ECE 101 – Linear Systems Fundamentals

Problem Set # 3A Solutions

(send questions/comments to psiegel@ucsd.edu)

1. SYNTHESIS

3.21 The Fourier coefficients are given, so we can plug them into the synthesis equation (3.38) to obtain the signal x(t):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} = a_1 e^{j\omega_o t} + a_{-1} e^{-j\omega_o t} + a_5 e^{j5\omega_o t} + a_{-5} e^{-j5\omega_o t}$$

$$= je^{j(\pi/4)t} + -je^{-j(\pi/4)t} + 2e^{j5(\pi/4)t} + 2e^{-j5(\pi/4)t}, \text{ noting that } \omega_o = 2\pi/T = 2\pi/8 = \pi/4.$$

Now use Euler's relation $\sin(\omega_o t) = (1/2j)[e^{j\omega_o t} - e^{j\omega_o t}]$ on the first 2 terms, and $\cos(\omega_o t) = (1/2)[e^{j\omega_o t} + e^{j\omega_o t}]$ on the second 2 terms, to obtain:

$$\Rightarrow x(t) = -2\sin(\pi t/4) + 4\cos(5\pi t/4)$$

To get x(t) into the desired form of purely cosine functions, note that the sin function is just the cos function shifted $\pi/2$ to the right. Thus, we have:

$$\boxed{x(t) = -2\cos(\pi t/4 - \pi/2) + 4\cos(5\pi t/4)}$$

3.23(c) Here the only non-zero Fourier coefficients are $a_{\pm1}$ and $a_{\pm2}$. Note that the non-zero coefficients are also purely imaginary numbers, so we should expect x(t) to be an odd function (i.e., a sum of sines, no cosines). To find x(t), we just plug the a_k's into the synthesis equation, as in the previous problem:

$$x(t) = a_1 e^{j\omega_o t} + a_{-1} e^{-j\omega_o t} + a_2 e^{j2\omega_o t} + a_{-2} e^{-j2\omega_o t} = je^{j(\pi/2)t} + -je^{-j(\pi/2)t} + 2je^{j2(\pi/2)t} + -2je^{-j2(\pi/2)t}$$

noting that $\omega_o = 2\pi/T = 2\pi/4 = \pi/2$. Again, using Euler's relation, we can write the complex exponentials as a sum of sines:

$$\Rightarrow \boxed{x(t) = -2\sin(\pi t/2) - 4\sin(\pi t)}$$

2. ANALYSIS

3.22(a), fig (b) Examining the graph, T = 6 $\Rightarrow \omega_o = \pi/3$. Note that the average value, over one period, of this function is 0.5. Thus, $a_0 = 0.5$. To compute the other a_k's, we need to use the analysis equation in (3.39) and do out the integral:

$$a_k = \frac{1}{6}\int_{-3}^{3} x(t) e^{-jk\omega_o t} dt = \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-jk(\pi/3)t}dt + \int_{-1}^{1} 1 e^{-jk(\pi/3)t}dt + \int_{1}^{2}(-t+2)e^{-jk(\pi/3)t}dt\right]$$

The first integral is solved by a substitution and by-parts integration:

$$\int_{-2}^{-1}(t+2)e^{-jk(\pi/3)t}dt \to u=t+2 \Rightarrow du=dt, t=u-2 \Rightarrow \int_{0}^{1}ue^{-jk(\pi/3)(u-2)}du$$

For convenience, let A = –jkπ/3. Then the integral becomes:

$$\int_{0}^{1}ue^{A(u-2)}du = e^{-2A}\left(\frac{u}{A}-\frac{1}{A^2}\right)e^{Au}\Big|_0^1 = e^{-2A}\left[\left(\frac{1}{A}-\frac{1}{A^2}\right)e^{A}+\frac{1}{A^2}\right] = \frac{1}{A}e^{-A}-\frac{1}{A^2}e^{-A}+\frac{1}{A^2}e^{-2A}$$

The second integral is simply integrating e^{At} over [-1,1] $\Rightarrow \frac{1}{A}e^{A}-\frac{1}{A}e^{-A}$

The third integral can be solved in the same way as the first, using substitution then integration by parts on the resulting ue^{-Au} form. The result is:

$$\int_{1}^{2}(-t+2)e^{At}dt = e^{2A}\left[\left(\frac{-1}{A}-\frac{1}{A^2}\right)e^{-A}+\frac{1}{A^2}\right] = \frac{-1}{A}e^{A}-\frac{1}{A^2}e^{A}+\frac{1}{A^2}e^{2A}$$

Putting the three integrals together and dividing by 1/T = 1/6, we have:

$$\frac{1}{6}\left[\frac{1}{A}e^{-A}-\frac{1}{A^2}e^{-A}+\frac{1}{A^2}e^{-2A}+\frac{1}{A}e^{A}-\frac{1}{A}e^{-A}+\frac{-1}{A}e^{A}-\frac{1}{A^2}e^{A}+\frac{1}{A^2}e^{2A}\right]$$
$$=\frac{1}{6}\left[\frac{1}{A^2}(e^{2A}+e^{-2A})-\frac{1}{A^2}(e^{A}+e^{-A})\right]$$

