UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2019
Linear Systems Fundamentals

SOLUTIONS TO FINAL EXAM

You are allowed four 2-sided sheets of notes.

No books, no other notes, no electronics.

PRINT YOUR NAME J. B. Joseph Fourier

SIGNATURE

Your signature confirms that you have completed this exam on your own and in accordance with the ECE 101 honor code.

Student ID Number Transform Student ID Number 101

Problem	Weight	Score
1	20 pts	20
2	20 pts	20
3	20 pts	20
4	20 pts	20
5	20 pts	20
6	20 pts	20
Total	120 pts	120

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Tables 3.1, 3.2, 4.1, 4.2, 5.1, 5.2, 9.1, and 9.2 are attached.
Good luck!

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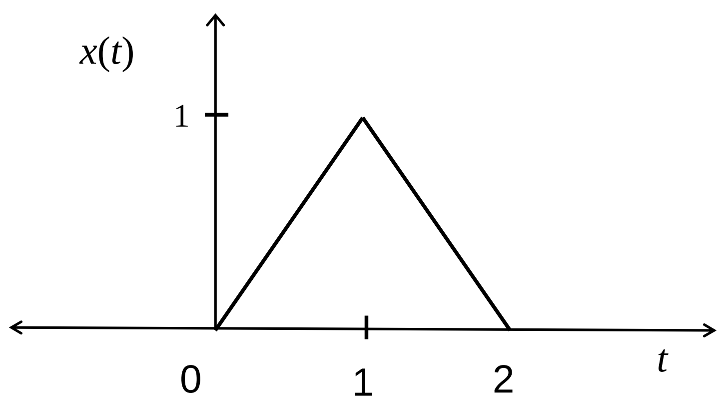
Problem 1 [CTFT] (20 points, 4pts each part)

Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ defined by

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following problems about $X(j\omega)$. You can answer all of them **without** actually computing $X(j\omega)$ by using properties of the CTFT. **Justify your answers.**

Here is a sketch of $x(t)$:



(a) Find $X(j0)$.

The analysis equation states: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$.

Evaluating at $\omega = 0$, we get $X(j0) = \int_{-\infty}^{\infty} x(t)dt = 1$.

(b) Evaluate $\int_{-\infty}^{\infty} X(j\omega)d\omega$

The synthesis equation states: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$.

Evaluating at $t = 0$, we get $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$.

So, $\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 0$.

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Problem 1 [CTFT] (cont.)

(c) Evaluate $\int_{-\infty}^{\infty} X(j\omega)2\frac{\sin(\omega)}{\omega}d\omega$.

From Table 4.2, we have the CT Fourier transform pair:

$$y(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} \longleftrightarrow Y(j\omega) = \frac{2\sin(\omega)}{\omega}$$

The convolution property (Table 4.1) states that:

$$z(t) = x(t) * y(t) \longleftrightarrow Z(j\omega) = X(j\omega)Y(j\omega).$$

Using the synthesis equation as in part (b), we get:

$$2\pi z(0) = \int_{-\infty}^{\infty} Z(j\omega)d\omega = \int_{-\infty}^{\infty} X(j\omega)2\frac{\sin(\omega)}{\omega}d\omega.$$

Evaluating the convolution $z(t)$ at $t = 0$, we flip $y(t)$, shift right by 0, multiply by $x(t)$ and integrate. This leads to:

$$z(0) = \int_0^1 tdt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}.$$

So, $\int_{-\infty}^{\infty} X(j\omega)2\frac{\sin(\omega)}{\omega}d\omega = 2\pi z(0) = \pi$.

(d) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

By Parseval's Relation (Table 4.1, 4.3.7):

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Using symmetry of $x(t)$ about its peak, we have

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = 2\int_0^1 t^2 dt = \frac{2}{3}t^3\Big|_0^1 = \frac{2}{3}.$$

So, $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi(\frac{2}{3}) = \frac{4\pi}{3}$.

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Problem 1 [CTFT] (cont.)

