Lecture 15

Convolution

Preview of today's lecture

- Convolution property
  - Convolution in time is multiplication in frequency
  - Use this fact to compute convolutions with less work!
- Multiplication property
  - Multiplication in time is convolution in frequency
  - Use this fact to explain windowing
- Bandwidth
  - Finite duration signals have infinite bandwidth
  - Different measures of bandwidth are used in practice

Fourier transform properties I $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \qquad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

	Time domain	Fourier transform
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(j\omega)$

Fourier transform properties 2 $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

	Time domain	Fourier transform
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert} x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$	

Fourier transform properties 3

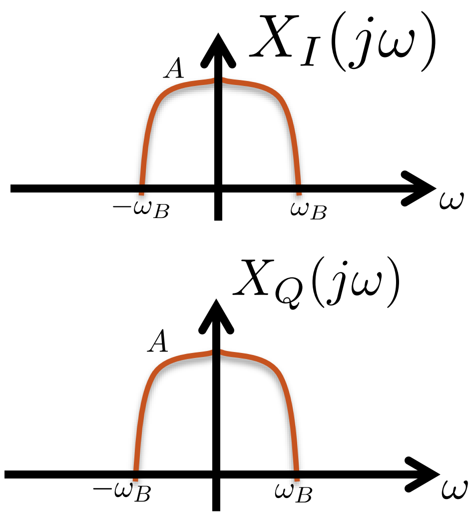
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

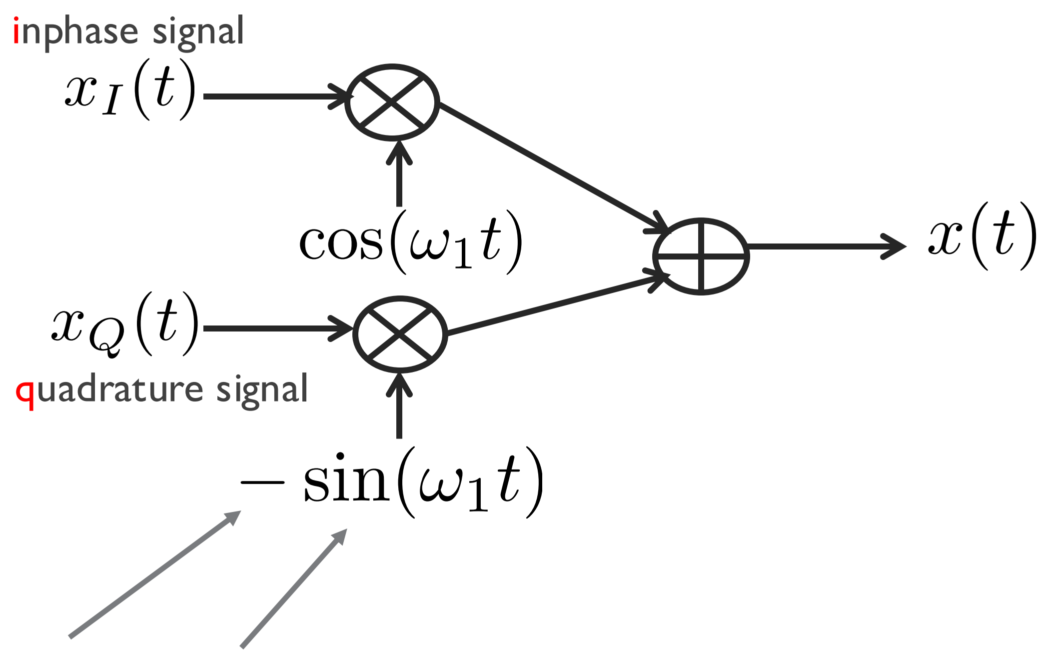
$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$$

	Time domain	Fourier transform
Convolution in time	$y(t) = h(t) * x(t)$	$Y(j\omega) = H(j\omega)X(j\omega)$
Multiplication in time	$y(t) = h(t)x(t)$	$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$

Practical application – Inphase and quadrature

- What if two information signals are sent as follows?





note the sign and sine here, which will make sense shortly

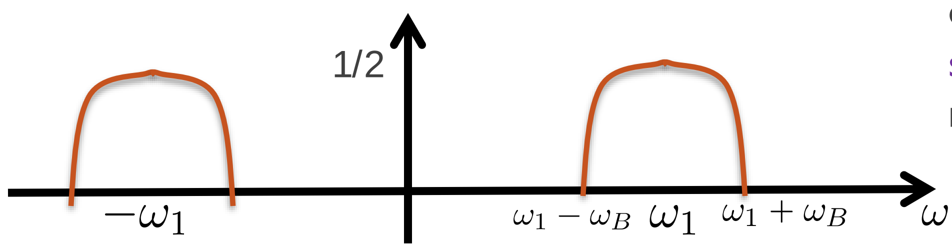
Practical application – Inphase and quadrature (cont.)

- What happens in the frequency domain?
  - Inphase term

$$\mathcal{F}\left\{x_I(t)\cos(\omega_1 t)\right\} = \frac{1}{2}X_I(j(\omega-\omega_1)) + \frac{1}{2}X_I(j(\omega+\omega_1))$$

  - Quadrature term

$$\mathcal{F}\left\{-x_Q(t)\sin(\omega_1 t)\right\} = \frac{j}{2}X_Q(j(\omega-\omega_1)) - \frac{j}{2}X_Q(j(\omega+\omega_1))$$



mixture of inphase and quadrature terms but not the same mixture at positive and negative frequencies

Practical application – Inphase and quadrature (cont.)

- What about demodulation?
  - Trig identities

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$
$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$
$$\sin u \cos v = \frac{1}{2}\left[\sin(u-v) + \sin(u+v)\right]$$

Can recover both inphase and quadrature!

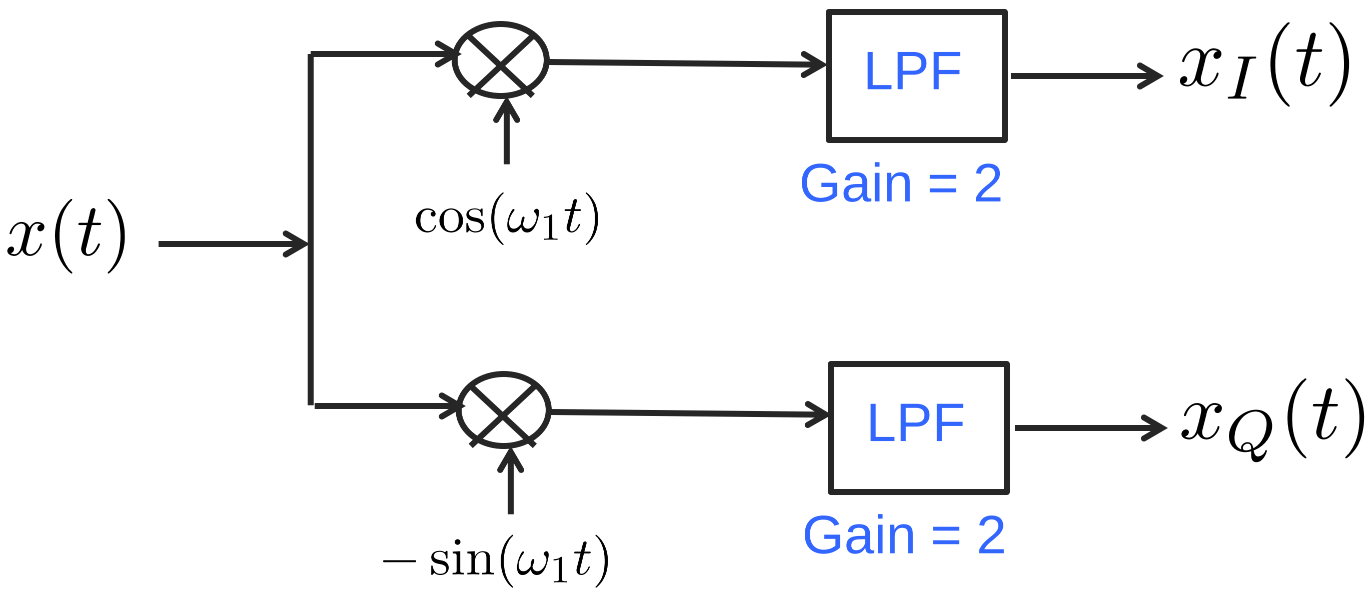
  - Applying the identities

filter out

$$x(t)\cos(\omega_1 t) = \frac{1}{2}x_I(t) + \boxed{\frac{1}{2}x_I(t)\cos(2\omega_1 t) - \frac{1}{2}x_Q(t)\sin(2\omega_1 t)}$$
$$x(t)\sin(\omega_1 t) = -\frac{1}{2}x_Q(t) + \boxed{\frac{1}{2}x_Q(t)\cos(2\omega_1 t) + \frac{1}{2}x_I(t)\sin(2\omega_1 t)}.$$

Practical application – Inphase and quadrature (cont.)

- IQ demodulator



Practical application – Inphase and quadrature (cont.)

- Why do we use complex signals?