Bringing in A = –jkπ/3 now, we can apply Euler's formula to obtain:

$$=\frac{1}{6}\left[\frac{1}{(-jk\pi/3)^2}(e^{2(-jk\pi/3)}+e^{-2(-jk\pi/3)})-\frac{1}{(-jk\pi/3)^2}(e^{(-jk\pi/3)}+e^{-(-jk\pi/3)})\right]$$
$$=\frac{1}{6}\left[\frac{9}{-k^2\pi^2}\left(e^{-2jk\pi/3}+e^{2jk\pi/3}-(e^{-jk\pi/3}+e^{jk\pi/3})\right)\right]=\frac{3}{k^2\pi^2}\left(\cos(\pi k/3)-\cos(2\pi k/3)\right)$$

So, the Fourier coefficients are given by:

$$\boxed{a_0 = 0.5,\ a_k = \frac{3}{k^2\pi^2}\left(\cos(\pi k/3)-\cos(2\pi k/3)\right),\ k \neq 0}$$

3.22(a), fig (d) By the graph, it is clear that the period T = 2 $\Rightarrow \omega_o = \pi$. To find the a_k's, we plug our x(t) into the analysis equation:

$\frac{1}{2}\int_{-0.5}^{1.5}[\delta(t)-2\delta(t-1)]e^{-jk\pi t}dt$, where we chose the period interval of -0.5 to 1.5 to avoid having the delta functions fall right at the end points of the integral. For k = 0, the integral simply evaluates to:

$$\frac{1}{2}\int_{-0.5}^{1.5}[\delta(t)-2\delta(t-1)]dt=\frac{1}{2}(1-2)=-0.5 \Rightarrow a_0=-0.5 .$$

For $k \neq 0$, we use the sifting property of delta functions to obtain:

$$\frac{1}{2}\int_{-0.5}^{1.5}[\delta(t)-2\delta(t-1)]e^{-jk\pi}dt=\frac{1}{2}\int_{-0.5}^{1.5}[e^{-j0k\pi}\delta(t)-2e^{-j1k\pi}\delta(t-1)]dt=\frac{1}{2}(1-2e^{-jk\pi})=0.5-(-1)^k$$

Thus, the Fourier coefficients are: $\boxed{a_0 = -0.5;\ \ a_k = 0.5 - (-1)^k,\ k \neq 0}$

3. DEDUCTION

3.26 x(t) has Fourier series coefficients $a_0 = 2$; $a_k = j(0.5)^{|k|}$, $k \neq 0$. We'll use the table of Fourier series properties (pg. 206) to find the answer to each question.

(a) **Is x(t) real?** Looking at the table, we see that x(t) is real if the Fourier coefficients are conjugate-symmetric, i.e., $a_k = a_{-k}^*$. Examine k = 1. $a_1 = 0.5j = a_{-1}$. If the coefficients were conjugate-symmetric, a_1 would equal a_{-1}^*, i.e., $a_1 = a_{-1}^* = (0.5j)^* = -0.5j$. But $a_1 \neq -0.5j$, so conjugate symmetry doesn't hold for the a_k's. Thus, **x(t) is not real**.

(b) **Is x(t) even?** x(t) is even if its Fourier coefficients are even, i.e. $a_k = a_{-k}$. As we see from the definition of the a_k's of this signal, $a_k = a_{-k}$, and thus the coefficients are even $\Rightarrow$ **x(t) is even**.

(c) **Is dx(t)/dt even?** If x(t) is an *even* periodic function (which by (b) we know is true here), it consists of a summation of purely cosines (since cosines are even), and no sine components. When we take the derivative, the cosines turn into sines, so that dx(t)/dt consists of a summation of *sines*, and no cosine components. The sine function is an odd function, so a sum of sines will also be odd. Thus, dx(t)/dt will be *odd*, i.e., **dx(t)/dt is not even**.

Another way to solve this is to again use the table of Fourier series properties. When x(t) has FS coefficients a_k, then the derivative dx(t)/dt has coefficients $jk\omega_o a_k$, i.e. the coefficients get multiplied by a factor of $jk\omega_o$. We can see that due to the new factor of k, the new coefficients no longer have even symmetry: $jk\omega_o * j(0.5)^{|k|} = -k\omega_o(0.5)^{|k|}$ is not an even function of k (in fact, it's odd). Thus, the signal dx(t)/dt is not even.

4. FILTERING

3.34 (b) From Section 3.8 (pg. 228), we know that if x(t) with FS coefficients a_k goes through an LTI system with frequency response $H(j\omega)$, the output of the system will have coefficients $H(jk\omega_o)a_k$, that is, the output's coefficients will be the input's coefficients multiplied by a set of weights $H(jk\omega_o)$.

