ECE 101

Problem Set #5C Solutions

1. **Problem 9.4**

$$x(t) = \begin{cases} e^t \sin 2t & t \le 0 \\ 0 & t > 0 \end{cases}$$

Approach #1:

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t \\
&= \int_{-\infty}^{0} e^t \sin 2t e^{-st}\,\mathrm{d}t \\
&= \int_{-\infty}^{0} e^t \frac{1}{2j}(e^{j2t} - e^{-j2t})e^{-st}\,\mathrm{d}t \\
&= \frac{1}{2j}\int_{-\infty}^{0} (e^{(1+2j-s)t} - e^{(1-2j-s)t})\,\mathrm{d}t, \quad s = \sigma + j\omega \\
&= \frac{1}{2j}\int_{-\infty}^{0} (e^{(1+2j-\sigma-j\omega)t} - e^{(1-2j-\sigma-j\omega)t})\,\mathrm{d}t \\
&= \frac{1}{2j}\left(\frac{1}{1+2j-\sigma-j\omega}e^{(1+2j-\sigma-j\omega)t} - \frac{1}{1-2j-\sigma-j\omega}e^{(1-2j-\sigma-j\omega)t}\right)\Big|_{-\infty}^{0} \\
&= \frac{1}{2j}\left(\frac{1}{1+2j-\sigma-j\omega} - \frac{1}{1-2j-\sigma-j\omega}\right), \quad \text{if} \quad 1-\sigma > 0 \\
&= \frac{1}{2j}\left(\frac{1}{1+2j-s} - \frac{1}{1-2j-s}\right) \\
&= \frac{1}{2j}\frac{-4j}{(s-1)^2+4} \\
&= \frac{-2}{(s-1)^2+4}
\end{aligned}$$

$$ROC = \{s \in \mathbb{C} | 1-\sigma > 0\} = \{s \in \mathbb{C} | Re\{s\} < 1\}$$

$X(s)$ has poles at $s = 1+2j$, $s = 1-2j$.

Approach #2:
From Table 9.2, we know

$$(e^{-\alpha t}\sin \omega_o t)u(t) \longleftrightarrow \frac{\omega_o}{(s+\alpha)^2+\omega_o^2}, \quad Re\{s\} > -\alpha$$

$x(t)$ can be written as

$$x(t) = e^t \sin 2t u(-t)$$

We construct another signal

$$x_1(t) = x(-t) = e^{-t}\sin(-2t)u(t) = -e^{-t}\sin(2t)u(t)$$

We know that

$$X_1(s) = -\frac{2}{(s+1)^2+4}, \quad Re\{s\} > -1$$

From Table 9.1, we know that

$$x(t) \longleftrightarrow X(s), \quad ROC = R_1$$

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{s}{a}), \quad ROC = \text{scaled}R_1$$

In our case,

$$a = -1, \quad X(s) = \frac{1}{|-1|}X_1(\frac{s}{-1}) = X_1(-s) = -\frac{2}{(1-s)^2+4}, \quad Re\{s\} < 1$$

$X(s)$ has poles at $s = 1+2j$, $s = 1-2j$.

2. **Problem 9.25** Sketch the magnitude of Fourier transform according to the pole-zero plot.

   (a) There are two zeros at $s = z_1 = a + j\omega_0$ and $s = z_2 = a - j\omega_0$. There is a pole at $s = p$. $a, p, \omega_0 \in \mathbb{R}$, and a is very small. The Laplace transform can be written as

$$X(s) = M\frac{(s - z_1)(s - z_2)}{s - p}$$

   The Fourier transform can be obtained by replacing s with $j\omega$

$$X(j\omega) = M\frac{(j\omega - z_1)(j\omega - z_2)}{j\omega - p} = M\frac{(j\omega - a - j\omega_0)(j\omega - a + j\omega_0)}{j\omega - p}$$

   The magnitude of Fourier transform is

$$|X(j\omega)| = |M|\frac{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|}{|j\omega - p|}$$

   The quantities $|j\omega - a - j\omega_0|, |j\omega - a + j\omega_0|, |j\omega - p|$ are the lengths of the vectors shown in Fig. 1.

