Lecture 9

Fourier series example and convergence

Preview of today's lecture

- Brief review, FS and orthogonality
- Rectangular pulse train
- Convergence and Gibb's phenomena

Explaining the FS domain

- Consider a periodic signal with period T=4 and FS coefficients

$$a_0 = 1 \qquad a_1 = a_{-1} = \frac{1}{2} \qquad a_2 = a^*_{-2} = \frac{j}{2}$$

- The fundamental frequency of the periodic signal is

$$\begin{aligned}\omega_0 &= \frac{2\pi}{T} \\ &= \frac{\pi}{2}\end{aligned}$$

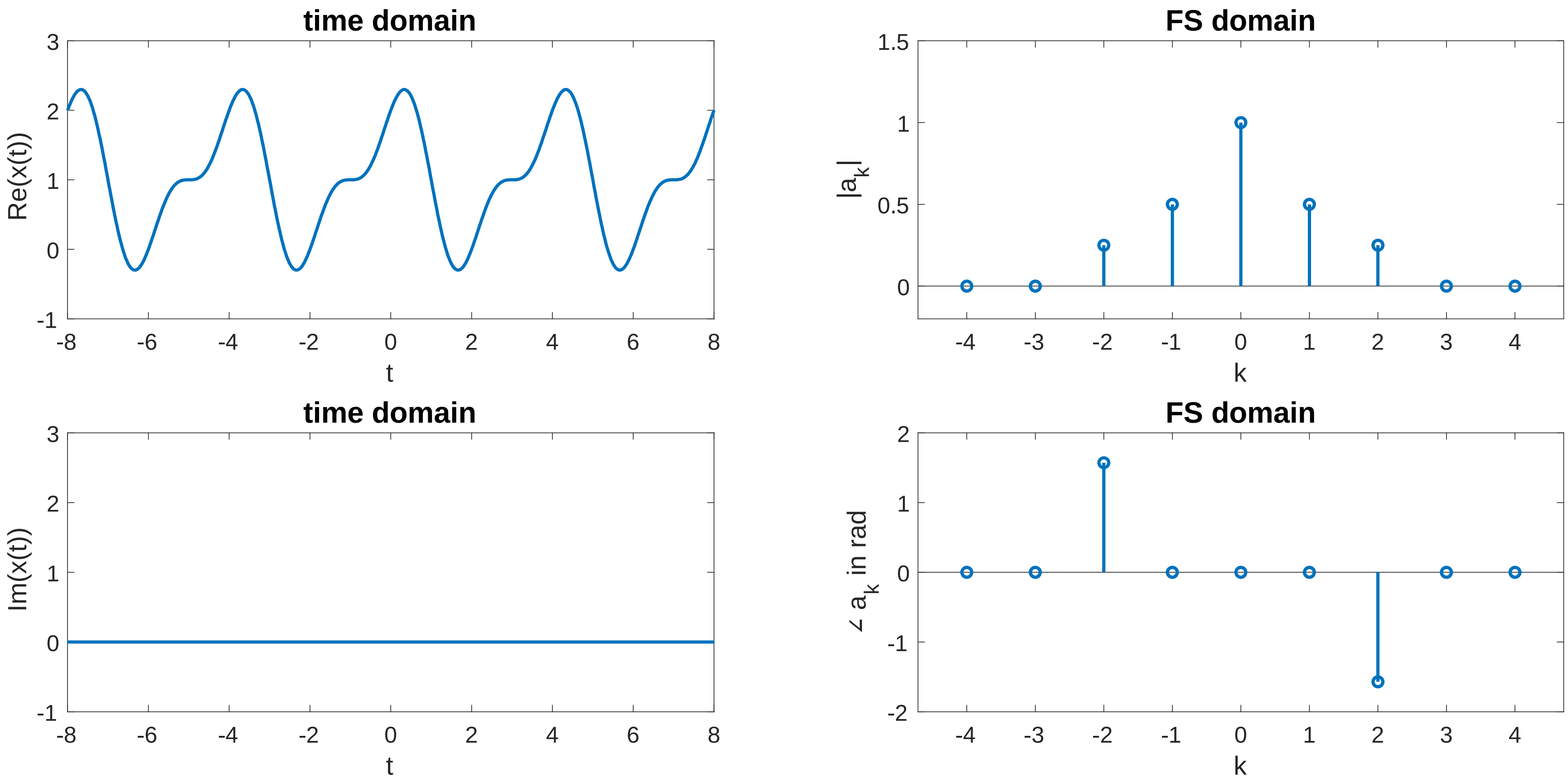
Explaining the FS domain

- The time domain signal is found from the synthesis equation

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\
&= 1 + \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \frac{1}{2}\frac{1}{2j}\left(e^{j2\omega_0 t} - e^{-j2\omega_0 t}\right) \\
&= 1 + \cos(\omega_0 t) + \frac{1}{2}\sin(2\omega_0 t)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_0 &= \frac{2\pi}{T} \\
&= \frac{\pi}{2}
\end{aligned}
$$


time domain
Re(x(t))
t
FS domain
$|a_k|$
k
time domain
Im(x(t))
t
FS domain
$\angle a_k$ in rad
k

Calculations

- In the time domain

- In the FS domain

$$|a_0| = 1 \qquad \angle a_0 = \angle a_1 = \angle a_{-1} = 0$$

$$|a_1| = |a_{-1}| = \frac{1}{2} \qquad \angle a_1 = \angle a_{-1} = 0$$

$$\angle a_2 = -\pi/2$$

$$|a_2| = |a_{-2}| = \frac{1}{4} \qquad \angle a_{-2} = \pi/2$$

Fourier series analysis and synthesis

FS **domain**

$\{a_k\}$ → **Fourier synthesis** →

time **domain**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

putting the frequencies together to build the time domain signal

---

time **domain**

$x(t)$ → **Fourier analysis** →

FS **domain**

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

figuring out what frequencies are used to build the time domain signal

Why does the analysis equation work?

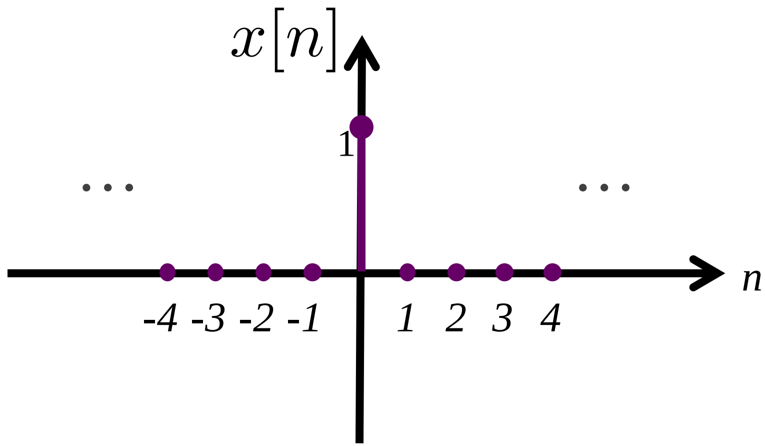
$$
\begin{aligned}
a_n &= \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt \\
&= \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} \int_0^T a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt
\end{aligned}
$$

