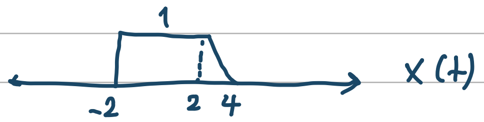
Discussion Session 4:

(1) plot the following signals



a) $x(2t+4)$

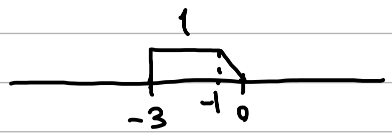
b) $x(\frac{-t}{2}+1)$

c) $3x(-t+1)+2$

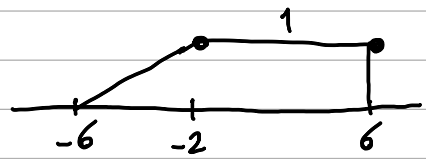
d) $-x(t)\,u(t-1)$

Answers:

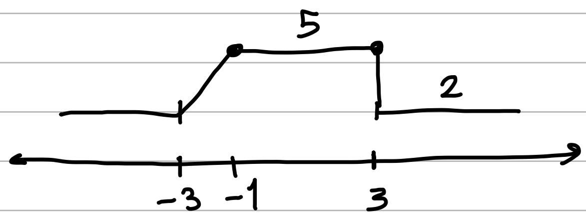
a)



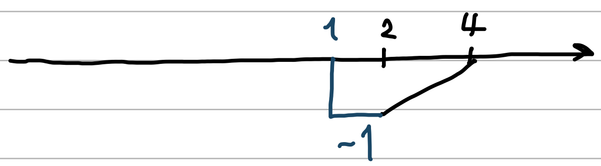
b)



c)



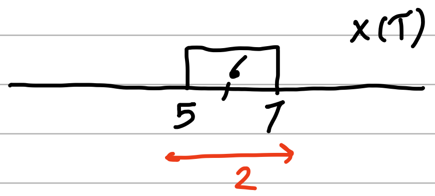
d)

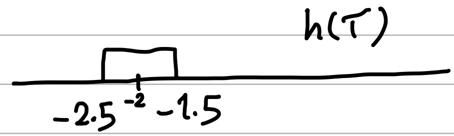


② Find $x(t) * h(t)$

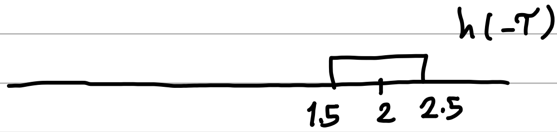
$x(t) = \text{rect}\left(\frac{t}{2} - 3\right)$ $\qquad h(t) = \text{rect}(t+2)$

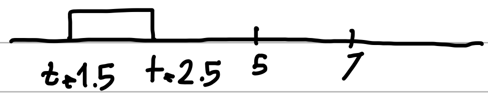
Answer:

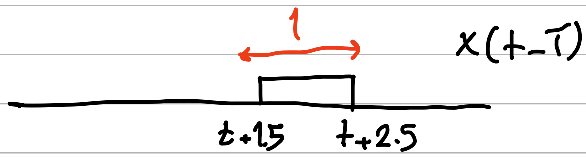




case 1:



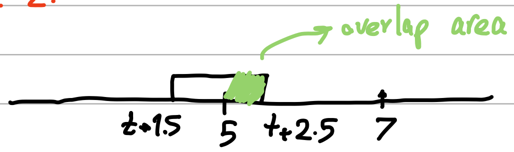




no overlap, $y(t) = 0$, $t+2.5 < 5$

$\longrightarrow t < 2.5$

case 2:



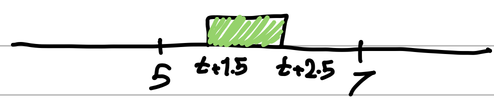
$t+1.5 < 5 < t+2.5 < 7$

① $t < 3.5$ $\qquad$ ② $t > 2.5$ $\qquad$ ③ $t < 4.5$

① ∩ ② ∩ ③ : $2.5 < t < 3.5$

$$y(t) = \int_{5}^{t+2.5} 1 \times 1 \, dT = t - 2.5$$

case 3:

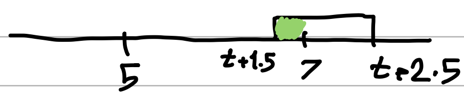


$5 < t+1.5 < t+2.5 < 7$

$t > 3.5$ $\qquad$ $t < 4.5$

$3.5 < t < 4.5$: $\quad y(t) = \int_{t+1.5}^{t+2.5} 1 \, dT = 1$

case 4:



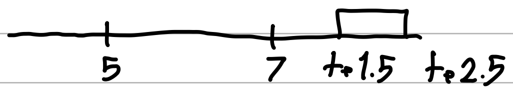
$5 < t+1.5 < 7 < t+2.5$

$t > 3.5$ ① $\qquad$ $t < 5.5$ ② $\qquad$ $t > 4.5$ ③

① ∩ ② ∩ ③ $\Rightarrow 4.5 < t < 5.5$

$$y(t) = \int_{t+1.5}^{7} 1 \, dT = 5.5 - t$$

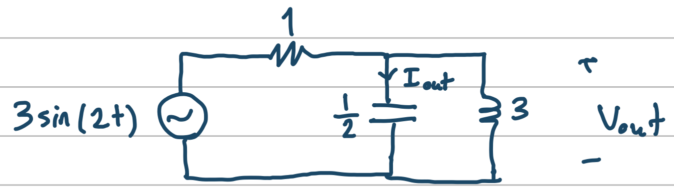
Case 5:



$y(t) = 0, \quad t+1.5 > 7 \longrightarrow t > 5.5$

$$y(t) = \begin{cases} 0 & t \leq 2.5 \\ t-2.5 & 2.5 \leq t \leq 3.5 \\ 1 & 3.5 \leq t \leq 4.5 \\ 5.5-t & 4.5 \leq t \leq 5.5 \\ 0 & t \geq 5.5 \end{cases}$$

③ Find V_{out} and I_{out} with phasor method



Answer

$Z_R = R = 1 \qquad Z_C = \frac{1}{j\omega C} = \frac{1}{j\,2\,\frac{1}{2}} = -j \qquad Z_L = j\omega L = j\omega L = 6j$

$$Z_C \parallel Z_L = \frac{(-j)(6j)}{-j+6j} = \frac{6}{5j} = -1.2j$$

$$Z_T = Z_R + (Z_C \parallel Z_L) = 1 - 1.2j$$

$$V_{in}(t) = 3\sin(2t) = 3\cos\left(2t - \frac{\pi}{2}\right) \longrightarrow V_{in} = 3e^{j\left(-\frac{\pi}{2}\right)} = -3j$$

$$V_{out} = \frac{Z_C \parallel Z_L}{Z_T} V_{in} = \frac{-1.2j}{1-1.2j}(-3j) = \frac{-3.6}{1-1.2j}$$

$$|V_{out}| = \frac{|-3.6|}{|1-1.2j|} = \frac{3.6}{\sqrt{1+(1.2)^2}} \qquad \measuredangle V_{out} = \measuredangle(-3.6) - \measuredangle(1-1.2j) = \pi - \arctan\left(\frac{-1.2}{1}\right) = \pi + \arctan(1.2)$$

$$V_{out} = \frac{3.6}{\sqrt{1+(1.2)^2}} \cos(2t + \pi + \arctan(1.2)) = \frac{-3.6}{\sqrt{1+(1.2)^2}} \cos(2t + \arctan(1.2))$$

$$I_{out} = \frac{V_{out}}{Z_C} = \frac{\frac{-3.6}{1-1.2j}}{-j} = \frac{-3.6j}{1-1.2j} \qquad |I_{out}| = \frac{3.6}{\sqrt{1+(1.2)^2}}$$

$$\measuredangle I_{out} = \measuredangle(-3.6j) - \measuredangle(1-1.2j) = \frac{-\pi}{2} - \arctan\left(\frac{-1.2}{1}\right) = \frac{-\pi}{2} + \arctan(1.2)$$

$$\rightarrow I_{out} = \frac{3.6}{\sqrt{1+(1.2)^2}} \cos\left(2t - \frac{\pi}{2} + \arctan(1.2)\right) = \frac{3.6}{\sqrt{1+(1.2)^2}} \sin(2t + \arctan(1.2))$$

④ LTI checking

a) $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(2t-\tau)\, d\tau$

b) $y(t) = \begin{cases} x(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$

Answer:

$x'(t) = x(t-t_0)$

$y'(t) = S(x'(t)) = \int_{-\infty}^{\infty} x'(\tau)\, h(2t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau-t_0)\, h(2t-\tau)\, d\tau$

taking $\tau - t_0 = z$, $d\tau = dz$, $\tau = z + t_0$

$y'(t) = \int_{-\infty}^{\infty} x(z)\, h(2t-(z+t_0))\, dz = \int_{-\infty}^{\infty} x(z)\, h(2t-z-t_0)\, dz$

