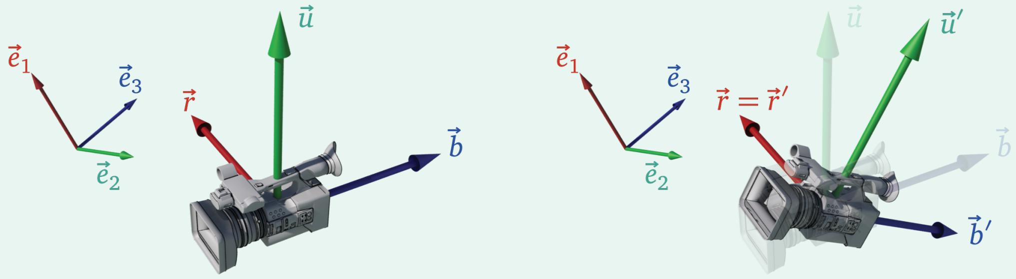
CSE 167 (WI 2025) Exercise 3 — Due 1/31/2025

Exercise 3.1 — 3 pts. Consider an object (such as the camera shown in the figure below), and consider an orthonormal basis $(\vec{r}, \vec{u}, \vec{b})$, where the "right vector" $\vec{r}$ points from left to right of the object, the "up vector" $\vec{u}$ points from bottom to top of the object, and the "back vector" $\vec{b}$ points from front to back of the object.

The rotation operator $R^{\vec{r},\theta}$ **rotates** this object and the orthonormal basis $(\vec{r}, \vec{u}, \vec{b})$ **by angle** θ **about the right vector** $\vec{r}$, producing a new orthonormal basis $(\vec{r}', \vec{u}', \vec{b}')$. In other words $R^{\vec{r},\theta}\vec{r} = \vec{r}', R^{\vec{r},\theta}\vec{u} = \vec{u}', R^{\vec{r},\theta}\vec{b} = \vec{b}'$.



(a) What is the rotation matrix $\mathbf{R}$ for the rotation operator $R^{\vec{r},\theta}$ represented under the basis $(\vec{r}, \vec{u}, \vec{b})$?

$$\begin{bmatrix} R^{\vec{r},\theta}\vec{r} & R^{\vec{r},\theta}\vec{u} & R^{\vec{r},\theta}\vec{b} \end{bmatrix} = \begin{bmatrix} \vec{r} & \vec{u} & \vec{b} \end{bmatrix} \begin{bmatrix} & & \\ & \mathbf{R} & \\ & & \end{bmatrix} \tag{1}$$

Write down the matrix $\mathbf{R}$ in terms of θ and trigonometry functions.

(b) Suppose there is a "world" orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Suppose $\vec{r}, \vec{u}, \vec{b}$ have coefficients $\mathbf{r}, \mathbf{u}, \mathbf{b}$ under this world basis; *i.e.*,

$$\vec{r} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{2}$$

Let $\vec{p} = \begin{bmatrix} \vec{r}' & \vec{u}' & \vec{b}' \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ be an arbitrary point on the rotated object (hence naturally expressed using the basis $(\vec{r}', \vec{u}', \vec{b}')$). If we represent the same vector $\vec{p}$ in terms of the world basis, obviously we get a different list of coefficients $\vec{p} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$.

Find the transformation matrix $\mathbf{M}$ such that

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} & & \\ & \mathbf{M} & \\ & & \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}. \tag{3}$$

Hint *You can use the result of* ***(a)****.*

Exercise 3.2 — 2 pts. What is the quaternion $q \in \mathbb{H}$ so that

$$q\mathbb{i}\overline{q} = \mathbb{j}, \quad q\mathbb{j}\overline{q} = \mathbb{k}, \quad q\mathbb{k}\overline{q} = \mathbb{i}, \quad \mathrm{Re}(q) > 0? \tag{4}$$

Hint *Think of some 3D rotation and its axis and angle.*