This is called the complex baseband signal

$$x_{bb}(t) = x_I(t) + jx_Q(t)$$

$\mathrm{Re}\{x_{bb}(t)\}$ $\mathrm{Im}\{x_{bb}(t)\}$

Complex signals become a convenient way to work with inphase and quadrature together, avoiding the need for matrix notation

Connections back to ECE 45

Lectures 2 - 3 working with signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Lectures 4 - 7 LTI systems in the time domain

Lectures 11, 17 LTI systems in the frequency domain

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Lectures 8 - 10 Fourier series

Lectures 11 - 16 Fourier transform

Fourier

Convolution property

Key points

- Convolution in time is multiplication in frequency
- Use this fact to compute convolutions

Convolution property

- If $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega) \quad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

- Then

$$\boxed{y(t) = h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)}$$

Convolution in time is multiplication in frequency

Proof of the convolution property

$$Y(j\omega) = \mathcal{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t}dt$$

Apply definition

$$= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dtd\tau$$

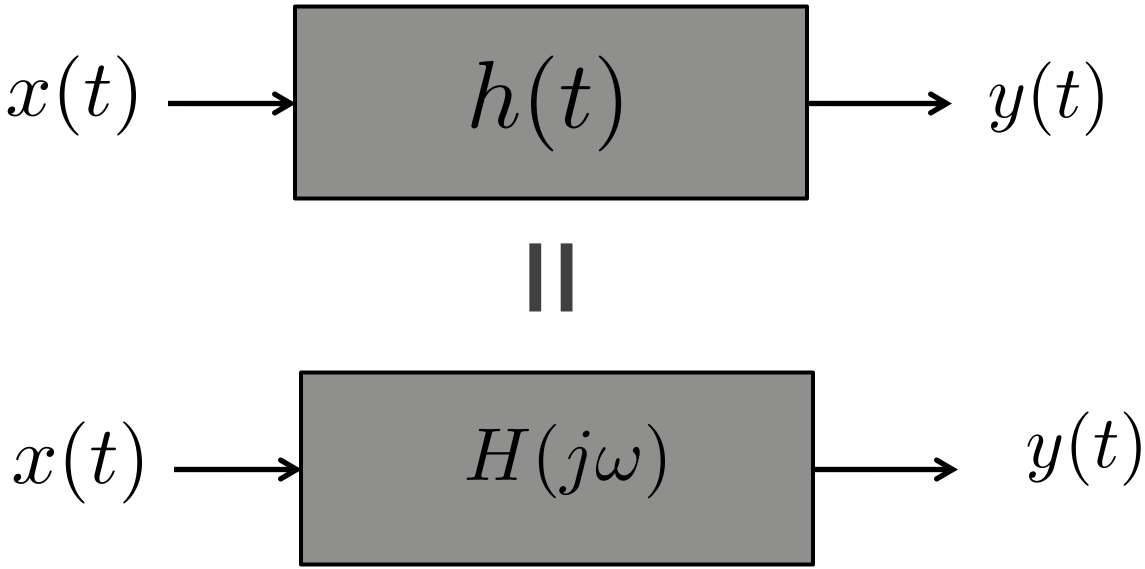
Exchange order of integration

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau$$

Time shift property

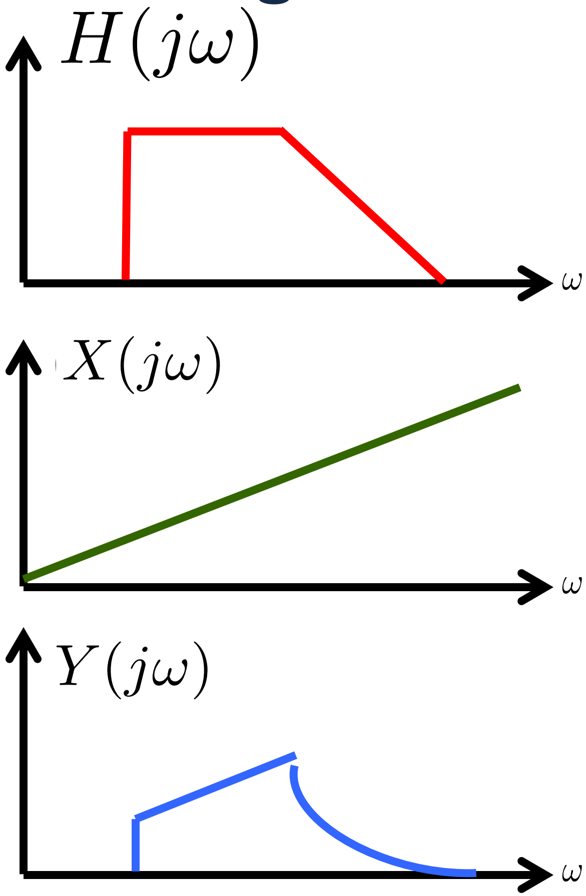
$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \quad = H(j\omega)X(j\omega)$$

Block diagrams



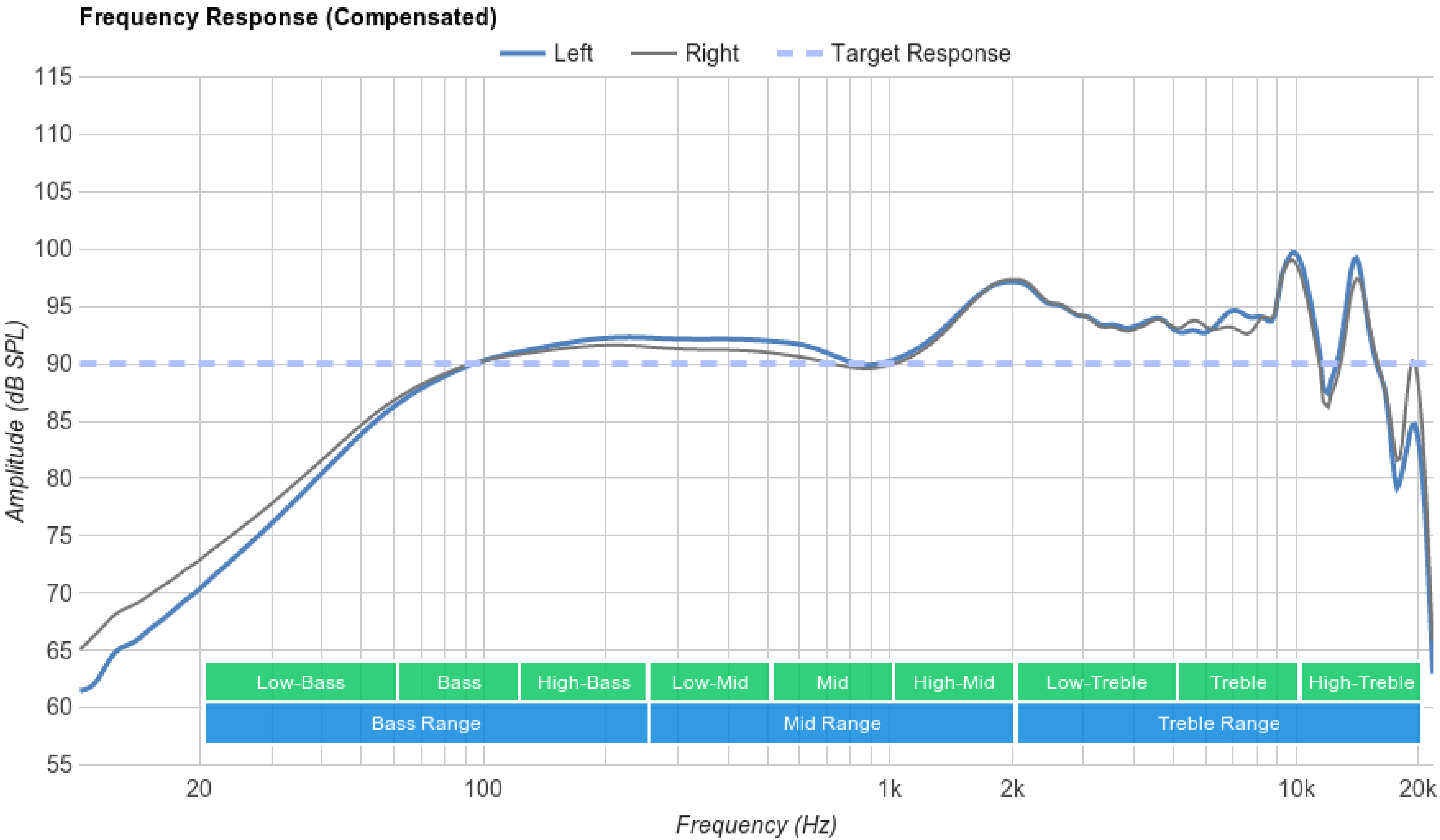
From a notational perspective, an LTI system may be described by the impulse response in the time or frequency domains

