Lecture 6

More convolution

Preview of today's lecture

- Convolution
  - Determine the output of an LTI system using the convolution
  - Compute the continuous-time convolution between two signals

- Convolution properties
  - Summarize key properties of convolution
  - Leverage properties of convolution to simplify its calculations

Review

- Output of an LTI system is completely characterized by the impulse response of the system

$$\delta(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow h(t)$$

- Input and output of an LTI system are related through convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11-12 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 13 - 17 Fourier transform

Fourier

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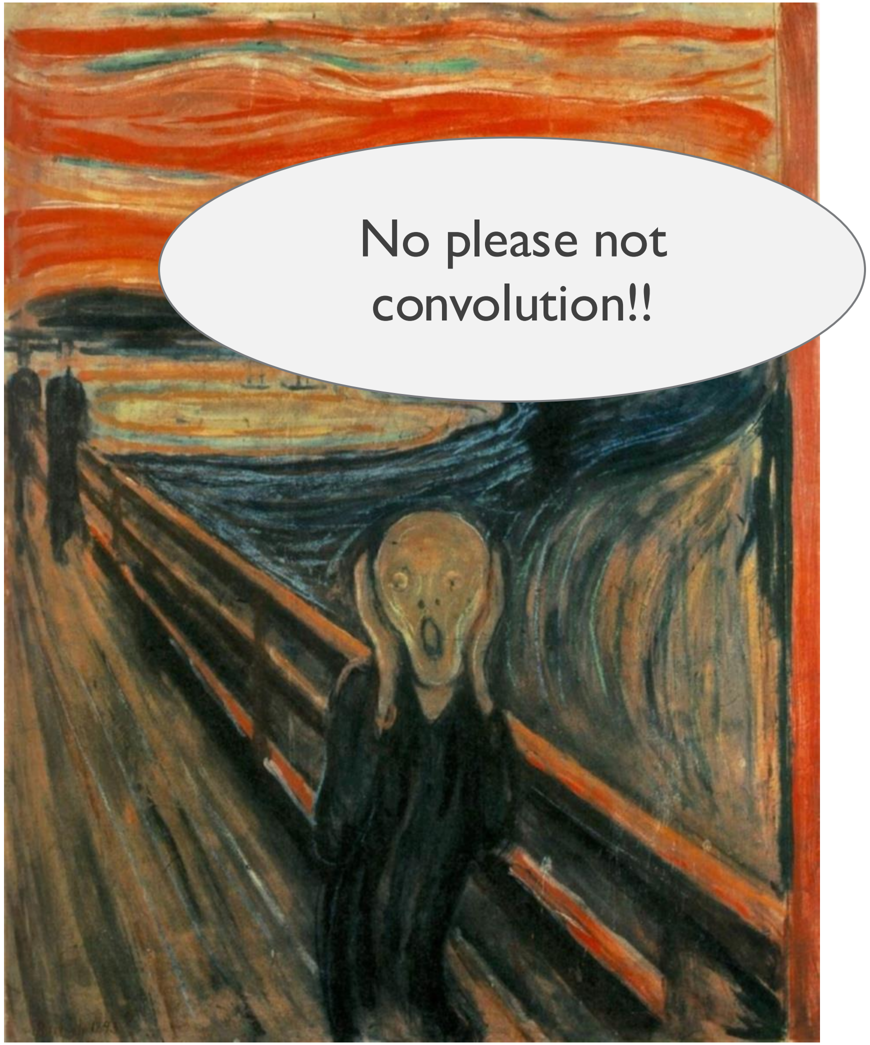
	Date	Theme	Topic	Readings	Out	In
1	1/7	Signals	Signals, systems, circuits and phasors	1.1	HW1	
2	1/9	Signals	Rectangle, step functions, signal transformations, periodic, even and odd	1.2	HW2	HW1
3	1/14	Signals	Exponential, sinusoids, complex exponentials, Dirac Delta	1.3 - 1.4		
4	1/16	LTI in time	Systems, linearity, time invariance	1.6.5, 1.6.6	HW3	HW2
5	1/21	LTI in time	Impulse response and convolution	2.2		
6	1/23	LTI in time	Convolution with a sinusoid, connection to phasors	2.2	HW4	HW3
7	1/28	LTI in time	Convolution properties	2.3		
8	1/30	Fourier series	Fourier series	3.1 - 3.3	HW5	HW4
	2/4		**Midterm 1**			
9	2/6	Fourier series	Fourier series convergence and properties	3.4	HW6	HW5
10	2/11	Fourier series	Fourier series properties	3.5		
11	2/13	LTI in frequency	Frequency response of LTI systems	3.9	HW7	HW6
12	2/18	LTI in frequency	Filters, bode plots	3.10, 6.2.3		
13	2/20	Fourier transform	Fourier transform	4.1-4.2	HW8	HW7
	2/25		**Midterm 2**			
14	2/27	Fourier transform	Fourier transform properties	4.3	HW9	HW8
15	3/4	Fourier transform	Rectangle and sinc functions	4.3		
16	3/6	Fourier transform	Convolution property	4.4	HW10	
17	3/11	Fourier transform	Multiplication property	4.5		
18	3/13	Sampling	Sampling theorem	7.1		HW10
	3/19		**Final exam Tuesday 3-6pm**			

Convolution

Learning objectives

- Determine the output of an LTI system using the convolution
- Compute the continuous-time convolution between two signals



https://commons.wikimedia.org/wiki/File:The_Scream.jpg with additions by Prof. Heath

Uncovering the convolution

- Consider an LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- The output can be computed from the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Basic convolution properties

Shorthand notation

- Commutative

$$y(t) = x(t) * h(t) = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau$$

Choose option that makes it easy!

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

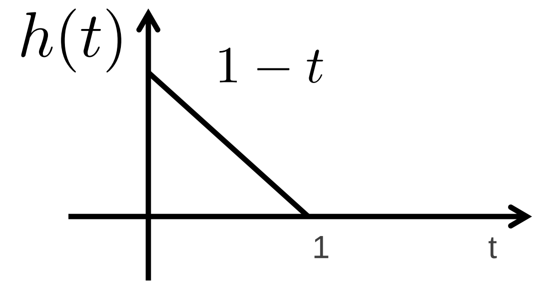
Use properties to simplify convolutions (more next lecture)

CT convolution example #1

This is a typical example using two finite length signals. Rectangles and triangle functions are common in examples / HW as they give results that are easy to integrate. It is important here to understand the different **intervals** in the convolutions.

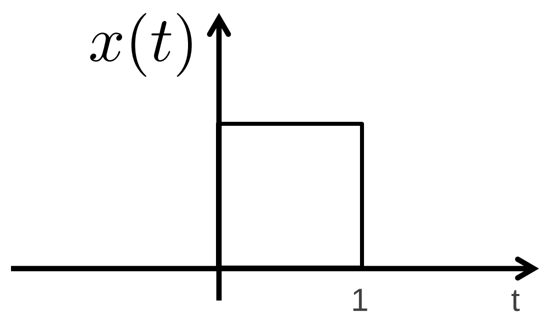
- Find the output of a system with impulse response

$$h(t) = (1-t)[u(t) - u(t-1)]$$



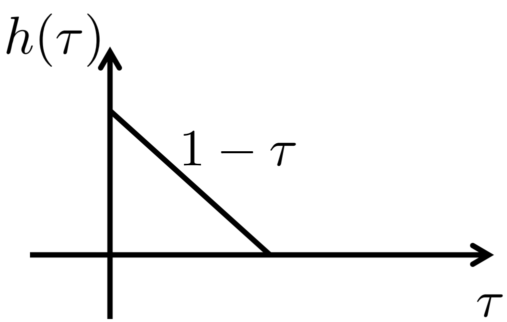
for the input

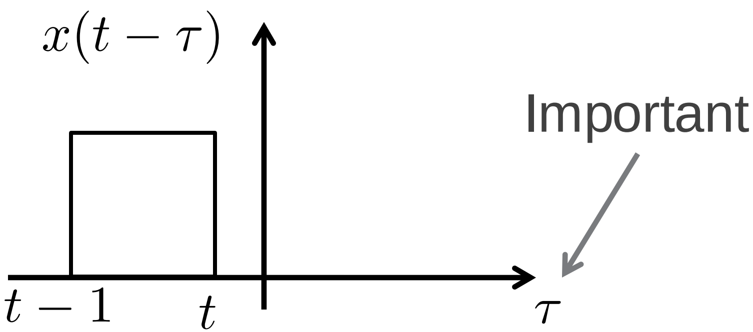
$$x(t) = u(t) - u(t-1)$$



CT convolution example #1: Graphical solution

- Plot one signal versus τ
- Flip the second signal and shift it by t
  - Here, plot it to the left of h(τ)
  - So plotted t has a negative value, usually



