Lecture 7

Eigenfunctions and differential equations

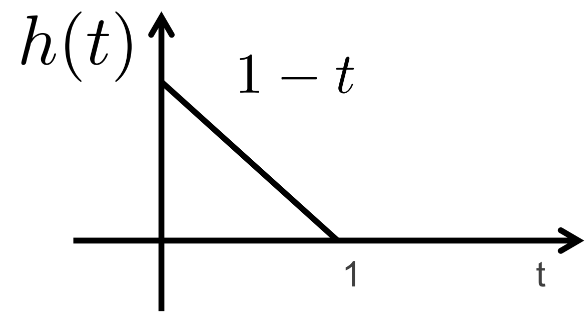
Preview of today's lecture

- Convolution review
- Eigenfunctions
- Connections to linear constant coefficient differential equations

Example convolution brute force

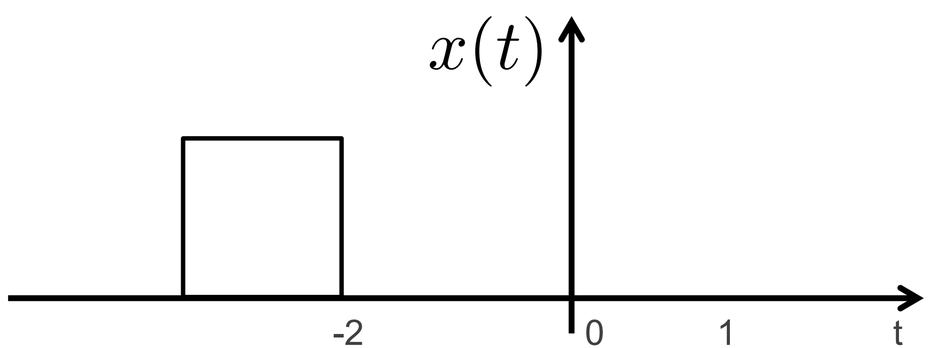
- Convolve the two signals

$$h(t) = (1-t)[u(t) - u(t-1)]$$



and

$$x(t) = \text{rect}\left(t + \frac{1}{2} + 2\right)$$



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Basic convolution properties

- Commutative

Shorthand notation

$$y(t) = x(t) * h(t) = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau$$

Choose option that makes it easy!

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

A B A B

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

Use properties to simplify convolutions

Convolution with the delta

- Convolution with delta functions is easy

$$\delta(t) * x(t) = x(t)$$

$$x(t) * \delta(t) = x(t)$$

- Convolution with shifted deltas is easy

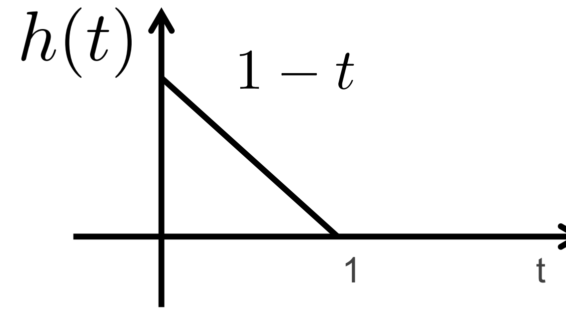
$$\boxed{x(t) * \delta(t - t_0) = x(t - t_0)}$$

Celebrate simplicity when faced with a convolution with a delta!