Visualizing the convolution property

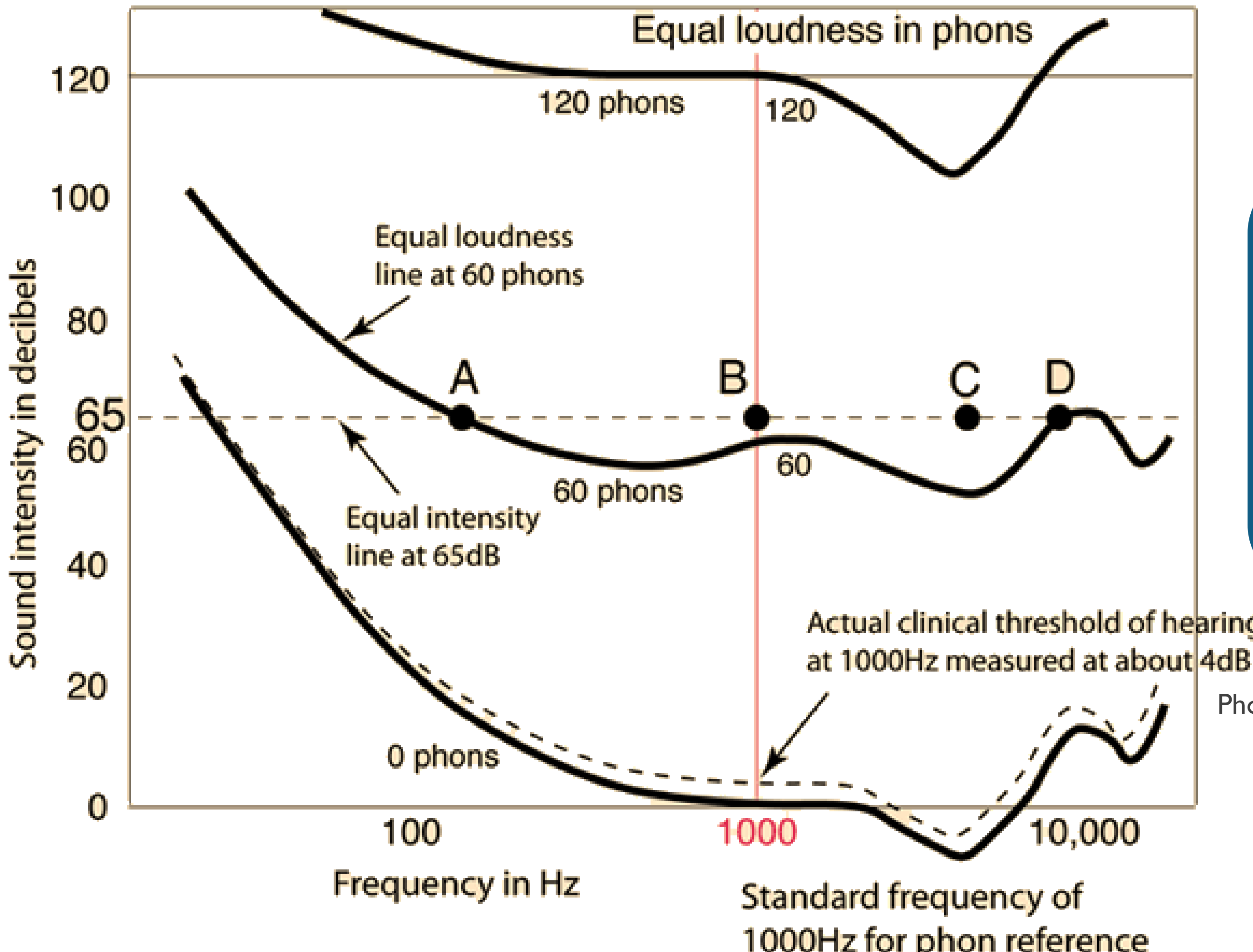


$$Y(j\omega) = H(j\omega)X(j\omega)$$

Direct multiplication at each frequency



Contributed by EE 313 student Erte Bablu from http://www.rtings.com/headphones/reviews/apple/wireless-airpods



Perceptual response to sound intensity is not uniform in frequency

Phon is sound referenced to 1kHz

http://hyperphysics.phy-astr.gsu.edu/hbase/Sound/earcrv.html

Using the convolution property to do convolutions

- Compute the following convolution

$$y(t) = h(t) * x(t)$$

- Convert the two signals into the frequency domain

$$H(j\omega) = \mathcal{F}\{h(t)\}$$
$$X(j\omega) = \mathcal{F}\{x(t)\}$$

- Compute the product

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Go from frequency domain back into the time domain

$$y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$$

Double sinc example

- Given where $\omega_i > 0$ and $\omega_c > 0$

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \qquad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

- Find

$$y(t) = h(t) * x(t)$$

Double sinc example (continued)

- Solve by going into the frequency domain
- First find

$$Y(j\omega) = H(j\omega)X(j\omega)$$

- Need to compute

$$\mathcal{F}\left\{\frac{\sin(\omega_i t)}{\pi t}\right\}\mathcal{F}\left\{\frac{\sin(\omega_c t)}{\pi t}\right\}$$

- But note that

$$\mathrm{sinc}\left(\frac{t}{2\pi}\right) = \frac{\sin(t/2)}{t/2} \quad \text{and} \quad \mathrm{sinc}\left(\frac{t}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\mathrm{rect}(\omega)$$

Double sinc example (continued)

- Using the scaling property

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- Write

$$\frac{\sin(\omega_i t)}{\pi t} = \frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}$$

- It follows that

$$\begin{aligned} \mathcal{F}\left\{\frac{\omega_i}{\pi} \frac{\sin(2\omega_i t/2)}{2\omega_i t/2}\right\} &= 2\pi \frac{\omega_i}{\pi} \frac{1}{|2\omega_i|} \mathrm{rect}(\omega/2\omega_i) \\ &= \mathrm{rect}(\omega/2\omega_i) \end{aligned}$$

Double sinc example (continued)

- The convolution is then

$$Y(j\omega) = \mathrm{rect}(\omega/2\omega_i)\mathrm{rect}(\omega/2\omega_c)$$

$$= \mathrm{rect}(\omega/2\min(\omega_c, \omega_i))$$

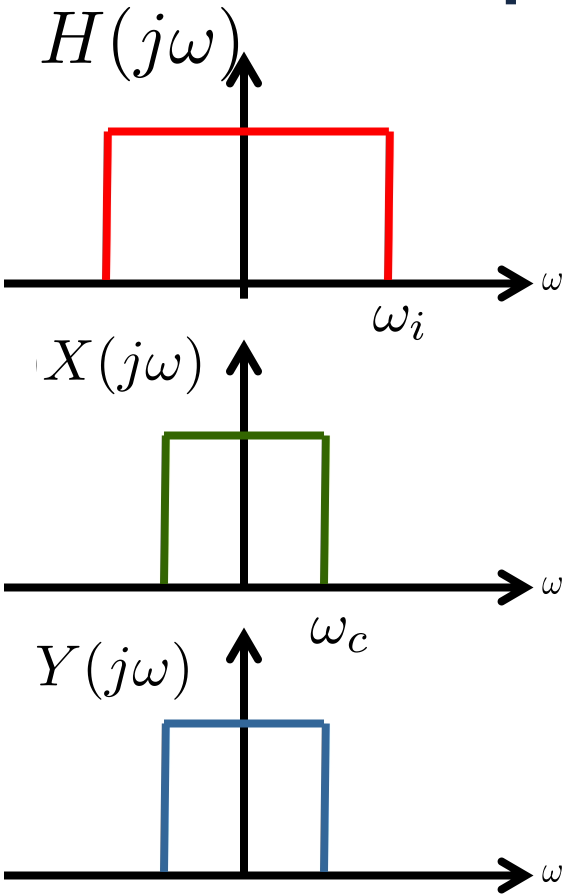
- Back in the time domain

$$y(t) = \frac{\sin(\min(\omega_i, \omega_c)t)}{\pi t}$$

This is a general result that sinc convolved with sinc gives sinc

Double sinc example (concluded)

Visualizing the effect in the frequency domain



$$Y(j\omega) = H(j\omega)X(j\omega)$$

Example where ω_i is bigger than ω_c

Summarizing the convolution property

- Convolution between two signals in time becomes the product of the Fourier transforms of those signals in the frequency domain

- Convolutions are easy to do in the frequency domain as they involve a simple point-wise multiplication

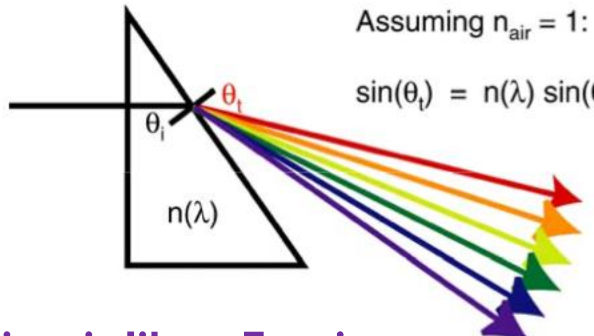
- The convolution property explains how the frequency response of a system directly effects the frequencies of the input signal to create the output signal

Fourier in practice

Key points

- Fourier concepts show up everywhere

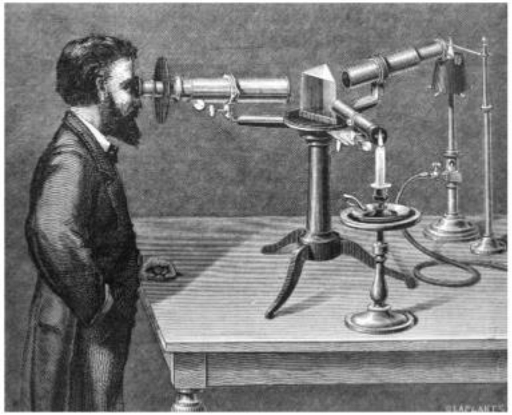
Spectrometer



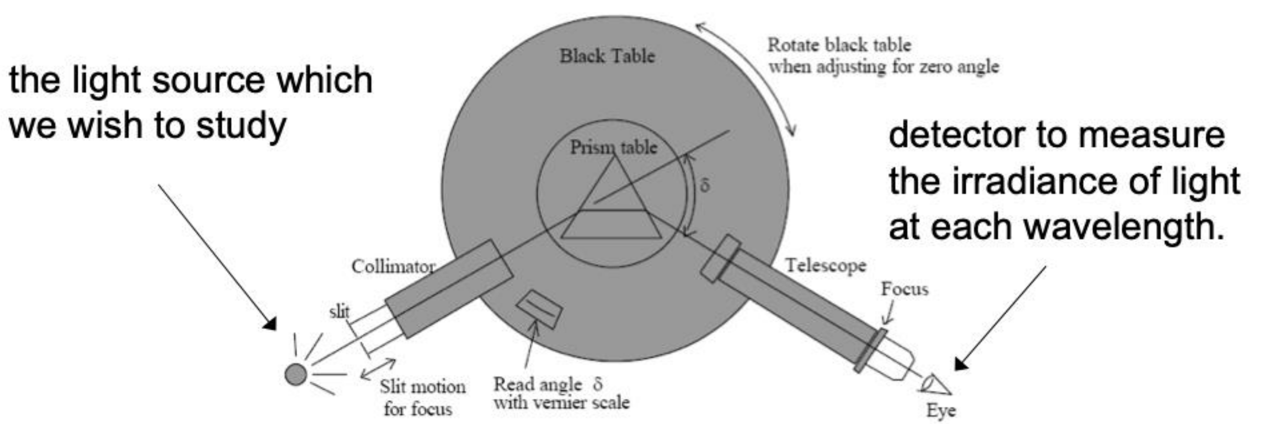
Assuming $n_{air} = 1$:

$\sin(\theta_t) = n(\lambda) \sin(\theta_i)$

Prism is like a Fourier operator

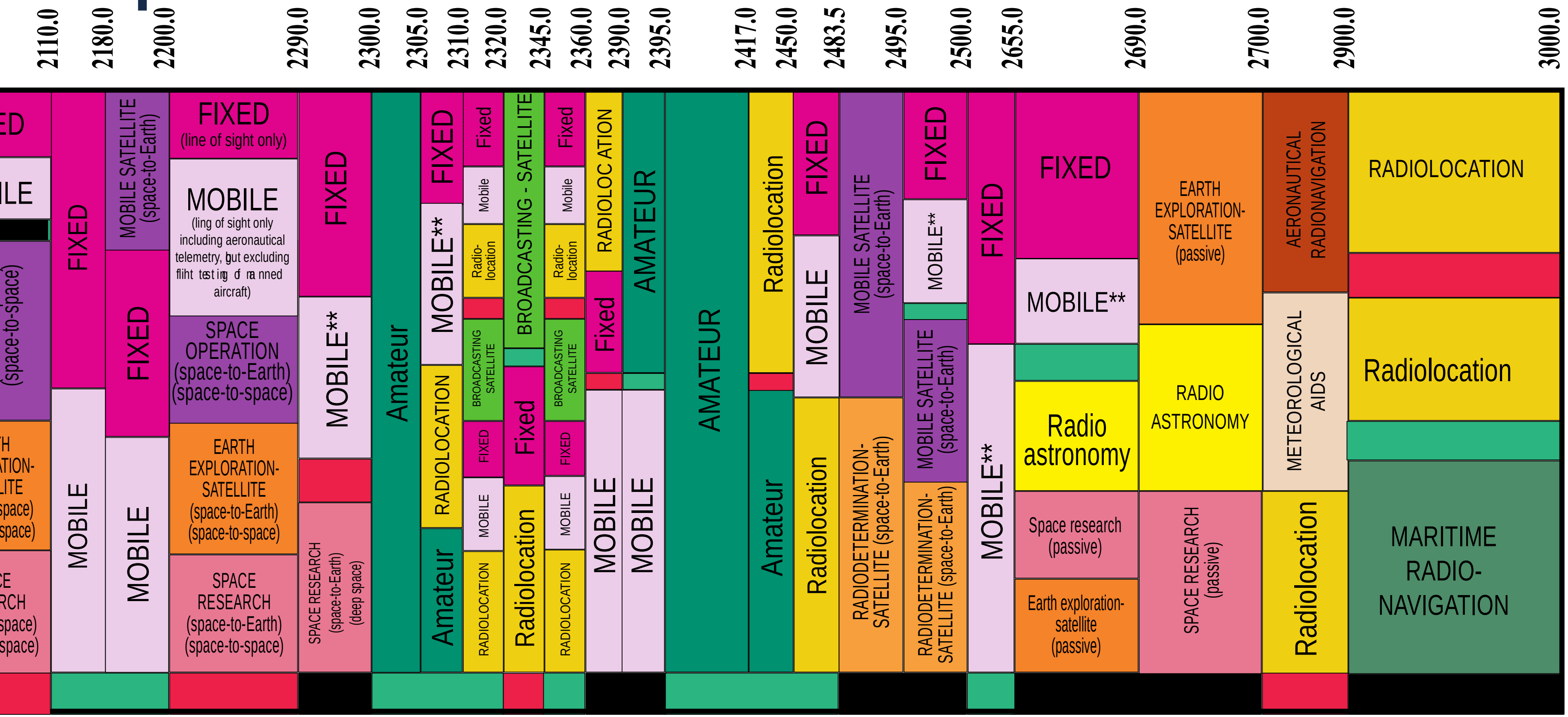


Robert Bunsen, 1859



Figures from https://www.brown.edu/research/labs/mittleman/sites/brown.edu.research.labs.mittleman/files/uploads/lecture19_0.pdf

Spectrum allocation



ISM – 2450.0± .50 MHz

3 GHz

https://www.ntia.doc.gov/files/ntia/publications/january_2016_spectrum_wall_chart.pdf

Communications

Spectrum for in-band signal

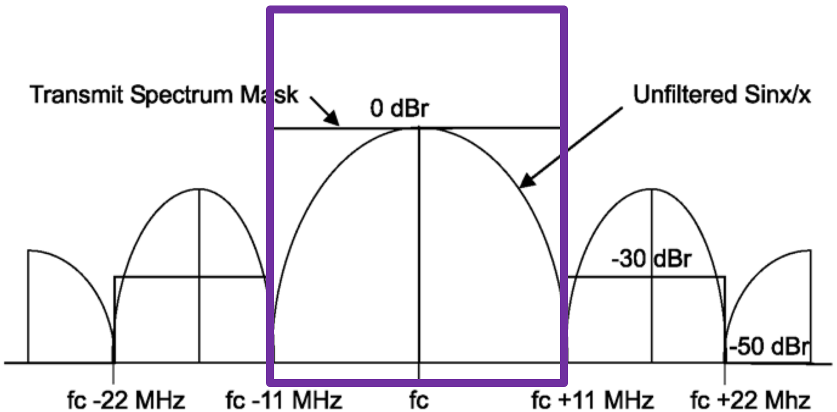
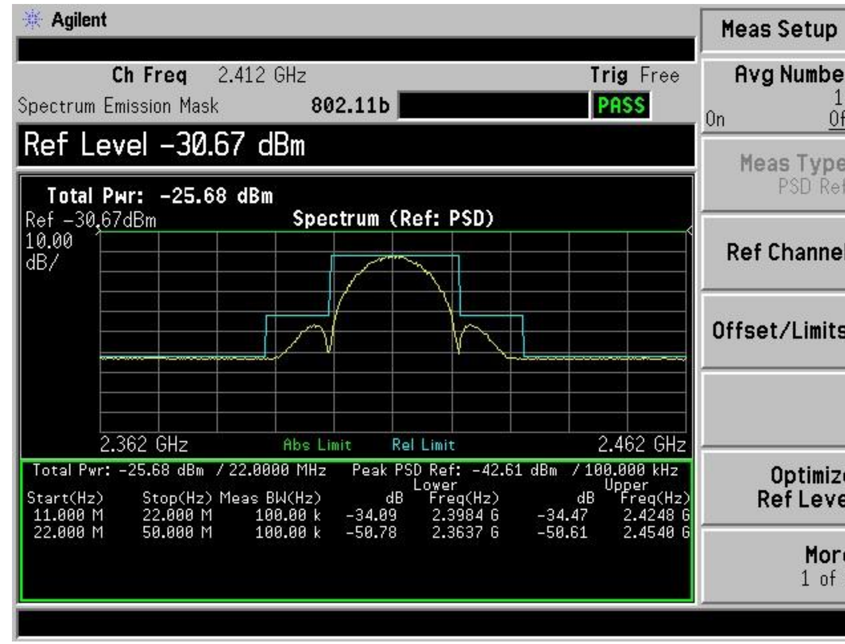


Figure 15-10—Transmit spectrum mask



Allowed out-of-band leakage

Transmit spectrum mask from IEEE 802.11-2016, 15.4.5.5 WiFi!
https://www.keysight.com/us/en/lib/resources/user-manuals/transmit-spectrum-mask-332766.html

Multiplication property

Key points

- Multiplication in time is convolution in frequency
- Use this fact to explain windowing

Multiplication property

- If $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$ $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$

- Then

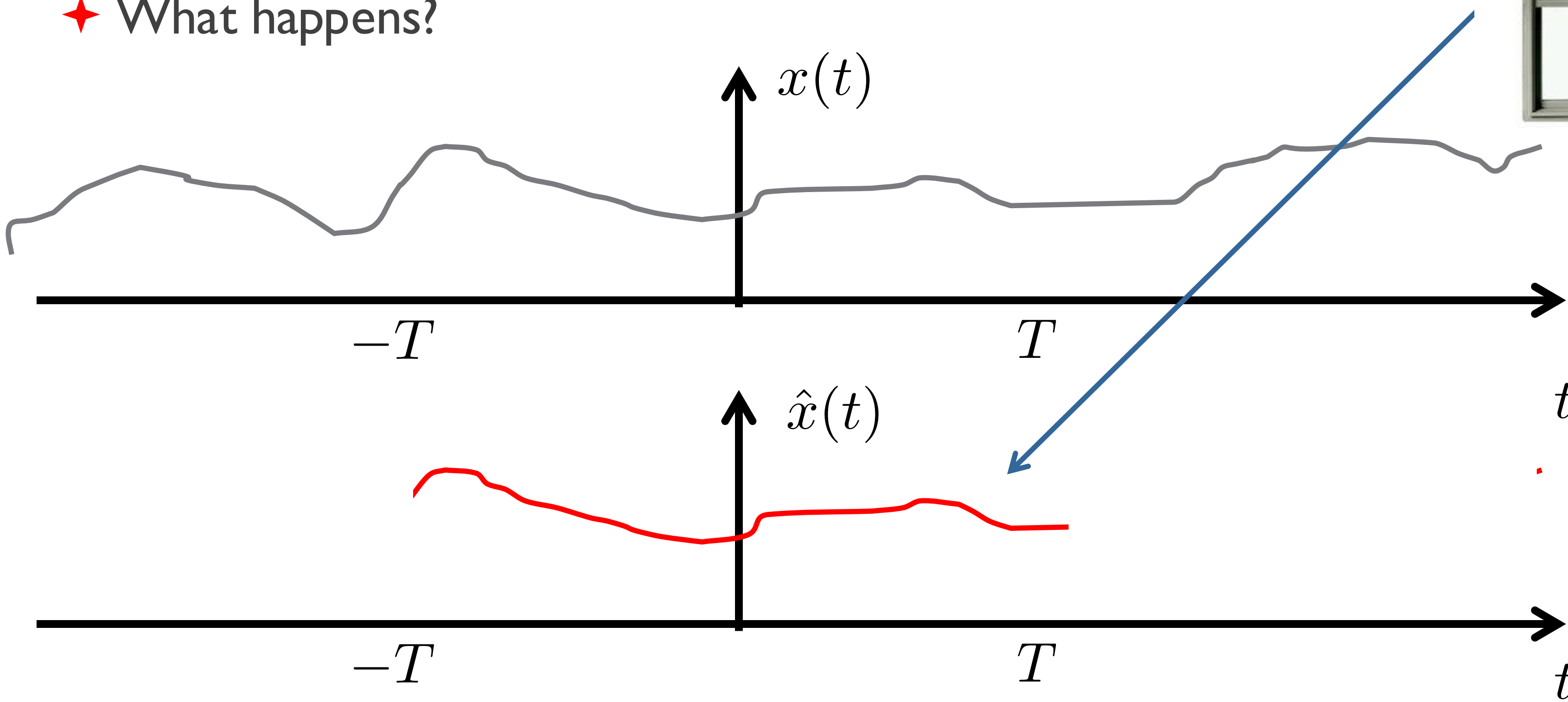
$$\boxed{y(t) = h(t)x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta}$$