Why does the analysis equation work?

$$
\begin{aligned}
a_n &= \sum_{k=-\infty}^{\infty} a_k \frac{1}{T} \int_0^T e^{j(k-n)\omega_0 t} dt \\
&= \begin{cases} 0 & k \neq n \\ T & k = n \end{cases} \\
&= T\delta[k-n]
\end{aligned}
$$

Discrete-time Kronecker delta function

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Also known as the unit-impulse function

note

$$\sum_{n=-\infty}^{\infty} \delta[n] = 1$$

Orthogonality of complex sinusoids

- Consider the following periodic signals

$$x(t) = e^{jk\omega_0} \qquad y(t) = e^{jn\omega_0}$$

- Compute the inner product between these two signals

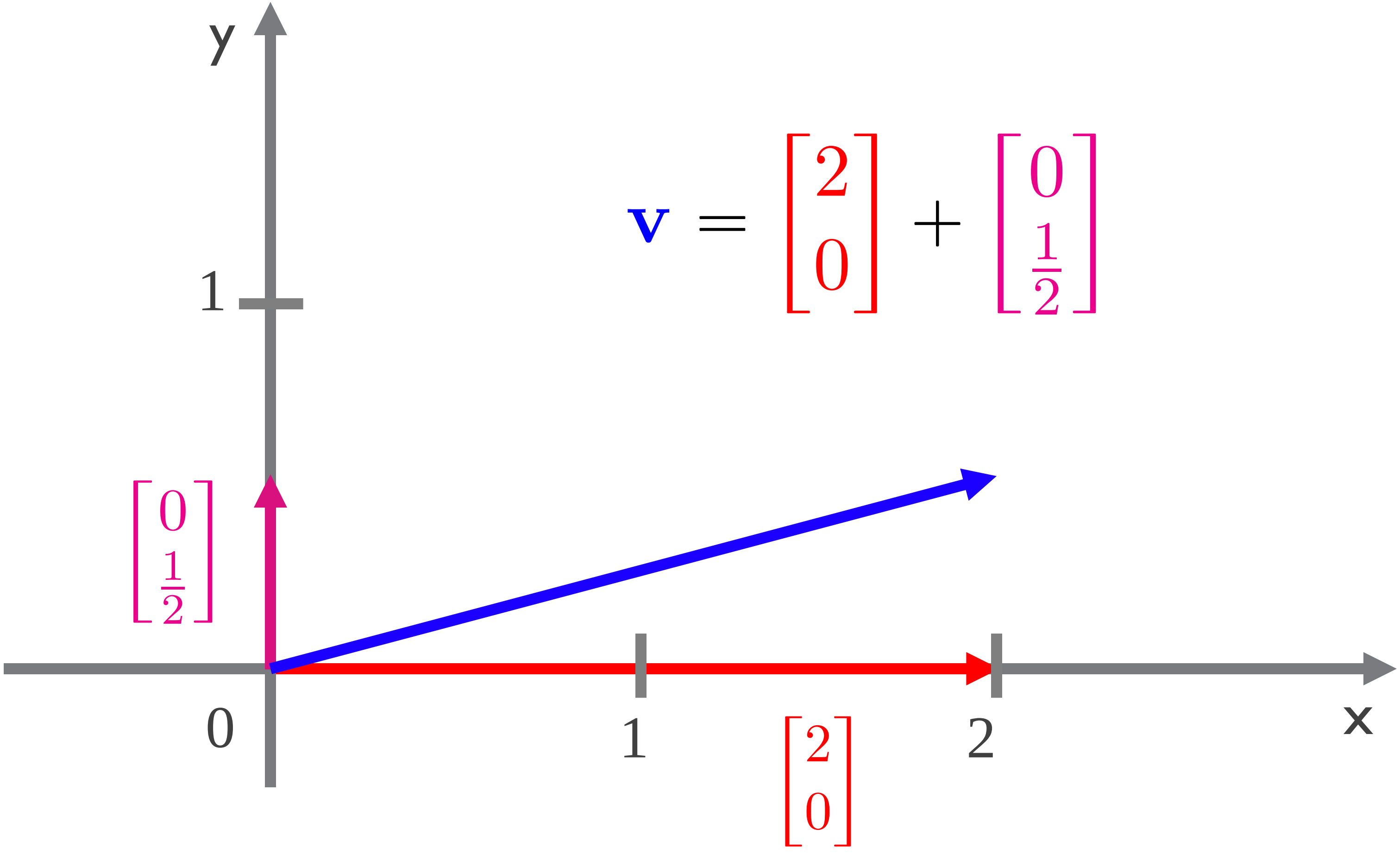
$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos((k-n)\omega_0 t) + j\sin((k-n)\omega_0 t) dt$$

$$= \begin{cases} T & k = n \\ 0 & k \neq n \end{cases} \Rightarrow \quad T\delta[k-n]$$

Complex sinusoids comprised of different harmonics of the fundamental frequency are orthogonal