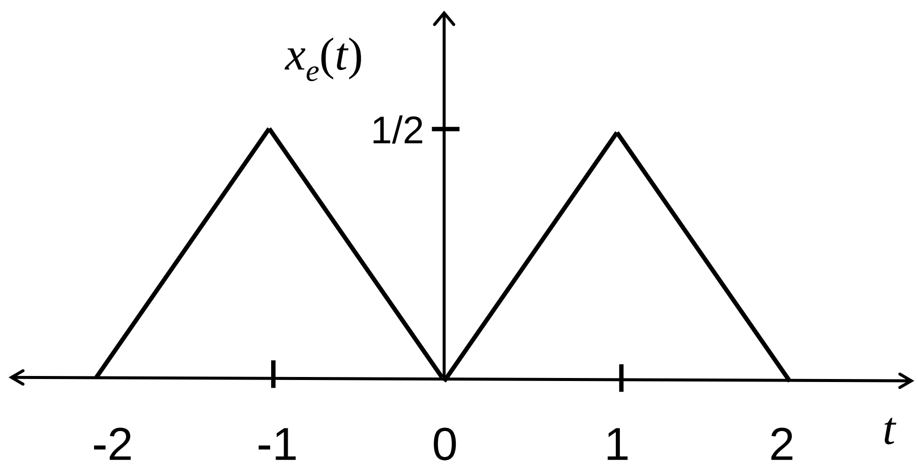
(e) Sketch the inverse Fourier transform of the real part of $X(j\omega)$, $\mathcal{R}e\{X(j\omega)\}$.

The Even-Odd Decomposition for Real Signal property (Table 4.1, 4.33), states:

$$x_e(t) = \mathcal{E}v\{x(t)\} \longleftrightarrow \mathcal{R}e\{X(j\omega)\}$$

where $x_e(t) = \frac{x(t)+x(-t)}{2}$.

For this signal $x(t)$, the even part $x_e(t) = \mathcal{E}v\{x(t)\}$ is:



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Problem 2 [DTFT] (20 points, 10pts each part)

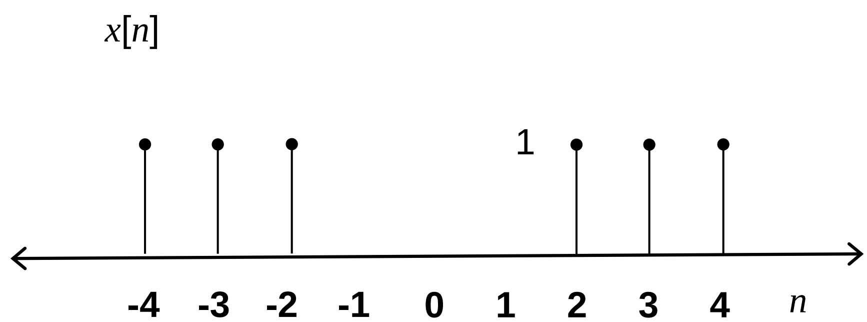
(a) (DTFT analysis)

Let $x[n]$ be the discrete-time signal defined by:

$$x[n] = \begin{cases} 1, & 2 \leq |n| \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

(i) (2 points)

Sketch $x[n]$ precisely.



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Problem 2 [DTFT] (cont.)

(a) (cont.)

Let $x[n]$ be the discrete-time signal defined by:

$$x[n] = \begin{cases} 1, & 2 \leq |n| \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) (8 points)

Determine the discrete-time Fourier transform of $x[n]$. Express the answer using trigonometric functions. Write your answer in the box below. **Justify your answer.**

From Table 5.2, we have the DT Fourier transform pair:

$$y(t) = \begin{cases} 1, & |n| \leq 1 \\ 0, & |n| > 1 \end{cases} \longleftrightarrow Y(e^{j\omega}) = \frac{\sin(3\omega/2)}{\sin(\omega/2)}$$

From the figure, we see that:

$$x[n] = y[n+3] + y[n-3].$$

From the Time Shifting property (Table 5.1, 5.3.3), we have:

$$\begin{aligned} X(e^{j\omega}) &= e^{j\omega 3} Y(e^{j\omega}) + e^{-j\omega 3} Y(e^{j\omega}) = (e^{j\omega 3} + e^{-j\omega 3}) Y(e^{j\omega}) \\ &= 2\cos(3\omega) Y(e^{j\omega}) = 2\cos(3\omega) \frac{\sin(3\omega/2)}{\sin(\omega/2)} \end{aligned}$$

Alternative solution:

$x[n] = \delta[n+4] + \delta[n+3] + \delta[n+2] + \delta[n-2] + \delta[n-3] + \delta[n-4]$

Table 5.2 contains the transform pair: $\delta[n - n_0] \longleftrightarrow e^{-j\omega n_0}$.

$$\begin{aligned} X(e^{-j\omega}) &= e^{j\omega 4} + e^{j\omega 3} + e^{j\omega 2} + e^{-j\omega 2} + e^{-j\omega 3} + e^{-j\omega 4} \\ &= 2\cos(2\omega) + 2\cos(3\omega) + 2\cos(4\omega) \end{aligned}$$

$$\boxed{X(e^{j\omega}) = 2\cos(3\omega)\frac{\sin(3\omega/2)}{\sin(\omega/2)} = 2(\cos(2\omega) + \cos(3\omega) + \cos(4\omega))}$$

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Problem 2 [DTFT] (cont.)

