ECE 101 – Linear Systems

Problem Set #0 Solutions

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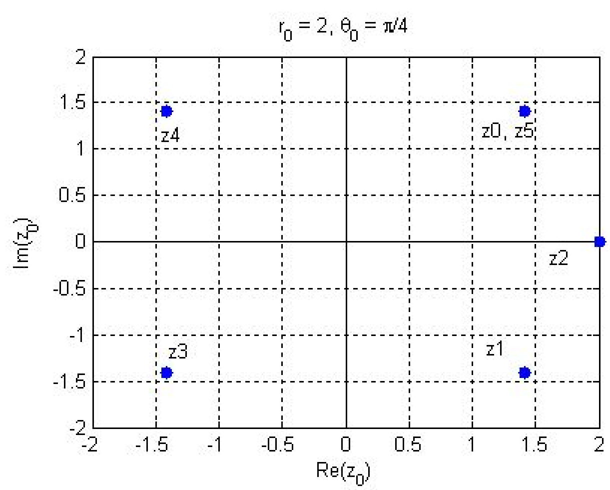
1.48 Using Euler's relation, we have:

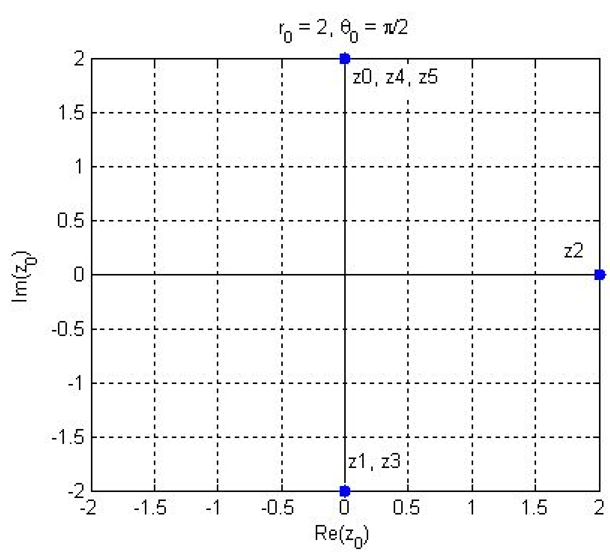
$$z_0 = r_0 e^{j\theta_0} = r_0 \cos\theta_0 + j r_0 \sin\theta_0 = x_0 + j y_0$$

Then z_1 through z_5 are:

(a) $z_1 = x_0 - jy_0$ (b) $z_2 = \sqrt{x_0^2 + y_0^2}$
(c) $z_3 = -x_0 - jy_0 = -z_0$ (d) $z_4 = -x_0 + jy_0$
(e) $z_5 = x_0 + jy_0 = z_0$ (recall, $e^{j\theta}$ is periodic with period 2π)

Plots:

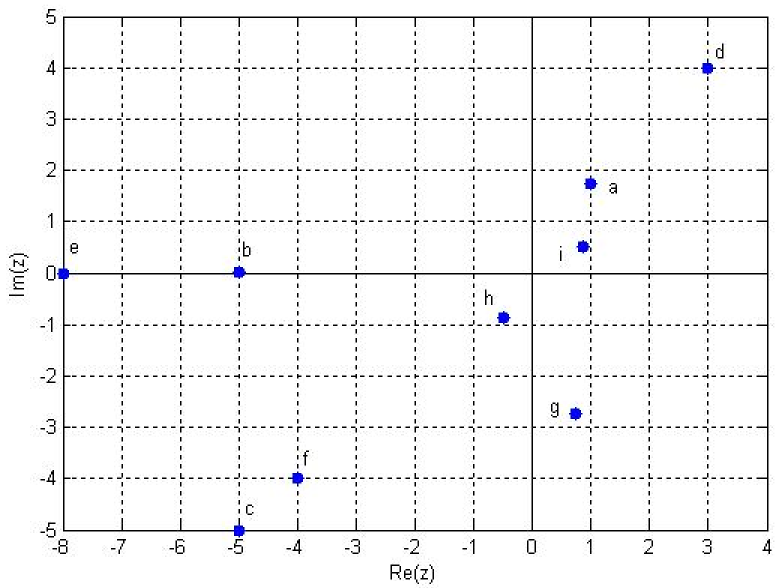




1.49 Once in polar form $re^{j\theta}$, the magnitude is given by r and angle given by θ.

(a) $2e^{j\pi/3}$ (b) $5e^{j\pi}$ (c) $5\sqrt{2}e^{j5\pi/4}$
(d) $5e^{jtan^{-1}(4/3)} = 5e^{j53.13^o}$ (e) $8e^{-j\pi}$ (f) $4\sqrt{2}e^{j5\pi/4}$
(g) $2\sqrt{2}e^{-j5\pi/12}$ (h) $e^{-j2\pi/3}$ (i) $e^{j\pi/6}$

Plot:



1.50
(a) $x = r\cos\theta$ and $y = r\sin\theta$

(b) $r = \sqrt{x^2 + y^2}$ and $\theta = \begin{cases} \arctan\frac{y}{x} & x \geq 0, \\ \arctan\frac{y}{x} + \pi & x < 0. \end{cases}$

Here we assume that the function arctan takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as its argument ranges over $[-\infty, \infty]$.