Connections to linear algebra

$$\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

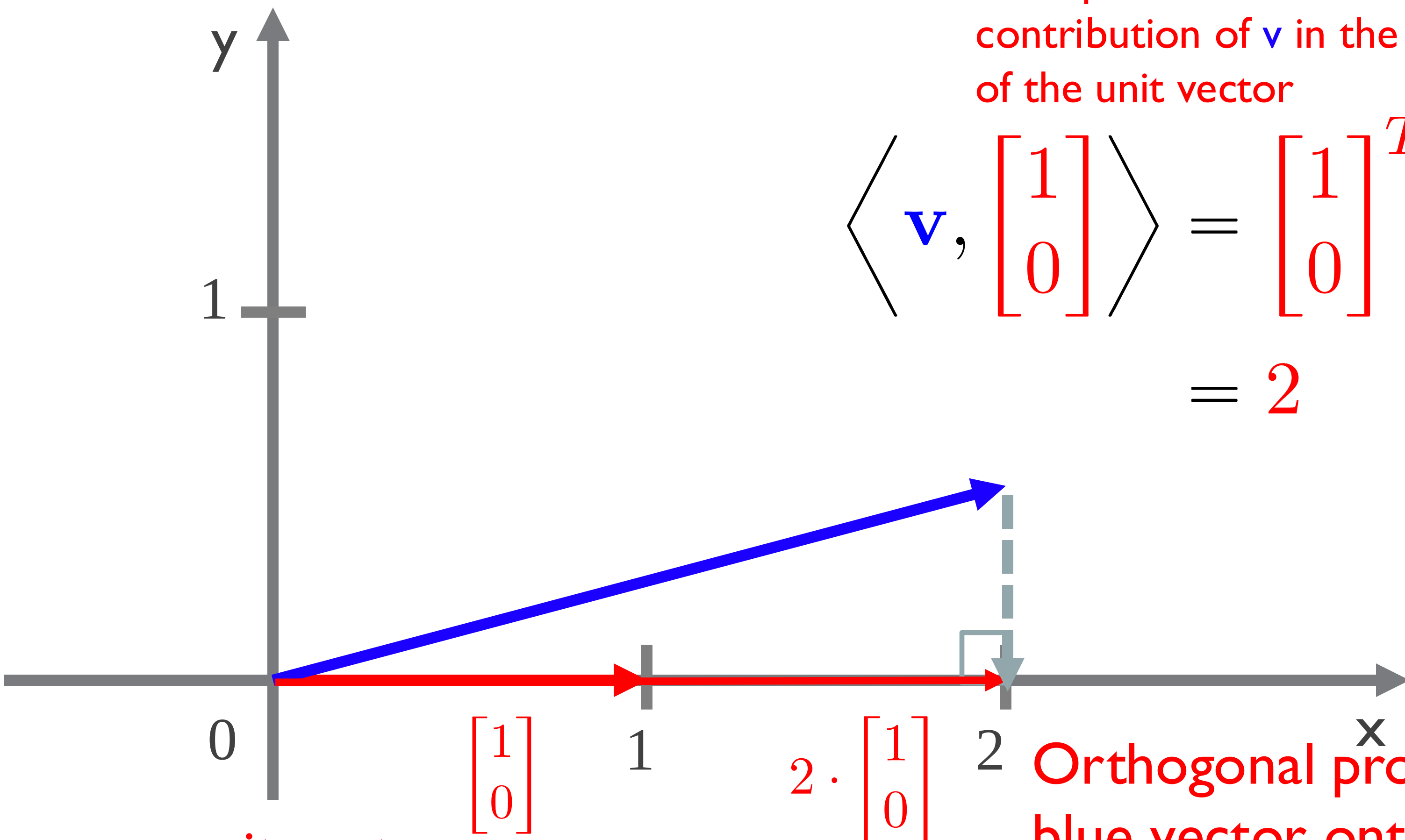


Connections to linear algebra

inner product measures the contribution of $\mathbf{v}$ in the direction of the unit vector

$$\left\langle \mathbf{v}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \mathbf{v}$$

$$= 2$$

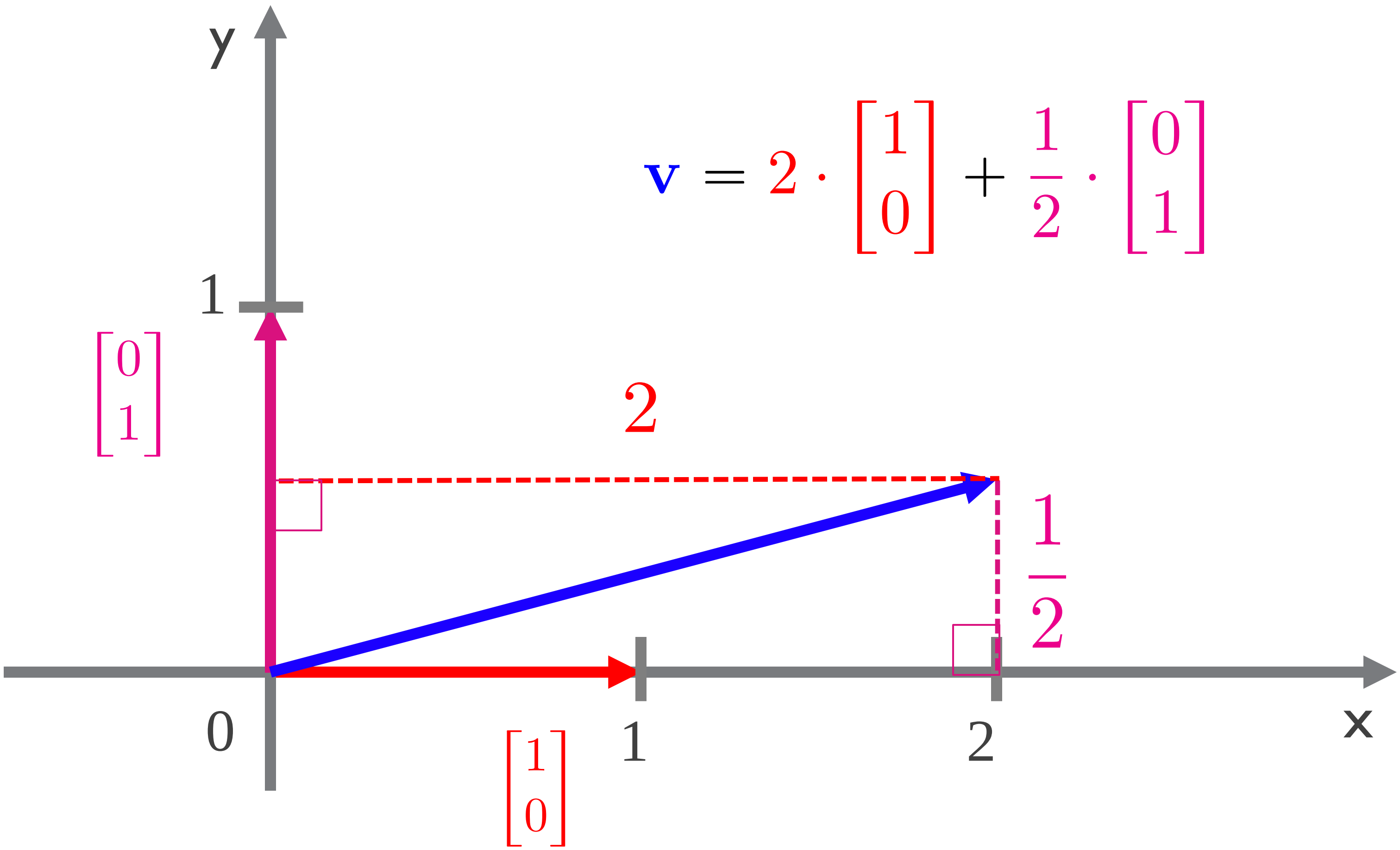


unit vector

Orthogonal projection of blue vector onto the x-axis

Connections to linear algebra

$$\mathbf{v} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Connecting the two pictures

coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad\qquad \mathbf{v} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

orthogonal components

periodic signal

2x1 vector

What do you need to know?

- BUILD A SIGNAL FROM THE FS COEFFICIENTS
- FIND THE FS COEFFICIENTS OF A PERIODIC SIGNAL
- KNOW THE FS EXPANSIONS OF SOME COMMON PERIODIC SIGNALS
- MAKE USE OF THE PROPERTIES TO SIMPLIFY THE CALCULATIONS

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

Fourier coefficients of a rectangular pulse train

Learning objectives

- Find the Fourier series coefficients of a classic example
- Use the results of this derivation in future lectures