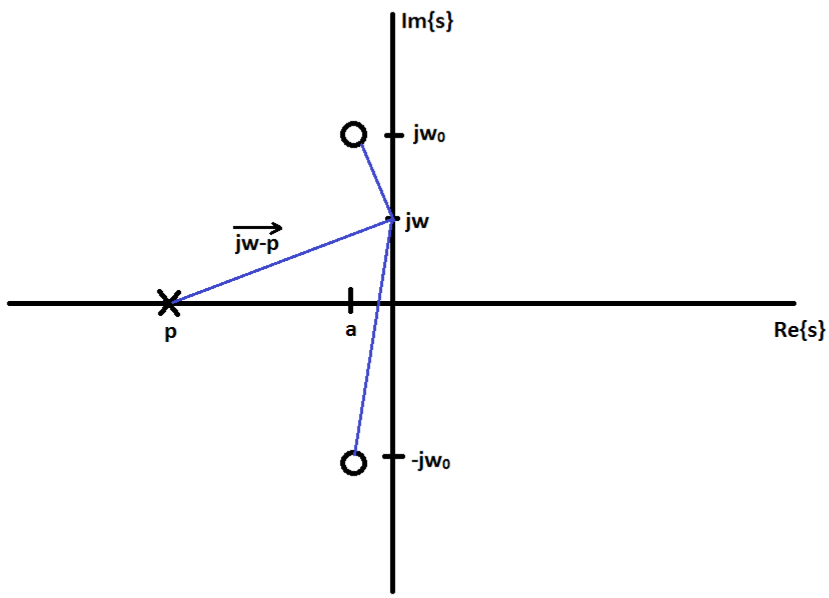


Figure 1: Problem 9.25(a) Pole-zero plot

   (1) As $\omega \to \infty$, $|j\omega - a - j\omega_0| \approx |j\omega - a + j\omega_0| \approx |j\omega - p| = L \to \infty$, so

$$|X(j\omega)| = |M|\frac{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|}{|j\omega - p|} \approx |M|\frac{L * L}{L} = |M||L| \to \infty$$

(2) As $\omega \to -\infty$, we have a similar situation, so

$$|X(j\omega)| = |M|\frac{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|}{|j\omega - p|} \approx |M|\frac{L * L}{L} = |M||L| \to \infty$$

(3) For $\omega = \omega_0$, $|X(j\omega)|$ has a minimum.

$$|X(j\omega)| = |M|\frac{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|}{|j\omega - p|} = |M|\frac{|-a||j2\omega_0 - a|}{|j\omega_0 - p|}$$

When $a \to 0$ (the zeros get close to the $j\omega$-axis), $|X(j\omega)| \to 0$.

(4) For $\omega = -\omega_0$, $|X(j\omega)|$ has another minimum.

$$|X(j\omega)| = |M|\frac{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|}{|j\omega - p|} = |M|\frac{|-j2\omega_0 - a||-a|}{|-j\omega_0 - p|}$$

which is equal to the magnitude when $\omega = \omega_0$.

(5) Between ω_0 and $-\omega_0$, for $\omega = 0$,

$$|X(j\omega)| = |M|\frac{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|}{|j\omega - p|} = |M|\frac{|-a - j\omega_0||-a + j\omega_0|}{|-p|} = |M|\frac{a^2 + \omega_0^2}{|p|}$$

When $a \to 0$, $|X(j\omega)| \to \frac{|M|\omega_0^2}{|p|} > 0$. So it is greater than the magnitude when $\omega = \pm\omega_0$. So it is a local maximum.

The magnitude of $X(j\omega)$ is shown as Fig. 2.