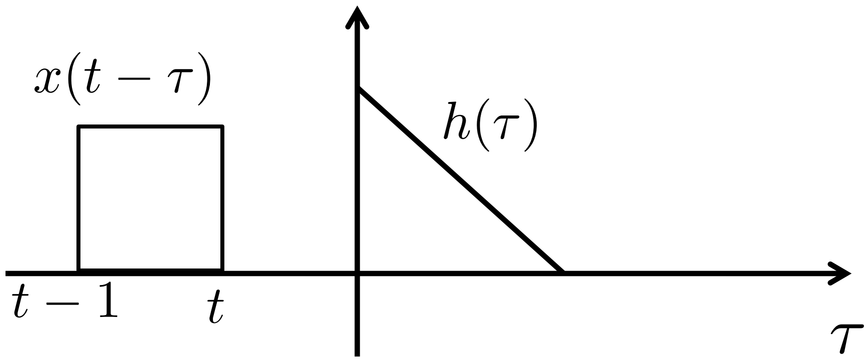


- There are 4 intervals (why?):

$t < 0$ $\quad 0 \leqslant t \leqslant 1$ $\quad 1 \leqslant t \leqslant 2$ $\quad 2 < t$

CT convolution example #1: First interval

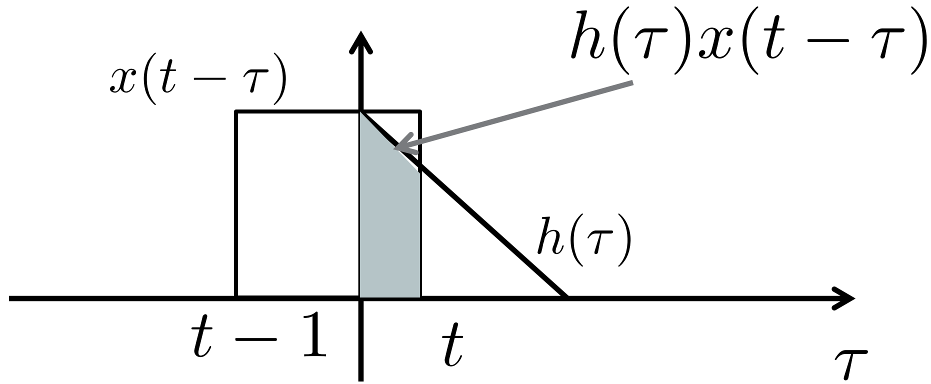
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



- First interval: there is no overlap!

$$t < 0 \qquad h(\tau)x(t-\tau) = 0 \quad \Rightarrow \quad y(t) = 0$$

CT convolution example #1: Second interval



- Second interval $\quad 0 \leqslant t \leqslant 1$

$$y(t) = \int_0^t (1-\tau)d\tau = \tau - \frac{\tau^2}{2}\Big]_0^t$$

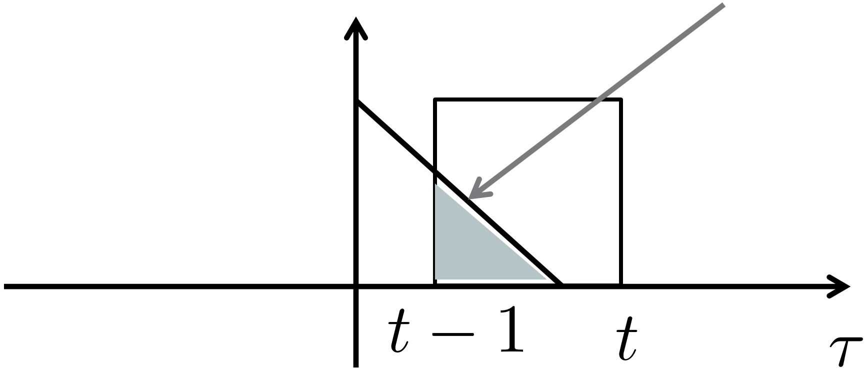
$$= t - \frac{t^2}{2}$$

CT convolution example #1: Third interval

- Third interval $1 \leqslant t \leqslant 2$

$$
\begin{aligned}
y(t) &= \int_{t-1}^{1} (1-\tau)\,d\tau \\
&= \tau - \frac{\tau^2}{2}\Big]_{t-1}^{1} \\
&= 1 - \frac{1}{2} - \left(t - 1 - \frac{(t-1)^2}{2}\right) \\
&= \frac{t^2}{2} - 2t + 2
\end{aligned}
$$

$h(\tau)x(t-\tau)$

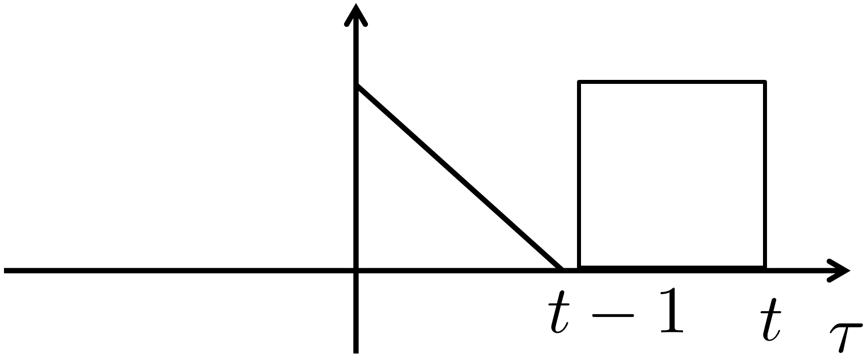


Integrating the same function, but with different integration limits

CT convolution example #1: Fourth and final interval

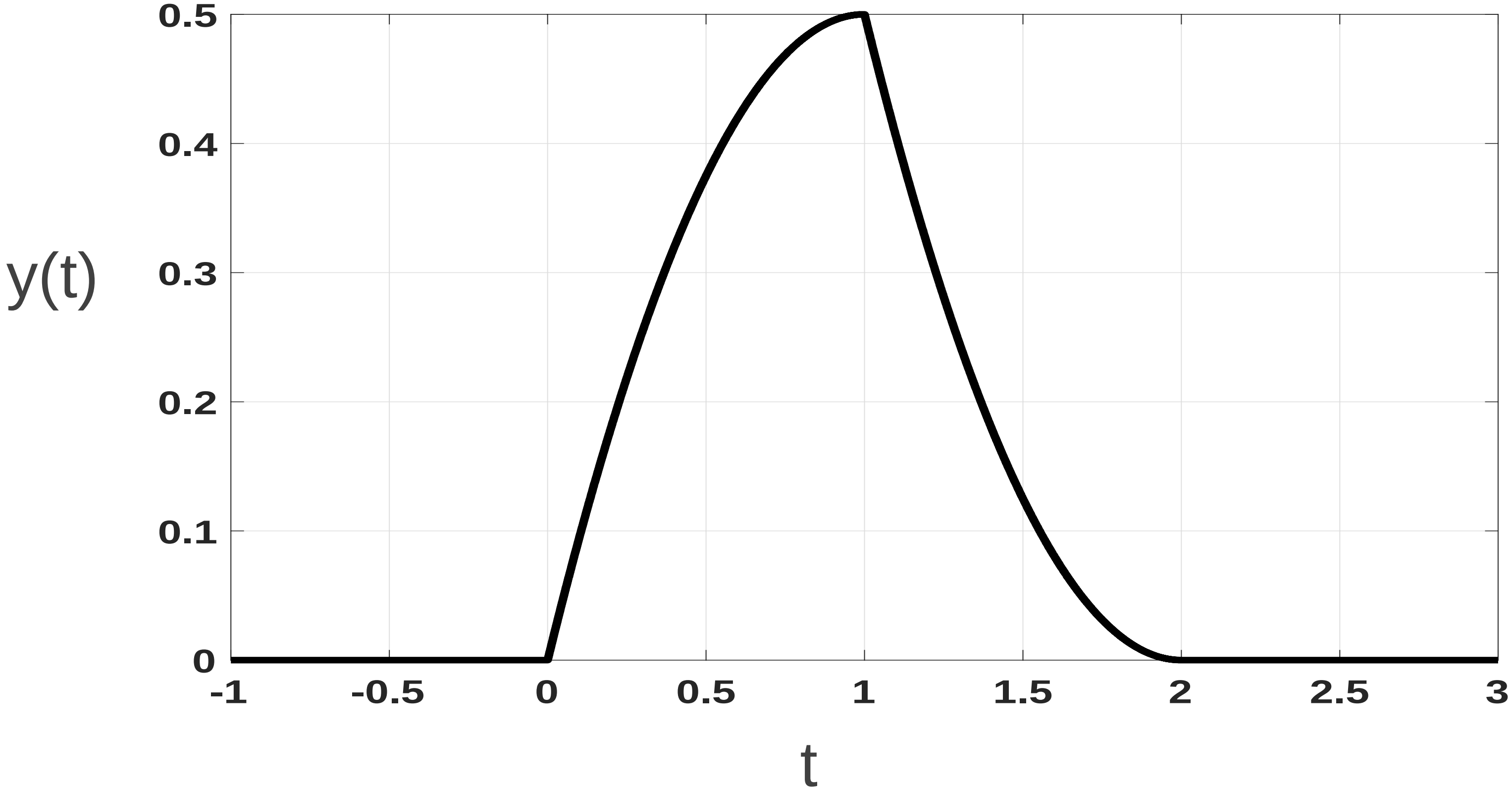
- Fourth interval $2 < t$

$$y(t) = 0$$



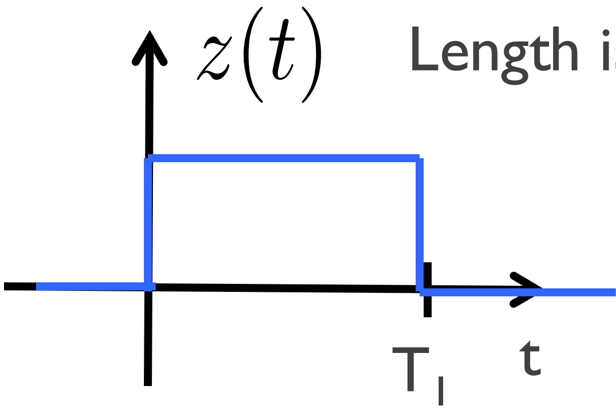
- Sanity checks
  - Check at $t=1$, the output should be the same for intervals 2 & 3 in order for it to be continuous
  - Same for t = 0 and t = 2 (should be zero there)
  - Duration of output should be $T_1 + T_2 = 1+1 = 2$

