# ECE 101: Linear Systems Fundamentals

Summer Session II 2020 - Lecture 6

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# Today's topics

- Causal LTI systems defined by difference or differential equations
- DT LTI systems and convolution sum
- Examples of DT convolution
- Convolution properties

Signals and Systems (2th Edition): sections 2.1, 2.3.1-3, 2.4

# Causal LTI Systems Defined by Difference or Differential Equations

# DT LTI systems defined by difference equations

- The general form of DT LTI systems defined by linear constant-coefficient difference equations is

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

where $x[n]$ is the input signal and $y[n]$ is the output signal.

- Example:

$$4y[n-3] + y[n-2] - 6y[n-1] + y[n] = x[n-1] + x[n]$$

# Causal DT LTI systems defined by difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- If $N = 0$ in the above difference equation, the system will be causal and the difference equation will be reduced to

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

# CT LTI systems defined by differential equations

- The general form of CT LTI systems defined by linear constant-coefficient differential equations is

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

  where $x(t)$ is the input signal and $y(t)$ is the output signal.

- Example:

$$4\frac{d^3y(t)}{dt^3} + \frac{d^2y(t)}{dt^2} - 6\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + x(t)$$

# Causal CT LTI systems defined by differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- If $N = 0$ in the above differential equation, the system will be causal and the differential equation will be reduced to

$$y(t) = \sum_{k=0}^{M} \frac{b_k}{a_0} \frac{d^k x(t)}{dt^k}$$

# Linear constant-coefficient difference and differential equations

- The linear constant-coefficient difference and differential equations provide an implicit specification of the systems.
- They describe a relationship between the input and the output, rather than an explicit expression for the system output as a function of the input.
- In order to obtain an explicit expression, we must solve the differential equation.

# Solving linear constant-coefficient difference and differential equations

- The solution to the linear constant-coefficient difference and differential equations consists of two parts, a particular solution plus a homogeneous solution.

$$y[n] = y_p[n] + y_h[n]$$
$$y(t) = y_p(t) + y_h(t)$$

- The homogeneous solutions, $y_h[n]$ and $y_h(t)$, are the solutions to the following equations.

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$
$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

# Auxiliary conditions - initial rest

- To completely determine the input-output relationship for the system, we need to identify auxiliary conditions.
- For causal LTI system we will use the condition of initial rest as the auxiliary condition - i.e.,

$$\text{if } x[n] = 0 \text{ for } n < n_0, \text{ then } y[n] = 0 \text{ for } n < n_0$$
$$\text{if } x(t) = 0 \text{ for } t < t_0, \text{ then } y(t) = 0 \text{ for } t < t_0$$

- Condition of initial rest means until something happens in the input signal, nothing is going to happen in the output signal.

# Example

- Example. Consider a *causal LTI* system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Determine y[n] if $x[n] = \delta[n]$.

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

$$y[0] = x[0] + \frac{1}{2}y[-1]$$

$x[n] = \delta[n] = 0$ for $n \leq -1$ . The condition of initial rest implies

that $y[n] = 0$ for $n \leq -1$ $\longrightarrow$ $y[-1] = 0$

# Example

- Example. Consider a *causal LTI* system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Determine y[n] if $x[n] = \delta[n]$.

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

$$y[0] = x[0] + \frac{1}{2}y[-1] = 1$$

$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2} \times 1$$

$$y[2] = x[2] + \frac{1}{2}y[1] = \frac{1}{2} \times \frac{1}{2}$$

$$y[3] = x[3] + \frac{1}{2}y[2] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$\vdots$$

looking at the pattern :

For $n \geq 0$, $y[n] = \left(\frac{1}{2}\right)^n$

# Example

- Example. Consider a *causal LTI* system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Determine y[n] if $x[n] = \delta[n]$.

For $n \geq 0$, $y[n] = \left(\frac{1}{2}\right)^n \equiv y[n] = \left(\frac{1}{2}\right)^n u[n]$

when $x[n] = \delta[n] \longrightarrow y[n] = \left(\frac{1}{2}\right)^n u[n]$

we also know that when $x[n] = \delta[n]$, the output $y[n] = h[n]$

$h[n]$ is the impulse response of an LTI system. It's the response of a DT LTI system when $x[n] = \delta[n]$

# Example

- Example. Consider a *causal LTI* system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Determine y[n] if $x[n] = \delta[n]$.