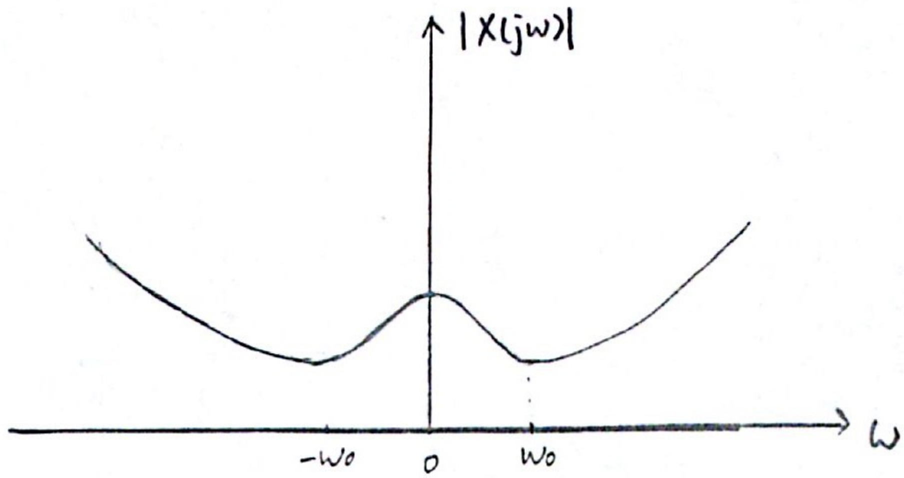


Figure 2: Problem 9.25(a) $|X(j\omega)|$

(b) There are two poles at $s = p_1 = a + j\omega_0$ and $s = p_2 = a - j\omega_0$. $a, \omega_0 \in \mathbb{R}$, and a is very small. The Laplace transform can be written as

$$X(s) = M\frac{1}{(s - p_1)(s - p_2)}$$

The Fourier transform can be obtained by replacing s with $j\omega$

$$X(j\omega) = M\frac{1}{(j\omega - p_1)(j\omega - p_2)} = M\frac{1}{(j\omega - a - j\omega_0)(j\omega - a + j\omega_0)}$$

The magnitude of Fourier transform is

$$|X(j\omega)| = |M|\frac{1}{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|}$$

The quantities $|j\omega - a - j\omega_0|, |j\omega - a + j\omega_0|$ are the lengths of the vectors shown in Fig. 3.

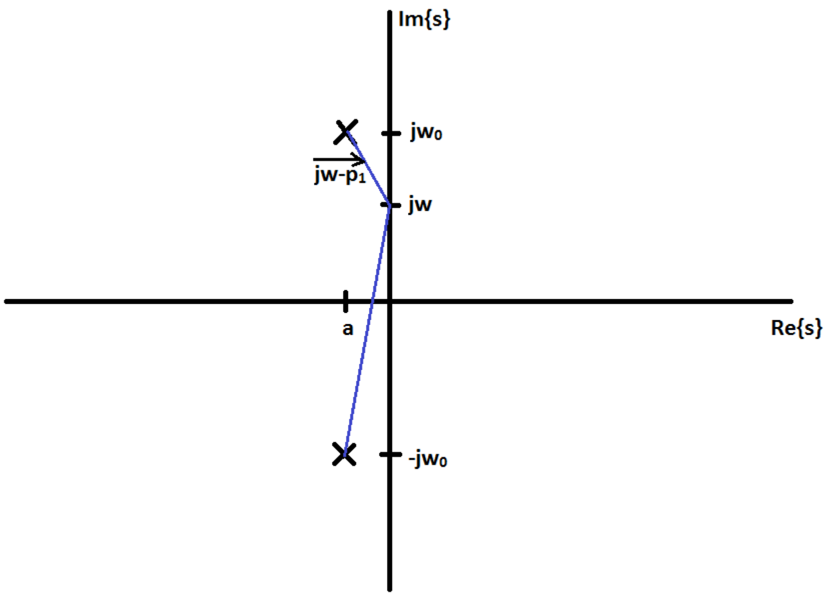


Figure 3: Problem 9.25(b) Pole-zero plot

(1) As $\omega \to \infty$, $|j\omega - a - j\omega_0| \approx |j\omega - a + j\omega_0| \approx |j\omega - p| = L \to \infty$, so

$$|X(j\omega)| = |M|\frac{1}{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|} \approx |M|\frac{1}{L * L} \to 0$$

(2) As $\omega \to -\infty$, we have a similar situation, so

$$|X(j\omega)| = |M|\frac{1}{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|} \approx |M|\frac{1}{L * L} \to 0$$

(3) For $\omega = \omega_0$, $|X(j\omega)|$ has a maximum.

$$|X(j\omega)| = |M|\frac{1}{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|} = |M|\frac{1}{|-a||j2\omega_0 - a|}$$

When $a \to 0$ (the poles get close to the $j\omega$-axis), $|X(j\omega)| \to \infty$.