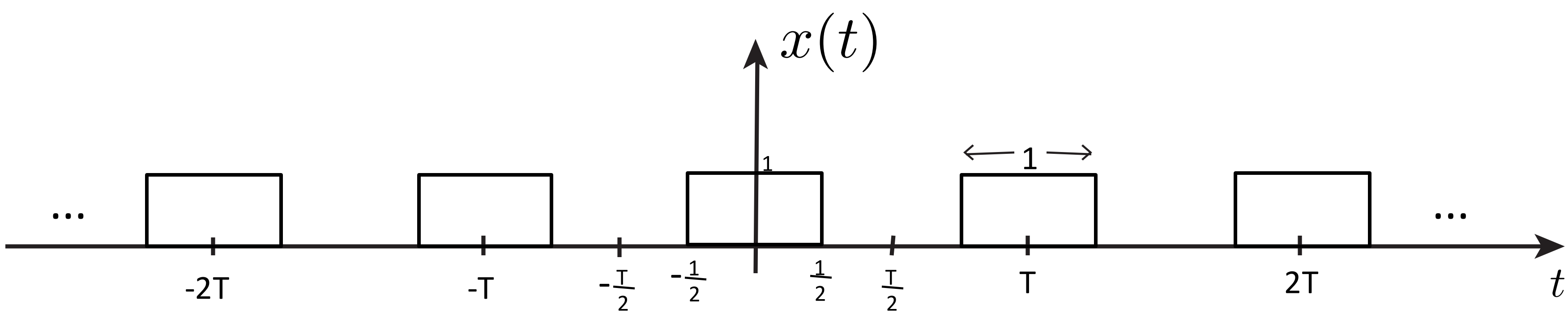
Pulse train I

This is an important reference example. It may not be covered in class. We will use the general result though in other example problems as this is an interesting and relevant signal used in circuits.

- Find the Fourier series coefficients of the unit pulse train

$$x(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \frac{1}{2} < |t| < \frac{T}{2} \end{cases}$$

and is repeated every T



Pulse train 2

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt, \quad \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-jk\omega_0 t}dt$$

$$= \frac{-1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{jk\omega_0 T}\left(e^{\frac{-jk\omega_0}{2}} - e^{\frac{jk\omega_0}{2}}\right)$$

$$= \frac{2}{k\omega_0 T}\frac{1}{2j}\left(e^{\frac{jk\omega_0}{2}} - e^{-\frac{jk\omega_0}{2}}\right)$$

Pulse train 3

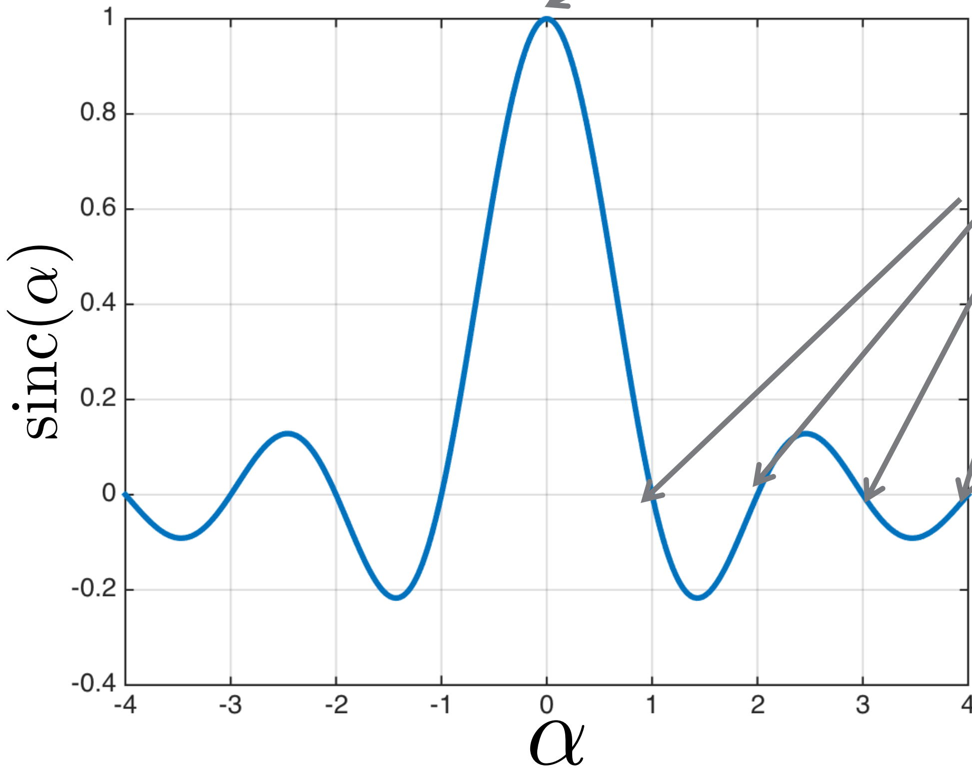
$$
\begin{aligned}
a_k &= \frac{1}{T}\frac{\sin\left(\frac{k\omega_0}{2}\right)}{\frac{k\omega_0}{2}} \\
&= \frac{1}{T}\frac{\sin\left(\pi\frac{k\omega_0}{2\pi}\right)}{\pi\frac{k\omega_0}{2\pi}} \\
&= \frac{1}{T}\operatorname{sinc}\left(\frac{k\omega_0}{2\pi}\right)
\end{aligned}
$$

We define the sinc as

$$
\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}
$$

Pulse train 4

Maximum value of 1, i.e. $\mathrm{sinc}(0) = 1$



Zero crossings at +/-1, +/- 2, ….

$$\mathrm{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

Be aware, sometimes sinc is defined like this

$$\mathrm{sinc}(\alpha) = \frac{\sin(\alpha)}{\alpha}$$

Pulse train 5

- What about k = 0?

$$a_0 = \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{x(t)}_{1}\, dt = \frac{1}{T}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{T}$$

