ECE 101: Linear Systems Fundamentals

Summer Session II 2020 - Lecture 5

Saharnaz Baghdadchi

University of California, San Diego

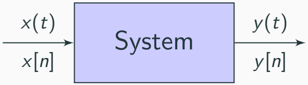
Today's topics

- Systems
- System properties
  1. Memoryless
  2. stable
  3. Causal
  4. invertible
  5. Time-invarying
  6. Linear (additive, homogeneous)

Signals and Systems (2th Edition): sections 1.5 and 1.6

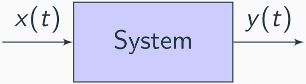
Systems

Definition of a System

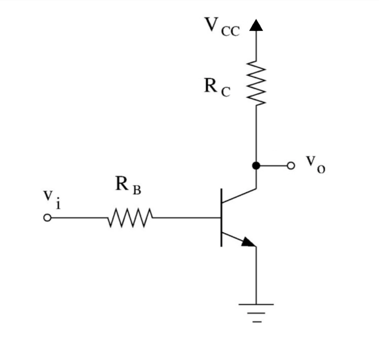


- System: any mapping of input signals to output signals

Continuous Time Systems: Examples



Electrical Circuits:



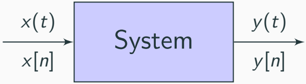
Discrete-time Systems: Examples

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

Sound Processing:

- Input $x[n]$ = input sound
- Output $y[n]$ = output sound

Important Mathematical Systems



- Delay: $x(t) \rightarrow y(t) = x(t - t_0)$ or $x[n] \rightarrow y[n] = x[n - n_0]$

- Scaling: $x(t) \rightarrow y(t) = ax(t)$ or $x[n] \rightarrow y[n] = ax[n]$

- Differentiator: $x(t) \rightarrow y(t) = \frac{d}{dt}x(t)$ or

  $x[n] \rightarrow y[n] = x[n] - x[n - 1]$

Important Mathematical Systems

- Integrator: $x(t) \to y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ or $x[n] \to y[n] = \sum_{k=-\infty}^{n} x[k]$

- What is the output of the DT integrator system to the input $x[n] = \delta[n]$?

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

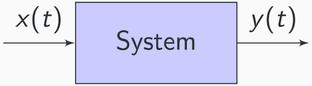
Note:

$$\sum_{k=-\infty}^{n} \delta[k] = \sum_{m=-\infty}^{n} \delta[m]$$, if we use $m = n - k$ then $\begin{cases} m = -\infty \to k = +\infty \\ m = n \to k = 0 \end{cases}$

$$\sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{\infty} \delta[n-k] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \dots = u[n]$$

System Properties

Properties of Systems: Memoryless

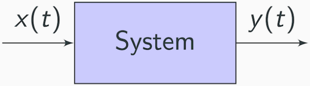


- A system is memoryless if output at time t (or n) only depends on the value of the input at time t (or n)

- Example 1: Is $y(t) = 2x(t) - x^2(t)$ memoryless?

Yes, because the output at time t only depends on the value of input signal at time t.

Properties of Systems: Memoryless

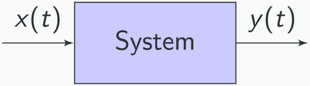


- A system is memoryless if output at time t (or n) only depends on the value of the input at time t (or n)

- Example 2: Is $y[n] = x[n-1]$ memoryless?

$y[n=0] = x[-1]$, $y[0] = x[-1]$

Properties of Systems: Memoryless



- A system is memoryless if output at time t (or n) only depends on the value of the input at time t (or n)

- Example 3: Is $y[n] = x[n] + y[n-1]$ memoryless? No

$$y[n-1] = x[n-1] + y[n-2]$$

$$y[n-2] = x[n-2] + y[n-3]$$

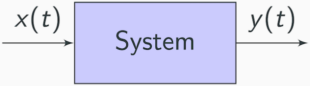
$$y[n-3] = x[n-3] + y[n-2]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y[n] = x[n] + x[n-2] + x[n-3] + \cdots\cdots = \sum_{k=-\infty}^{n} x[k]$$

The system has memory.

