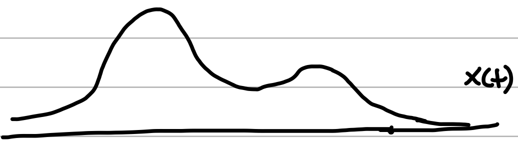
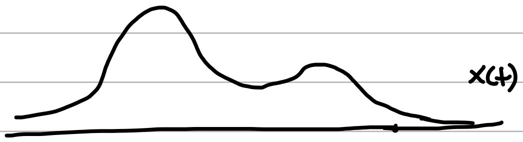
# Discussion 10:

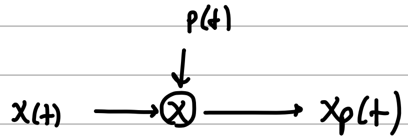
Sampling and reconstruction

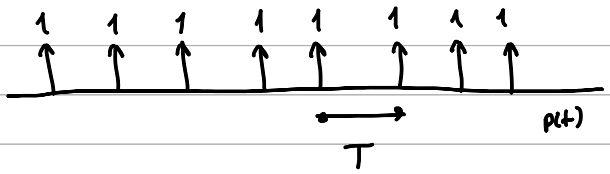
Imagine you have a signal $x(t)$ $\longrightarrow$




we want to sample $x(t)$, with sampling period $T$.



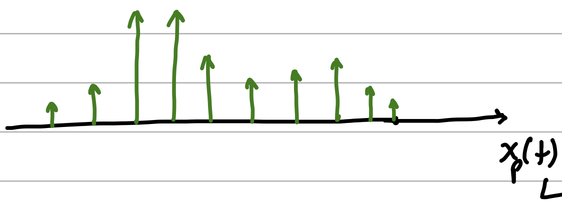




$\Rightarrow$ Sampling function (periodic impulse train)

$T =$ sampling period

$w_s = \frac{2\pi}{T}$ = sampling frequency




$\longrightarrow$ train of samples

$$p(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)\, p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)$$

What happen in frequency domain?

Time $\longleftrightarrow$ Freq

$$x(t)\, p(t) \longleftrightarrow \frac{1}{2\pi} X(jw) * P(jw)$$

What is $P(jw) = F(p(t))$

$p(t)$ is a periodic function with period $T$. So its Fourier transform is

$$P(jw) = 2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta(w - k \underbrace{\tfrac{2\pi}{T}}_{w_s})$$

Where $a_k$ is fourier series coefficient of train of delta functions, which is $\frac{1}{T}$

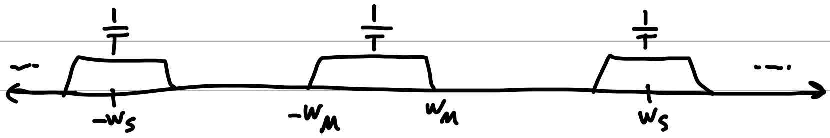
$\longrightarrow P(jw) = \frac{2\pi}{T} \sum \delta(w - kw_s)$ , $w_s = \frac{2\pi}{T}$

$$\Rightarrow X_p(jw) = \frac{1}{2\pi} X(jw) * P(jw) = \frac{1}{2\pi}\left[X(jw) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(w - kw_s)\right]$$

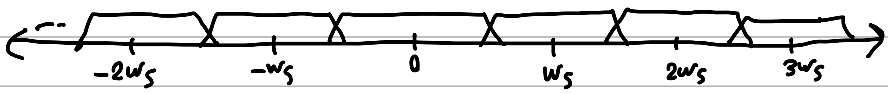
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(w - kw_s))$$

So if $X(jw)$ is


$X_p(jw)$ will be




OR




Depending on $w_M$ and $w_s$, we can have overlap (☹) or no overlap (☺)