- As an side, for the sinc function

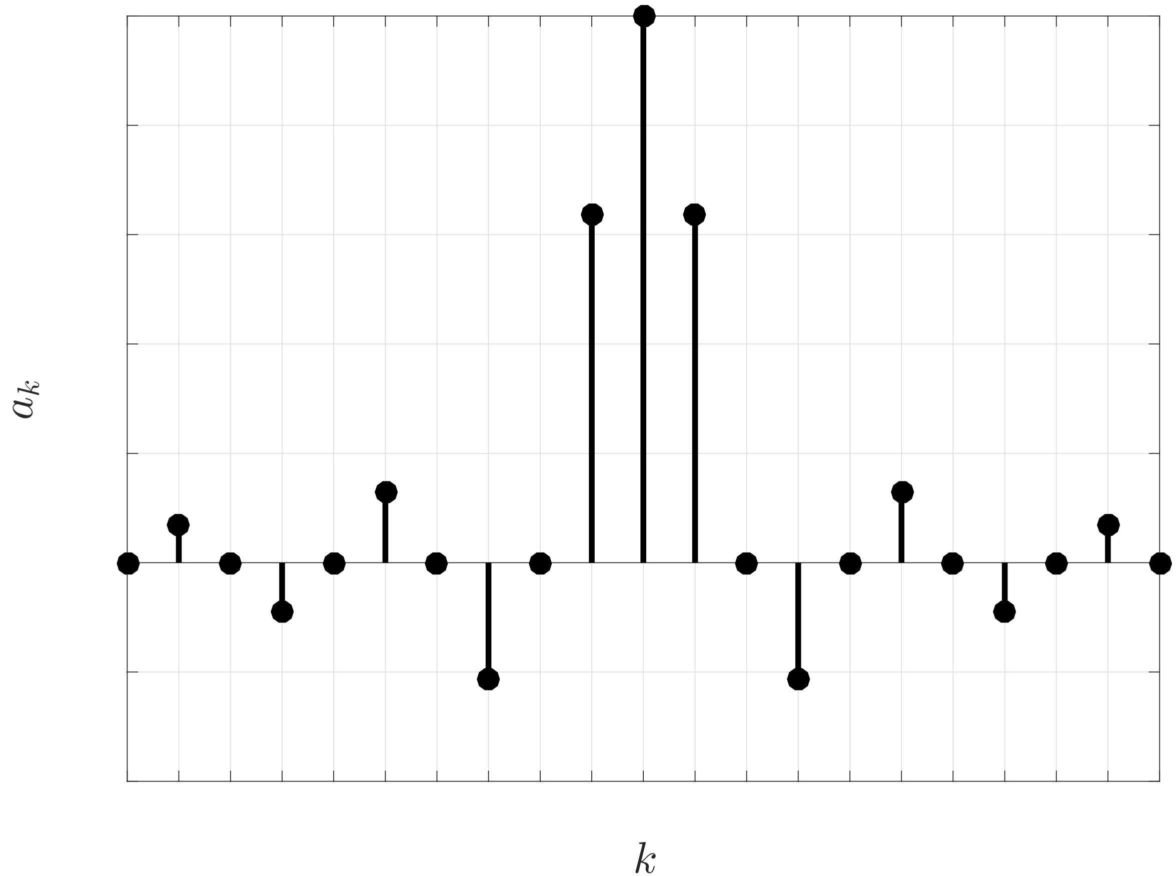
$$\lim_{t\to\infty} \frac{\sin(\pi t)}{\pi t} = \lim_{t\to\infty} \frac{\pi\cos(\pi t)}{\pi} = 1$$

- Therefore the following holds for all values of k

$$a_k = \frac{1}{T}\mathrm{sinc}\left(\frac{k\omega_0}{2\pi}\right) = \frac{\omega_0}{2\pi}\mathrm{sinc}\left(\frac{k\omega_0}{2\pi}\right)$$

Pulse train 6

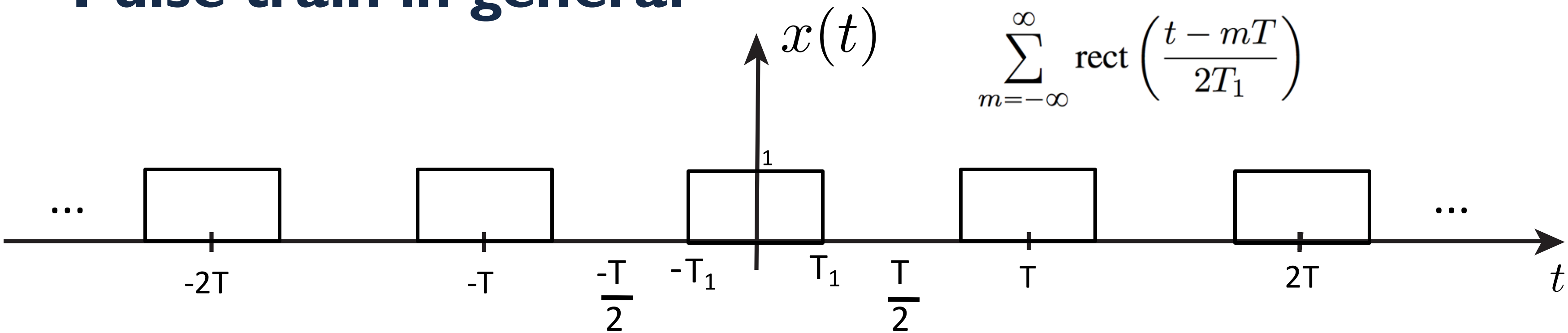
Example with T=2



Pulse train in general

From O&W Example 3.5

$$\sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{t-mT}{2T_1}\right)$$



From the book

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \qquad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$a_k = \frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

with fundamental frequency

$$a_k = \frac{2T_1}{T} \text{sinc}\left(\frac{k2T_1}{T}\right)$$

simplified

Sufficient conditions for a periodic signal to have a Fourier series representation

Learning objectives

- Understand the Gibbs phenomena
- Determine whether a periodic signal satisfy Dirichlet conditions

The issues in a nutshell

- If we approximate the Fourier Series with a finite number of terms, is that a good approximation of the original signals?

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \xrightarrow{?} x(t)$$

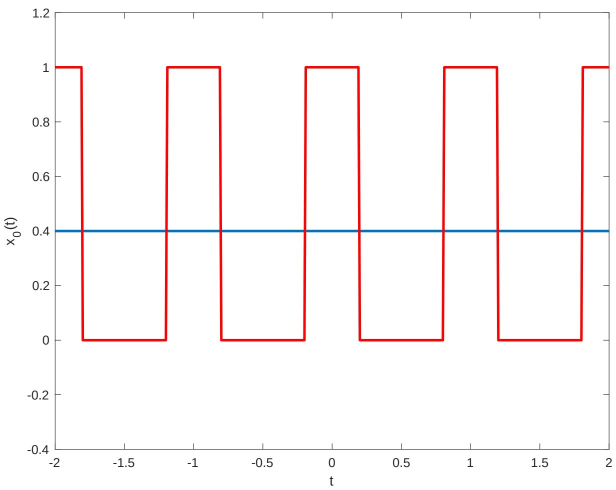
- What does it mean for a signal to have a Fourier Series representation?

Synthesizing a square wave

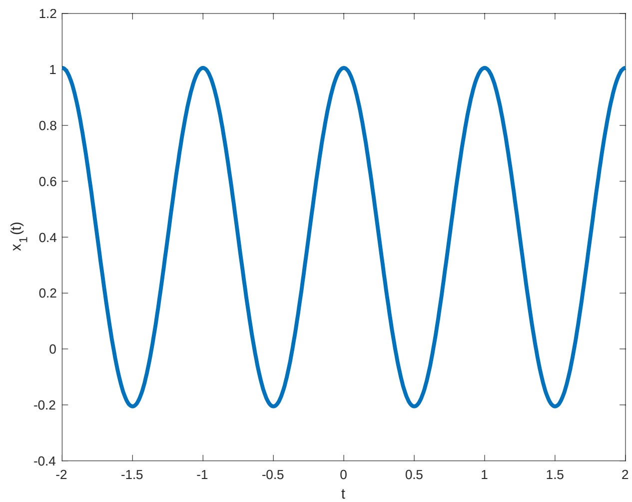
$$x(t) = \frac{T_1}{T} + 2\sum_{k=0}^{\infty} \frac{\sin(k\omega_0 T_1/2)}{k\pi}\cos(k\omega_0 t)$$