Recall this convolution

- Find the output of a system with impulse response

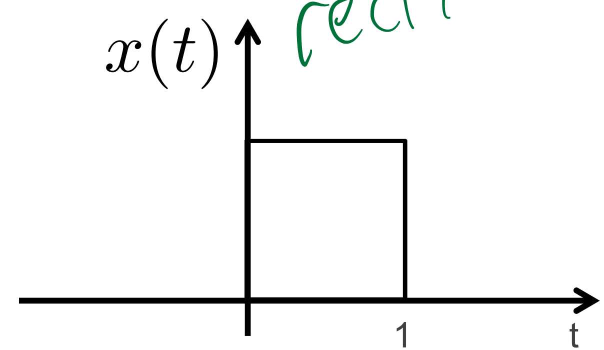
$$h(t) = (1 - t)[u(t) - u(t-1)]$$



for the input

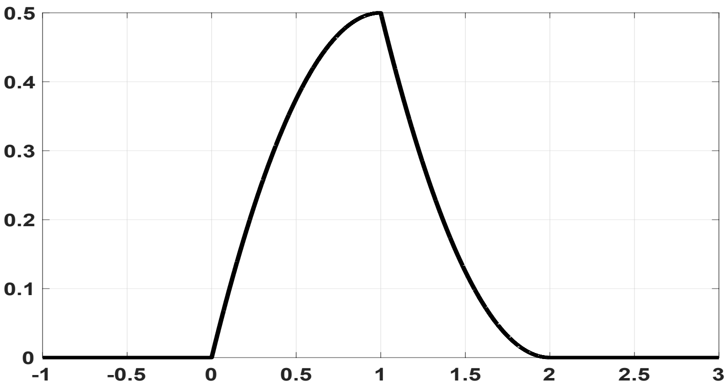
$$x(t) = u(t) - u(t-1)$$

$\mathrm{rect}(t - 1/2)$



Solution to the convolution

$$y(t) = \begin{cases} 0 & t < 0 \\ t - \frac{t^2}{2} & 0 \leqslant t \leqslant 1 \\ \frac{t^2}{2} - 2t + 2 & 1 \leqslant t \leqslant 2 \\ 0 & 2 < t \end{cases}$$

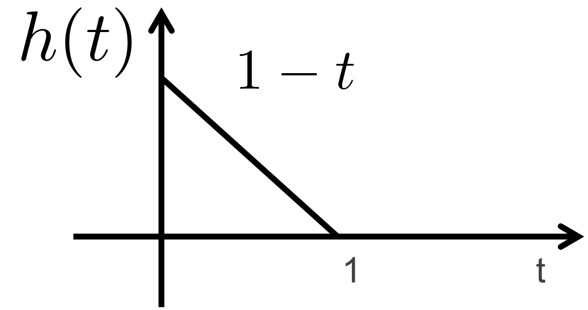


For later use, let us refer to this convolution as $f(t)$

Now solve the convolution with the properties

- Convolve the two signals

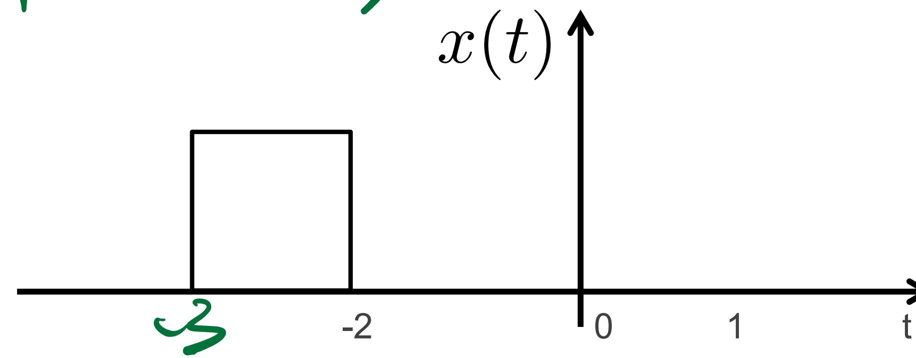
$$h(t) = (1-t)[u(t) - u(t-1)]$$



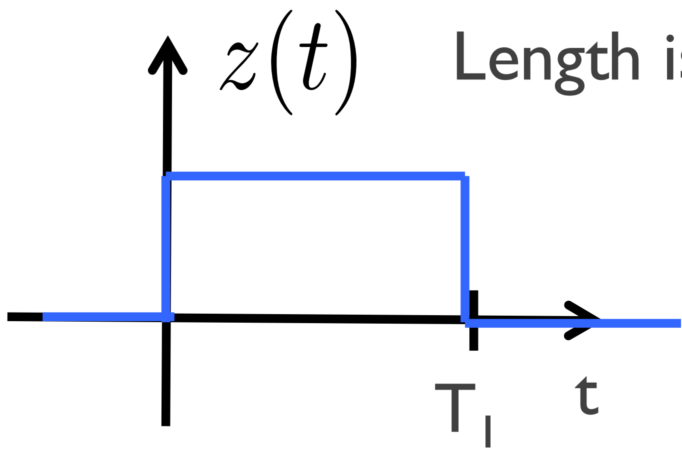
and

rect(t+2+1/2)

