Problem Set #1: Basic Properties of Signals

Signal Power / Energy

1.3. Determine the values of P_∞ and E_∞ for each of the following signals:

(a) $x_1(t) = e^{-2t}u(t)$ **(b)** $x_2(t) = e^{j(2t+\pi/4)}$ **(c)** $x_3(t) = \cos(t)$

(d) $x_1[n] = (\frac{1}{2})^n u[n]$ **(e)** $x_2[n] = e^{j(\pi/2n+\pi/8)}$ **(f)** $x_3[n] = \cos(\frac{\pi}{4}n)$

1.3(c): $x_3(t) = \cos(t)$

$$E_\infty = \int_{-\infty}^{\infty} \cos^2(t)\,dt = \infty$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(t)\,dt = \frac{1}{2}$$

1.3(e): $x_2[n] = e^{j\left(\frac{\pi}{2}n+\frac{\pi}{8}\right)}$

$$E_\infty = \sum_{n=-\infty}^{\infty} |x_2[n]|^2 = \infty$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x_2[n]|^2 = 1$$

1.13. Consider the continuous-time signal

$$x(t) = \delta(t+2) - \delta(t-2).$$

Calculate the value of E_∞ for the signal

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

1.13: $x(t) = \delta(t+2) - \delta(t-2), \quad y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = u(t+2) - u(t-2)$$

$$y(t) = \begin{cases} 1, & -2 \le t < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-2}^{2} 1\,dt = \boxed{4}$$

Signal Transformations

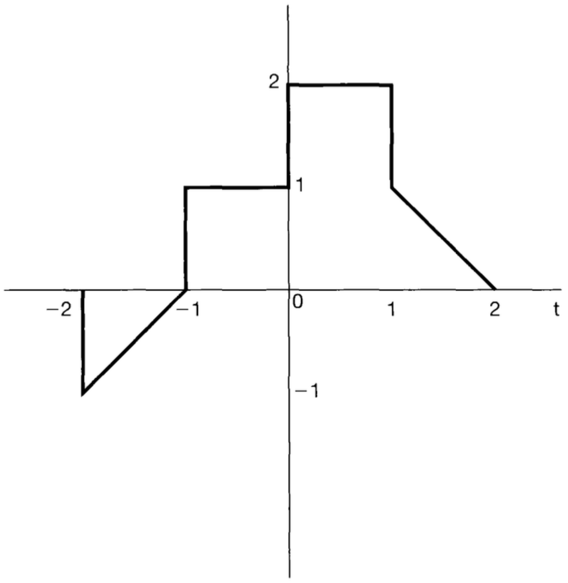
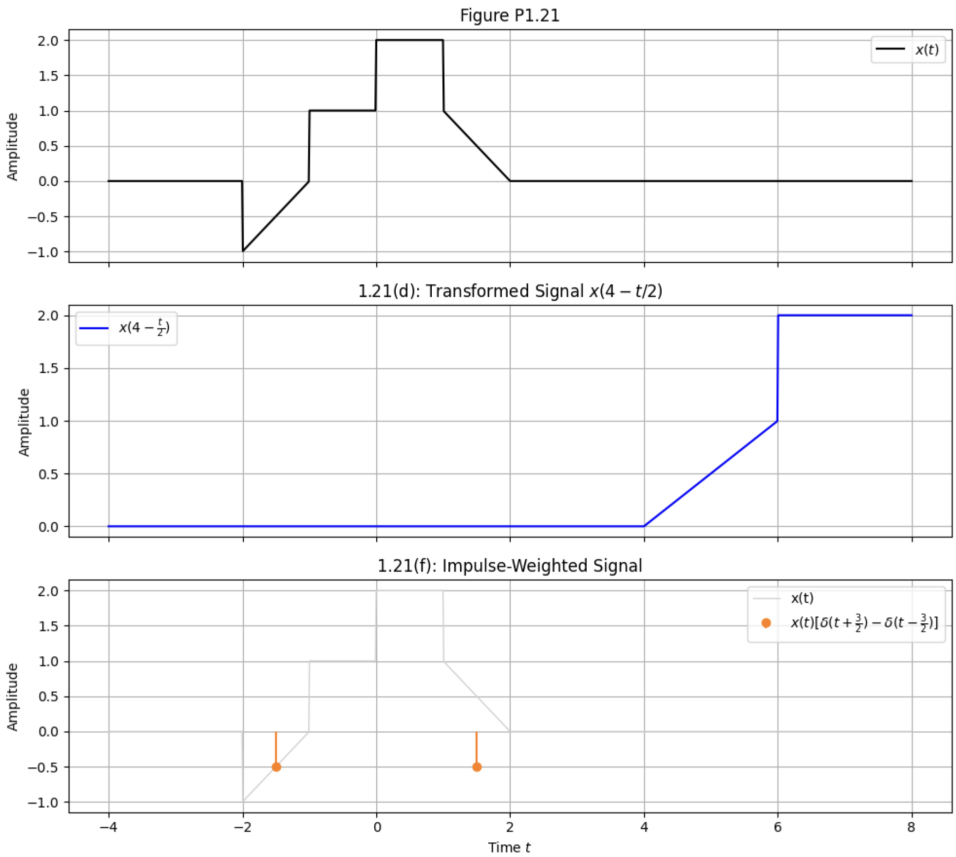


