ECE 101 Linear Systems

Problem Set 5 Solutions

Problem 1: 9.7

The given Laplace transform can be written as follows

$$\frac{s-1}{(s+2)(s+3)\left(s+\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)\left(s+\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)}. \tag{1}$$

Hence, the poles of the above Laplace transform are $s=-2$, $s=-3$, $s=-\frac{1}{2}-\frac{\sqrt{3}}{2}j$ and $s=-\frac{1}{2}+\frac{\sqrt{3}}{2}j$. We can choose the following 4 regions based on the real value of s such that the poles are not contained in the ROC:

- $\mathrm{Re}(s) > -\frac{1}{2}$
- $-2 < \mathrm{Re}(s) < -\frac{1}{2}$
- $-3 < \mathrm{Re}(s) < -2$
- $\mathrm{Re}(s) < -3$

For each these regions determines a signal, hence there are 4 signals that have the given Laplace transform.

Problem 2: 9.21

9.21(a)

Given $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$ which is a sum of two right sided signals $e^{-2t}u(t)$ and $e^{-3t}u(t)$. The Laplace transform of $e^{-2t}u(t)$ is $\frac{1}{s+2}$ with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > -2\}$. Similarly, the Laplace transform of $e^{-3t}u(t)$ is $\frac{1}{s+3}$ with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > -3\}$. Using the property of Laplace transform for sum of signals we have

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{s^2+5s+6}, \tag{2}$$

and ROC is given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > -2\}$. The poles are located at $s=-2$ and $s=-3$ and the zeros are located at $s = -\frac{5}{2} = -2.5$.

9.21(e)

Given $x(t) = |t|e^{-2|t|} = -te^{2t}u(-t) + te^{-2t}u(t)$, which is a sum of left-sided signal $-te^{2t}u(-t)$ and right-sided signal $te^{-2t}u(t)$. The Laplace transform of $-e^{2t}u(-t)$ is $\frac{1}{s-2}$ with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) < 2\}$. Now using the differentiation property we have

$$(-t)(-e^{2t}u(-t)) = te^{2t}u(-t) \longleftrightarrow \frac{d}{ds}\left(\frac{1}{s-2}\right) = -\frac{1}{(s-2)^2} \tag{3}$$

with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) < 2\}$. Repeating the same steps for the right-sided signal $e^{-2t}u(t)$ and using the differentiation property we have

$$(-t)(e^{-2t}u(t)) = -te^{-2t}u(t) \longleftrightarrow \frac{d}{ds}\left(\frac{1}{s+2}\right) = -\frac{1}{(s+2)^2} \tag{4}$$

with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > -2\}$. Using the property of Laplace transform for sum of signals we have

$$X(s) = \frac{1}{(s-2)^2} + \frac{1}{(s+2)^2} \tag{5}$$

$$= \frac{2s^2+8}{(s^2-4)^2} \tag{6}$$

with ROC given by $\{s \in \mathbb{C} : -2 < \mathrm{Re}(s) < 2\}$. The poles are located at $s = 2$ and $s = -2$ and the zeros are located at $s = 2j$ and $s = -2j$.

9.21(g)

Given $x(t) = 1$ for $0 \le t \le 1$ and 0 otherwise. The Laplace transform for $s \neq 0$ is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{7}$$

$$= \int_0^1 e^{-st}dt \tag{8}$$

$$= \left.\frac{e^{-st}}{-s}\right|_0^1 = \frac{1-e^{-s}}{s} \tag{9}$$

and for $s = 0$ we have $X(s) = 1$ and hence the ROC is given by entire s plane.

Problem 3: 9.22

9.22(a)

Using the Laplace transform pair table 9.2 we have $x(t) = \frac{1}{3}\sin(3t)u(t)$.

9.22(e)

Given

$$X(s) = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)}, \tag{10}$$

with ROC given by $\{s \in \mathbb{C} : -3 < \mathrm{Re}(s) < -2\}$. Now using partial fraction method we have

$$X(s) = \frac{A}{s+2} + \frac{B}{s+3} \tag{11}$$

which implies $s+1 = A(s+3) + B(s+2)$. Substituting $s = -2$ we have $A = -1$ and substituting $s = -3$ we have $B = 2$. Hence, the signal is two sided. Since $\{s \in \mathbb{C} : \mathrm{Re}(s) < -2\}$ is contained in ROC we have

$$\frac{1}{s+2} \longleftrightarrow -e^{-2t}u(-t) \tag{12}$$

and Since $\{s \in \mathbb{C} : \mathrm{Re}(s) > -3\}$ is contained in ROC we have

$$\frac{1}{s+3} \longleftrightarrow e^{-3t}u(t) \tag{13}$$

. Therefore, $x(t) = e^{-2t}u(-t) + 2e^{-3t}u(t)$.