(b) (DTFT synthesis)

The Fourier transform of a discrete-time signal $x[n]$ is given by

$$X(e^{j\omega}) = e^{-j\omega/4} \text{ for } 0 \leq \omega \leq 2\pi.$$

Determine $x[n]$.

Express your answer in rectangular form, $x[n] = R(n) + jI(n)$, where $R(n)$ and $I(n)$ are real-valued functions of n.

Write your answer in the box below. **Justify your answer.**

The transform pair $\delta[n - n_0] \leftrightarrow e^{j\omega n_0}$ requires n_0 to be an integer, so it is not applicable here.

Instead, we apply the synthesis equation of the DTFT:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_0^{2\pi} e^{-j\omega/4}e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{j\omega(n-\frac{1}{4})}d\omega \\
&= \frac{1}{2\pi j(n-\frac{1}{4})}e^{j\omega(n-\frac{1}{4})}\Big|_0^{2\pi} \\
&= \frac{1}{2\pi j(n-\frac{1}{4})}(e^{j(2\pi n-\frac{\pi}{2})} - 1) \\
&= \frac{1}{2\pi j(n-\frac{1}{4})}(-j-1) \text{ using } e^{j2\pi n} = 1 \text{ and } e^{-j\frac{\pi}{2}} = -j \\
&= \frac{1}{2\pi(n-\frac{1}{4})}(-1+j)
\end{aligned}$$

$$\boxed{x[n] = \frac{1}{2\pi(n-\frac{1}{4})}(-1+j)}$$

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Problem 3 [Filtering] (20 points, 4pts each part)

Consider a causal and stable LTI system whose input $x[n]$ and output $y[n]$ are related by the first-order difference equation

$$y[n] + \frac{1}{2}y[n-1] = x[n]$$

(a) Determine the frequency response $H(e^{j\omega})$ of the system.
Justify your answer.

There are several ways to solve this problem.

Using either the eigenfunction property $e^{j\omega n} \to H(e^{j\omega})e^{j\omega n}$ we get:

$$H(e^{j\omega})(e^{j\omega n} + \frac{1}{2}e^{-j\omega}H(e^{j\omega})e^{j\omega n} = e^{j\omega n}$$

or

$$H(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

This can also be obtained by applying the DTFT to both sides of the equation and using the Time Shifting property (Table 5.1, 5.3.3):

$$Y(e^{j\omega}) + \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

which implies, by the comvolution property (Table 5.1, 5.4),

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+\frac{1}{2}e^{-j\omega}}.$$

Finally, we could use the fact that for the causal, stable LTI system defined by the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

the frequency response is given by:

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

with $a_0 = 1$, $a_1 = \frac{1}{2}$, and $b_0 = 1$.

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Problem 3 [DT Filtering] (cont.)

(b) Evaluate $H(e^{j\omega})$ at $\omega = 0$ and $\omega = \pi$.

Does the system represent a low-pass or a high-pass filter?

$$H(e^{j0}) = \frac{1}{1 + \frac{1}{2}e^{-j\omega}}\bigg|_{\omega=0} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}.$$

$$H(e^{j\pi}) = \frac{1}{1 + \frac{1}{2}e^{-j\omega}}\bigg|_{\omega=\pi} = \frac{1}{1-\frac{1}{2}} = 2.$$

$H(e^{j0}) = \frac{2}{3}$	Low-pass	High-Pass
$H(e^{j\pi}) = 2$	☐	☒

(c) Determine the impulse response $h[n]$ of the system.
Justify your answer.

In the table of DTFT pairs (Table 5.2), we find:

$$x[n] = a^n u[n],\ |a| < 1 \longleftrightarrow X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

Here $a = -\frac{1}{2}$, so

$$h[n] = \left(-\frac{1}{2}\right)^n u[n].$$

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Problem 3 [DT Filtering] (cont.)

