(\frac{t}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \cdot \mathrm{rect}(-\omega)$$

$$= 2\pi \cdot \mathrm{rect}(j\omega)x(t - t_0)$$

Since rect is an even function

Symmetry, even, and odd

Key points

- Connect signal properties like symmetric, even and odd in both time and frequency domains

Even and odd

- Can decompose a signal into even and odd components

$$x(t) = e(t) + o(t)$$

- Even part is

$$e(t) = \frac{1}{2}(x(t) + x^*(-t))$$

- Odd part is

$$o(t) = \frac{1}{2}(x(t) - x^*(-t))$$

- Can similarly decompose

$$X(j\omega) = E(j\omega) + O(j\omega)$$

Connecting the properties

$$x(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

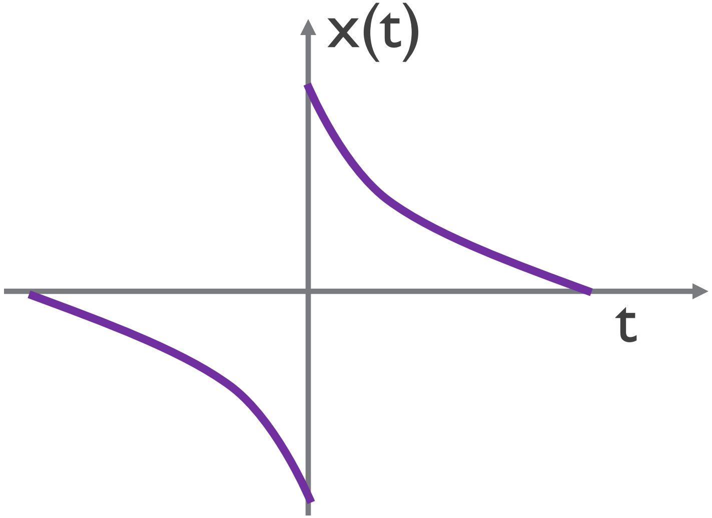
$$X(j\omega) = \mathrm{Re}\{E(j\omega)\} + j\mathrm{Im}\{E(j\omega)\} + \mathrm{Re}\{O(j\omega)\} + j\mathrm{Im}\{O(j\omega)\}$$

- Implications
  - Conjugate symmetry $x(t)$ is real $\rightarrow X(j\omega) = X^*(-j\omega)$
  - Real signals are even in amplitude since $|X(j\omega)| = |X(-j\omega)|$

Even / odd example

$$x(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

$$X(j\omega) = \cancel{\mathrm{Re}\{E(j\omega)\}} + \cancel{j\mathrm{Im}\{E(j\omega)\}} + \cancel{\mathrm{Re}\{O(j\omega)\}} + j\mathrm{Im}\{O(j\omega)\}$$



A real signal x(t) has the graph above

Answer the following questions:

$\mathrm{Re}\{X(j\omega)\} = 0$ Yes

$\mathrm{Im}\{X(j\omega)\} = 0$ No

Summarizing symmetry

$$x(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

$$X(j\omega) = \mathrm{Re}\{E(j\omega)\} + j\mathrm{Im}\{E(j\omega)\} + \mathrm{Re}\{O(j\omega)\} + j\mathrm{Im}\{O(j\omega)\}$$

- Key symmetry equation relates real, imaginary, even, and odd in the time and frequency domains
- Can determine signal characteristics in one domain by inspecting the other domain