$$x(t) = \mathrm{rect}\left(t - \frac{1}{2} - 2\right)$$

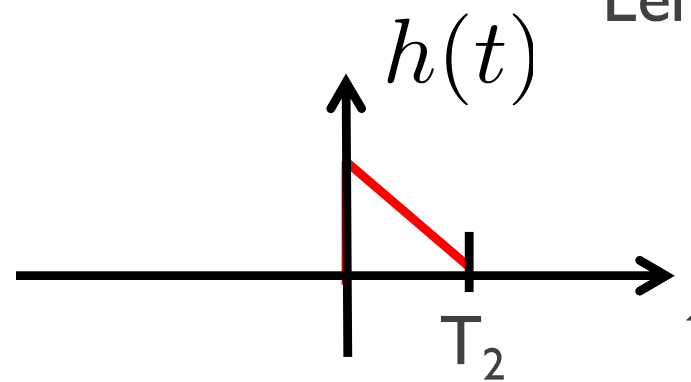


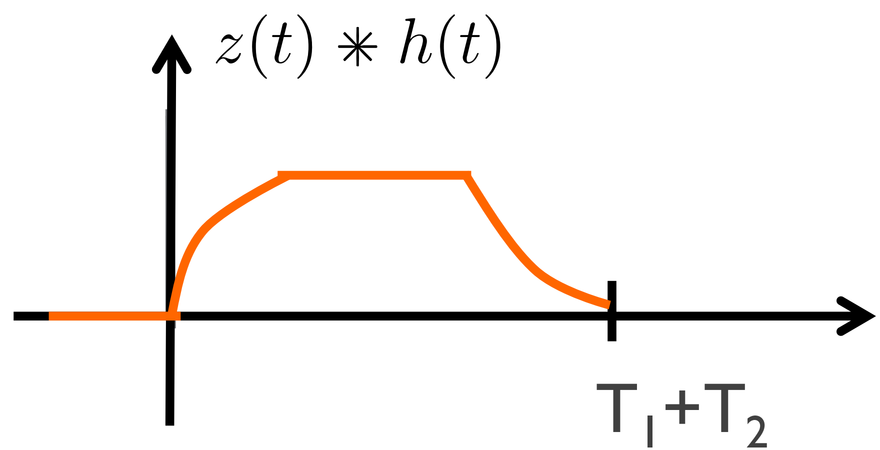
Length / duration of a convolution in continuous time