Figure P1.21

1.21. A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals:

(a) $x(t-1)$ **(b)** $x(2-t)$ **(c)** $x(2t+1)$
(d) $x(4-\frac{t}{2})$ **(e)** $[x(t)+x(-t)]u(t)$ **(f)** $x(t)[\delta(t+\frac{3}{2})-\delta(t-\frac{3}{2})]$

1.21(d): $x\left(4-\frac{t}{2}\right)$
1.21(f): $x(t)\left[\delta\left(t+\frac{3}{2}\right)-\delta\left(t-\frac{3}{2}\right)\right]$



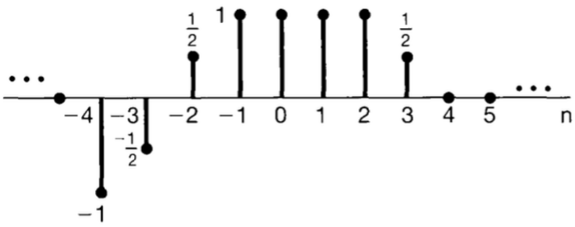
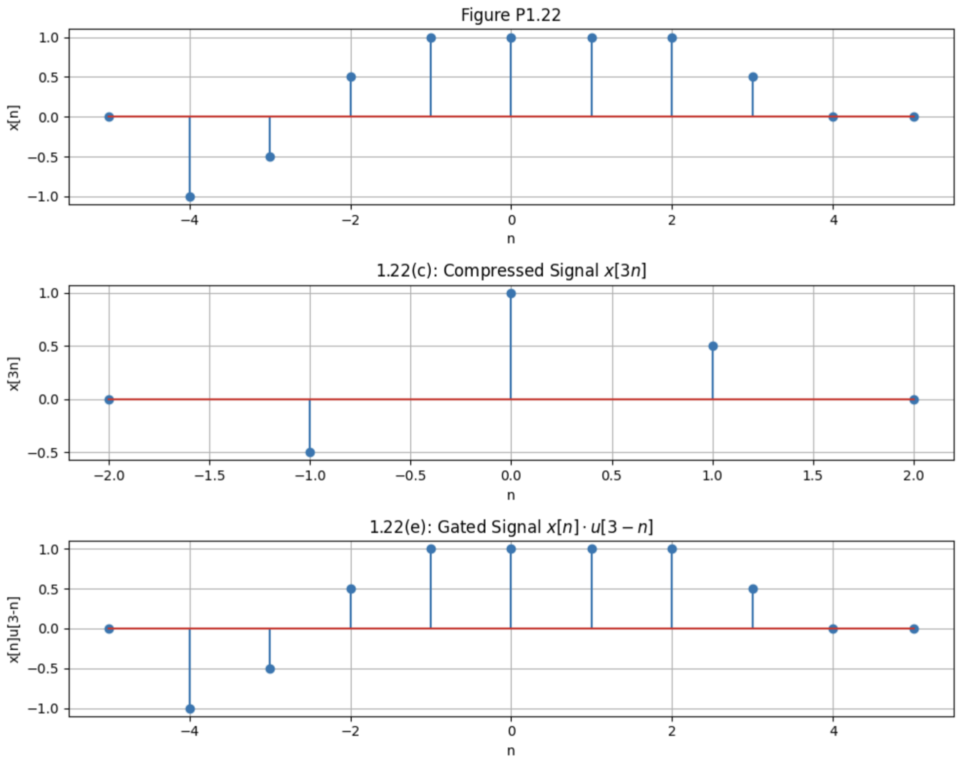


