ECE 45 – Circuits and Systems
Winter 2025

Homework #8
Due: February 27 at 11:59pm, submitted via GradeScope.

You can make multiple upload attempts to experiment with the system and the best way to upload. You must correctly mark the answers to the problems in GradeScope, e.g. problem 1, problem 2, problem 3, to get full credit. Note that you must tag your problems when uploading to GradeScope or they will not be graded and you will not receive credit. Any regrade requests must be placed through GradeScope within one week of the return of the homework.

Remember, discussion of homework questions is encouraged. Please be absolutely sure to submit your own independent homework solution.

1. (70 %) Compute the following Fourier transform or inverse Fourier transform as requested. You may compute these using the integral form or the common transform pairs and properties (but be sure to explain your work).

   (a) $x(t) = 1 - |t|$ for $t \in [0, 1]$ otherwise $x(t) = 0$
   Solution: The Fourier transform is given by:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

   Substituting $x(t)$:

$$\begin{aligned}
X(j\omega) &= \int_0^1 (1-t)e^{-j\omega t}dt \\
&= \int_0^1 e^{-j\omega t}dt - \int_0^1 te^{-j\omega t}dt \\
&= \frac{e^{-j\omega t}}{-j\omega}\Big|_0^1 - \frac{te^{-j\omega t}}{-j\omega}\Big|_0^1 + \int_0^1 \frac{e^{-j\omega t}}{-j\omega}dt \\
&= \frac{1-e^{-j\omega}}{j\omega} + \frac{e^{-j\omega}}{j\omega} + \frac{e^{-j\omega t}}{(-j\omega)^2}\Big|_0^1 \\
&= \frac{1}{j\omega} + \frac{1-e^{-j\omega}}{\omega^2}
\end{aligned}$$

   (b) $x(t) = \frac{\sin(3\pi t)}{t}$
   Solution:

$$x(t) = \frac{\sin(3\pi t)}{t} = 3\pi\frac{\sin(3\pi t)}{3\pi t} = 3\pi \text{sinc}(3t)$$

$$
\begin{aligned}
x(t) = sinc(t) &\Longleftrightarrow \text{rect}(\frac{w}{2\pi}) \\
x(t) = sinc(3t) &\Longleftrightarrow \frac{1}{3}\text{rect}(\frac{w/3}{2\pi}) \\
x(t) = 3\pi sinc(3t) &\Longleftrightarrow \frac{3\pi}{3}\text{rect}(\frac{w/3}{2\pi}) \\
x(t) = 3\pi sinc(3t) &\Longleftrightarrow \pi\text{rect}(\frac{w}{6\pi})
\end{aligned}
$$

(c) $x(t) = \sum_{n=-\infty}^{\infty} \text{rect}(2t-n)$

Solution:

From previous lectures, $x(t) \overset{FS}{\longleftrightarrow} \frac{2T_1}{T}\text{sinc}\left(\frac{k2T_1}{T}\right)$, where $T_1 = \frac{1}{4}$, $T = \frac{1}{2}$, and $\omega_0 = 4\pi$. Using the property $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$:

$$
\begin{aligned}
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} \text{sinc}(k)\delta(\omega - 4\pi k) \\
&= 2\pi\delta(\omega)
\end{aligned}
$$

(d) Fourier transform of $x(t)$, which is **periodic** with $T = 10$ signal and first period $t \in [-5, 5)$ given by

$$
x(t) = \text{rect}\,(t-1) - \text{rect}\,(t+1)\,.
$$

Solution: Using the Fourier transform of periodic signals:

$$
x(t) \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)
$$

where

$$
a_k = \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right)\left(e^{-jk\omega_0} - e^{jk\omega_0}\right), \quad \omega_0 = \frac{\pi}{5}
$$

Thus, the Fourier transform becomes:

$$
\begin{aligned}
x(t) &\leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right)\left(e^{-jk\frac{\pi}{5}} - e^{jk\frac{\pi}{5}}\right)\delta\left(\omega - k\frac{\pi}{5}\right) \\
x(t) &\leftrightarrow \frac{-2\pi j}{5} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{10}\right)\sin(\frac{k\pi}{5})\delta\left(\omega - k\frac{\pi}{5}\right)
\end{aligned}
$$

(e) Inverse Fourier transform of $Y(j\omega) = e^{j\omega^2}\delta(\omega+1) + e^{-j\omega^2}\delta(\omega-1) + e^{j\omega/2}\frac{1}{\frac{1}{2}+j\omega}$.

