Discussion Session 7

Review on Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jw_n t} = \sum_{n=-\infty}^{\infty} F_n e^{jnw_0 t} \qquad w_0 = \frac{2\pi}{T} : \text{fundamental frequency}$$

$$F_n = \frac{1}{T}\int_T f(t)\, e^{-jw_n t}\, dt \qquad F_n : \text{Fourier coefficient}$$

Example : $T = 2\pi \quad f(x) = e^{ax}$ for $x \in (-\pi, \pi)$

Answer: $w_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \longrightarrow w_n = n w_0$

$$F_n = \frac{1}{T}\int_T f(t)\, e^{-jnw_0 t} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-jnt}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-jn)t}\, dt = \frac{1}{2\pi(a-jn)}\Big[e^{(a-jn)\pi}$$

$$- e^{(a-jn)-\pi}\Big] = \frac{1}{2\pi(a-jn)}\Big[e^{a\pi}\underbrace{e^{-jn\pi}}_{(-1)^n} - e^{-a\pi}\underbrace{e^{jn\pi}}_{(-1)^n}\Big] = \frac{(-1)^n}{2\pi(a-jn)}\left(e^{a\pi} - e^{-a\pi}\right)$$

Example : $T = 2\pi$, $f(t) = 1 + \sin^2(t)$ for $t \in (-\pi, \pi)$

Answer: $f(t) = 1 + \sin^2(t) = 1 + \frac{1}{2}\left(1 - \cos(2t)\right) = \frac{3}{2} - \frac{1}{2}\cos(2t) = \frac{3}{2} - \frac{1}{2}\Big[\frac{1}{2}e^{j2t}$

$+ \frac{1}{2}e^{-j2t}\Big] = \frac{3}{2} - \frac{1}{4}e^{j2t} - \frac{1}{4}e^{-j2t}$, $w_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$, $w_n = n w_0 = n \times 1 = n$

$$F_n = \frac{1}{T}\int_T f(t)\, e^{-jnw_0 t}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\frac{3}{2} - \frac{1}{4}e^{j2t} - \frac{1}{4}e^{-j2t}\Big] e^{-jnt}\, dt$$

Solving this will give you $F_0 = \frac{3}{2} \quad F_2 = F_{-2} = \frac{-1}{4}$

Alternetively: $f(t) = \frac{3}{2} - \frac{1}{4}e^{j2t} - \frac{1}{4}e^{-j2t} = \sum_{n=-\infty}^{\infty} F_n e^{jnw_0 t} = \sum_{n=-\infty}^{\infty} F_n e^{jnt}$

By comparing the right and left side of equation, you can find $F_0 = \frac{3}{2}$, $F_2 = F_{-2} = \frac{-1}{4}$

Fourier Coefficient and signal transformation

$$f(t) \xleftrightarrow{FS} F_n \qquad \text{period} = T$$

$$f(t-t_0) \xleftrightarrow{FS} F_n\, e^{-jnw_0 t_0}$$

$$f(-t) \xleftrightarrow{FS} F_{-n}$$

$$f(at) \longleftrightarrow F_n \qquad \text{period} = \frac{T}{a}$$

$$\frac{d}{dt} f(t) \longleftrightarrow (jkw_0)\, F_n$$

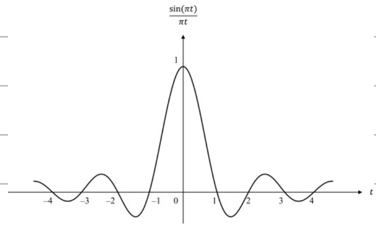
$$f^*(t) \longleftrightarrow F^*_{-n}$$

$$f_1(t) \xleftrightarrow{FS} a_n \qquad f_2(t) \xleftrightarrow{FS} b_k$$

$$A f_1(t) + B f_2(t) \xleftrightarrow{FS} A a_k + B b_k$$

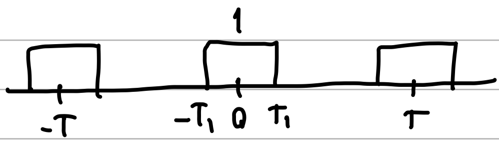
$$f_1(t)\, f_2(t) \xleftrightarrow{FS} a_k * b_k$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

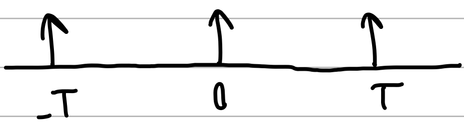


Famous fourier series pairs:

$$\text{pulse train} = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\frac{t-kT}{2T_1}\right) \xleftrightarrow{FS} a_k = \frac{2T_1}{T} \operatorname{sinc}\left(\frac{2T_1}{T} k\right)$$



$$\text{Delta train} = \sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{FS} a_k = \frac{1}{T}$$



$$\cos(\ell w_0 t) = \cos\left(\ell \frac{2\pi}{T} t\right) \xleftrightarrow{FS} a_k = \begin{cases} 0 & k \neq \ell \\ \frac{1}{2} & k = \pm \ell \end{cases}$$

$$\sin(\ell w_0 t) = \sin\left(\ell \frac{2\pi}{T} t\right) \xleftrightarrow{FS} \begin{cases} 0 & k \neq \ell \\ \frac{1}{2j} & k = \ell \\ \frac{-1}{2j} & k = -\ell \end{cases}$$

Example :

What is $x(t)$ if its coefficients are $a_k = e^{-jk\frac{\pi}{4}} \frac{\sin(k\frac{\pi}{3})}{2k}$ with $T=12$.

