UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2018
Linear Systems Fundamentals

SOLUTIONS TO MIDTERM EXAM

You are allowed one 2-sided sheet of notes.

No books, no other notes, no calculators.

PRINT YOUR NAME ____

Signature ____

Your signature confirms that you have completed this exam on your own and in accordance with the ECE 101 honor code.

Student ID Number ____

Problem	Weight	Score
1	30 pts	
2	34 pts	
3	36 pts	
Total	100 pts	

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Table 3.1 from the textbook is attached to the exam.

Good luck!

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Problem 1 (30 pts)

(a) (10 pts)

Determine three distinct complex numbers that satisfy the equation

$$z^3 = j.$$

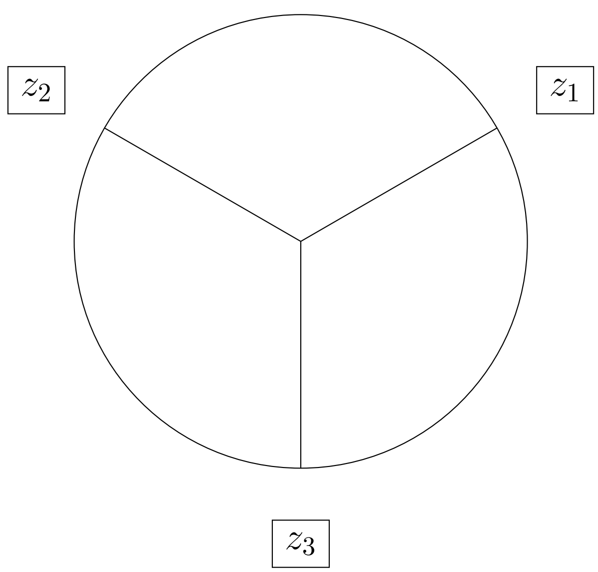
Write your answers in polar form and rectangular form.

The three distinct solutions of $z^3 = 1$ are $1, e^{j\frac{2\pi}{3}}, e^{j\frac{4\pi}{3}}$.
The solutions to $z^3 = j = e^{j\frac{\pi}{2}}$ are:

$$z_1 = e^{j\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}j.$$

$$z_2 = z_1 e^{j\frac{2\pi}{3}} = e^{j\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}j$$

$$z_3 = z_1 e^{j\frac{4\pi}{3}} = e^{j\frac{3\pi}{2}} = -j.$$



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Problem 1 (cont.)

(b) (10 pts)

Determine the fundamental period of the discrete-time signal

$$x[n] = e^{j\frac{3\pi}{2}n} + \cos\left(\frac{\pi}{3}n\right).$$

Let $x_1[n] = e^{j\frac{3\pi}{2}n} = e^{j2\pi\left(\frac{3}{4}\right)n}$. The fundamental period is $N_1 = 4$.

Let $x_2[n] = \cos\left(\frac{\pi}{3}n\right) = \cos\left(\frac{2\pi}{6}n\right)$. The fundamental period is $N_2 = 6$.

The fundamental period N of $x[n] = x_1[n] + x_2[n]$ is the least common multiple of N_1 and N_2.

That is, $N = \text{lcm}\{N_1, N_2\} = \text{lcm}\{4, 6\} = 12$.

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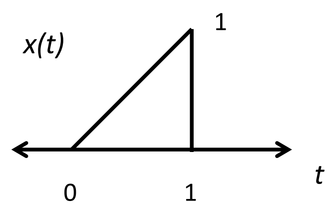
Problem 1 (cont.)

(c) (10 pts)

Let $x(t)$ be the continuous-time signal defined by

$$x(t) = \begin{cases} 0 \ , & t < 0 \\ t \ , & 0 \leq t < 1 \\ 0 \ , & t \geq 1. \end{cases}$$

Precisely sketch the signal $y(t) = x(-2t + 3)$.

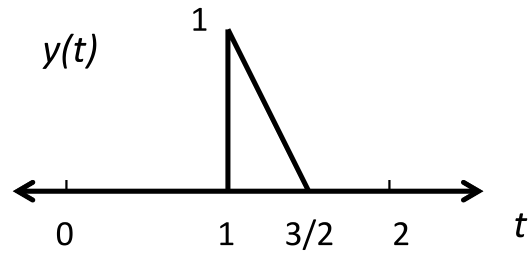


The signal $y(t) = x(-2t+3) = x(-2t-(-3))$ can obtained from $x(t)$ by either of the following approaches:

- shift by -3 then scale by -2
- scale by -2 then shift by $\frac{-3}{-2} = \frac{3}{2}$.