(4) For $\omega = -\omega_0$, $|X(j\omega)|$ has another maximum.

$$|X(j\omega)| = |M| \frac{1}{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|} = |M| \frac{1}{|-j2\omega_0 - a||-a|}$$

which is equal to the magnitude when $\omega = \omega_0$.

(5) Between ω_0 and $-\omega_0$, for $\omega = 0$,

$$|X(j\omega)| = |M| \frac{1}{|j\omega - a - j\omega_0||j\omega - a + j\omega_0|} = \frac{|M|}{|-a - j\omega_0||-a + j\omega_0|} = \frac{|M|}{a^2 + \omega_0^2}$$

When $a \to 0$, $|X(j\omega)| \to \frac{|M|}{\omega_0^2} << \infty$. So it is less than the magnitude when $\omega = \pm\omega_0$. So it is a local minimum.

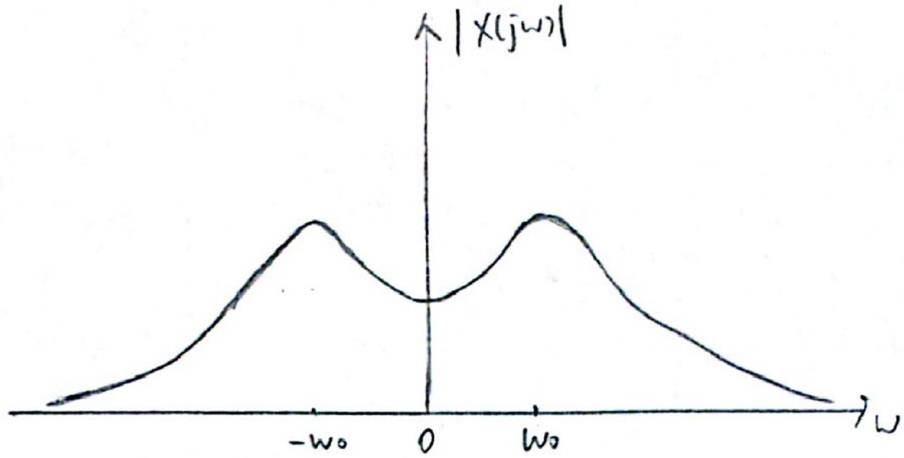


Figure 4: Problem 9.25(b) $|X(j\omega)|$

3. **Problem 9.33**

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$$

Its Laplace transform is

$$X(s) = \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{(s+1)(s-1)}, \quad -1 < Re\{s\} < 1$$

$$H(s) = \frac{s+1}{s^2+2s+2}$$

Then

$$Y(s) = X(s)H(s) = \frac{-2}{(s+1)(s-1)} \frac{s+1}{s^2+2s+2} = \frac{-2}{(s-1)(s^2+2s+2)}$$

Let

$$Y(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2} = \frac{A(s^2+2s+2) + (Bs+C)(s-1)}{(s-1)(s^2+2s+2)}$$

Then

$$A(s^2+2s+2) + (Bs+C)(s-1) = -2$$

$$\begin{cases} A + B = 0 \\ 2A - B + C = 0 \\ 2A - C = -2 \end{cases}$$

We get $A = -\frac{2}{5}, B = \frac{2}{5}, C = \frac{6}{5}$.

$$Y(s) = \frac{-2/5}{s-1} + \frac{2/5s + 6/5}{s^2+2s+2} = \frac{-2/5}{s-1} + \frac{2/5(s+1) + 4/5}{(s+1)^2+1} = \frac{-2/5}{s-1} + \frac{2/5(s+1)}{(s+1)^2+1} + \frac{4/5}{(s+1)^2+1}$$

From Table 9.2, and since $-1 < Re\{s\} < 1$, we get

$$y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos t u(t) + \frac{4}{5}e^{-t}\sin t u(t)$$