Length is T_1

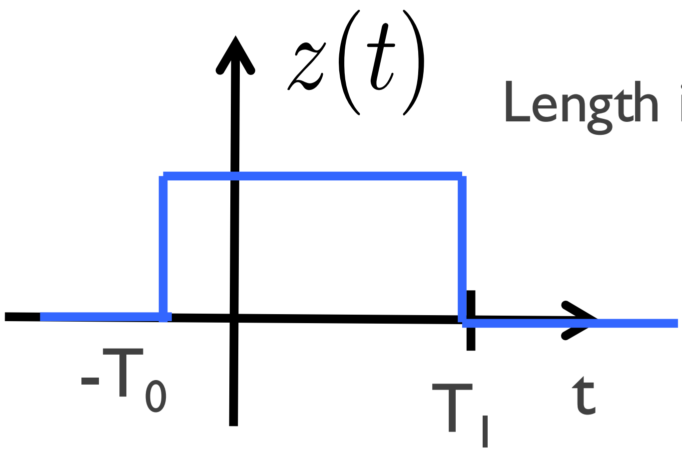
Length is T_2



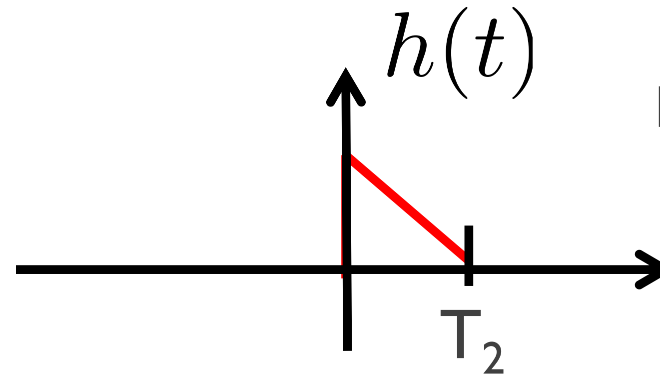


Length is T_1+T_2

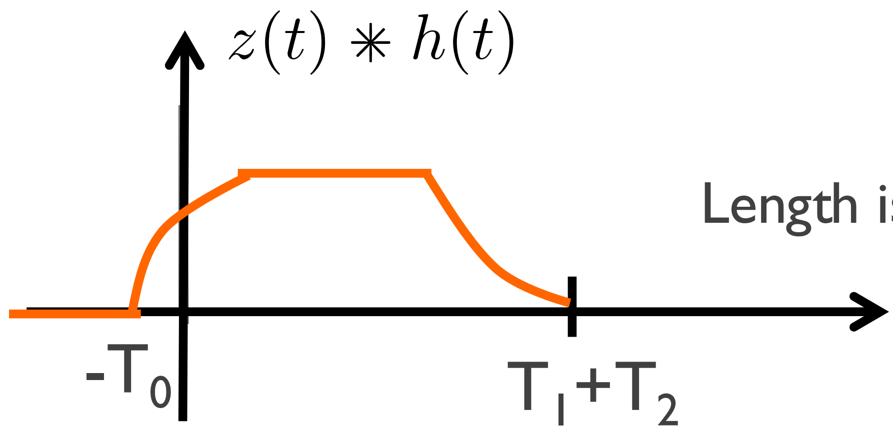
Length / duration of a convolution in continuous time



Length is $T_{0+}T_1$



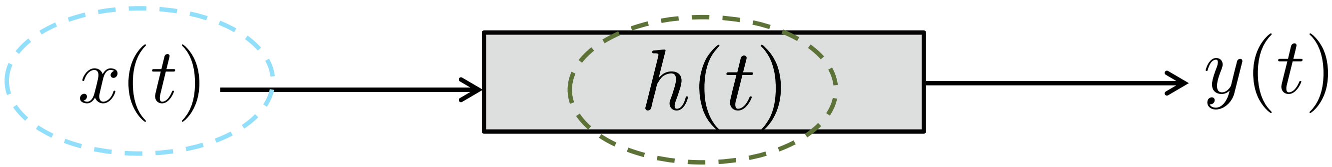
Length is T_2



Length is $T_0+T_1+T_2$

Connections back to ECE 45

Lectures 2 - 3 working with signals



Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

Eigenfunctions of LTI systems

Learning objectives

- Characterize the eigenfunctions of CT LTI systems

LTI systems

- LTI systems are characterized by their impulse responses

- Output is the convolution of the input and the impulse response

$$y(t) = x(t) * h(t)$$

- Certain special functions called eigenfunctions pass through *almost* untouched by the convolution

Complex exponentials are special signals

- Consider an LTI system with impulse response $h(t)$
- Input into the system a complex exponential $x(t) = e^{st}$
- Recall that $s = r + j\omega$ is a complex number

Convolution with a complex exponential

$$
\begin{aligned}
y(t) &= h(t) * e^{st} \\
&= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau} d\tau \\
&= e^{st} \boxed{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau} \\
&= e^{st} H(s)
\end{aligned}
$$

Note for the future: *H(s)* is the Laplace transform of the impulse response

H(s) also called the transfer function

Eigenfunctions of a CT LTI system

$$x(t) = e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = H(s)e^{st}$$

Complex exponential with complex "frequency" *s*

Eigenvalue (a complex scalar)

eigenfunction

- CT complex exponentials are eigenfunctions of LTI systems
  - Eigen comes from the German word "own" or "self"
  - Eigenfunction "passes through" the LTI system
  - Attenuated and scaled according to $H(s)$ (system response)

Eigenfunctions are *easy* to convolve

Cautionary note!!

