University of California San Diego

ECE 101 - Linear Systems Fundamentals
Final Exam, Fall 2024

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11:30 am - 2:15 pm, December 11, 2024

Please read the exam rules. NO NOT BEGIN UNTIL TOLD!

- The exams can be done in a blue book or white A4 paper we provide. Remember to write your name and PID on your bluebook or every page of your A4 papers.

- On the first page of your solutions, please write out and sign the following pledge: "*I pledge my honor that I have not violated the honor code or the rules specified by the instructor during this examination.*"

- This final exam is closed book, closed notes, and closed external links. No MATLAB is required. No calculators are permitted.

- Except for the two pages of 2-sided cheat sheets, you are not allowed to use any other materials during the exam.

- You cannot communicate with anyone during the exam. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing.*

- Please write your solutions as clear as possible. You should also provide sufficient details for your solutions. We will give your partial credits even if the final answer is incorrect but the details make sense. Please be rigorous and to the point in your answers.

- **Gradescope scans**: You will scan and upload your solutions to Gradescope at the end of the exam. You will also submit your hard-copy solutions in case of upload failure.

- **There are six problems in total with 120 points**. Some problems might be harder than others, and there is no particular order. Good luck!

Problem 1: Multiple Choices with Justification (24 pts; 6 pts each)

Select the best answer from the options provided. Please provide a clear justification for your answer. **A correct selection with no/insufficient justification gets at most half credit**.

1. Consider an LTI system that consists of the cascade shown in Figure 1 below

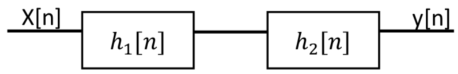

Fig. 1: Problem 1.1

where the impulse responses and the input signal are

$$h_1[n] = \beta^n \cos\left(\frac{\pi n}{4}\right), \quad h_2[n] = \alpha^n u[n], \quad x[n] = \delta[n] - \alpha\delta[n-1]$$

with $0 < |\beta| < 1$ and $0 < |\alpha| < 1$. What is the output $y[n]$?

(a) $y[n] = \beta^n \cos(\frac{\pi n}{4})$

(b) $y[n] = \beta^{n-1} \cos(\frac{\pi(n-1)}{4})$

(c) $y[n] = \beta^n \alpha^n \cos(\frac{\pi n}{4}) u[n]$

(d) $y[n] = \beta^n \alpha^n \cos(\frac{\pi n}{4})$

2. Consider a discrete-time LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

What is the output of the system for input $x[n] = 3^n$?

(a) $y[n] = \left(\frac{1}{2}\right) 3^n$

(b) $y[n] = \left(\frac{6}{5}\right) 3^n$

(c) $y[n] = \left(\frac{1}{2}\right) (3)^n u[n]$

(d) $y[n] = \left(\frac{6}{5}\right) (3)^n u[n]$

3. Consider a discrete-time LTI system whose frequency response is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{5\pi}{6} \\ 0, & \frac{5\pi}{6} < |\omega| < \pi \end{cases}$$

What is the output of this system to a periodic input signal with a fundamental period of $N = 3$ and Fourier series coefficients $a_0 = 1$, $a_1 = j$, $a_{-1} = -j$.

(a) $y[n] = 1$

(b) $y[n] = 1 + \frac{1}{2} \cos(\frac{2\pi}{3} n)$

(c) $y[n] = 1 - 2\sin(\frac{2\pi}{3} n)$

(d) $y[n] = j\sin(\frac{\pi}{3} n)$

4. Let $x(t)$ be a continuous-time signal. Let $x_p(t) = x(t)p(t)$ be the signal obtained by sampling with a continuous-time impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ with sampling period T corresponding to a sampling frequency $f_s = 10$ kHz. Let $x_d[n] = x(nT)$ be the discrete-time signal corresponding to the samples of $x(t)$ at integer multiples of T. Let $X_p(j\omega)$ be the CT Fourier transform of $x_p(t)$ and let $X_d(e^{j\Omega})$ be the DT Fourier transform of $x_d[n]$.
   The value of $X_d(e^{j\Omega})$ at $\Omega = \pi$ is equal to the value of $X_p(j\omega)$ at:
   (a) $\omega = 10^4\pi$
   (b) $\omega = 10^{-4}\pi$
   (c) $\omega = 2 \times 10^4\pi$
   (d) $\omega = 0.5 \times 10^{-4}\pi$