9.22(g)

Given

$$X(s) = \frac{s^2 - s + 1}{(s+1)^2} = 1 - \frac{3s}{(s+1)^2} \tag{14}$$

with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. From table 9.2 we have

$$tu(t) \longleftrightarrow \frac{1}{s^2} \tag{15}$$

with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. Using shifting property we have

$$e^{-t}tu(t) \longleftrightarrow \frac{1}{(s+1)^2} \tag{16}$$

with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > -1\}$. Using the differentiation property we have

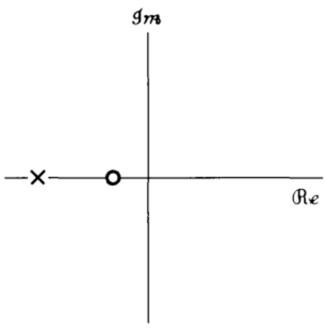
$$\frac{d}{dt}\left(e^{-t}tu(t)\right) = e^{-t}u(t) - e^{-t}tu(t) \longleftrightarrow \frac{s}{(s+1)^2} \tag{17}$$

with ROC given by $\{s \in \mathbb{C} : \mathrm{Re}(s) > -1\}$. Therefore, $x(t) = \delta(t) - 3e^{-t}u(t) + 3e^{-t}tu(t)$.

1 Problem 4: Geometric evaluation of the Fourier Transform

9.25. By considering the geometric determination of the Fourier transform, as developed in Section 9.4, sketch, for each of the pole-zero plots in Figure P9.25, the magnitude of the associated Fourier transform.

9.25.(c).



(c)

According to the plot, H(s) has a zero and a pole in the negative half plane. Let's assume the zero is z = -a, and the pole is p = -b such that $b > a > 0$. (E.g. z = -1, p = -2)

Now in order to calculate $|H(jw)|$ one must note that H(jw) is H(s) restricted to the jw axis. In this case:

$H(s) = \frac{s+a}{s+b}$

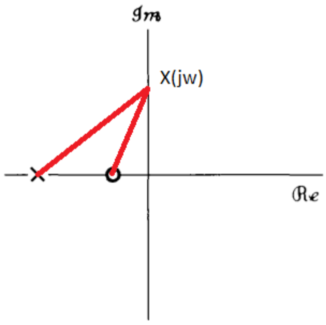
$H(jw) = \frac{jw+a}{jw+b}$

$|H(jw)| = \frac{|jw+a|}{|jw+b|}$

This is shown in the following figure as the distance between H(jw) and -a and -b. We can now examine the $|H(jw)$ equation:

$lim_{w \to \infty} |H(jw)| = lim_{w \to \infty} \frac{|jw+a|}{|jw+b|} = \frac{|jw|}{|jw|} = 1$

In other words, as w increases, the two red lines in the figure become closer



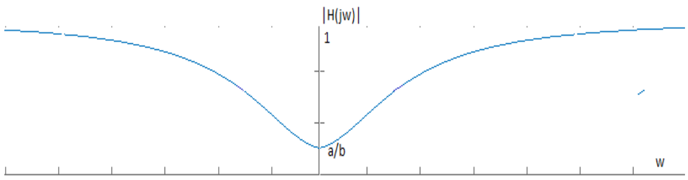
(c)

and closer together, until the have approximately the same length for very large w (high on the jw axis).

On the other hand, the minimum ratio of the length of the two red lines is reached when $w = 0$. In other words:

$lim_{w \to 0} |H(jw)| = lim_{w \to 0} \frac{|jw+a|}{|jw+b|} = \frac{|a|}{|b|} = \frac{a}{b}$

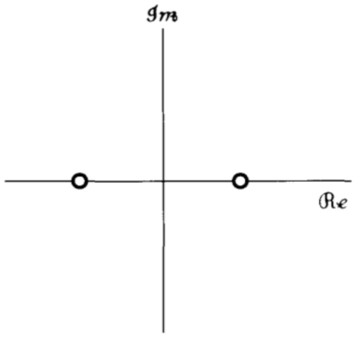
This result is shown in the following figure:



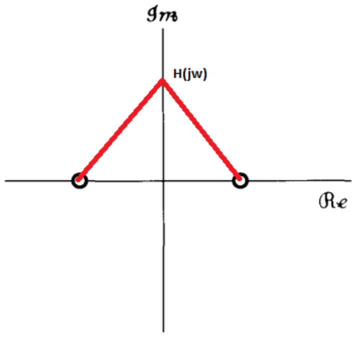
9.25.(f). In this part, H(s) has two zeros with the same magnitude and different signs. Let's define H(s) as the following:

$H(s) = (s + a)(s - a)$ such that $a > 0$.

