UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Spring 2025
Linear Systems Fundamentals

SOLUTIONS TO MIDTERM EXAM

You are allowed one 2-sided sheet of notes.

No books, no other notes, no calculators.

PRINT YOUR NAME Leopold Kronecker

Signature $\delta[n]$

Student ID Number $A0001000$

Your signature confirms that you have completed this exam in accordance with the ECE 101 Academic Integrity Agreement.

Problem	Weight	Score
1	32 pts	32
2	34 pts	34
3	34 pts	34
Total	100 pts	100

Please do not begin until told.
Show your work.
Use back of previous page and attached scratch sheets as needed.
Tables 3.1 and 3.2 from the textbook are attached to the exam.
Scan, upload, and tag your solutions on Gradescope at the end of the exam.

Good luck!

Name/Student ID: ____________________

Problem 1 [DT Signals and Systems] (32 points)

Consider the discrete-time system whose output $y[n]$ in response to input $x[n]$ is given by:

$$y[n] = x[2n-1] + x[3n-1].$$

(a) (10 points)

Determine and sketch precisely the impulse response $h[n]$ of the system.

(b) (10 points)

Determine and sketch precisely the step response $s[n]$ of the system.

(c) (12 points)

Check the appropriate box indicating whether or not the system satisfies the specified property.

Justify your answers by either proving that the property holds for the system, or giving a counterexample with specific signals if it does not.

True	False	
☐	☐	Invertible
☐	☐	Stable
☐	☐	Linear
☐	☐	Time-invariant

Do not write your answers on this page.

Write your answers to parts (a), (b), and (c) on the following pages.

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Problem 1 [DT Signals and Systems] (cont.)

Consider the discrete-time system whose output $y[n]$ in response to input $x[n]$ is given by: $y[n] = x[2n-1] + x[3n-1]$.

(a) Determine and sketch precisely the impulse response $h[n]$ of the system.

$h[n] = \delta[2n-1] + \delta[3n-1]$

$\delta[2n-k]$ is obtained from $\delta[n]$ by "shift by k" and "decimate by 2."

So, as dervied in lecture,

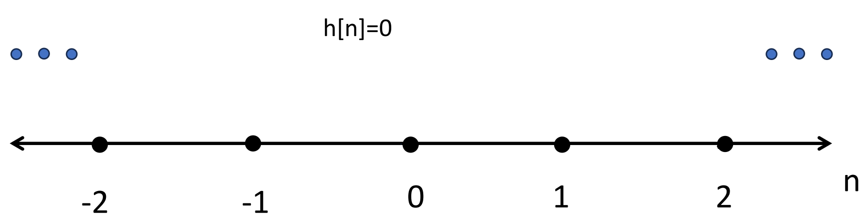
$$\delta[2n-k] = \begin{cases} 0, & k \text{ odd} \\ \delta[m] & k = 2m,\ m \in \mathbb{Z}. \end{cases}$$

Similarly, $\delta[3n-k]$ is obtained by "shift by k" and "decimate by 3."

So,

$$\delta[3n-k] = \begin{cases} 0, & k \text{ odd} \\ \delta[m] & k = 3m,\ m \in \mathbb{Z} \end{cases}$$

Here $k = 1$, and we have $h[n] = 0$, for all n.



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Problem 1 [DT Signals and Systems] (cont.)

Consider the discrete-time system whose output $y[n]$ in response to input $x[n]$ is given by: $y[n] = x[2n-1] + x[3n-1]$.

(b) Determine and sketch precisely the step response $s[n]$ of the system.

Set $x[n] = u[n]$.

Then, $s[n] = u[2n-1] + u[3n-1]$

$u[2n-k]$ is obtained from $u[n]$ by "shift by k" and "decimate by 2."