- Note that

$$e^{st} \neq e^{st} u(t)$$

- Only everlasting exponentials are true eigenfunctions

☹

Convolution with a causal complex exponential

$$y(t) = h(t) * e^{st}u(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}u(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}u(t-\tau)d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}u(t-\tau)d\tau$$

$$= e^{st}\boxed{\int_{-\infty}^{t} h(\tau)e^{-s\tau}d\tau}$$

$u(t-\tau) = 1$

$t-\tau \geq 0$

$t \geq \tau$

$\tau \leq t$

As $t \rightarrow \infty$ this converges to $H(s)$ which is the "steady state" assumption

Eigenfunctions in other contexts

- Linear algebra

eigenvalue

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

matrix eigenvector

- Eigenfaces are eigenvectors used in human face recognition

original faces

eigenfaces

http://www.pages.drexel.edu/~sis26/Eigenface%20Tutorial.htm

Example: Constant input

- Consider $x(t) = c$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

- This is just a trivial exponential function with $s = 0$

$= r + j\omega$

$$x(t) = ce^{0\,t}$$

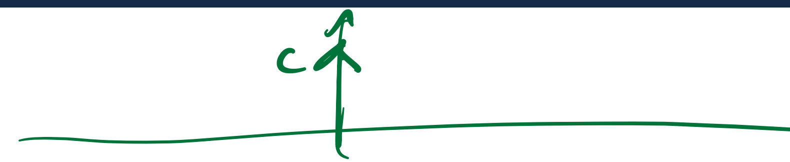
$\int_{-\infty}^{\infty} h(\tau)\,d\tau$

- Hence we can use the eigenfunction proprty

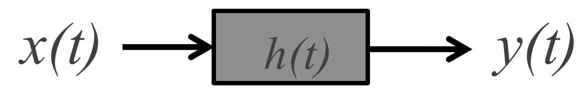
$\int_{-\infty}^{\infty} h(\tau)e^{-0\cdot\tau}\,d\tau$

$$y(t) = H(0)c$$

Example: Delta function



- Consider $x(t) = c\,\delta(t)$



- Not a constant signal, cannot use the eigenfunction property

- But we can use the fact that the impulse response is by definition the response to an impulse

$$y(t) = c\,h(t)$$

Be careful not to confuse constant signals with delta functions

Example: Complex sinusoids

- Consider $x(t) = e^{j\omega t}$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- This is just a trivial exponential function with $\boxed{s = j\omega}$

$$H(\cancel{\sigma}+j\omega)\, e^{(\overset{0}{\cancel{\sigma}}+j\omega)t}$$

- Hence

$$y(t) = \boxed{H(j\omega)}e^{j\omega t}$$

Frequency response

Frequency response

- Frequency response is used to characterize LTI systems

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

- The frequency response is computed from the impulse response

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

- This is the Fourier transform of the impulse response of the system

Example: Cosine

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Consider $x(t) = \cos(\omega t)$

Eulers

- Decomposing using Euler's identity

$$x(t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

- Gives the output

$$y(t) = \frac{1}{2}H(j\omega)e^{j\omega t} + \frac{1}{2}H(-j\omega)e^{-j\omega t}$$

- Can be simplified further in some cases

Eigenfunctions in summary

$$x(t) = e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = H(s)e^{st}$$

CT complex exponential with complex s

Eigenvalue
(a complex scalar)

Eigenfunction

Convolution is easy with eigenfunctions!