The process of finding the absolute value of the magnitude of H(jw) is the same as the previous part. Consider the following figure:
Again For $w \to \infty$:



(f)

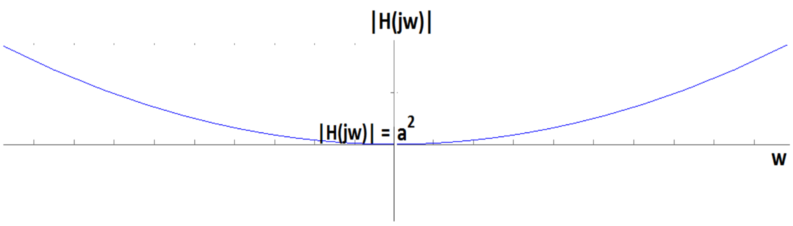


(f)

$$|H(jw)| = |(jw+a)(jw-a)| = |(jw+a)|.|(jw-a)|$$

$$lim_{w \to \infty} |H(jw)| = \infty$$

$$lim_{w \to 0} |H(jw)| = a^2$$



2 Problem 5: Laplace Transform and differential equations

9.31. Consider a continuous-time LTI system for which the input x(t) and output y(t) are related by the differential equation

$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$

Let X(s) and Y(s) denote Laplace transforms of x(t) and y(t), respectively, and let H(s) denote the Laplace transform of h(t), the system impulse response.

9.31.(a). Determine H(s) as a ratio of two polynomials in s. Sketch the pole-zero pattern of H(s).

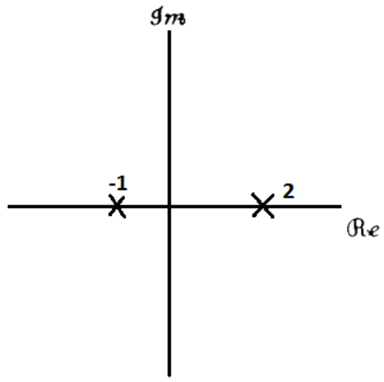
In order to determine H(s) we calculate the Laplace transform of both sides of the equation. For each $\frac{d^n}{dt^n}$ in the equation we can substitute s^n as taking Laplace from both sides:

$s^2Y(s) - sY(s) - 2Y(s) = X(s)$

$(s^2 - s - 2)Y(s) = X(s)$

$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2-s-2} = \frac{1}{(s-2)(s+1)}$

Therefore, by putting $(s - 2)(s + 1) = 0$, we can conclude that H(s) has two poles at s = 2 and s = -1. You can see the plot in the following:



9.31.(b). Determine h(t) for each of the following cases:

1. The system is stable.

$H(s) = \frac{1}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)}$

From the above equation we have:

$A(s+1) + B(s-2) = 1 \quad$ or $\quad (A+B)s + (A-2B) = 1$

$A + B = 0, \quad A - 2B = 1 \quad$ which gives us $\quad A = \frac{1}{3}, \quad B = \frac{-1}{3}$

Thus, $\quad H(s) = \frac{1}{3}(\frac{1}{(s-2)} - \frac{1}{(s+1)})$

By checking the Laplace transform table, we can calculate h(t) as the following:

$h(t) = L^{-1}\{H(s)\} = \frac{1}{3}(L^{-1}\{\frac{1}{(s-2)}\} - L^{-1}\{\frac{1}{(s+1)})\}$

Because the system is stable, the ROC of H(s) must contain the jw axis, which means that its ROC is $-1 < s < 2$. This means that ROC is in the right side of the pole in -1 (right hand h(t) for $L^{-1}\{\frac{1}{(s+1)})\}$ which contains u(t)). On the other hand the ROC is on the left side of the pole in 2 (left hand h(t) for $\frac{1}{3}(L^{-1}\{\frac{1}{(s-2)}\}$ which contains -u(-t)).

$L^{-1}\{H(s)\} = \frac{1}{3}(-e^{2t}u(-t) - e^{-t}u(t)) = \frac{-1}{3}(e^{2t}u(-t) + e^{-t}u(t))$

2. The system is causal.

If the system is causal, the ROC must be on the right hand side of the s plane ($ROC : s > 2$) which means that it is on the right side of both poles and both parts should be on the right hand in time domain as well. Therefore:

$L^{-1}\{H(s)\} = \frac{1}{3}(L^{-1}\{\frac{1}{(s-2)}\} - L^{-1}\{\frac{1}{(s+1)})\} = \frac{1}{3}(e^{2t}u(t) - e^{-t}u(t)) = \frac{1}{3}(e^{2t} - e^{-t})u(t)$

3. The system is neither stable nor causal.

If the system is neither stable nor causal, the ROC should not contain jw axis and it cannot be on the right hand side of the s plane. Therefore, $ROC : s < -1$ and both parts will have -u(-t) in their inverse Laplace transforms.

$L^{-1}\{H(s)\} = \frac{1}{3}(L^{-1}\{\frac{1}{(s-2)}\} - L^{-1}\{\frac{1}{(s+1)})\} = \frac{1}{3}(-e^{2t}u(-t) - (-e^{-t}u(-t))) = \frac{1}{3}(-e^{2t} + e^{-t})u(-t)$