CT convolution example #1: Plot of y(t)

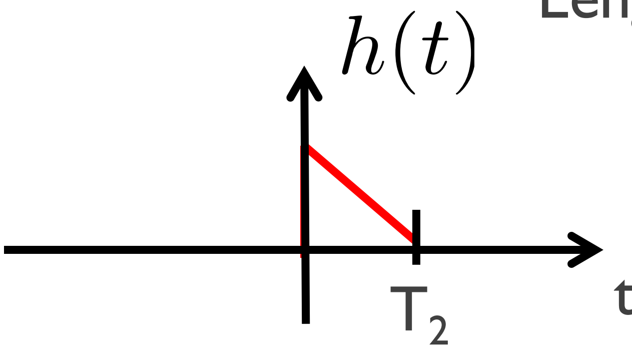


Length / duration of a convolution in continuous time

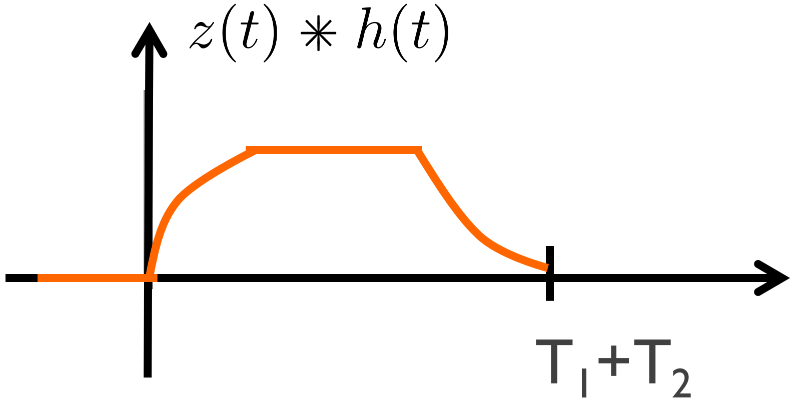
Length is T_1



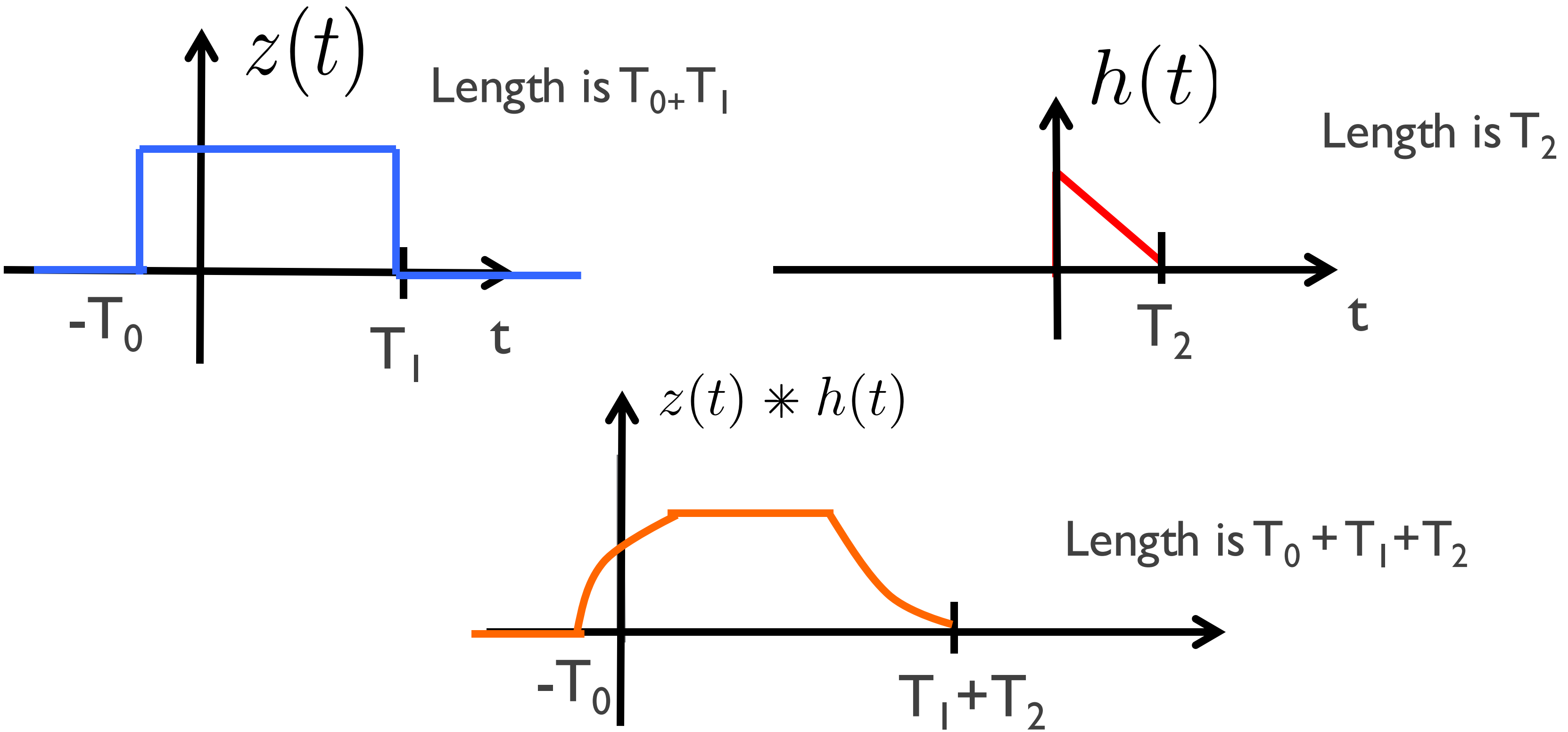
Length is T_2



Length is T_1+T_2



Length / duration of a convolution in continuous time

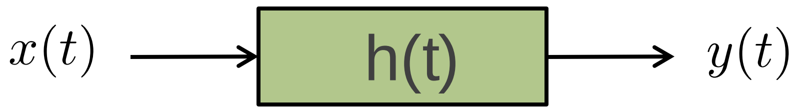


Animation example

http://www.cse.yorku.ca/~asif/spc/ConvolutionIntegral_Final3.swf

CT convolution example #2

This is an example with a simple input that consists of a few delta functions. The convolution is easy to compute using the LTI property of convolution and the definition of impulse response.



- Determine and sketch the convolution of the following input

$$x(t) = \delta(t) + 2\delta(t-1)$$

and system with impulse response

$$h(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1, \\ 2-t, & 1 < t \leqslant 2, \\ 0, & \text{elsewere} \end{cases}$$