Figure P1.22

1.22. A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals:

(a) $x[n-4]$ **(b)** $x[3-n]$ **(c)** $x[3n]$
(d) $x[3n+1]$ **(e)** $x[n]u[3-n]$ **(f)** $x[n-2]\delta[n-2]$
(g) $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]$ **(h)** $x[(n-1)^2]$

1.22(c): $x[3n]$
1.22(e): $x[n] \cdot u[3-n]$



Even/Odd Decomposition

1.23. Determine and sketch the even and odd parts of the signals depicted in Figure P1.23. Label your sketches carefully.

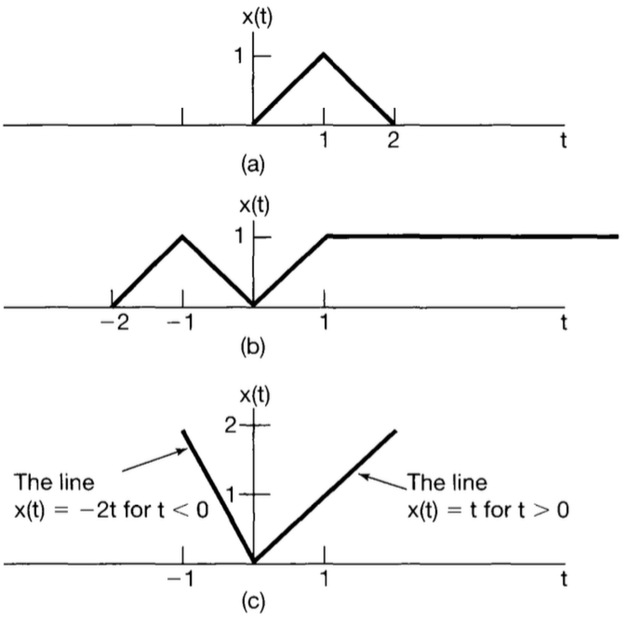
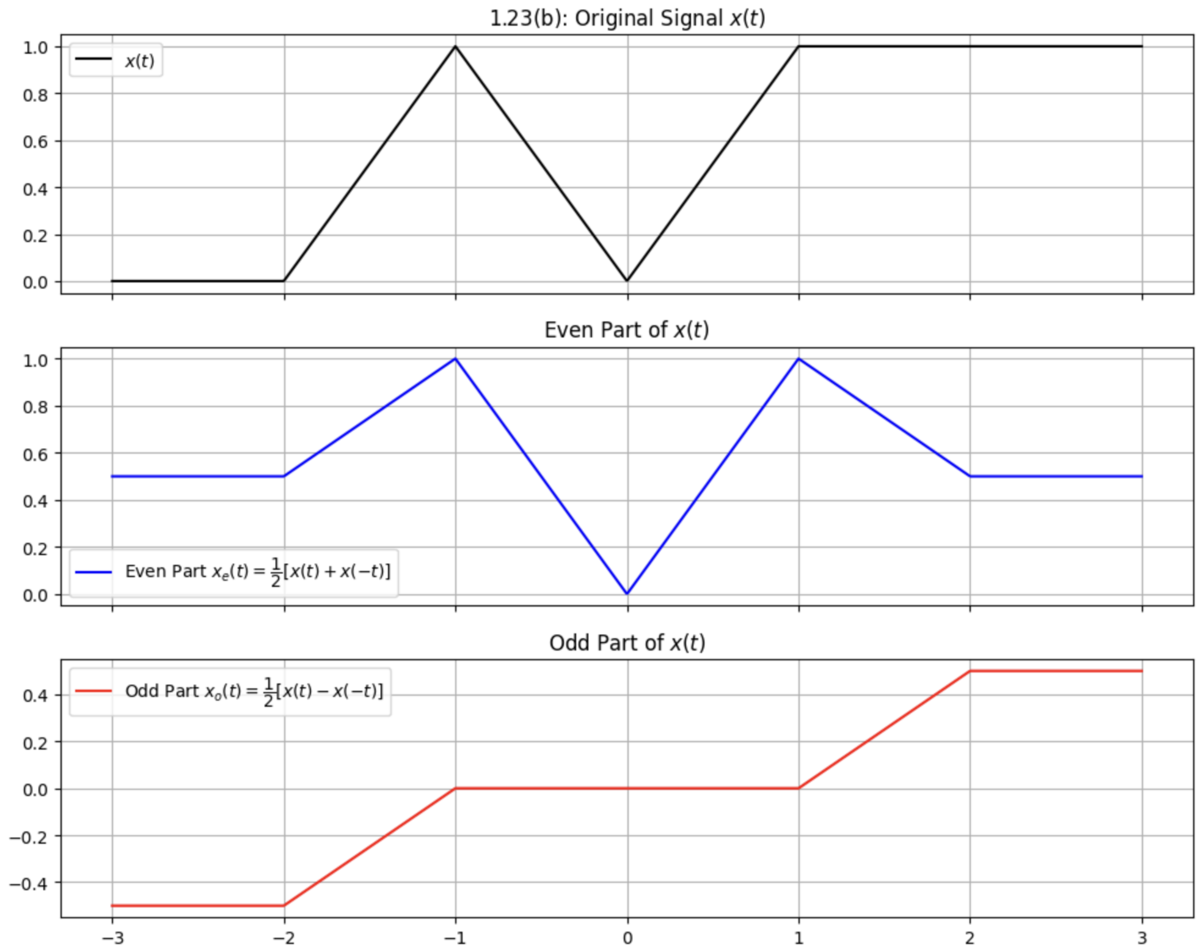
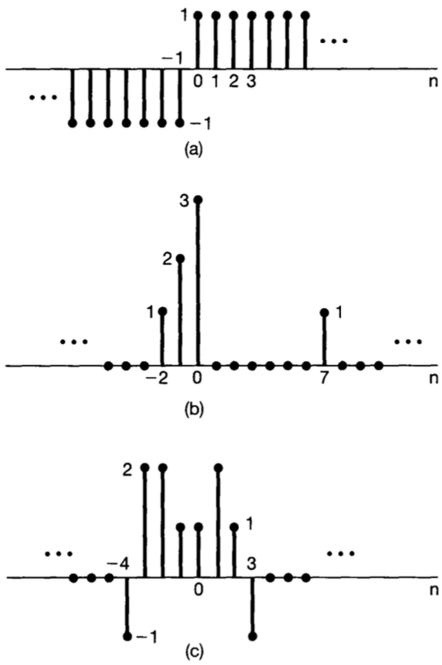