(d) Determine the step response $s[n]$ of the system. The step response is related to the impulse response by $s[n] = u[n] * h[n] = \sum_{-\infty}^{n} h[k]$.

For $n \geq 0$,

$$\begin{aligned} s[n] &= \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \\ &= \frac{\left(-\frac{1}{2}\right)^{n+1} - 1}{\left(-\frac{1}{2}\right) - 1} \\ &= \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{n+1}\right) \end{aligned}$$

Therefore,

$$s[n] = \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{n+1}\right) u[n].$$

(e) Let $x[n] = (-1)^n$ be the system input. Determine the output $y[n]$.

By the eigenfunction property,

$$x[n] = (-1)^n = (e^{j\pi})^n \rightarrow y[n] = H(e^{j\pi})(e^{j\pi})^n = H(e^{j\pi})(-1)^n.$$

From part (b), we have $H(e^{j\pi}) = 2$, so

$$y[n] = 2(-1)^n.$$

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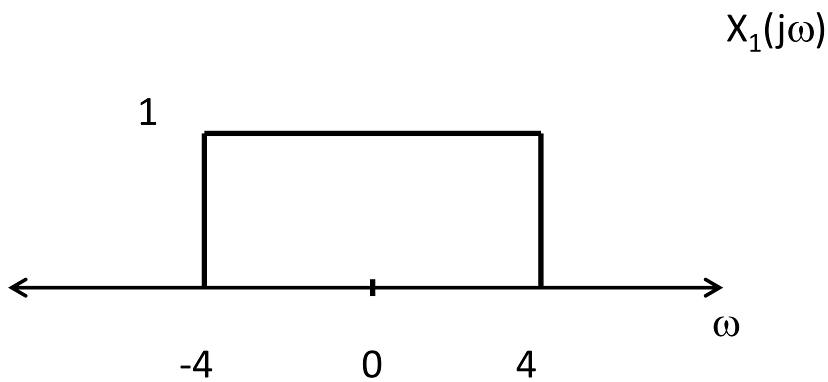
Problem 4 [Amplitude Modulation] (20 pts, 4pts each part)

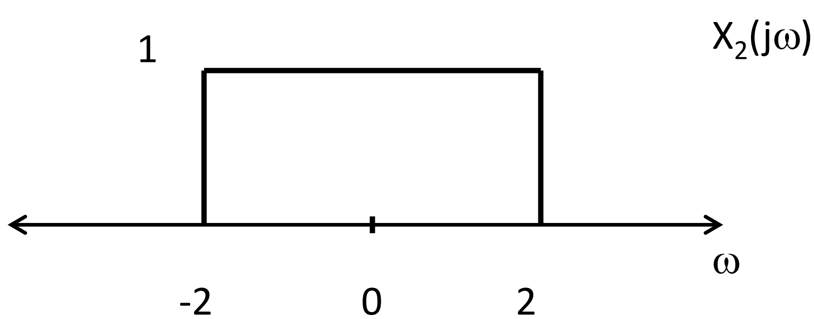
Consider the signals $x_1(t) = \frac{\sin(4t)}{\pi t}$ and $x_2(t) = \frac{\sin(2t)}{\pi t}$.

(a) Determine their Fourier transforms $X_1(j\omega)$ and $X_2(j\omega)$.
Sketch them precisely. From Table 4.2:

$$X_1(j\omega) = \begin{cases} 1, & |\omega| < 4 \\ 0, & |\omega| > 4 \end{cases}$$

$$X_2(j\omega) = \begin{cases} 1, & |\omega| < 2 \\ 0, & |\omega| > 2 \end{cases}$$





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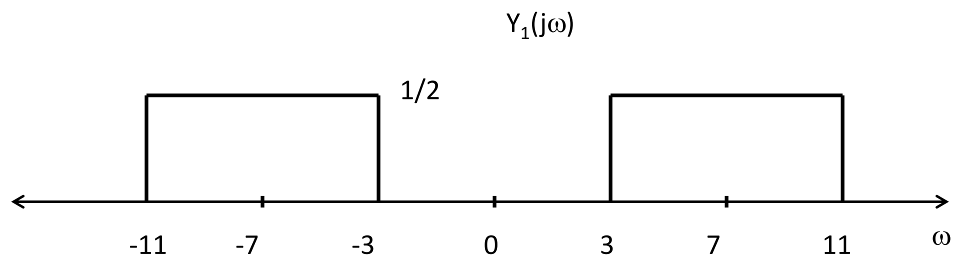
Problem 4 [Amplitude Modulation] (cont.)