First, we need the frequency response $H(j\omega)=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}dt$, given by equation (3.121) of the text:

$$H(j\omega)=\int_{-\infty}^{\infty}e^{-4|t|}e^{-j\omega t}dt=\int_{-\infty}^{0}e^{4t}e^{-j\omega t}dt+\int_{0}^{\infty}e^{-4t}e^{-j\omega t}dt=\frac{1}{4-j\omega}+\frac{1}{4+j\omega}$$

In problem (b), the signal is periodic with T = 2 (it's an alternating impulse train), so $\omega_o = \pi$. Using the same type of analysis we did for Problem 3.22(a), fig(d), we have that $a_0 = 0$, and $a_k = 0.5 - 0.5(-1)^k$, $k \neq 0$ $\Rightarrow$ **$a_k = 0$, k even; $a_k = 1$, k odd**.

Now we use the relationship that the output of the LTI system has Fourier coefficients $H(jk\omega_o)a_k$:

$$H(jk\omega_o)a_k = H(j\pi k)a_k = \left(\frac{1}{4-j\pi k}+\frac{1}{4+j\pi k}\right)a_k \Rightarrow \boxed{\left(\frac{1}{4-j\pi k}+\frac{1}{4+j\pi k}\right),\ k\text{ odd};\ \ 0,\ k\text{ even}}.$$

(c) By the graph, x(t) is periodic with T = 1 $\Rightarrow \omega_o = 2\pi$. The average value over one period is ½, so $a_0 = 0.5$. For $k \neq 0$, a_k is given by:

$$a_k = \int_{-0.25}^{0.25} 1e^{-jk2\pi t}dt = \frac{-1}{2\pi jk}e^{-jk2\pi t}\Big|_{-0.25}^{0.25} = \frac{-1}{2\pi jk}(e^{-jk\pi/2}-e^{jk\pi/2}) = \frac{1}{\pi k}\frac{1}{2j}(e^{jk\pi/2}-e^{-jk\pi/2}) = \frac{1}{\pi k}\sin(\pi k/2)$$

Note that for k even, $k \neq 0$, $a_k = \frac{1}{\pi k}\sin(\pi k/2) = 0$. Then the output FS coefficients are $H(jk\omega_o)a_k$:

$$H(jk\omega_o)a_k = H(j2\pi k)a_k = \left(\frac{1}{4-j2\pi k}+\frac{1}{4+j2\pi k}\right)a_k$$

$$\Rightarrow \boxed{\left(\frac{1}{4-j2\pi k}+\frac{1}{4+j2\pi k}\right)\frac{1}{\pi k}\sin(\pi k/2),\ k\text{ odd};\ \ 0.25,\ k=0;\ \ 0,\ k\text{ even},\ k\neq 0}.$$

5. SOLVING

3.41 In this problem, we want to derive x(t) from a number of "clues." What we have to work with is:

(i) T = 3 (ii) $a_k = a_{-k}$ (iii) $a_k = a_{k+2}$

(iv) $\int_{-0.5}^{0.5} x(t)dt = 1$ (v) $\int_{1}^{2} x(t)dt = 2$

From (ii), we know x(t) is even. If $a_k \leftrightarrow x(t)$ are a FS pair, then $a_{k+2} \leftrightarrow e^{-j2\omega_o t}x(t)$, by the frequency shifting property (see the table on pg. 206). In other words, if we modify a_k by adding 2 to each index k, we modify x(t) by multiplying it by a complex exponential, $e^{-j2\omega_o t}$. Now by (iii), $a_k = a_{k+2}$ for all k, so the corresponding signals also have to match, that is

$$x(t) = e^{-j2\omega_o t}x(t), \text{ for all time } t$$

Note that this must be true for *all time t*. Thus, $e^{-j2\omega_o t} = 1$ must hold, or else x(t) must be 0. Note that this implies

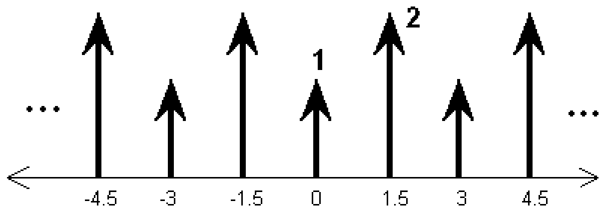
$$1 = e^{-j2\omega_o t} = e^{-j(4\pi/3)t} \Rightarrow$$ *t must be a multiple of 3/2* for this to be true.

So, the only places x(t) can be non-zero are at multiples of 3/2, i.e., t = 0, ±1.5, ±3, ±4.5, … .

Now (iv) implies that x(t) has to integrate to 1 over the range -0.5 to 0.5. x(t) = 0 in this range, except at t = 0. The only way we can have non-zero area at a single time t is if there's a delta function at that time. Thus, we can conclude from (iv) that there must be a δ(t) at time t = 0.

Furthermore, (v) implies that x(t) has to integrate to 2 over the range 1 to 2. x(t) = 0 in this range, except at t = 1.5. As before, the only way to have non-zero area under the curve at a single time instance t is if there's a delta function at that time. Thus, we know there must be a δ(t – 3/2) at t = 3/2. Since (v) states that the integral evaluates to 2, we need a scaled delta function: 2δ(t – 3/2).

Since x(t) is even and periodic with period 3, we now know all of x(t); it is just these two delta functions repeated over and over:



This can be written as:

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty}[\delta(t-3k) + 2\delta(t-3k-1.5)]}$$