Solution: Utilizing Fourier pairs and time shifting:

$$
\begin{aligned}
e^{j\omega^2}\delta(\omega+1) = e^{j}\delta(\omega+1) &\Longleftrightarrow \frac{e^{j}e^{-jt}}{2\pi} \\
e^{-j\omega^2}\delta(\omega-1) = e^{-j}\delta(\omega-1) &\Longleftrightarrow \frac{e^{-j}e^{jt}}{2\pi}
\end{aligned}
$$

Also,

$$\frac{1}{\frac{1}{2}+j\omega} \Longleftrightarrow e^{-\frac{1}{2}t}u(t)$$
$$e^{j\omega/2}\frac{1}{\frac{1}{2}+j\omega} \Longleftrightarrow e^{-\frac{1}{2}(t+\frac{1}{2})}u(t+\frac{1}{2})$$

combine:

$$x(t) = \frac{e^{j1-j1t}}{2\pi} + \frac{e^{-j1+jt}}{2\pi} + e^{-\frac{1}{2}(t+\frac{1}{2})}u(t+\frac{1}{2})$$

(f) Inverse Fourier transform of $Q(j\omega) = 2e^{j2\omega}\text{rect}(2\omega)$.
Solution: From the lecture note, we know that

$$\text{rect}(\frac{\omega}{\mathcal{B}}) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\mathcal{B}}{2\pi}\text{sinc}(\frac{\mathcal{B}t}{2\pi})$$

Therefore,

$$\text{rect}(2\omega) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{4\pi}\text{sinc}(\frac{t}{4\pi})$$

Moreover, we know

$$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$$

Thus,

$$\begin{aligned} 2e^{j2\omega}\text{rect}(2\omega) &\leftrightarrow 2\frac{1}{4\pi}\text{sinc}(\frac{t+2}{4\pi}) \\ &= \frac{1}{2\pi}\text{sinc}(\frac{t+2}{4\pi}) \end{aligned}$$

2. (10 %) Find an example of a signal that does not satisfy the Dirichlet sufficient conditions for a Fourier transform to exist, and is not an example in our textbook. Describe the signal and give the details about how you found it (e.g., searched online, used chatGPT with the following prompts, etc but not asking your friend).

Solution: (An example provided by Perplexity using the prompt "Find an example of a signal that does not satisfy the Dirichlet sufficient conditions for a Fourier transform to exist. Give the derivation of the Fourier transform.")

The Fourier transform of $\frac{1}{\sqrt{|t|}}$ does not satisfy the absolute integrability condition of the Dirichlet criteria ($\int_{-\infty}^{\infty}|x(t)|\,dt$ diverges), but it still has a valid Fourier transform under distribution theory. The result is:

$$\mathcal{F}\left\{\frac{1}{\sqrt{|t|}}\right\}(\omega) = \sqrt{\frac{2\pi}{|\omega|}}$$

Derivation

1. Symmetry and Simplification:

$$\mathcal{F}\left\{\frac{1}{\sqrt{|t|}}\right\}(\omega) = 2\int_0^{\infty}\frac{\cos(\omega t)}{\sqrt{t}}\,dt$$

(Exploiting even symmetry of $\frac{1}{\sqrt{|t|}}$).

2. Substitution: Let $u = \omega t$, so $t = u/\omega$, $dt = du/\omega$. The integral becomes:

$$\frac{2}{\sqrt{|\omega|}} \int_0^{\infty} \frac{\cos(u)}{\sqrt{u}}\, du$$

3. Fresnel Integral Evaluation: Let $u = x^2$, $du = 2x\, dx$:

$$\begin{aligned}\int_0^{\infty} \frac{\cos(u)}{\sqrt{u}}\, du &= 2 \int_0^{\infty} \cos(x^2)\, dx \\ &= 2 \cdot \sqrt{\frac{\pi}{8}} \\ &= \sqrt{\frac{\pi}{2}}\end{aligned}$$

(Using the Fresnel integral result $\int_0^{\infty} \cos(x^2)\, dx = \sqrt{\frac{\pi}{8}}$).

4. Final Result: Combining steps:

$$\frac{2}{\sqrt{|\omega|}} \cdot \sqrt{\frac{\pi}{2}} = \sqrt{\frac{2\pi}{|\omega|}}$$

Summary

The Dirichlet conditions are not satisfied due to the divergence of $\int_{-\infty}^{\infty} \frac{1}{\sqrt{|t|}}\, dt$. The Fourier transform exists in the distributional sense, leveraging symmetry and substitutions to evaluate otherwise divergent integrals.

3. (20 %) Based on the solutions, correct your previous week's homework using a colored pen (or annotation) so it's obvious what you've corrected. If you got a problem exactly right, just use a red check mark to indicate as such.