Product in time is convolution in frequency

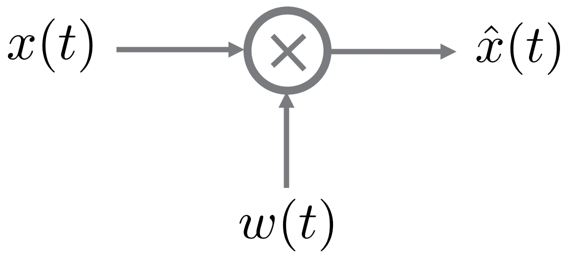
Implication of product property

- Suppose that you have a signal x(t)
  - But you only measure x(t) from –T….T
  - What happens?

Have only a window of the data



Windowing the spectrum



- The observed signal can be written as

$$\hat{x}(t) = \underbrace{\mathrm{rect}(t/(2T))}_{w(t)}x(t)$$

- In the frequency domain

$$\hat{X}(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)X(j(\omega-\theta))d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2T\mathrm{sinc}\left(\frac{2T\theta}{2\pi}\right)X(j(\omega-\theta))d\theta$$

Spectrum is filtered by the sinc function

Example – windowing a cosine

$$x(t) = \cos(\omega_c t) \longrightarrow \otimes \longrightarrow \hat{x}(t)$$

$$w(t) = \mathrm{rect}(t/(2T))$$

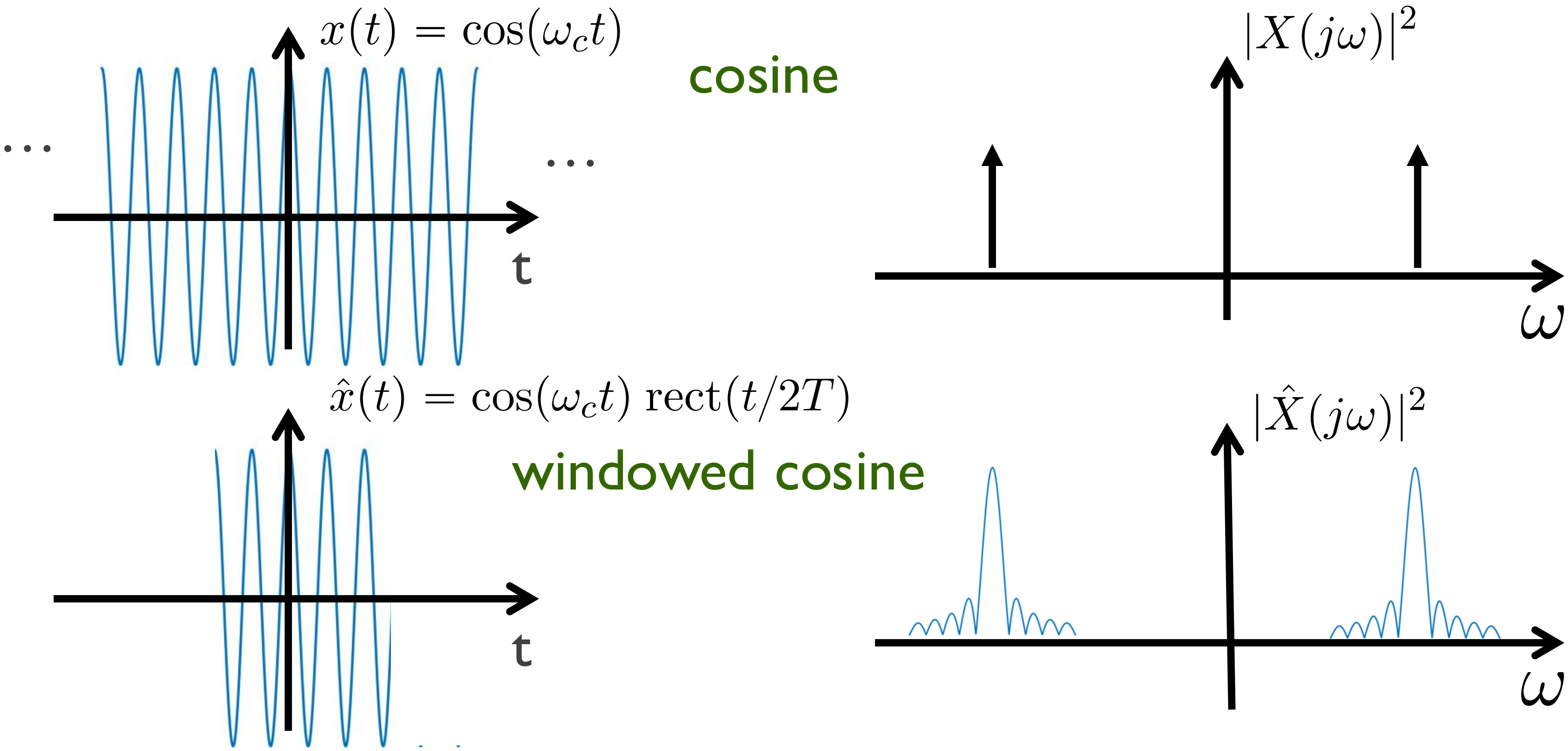
$$X(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

$$W(j\omega) = 2T\mathrm{sinc}\left(\frac{2T\omega}{2\pi}\right)$$

$$\hat{X}(j\omega) = \frac{T}{\pi}\mathrm{sinc}\left(\frac{2T(\omega - \omega_c)}{2\pi}\right) + \frac{T}{\pi}\mathrm{sinc}\left(\frac{2T(\omega + \omega_c)}{2\pi}\right)$$

Impulses get smeared due to windowing

Intuition on windowing



Impact of windowing on resolution

- Suppose that we window a sum of two cosines

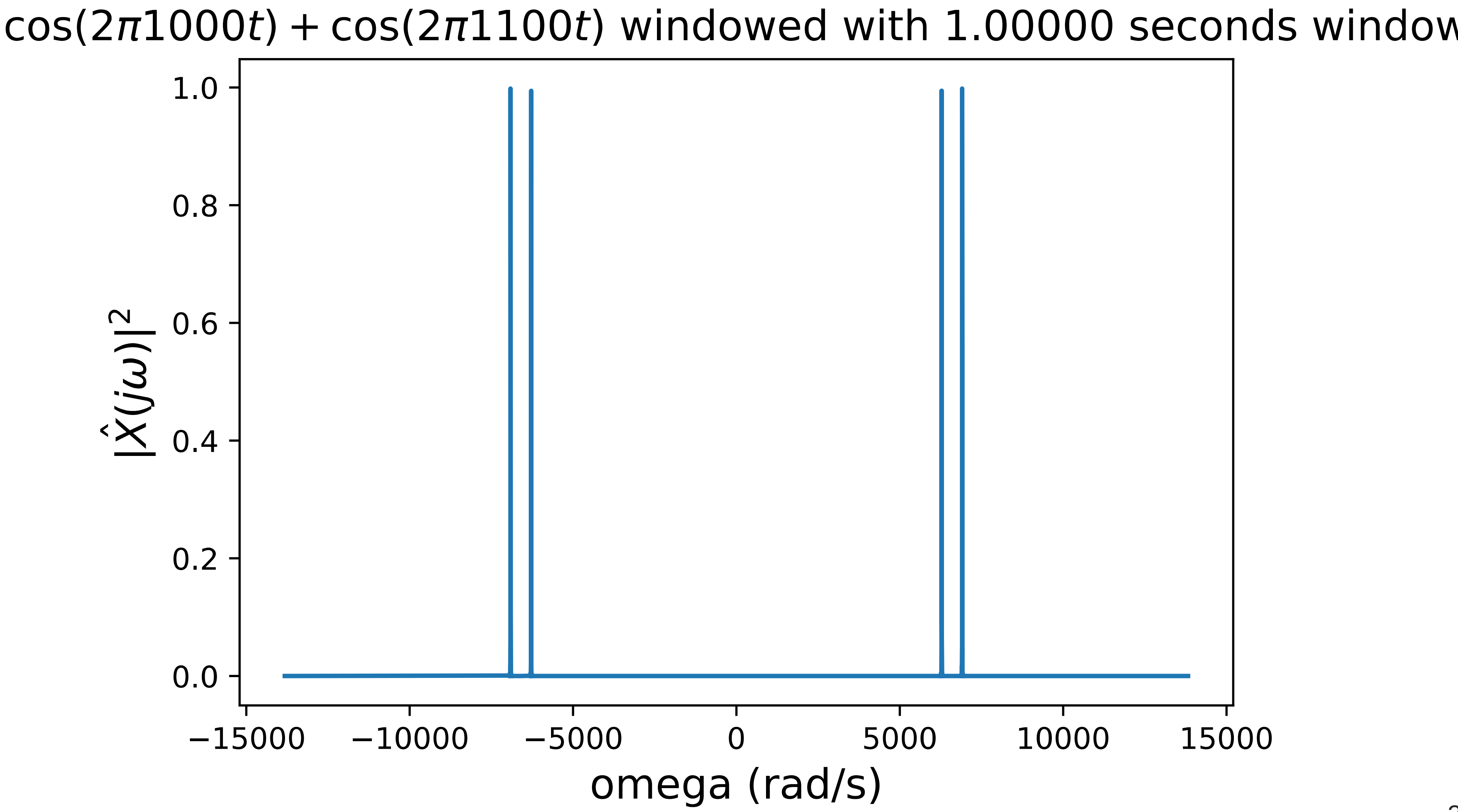
$$\cos(2\pi 1000t) + \frac{1}{2}\cos(2\pi 1100t) \longrightarrow \bigotimes \longrightarrow \hat{x}(t)$$