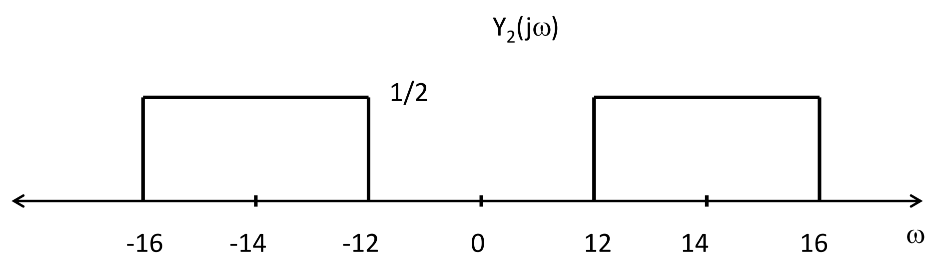
(b) Let $y_1(t) = x_1(t)\cos(7t)$. Express $Y_1(j\omega)$ in terms of $X_1(j\omega)$. Sketch $Y_1(j\omega)$ precisely.

$$Y_1(j\omega) = \frac{1}{2}X_1(j(\omega-7)) + \frac{1}{2}X_1(j(\omega+7))$$



(c) Let $y_2(t) = x_2(t)\cos(14t)$. Express $Y_2(j\omega)$ in terms of $X_2(j\omega)$. Sketch $Y_2(j\omega)$ precisely.

$$Y_2(j\omega) = \frac{1}{2}X_2(j(\omega-14)) + \frac{1}{2}X_2(j(\omega+14))$$



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Problem 4 [Amplitude Modulation] (cont.)

(d) The signal $y(t) = y_1(t) + y_2(t)$ is transmitted. It is demodulated using multiplication by $\cos(7t)$ to form the signal $z(t) = y(t)\cos(7t)$.

Express the Fourier Transform $Z(j\omega)$ in terms of $Y_1(j\omega)$ and $Y_2(j\omega)$.

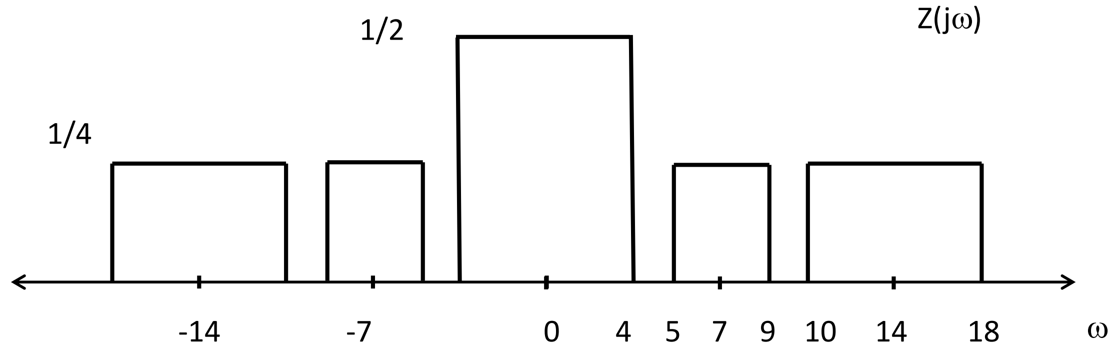
Express $Z(j\omega)$ in terms of $X_1(j\omega)$ and $X_2(j\omega)$.

Sketch precisely the portion of $Z(j\omega)$ in the frequency range $[-18, 18]$.

$$Z(j\omega) = \frac{1}{2}Y_1(j(\omega-7)) + \frac{1}{2}Y_1(j(\omega+7)) + \frac{1}{2}Y_2(j(\omega-7)) + \frac{1}{2}Y_2(j(\omega+7))$$

$$\begin{aligned} Z(j\omega) = &\frac{1}{4}X_1(j(\omega-14)) + \frac{1}{2}X_1(j\omega) + \frac{1}{4}X_1(j(\omega+14)) \\ &+ \frac{1}{4}X_2(j(\omega-21)) + \frac{1}{4}X_2(j(\omega-7)) + \frac{1}{4}X_2(j(\omega+7)) + \frac{1}{4}X_2(j(\omega+21)) \end{aligned}$$

In the range $[-18, 18]$, the transform $Z(j\omega)$ is as shown below:



(e) Can $x_1(t)$ be recovered from $z(t)$? If not, why not.
If so, describe precisely how it can be recovered.