Problem 2: CT Fourier Transform [20 pts; 5 pts each]

Let $x(t) = u(t+4) - u(t-2)$, where $u(t)$ denotes the usual continuous-time (CT) step signal. Let $X(j\omega)$ denote its Fourier transform. In the following problems, refer to the attached tables, as necessary, to explain your solutions

1. Determine $X(j\omega)$.

2. Determine the value of
$$\int_{-\infty}^{\infty} X(j\omega) d\omega.$$

3. Determine the value of
$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

4. Determine the value of $Y(j\frac{\pi}{4})$ where $Y(j\omega)$ is the Fourier transform of
$$y(t) = \frac{dx(t)}{dt}.$$

Problem 3: Sampling Theory (15 pts; 5 pts each)

For each of the following signals, determine if a sampling period of $T = 0.1$ seconds is sufficiently small to avoid aliasing and allow reconstruction of the signal from its samples.

Justify your answers by explicit reference to the Fourier transforms of the signals.

1. Signal 1:
$$x_1(t) = \begin{cases} 1, & |t| < 1/5 \\ 0, & |t| \geq 1/5 \end{cases}$$

2. Signal 2:
$$x_2(t) = \cos(4\pi t)e^{-j4\pi t}$$

3. Signal 3:
$$x_3(t) = \frac{\sin^2(4\pi t)}{\pi t}$$

Problem 4: Amplitude Modulation (22 pts)

Consider the signals
$$x_1(t) = \frac{\sin(20t)}{\pi t}, \qquad \text{and} \qquad x_2(t) = \frac{\sin(10t)}{\pi t}$$

1. Determine the Fourier transforms $X_1(j\omega)$ and $X_2(j\omega)$. Sketch them precisely. [5 pts]

2. Let $y_1(t) = x_1(t)\cos(40t)$ and $y_2(t) = x_2(t)\cos(80t)$. Compute the Fourier transform $Y(j\omega)$ of $y(t) = y_1(t) + y_2(t)$ in terms of $X_1(j\omega)$ and $X_2(j\omega)$, and then sketch $Y(j\omega)$ precisely. [5 pts]

3. The signal $y(t)$ is demodulated using multiplication by $\cos(40t)$ to form the signal $z(t) = y(t)\cos(40t)$. [7 pts]
   - Express the Fourier transform $Z(j\omega)$ in terms of $Y_1(j\omega)$ and $Y_2(j\omega)$.
   - Then express $Z(j\omega)$ in terms of $X_1(j\omega)$ and $X_2(j\omega)$.
   - Sketch precisely the portion of $Z(j\omega)$ in the frequency range $[-100, 100]$.

4. Describe a scheme to recover $x_2(t)$ from the signal $y(t)$. [5 pts]

Problem 5: Laplace Transform (18 pts)

Let

$$H(s) = \frac{s}{s^2 + 3s + 2} \tag{1}$$

1. Determine the number N of distinct signals that have Laplace transform expressed as $H(s)$ in their region of convergence [5 pts].

2. Determine the signal $h(t)$ with Laplace transform $H(s)$ that has a Fourier transform. [7 pts]

3. Let $H(s)$ denote the system function of a stable LTI system. Sketch $|H(j\omega)|$ approximately. Is the system low-pass, high-pass, bandpass, or all-pass in nature? [6 pts]

Problem 6: Laplace Transform and LTI Systems (21 pts)

Consider an LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} \tag{2}$$

1. Determine the system function $H(s)$ and the possible regions of convergence (ROC). For each region of convergence, determine whether the corresponding system is causal or stable. [6 pts]

2. Determine the poles and zeros of $H(s)$ and sketch precisely the pole-zero plot. [4 pts]

3. Let $H(s)$ serially concatenated with another causal system $H_2(s)$ that has impulse response

$$h_2(t) = \frac{3}{2}e^{t}u(t) - \frac{1}{2}e^{-t}u(t).$$

   Determine the transfer function $G(s)$ of the system obtained from this concatenation of $H(s)$ and $H_2(s)$. [4 pts]