$$\mathrm{rect}(t/(2T))$$

1000 Hz or 6,280 rad/s

1100 Hz or 6,911 rad/s

What is the impact of the window size T on the ability to resolve the sinusoids?


$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 1.00000 seconds window
$|\hat{X}(j\omega)|^2$
1.0
0.8
0.6
0.4
0.2
0.0
−15000
−10000
−5000
0
5000
10000
15000
omega (rad/s)

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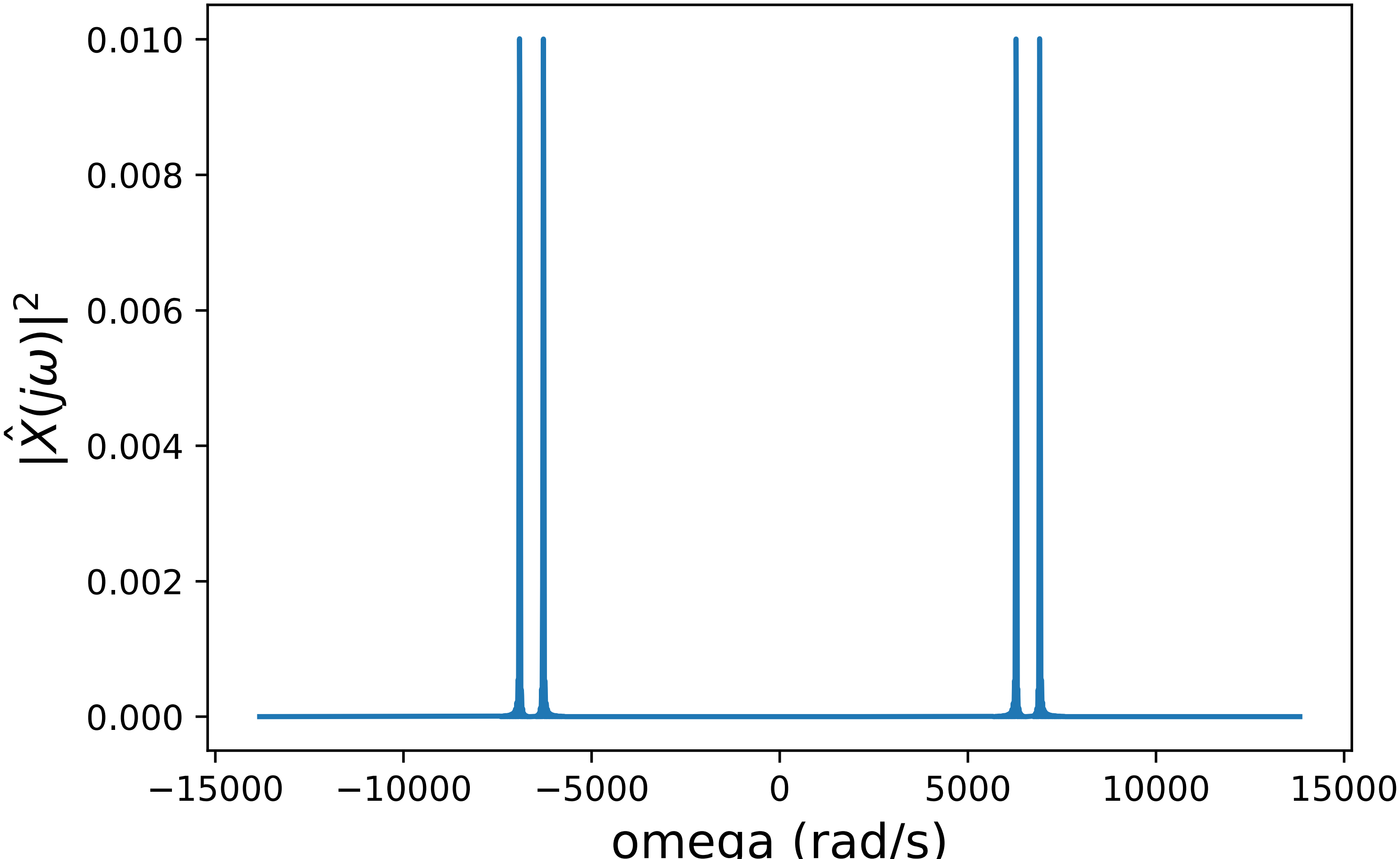
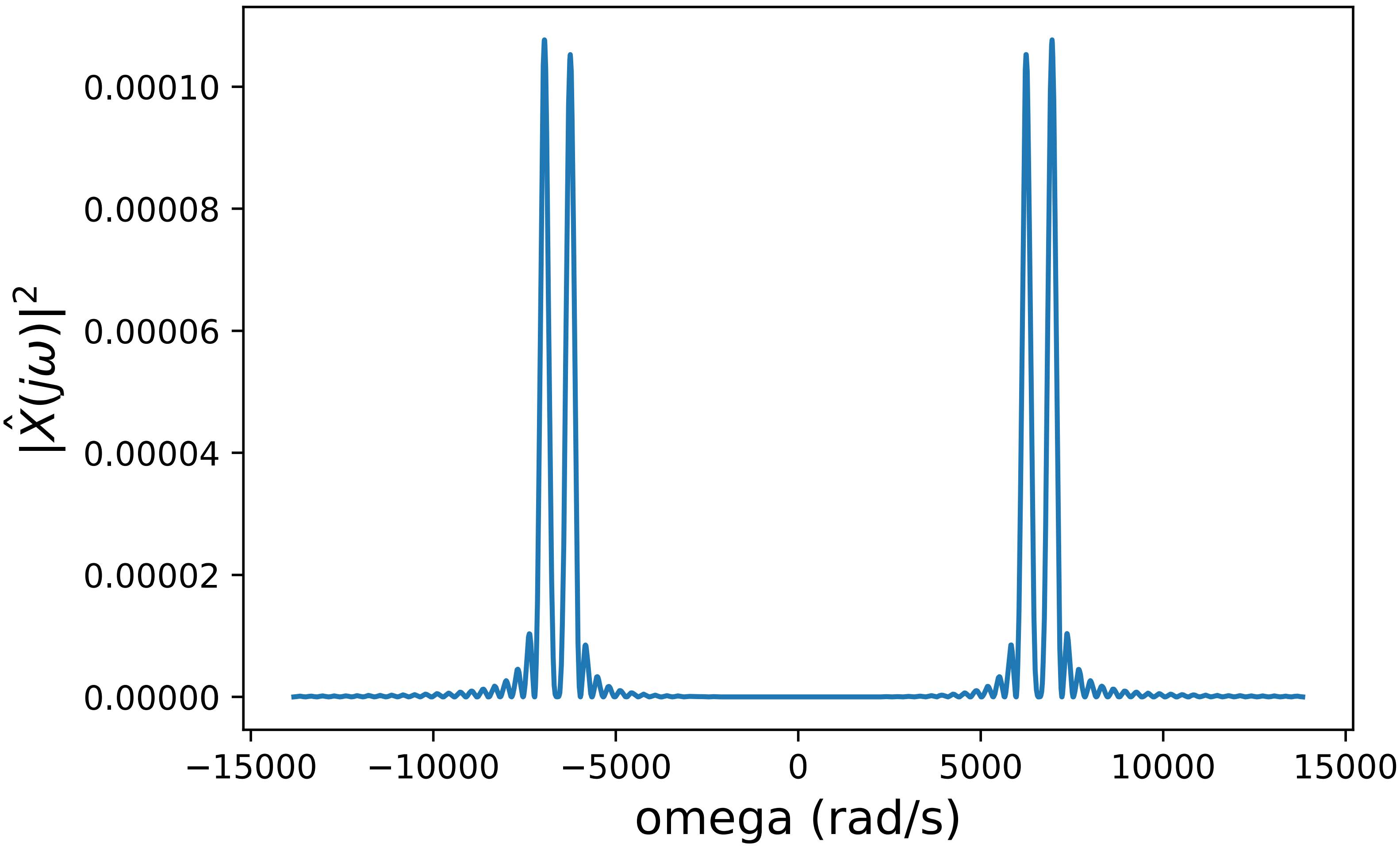
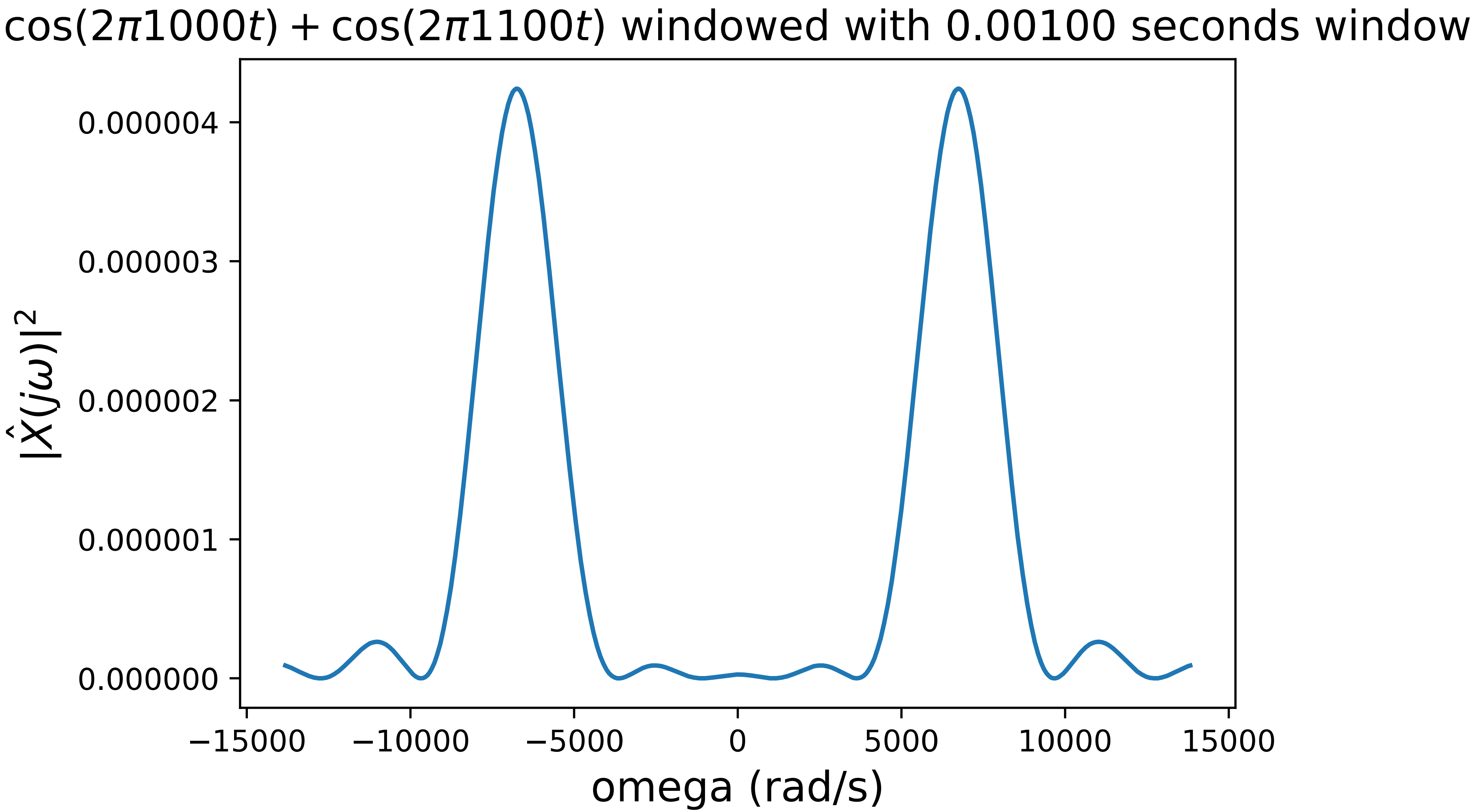
$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 0.10000 seconds window