Properties of Systems: Memoryless



$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + \cdots$

- A system is memoryless if output at time t (or n) only depends on the value of the input at time t (or n)

- Example 3: Is $y[n] = x[n] + y[n-1]$ memoryless? No

1 $\quad y[n] = \sum_{k=-\infty}^{n} x[k] \quad$ or $\quad y[n] = \sum_{k=0}^{\infty} x[n-k]$

1 : $\quad y[n] = \sum_{m=-\infty}^{n} x[m]$, $\quad m = n-k$

$\begin{cases} m = -\infty \rightarrow k = +\infty \\ m = n \rightarrow k = 0 \end{cases}$

$y[n] = \sum_{k=0}^{\infty} x[n-k]$

Bounded signals

- Definition: A signal $x(t)$ (and $x[n]$) is **bounded** if there exists a constant $B < \infty$ such that $|x(t)| < B$ for all t.
- Example: Is $x(t) = tu(t)$ bounded? What about $x[n] = e^{jn}$?

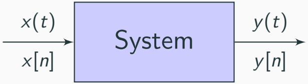
$x(t) = t\,u(t)$

$|u(t)| \leq 1$ $\qquad$ $|t\,u(t)| \rightarrow \infty$ as $t \rightarrow \infty$

so, $x(t) = t\,u(t)$ is not bounded.

$x[n] = e^{jn}$ $\qquad$ $|x[n]| = |e^{jn}| = 1 < \infty \Rightarrow$ bounded

Properties of Systems: BIBO Stable



- BIBO Stability: A system is Bounded-Input Bounded-Output (BIBO) stable if all the bounded inputs result in bounded outputs.

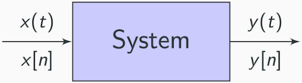
- That is, if

  $$|x(t)| \leq B, \text{ for some } B < \infty$$

  then,

  $$|y(t)| \leq C, \text{ for some } C < \infty$$

Properties of Systems: BIBO Stable



- BIBO Stability: A system is Bounded-Input Bounded-Output (BIBO) stable if all the bounded inputs result in bounded outputs.
- Example 1: Is $y(t) = 2x^2(t-1) + x(2t)$ BIBO stable?

if $|x(t)| \leq B$ for some $B < \infty$

$$|y(t)| = |2x^2(t-1) + x(2t)|$$

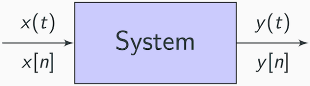
$$\leq |2x^2(t-1)| + |x(2t)|$$

$$\leq 2|x^2(t-1)| + |x(2t)|$$

$$\leq 2B^2 + B < \infty$$

$\longrightarrow$ This system is BIBO stable.

Properties of Systems: BIBO Stable



- BIBO Stability: We say that a system is Bounded-Input Bounded-Output (BIBO) stable if all the bounded inputs result in bounded outputs.

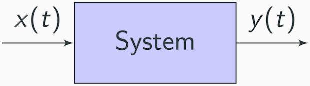
- Example 2: Is the integrator system $y[n] = \sum_{k=-\infty}^{n} x[k]$ BIBO stable?

$x[n] = u[n]$, $|x[n]| \leqslant 1$

$$|y[n]| = \left| \sum_{k=-\infty}^{n} x[k] \right| = \sum_{k=-\infty}^{n} u[k] = \sum_{k=0}^{n} u[k] = \sum_{k=0}^{n} 1 = n+1$$

$|y[n]| = n+1$, as $n \longrightarrow \infty$, $|y[n]| \rightarrow \infty \Rightarrow$ system is not stable

Properties of Systems: Causality



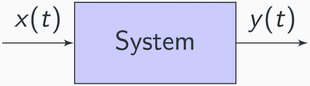
- A system is *causal* if output at time t (or n) only depends on the input at time $s \leq t$ (or $k \leq n$)

- Example 1: Is the delay system $y(t) = x(t - t_0)$ causal?

if $t_0 \geq 0$, the system is causal. The output depends on the past values of the input signal.

for example: $y(t) = x(t-1)$

Properties of Systems: Causality

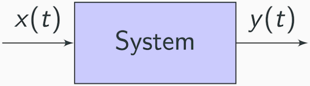


- A system is *causal* if output at time t (or n) only depends on the input at time $s \leq t$ (or $k \leq n$)
- Example 2: Is the integrator system $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ causal? Yes.