Yes. Apply a low-pass filter with cut-off frequency ω_{co} satisfying $4 < \omega_{co} < 5$ and with gain 2.

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Problem 5 [Sampling Theory] (20 pts)

(a) (10 points, 5pts each part)

Let $x(t)$ be a continuous-time band-limited signal with

$$X(j\omega) = 0 \text{ for } |\omega| > W$$

For each of the signals below, indicate the conditions that, according to the sampling theorem, the sampling frequency ω_s and sampling period T should satisfy so that the signal can be reconstructed from its samples.

Justify your answers by explicit reference to the Fourier Transforms $Y(j\omega)$ of the signals.

(i) $y(t) = x(t) * x(t-1)$. Uisng the Time Shifting property and Convolution property, we have:

$$Y(j\omega) = X(j\omega)e^{-j\omega}X(j\omega) = e^{-j\omega}X(j\omega)^2.$$

Clearly, $W_{\max} = W$.

Condition on $\omega_s:\ \omega_s > 2W$	Condition on $T:\ T < \frac{\pi}{W_{\max}} = \frac{\pi}{W}$

(ii) $y(t) = x(t)x(-4t)$.

Using the Multiplication property, Time Scaling property, and Time Reversal property, we have:

$$Y(j\omega) = \frac{1}{2\pi}X(j\omega) * \frac{1}{4}X(-\frac{j\omega}{4})$$

The first factor has maximum frequency W and the second factor has maximum frequency $4W$.

The convolution therefore has maximum frequency $W_{\max} = 5W$.

Condition on $\omega_s:\ \omega_s > 10W$	Condition on $T:\ T < \frac{\pi}{W_{\max}} = \frac{\pi}{5W}$

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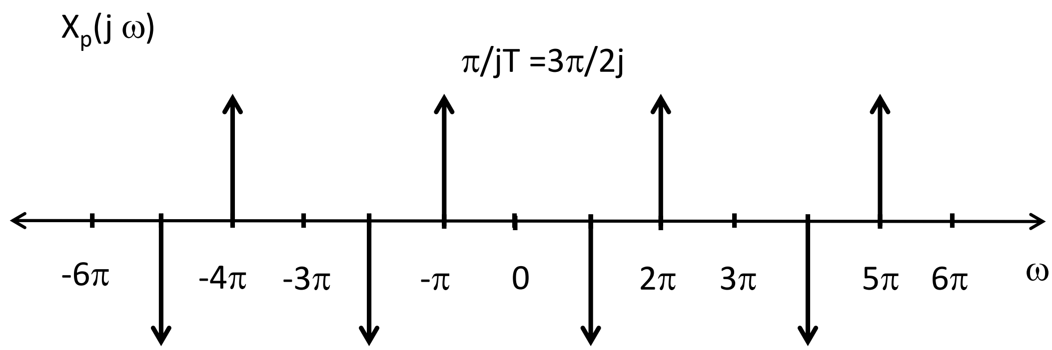
Problem 5 [Sampling Theory] (cont.)

(b) (10 points, 5pts each part)

Consider the signal $x(t) = \sin(2\pi t)$. Let $x_p(t)$ be obtained by impulse-train sampling $x(t)$ with sampling frequency $\omega_s = 3\pi$.

(i) Let $X_p(j\omega)$ be the Fourier transform of the signal $x_p(t)$. Sketch $X_p(j\omega)$ precisely in the frequency range $[-6\pi, 6\pi]$.

$X(j\omega) = \frac{\pi}{j}\delta(\omega - 2\pi) - \frac{\pi}{j}\delta(\omega + 2\pi)$
$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k3\pi))$, where $T = \frac{2\pi}{\omega_s} = \frac{2}{3}$.



(ii) Suppose $x_p(t)$ is passed through a low-pass filter with gain $T = 2\pi/\omega_s = 2/3$ and cutoff frequency $\omega_c = \omega_s/2 = 3\pi/2$. Determine the output $y(t)$ of the low-pass filter and determine whether aliasing occurred.

From part (i), we see that the transform of the filter output is

$$Y(j\omega) = \frac{\pi}{j}\delta(\omega + \pi) - \frac{\pi}{j}\delta(\omega - \pi)$$

So,

$$y(t) = -\sin(\pi t).$$

Therefore, aliasing (including a sign reversal) has occurred.

