Discussion Notes 6

Let $f(t)$ be a periodic signal with period T, and frequency coefficient F_n.

$f(t) \longleftrightarrow F_n$

① time shift: $f'(t) = f(t-t_0) \longleftrightarrow F_n' = F_n e^{-jnw_0 t_0}$

② time reversal: $f'(t) = f(-t) \longleftrightarrow F_n' = F_{-n}$

That is why if $f(t)$ is even fanction $(f'(t) = f(-t) = f(t))$, we will have even frequency coeffient $(F_n' = F_{-n} = F_n)$. The same for odd functions.

③ time scaling: $f'(t) = f(at) \longleftrightarrow F_n' = F_n \quad T_{new} = \frac{T}{\alpha}$

④ derivative: $f'(t) = \frac{d}{dt} f(t) \longleftrightarrow F_n' = (jkw_0) F_n$

⑤ conjugate: $f'(t) = f^*(t) \longleftrightarrow F_n' = F_{-n}^*$

That is why if $f(t)$ is real functions $(f^*(t) = f(t))$, we will have conjugate symmetry $(F_n' = F_{-n}^* = F_n)$. If $f(t)$ is pure imaginary, we will have $f'(t) = f^*(t) = -f(t)$

$$F_n' = F_{-n}^* = -F_n$$

Combinig ② and ⑤ says that

(6a) If $f(t)$ is real and even, F_n is real and even.

(6b) If $f(t)$ is real and odd, F_n is imaginary and odd.

For $f_1(t)$ and $f_2(t)$ with period of T, and frequency response a_n and b_n:

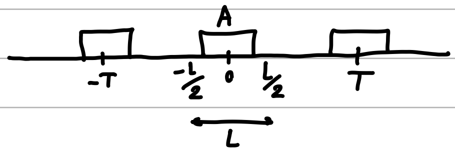
$f_1(t) \longleftrightarrow a_n \qquad f_2(t) \longleftrightarrow b_n$

① linearity: $f'(t) = A f_1(t) + B f_2(t) \longleftrightarrow F_n' = A a_n + B b_n$

② multiplication: $f'(t) = f_1(t) f_2(t) \longleftrightarrow F_n' = \sum_{l=-\infty}^{\infty} a_l b_{n-k} = a_n * b_n$

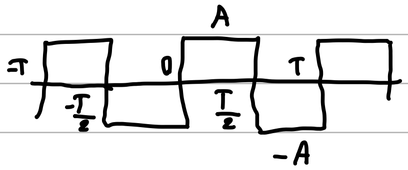
Some useful fourier Series pairs

pulse train



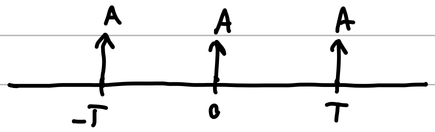
$$F_n = \begin{cases} n=0: & \dfrac{AL}{T} \\ n\neq 0: & \dfrac{A\sin\left(\frac{\pi L}{T}n\right)}{n\pi} = \dfrac{AL}{T}\,\mathrm{sinc}\left(\frac{L}{T}n\right) \end{cases}$$

Square Wave



$$F_n = \begin{cases} n=0 & : 0 \\ n=\text{even} & : 0 \\ n=\text{odd} & : \dfrac{2A}{jn\pi} \end{cases}$$

delta train



$$F_n = \frac{A}{T}$$

parseval theorem: $$P_f = \frac{1}{T}\int_T |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Example #1:

Consider two periodic signal with Fourier Series coefficient $x(t) \overset{FS}{\longleftrightarrow} a_k$ and $y(t) \overset{FS}{\longleftrightarrow} b_k$. As usual, assume that $x(t)$ has period T and thus fundamental frequency $w_o = \frac{2\pi}{T}$. Use the properties of fourier series to answer following questions:

a) Find the period and FS coefficients of $y(t) = x(3t-5)$

b) Express $y(t)$ in terms of $x(t)$ if $b_k = 1 + (a_k - a^*_{-k})$

c) Find the period and FS coefficients of $y(t) = x(1-t)\, x(t)$.

d) Compute $\frac{1}{T}\int_T \left| e^{jw_o t}\, x(3t-5)\right|^2 dt$

Answer:

a) Consider a new function $z(t) = x(t-5)$ with FS coef c_k:

$$z(t) = x(t-5) \overset{FS}{\longleftrightarrow} c_k = a_k e^{-jkw_o 5}$$

Then using $z(t)$, we can write $y(t) = z(3t) = x(3t-5)$.

$$y(t) = z(3t) \overset{FS}{\longleftrightarrow} b_k = c_k = a_k{}^{-jk5w_o}$$

with the fundamental frequency of $w_{y_0} = 3w_o$. The period $T_y = \frac{T}{3}$.

b) We know that

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) \overset{FS}{\longleftrightarrow} \frac{1}{T}$$

$$x^*(t) \overset{FS}{\longleftrightarrow} a^*_{-k}$$

using the linear property of fourier transform, if

$$b_k = 1 + a_k - a^*_{-k}$$

then

$$y(t) = T_o \sum_{k=-\infty}^{\infty} \delta(t-kT_o) + x(t) - x^*(t) \quad \text{with time period} = T$$

c) $x(t) \xleftrightarrow{FS} a_k$

like part a, we set $z(t) = x(t+1)$ with fourier with FS coefficient c_k.

$$z(t) = x(t+1) \longrightarrow c_k = e^{-jkw_0(-1)} a_k = e^{jkw_0} a_k$$

So then $x(1-t) = z(-t)$, then

$$x(1-t) = z(-t) \longrightarrow b_k = c_{-k} = e^{j(-k)w_0} a_{-k}$$

$$= e^{-jkw_0} a_{-k}$$

Now $y(t) = x(t)\, x(1-t)$, then

$$y(t) = x(t)\, x(1-t) \xleftrightarrow{FS} \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$$

$$= \sum_{l=-\infty}^{\infty} a_k\, a_{-(k-l)}\, e^{-j(k-l)w_0}$$

$$= \sum_{l=-\infty}^{\infty} a_k\, a_{l-k}\, e^{-jw_0(k-l)}$$

with period $= T$

d) We know $x(t) \xleftrightarrow{FS} a_k$

$$x(3t-5) \xleftrightarrow{FS} e^{-5jw_0 k} a_k$$

With parsevel theorem,

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

As $|e^{jT}| = 1$, we have

$$\frac{1}{T}\int_T |e^{jw_0 t} x(3t-5)|^2 dt = \frac{1}{T}\int_T |x(3t-5)|^2 dt = \sum_{k=-\infty}^{\infty} |e^{-5jw_0 k} a_k|^2$$

$$= \sum_{k=-\infty}^{\infty} |e^{-5jw_0 k}|^2\, |a_k|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2$$