Either way, the resulting signal is given by:

$$y(t) = \begin{cases} 0, & t \leq 1 \\ -2t+3, & 1 < t \leq \frac{3}{2} \\ 0, & t > \frac{3}{2}. \end{cases}$$



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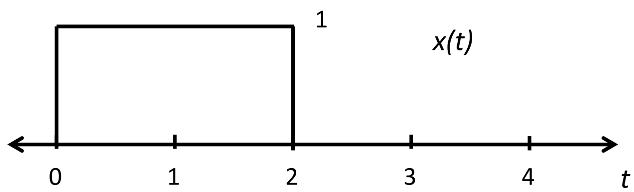
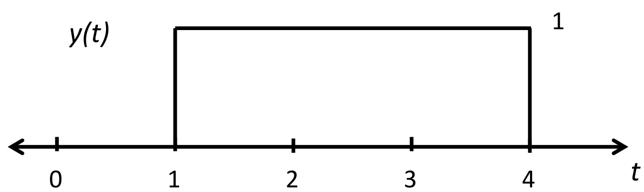
Problem 2 (34 pts)

(a) (12 pts)

Let $x(t) = u(t) - u(t-2)$ and $y(t) = u(t-1) - u(t-4)$.

Determine and precisely sketch the convolution $z(t) = x(t) * y(t)$.



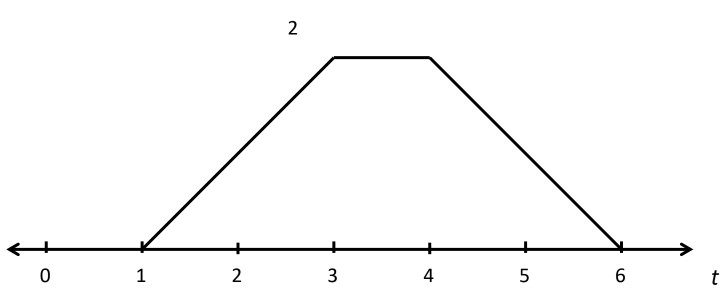


We choose to write the convolution as follows (using commutativity):

$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} y(\tau)x(t-\tau)d\tau$

The result of the convolution is:

$t < 1$	$z(t) = 0$
$1 \le t < 3$	$z(t) = \int_1^t dt = t - 1$
$3 \le t < 4$	$z(t) = \int_{t-2}^t dt = 2$
$4 \le t < 6$	$z(t) = \int_{t-2}^4 dt = 6 - t$
$t \ge 6$	$z(t) = 0$



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Problem 2 (cont.)

(b) (12 pts; 4pts each part)

Consider the discrete-time LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n-1] + \left(-\frac{1}{4}\right)^{n+1} u[n]$$

For each property below, indicate whether or not the system satisfies the specified property.

Justify your answers citing properties of this particular $h[n]$.

True	False	
☐	☒	Memoryless

Memoryless $\Leftrightarrow h[n] = a\delta[n]$ or $h[n] = 0, n \neq 0$.
But $h[1] = \frac{1}{2} + \frac{1}{16} \neq 0$.

True	False	
☒	☐	Causal

Causal $\Leftrightarrow h[n] = 0, \quad n < 0$.
Now $u[n-1] = 0, \quad n < 1$ and $u[n] = 0, \quad n < 0$.
So, $h[n] = 0, \quad n < 0$.

True	False	
☒	☐	Stable

Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
By the triangle inequality,

$\sum_{n=-\infty}^{\infty} |h[n]| \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n+1}$

$\leq \left(\frac{1}{1-\frac{1}{2}}\right) + \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right)$

$= 2 + \frac{1}{3} < \infty$.

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Problem 2 (cont.)

(c) (10 points)

Consider the system S defined by:

$$y[n] = \begin{cases} x[n/2], & n \text{ even} \\ n, & n \text{ odd.} \end{cases}$$

Determine if the system is invertible. (Justify your answer.)

If it is invertible, write an expression for the inverse system.

The output sequence looks like:

n	$\ldots$	-3	-2	-1	0	1	2	3	$\ldots$
$y[n]$	$\ldots$	-3	$x[-1]$	-1	$x[0]$	1	$x[1]$	3	$\ldots$

Clearly, the value $x[k]$ sits in position $n = 2k$ in $y[n]$.

Thus, the system is invertible, and the inverse system can be expressed as:

$$z[n] = y[2n].$$

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Problem 3 (36 points)

(a) (12 points)

Consider three discrete-time systems S_1, S_2, and S_3. For each one, an input-output pair is shown below.

$$\begin{aligned} S_1 &: \; j^n \rightarrow j^{n+1} \\ S_2 &: \; e^{j5n} \rightarrow \cos(5n) \\ S_3 &: \; e^{j\pi n/3} \rightarrow 2e^{j7\pi n/3} \end{aligned}$$

For each system, determine whether the given information is sufficient to conclude that the system is definitely **not** LTI.