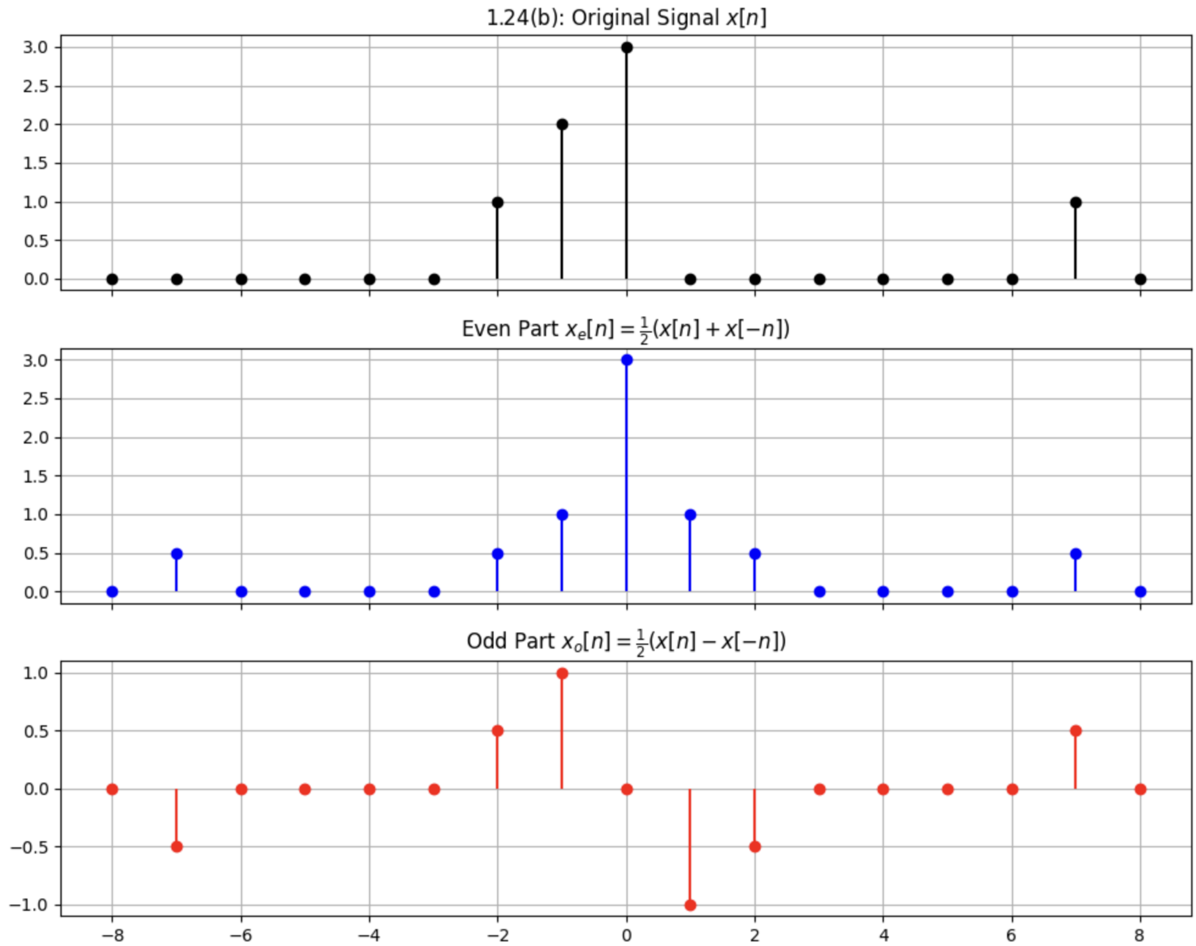
Figure P1.23





(a)
(b)
(c)

Figure P1.24



1.24(b): Original Signal $x[n]$
Even Part $x_e[n] = \frac{1}{2}(x[n] + x[-n])$
Odd Part $x_o[n] = \frac{1}{2}(x[n] - x[-n])$