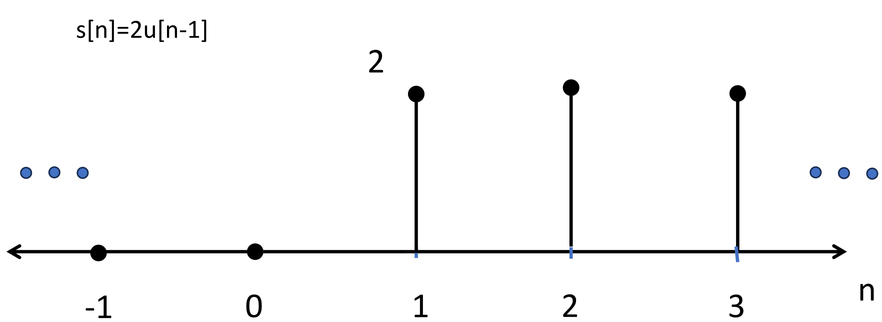
So, $u[2n-k] = 1$ for $2n \geq k$, or $u[2n-k] = u[n - \lceil k/2 \rceil]$.

Similarly, $u[3n-k]$ is obtained by "shift by k" and "decimate by 3."

So, $u[3n-k] = u[n - \lceil k/3 \rceil]$.

In particular,

$$\begin{aligned} s[n] &= u[2n-1] + u[3n-1] \\ &= 2u[n-1] \end{aligned}$$



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Problem 1 [DT Signals and Systems] (cont.)

Consider the discrete-time system whose output $y[n]$ in response to input $x[n]$ is given by: $y[n] = x[2n-1] + x[3n-1]$.

(c) Check the appropriate box indicating whether or not the system satisfies the specified property. **Justify your answers by either proving the property holds or giving a counterexample with specific signals.**

True	False	
☐	☒	Invertible

Referring to part (a), since $x[n] = \delta[n]$ and $x[n] = 0$ both produce the same output $y[n] = 0$, the system is not invertible.

True	False	
☒	☐	Stable

Suppose $x[n] \leq B$. For all n, the output $y[n]$ satisfies

$$\begin{aligned} |y[n]| &= |x[2n-1] + x[3n-1]| \\ &\leq |x[2n-1]| + |x[3n-1]| \\ &\leq B + B \leq 2B < \infty \end{aligned}$$

Bounded inputs produces a bounded outputs. The system is stable.

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Problem 1 [DT Signals and Systems] (cont.)

Consider the discrete-time system whose output $y[n]$ in response to input $x[n]$ is given by: $y[n] = x[2n-1] + x[3n-1]$.

(c) Check the appropriate box indicating whether or not the system satisfies the specified property. **Justify your answers by either proving the property holds or giving a counterexample with specific signals.**

True	False	
☒	☐	Linear

Suppose input $x_1[n]$ produces output $y_1[n] = x_1[2n-1] + x_1[3n-1]$ and input $x_2[n]$ produces output $y_2[n] = x_2[2n-1] + x_2[3n-1]$. The input $x_3[n] = x_1[n] + x_2[n]$ produces the output

$$\begin{aligned} y_3[n] &= x_3[2n-1] + x_3[3n-1] \\ &= x_1[2n-1] + x_2[2n-1] + x_1[3n-1] + x_2[3n-1] \\ &= y_1[n] + y_2[n]. \end{aligned}$$

so the system is additive. Similarly, the input $x_4[n] = ax[n]$ produces output

$$y_4[n] = x_4[2n-1] + x_4[3n-1] = ax[2n-1] + ax[3n-1] = ay[n].$$

So, the system is scalable. Therefore, it is linear.

☐ ☒ Time-invariant

From part (a), $x[n] = \delta[n]$ produces output $y[n] = 0$.

Let $x_5[n] = x[n-1] = \delta[n-1]$.

It produces the output

$y_5[n] = x_5[2n-1] + x_5[3n-1] = \delta[2n-2] + \delta[3n-2] = \delta[n-1]$.

Since $y_5[n] \neq y[n-1]$, the system in not time-invariant.

Name/Student ID: ____

Problem 2 [DT - LTI Systems] (34 points)

Let S be the discrete-time LTI system with impulse response:

$$h[n] = \left(\frac{1}{4}\right)^n u[n].$$

(a) (10 points)

Determine the step response $s[n]$ of system S.

(b) (12 points)

Determine the transfer function $H(z)$ of S and express it as a ratio of polynomials in z. Does the frequency response of the system vanish at any frequencies ω? If so, determine the frequencies. If not, explain why not.