Introduction to continuous-time systems as differential equations

Learning objectives

- Define a linear constant coefficient differential equation
- Formulate differential equations for circuits problems

Systems described with differential equations

- Many practical systems are described by differential equations
  - RLC circuits and filters
  - Mechanical systems
  - Heat transfer systems
  - Chemical systems

Linear constant coefficient differential equations have many connections to LTI systems

A simple differential equation example

$x(t) \rightarrow \boxed{\quad} \rightarrow y(t)$

Constant coefficients

$$a_0\, y(t) + a_1\, \frac{d}{dt} y(t) = b_0\, x(t) + b_1\, \frac{d}{dt} x(t)$$

output

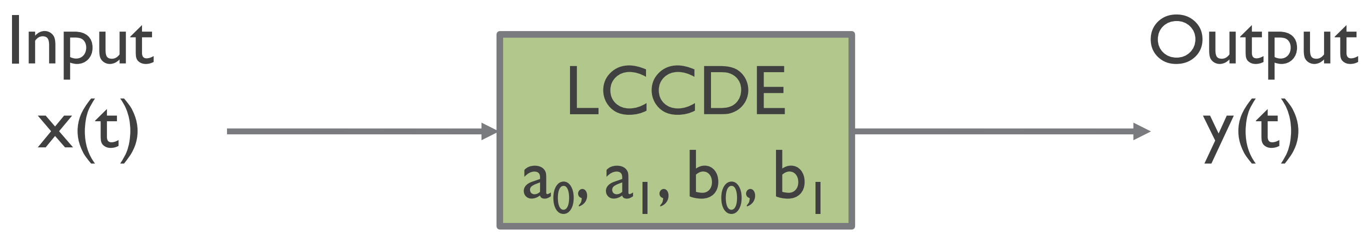
input

derivative of output

derivative of input

Example of a linear constant coefficient differential equation

Connecting differential equations to systems



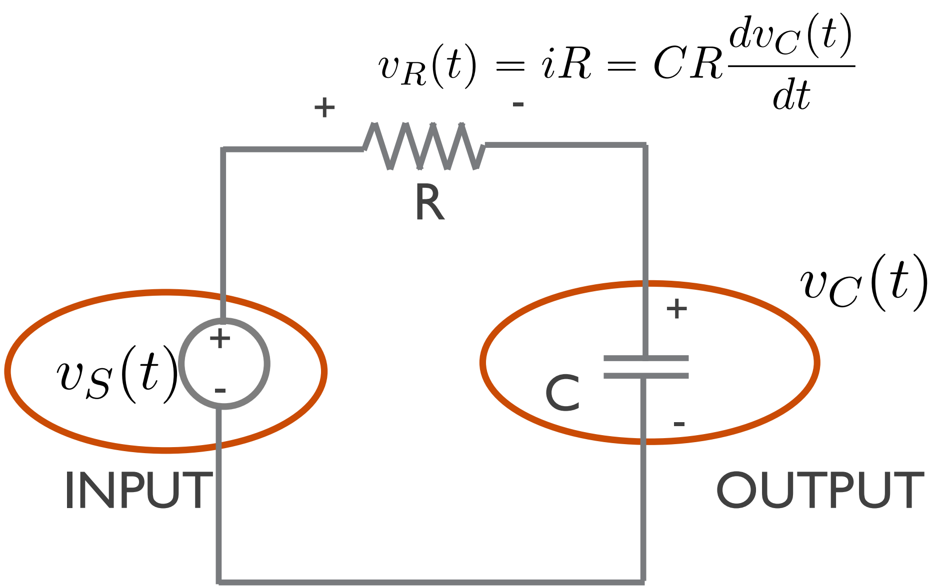
- Input and output are related through the linear constant coefficient differential equation (LCCDE)

$$a_0\, y(t) + a_1\, \frac{d}{dt}y(t) = b_0\, x(t) + b_1\, \frac{d}{dt}x(t)$$

- Solution to the differential equation provides a formula for the output y(t) as a function of the input x(t) and system parameters

Circuit examples

- RC lowpass filter as a differential equation
  - Source voltage as the input
  - Capacitor voltage as the output