The system is causal because the integral evaluates $x(\tau)$ from $-\infty$ to t.

The output depends on the past and current values of $x(t)$.

Properties of Systems: Causality



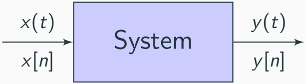
- A system is *causal* if output at time t (or n) only depends on the input at time $s \leq t$ (or $k \leq n$)

- Example 3: Is $y[n] = x[-n]$ causal?

$y[n=-1] = y[-1] = x[1]$

The system is not causal, because the output at $n=-1$ depends on the input at $n=1$ (a future sample of $x[n]$)

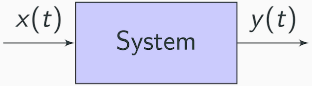
Properties of Systems: Invertible



- A system is invertible if distinct inputs produce distinct outputs

- In other words, a system is invertible if there exists a one-to-one mapping from the set of input signals to the set of output signals.
    - To show that a system is invertible: find the inversion formula
    - To show that a system is not invertible: show that two signals map to the same output

Properties of Systems: Invertible



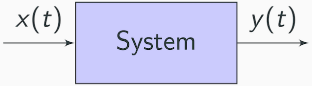
- A system is invertible if distinct inputs produce distinct outputs
- Example 1: Is $y(t) = x(2t)$ invertible? Yes, the inversion formula is $x(t) = y(t/2)$

$$t' = 2t \longrightarrow t = \frac{1}{2}t'$$

$$x(2t) = x(t')$$

$$y(t) = y\left(\frac{t'}{2}\right) \longrightarrow x(t') = y\left(\frac{t'}{2}\right) \longrightarrow \boxed{x(t) = y(t/2)}$$

Properties of Systems: Invertible



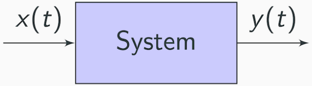
- A system is invertible if distinct inputs produce distinct outputs

- Example 2: Is the system $y[n] = x[n] + y[n-1]$ invertible?

$$x[n] = y[n] - y[n-1]$$

The system is invertible

Properties of Systems: Invertible



- A system is invertible if distinct inputs produce distinct outputs

- Example 3: Is $y(t) = x^2(t)$ invertible?

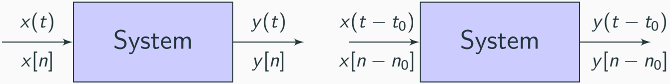
$x_1(t) = 1$ for all t

$x_2(t) = -1$ for all t

$y_1(t) = y_2(t) = 1$ for all $t \Rightarrow$

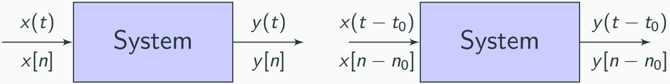
the system is not invertible

Properties of Systems: Time-Invariant (TI)



- A system is TI if **for all inputs** $x(t)$ (or $x[n]$) and **all** time-shifts t_0 (or n_0) the output to $x(t - t_0)$ (or $x[n - n_0]$) is $y(t - t_0)$ (or $y[n - n_0]$)

Properties of Systems: Time-Invariant (TI)



- To check for TI property in a system:

  1. define $x_1[n] = x[n - n_0]$
  2. find $y_1[n]$ (the output of the system when input is $x_1[n]$)
  3. find $y[n - n_0]$
  4. compare $y_1[n]$ with $y[n - n_0]$. If they are equal, the system is time-invariant.

Properties of Systems: Time-Invariant (TI)

- Example 1: Is $y(t) = sin(x(t))$ Time-Invariant?

① $x_1(t) = x(t - t_0)$

② $y_1(t) = \sin(x_1(t)) = \sin(x(t - t_0))$

③ $y(t - t_0) = \sin(x(t - t_0))$

④ $y_1(t) = y(t - t_0)$

The system is time-invariant.

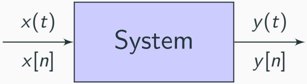
Properties of Systems: Time-Invariant (TI)

- Example 2. Is the system $y[n] = nx[n]$ Time-Invariant?

$$\begin{aligned} x_1[n] &= x[n - n_0] \\ y_1[n] &= nx_1[n] = nx[n - n_0] \\ y[n - n_0] &= (n - n_0)x[n - n_0] \\ y_1[n] &\neq y[n - n_0] \end{aligned}$$

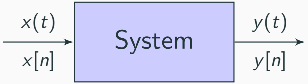
The system is not time-invariant.