Suppose we synthesize a square wave (T_1=0.4T) with a finite number of terms

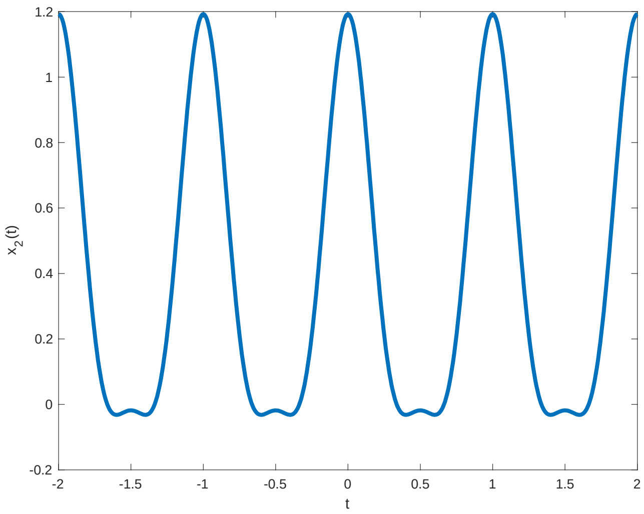
$$\boxed{x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}}$$



DC



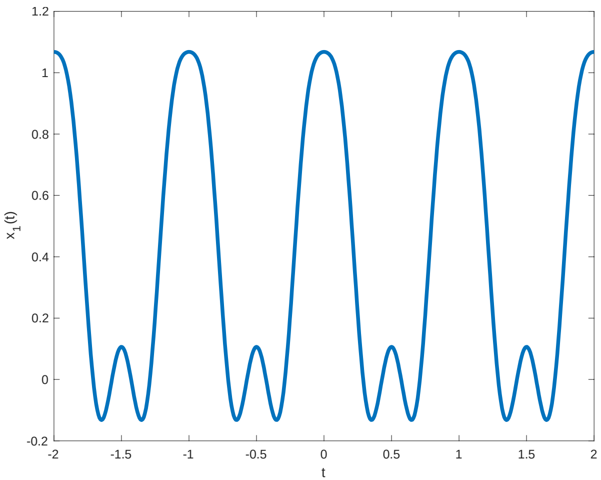
N=1 (k is -1,0,1)



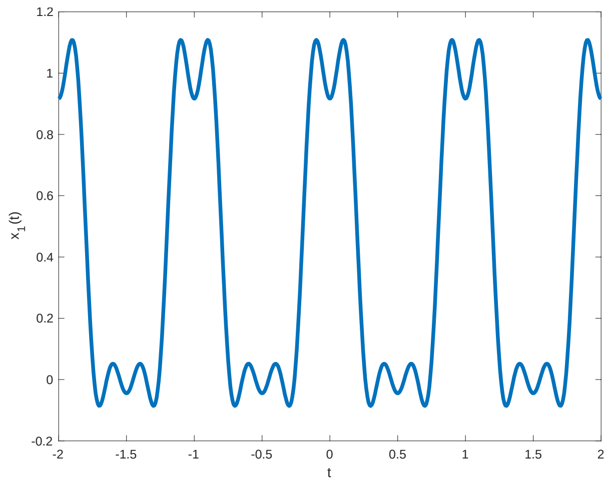
N=2 (k is -2,…,2)

Adding more terms

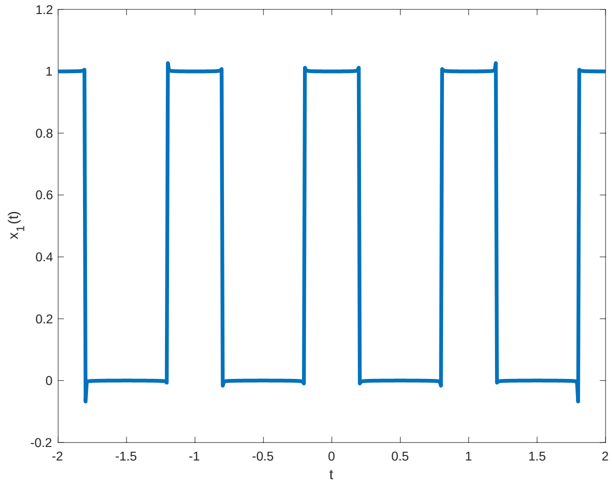
$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$



N= 3 (k is -3,…,3)

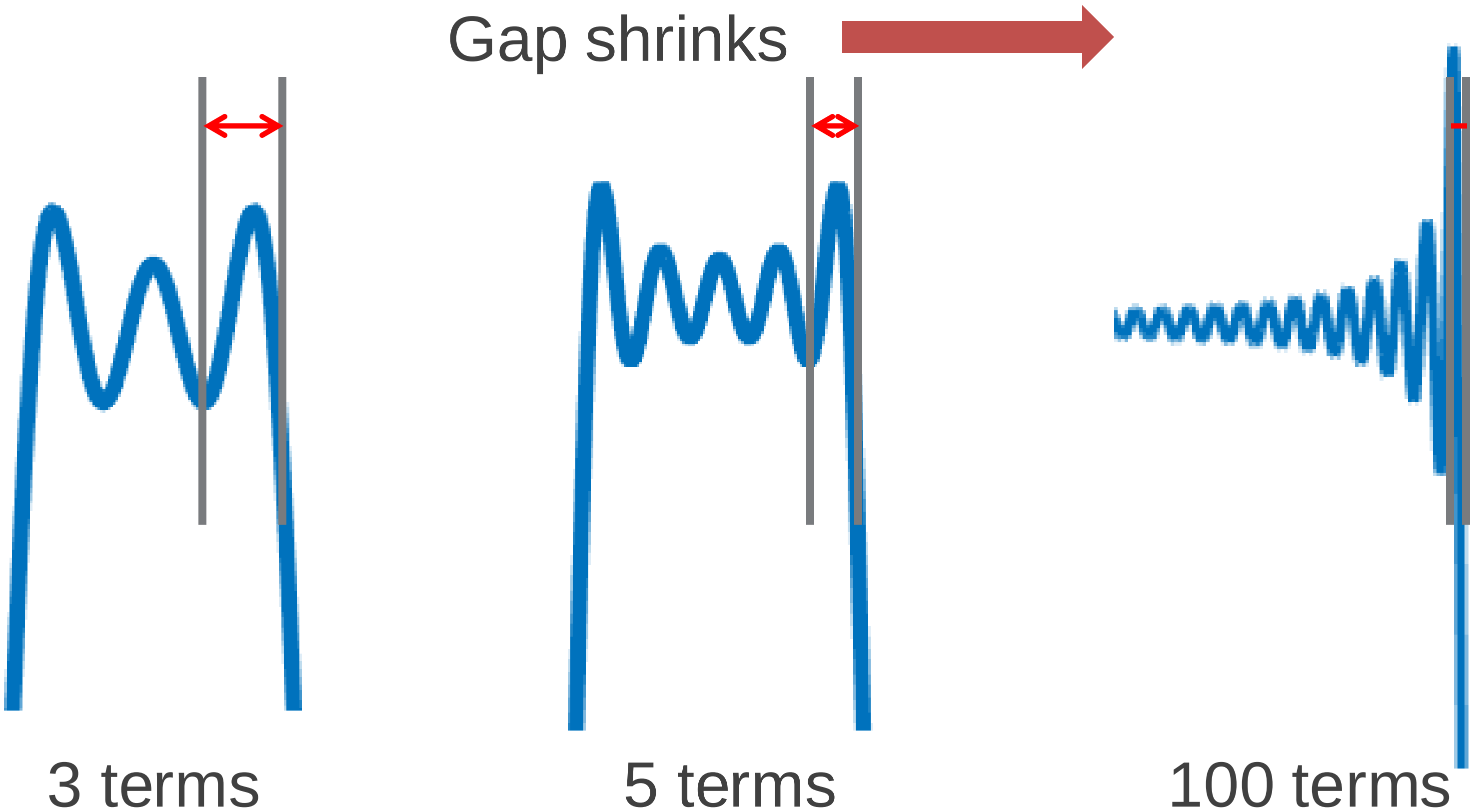


N=4 (k is -4,…,4)