Current through capacitor

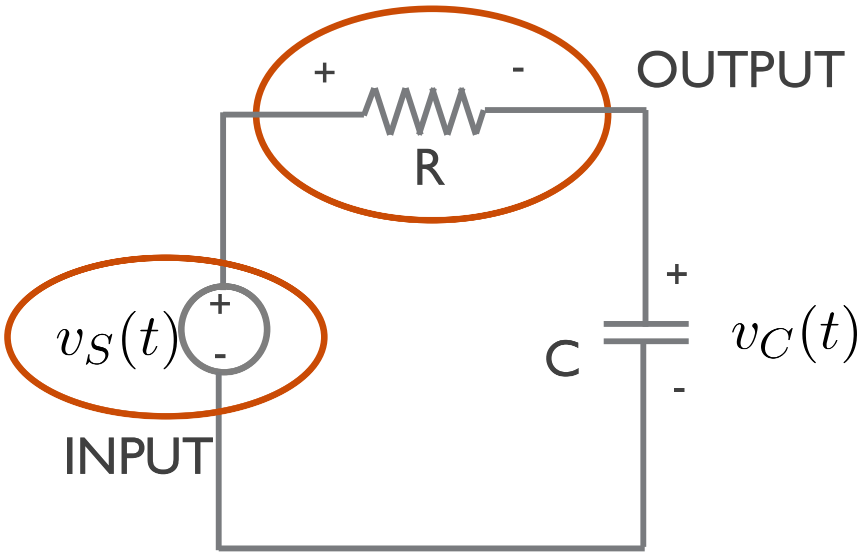
$$i = C\frac{dv_C(t)}{dt}$$

Resulting differential equation

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$

Circuit examples

- RC highpass filter
  - Source voltage as the input as a differential equation
  - Resistor voltage as the output



$$v_R(t) = iR = RC\frac{dv_C(t)}{dt}$$

$$\Rightarrow \frac{1}{RC}\int v_R(t)dt = v_C(t)$$

$$v_R(t) + \frac{1}{RC}\int v_R(t)dt = v_S(t)$$

$$\frac{dv_R(t)}{dt} + \frac{1}{RC}v_R(t) = v_S(t)$$

Resulting differential equation

$$RC\frac{dv_R(t)}{dt} + v_R(t) = RCv_S(t)$$

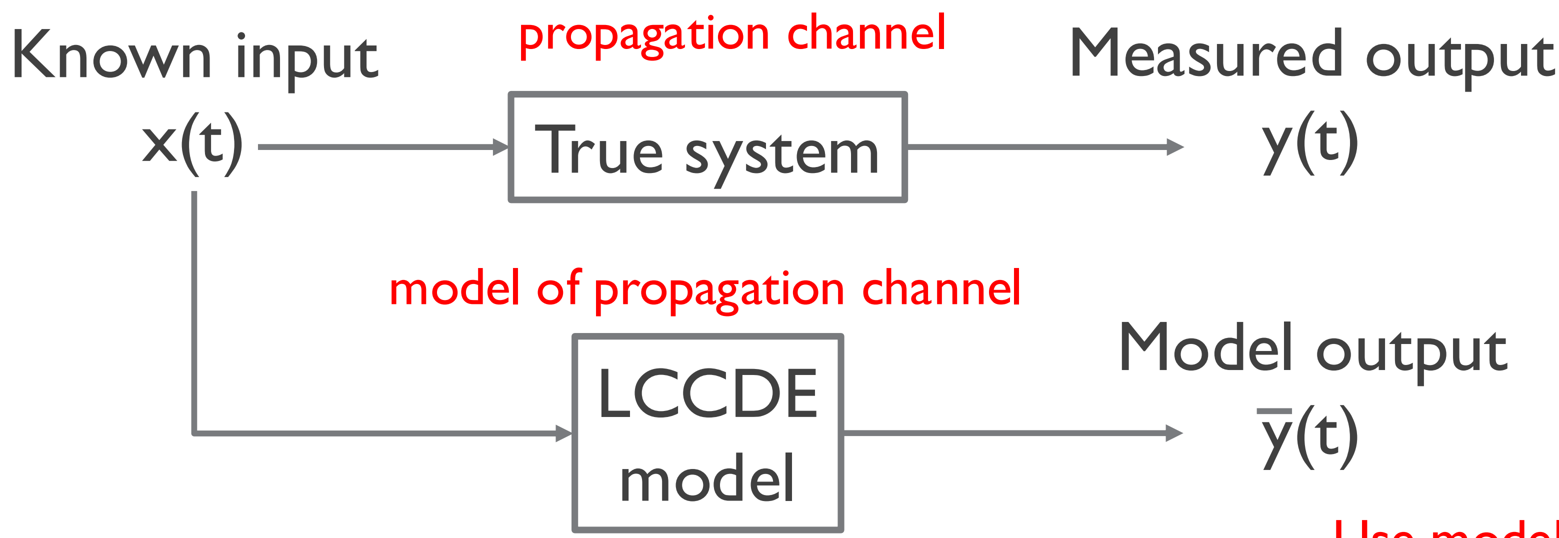
Applications of DEs in Electrical Engineering