4. **Problem 9.36**

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2} = \frac{2(s+3)(s-1)}{(s+1)(s+2)}$$

$$H_1(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

(a)

$$H(s) = H_1(s)(2s^2 + 4s - 6)$$

Since

$$H(s) = \frac{Y(s)}{X(s)}, H_1(s) = \frac{Y_1(s)}{X(s)}$$

$$Y(s) = Y_1(s)(2s^2 + 4s - 6)$$

which means

$$y(t) = 2\frac{d^2 y_1(t)}{dt^2} + 4\frac{dy_1(t)}{dt} - 6y_1(t)$$

(b) From S_1's block diagram,

$$Y_1(s) = F(s)\frac{1}{s}$$

$$F(s) = sY_1(s)$$

$$f(t) = \frac{dy(t)}{dt}$$

(c)

$$Y_1(s) = E(s)\frac{1}{s^2}$$

$$E(s) = s^2 Y_1(s)$$

$$e(t) = \frac{d^2 y(t)}{dt^2}$$

(d)

$$y(t) = 2e(t) + 4f(t) - 6y_1(t)$$

(e) The block diagram of $H(s)$ is shown in Fig. 5.

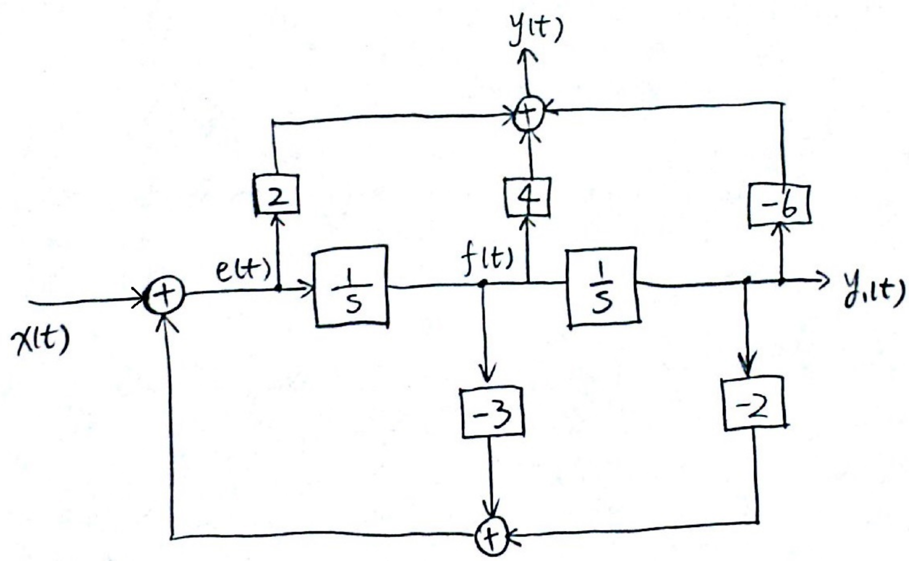


Figure 5: Problem 9.36(e)

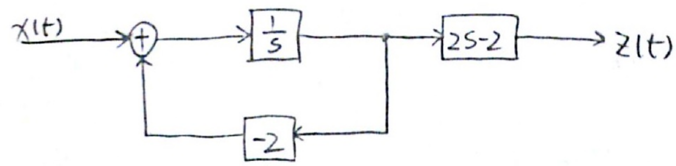
(f) The system can be represented as the cascade of two systems.

$$H(s) = \left(\frac{2(s-1)}{s+2}\right)\left(\frac{s+3}{s+1}\right) = H_1(s)H_2(s)$$

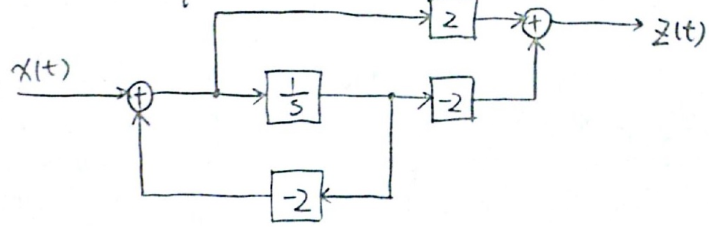
where

$$H_1(s) = \frac{2(s-1)}{s+2}$$
$$H_2(s) = \frac{s+3}{s+1}$$

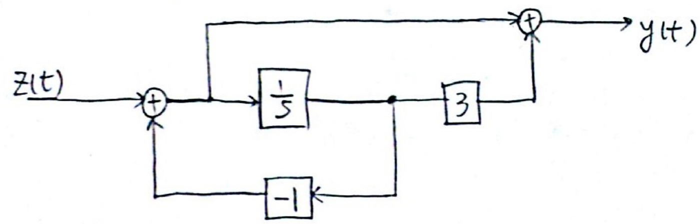
$$H_1(s) = \frac{2(s-1)}{s+2}$$



which is equivalent to



$$H_2(s) = \frac{s+3}{s+1}$$



H(s)