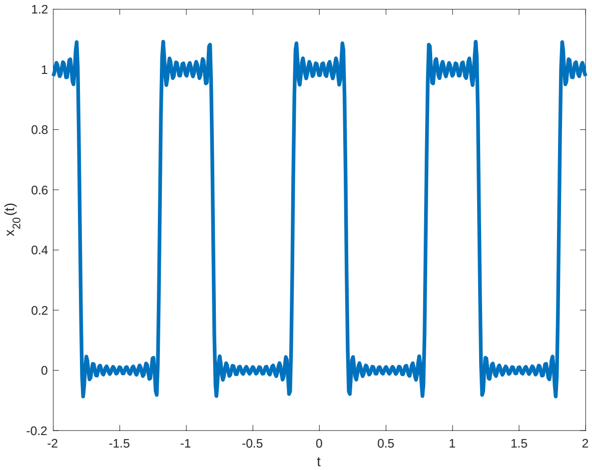
N=1000
(k is -1000,…,1000)

Zooming in on the ripple

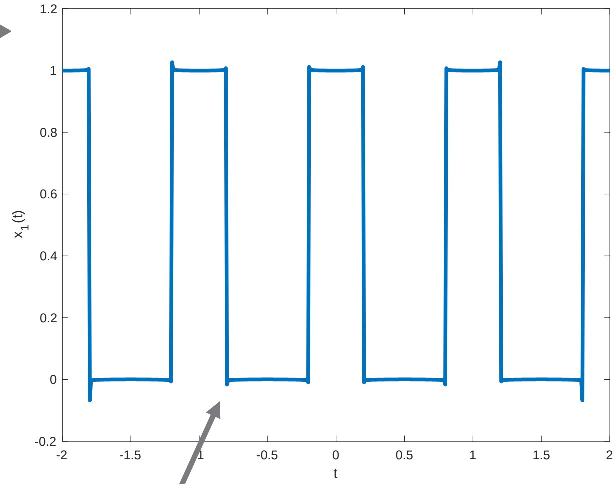


Gap goes to zero for very large number of terms

Gibbs phenomena



Ripple becomes smaller



The overshoot at the point of discontinuity is known as Gibbs phenomena

UC San Diego

Sufficient condition to have a Fourier series

A periodic signal x(t) that satisfies the Dirichlet ("Diri-klay") conditions

(1) Absolute integrability

(2) Finite number of minima and maxima for a given time period

(3) Finite number of discontinuities for a period T

has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where equality holds for all t except possibly at the points of discontinuity

Gibbs phenomena occurs at those points

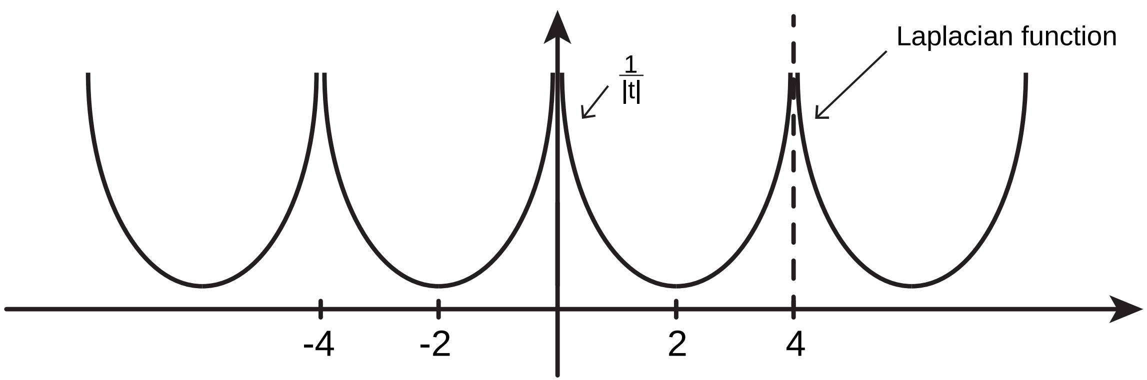
http://www.dictionary.com/browse/dirichlet

#1: Absolute integrability

$$\int_T |x(t)|dt < \infty$$

- Example of violation

$$x(t) = \begin{cases} \frac{1}{|t|}, & t \in (-2, 2) \\ \text{repeat for all T} \end{cases}$$

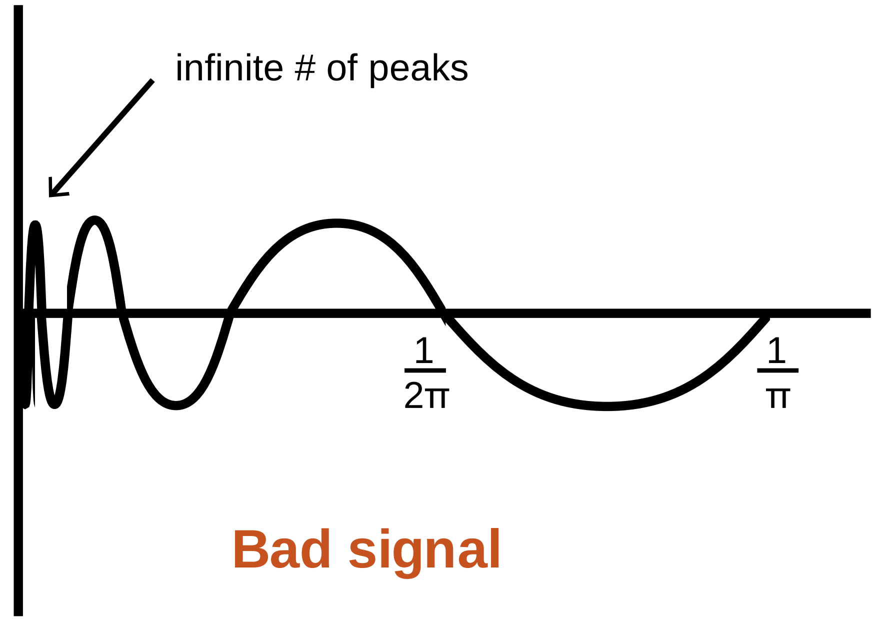


Bad signal

#2: Finite number of min and max for a given period

- Example of violation (just one period shown)

