UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2020
Linear Systems Fundamentals

SOLUTIONS TO MIDTERM EXAM

Instructions

- Open Canvas website, no electronics for problem solving.
- Tables 3.1 and 3.2 from the textbook are attached to the exam.
- No collaboration or external help of any kind, either in person or via web, phone, or any other device, is allowed. You may be asked to explain your answers over a Zoom call when the exam is graded.
- Time allowed: from 5:30pm PST Thursday to 11:59pm PST Friday
- Write your solutions in the applicable space in the exam. **Justify your answers.** If extra pages are needed to show all of your work, use the scratch pages at the end of the exam and submit them.
- Upload your solutions via Gradescope

PRINT YOUR NAME Marc-Antoine Parseval des Chênes

Signature $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Student ID Number $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$

Your signature confirms that you have completed this exam on your own and in accordance with the Academic Integrity Agreement.

Problem	Weight	Score
1	16 pts	16
2	24 pts	24
3	30 pts	30
4	30 pts	30
Total	100 pts	100

Good luck!

Name/Student ID: ____________________

Problem 1 (16 points: 2 parts, 8 points each

(a) (8 pts)

Let $x(t) = t\,(u(t) - u(t-1))$. Define $y(t) = x((-3t) - 2)$.

Describe how to obtain $y(t)$ from $x(t)$ by the following rules:

Approach 1:

First shift $x(t)$ by $b =$ ___2___
($b > 0$ means shift right by $b < 0$ means shift left by $|b|$)

Then (circle one) **stretch** or $\boxed{\textbf{compress}}$ the signal by $a =$ ___-3___
($a < 0$ means flip; $|a| \geq 1$ is the stretch or compress factor)

or

Approach 2:

First (circle one) **stretch** or $\boxed{\textbf{compress}}$ the signal by $c =$ ___-3___
($c < 0$ means flip; $|c| \geq 1$ is the stretch or compress factor)

Then shift $x(t)$ by $d =$ ___$-2/3$___
($d > 0$ means shift right by $d < 0$ means shift left by $|d|$)

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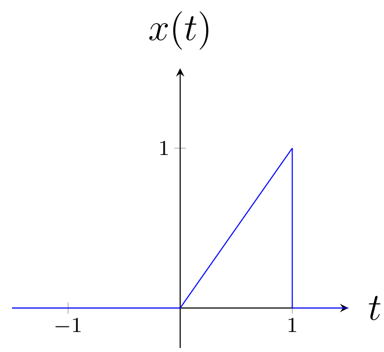
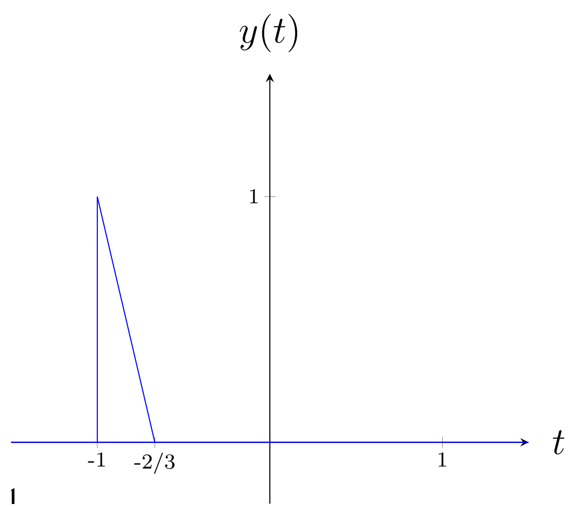
Problem 1 (cont.)

(b) (8 pts) Let $x(t) = t\,(u(t) - u(t-1))$. Define $y(t) = x((-3t) - 2)$.

Sketch precisely $y(t)$.

$$\begin{aligned} y(t) &= (-3t-2)(u(-3t-2) - u(-3t-3)) \\ &= (-3t-2)(u(t+1) - u(t+\frac{2}{3})) \end{aligned}$$





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Problem 2 (24 points: 3 parts, 8 points each)

Let $x[n] = e^{j\frac{\pi}{3}(n-1)} + \sum_{k=-\infty}^{\infty} \delta[n-6k]$.