4. Let the differential equation (2) denote a casual and stable LTI system. What is its step response $y(t)$ (i.e., given a step input signal $x(t) = u(t)$) [7 pts]

Table 3.1 PROPERTIES OF THE CONTINUOUS-TIME FOURIER SERIES

$x(t)$ and $y(t)$ are periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$

Section	Property	Periodic Signal	Fourier Series Coefficients
	both $x(t)$ and $y(t)$ periodic with period T and frequency $\omega_0 = 2\pi/T$	$x(t)$ $y(t)$	a_k b_k
3.5.1	Linearity	$A\,x(t) + B\,y(t)$	$A\,a_k + B\,b_k$
3.5.2	Time Shifting	$x(t - t_0)$	$a_k\, e^{-jk\omega_0 t_0}$
	Frequency Shifting	$\exp[jM\omega_0 t]\, x(t)$	a_{k-M}
3.5.6	Conjugation	$x^*(t)$	a^*_{-k}
3.5.3	Time Reversal	$x(-t)$	a_{-k}
3.5.4	Time Scaling	$x(\alpha t),\ \alpha > 0$ (periodic with period T/α)	a_k
	Periodic Convolution	$\int_T x(\tau) y(t-\tau) d\tau$	$T\,a_k\,b_k$
3.5.5	Multiplication	$x(t)\,y(t)$	$\sum_{\ell=-\infty}^{\infty} a_\ell\, b_{k-\ell}$
	Differentiation	$dx(t)/dt$	$j\,k\,\omega_0\,a_k$
	Integration (requires $a_0 = 0$)	$\int_{-\infty}^{t} x(t')\,dt'$	$a_k/(j\,k\,\omega_0)$
3.5.6	Conjugate Symmetry	$x(t)$ is real	$\begin{cases} \lvert a_k\rvert = \lvert a_{-k}\rvert \\ a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \angle a_k = -\angle a_{-k} \end{cases}$
3.5.6	real and even signals	$x(t)$ real and even	a_k purely real and even
3.5.6	real and odd signals	$x(t)$ real and odd	a_k purely imaginary and odd
3.5.6	Even Decomposition ($x(t)$ is real)	$x_e(t) = \text{Ev}\,\{x(t)\}$	$\mathfrak{Re}\,\{a_k\}$
3.5.6	Odd Decomposition ($x(t)$ is real)	$x_o(t) = \text{Od}\,\{x(t)\}$	$j\mathfrak{Im}\,\{a_k\}$

Parseval's Relation for Periodic Signals $\quad (1/T)\int_T \lvert x(t)\rvert^2\,dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$

Table 3.2 PROPERTIES OF THE DISCRETE-TIME FOURIER SERIES

$x[n]$ and $y[n]$ are periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$

Property	Periodic Signal	Fourier Series Coefficients
both $x[n]$ and $y[n]$ periodic with period N and frequency $\omega_0 = 2\pi/N$	$x[n]$ $y[n]$	a_k (periodic, period N) b_k (periodic, period N)
Linearity	$A\,x[n] + B\,y[n]$	$A\,a_k + B\,b_k$
Time Shifting	$x[n-n_0]$	$a_k\,e^{-jk\omega_0 n_0}$
Frequency Shifting	$\exp[jM\omega_0 n]\,x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & n \text{ Mod } m = 0 \\ 0, & n \text{ Mod } m \neq 0 \end{cases}$	$\dfrac{a_k}{m}$ (periodic, period mN)
Periodic Convolution	$\displaystyle\sum_{r=<N>} x[r]y[n-r]$	$N\,a_k\,b_k$
Multiplication	$x[n]\,y[n]$	$\displaystyle\sum_{\ell=<N>} a_\ell\,b_{k-\ell}$
First Difference	$x[n] - x[n-1]$	$(1-e^{-j\,k\,\omega_0})a_k$
Running Sum (requires $a_0 = 0$)	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\left(\dfrac{1}{1-e^{-j\,k\,\omega_0}}\right) a_k$
Conjugate Symmetry	$x[n]$ is real	$\begin{cases} \lvert a_k\rvert = \lvert a_{-k}\rvert \\ a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \angle a_k = -\angle a_{-k} \end{cases}$
real and even signals	$x[n]$ real and even	a_k purely real and even
real and odd signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even Decomposition ($x[n]$ is real)	$x_e[n] = \text{Ev}\,\{x[n]\}$	$\mathfrak{Re}\,\{a_k\}$
Odd Decomposition ($x[n]$ is real)	$x_o[n] = \text{Od}\,\{x[n]\}$	$j\mathfrak{Im}\,\{a_k\}$