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Problem 6 [Laplace Transform](20 points)

(a) (8 points)

Let S be a **causal** LTI system whose input-output relationship is described by the differential equation:

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+5y(t) = \frac{d^2x(t)}{dt^2} - x(t).$$

(i) Determine the system function $H(s)$.

The system function is given by:

$$\begin{aligned} H(s) &= \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \\ &= \frac{s^2-1}{s^2+2s+5} \\ &= \frac{(s-1)(s+1)}{(s-(-1+2j))(s-(-1-2j))}. \end{aligned}$$

So there are first order poles at $s = -1+2j, -1-2j$ and first-order zeros at $s = 1, -1$.

$$\boxed{H(s) = \frac{s^2-1}{s^2+2s+5} = \frac{(s-1)(s+1)}{(s-(-1+2j))(s-(-1-2j))}}$$

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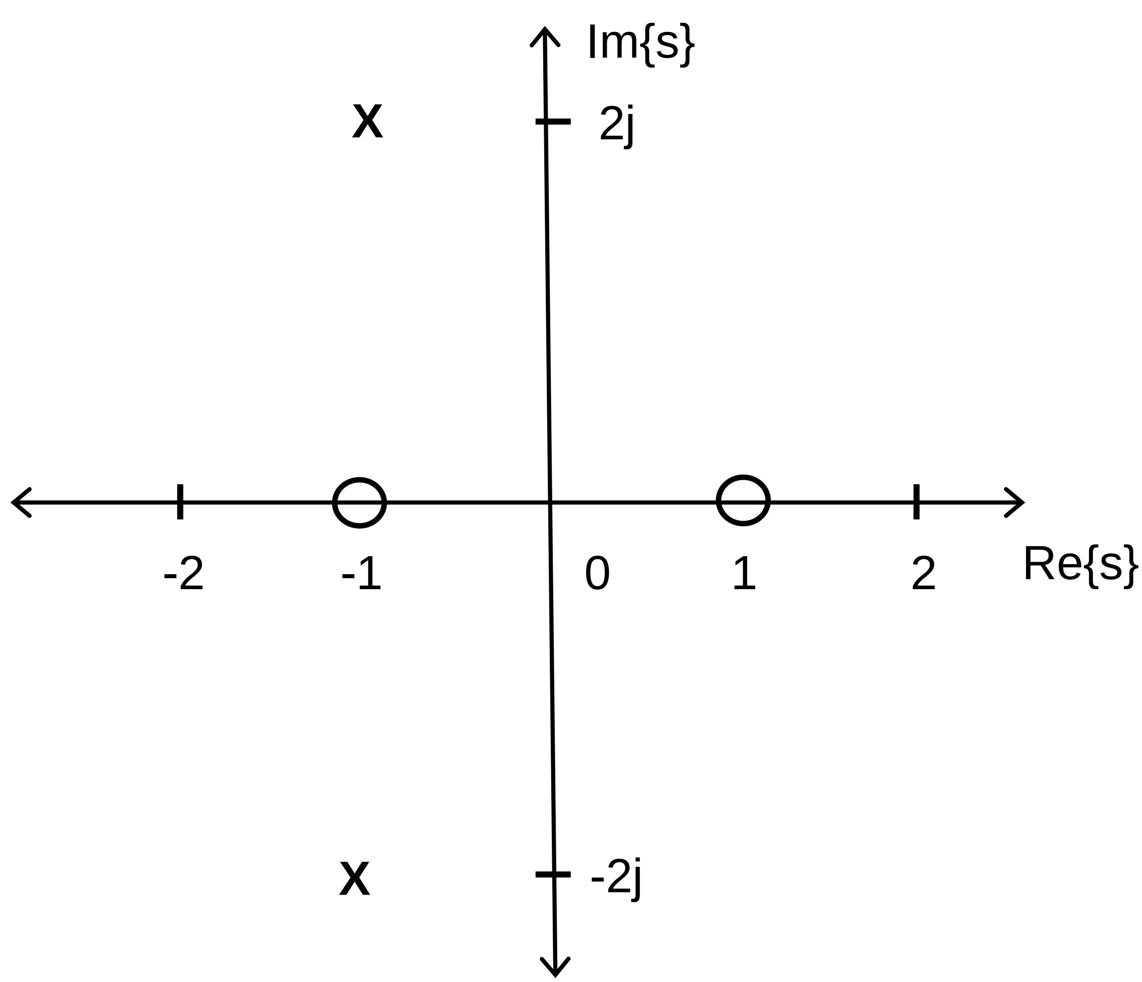
Problem 6 [Laplace Transform](cont.)