In the causal LTI system defined by

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

the impulse response is

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

# Recursive and non-recursive difference equations

- The difference equation describing the DT causal LTI systems can be rearranged in the form

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

- When $N \geq 1$ in the above equation, the equation is called a recursive equation, since it specifies a recursive procedure for determining the output in terms of the input and previous outputs.

- Such systems are commonly referred to as infinite impulse response (IIR) systems.

# Recursive and non-recursive difference equations

- The difference equation describing the DT causal LTI systems can be rearranged in the form

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

- When $N = 0$ in the above equation, the equation is called a nonrecursive equation, since we do not recursively use previously computed values of the output to compute the present value of the output.

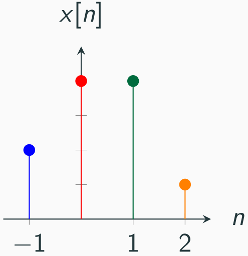
- Such systems are commonly referred to as finite impulse response (FIR) systems.

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

# DT LTI systems and convolution sum

# Discrete-time LTI Systems

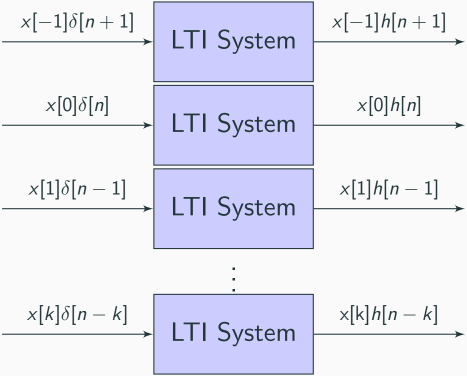
- Definition: We refer to the response (output) of the system to the input $x[n] = \delta[n]$ as **impulse response** of the system.
- Denote the impulse response by $h[n]$.
- Given any signal $x[n]$




- we can write it as:
  $x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$
- In short: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ (representation property)

# Discrete-time LTI Systems

- By Linearity and Time-Invariant property:




- By Linearity:

# Discrete-time LTI Systems: The Main Result

**Response of DT LTI System to an Arbitrary Input**

Let $h[n]$ be the impulse response of an LTI system. Then for any input $x[n]$, the output is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

- The above sum is called **the convolution (sum)** of $x[n]$ and $h[n]$. Denote it by $y[n] \stackrel{def}{=} x[n] * h[n]$

# Convolution is only for LTI systems

- The most important fact to remember about convolution is that **convolution is valid only when the system is linear time-invariant**.

- If the system is not linear, then

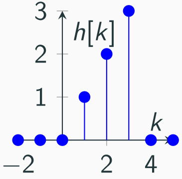
$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \not\longrightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

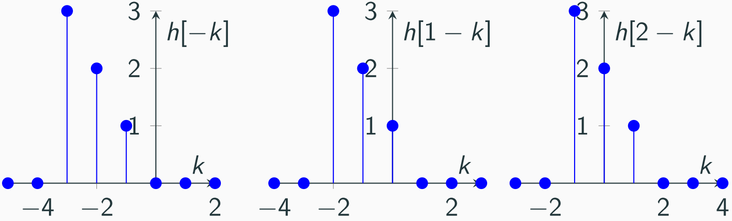
- If the system is not time-invariant, then even if $\delta[n] \rightarrow h[n]$, we can still have

$$\delta[n-k] \not\longrightarrow h[n-k].$$

# How to compute convolution?

- Fix time $n$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- As a function of $k$: $h[n-k]$ is $h[k]$ flipped and shifted to the right by $n$:



# How to compute convolution?

- Fix time $n$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- As a function of $k$: $h[n-k]$ is $h[k]$ flipped and shifted to the right by $n$:
- Output at time $n$: sample by sample multiply $x[k]$ by $h[n-k]$ and then add.

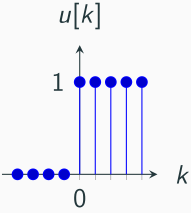
# question

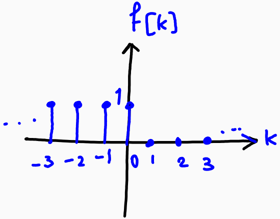
- Considering $u[k]$, the unit step signal shown below, which option correctly shows $u[n-k]$ for n=3?