Parseval's Relation for Periodic Signals $\quad \displaystyle\frac{1}{N}\sum_{n=<N>} \lvert x[n]\rvert^2 = \sum_{k=<N>} \lvert a_k\rvert^2$

Table 4.1 PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

Property	Aperiodic Signal	Fourier Transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$a\,x(t)+b\,y(t)$	$a\,X(j\omega)+b\,Y(j\omega)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$\exp(j\omega_0 t)\,x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)\,y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation (time)	$dx(t)/dt$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(t')\,dt'$	$\frac{1}{j\omega}X(j\omega)+\pi\,X(0)\delta(\omega)$
Differentiation (frequency)	$t\,x(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry	$x(t)$ is real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathfrak{Re}\{X(j\omega)\}=\mathfrak{Re}\{X(-j\omega)\} \\ \mathfrak{Im}\{X(j\omega)\}=-\mathfrak{Im}\{X(-j\omega)\} \\ \|X(j\omega)\|=\|X(-j\omega)\| \\ \angle X(j\omega)=-\angle X(-j\omega) \end{cases}$
real and even signals	$x(t)$ real and even	$X(j\omega)$ purely real and even
real and odd signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even Decomposition ($x(t)$ is real)	$x_e(t)=\mathrm{Ev}\,\{x(t)\}$	$\mathfrak{Re}\,\{X(j\omega)\}$
Odd Decomposition ($x(t)$ is real)	$x_o(t)=\mathrm{Od}\,\{x(t)\}$	$j\mathfrak{Im}\,\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{\infty}|x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega$

Table 4.2 CONTINUOUS-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$ otherwise
$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = 1/2$ $a_k = 0$ otherwise
$\sin(\omega_0 t)$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = -j/2$ $a_k = 0$ otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \quad k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq T/2 \end{cases}$ and $\quad x(t+T) = x(t)$	$\sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	aperiodic
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	aperiodic
$\delta(t)$	1	aperiodic
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	aperiodic
$\delta(t - t_0)$	$e^{-j\omega t_0}$	aperiodic
$e^{-at}u(t), \ \mathfrak{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	aperiodic
$t\, e^{-at}u(t), \ \mathfrak{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	aperiodic
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t),$ $\mathfrak{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	aperiodic

Table 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Discrete Fourier Transform always has a period of 2π.

Property	Aperiodic Signal	Fourier Transform
	$x[n]$ $y[n]$	$X(e^{j\omega})$ $Y(e^{j\omega})$
Linearity	$a\,x[n] + b\,y[n]$	$a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$
Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$\exp[j\omega_0 n]\,x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & n \text{ Mod } k = 0 \\ 0, & n \text{ Mod } k \neq 0 \end{cases}$	$X(e^{j\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})\,Y(e^{j\omega})$
Multiplication	$x[n]\,y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\,Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1-e^{-j\omega})\,X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi\,X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$
Differentiation (frequency)	$n\,x[n]$	$j\frac{d}{d\omega}X(e^{j\omega})$
Conjugate Symmetry	$x[n]$ is real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathfrak{Re}\{X(e^{j\omega})\} = \mathfrak{Re}\{X(e^{-j\omega})\} \\ \mathfrak{Im}\{X(e^{-j\omega})\} = -\mathfrak{Im}\{X(e^{j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \measuredangle X(e^{j\omega}) = -\measuredangle X(e^{-j\omega}) \end{cases}$
real and even signals	$x[n]$ real and even	$X(e^{j\omega})$ purely real and even
real and odd signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even Decomposition ($x[n]$ is real)	$x_e[n] = \text{Ev}\,\{x[n]\}$	$\mathfrak{Re}\left\{X(e^{j\omega})\right\}$
Odd Decomposition ($x[n]$ is real)	$x_o[n] = \text{Od}\,\{x[n]\}$	$j\mathfrak{Im}\left\{X(e^{j\omega})\right\}$

Parseval's Relation for Aperiodic Signals $\displaystyle\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2\,d\omega$

Table 5.2 DISCRETE-TIME FOURIER TRANSFORM PAIRS

In the pairs given below, if the signal is periodic, then $\omega_0 = 2\pi m/N$, where $m, N \in \mathbb{Z}$.
If $\omega_0/(2\pi)$ is irrational, then the signal is aperiodic.
Discrete Fourier Transform always has a period of 2π.