(a) (cont.)

(ii) Sketch precisely the pole-zero plot associated with the system function $H(s)$. Indicate the order of all poles and zeros.

First order poles at $s = -1 + 2j, -1 - 2j$.

First-order zeros at $s = 1, -1$.



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Problem 6 [Laplace Transform] (cont.)

(a) (cont.)

(iii) Determine the region of convergence (ROC) associated with $H(s)$. **Justify your answer.**

The system is causal and the system function is rational. The ROC must be the right half-plane to the right of the right-most pole. Therefore,

$$ROC = \{s | \mathcal{R}e\{s\} > -1\}.$$

(iv) Is the systems stable? Check the box. **Justify your answer.**

Yes	No
☒	☐

The system is stable because the ROC contains the $j\omega$-axis.

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Problem 6 [Laplace Transform] (cont.)

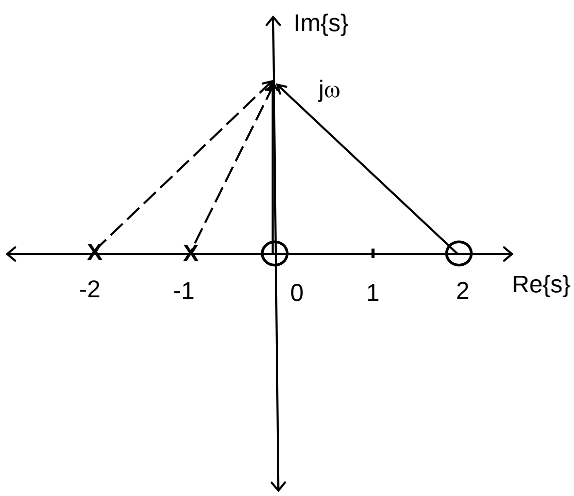
(b) (6 points)

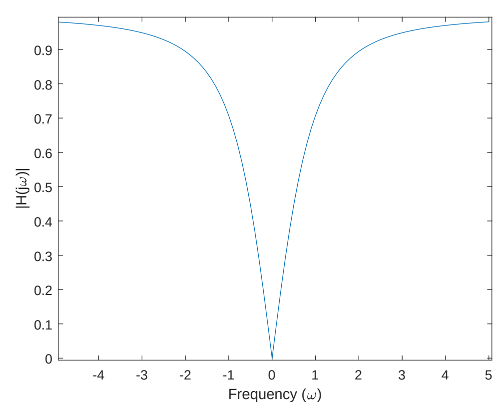
Let S be a **causal, stable** LTI system with system function

$$H(s) = \frac{s(s-2)}{(s+1)(s+2)}.$$

Let $H(j\omega)$ be the frequency response of the system.

(i) Sketch the magnitude of the frequency response $\left|H(j\omega)\right|$.





(ii) Is the frequency response lowpass, highpass, bandpass, or all-pass? Indicate your answer below by checking the appropriate box.

Low-pass	High-pass	Band-pass	All-pass
☐	☒	☐	☐

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Problem 6 [Laplace Transform] (cont.)

(c) (6 points)

Let S be a **stable** LTI system with system function

$$H(s) = \frac{s}{s^2+2s-3} = \frac{s}{(s-1)(s+3)}.$$

Determine the impulse response $h(t)$.

First-order poles at $s = 1, -3$ and a first-order zero at $s = 0$.

Since S is stable, the ROC contains the $j\omega$-axis.

The ROC is $\{s| -3 < \mathcal{R}e\{s\} < 1\}$.

Note that the system is not causal.

Using partial fraction expansion,

$$\begin{aligned} H(s) &= \frac{A}{s-1} + \frac{B}{s+3} \\ &= \frac{1/4}{s-1} + \frac{3/4}{s+3} \end{aligned}$$

since $A + B = 1$ and $3A - B = 0$.

The impulse response is

$$h(t) = -\frac{1}{4}e^t u(-t) + \frac{3}{4}e^{-3t}u(t).$$

$$\boxed{h(t) = -\frac{1}{4}e^t u(-t) + \frac{3}{4}e^{-3t}u(t)}$$

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