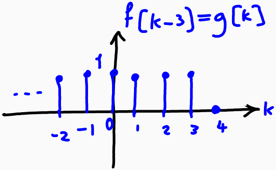
$g[k] = u[3-k] = u[-k+3]$

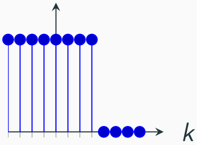
$f[k] = u[-k]$

$g[k] = f[k-n] = u[-k+n]$

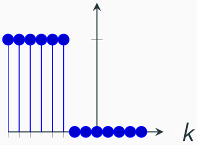



(A)




(B)

Both A and B

(C)

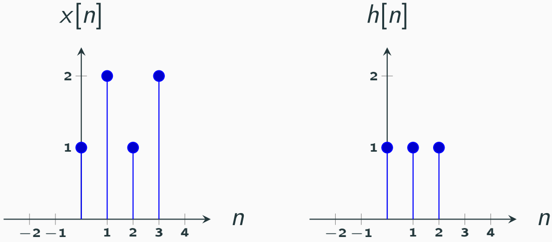
# How to graphically compute the convolution?

- To compute DT convolution, there are three basic steps:
  1. Flip
  2. Shift
  3. Multiply and Add

# How to graphically compute convolution?

**Example 1.** Consider the signal $x[n]$ and the impulse response $h[n]$ shown below. Find $y[n] = x[n] * h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



# How to graphically compute convolution?

**Example 1.** Consider the signal $x[n]$ and the impulse response $h[n]$ shown below. Find $y[n] = x[n] * h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Let's compute the output $y[n]$ point by point.

# How to graphically compute convolution?

**Example 1.** Consider the signal $x[n]$ and the impulse response $h[n]$ shown below. Find $y[n] = x[n] * h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

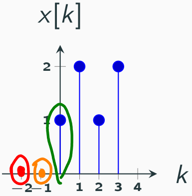
- To calculate $\mathbf{y[0]}$:

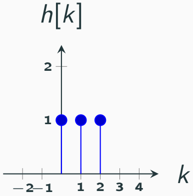
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\ h[0-k] = \sum_{k=-\infty}^{\infty} x[k]\ h[-k]$$

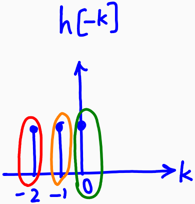
- Plot $x[k]$ and $h[k]$
- Flip $h[k]$ to get $h[-k]$
- Shift $h[-k]$ by 0 to get $h[0-k] = h[-k]$
- Multiply-add to get $\sum_{k=-\infty}^{\infty} x[k]\ h[-k]$

# convolution example

**Example 1.** Find $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$



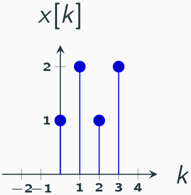

$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[-k]$$

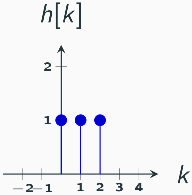
$$y[0] = 0 \times 1 + 0 \times 1 + 1 \times 1 + 2 \times 0 + 1 \times 0 + 2 \times 0 = 1$$

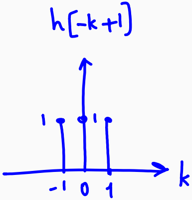
$$y[0] = 1$$

# convolution example

**Example 1.** Find $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$



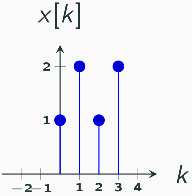

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]\, h[-k+1]$$

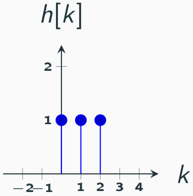
$$= 0 \times 1 + 1 \times 1 + 2 \times 1 + 1 \times 0 + 2 \times 0 = 3$$

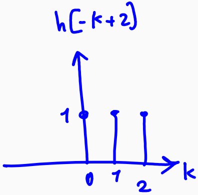
$$y[1] = 3$$

# convolution example

**Example 1.** Find $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$



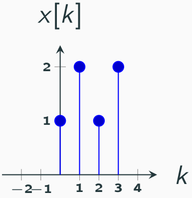

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]\, h[-k+2] = \sum_{k=-\infty}^{+\infty} x[k]\, h[2-k]$$

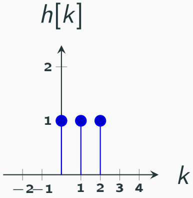
$$= 1 \times 1 + 2 \times 1 + 1 \times 1 + 2 \times 0 = 4$$