Answer: $a_k = e^{-jk\frac{\pi}{4}} \frac{\sin(k\frac{\pi}{3})}{2k} = e^{-jk\frac{\pi}{4}} \frac{\sin(k\frac{\pi}{3})}{k\frac{\pi}{3}} \times \frac{\pi}{3\times 2} = e^{-jk\frac{\pi}{4}} \frac{\pi}{6} \text{sinc}(\frac{k}{3})$

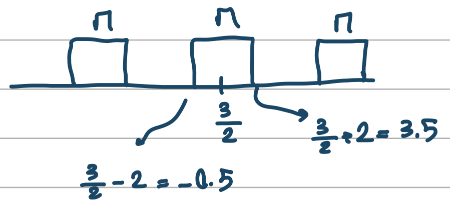
$= e^{-jk\frac{\pi}{4}} \frac{\pi}{6} \left[3\times\frac{1}{3} \text{sinc}(\frac{k}{3})\right] = e^{-jk\frac{\pi}{4}} \frac{\pi}{2} \left\{\frac{1}{3} \text{sinc}(\frac{k}{3})\right\}$

As $\sum_{l=-\infty}^{\infty} \text{rect}\left(\frac{t-lT}{2T_1}\right) \xleftrightarrow{FS} \frac{2T_1}{T} \text{sinc}\left(\frac{2T_1}{T}k\right)$, so $\frac{1}{3}\text{sinc}\left(\frac{k}{3}\right)$ is FS

of train of pulse with $\frac{2T_1}{T} = \frac{2T_1}{12} = \frac{1}{3} \longrightarrow T_1 = 2$

$T = 12 \longrightarrow w_0 = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}$, $e^{-jk\frac{\pi}{4}} = e^{-jk\underbrace{\frac{\pi}{6}}_{w_0}\times t_0}$ $\quad t_0 = \frac{3}{2}$

So the output is the train pulse, shifted by $\frac{3}{2}$ and multiplied by $\frac{\pi}{2}$.



$$= \frac{\pi}{2} \sum_{l=-\infty}^{\infty} \text{rect}\left(\frac{t-\frac{3}{2}-12l}{4}\right)$$

Fourier Transform:

For a continuous-time function $f(t)$, the fourier transform is defined as:

$$F(jw) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

The inverse fourier transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, e^{jwt}\, dt$$

Example: $x(t) = e^{-a|t|}$ for $a>0$. Find FT.

$X(jw) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt = \int_{-\infty}^{0} e^{-a(-t)} e^{-jwt}\, dt + \int_{0}^{\infty} e^{-at} e^{-jwt}\, dt =$

$= \int_{-\infty}^{0} e^{(a-jw)t}\, dt + \int_{0}^{\infty} e^{-(a+jw)t}\, dt = \frac{1}{a-jw}[1-0] + \frac{1}{-(a+jw)}[0-1] =$

$$= \frac{1}{a-jw} + \frac{1}{a+jw} = \frac{a+jw+a-jw}{(a-jw)(a+jw)} = \frac{2a}{a^2+w^2}$$

Example: $x(t) = t e^{-2t} u(t)$. Find FT.

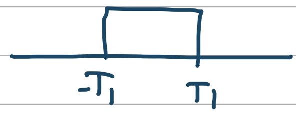
$$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt = \int_{0}^{\infty} t\, e^{-2t}\, e^{-jwt}\, dt = \int_{0}^{\infty} t\, e^{-(2+jw)t}\, dt$$

$v=t,\ dv=dt,\ du = e^{-(2+jw)t}\, dt \longrightarrow u = \frac{1}{-(2+jw)} e^{-(2+jw)t}$ $\quad \int_a^b v\, du = vu\Big|_a^b - \int_a^b u\, dv$

$$\longrightarrow X(jw) = \left[t \frac{1}{-(2+jw)} e^{-(2+jw)t} \right]_0^{\infty} - \int_0^{\infty} 1\, \frac{1}{-(2+jw)} e^{-(2+jw)t}\, dt$$

$$= -\frac{1}{(2+jw)^2} \left[e^{-(2+jw)t} \right]_{t=0}^{t=\infty} = \frac{1}{(2+jw)^2}$$

Example: $x(t) = \text{rect}\left(\frac{t}{2T_1}\right)$



Answer: $X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt = \int_{-T_1}^{T_1} 1\, e^{-jwt} = \frac{1}{-jw} e^{-jwt}\Big|_{t=-T_1}^{t=T_1}$

$$= \frac{1}{-jw}\left[e^{-jwT_1} - e^{jwT_1} \right] = \frac{1}{-jw}\left[-2j \sin(wT_1) \right] = \frac{2\sin(wT_1)}{w}$$

$$= 2T_1 \frac{\sin(wT_1)}{wT_1} = 2T_1 \operatorname{sinc}\left(\frac{wT_1}{\pi}\right)$$