(a) (8 pts)

Determine the fundamental period of $x[n]$.

$x_1[n] = e^{j\frac{\pi}{3}(n-1)} = e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{3}n} = e^{-j\frac{\pi}{3}} e^{j\frac{2\pi}{6}n}$

Fundamental period $N_1 = 6$.

$x_2[n] = \sum_{k=-\infty}^{\infty} \delta[n-6k]$

Periodic impulse train, fundamental period $N_2 = 6$.

$x[n] = x_1[n] + x_2[n]$

Fundamental period $N = \mathrm{lcm}\{N_1, N_2\} = \mathrm{lcm}\{6, 6\} = 6$.

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Problem 2 (cont.)

Let $x[n] = e^{j\frac{\pi}{3}(n-1)} + \sum_{k=-\infty}^{\infty} \delta[n-6k]$.

(b) (8 pts)

Determine if $x[n]$ is even, odd, or neither.

$x_1[n] = e^{j\frac{\pi}{3}(n-1)} = e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{3}n}$
$\mathcal{E}v\{x_1[n]\} = e^{-j\frac{\pi}{3}}(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n})/2 = e^{-j\frac{\pi}{3}} \cos(\frac{\pi}{3}n)$
$\mathcal{O}d\{x_1[n]\} = e^{-j\frac{\pi}{3}}(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n})/2 = je^{-j\frac{\pi}{3}} \sin(\frac{\pi}{3}n)$

$x_2[n] = \sum_{k=-\infty}^{\infty} \delta[n-6k]$
$\mathcal{E}v\{x_2[n]\} = \sum_{k=-\infty}^{\infty} \delta[n-6k] = x_2[n]$
$\mathcal{O}d\{x_2[n]\} = 0$

So,
$\mathcal{E}v\{x[n]\} = \mathcal{E}v\{x_1[n]\} + \mathcal{E}v\{x_2[n]\} = e^{-j\frac{\pi}{3}} \cos(\frac{\pi}{3}n) + \sum_{k=-\infty}^{\infty} \delta[n-6k] \neq 0$
$\mathcal{O}d\{x[n]\} = \mathcal{O}d\{x_1[n]\} + \mathcal{O}d\{x_2[n]\} = je^{-j\frac{\pi}{3}} \sin(\frac{\pi}{3}n) \neq 0$

This implies that $x[n]$ is neither even nor odd.

Alternatively:
$x[1] = 1 + 0 = 1$ and $x[-1] = e^{-j2\frac{\pi}{3}} + 0 = e^{-j2\frac{\pi}{3}}$, so $x[1] \neq x[-1]$.
This impliles $x[n]$ is not even.
$x[0] = e^{-j\frac{\pi}{3}} + 1 \neq 0$
so $x[n]$ is not odd.
Therefore, $x[n]$ is neither even nor odd.

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Problem 2 (cont.

Let $x[n] = e^{j\frac{\pi}{3}(n-1)} + \sum_{k=-\infty}^{\infty} \delta[n-6k]$.

(c) (8 pts)

Determine the discrete-time Fourier series (DTFS) coefficients of $x[n]$.

The DTFS coefficients of $x[n]$ are periodic with period $N = 6$.

The DTFS coefficients in one period of

$$x_1[n] = e^{j\frac{\pi}{3}(n-1)} = e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{3}n}$$

are

$$a_0 = 0, a_1 = e^{-j\frac{\pi}{3}}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0.$$

The DTFS coefficients of

$$x_2[n] = \sum_{k=-\infty}^{\infty} \delta[n-6k]$$

are

$$a_k = \frac{1}{6},\ \forall k.$$

So, the DTFS coefficients of $x[n] = x_1[n] + x_2[n]$ in one period are obtained by summing these:

$$a_0 = \frac{1}{6}, a_1 = e^{-j\frac{\pi}{3}} + \frac{1}{6}, a_2 = \frac{1}{6}, a_3 = \frac{1}{6}, a_4 = \frac{1}{6}, a_5 = \frac{1}{6}.$$

or, using the fact that $e^{-j\frac{\pi}{3}} = \frac{1}{2} - j\frac{\sqrt{3}}{2}$,

$$a_0 = \frac{1}{6}, a_1 = \frac{2}{3} - j\frac{\sqrt{3}}{2}, a_2 = \frac{1}{6}, a_3 = \frac{1}{6}, a_4 = \frac{1}{6}, a_5 = \frac{1}{6}.$$