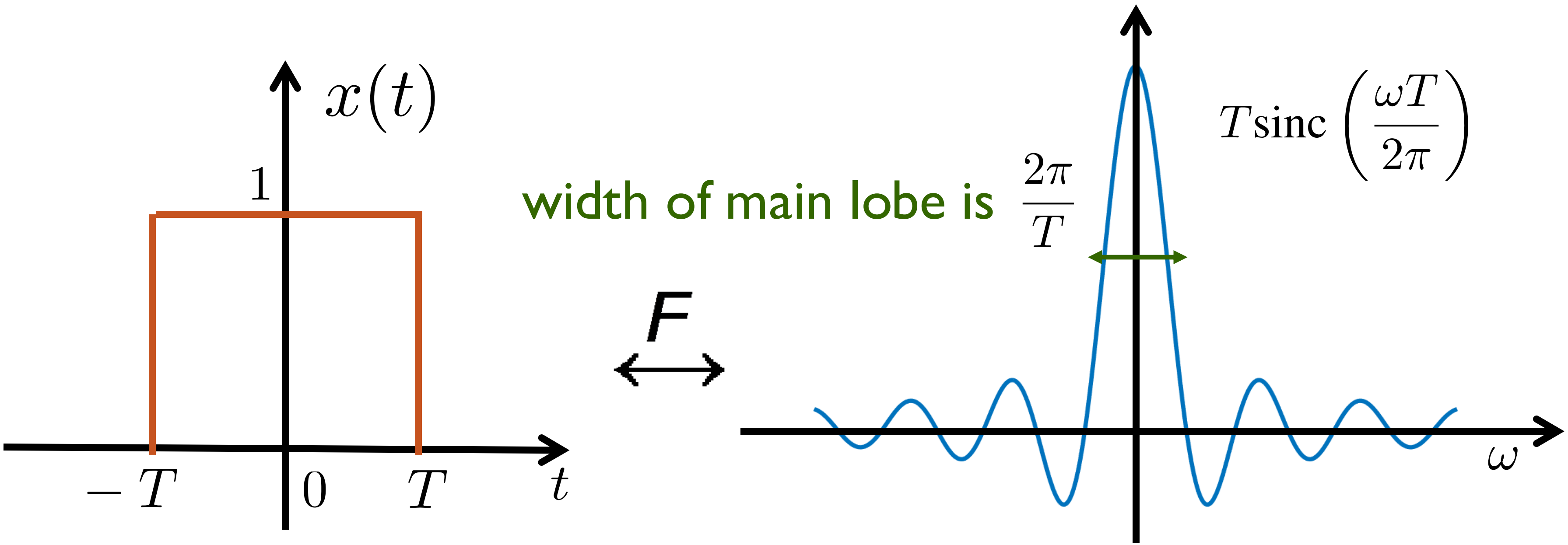


$\cos(2\pi 1000t) + \cos(2\pi 1100t)$ windowed with 0.01000 seconds window




cos(2π1000t) + cos(2π1100t) windowed with 0.00100 seconds window
$|\hat{X}(j\omega)|^2$
0.000004
0.000003
0.000002
0.000001
0.000000
−15000
−10000
−5000
0
5000
10000
15000
omega (rad/s)

How much time is needed to resolve these cosines?



Zeros at $\pm\,\pi/T, \pm\pi/T, \ldots$

One main lobe of separation

$$\frac{2\pi}{T} = |\omega_1 - \omega_2| \quad \Rightarrow \quad T = \frac{2\pi}{|\omega_1 - \omega_2|} = \frac{2\pi}{2\pi 100} = 0.01\,\text{s}$$

Summarizing the multiplication property

- Product between two signals in time becomes the (scaled) convolution of the Fourier transforms of those signals in the frequency domain

- Truncating a real signal for analysis, called windowing, leads to a distortion of the original signal's Fourier transform

- The ability to resolve different frequencies in a signal improves as the observation window grows longer

Bandwidth

Key points

- Finite duration signals have infinite bandwidth
- Different measures of bandwidth are used in practice

Isolation in time and frequency

- From the windowing theorem

sinc has infinite duration

$$x(t)\text{rect}(t/2T) \Rightarrow \frac{2T}{2\pi}\left[X(j\omega) * \text{sinc}(2T\omega/2\pi)\right]$$

Finite duration in time

Infinite duration in frequency

- From the convolution theorem

sinc has infinite duration

$$X(j\omega)\text{rect}(\omega/2B) \Rightarrow x(t) * 2B\text{sinc}(2Bt/2\pi)$$

Finite duration in frequency

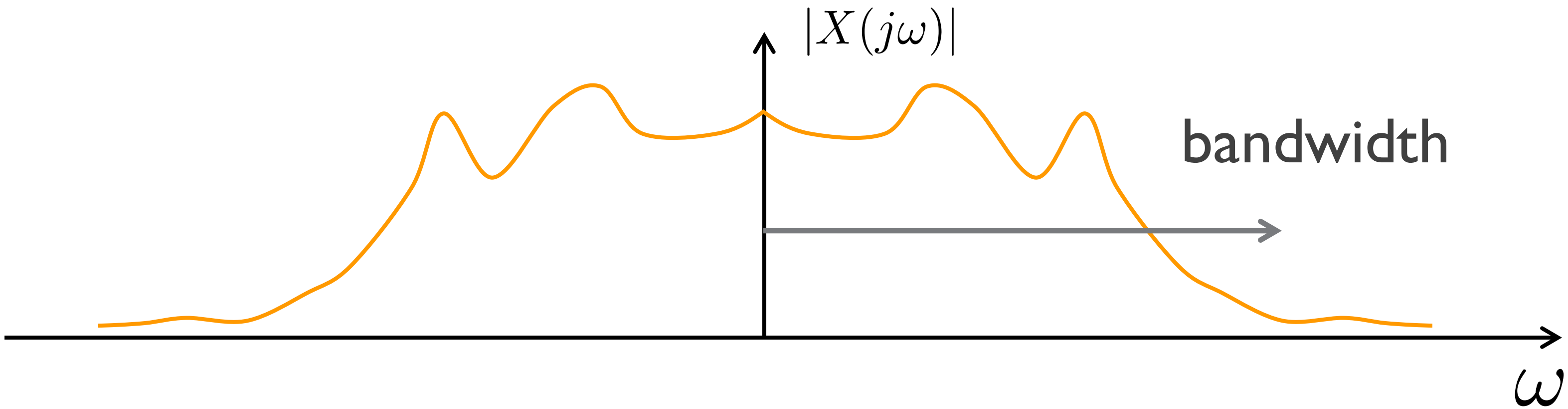
Infinite duration in time

Windowing and convolution have impact on the spectrum of practical signals (infinite) and the impulse response of an ideal low-pass filter (infinite)