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=<N>} a_k e^{j(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\omega - \omega_0 - 2\pi\ell\right)$	$a_k = \begin{cases} 1, & k = m + qN \text{ where } q \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$\cos\omega_0 n$	$\pi \sum_{\ell=-\infty}^{\infty} \{\delta\left(\omega - \omega_0 - 2\pi\ell\right) + \delta\left(\omega + \omega_0 - 2\pi\ell\right)\}$	$a_k = \begin{cases} 1/2, & k = \pm m + qN \text{ where } q \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$\sin\omega_0 n$	$-j\pi \sum_{\ell=-\infty}^{\infty} \{\delta\left(\omega - \omega_0 - 2\pi\ell\right) - \delta\left(\omega + \omega_0 - 2\pi\ell\right)\}$	$a_k = \begin{cases} -j/2, & k = m + qN \text{ where } q \in \mathbb{Z} \\ j/2, & k = -m + qN \\ 0, & \text{otherwise} \end{cases}$
$x[n] = 1$	$2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\omega - 2\pi\ell\right)$	$a_k = \begin{cases} 1, & k = qN \text{ where } q \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + 1/2)]}{N \sin[2\pi k/2N]}$, k Mod $N \neq 0$; $a_k = (2N_1 + 1)/N$, k Mod $N = 0$
$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \ \|a\| < 1$	$\frac{1}{1 - ae^{-j\omega}}$	aperiodic
$x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$	aperiodic
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\text{sinc}\left(\frac{Wn}{\pi}\right)$, $0 < W < \pi$	$X\left(e^{j\omega}\right) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$	aperiodic
$\delta[n]$	1	aperiodic
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$	aperiodic
$\delta[n - n_0]$	$e^{-j\omega n_0}$	aperiodic
$(n+1)a^n u[n], \ \|a\| < 1$	$\frac{1}{(1 - a\,e^{-j\omega})^2}$	aperiodic
$\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n], \ \|a\| < 1$	$\frac{1}{(1 - a\,e^{-j\omega})^r}$	aperiodic

Table 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a\,x_1(t) + b\,x_2(t)$	$a\,X_1(s) + b\,X_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-s\,t_0} X(s)$	R
Shifting in s	$e^{s_0\,t}\,x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
Time scaling	$x(a\,t)$	$\frac{1}{\lvert a \rvert} X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)\,X_2(s)$	At least $R_1 \cap R_2$
Differentiation	$\frac{d}{dt}x(t)$	$s\,X(s)$	At least R
Differentiation in s	$-t\,x(t)$	$\frac{d}{ds}X(s)$	R
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathfrak{Re}\{s\} > 0\}$

Initial Value Theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then $x(0^+) = \lim_{s\to\infty} s\,X(s)$.

Final Value Theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then $\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s)$.

Table 9.2 LAPLACE TRANSFORM PAIRS

Signal	Laplace Transform	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathfrak{Re}\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathfrak{Re}\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	s^{-n}	$\mathfrak{Re}\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	s^{-n}	$\mathfrak{Re}\{s\} < 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathfrak{Re}\{s\} > -\alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathfrak{Re}\{s\} < -\alpha$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathfrak{Re}\{s\} > -\alpha$
$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathfrak{Re}\{s\} < -\alpha$
$\delta(t-T)$	e^{-sT}	All s
$[\cos\omega_0 t]\,u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > 0$
$[\sin\omega_0 t]\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > 0$
$[e^{-\alpha t}\cos\omega_0 t]\,u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > -\alpha$
$[e^{-\alpha t}\sin\omega_0 t]\,u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathfrak{Re}\{s\} > -\alpha$
$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	all s
$u_{-n}(t) = u(t) * \ldots * u(t)$ n times	s^{-n}	$\mathfrak{Re}\{s\} > 0$