$y(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, h(2(t-t_0)-\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, h(2t-\tau-2t_0)\, d\tau$

$y'(t) \neq y(t-t_0) \longrightarrow$ Time variant

$x'(t) = a\, x_1(t) + b\, x_2(t)$

$y'(t) = \int_{-\infty}^{\infty} x'(\tau)\, h(2t-\tau)\, d\tau = \int_{-\infty}^{\infty} (a x_1(\tau) + b x_2(\tau))\, h(2t-\tau)\, d\tau$

$= a \int_{-\infty}^{\infty} x_1(\tau)\, h(2t-\tau)\, d\tau + b \int_{-\infty}^{\infty} x_2(\tau)\, h(2t-\tau)\, d\tau$

$= a\, y_1(t) + b\, y_2(t)$

Linea ✓

b) Checking for LTI

$x'(t) = x(t-t_0)$

$$y'(t) = \begin{cases} x'(t) & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} x(t-t_0) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$y(t-t_0) = \begin{cases} x(t-t_0) & t-t_0 \geq 0 \\ 0 & t-t_0 < 0 \end{cases}$$

As $y'(t) \neq y(t-t_0) \longrightarrow$ Time invariant.

Checking for Linearity:

$x'(t) = a x_1(t) + b x_2(t)$

$$y'(t) = \begin{cases} x'(t) & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} a x_1(t) + b x_2(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= a \begin{cases} x_1(t) & t \geq 0 \\ 0 & t < 0 \end{cases} + b \begin{cases} x_2(t) & t \geq 0 \\ 0 & t < 0 \end{cases} = a\, y_1(t) + b\, y_2(t)$$

Transfer Function: Laplace transform of the impulse response.

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$x(t) = e^{st} \longrightarrow \boxed{LTI} \longrightarrow y(t) = H(s)\, e^{st}$$

Eigen function of a LTI system

Some properties of laplace: ($x(t)$: a caus

$L\{a f(t) + b g(t)\} = a F(s) + b G(s)$ Linearity

$L\{f(t-T)\, u(t-T)\} = e^{-Ts} F(s)$ Time Shifting

$L\{e^{at} f(t)\} = F(s-a)$ Frequency Shifting

$L\{f'(t)\} = s F(s) - f(0^+)$ Differentiation in time domain

$L\{f''(t)\} = sL\{f'(t)\} - f'(0^+) = s^2 F(s) - s F(0^+) - f'(0^+)$

$L\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}$ Integration in time domain

$L\{f(t) * g(t)\} = F(s)\, G(s)$ Convolution Theorem

$L\{t f(t)\} = \frac{-d}{ds} F(s)$ Differentiation is s domain

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$$\begin{cases} L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} u(t) \\ L^{-1}\left\{\frac{1}{(s+a)^2}\right\} = t e^{-at} u(t) \end{cases}$$

Partial Fraction: $\frac{N(s)}{D(s)}$ (degree of $N(s)$ is less than degree of $D(s)$)

Example: $y(s) = \frac{3}{s(s+2)}$ (no repeated factor $(s-a)^2$, no irreducible quatratic factor (s^2+w^2))

$$= \frac{A}{s} + \frac{B}{s+2}$$

$$\left.\begin{aligned} A &= s y(s)\Big|_{s=0} = \left(\frac{3}{s+2}\right)\Big|_{s=0} = \frac{3}{2} \\ B &= (s+2)\, y(s)\Big|_{s=-2} = \frac{3}{s}\Big|_{s=-2} = \frac{-3}{2} \end{aligned}\right\} \longrightarrow \frac{3}{s(s+2)} = \frac{3/2}{s} - \frac{3/2}{s+2}$$

Example #2: $y(s) = \frac{5s+7}{(s+1)^2}$ (Repeated linear factor)

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$B = \left((s+1)^2\, Y(s)\right)\Big|_{s=-1} = 5s+7\Big|_{s=-1} = 2$$

$$A = \left(\frac{d}{ds}\,(s+1)^2\, y(s)\right)\Big|_{s=-1} = \left(\frac{d}{ds}\,(5s+7)\right)\Big|_{s=-2} = 5$$

Example #3: what is $y(t)$ in example #2?

$$y(s) = \frac{5}{s+1} + \frac{2}{(s+1)^2}$$

$$\frac{5}{s+1} \rightarrow 5e^{-t}\,u(t)$$

$$\frac{2}{(s+1)^2} \rightarrow 2te^{-t}\,u(t)$$

$$\longrightarrow y(t) = (5+2t)\, e^{-t}\, u(t)$$