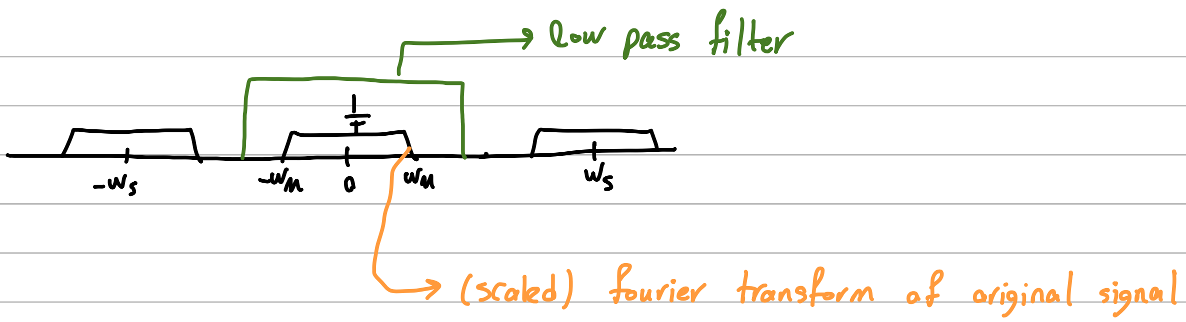
When we have no overlap? $w_M < w_s - w_M \longrightarrow w_s > 2w_M$

$w_M$ = Nyquist frequency

$2w_M$ = Nyquist rate

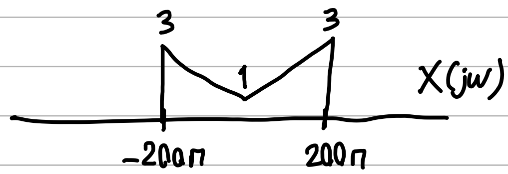
So if the sampling frequency ($w_s$) is larger than Nyquist rat ($2w_M$), we will have no overlap, no distortion, no aliasing, ☑

Why we are interested in no overlap? We want to able to reconstruct the original message $x(t)$ from the train of samples $x_p(t)$.




Using a low pass filter, you can recover $X(jw)$ from $X_p(jw)$ if there is no aliasing.

Example: For $x(t)$ and its FT, $X(jw)$,




a) Find Nyquist frequency and rate.

b) What is the biggest sampling period, T, that leads to no distortion.

c) Plot $X_p(jw)$ for $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$, for $T_s = \frac{T}{3}$ ($T$ calculated in part b)

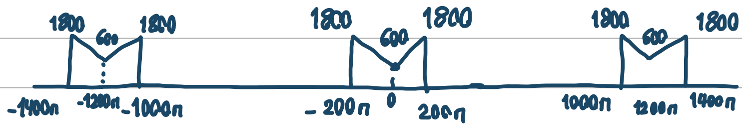
Answer:

a) Nyquist freq $= w_m = 200\pi$, Nyquist rate $= 2w_m = 400\pi$.

b) $w_s \geq 2w_m \longrightarrow \frac{2\pi}{T} \geq 400\pi \longrightarrow T \leq \frac{1}{200} \longrightarrow$ Biggest possible without distortion $= \frac{1}{200}$

c) $T_s = \frac{T}{3} = \frac{1}{600}$, $w_s = \frac{2\pi}{T_s} = 1200\pi$

$X_p(jw)$



# Bode Plot Review

Step ①: Write $H(jw)$ in this format: $$H(jw) = A\,\frac{\left(1+\frac{jw}{z_1}\right)\cdots\left(1+\frac{jw}{z_n}\right)}{\left(1+\frac{jw}{p_1}\right)\cdots\left(1+\frac{jw}{p_m}\right)}$$

Step ② Identify all breaking points.

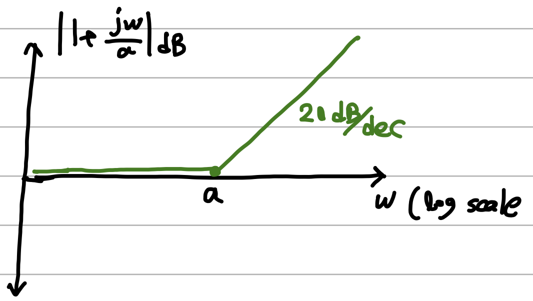
- For magnitude: $\{z_k\}_{k=1}^{n}$ and $\{p_l\}_{l=1}^{m}$.
- For phase: $\left\{\frac{z_k}{10}, 10z_k\right\}_{k=1}^{n}$ and $\left\{\frac{p_l}{10}, 10p_l\right\}_{l=1}^{m}$.