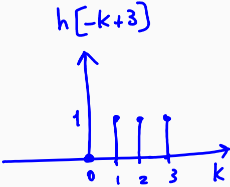
$$y[2] = 4$$

# convolution example

**Example 1.** Find $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$



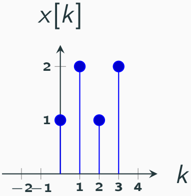

$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]\,h[-k+3]$$

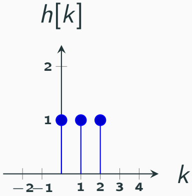
$$= 1\times 0 + 2\times 1 + 1\times 1 + 2\times 1 = 5$$

$$y[3] = 5$$

# convolution example

**Example 1.** Find $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

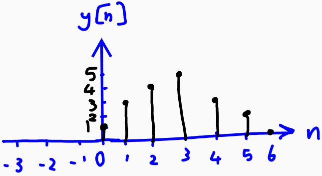



$y[4] = 3$

$y[5] = 2$

$y[6] = 0$



# convolution example

**Example 2.** Find $y[n] = x[n] * h[n]$ for $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$, where $0 < |\beta| < 1$ and $0 < |\alpha| < 1$, and $\alpha \neq \beta$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{+\infty} \alpha^k u[k]\, \underbrace{\beta^{n-k} u[n-k]}_{h[n-k]}$$

$$= \sum_{k=0}^{\infty} \alpha^k \times 1 \times \beta^{n-k} u[n-k]$$

$$u[n-k] = \begin{cases} 1 & n-k \geq 0 \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & k \leq n \\ 0 & k > n \end{cases}$$

$$\boxed{y[n] = \sum_{k=0}^{n} \alpha^k \times 1 \times \beta^{n-k} \times 1 \quad \text{for } n \geq 0}$$

$$\boxed{y[n] = 0 \quad \text{for } n < 0}$$

# convolution example

**Example 2.** Find $y[n] = x[n] * h[n]$ for $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$, where $0 < |\beta| < 1$ and $0 < |\alpha| < 1$, and $\alpha \neq \beta$.

$$\beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k = \beta^n \sum_{m=0}^{n} \left(\frac{\alpha}{\beta}\right)^m = \beta^n \left( \frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1}}{1 - \frac{\alpha}{\beta}} \right)$$

**Example 2.** Find $y[n] = x[n] * h[n]$ for $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$, where $0 < |\beta| < 1$ and $0 < |\alpha| < 1$, and $\alpha \neq \beta$.

$$y[n] = \begin{cases} \sum_{k=0}^{n} \alpha^k \beta^n \beta^{-k} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$y[n] = \begin{cases} \beta^n \boxed{\sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$y[n] = \begin{cases} \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$, \boxed{y[n] = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]}$$

# convolution example

**Example 2.** Find $y[n] = x[n] * h[n]$ for $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$, where $0 < |\beta| < 1$ and $0 < |\alpha| < 1$, and $\alpha \neq \beta$.

$$\beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k = \beta^n \sum_{m=0}^{n} \left(\frac{\alpha}{\beta}\right)^m = \beta^n \left( \frac{1 - \left(\frac{\alpha}{\beta}\right)^{n+1}}{1 - \frac{\alpha}{\beta}} \right)$$

# Properties of Convolution

# Properties of Convolution

- ***Commutative***:

  For any signals $x(t)$ and $h(t)$, $x(t) * h(t) = h(t) * x(t)$.

  For any signals $x[n]$ and $h[n]$, $x[n] * h[n] = h[n] * x[n]$.

- Proof DT:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \overset{(n-k)\to\ell}{=} \sum_{\ell=-\infty}^{\infty} x[n-\ell]h[\ell] = h[n] * x[n]$$

- Proof of CT is similar.

# Properties of Convolution

- ***Associative***: For any three signals $x[n]$, $h_1[n]$, and $h_2[n]$:

$$\Big(x[n] * h_1[n]\Big) * h_2[n] = x[n] * \Big(h_1[n] * h_2[n]\Big)$$

and similarly:

$$\Big(x(t) * h_1(t)\Big) * h_2(t) = x(t) * \Big(h_1(t) * h_2(t)\Big)$$

.

- ***Distributive***: For any three signals $x[n]$, $h_1[n]$, and $h_2[n]$:

$$x[n] * \Big(h_1[n] + h_2[n]\Big) = x[n] * h_1[n] + x[n] * h_2[n]$$

and similarly:

$$x(t) * \Big(h_1(t) + h_2(t)\Big) = x(t) * h_1(t) + x(t) * h_2(t)$$

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