Properties of Systems: Linear



- A system is linear if it is **additive** and **scalable**.

Properties of Systems: Linear



- A system is **additive** if whenever

$$x_1(t) \longrightarrow y_1(t) \text{ and } x_2(t) \longrightarrow y_2(t),$$

then

$$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t).$$

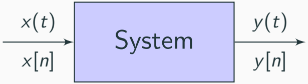
- A system is **scalable** if whenever

$$x(t) \longrightarrow y(t),$$

then

$$ax(t) \longrightarrow ay(t), \quad \text{for all } a \in \mathbb{C}.$$

Properties of Systems: Linear

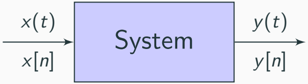


- **Note:** Suppose the system is scalable. Then

$$x(t) = 0, \forall t \longrightarrow y(t) = 0, \forall t.$$

- In linear systems, if input signal $x(t) = 0$ ($x[n] = 0$) is applied to the system, the output will be $y(t) = 0$ ($y[n] = 0$).

Properties of Systems: Linear



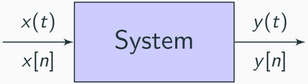
- Homogeneity test for linear systems:
- Linear systems are scalable that means

$$x(t) \to y(t)$$
$$ax(t) \to ay(t) \quad \text{for } a \in \mathbb{R}.$$

- If $x(t) = 0$ for every $a \in \mathbb{R}$, $ax(t) = x(t) = 0$. This implies

$$ay(t) = y(t) \quad \text{for } a \in \mathbb{R}.$$
$$y(t) = 0 \quad \text{for every t}$$

Properties of Systems: Linear



- A system is linear if it is **additive** and **scalable**.
- An equivalent condition is that **linear superposition holds**, i.e., whenever

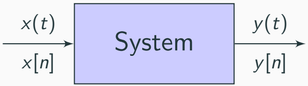
$$x_1(t) \longrightarrow y_1(t) \text{ and } x_2(t) \longrightarrow y_2(t),$$

then

$$ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t).$$

for all $a, b \in \mathbb{C}$.

Properties of Systems: Linear



- To check for linearity of a system:

  1. define $x_3[n] = ax_1[n] + bx_2[n]$
  2. find $y_3[n]$ (the output of the system when input is $x_3[n]$)
  3. find $y_4[n] = ay_1[n] + by_2[n]$
  4. compare $y_3[n]$ with $y_4[n]$. If they are equal, the system is linear

Properties of Systems: Linear

- Example 1: Is the integrator system $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ linear?

1 $x_3(t) = a_1 x_1(t) + bx_2(t)$, a and b $\in C$

2 $y_3(t) = \int_{-\infty}^{t} x_3(\tau)d\tau = \int_{-\infty}^{t} \left(a x_1(\tau) + bx_2(\tau)\right)d\tau$

$= \int_{-\infty}^{t} a x_1(\tau)d\tau + \int_{-\infty}^{t} b x_2(\tau)d\tau$

$= a y_1(t) + by_2(t)$

3 $y_4(t) = a y_1(t) + by_2(t)$

4 $y_3(t) = y_4(t) \Longrightarrow$ the system is linear

Properties of Systems: Linear

- Example 2: Is the following system linear?

$$y[n] = (x[2n])^2$$

(1) $x_3[n] = a x_1[n] + b x_2[n]$

(2) $y_3[n] = \left(x_3[2n]\right)^2 = \left(a x_1[2n] + b x_2[2n]\right)^2$

$= a^2 x_1^2[2n] + b^2 x_2^2[2n] + 2ab x_1[2n] x_2[2n]$

$y_1[n] = \left(x_1[2n]\right)^2$

$y_2[n] = \left(x_2[2n]\right)^2$

(3) $y_4[n] = a y_1[n] + b y_2[n] = a x_1^2[2n] + b x_2^2[2n]$

(4) $y_3[n] \neq y_4[n] \Longrightarrow$ the system is not linear