Periodicity

1.25. Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = 3\cos(4t + \frac{\pi}{3})$ **(b)** $x(t) = e^{j(\pi t - 1)}$

(c) $x(t) = [\cos(2t - \frac{\pi}{3})]^2$ **(d)** $x(t) = \mathcal{E}v\{\cos(4\pi t)u(t)\}$

(e) $x(t) = \mathcal{E}v\{\sin(4\pi t)u(t)\}$ **(f)** $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)}u(2t-n)$

1.25(e): $x(t) = \mathcal{E}\{\sin(4\pi t)u(t)\}$ which refers to the even part of the signal:

$$x_e(t) = \frac{1}{2}\left[\sin(4\pi t)u(t) + \sin(-4\pi t)u(-t)\right] = \frac{1}{2}\left[\sin(4\pi t)u(t) - \sin(4\pi t)u(-t)\right]$$

$$= \frac{1}{2}\sin(4\pi t)\left[u(t) - u(-t)\right] = \frac{1}{2}\,\mathrm{sgn}(t)\sin(4\pi t)$$

This signal is **not periodic** because the sign function $\mathrm{sgn}(t)$ is not periodic.

$$\boxed{\text{Not periodic}}$$

1.26. Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x[n] = \sin(\frac{6\pi}{7}n + 1)$ **(b)** $x[n] = \cos(\frac{n}{8} - \pi)$ **(c)** $x[n] = \cos(\frac{\pi}{8}n^2)$

(d) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$ **(e)** $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

1.26(d): $x[n] = \cos\left(\frac{\pi}{2}n\right) \cdot \cos\left(\frac{\pi}{4}n\right)$

Use the product-to-sum identity:

$$\cos A \cdot \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Apply this to simplify:

$$x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{4}n - \frac{\pi}{2}n\right) + \cos\left(\frac{\pi}{4}n + \frac{\pi}{2}n\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(-\frac{\pi}{4}n\right) + \cos\left(\frac{3\pi}{4}n\right)\right] = \frac{1}{2}\left[\cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{3\pi}{4}n\right)\right]$$

Determine the period of each cosine:

- $\cos\left(\frac{\pi}{4}n\right)$: period $N_1 = 8$
- $\cos\left(\frac{3\pi}{4}n\right)$: period $N_2 = 8$

Since both components are periodic with period 8, their sum is periodic with:

$$\boxed{\text{Fundamental period } N = 8}$$

1.32. Let $x(t)$ be a continuous-time signal, and let

$$y_1(t) = x(2t) \text{ and } y_2(t) = x(t/2).$$

The signal $y_1(t)$ represents a speeded up version of $x(t)$ in the sense that the duration of the signal is cut in half. Similarly, $y_2(t)$ represents a slowed down version of $x(t)$ in the sense that the duration of the signal is doubled. Consider the following statements:

(1) If $x(t)$ is periodic, then $y_1(t)$ is periodic.
(2) If $y_1(t)$ is periodic, then $x(t)$ is periodic.
(3) If $x(t)$ is periodic, then $y_2(t)$ is periodic.
(4) If $y_2(t)$ is periodic, then $x(t)$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

1.32:

1. If $x(t)$ is periodic, then $y_1(t) = x(2t)$ is also periodic.

   Reasoning: If $x(t)$ is periodic with fundamental period T, then

   $$y_1(t) = x(2t) \quad \text{has period} \quad T' = \frac{T}{2}$$

   $$\boxed{\text{True. Fundamental period of } y_1(t) \text{ is } \frac{T}{2}}$$

2. If $y_1(t)$ is periodic, then $x(t)$ is periodic.

   Counterexample: Let $x(t) = \cos(\pi t^2)$. Then $y_1(t) = x(2t) = \cos(4\pi t^2)$, which is not periodic — but neither is $x(t)$. A better example: let $x(t)$ be non-periodic but constructed so $x(2t)$ ends up constant or periodic by coincidence.