(c) The values r and $\tan\theta$ alone do not uniquely determine x and y. Non-zero complex numbers z and $-z$ have the same value of r and $\tan\theta$.

1.51 By Euler's relation, we have

$$\text{(i) } e^{j\theta} = \cos\theta + j\sin\theta \qquad \text{and} \qquad \text{(ii) } e^{-j\theta} = \cos\theta - j\sin\theta$$

(a) Summing (i) + (ii) yields $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$.

(b) Subtracting (i) - (ii) yields $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$.

(c) Squaring (a) yields $\cos^2\theta = \frac{1}{4}(e^{j2\theta} + 2 + e^{-j2\theta}) = \frac{1}{2}(1 + \cos 2\theta)$

(d) Applying (b) yields

$$\begin{aligned}
(\sin\theta)(\sin\phi) &= -\frac{1}{4}(e^{j(\theta+\phi)} - e^{j(\theta-\phi)} - e^{-j(\theta-\phi)} + e^{-j(\theta+\phi)}) \\
&= \frac{1}{2}\cos(\theta-\phi) - \frac{1}{2}\cos(\theta+\phi).
\end{aligned}$$

(e) From Euler's relation: $e^{j(\theta+\phi)} = \cos(\theta+\phi) + j\sin(\theta+\phi)$. We can also write

$$\begin{aligned}
e^{j(\theta+\phi)} &= e^{j\theta}e^{j\phi} \\
&= (\cos\theta + j\sin\theta)(\cos\phi + j\sin\phi) \\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) + j(\sin\theta\cos\phi + \cos\theta\sin\phi).
\end{aligned}$$

Equating the real and imaginary parts of these two representations of $e^{j(\theta+\phi)}$ gives us the "sum identities":

$$\begin{aligned}
\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi.
\end{aligned}$$

1.52

(a) $zz^* = (re^{j\theta})(re^{-j\theta}) = r^2$.

(b) $\frac{z}{z^*} = \frac{re^{j\theta}}{re^{-j\theta}} = \frac{e^{j\theta}}{e^{-j\theta}} = e^{j2\theta}$.

(c) $z + z^* = x + jy + x - jy = 2x = 2Re\{z\}$

(d) $z - z^* = x + jy - x + jy = 2jy = 2jIm\{z\}$

(e) $(z_1+z_2)^* = (x_1+jy_1+x_2+jy_2)^* = (x_1+x_2) - j(y_1+y_2) = (x_1-jy_1)+(x_2-jy_2) = z_1^*+z_2^*$

(f)

$$\begin{aligned}
(az_1z_2)^* &= (ar_1e^{j\theta_1}r_2e^{j\theta_2})^* = (ar_1r_2e^{j(\theta_1+\theta_2)})^* \\
&= ar_1r_2e^{-j(\theta_1+\theta_2)} = ar_1e^{-j\theta_1}r_2e^{-j\theta_2} = az_1^*z_2^*.
\end{aligned}$$

(g) $Re(\frac{z_1}{z_2}) = \frac{1}{2}\left(\frac{z_1}{z_2} + (\frac{z_1}{z_2})^*\right) = \frac{1}{2}\left(\frac{z_1}{z_2} + \frac{z_1^*}{z_2^*}\right) = \frac{1}{2}[\frac{z_1 z_2 * 2 + z_1^* z_2}{z_2 z_2^*}]$.

1.53
(a) $(e^z)^* = (e^{x+jy})^* = (e^x e^{jy})^* = e^x e^{-jy} = e^{x-jy} = e^{z^*}$

(b) $z_1 z_2^* + z_1^* z_2 = z_1 z_2^* + (z_1 z_2^*)^* = 2Re\{z_1 z_2^*\}$.
Also, $Re\{z_1^* z_2\} = Re\{(z_1 z_2^*)^*\} = Re\{z_1 z_2^*\}$.

(c) $|z| = \left|re^{j\theta}\right| = |r| = \left|re^{-j\theta}\right| = |z^*|$.
In words, taking the conjugate means flipping across the real axis; this negates the angle, but does not affect the magnitude.

(d) $|z_1 z_2| = \left|r_1 r_2 e^{j(\theta_1+\theta_2)}\right| = |r_1 r_2| = |r_1|\,|r_2| = |z_1|\,|z_2|$

(e)
$Re\{z\} = x \le \sqrt{x^2+y^2} = r = |z|$. Similarly, $Re\{z\} = x \ge -\sqrt{x^2+y^2} = -r = -|z|$.
$Im\{z\} = y \le \sqrt{x^2+y^2} = r = |z|$. Similarly, $Im\{z\} = y \ge -\sqrt{x^2+y^2} = -r = -|z|$.