- DE may result directly from mathematical fundamentals
  - Electromagnetics, e.g. Maxwell´s equations
  - Passive circuits (RC, RLC examples)

- DE may be used as to model observed phenomena
  - Attenuation on a wire or cable
  - Wireless propagation channels
  - Spectrum utilization
  - Control systems

A common modeling problem

In red: how this works in wireless



Use model to equalize and recover transmitted data

Find the coefficients of the LCCDE such that the model output is a good approximation of the measured output

Why LCCDE's as models?

- Describe a range of phenomena with a few coefficients
- Provide a convenient way to represent LTI systems with long impulse responses (under certain conditions that will explained)
- Can be realized using passive circuits or op-amps
- Solutions to LCCDEs are well understood

N^{th} derivative

General LCCDE relating input and output

$$a_N \frac{d^N y}{dt^N} + a_{N-1} \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t)$$

$$= \; b_M \frac{d^M x}{dt^M} + b_{M-1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x(t)$$

Order of the DE is *max(N,M)*

Solving a differential equation normally involves finding a solution for *y(t)* without all of the derivatives

Differential equations and convolution

- Solution can be decomposed into two parts

$$y(t) = \cancel{y_h}(t) + y_p(t)$$

Homogenous solution

- Depends on the initial conditions not the input
- Must be zero or "at rest" for the system to be LTI

Particular solution

$$x(t) * h(t)$$

- Depends on the input signal and the impulse response of the system

LCCDEs describe LTI systems (when at rest)

What about an LTI system described by a LCCDE?

- Recall that $\frac{d^N}{dt^N} e^{st} = s^N e^{st}$
- Because the system is LTI, it follows that $y(t) = H(s)e^{st}$

$s^k e^{st}$

$s^k e^{st}$

- Inserting into the differential equation $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

$$H(s)\cancel{e^{st}} \sum_{k=0}^{N} a_k s^k = \cancel{e^{st}} \sum_{k=0}^{M} b_k s^k \quad \Rightarrow \quad \boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}$$

Note where the LCCDE coefficients occur

What about the frequency response?

- Recall that the frequency response is determined from $x(t) = e^{j\omega t}$
- Set $s = j\omega$

$(j\omega)^k$

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \Rightarrow \boxed{H(j\omega) = \frac{\sum_{k=0}^{M} b_k \cancel{s^k}}{\sum_{k=0}^{N} a_k \cancel{s^k}}}$$

$(j\omega)^k$

- This is the most common form of the frequency response and will be studied in future lectures, e.g. Bode Plots

Find the frequency response

- Consider the LCCDE that describes an LTI system

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = x(t)$$

- Find the frequency response

$$H(s) = \frac{1}{\frac{1}{2} + s}$$

$$= \frac{2}{1 + 2s}$$

$$\boxed{H(j\omega) = \frac{2}{1 + j2\omega}}$$

Differential equations summary

- Assuming that the system is at rest, linear constant coefficient differential equations are an example of LTI systems

- RLC circuits are described by differential equations

- Systems described by differential equations are among the most widely used in engineering

- The frequency response of a LCCDE has a nice form that depends on the coefficients of the differential equation