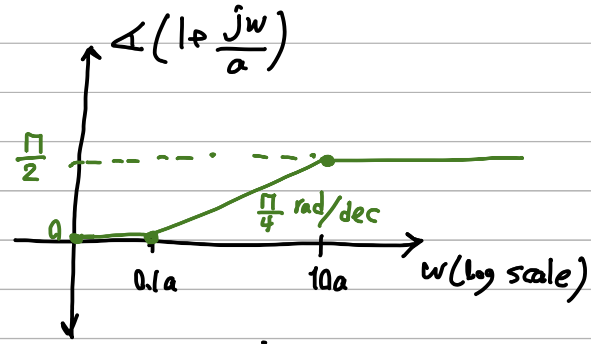
Step ③: Plot the individual Bode plot for each term

$$|H(jw)|_{dB} = 20\log|H(jw)| = 20\log|A| + 20\log\left|1+\frac{jw}{z_1}\right| + \cdots + 20\log\left|1+\frac{jw}{z_n}\right|$$
$$- 20\log\left|1+\frac{jw}{p_1}\right| - \cdots - 20\log\left|1+\frac{jw}{p_m}\right|$$

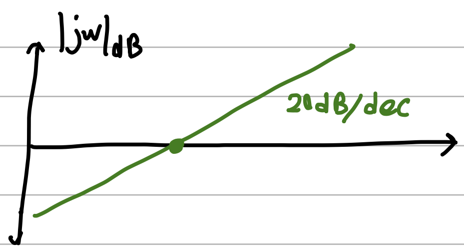
$$\measuredangle H(jw) = \measuredangle A + \measuredangle\left(1+\frac{jw}{z_1}\right) + \cdots + \measuredangle\left(1+\frac{jw}{z_n}\right)$$
$$- \measuredangle\left(1+\frac{jw}{p_1}\right) - \cdots - \measuredangle\left(1+\frac{jw}{p_m}\right)$$

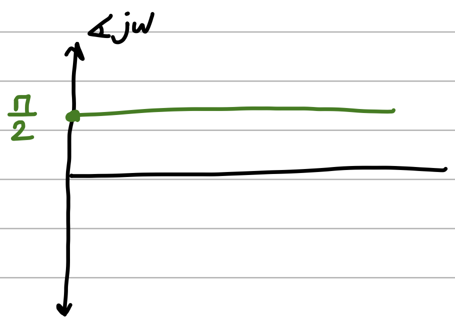
for a general case: $\left|1+\frac{jw}{a}\right|_{dB}$ and $\measuredangle\left(1+\frac{jw}{a}\right)$




For $|jw|_{dB}$ and $\measuredangle jw$



step ④: Sum the result.

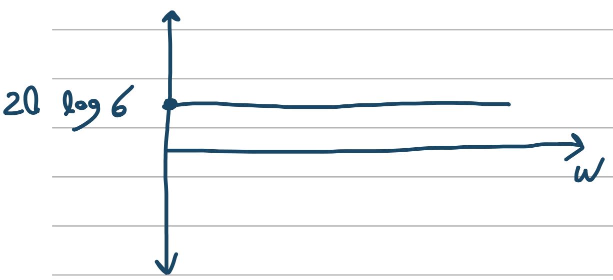
Step ⑤: Label all breaking points, slops, and transition points

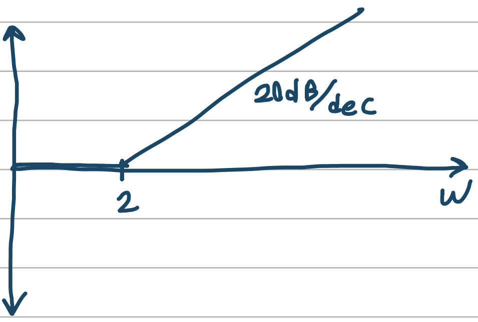
Example: $H(jw) = \dfrac{3(jw+2)}{(1+\frac{jw}{5})^2}$