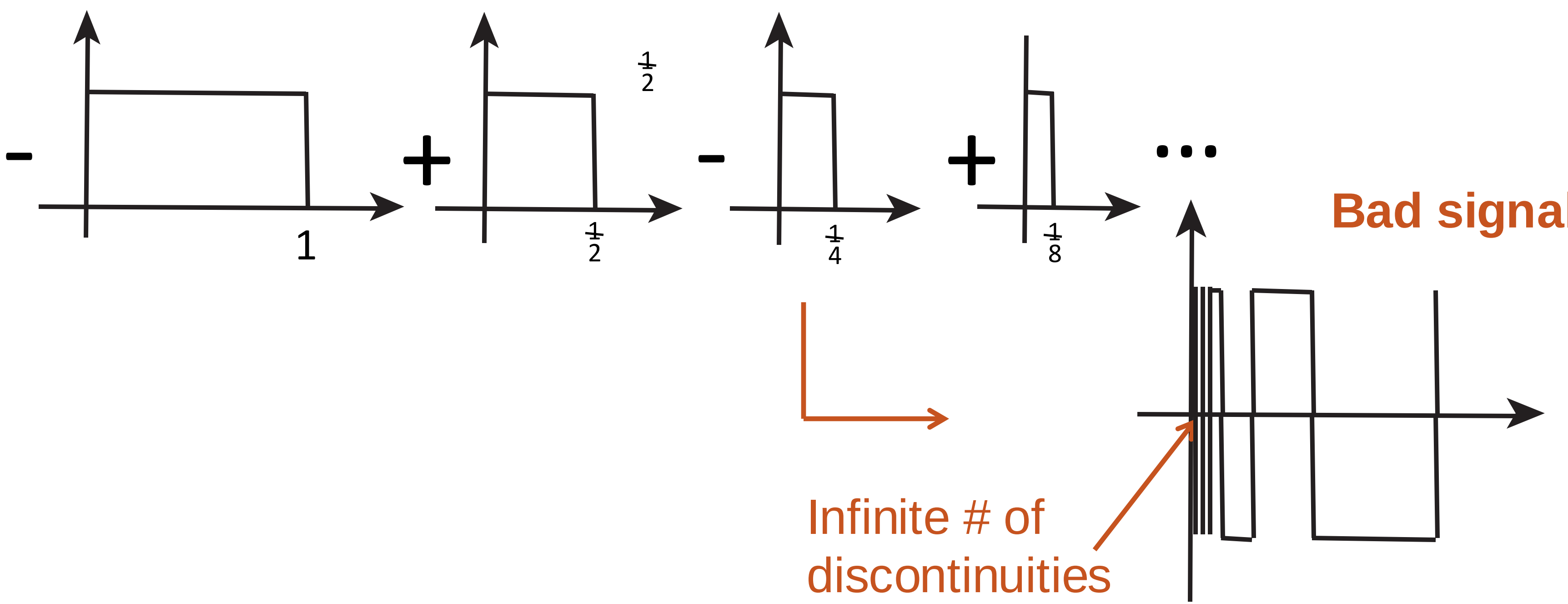
$$x(t) = \begin{cases} \sin\left(\frac{1}{t}\right), & t \in \left(0, \frac{1}{\square}\right) \\ \text{repeat every } \frac{1}{\square} \text{ secs} & \end{cases}$$



#3: Finite number of discontinuities for a period T

- Example of violating signal (shown and defined over one period)

$$x(t) = \sum_{k=0}^{\infty} \left[u\left(t - \frac{1}{2^k}\right) - u(t) \right] (-1)^k$$



Summary of Fourier series conditions

- Not every periodic signal has a Fourier series representation but…

- A large class of signals do have such representations if they satisfy the Dirichlet conditions
  - Fortunately this includes all practical signals (e.g. can not create signals with an infinite number of points of discontinuity)

- The Fourier series expansion of a signal may have a few points where equality is not satisfied, known as Gibbs phenomena

Working with the Fourier series

Learning objectives

- Compute the output of an LTI system to a periodic input
- Use the Fourier series coefficients of these signals in other problems

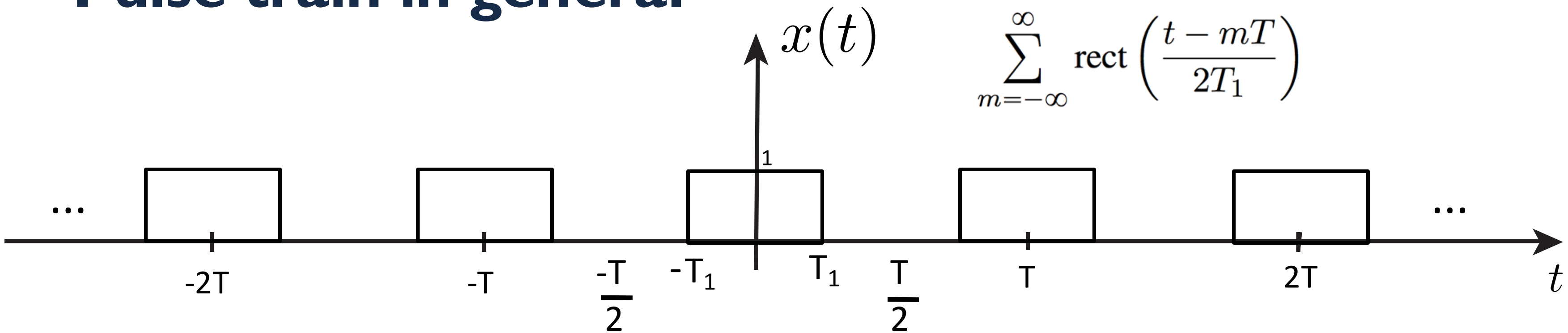
Basic signals

	Time domain *x(t)*	Fourier coefficients a_k
Constant (periodic for any T)	c	$c\delta[k]$
Cosine	$\cos(\omega_0 t)$	$\frac{1}{2}\delta[k-1]+\frac{1}{2}\delta[k+1]$
Sine	$\sin(\omega_0 t)$	$\frac{1}{2j}\delta[k-1]-\frac{1}{2j}\delta[k+1]$
Impulse train	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}$

Pulse train in general

From O&W Example 3.5

$$\sum_{m=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-mT}{2T_1}\right)$$



From the book

$$a_k = \frac{\sin\left(\pi k \frac{2T_1}{T}\right)}{k\pi} \qquad k \neq 0$$

$$a_0 = \frac{2T_1}{T}$$

Rewritten using the sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$a_k = \frac{\omega_0 T_1}{\pi} \mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

with fundamental frequency

$$a_k = \frac{2T_1}{T} \mathrm{sinc}\left(\frac{k2T_1}{T}\right)$$

simplified

Output of an LTI system

$$\int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}$$

$$e^{st} \xrightarrow{\text{LTI System}} \overbrace{H(s)}e^{st}$$

$$\alpha e^{s_1 t} + \beta e^{s_2 t} \xrightarrow{\text{LTI System}} \alpha H(s_1)e^{s_1 t} + \beta H(s_2)e^{s_2 t}$$

New FS coefficients

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \xrightarrow{\text{LTI System}} \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k)e^{j\omega_0 kt}$$

Fourier series coefficients are modified by the frequency response of the system