CT convolution example #2: Solution approach

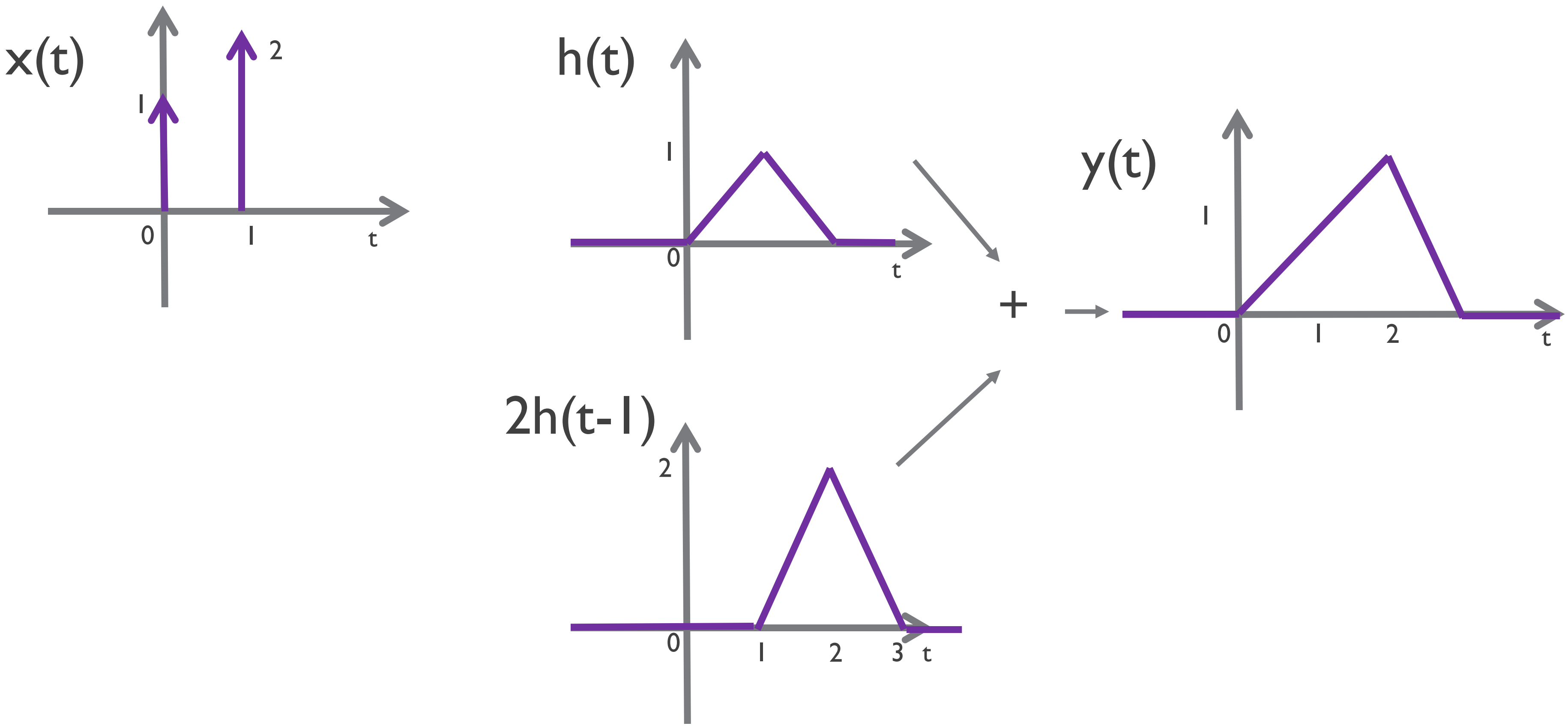
- Recall the property

$$\delta(t - T_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t - T_0)$$

- Therefore

$$x(t) = \delta(t) + 2\delta(t-1) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) + 2h(t-1)$$

CT convolution example #2: Solution sketch

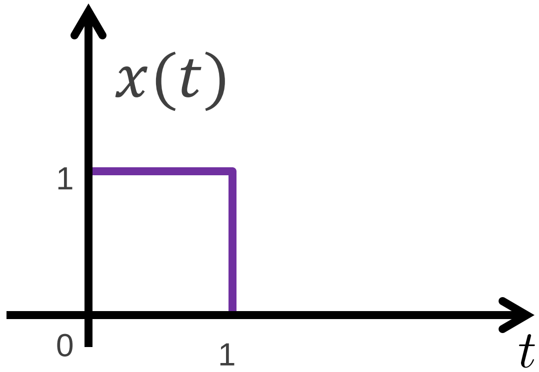


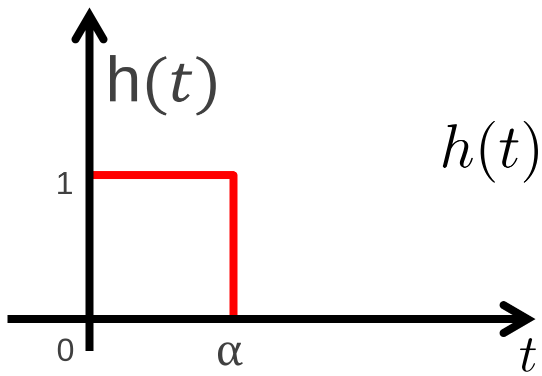
CT convolution example #3

This is an important example involving the convolution of two rectangles of different widths. The solution will be different depending on the widths of the rectangles. Notice what happens in each interval.

- Determine and sketch $y(t) = x(t) * h(t)$, where

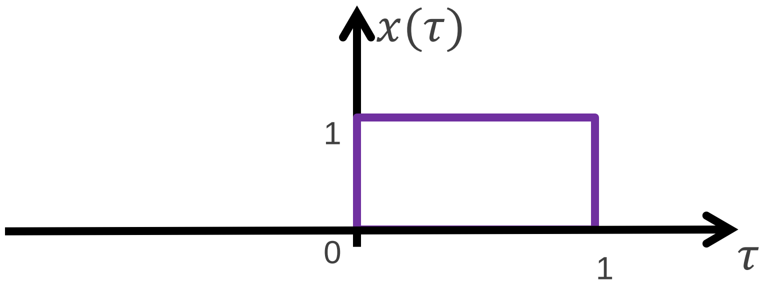
$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

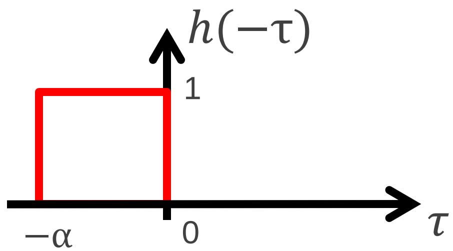




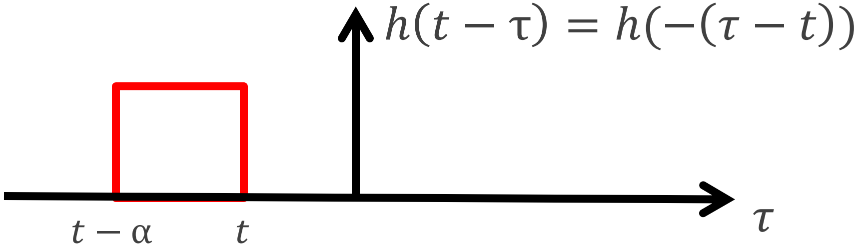
$$h(t) = x(t/\alpha) \qquad 0 < \alpha \le 1$$

CT convolution example #3: Solution



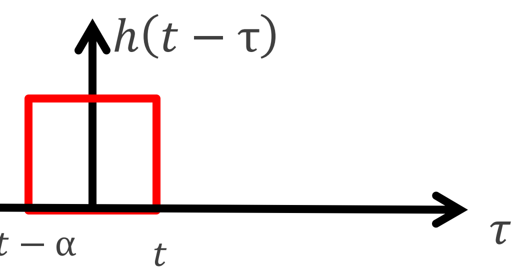


$t < 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0 \quad t < 0$$

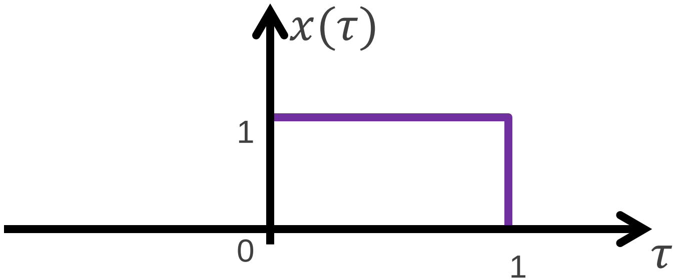
$t \in [0, \alpha]$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} 1 d\tau \quad t \in [0, \alpha]$$

$$= t \quad t \in [0, \alpha]$$

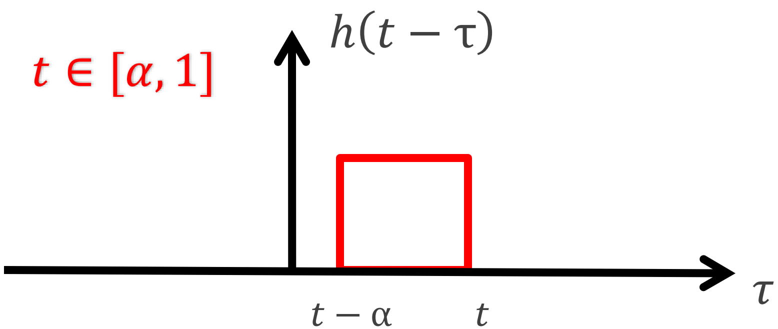
CT convolution example #3: Solution



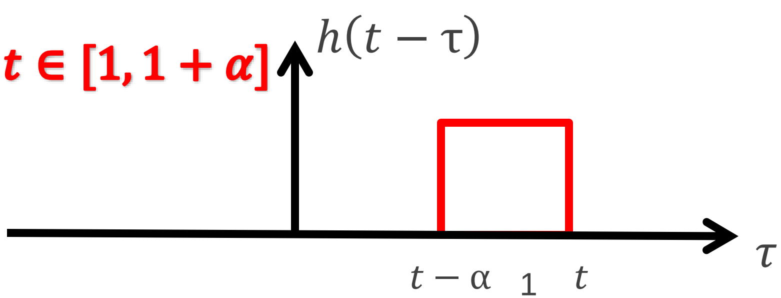
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{t} 1 d\tau \quad t \in [\alpha, 1]$$