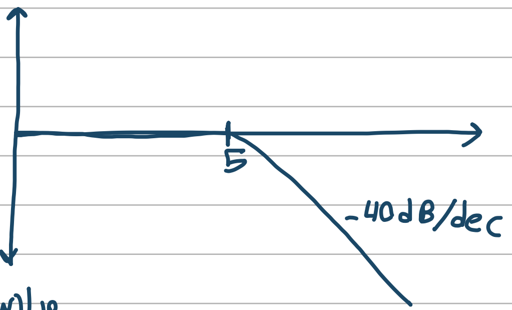
$$H(jw) = \frac{3(jw+2)}{(1+\frac{jw}{5})^2} = \frac{3\times 2\times(1+\frac{jw}{2})}{(1+\frac{jw}{5})^2} = 6\,\frac{1+\frac{jw}{2}}{(1+\frac{jw}{5})^2}$$

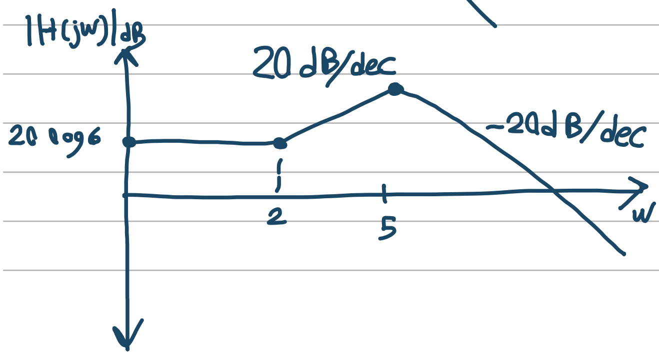
Magnitude: $|H(jw)|_{dB} = 20\log 6 + 20\log\left|1+\frac{jw}{2}\right| - 20\log\left|(1+\frac{jw}{5})^2\right|$

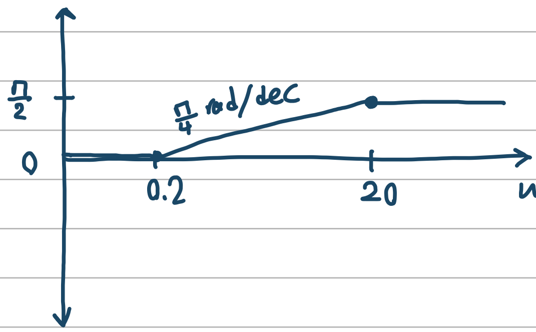
$= 20\log 6 + 20\log\left|1+\frac{jw}{2}\right| - 40\log\left|1+\frac{jw}{5}\right|$

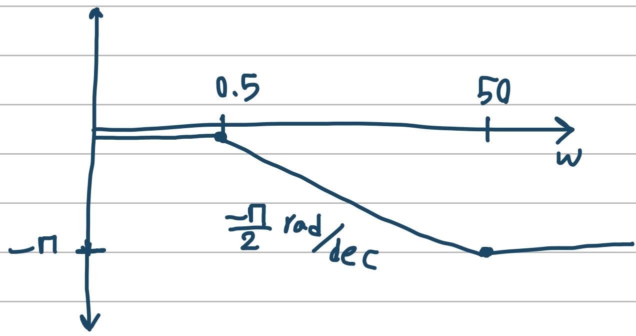


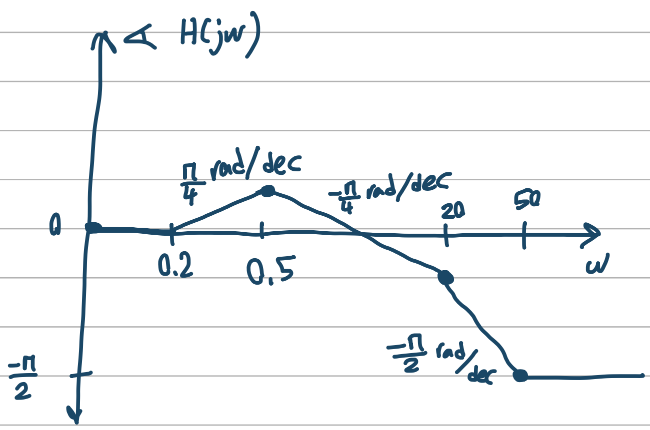



Example: $\dfrac{\left(1+\frac{j\omega}{2}\right)}{(j\omega)^2\left(1+\frac{j\omega}{10}\right)}$