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Problem 3 (30 points: 3 parts, 10 points each)

Consider the discrete-time linear time-invariant (LTI) system S with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1].$$

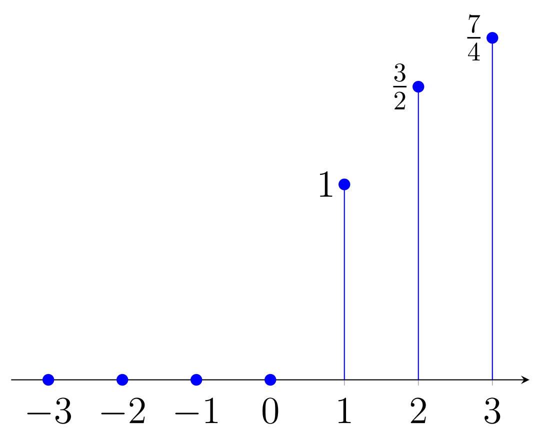
(a) (10 pts)

Determine the step response $s[n]$ of the system S, and sketch $s[n]$ precisely in the interval $-3 \leq n \leq 3$.

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k] = \left(2 - \left(\frac{1}{2}\right)^{n-1}\right) u[n-1]$$

So, on $[-3, 3]$,

$$s[n] = \begin{cases} 0, & -3 \leq n \leq 0 \\ 1, & n = 1 \\ 3/2, & n = 2 \\ 7/4, & n = 3 \end{cases}$$



Alternatively (the long way):

$$\begin{aligned} s[n] &= u[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} u[k] \left(\frac{1}{2}\right)^{n-k-1} u[n-k-1] = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{n-k-1} u[n-1] \\ &= \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{k} u[n-1] = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} u[n-1] \\ &= \left(2 - \left(\frac{1}{2}\right)^{n-1}\right) u[n-1] \end{aligned}$$

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Problem 3 (cont.)

Consider the discrete-time linear time-invariant (LTI) system S with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1].$$

(b) (10 pts)

Indicate whether the system S satisfies each listed property.

Justify your answers with *specific* reference to the system S.

True	False	
☐	☒	Memoryless

Justification:

$h[n]$ is not of the form $a\delta[n]$, since $h[1] = 1$.
So, S is not memoryless.

True	False	
☒	☐	Causal

Justification:

$h[n] = 0$ for $n \leq 0$ because $u[n-1] = 0$ for $n \leq 0$.
So, S is causal.

True	False	
☒	☐	Stable

Justification:

$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} = \frac{1}{1-\frac{1}{2}} = 2 < \infty.$
So, S is stable.

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Problem 3 (cont.)

Consider the discrete-time linear time-invariant (LTI) system S with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1].$$

(c) (10 pts)

The system S is invertible. Determine the impulse response $g[n]$ of the inverse system.

Note that

$$h[n] - \frac{1}{2}h[n-1] = h[n] * (\delta[n] - \frac{1}{2}\delta[n-1]) = \delta[n-1]$$

so

$$(h[n] * (\delta[n] - \frac{1}{2}\delta[n-1])) * \delta[n+1] = \delta[n]$$

or

$$h[n] * ((\delta[n] - \frac{1}{2}\delta[n-1]) * \delta[n+1]) =$$
$$h[n] * (\delta[n+1] - \frac{1}{2}\delta[n]) = \delta[n].$$

So,

$$g[n] = \delta[n+1] - \frac{1}{2}\delta[n].$$

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Problem 4 (30 points: 3 parts, 10 points each)

The signal $x[n] = 1 + \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{3}n)$ is passed through the discrete-time filter whose frequency response is given by:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{2\pi}{3} \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10 pts)

Determine the discrete-time Fourier series (DTFS) of the signal $x[n]$.