(f) From (b) and (e), $|z_1 z_2^* + z_1^* z_2| = |2Re\{z_1 z_2^*\}| \le 2|z_1 z_2^*| = 2|z_1 z_2|$.

(g)

$$\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)(z_1+z_2)^* \\
&= (z_1+z_2)(z_1^*+z_2^*) \\
&= z_1 z_1^* + z_1 z_2^* + z_2 z_1^* + z_2 z_2^* \\
&= |z_1|^2 + 2Re\{z_1 z_2^*\} + |z_2|^2
\end{aligned}$$

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From this and from (b), (c), (d), and (e), we get

$$|z_1|^2 - 2|z_1||z_2| + |z_2|^2 \le |z_1+z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

or

$$(|z_1| - |z_2|)^2 \le |z_1+z_2|^2 \le (|z_1| + |z_2|)^2.$$

1.54 (a) Clearly, for $\alpha = 1$, we have

$$\textstyle\sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{N-1} 1 = N$$

For $\alpha \neq 1$, if we multiply the sum by $(1-\alpha)$, we obtain:

$$(1-\alpha)\textstyle\sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{N-1} \alpha^n - \sum_{n=0}^{N-1} \alpha^{n+1} = 1 - \alpha^N$$

since the α^1 through α^{N-1} terms cancel out. Dividing both sides by $(1-\alpha)$, we obtain the desired result:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

(b) Since $|a| < 1$, we know that

$$\lim_{N\to\infty} \alpha^N = 0.$$

Referring to part (a), we

$$\sum_{n=0}^{\infty} \alpha^n = \lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{1-\alpha^N}{1-\alpha} = \frac{1}{1-\alpha}.$$

(c) Since $|a| < 1$, the identity in part (b) holds. We differentiate both sides of part (b) with respect to α to get

$$\frac{d}{d\alpha}\left(\sum_{n=0}^{\infty} \alpha^n\right) = \frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\right).$$

Evaluating the derivatives, we get

$$\sum_{n=0}^{\infty} n\alpha^{n-1} = \frac{1}{(1-\alpha)^2}.$$

So,

$$\sum_{n=0}^{\infty} n\alpha^{n} = \frac{\alpha}{(1-\alpha)^2}.$$

(d) Again, since $|a| < 1$, the identity in part (b) holds. We use it to rewrite and evaluate the given summation as

$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha}.$$

1.55

(a) $\sum_{n=0}^{9} e^{j\pi n/2} = \frac{1-e^{j\pi 10/2}}{1-e^{j\pi/2}} = \frac{2}{1-j} = \frac{2(1+j)}{(1-j)(1+j)} = 1+j$

(b) $\sum_{n=-2}^{7} e^{j\pi n/2} = \sum_{m=0}^{9} e^{j\pi(m-2)/2}$, using the substitution $m = n+2 \Rightarrow n = m-2$. This sum equals

$$e^{-j2\pi/2} \sum_{m=0}^{9} e^{j\pi m/2} = e^{-j2\pi/2}(1+j) = -(1+j).$$

(c) Here, use the result from Problem 1.54(b):

$$\sum_{n=0}^{\infty}(\tfrac{1}{2})^n e^{j\pi n/2} = \frac{1}{1-\frac{1}{2}e^{j\pi/2}} = 0.8 + 0.4j$$

(d) $\sum_{n=2}^{\infty}(\frac{1}{2})^n e^{j\pi n/2} = (\frac{1}{2})^2 e^{j2\pi/2} \sum_{m=0}^{\infty}(\frac{1}{2})^m e^{j\pi m/2}$, using the substitution $m = n–2 \Rightarrow n = m+2$. This sume equals

$$= (0.25)(-1) \sum_{m=0}^{\infty}(\tfrac{1}{2})^m e^{j\pi m/2} = (-0.25)(0.8 + 0.4j) = -0.2 - 0.1j$$

(e) Using the identity we proved in 1.51(a), we have

$$\sum_{n=0}^{9} \cos(\pi n/2) = \tfrac{1}{2}\sum_{n=0}^{9} e^{j\pi n/2} + \tfrac{1}{2}\sum_{n=0}^{9} e^{-j\pi n/2} = \tfrac{1}{2}(1+j) + \tfrac{1}{2}(1-j) = 1$$

1.56

(c) $\int_2^8 e^{j\pi t/2} dt = \frac{1}{j\pi/2} e^{j\pi t/2} \,|_2^8 = \frac{2}{j\pi}(e^{j4\pi} - e^{j\pi}) = -\frac{4}{\pi}j$

(f) Using the identity from 1.51(b),

$$\int_0^{\infty} e^{-2t}\sin(3t)dt = \int_0^{\infty}\left[\frac{e^{-(2-3j)t} - e^{-(2+3j)t}}{2j}\right] dt = \frac{1/2j}{2-3j} - \frac{1/2j}{2+3j} = \frac{3}{13}$$