$$= \alpha \quad t \in [\alpha, 1]$$

$t \in [\alpha, 1]$



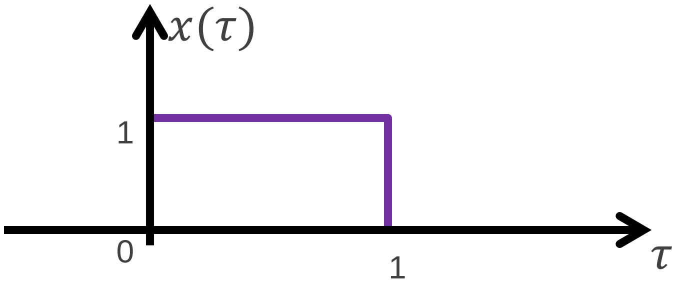
$t \in [1, 1+\alpha]$



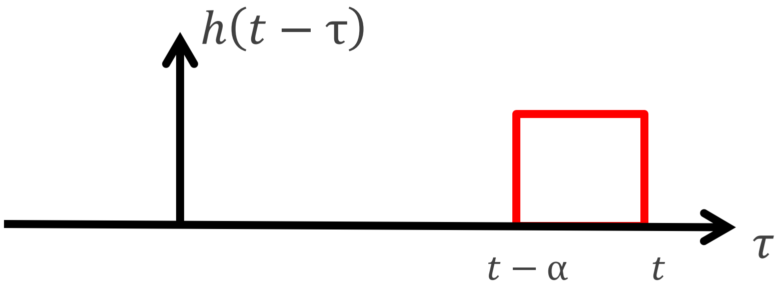
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-\alpha}^{1} 1 d\tau \quad t \in [1, 1+\alpha]$$

$$= 1-(t-\alpha) \quad t \in [1, 1+\alpha]$$

CT convolution example #3: Solution



$t \in [1+\alpha, \infty]$

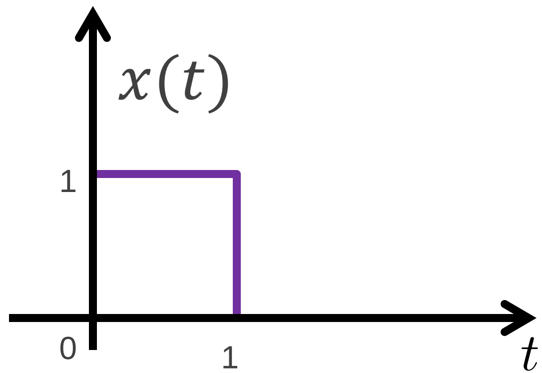


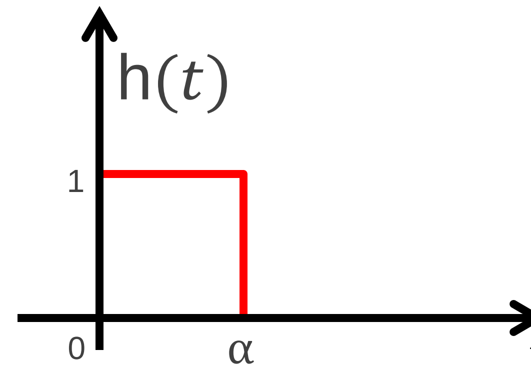
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{1}^{t} 0 d\tau \quad t \in [\alpha+1, \infty]$$

$$= 0 \quad t \in [\alpha+1, \infty]$$

CT convolution example #3: Solution (summary)

$$y(t) = x(t) * h(t)$$





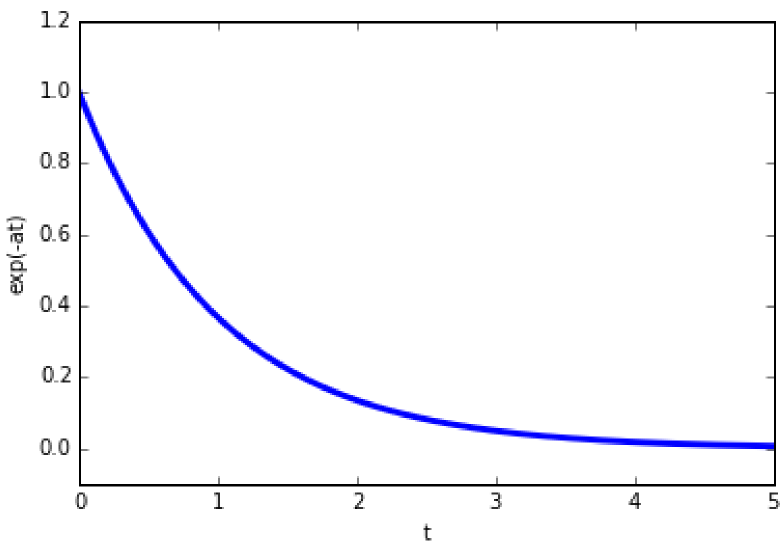
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} t & < & 0 & 0 \\ t & \in & [0,\alpha] & t \\ t & \in & [\alpha,1] & \alpha \\ t & \in & [1,1+\alpha] & 1-(t-\alpha) \\ t & > & 1+\alpha & 0 \end{cases}$$

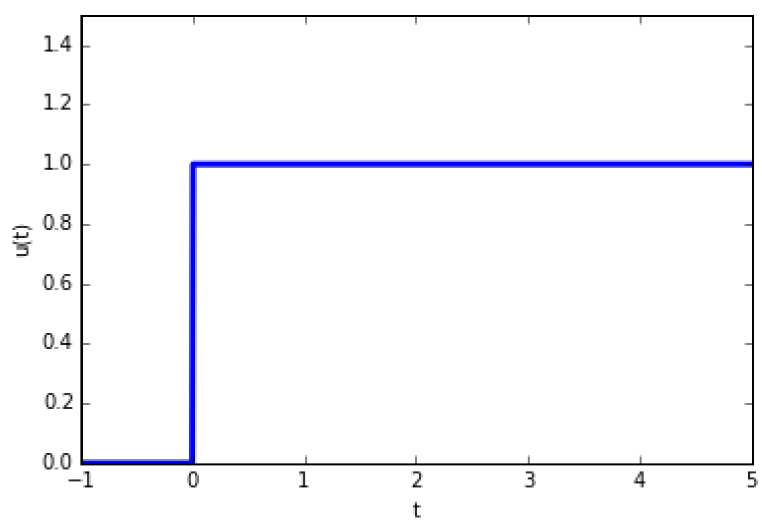
CT convolution example #4

This is an important example of convolution to causal signals together. It will be solved by using the graphical approach to help set up the integrals.

- Determine and sketch $y(t) = x(t) * h(t)$, where

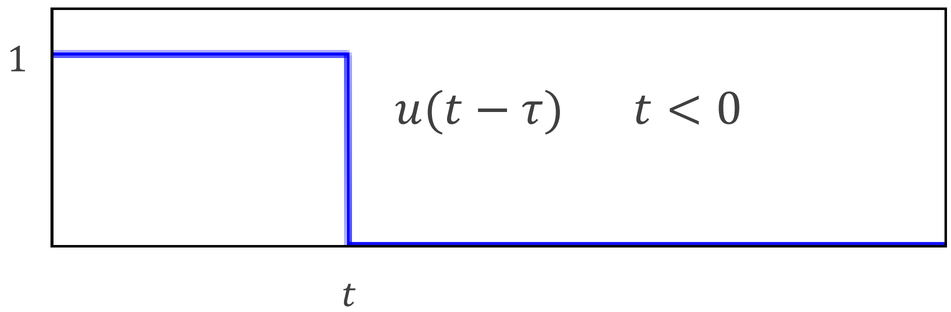
$$x(t) = e^{-at}u(t), \qquad a > 0 \qquad\qquad h(t) = u(t)$$