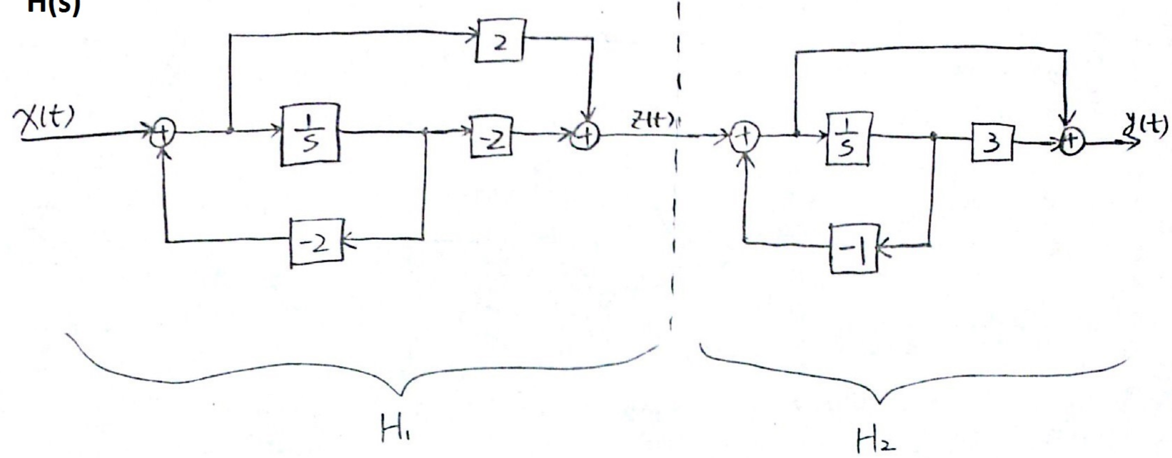


Figure 6: Problem 9.36(f)

(g)

$$H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1}$$

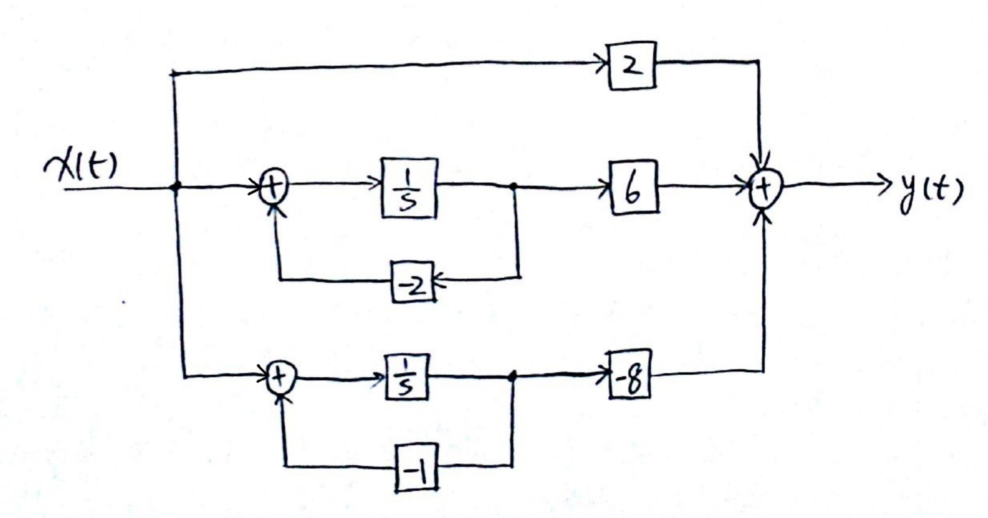


Figure 7: Problem 9.36(g)