(c) (12 points) For each of the following input signals $x[n]$, determine the corresponding output signal $y[n]$. Determine $y[0]$ and express it as a complex number in Cartesian form.

(i) $x[n] = 1$

(ii) $x[n] = (\frac{j}{4})^n$

Do not write your answers on this page.

Write your answers to parts (a), (b), and (c) on the following pages.

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Problem 2 [DT - LTI Systems](cont.)

Let S be the DT - LTI system with impulse response: $h[n] = \left(\frac{1}{4}\right)^n u[n]$.

(a) Determine the step response $s[n]$ of system S.

$$
\begin{aligned}
s[n] &= \sum_{k=-\infty}^{n} h[k] \\
&= \sum_{k=-\infty}^{n} (1/4)^k u[k] \\
&= \sum_{k=0}^{n} (1/4)^k \\
&= \begin{cases} 0, & n < 0 \\ \frac{1-(1/4)^{n+1}}{1-(1/4)} & \end{cases} \\
&= \frac{4}{3}(1 - (1/4)^{n+1}) u[n]
\end{aligned}
$$

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Problem 2 [DT - LTI Systems] (cont.)

Let S be the DT - LTI system with impulse response: $h[n] = \left(\frac{1}{4}\right)^n u[n]$.

(b) Determine the transfer function $H(z)$ of S and express it as a ratio of polynomials in z. Does the frequency response of the system vanish at any frequencies ω? If so, determine the frequencies. If not, explain why not.

The transfer function $H(z)$ is given by

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} ({}^1\!/\!{}_4)^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} ({}^1\!/\!{}_{4z})^n \\ &= \frac{1}{1-({}^1\!/\!{}_{4z})} \\ &= \frac{4z}{-1+4z} \end{aligned}$$

The frequency response $H(e^{j\omega})$ is then

$$H(e^{j\omega}) = \frac{4e^{j\omega}}{-1+4e^{j\omega}}.$$

The denominator has bounded magnitude since

$$|-1+4e^{j\omega}| \le |-1| + 4|e^{j\omega}| \le 1+4 = 5.$$

The numerator satisfies $|4e^{j\omega}| = 4$, so it is never zero.

So, the frequency response $H(e^{j\omega})$ does not vanish at any frequency ω.

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Problem 2 [DT - LTI Systems] (cont.)

Let S be the DT - LTI system with impulse response: $h[n] = \left(\frac{1}{4}\right)^n u[n]$.

(c) For each of the following input signals $x[n]$, determine the corresponding output signal $y[n]$. Determine $y[0]$ and express it as a complex number in Cartesian form.

(i) $x[n] = 1$ Refer to the eigenfunction property: $z^n \longrightarrow H(z)z^n$.

From part (b),
$H(1) = \frac{4z}{4z-1}\Big|_{z=1} = \frac{4}{3}$.
So,
$x[n] = 1 \longrightarrow y[n] = \frac{4}{3}$.
and
$y[0] = \frac{4}{3}$.

(ii) $x[n] = (\frac{j}{4})^n$

Refer again to the eigenfunction property: $z^n \longrightarrow H(z)z^n$.
From part (b),

$$\begin{aligned} H(\frac{j}{4}) &= \frac{4z}{4z-1}\Big|_{z=\frac{j}{4}} \\ &= \frac{j}{j-1} \\ &= \frac{1-j}{2}. \end{aligned}$$

So
$x[n] = (\frac{j}{4})^n \longrightarrow y[n] = \frac{1-j}{2}(\frac{j}{4})^n$
and
$y[0] = \frac{1-j}{2} = \frac{1}{2} - \frac{j}{2}$.

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Problem 3 [DTFS and DT Filtering] (34 points)

(a) (12 points)

Let $x[n]$ be a **real-valued** discrete-time periodic signal with fundamental period $N = 4$ and Fourier Series coefficients a_k. Assume

$$a_{-1} = \frac{1}{4}(1-j), a_2 > 0, \sum_{n=0}^{3} x[n] = 0, \sum_{n=0}^{3} |x[n]|^2 = 2.$$

Determine the coefficients a_0, a_1, a_2, a_3.