$$\begin{aligned} 1 + \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{3}n) &= 1 + \frac{1}{2}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right) \cdot \frac{1}{2j}\left(e^{j\ \frac{\pi}{3}n} - e^{-j\ \frac{\pi}{3}n}\right) \\ &= 1 + \frac{1}{4j}\left(e^{j\frac{5\pi}{6}n} - e^{j\frac{\pi}{6}n} + e^{-j\frac{\pi}{6}n} - e^{-j\frac{5\pi}{6}n}\right) \\ &= 1 + \frac{1}{4j}\left(e^{j2\pi\frac{5}{12}n} - e^{j2\pi\frac{1}{12}n} + e^{-j2\pi\frac{1}{12}n} - e^{-j2\pi\frac{5\pi}{12}n}\right) \end{aligned}$$

So, the fundamental period of $x[n]$ is $N = 12$.

Since $e^{-j2\pi\frac{1}{12}n} = e^{j2\pi\frac{11}{12}n}$ and $e^{-j2\pi\frac{5\pi}{12}n} = e^{j2\pi\frac{7}{12}n}$, we see that theDTFS coefficients of $x[n]$ in the period $-5 \le k \le 6$ are:

$$a_k = \begin{cases} 1, & k = 0 \\ -\frac{1}{4j}, & k = 1 \\ \frac{1}{4j}, & k = -1 \\ \frac{1}{4j}, & k = 5 \\ -\frac{1}{4j}, & k = -5 \\ 0, & k = -4, -3, -2, 2, 3, 4, 6 \end{cases}$$

The DTFS representation of $x[n]$ is then

$$x[n] = \sum_{k=-5}^{6} a_k e^{jk\frac{2\pi}{12}n}.$$

Alternate solution:

$x[n] = 1 + \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{3}n) = 1 + \frac{1}{2}(\sin(\frac{5\pi}{6}n) - \sin(\frac{\pi}{6}n))$

Then proceed similarly.

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Problem 4 (cont.)

The signal $x[n] = 1 + \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{3}n)$ is passed through the discrete-time filter whose frequency response is given by:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{2\pi}{3} \\ 0, & \text{otherwise.} \end{cases}$$

(b) (10 pts)

Determine the output signal $y[n]$ produced by the input signal $x[n]$. Write it in a form that does not involve complex exponential signals.

$$y[n] = \sum_{k=-5}^{6} b_k e^{jk\frac{2\pi}{12}n}$$

where

$$b_k = a_k H(e^{jk\frac{2\pi}{12}}), \forall k.$$

From the definition of $H(e^{j\omega})$, we see that

$$H(e^{jk\frac{2\pi}{12}}) = \begin{cases} 1, & |k| \leq 4 \\ 0, & 5 \leq |k| \leq 6 \end{cases}$$

$$b_k = \begin{cases} 1, & k = 0 \\ -\frac{1}{4j}, & k = 1 \\ \frac{1}{4j}, & k = -1 \\ 0, & 2 \leq |k| \leq 6 \end{cases}$$

Thus,

$$\begin{aligned} y[n] &= 1 + \frac{1}{4j}\left(e^{-j\frac{2\pi}{12}n} - e^{j\frac{2\pi}{12}n}\right) \\ &= 1 - \frac{1}{2}\sin(\frac{\pi}{6}n). \end{aligned}$$

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Problem 4 (cont.)

The signal $x[n] = 1 + \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{3}n)$ is passed through the discrete-time filter whose frequency response is given by:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{2\pi}{3} \\ 0, & \text{otherwise.} \end{cases}$$

(c) (10 pts)

Let b_k denote the DTFS of the output signal $y[n]$. Determine the value of b_{11}, b_{18}, and b_{52} (not the rock band) . Express your answers numerically, as well as in terms of $b_0, b_1, \ldots, b_N$ where N is the fundamental period of $y[n]$.

From part (b), we see that $y[n]$ has fundamental period $N = 12$. Thus, the DTFS coefficients have period $N = 12$ also.

Therefore,

$$b_{11} = b_{-1} = \frac{1}{4j}$$
$$b_{18} = b_6 = 0$$
$$b_{52} = b_4 = 0$$

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