The eigenfunction properrty of LTI systems requires that:

$$z^n \rightarrow H(z)z^n$$

where $H(z)$ is the system function.

S_1: $j^n \rightarrow j^{n+1} = j(j^n)$.

This system could be LTI, with $H(j) = j$.

S_2: $e^{j5n} \rightarrow \cos(5n) = \frac{1}{2}e^{j5n} + \frac{1}{2}e^{-j5n}$.

Note that $e^{-j5n} \neq ae^{j5n}$ for any complex a.

So, this system is **definitely not** LTI.

S_3: $e^{j\pi n/3} \rightarrow 2e^{j7\pi n/3} = 2e^{j\pi n/3}e^{j6\pi n/3} = 2e^{j\pi n/3}$.

So, this system could be LTI, with $H(e^{j\pi/3}) = 2$.

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Problem 3 (cont.)

(b) (12 points)

Let $x(t)$ be a **real-valued** continuous-time signal with fundamental period $T = 4$. Denote the Fourier series coefficients of $x(t)$ by a_k. Suppose $x(t)$ satisfies the following properties:

(i) $\int_1^5 x(t)dt = 4$

(ii) $a_1 = 2$

(iii) $a_2 = e^{j\frac{\pi}{3}}$

(iv) $a_k = 0,\ k > 2.$

Express $x(t)$ in the form $x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$

$T = 4 \Rightarrow \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

Recall: $a_0 = \frac{1}{T}\int_T x(t)dt.$

So, $a_0 = \frac{1}{4}\int_1^5 x(t)dt = 1.$

$x(t)$ real-valued $\Rightarrow a_k = a_{-k}^*$ (conjugate symmetry)

$a_1 = a_{-1}^* = 2 \Rightarrow a_{-1} = 2$

$a_2 = a_{-2}^* = e^{j\frac{\pi}{3}} \Rightarrow a_{-2} = e^{-j\frac{\pi}{3}}$

$a_k = a_{-k}^* = 0, k > 2 \Rightarrow a_k = 0, |k| > 2$

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\
&= a_0 + a_1 e^{j\frac{\pi}{2}t} + a_{-1}e^{-j\frac{\pi}{2}t} + a_2 e^{j\pi t} + a_{-2}e^{-j\pi t} \\
&= 1 + 4\left(\frac{e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}}{2}\right) + 2\left(\frac{e^{j(\pi t + \frac{\pi}{3})} + e^{-j(\pi t + \frac{\pi}{3})}}{2}\right) \\
&= 1 + 4\cos\left(\frac{\pi}{2}t\right) + 2\cos\left(\pi t + \frac{\pi}{3}\right)
\end{aligned}
$$

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Problem 3 (cont.)

(c) (12 points)

The signal $x(t) = 1 + \cos(3\pi t)\sin(\pi t)$ is passed through the continuous-time filter whose frequency response is given by:

$$H(j\omega) = \begin{cases} 1 - \frac{|\omega|}{3\pi}, & |\omega| \leq 3\pi \\ 0, & \text{otherwise.} \end{cases}$$

Determine the output signal $y(t)$.

$$\begin{aligned} x(t) &= 1 + \cos(3\pi t)\sin(\pi t) \\ &= 1 + \left(\frac{1}{2}e^{j3\pi t} + \frac{1}{2}e^{-j3\pi t}\right)\left(\frac{1}{2j}e^{j\pi t} - \frac{1}{2j}e^{-j\pi t}\right) \\ &= 1 + \frac{1}{4j}\left(e^{j4\pi t} + e^{-j2\pi t} - e^{j2\pi t} - e^{-j4\pi t}\right) \end{aligned}$$

The filter acts on complex exponentials according to the property:

$$e^{j\omega t} \to H(j\omega)e^{j\omega t}.$$

So, regrouping terms,

$$\begin{aligned} y(t) = H(j0)1 &+ \frac{1}{4j}\left(H(-j2\pi)e^{-j2\pi t} - H(j2\pi)e^{j2\pi t}\right) \\ &+ \frac{1}{4j}\left(H(j4\pi)e^{j4\pi t} - H(-j4\pi)e^{-j4\pi t}\right) \end{aligned}$$

Note that

$H(j0) = 1$, $H(j2\pi) = H(-j2\pi) = \frac{1}{3}$, and $H(j4\pi) = H(-j4\pi) = 0$,

so we get

$$\begin{aligned} y(t) &= 1 + \frac{1}{12j}\left(e^{-j2\pi t} - e^{j2\pi t}\right) \\ &= 1 - \frac{1}{6}\sin(2\pi t). \end{aligned}$$

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