(b) (12 points)

Consider a discrete-time signal $x[n]$ with fundamental period $N = 4$. Assume $x[0] = 0, x[1] = 1, x[2] = -1, x[3] = 0$. Determine its Fourier series coefficients a_k, for $k = 0, 1, 2, 3$.

(c) (10 points)

The signal $x[n]$ of part (b) is passed through an LTI system with frequency response $H(e^{j\omega}) = e^{-j\omega}$. Express the Fourier series coefficients b_k of the output signal $y[n]$ in terms of the Fourier coefficients a_k of $x[n]$, and evaluate b_k, for $k = 0, 1, 2, 3$. Determine the signal $y[n]$ **without using the synthesis equation**.

Do not write your answers on this page.

Write your answers to parts (a), (b), and (c) on the following pages.

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Problem 3 [DTFS and DT Filtering] (cont.)

(a) Let $x[n]$ be a **real-valued** discrete-time periodic signal with fundamental period $N = 4$ and Fourier Series coefficients a_k. Assume

$$a_{-1} = \frac{1}{4}(1-j), a_2 > 0, \sum_{n=0}^{3} x[n] = 0, \sum_{n=0}^{3} |x[n]|^2 = 2.$$

Determine the coefficients a_0, a_1, a_2, a_3.

For a_1:

Use the Conjugate Symmetry for Real Signals property:

$a_{-1} = a_1^* = \frac{1}{4}(1-j)$ so $a_1 = \frac{1}{4}(1+j)$.

For a_3: Use the periodicity property: $a_{-1} = a_3$, so $a_3 = \frac{1}{4}(1-j)$

For a_0:

Use the Analysis Equation for $k = 0$: $a_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = 0$.

For a_2:

Use Parseval's Relation:

$\sum_{k=0}^{3} |a_k|^2 = \frac{1}{4}\sum_{n=0}^{3} |x[n]|^2 = \frac{2}{4} = \frac{1}{2}$.

$\sum_{k=0}^{3} |a_k|^2 = 0 + |\frac{1}{4}(1+j)|^2 + |a_2|^2 + |\frac{1}{4}(1-j)|^2 = \frac{1}{2}$

Convclude $|a_2|^2 = \frac{1}{4}$, so, since $a_2 > 0$, $a_2 = \frac{1}{2}$.

Final answer:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= \frac{1}{4}(1+j) \\ a_2 &= \frac{1}{2} \\ a_3 &= \frac{1}{4}(1-j) \end{aligned}$$

To confirm that a_2 has to be real:

Use the Conjugate Symmetry for Real Signals property: $a_{-2} = a_2^*$.

From the periodicity property: $a_{-2} = a_2$, so $a_2 = a_2^*$, implying a_2 is **real**.

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Problem 3 [DTFS and DT Filtering] (cont.)

(b) Consider a discrete-time signal $x[n]$ with fundamental period $N = 4$. Assume $x[0] = 0, x[1] = 1, x[2] = -1, x[3] = 0$. Determine its Fourier series coefficients a_k, for $k = 0, 1, 2, 3$.

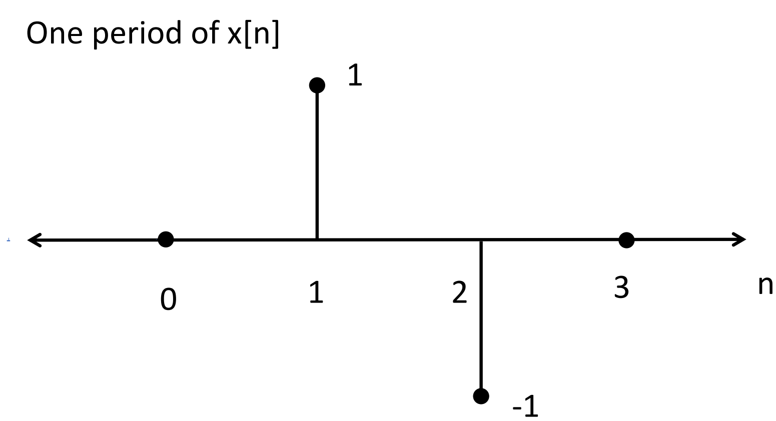
The fundamental frequency is $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$.