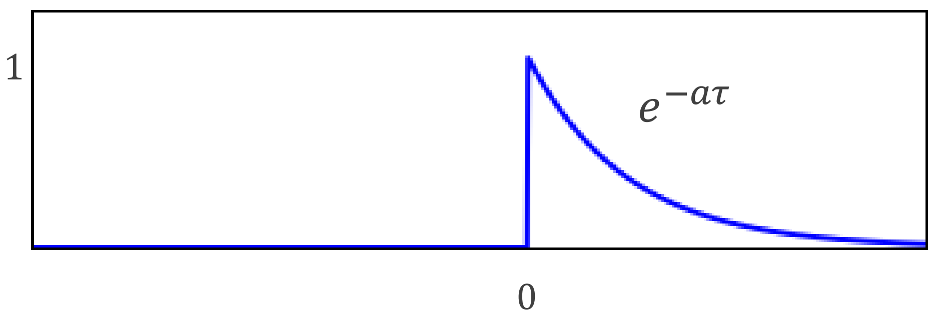




CT convolution example #4: Solution

$$y(t) = x(t) * h(t)$$

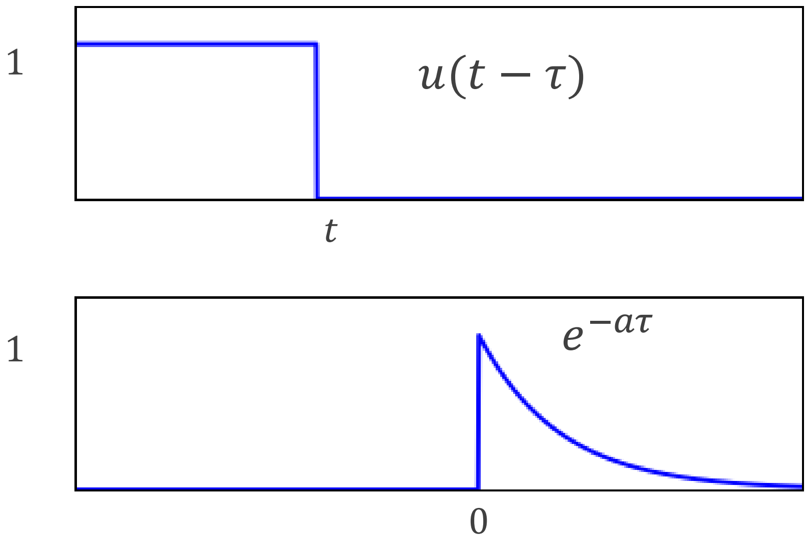




$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-a\tau}u(t-\tau)d\tau \end{aligned}$$

CT convolution example #4: Solution

For $t < 0$

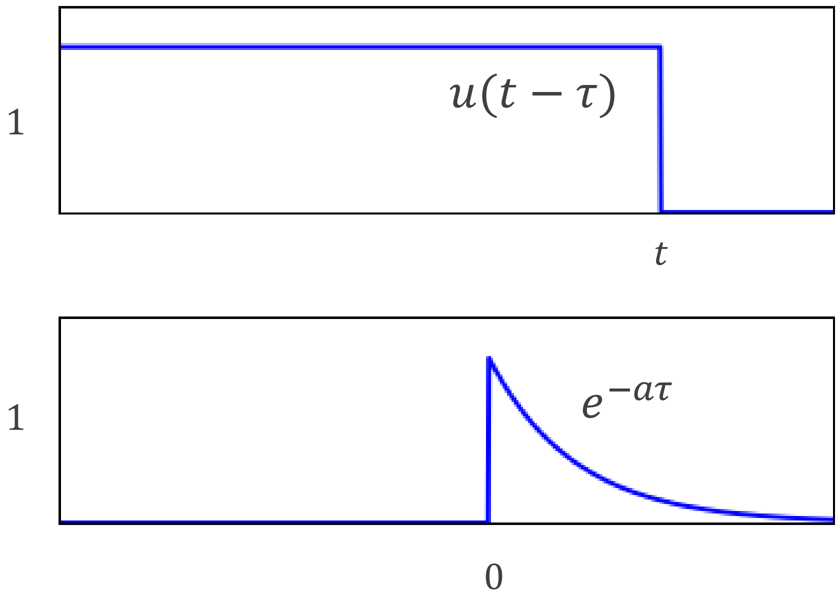


$$x(t) * h(t) = \int_{-\infty}^{\infty} 0 d\tau$$

$$= 0$$

CT convolution example #4: Solution

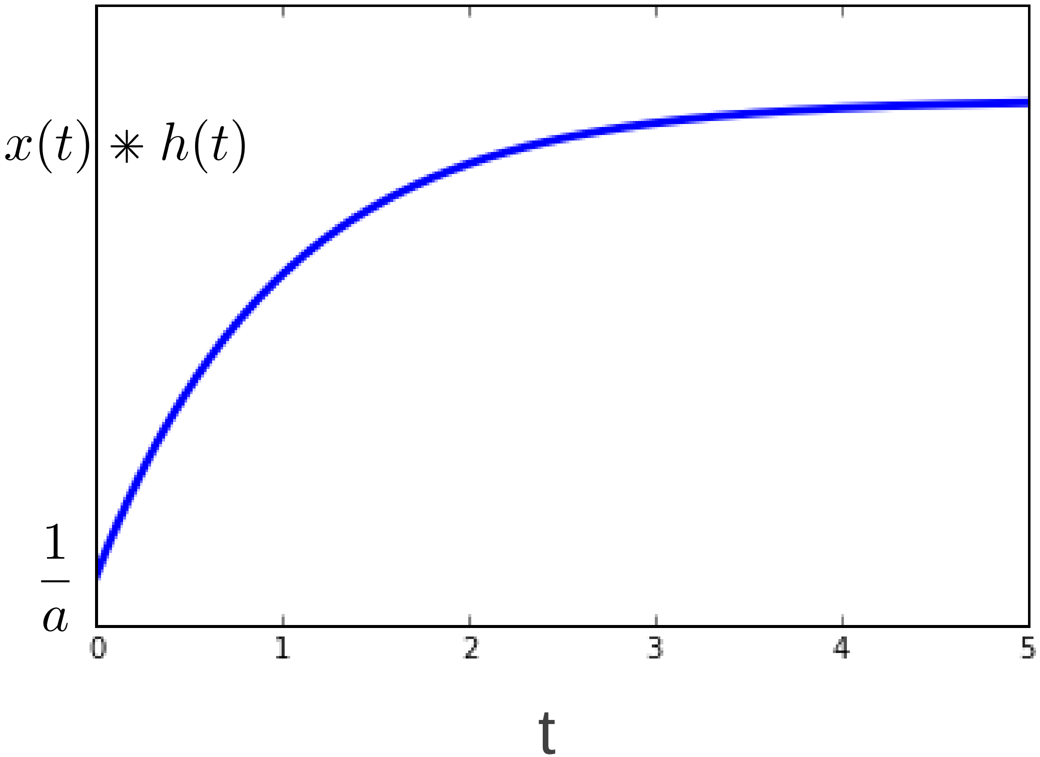
For $t \geq 0$



$$x(t) * h(t) = \int_0^t e^{-a\tau} u(t-\tau) d\tau$$

$$= \frac{1 - e^{-at}}{a}$$

CT convolution example #4: Solution



1/a (e.g. for a>0)

$$x(t) * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1-e^{-at}}{a} & t \geqslant 0 \end{cases}$$

$$= \frac{1 - e^{-at}}{a} u(t)$$

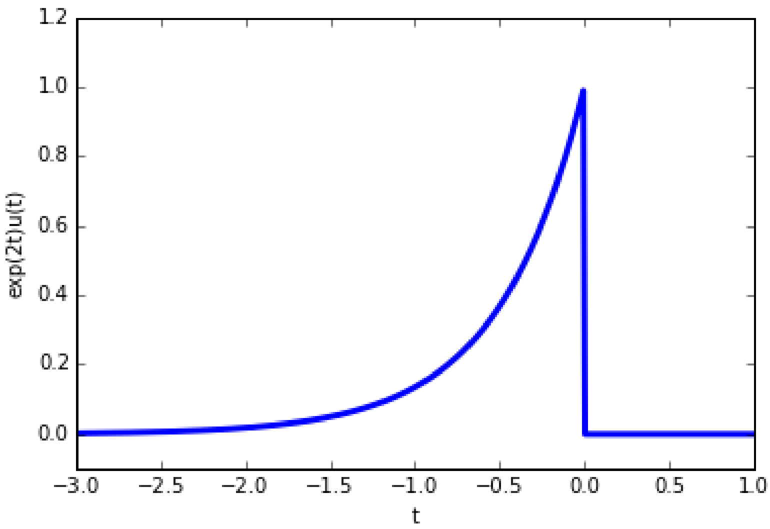
CT convolution example #5

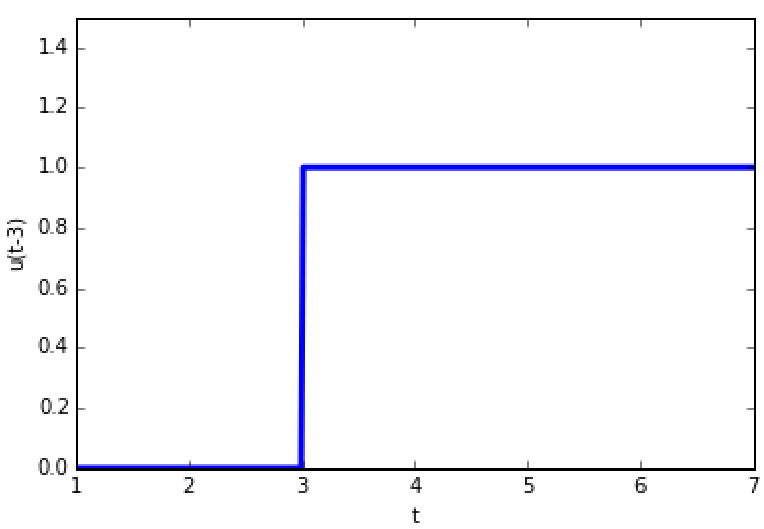
This is an example where an anti-causal and a causal signal are convolved together, unlike the previous examples. Notice the difference intervals in this case.