Bandwidth of a practical signal

- If time duration is finite → bandwidth is infinite
  - For any practical signal, the absolute bandwidth is infinite

- Define a "bandwidth" to measure the extent of frequency content



Common definitions of bandwidth

- Fractional containment bandwidth
  - Bandwidth such that a fraction of energy is contained
  - Solve for ω_B such that

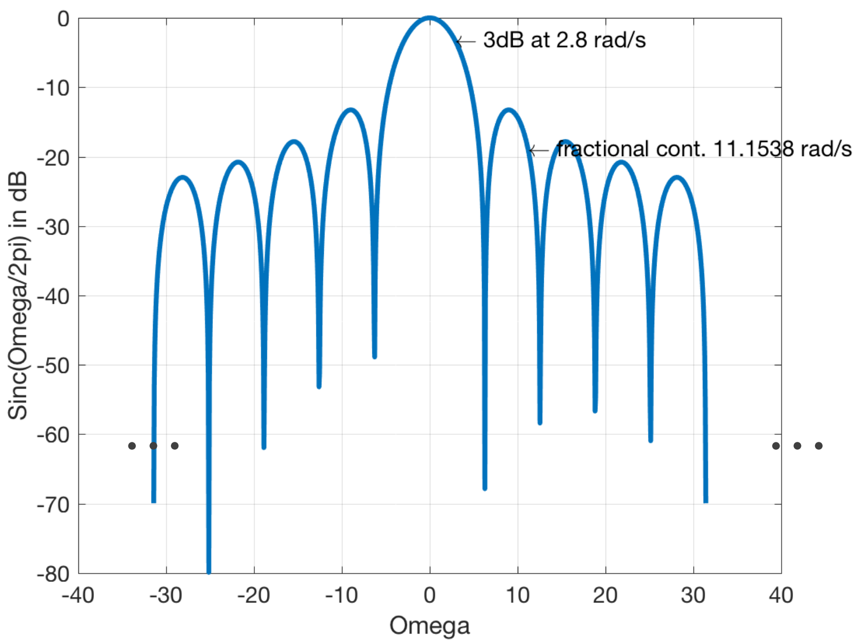
$$\int_{-\omega_B}^{\omega_B} |X(j\omega)|^2 d\omega \geqslant (1-\epsilon) \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- 3dB bandwidth (or half-power bandwidth)
  - Bandwidth where the signal achieves half the peak value
  - Makes the most sense with simple filters

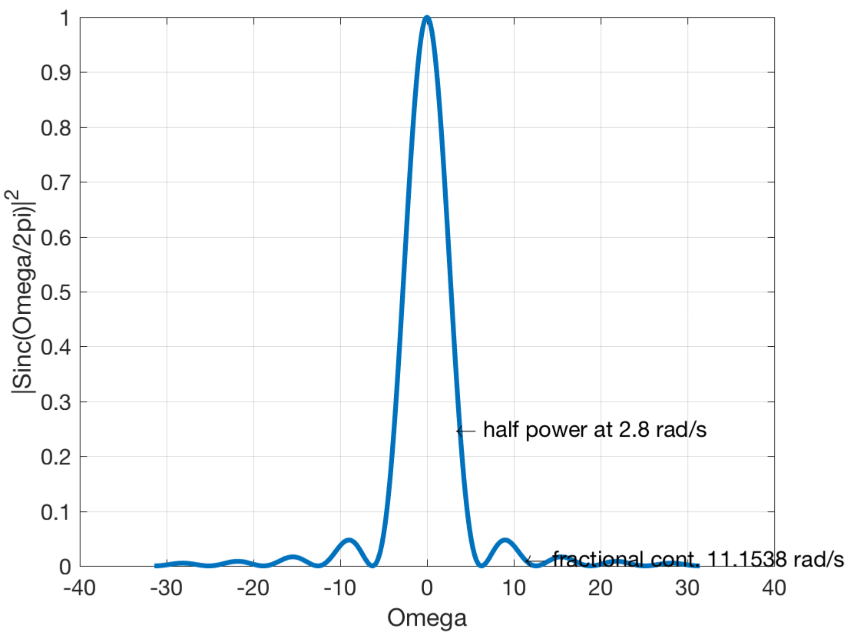
$$|X(j\omega_B)|^2 = \frac{1}{2} \max_{\omega} |X(j\omega)|^2$$

Examples of bandwidth

In dB



In linear scale (magnitude squared)



- Bandwidth of $\mathrm{sinc}(\omega/2\pi)$ with ½ power or 95% containment

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Example fractional containment calculation

- Consider the following facts about Gaussian signals (proof of these facts is beyond the scope of this course)

1) Gaussian is its own Fourier transform $$e^{-t^2} \overset{\mathcal{F}}{\longleftrightarrow} \sqrt{\pi}e^{-\frac{\omega^2}{4}}$$

2) Integral of tail is "known" $$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt = Q(x) = \frac{1}{2}\mathrm{erfc}(x/\sqrt{2})$$

3) Unit area $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-t^2/2}dt = 1$$

Gaussian distribution is a big part of probability and statistics

Example fractional containment calculation (cont.)

- Consider signal $x(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$

- Find an expression for the fractional containment bandwidth

$$\int_{-\omega_B}^{\omega_B} |X(j\omega)|^2 d\omega \geqslant (1-\epsilon) \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- Note that

$$|X(j\omega)| = e^{-\frac{\omega^2}{4}}$$

$$|X(j\omega)|^2 = e^{-\frac{\omega^2}{2}}$$

Example fractional containment calculation (cont.)

$$\int_{-\omega_B}^{\omega_B} e^{-\frac{\omega^2}{2}} d\omega = (1-\epsilon) \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega$$

- For the RHS note that

$$\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega = 1$$

- For the LHS

$$\begin{aligned}
\int_{-\omega_B}^{\omega_B} e^{-\frac{\omega^2}{2}} d\omega &= \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \int_{\omega_B}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - \int_{-\infty}^{-\omega_B} e^{-\frac{\omega^2}{2}} d\omega \\
&= \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} d\omega - 2\int_{\omega_B}^{\infty} e^{-\frac{\omega^2}{2}} d\omega \\
&= 1 - 2Q(\omega_B)
\end{aligned}$$

Example fractional containment calculation (cont.)

- Simplifying, we need to solve

$$1 - 2Q(\omega_B) = 1 - \epsilon$$

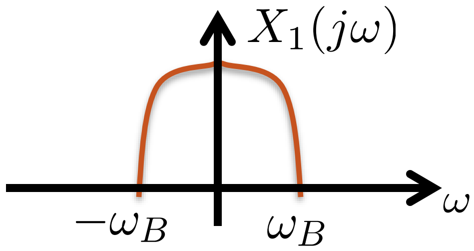
- Rearranging terms

$$Q(\omega_B) = \epsilon/2$$

$$\Rightarrow \quad \omega_B = Q^{-1}(\omega_B)$$

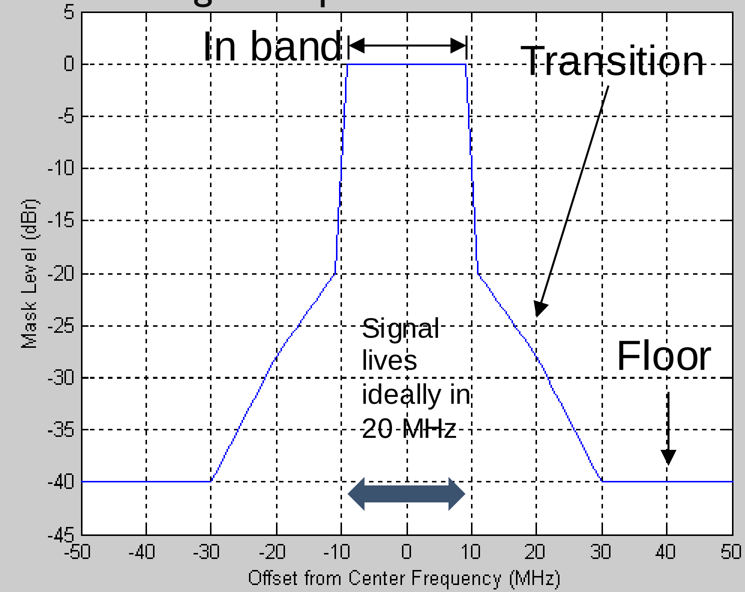
inverse Q function, available in some form (maybe with a different name) in Excel, Python, MATLAB, etc.

Spectrum masks



Since communication spectrum is not exactly band limited, the allowed profile is called a spectrum mask

IEEE 802.11g example



In Band: encompasses the desired signal
Transition: bounds adjacent channel interference
Floor: bounds other channel interference

Summarizing bandwidth

- Bandwidth is a measure of the extent of the non-zero frequency components present in a signal
- Practical signals always have infinite bandwidth due to be generated in a finite amount of time, a result of the windowing property
- There are different ways to define the bandwidth of a practical signal based on determining when the frequencies are sufficiently small