Using the Analysis Equation:

$$\begin{aligned} a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk\frac{\pi}{2}n} \\ &= \frac{1}{4}[0 + e^{-jk\frac{\pi}{2}} - e^{-jk\pi} + 0] \\ &= \frac{1}{4}[(-j)^k - (-1)^k]. \end{aligned}$$

So,

$$\begin{aligned} a_0 &= \frac{1}{4}(1-1) = 0. \\ a_1 &= \frac{1}{4}(-j-(-1)) = \frac{1}{4}(1-j). \\ a_2 &= \frac{1}{4}((-j)^2-(-1)^2) = \frac{1}{4}(-1-1) = -\frac{1}{2}. \\ a_3 &= \frac{1}{4}((-j)^3-(-1)^3) = \frac{1}{4}(1+j). \end{aligned}$$



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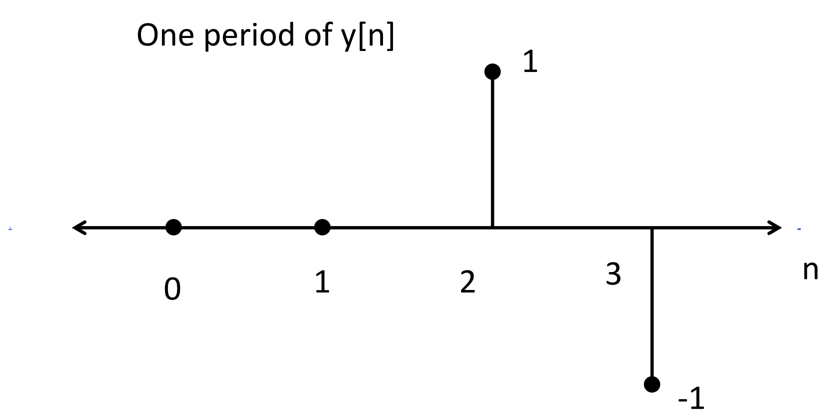
Problem 3 [DTFS and DT Filtering] (cont.)

(c) The signal $x[n]$ of part (b) is passed through an LTI system with frequency response $H(e^{j\omega}) = e^{-j\omega}$. Express the Fourier series coefficients b_k of the output signal $y[n]$ in terms of the Fourier coefficients a_k of $x[n]$, and evaluate b_k, for $k = 0, 1, 2, 3$. Determine the signal $y[n]$ **without using the synthesis equation**.

From the eigenfunction property of LTI systems, the output is
$y[n] = \sum_{k=0}^{3} b_k e^{jk\frac{\pi}{2}n} = \sum_{k=0}^{3} H(e^{jk\frac{\pi}{2}}) a_k e^{jk\frac{\pi}{2}n}$.
Here $H(e^{j\omega}) = e^{-j\omega}$, so, for k=0, 1, 2, 3:
$H(1) = 1$.
$H(e^{j\frac{\pi}{2}}) = e^{-j\frac{\pi}{2}} = -j$.
$H(e^{j\pi}) = e^{-j\pi} = -1$.
$H(e^{j3\frac{\pi}{2}}) = e^{-j\frac{3\pi}{2}} = j$.
From part (b)
$b_0 = 0$.
$b_1 = -j\frac{1}{4}(1-j) = \frac{1}{4}(-1-j)$.
$b_2 = (-1)(-\frac{1}{2}) = \frac{1}{2}$.
$b_3 = j\frac{1}{4}(1+j) = \frac{1}{4}(-1+j)$.
Their fundamental period is 4, so b_k are Fourier series coefficients of $y[n]$.
$H(e^{j\omega}) = e^{-j\omega}$ corresponds to a time delay of $n_0 = 1$, so $y[n] = x[n-1]$.
Therefore $y[0] = 0, x[1] = 0, x[2] = 1, x[3] = -1$.



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