- Determine and sketch $y(t) = x(t) * h(t)$, where

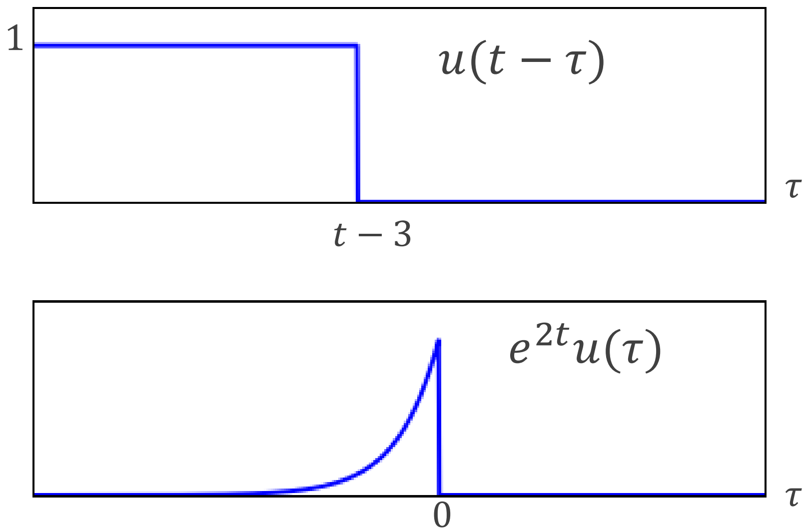
$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t-3)$$





CT convolution example #5: Solution

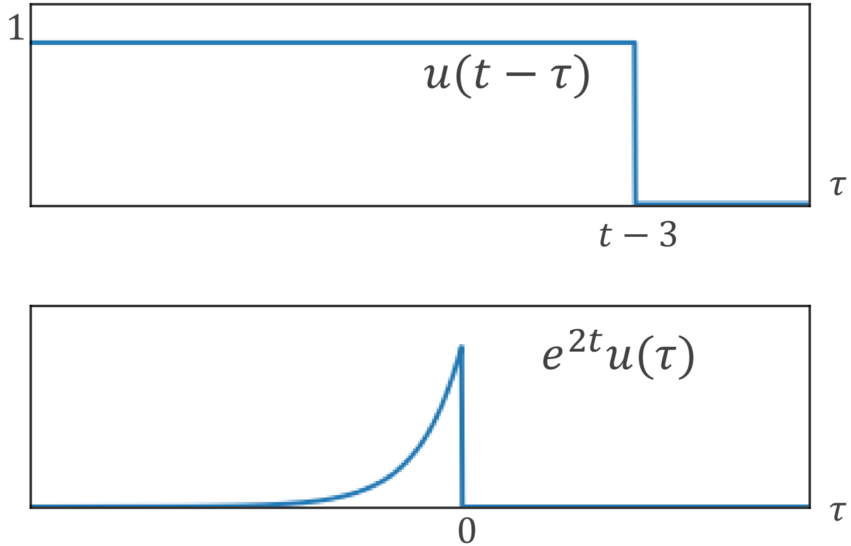


For t-3 < 0

$$x(t) * h(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau$$

$$= \frac{e^{2(t-3)}}{2}$$

CT convolution example #5: Solution



For t-3 > 0

$$x(t) * h(t) = \int_{-\infty}^{0} e^{2\tau} d\tau$$

$$= \frac{1}{2}$$

CT convolution example #5: Solution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \begin{cases} \frac{e^{2(t-3)}}{2} & t < 3 \\ \frac{1}{2} & t \geqslant 3 \end{cases}$$

Basic convolution properties

- Commutative

$$y(t) = x(t) * h(t) \quad = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) \quad = \int h(\tau)x(t-\tau)d\tau$$

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

Use properties to simplify convolutions

Words of wisdom on convolution

- Convolution is a fact of life
  - All real world LTI systems generate outputs in the time domain given by the convolution sum or integral (yes RLC circuits are doing convolution for you)
- You need to practice convolutions
  - There are many examples in the book and videos online
- In future lectures, we will cover tools and ideas that allow us to avoid computing convolutions (most of the time)
  - But you still need to learn how to do it in the time domain (don't wait)

How does research in ECE (especially signal processing) get done at a university?

Or do professors really write all of those papers they publish?

UC San Diego

Composition of authors on a paper

2020 IEEE 21st International Workshop on Signal Processing Advances in Wireless Communications (SPAWC)

Deep Learning-based Carrier Frequency Offset Estimation with One-Bit ADCs

Graduate research assistant (GRA)

Industry collaborator (recent PhD)

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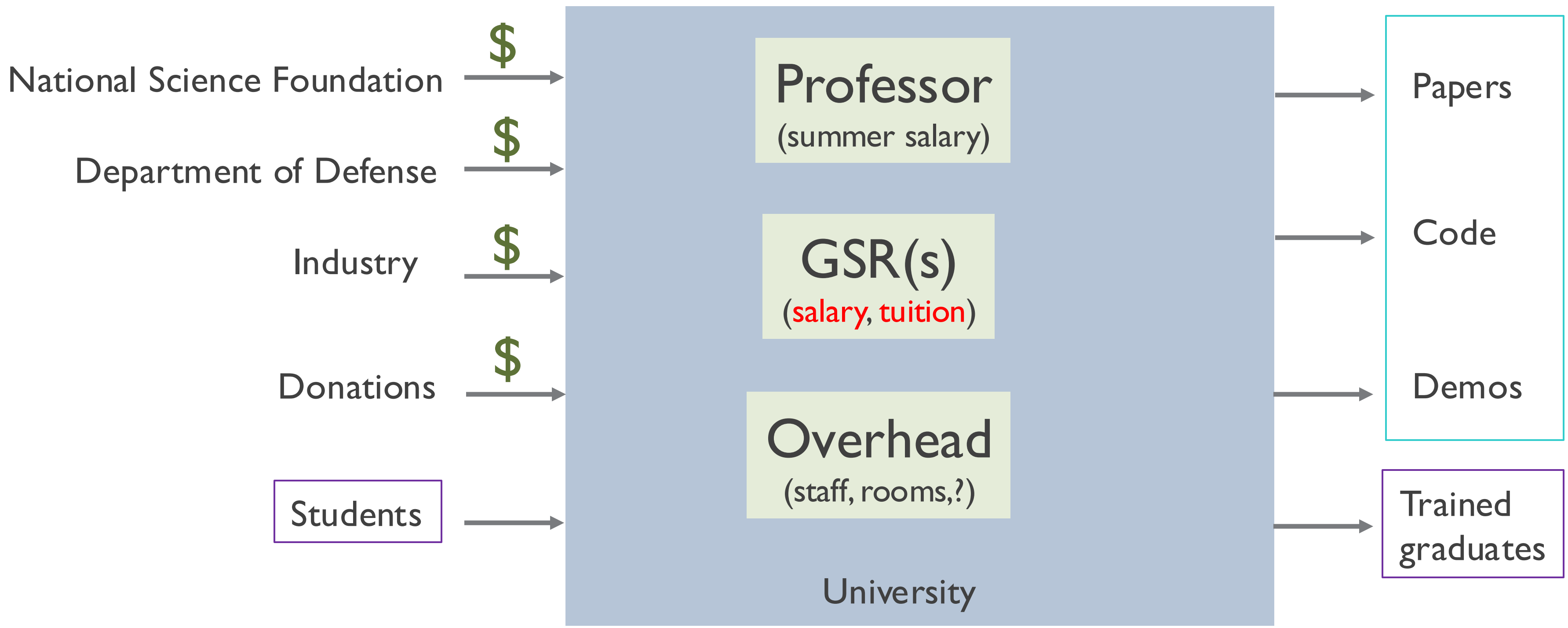
Professor

Industry collaborator (lab director)

Percent effort

Graduate research assistant (GRA)	Professor	Industry collaborator (recent PhD)	Industry collaborator (lab director)
75%	15%	8%	2%

Summary of system



Note that GSRs (PhD bound) are paid to do the research and attend school

How the work gets done 1/2

- Professors
  - Identify technical directions
  - Raise funding to support research
  - Provide input, on publications and presentations
  - General mentoring
- GRAs
  - Understand the state-of-the-art
  - Identify problems that can be solved
  - Develop solutions, simulate results, build prototype, etc
  - Create conference paper, journal paper, dissertation, etc

Type of work that may be done by a USR

How the work gets done 2/2

- Industry collaborators
  - Give insight into the state-of-the-art
  - Suggest important problems to solve
  - Fund the research (sometimes)

- Academic collaborators (usually Professor or Professor + GRA)
  - Contribute to the development of the research, e.g. bringing a complementary set of expertise or domain knowledge
  - Collaborate on the research results

Collaboration is increasingly important to identify and solve important problems

Convolution properties

Learning objectives

- Summarize key properties of convolution
- Exploit the properties to simplify convolution

Basic convolution properties

- Commutative

Shorthand notation

$$y(t) = x(t) * h(t) = \int x(\tau)h(t-\tau)d\tau$$

$$= h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau$$

Choose option that makes it easy!

- Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

- Distributive

$$f(t) * (h(t) + g(t)) = f(t) * h(t) + f(t) * g(t)$$

Use properties to simplify convolutions

Convolution with the delta

- Convolution with delta functions is easy

$$\delta(t) * x(t) = x(t)$$

$$x(t) * \delta(t) = x(t)$$

- Convolution with shifted deltas is easy

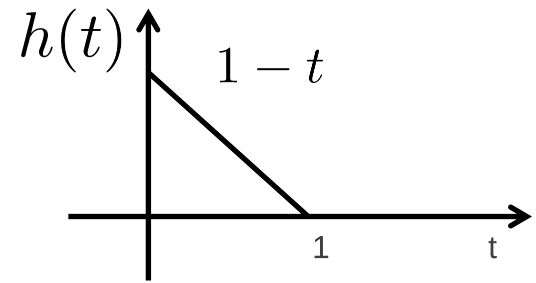
$$x(t) * \delta(t - t_0) = x(t - t_0)$$

Celebrate simplicity when faced with a convolution with a delta!

Recall this convolution

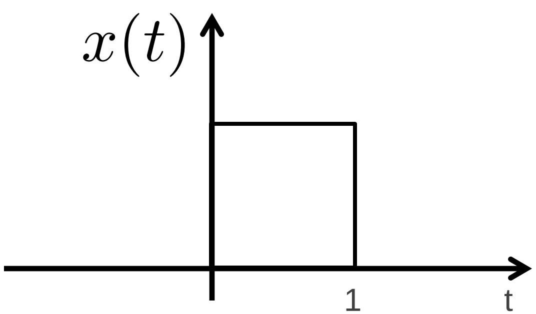
- Find the output of a system with impulse response

$$h(t) = (1-t)[u(t) - u(t-1)]$$



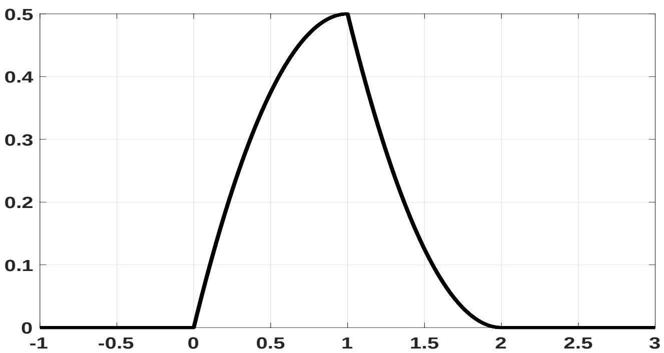
for the input

$$x(t) = u(t) - u(t-1)$$



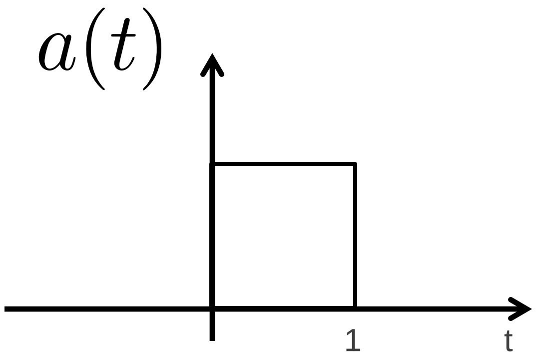
Solution to the convolution

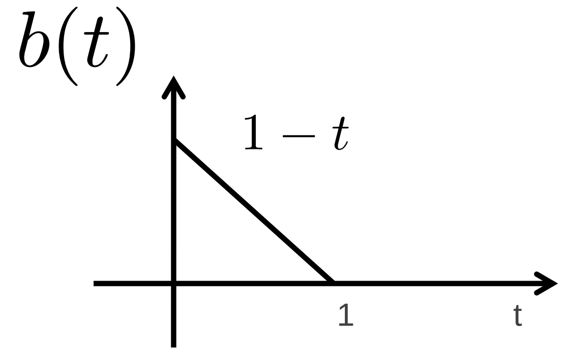
$$y(t) = \begin{cases} 0 & t < 0 \\ t - \frac{t^2}{2} & 0 \leqslant t \leqslant 1 \\ \frac{t^2}{2} - 2t + 2 & 1 \leqslant t \leqslant 2 \\ 0 & 2 < t \end{cases}$$



For later use, let us refer to this convolution as $f(t)$

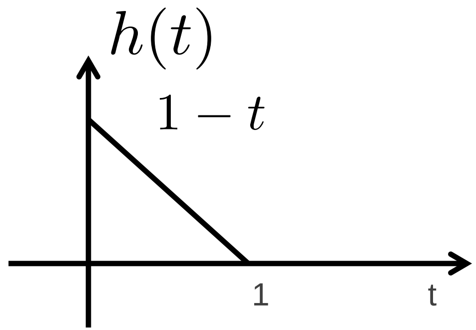
Use the properties to solve the following integral

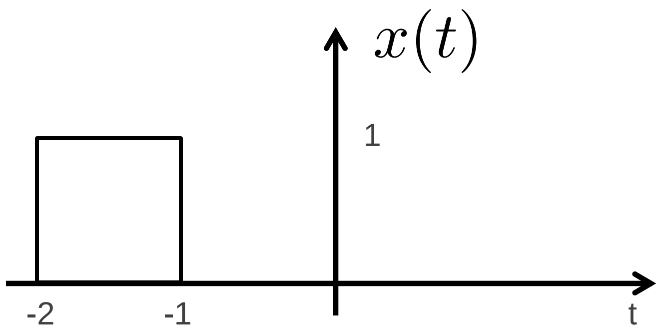




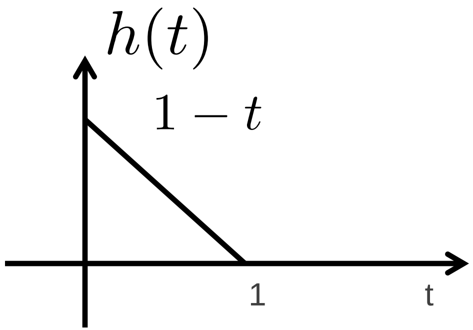
$$c(\gamma) = \int_{-\infty}^{\infty} b(q)a(\gamma - q)dq$$

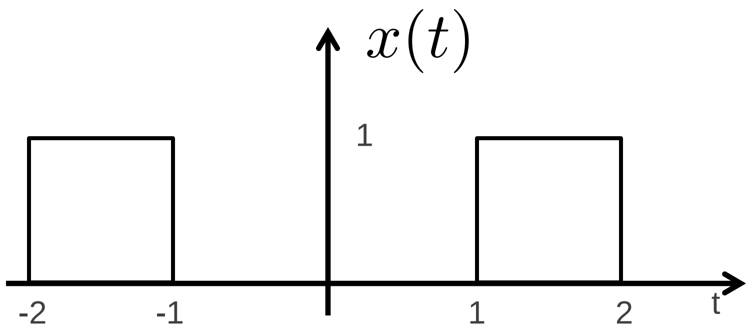
Use the properties to convolve the following



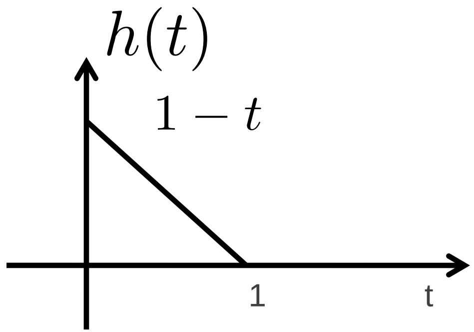


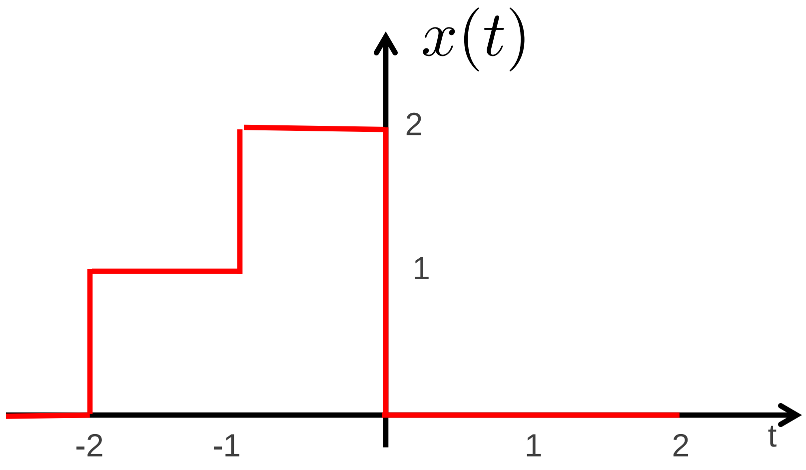
Use the properties to convolve the following





Use the properties to convolve the following





Summary about properties

- Convolution satisfies commutative, associative and distributed properties
  - Properties follow primarily from the integral structure
  - Use these properties to simplify the calculation of a convolution