   $$\boxed{\text{False. No guarantee that } x(t) \text{ is periodic even if } y_1(t) \text{ is}}$$

3. If $x(t)$ is periodic, then $y_2(t) = x\left(\frac{t}{2}\right)$ is also periodic.

   Reasoning: If $x(t)$ is periodic with period T, then

   $$y_2(t) = x\left(\frac{t}{2}\right) \quad \text{has period} \quad T' = 2T$$

   $$\boxed{\text{True. Fundamental period of } y_2(t) \text{ is } 2T}$$

4. If $y_2(t)$ is periodic, then $x(t)$ is periodic.

   Counterexample: Let $x(t) = \cos(\pi t^2)$, which is non-periodic, but $y_2(t) = x(t/2) = \cos(\pi(t/2)^2) = \cos\left(\frac{\pi}{4}t^2\right)$, which is also non-periodic — but this shows the relationship doesn't work in reverse.

   Alternatively, consider $x(t)$ defined only on $[0, 1)$ and extended arbitrarily — then $y_2(t)$ can appear periodic even when $x(t)$ isn't.

   $$\boxed{\text{False. Periodicity of } y_2(t) \text{ does not imply periodicity of } x(t)}$$

Exponentials and Periodicity

1.9. Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

(a) $x_1(t) = je^{j10t}$ **(b)** $x_2(t) = e^{(-1+j)t}$ **(c)** $x_3[n] = e^{j7\pi n}$
(d) $x_4[n] = 3e^{j3\pi(n+1/2)/5}$ **(e)** $x_5[n] = 3e^{j3/5(n+1/2)}$

1.9(b): $x_2(t) = e^{(-1+j)t} = e^{-t} \cdot e^{jt}$
The signal includes an exponentially decaying term e^{-t}, which causes the magnitude to decrease over time. Although e^{jt} is periodic, the presence of the exponential decay term means the full signal is not periodic.

$$\boxed{\text{Not periodic}}$$

1.9(d): $x_4[n] = 3e^{j\cdot\frac{3\pi}{5}(n+\frac{1}{2})} = 3e^{j\cdot\frac{3\pi}{10}} \cdot e^{j\cdot\frac{3\pi}{5}n}$

The constant phase factor $e^{j\cdot\frac{3\pi}{10}}$ does not affect periodicity.

We examine the term $e^{j\cdot\frac{3\pi}{5}n}$. Since:

$$\frac{3\pi}{5} \div 2\pi = \frac{3}{10} \in Q \quad \Rightarrow \quad \text{periodic}$$

The fundamental period is the smallest integer N such that:

$$\frac{3\pi}{5}N = 2\pi k \quad \Rightarrow \quad N = \frac{10k}{3}$$

The smallest such N occurs when $k = 3$, giving:

$$N = \frac{10 \cdot 3}{3} = 10$$

$$\boxed{\text{Periodic, with fundamental period } N = 10}$$

Impulse and Step Functions

1.14. Consider a periodic signal

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. The derivative of this signal is related to the "impulse train"

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

with period $T = 2$. It can be shown that

$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2).$$

Determine the values of A_1, t_1, A_2, and t_2.

The derivative $\frac{dx(t)}{dt}$ is zero almost everywhere, except at the discontinuities, where the derivative becomes a weighted impulse (Dirac delta) corresponding to the size of the jump.

Discontinuities and jumps:

- At $t = 0$: jump from -2 (left limit) to 1 (right limit), i.e., jump of $+3$.
- At $t = 1$: jump from 1 to -2, i.e., jump of -3.

Let the impulse train be:

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

Then:

$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2)$$

where the impulses appear every period $T = 2$, but shifted by the location of the discontinuities within the period.

Final values: $\boxed{A_1 = 3, \quad t_1 = 0, \quad A_2 = -3, \quad t_2 = 1}$