ES 249 Problem Set 2

Andrew T. Sullivan

8 March 2019

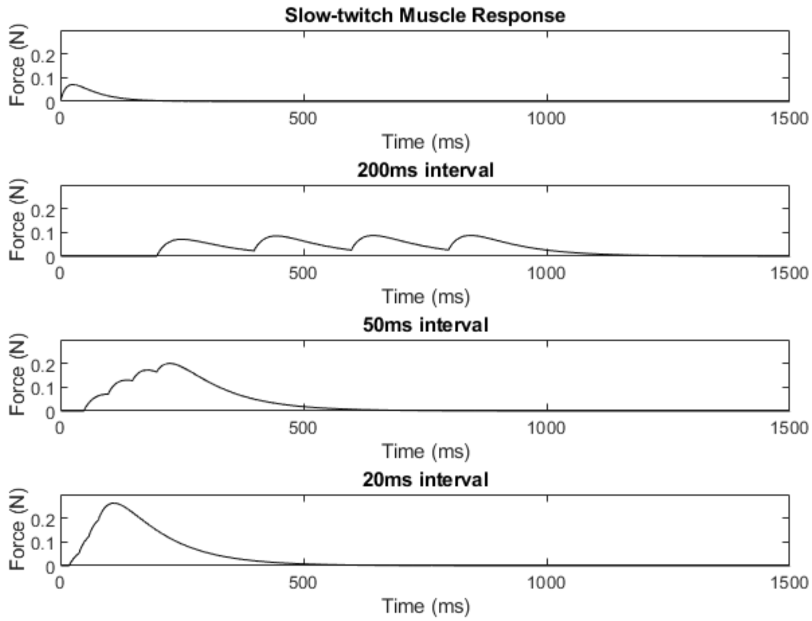

Figure 1: a) shows the impulse-response function of a slow twitch muscle as a function of time. The following three subfigures show the cumulative response to four impulses separated by b) 200 ms, c) 50 ms, and d) 20 ms.

Question 1 Figure 1 contains four panels with different muscle activation patterns. In **a)**, the unitary (twitch) response of a slow-twitch muscle is plotted as the function $h(t) = K(e^{-t/\tau_1} - e^{-t/\tau_2})$. Here, we use values of 50 ms, 15 ms, and 0.17 N for τ_1, τ_2, and K, respectively. The following three subpanels show the summation pattern of four twitch responses for impulses separated by 200 ms (**b)**), 50 ms (**c)**), and 20 ms (**d)**) obtained by convolving the unitary response with an impulse train constructed with the above constraints. It is clear from these plots that impulses separated by shorter time periods converge to a linear summation wherein the unitary response is scaled by the number of impulses. At longer time periods, the summation is smaller as the preceding response peaks will have already passed.

Question 2

a) As mentioned in the question, use of the calculus of variations to find an optimal function that minimizes the cumulative squared jerk, the third time derivative of position, reveals a fifth order polynomial for position:

$$x(t) = c_5t^5 + c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$$

The basis of this method will be discussed later in order to solve for minimal cumulative squared acceleration and snap, but is based on constructing an arbitrary perturbation of $x(t) \to x(t) + \varepsilon\eta(t)$, and differentiating the cost function, $H(x(t))$ with respect to the perturbation parameter ε. Evaluating this derivative for the cost function $H(x(t)+\varepsilon\eta(t)) = \int(\dddot{x}(t)+\varepsilon\dddot{\eta}(t))^2dt$ at $\varepsilon = 0$, using integration by parts three times so the integral is in terms of $\eta(t)$ rather than its derivatives (with boundary conditions such that $\eta(t)$ and all its derivatives are zero at the endpoints (t_0 and t_f)), setting this integral equal to zero, and using the fundamental theorem of the calculus of variations, i.e. that if $\int_a^b f(x)h(x)dx = 0$ for all continuous and bounded functions $h(x)$, then $f(x)$ is zero over the interval from a to b. Since $\eta(t)$ can be any arbitrary function, this implies that $x^{(VI)}(t) = 0$, and so $x(t)$ is a fifth-order polynomial. Using proper initial and boundary conditions, we can solve for the six coefficients in the polynomial.

Here, we use the boundary conditions $x(0) = v(0) = a(0) = v(t_f) = a(t_f) = 0$ and $x(t_f) = x_f$ for some smooth motion from an initial position of 0 at time zero, and a final position x_f after time t_f. We require the object to start and end at rest, hence the bounds on velocity, and remain at rest after, forcing acceleration to zero as well. In order to solve for the coefficients, we use these six boundary conditions to construct a matrix equation and invert this equation to isolate the coefficients. The velocity is given as the derivative of position, and acceleration is the derivative of velocity, so:

$$v(t) = 5c_5t^4 + 4c_4t^3 + 3c_3t^2 + 2c_2t + c_1$$
$$a(t) = 20c_5t^3 + 12c_4t^2 + 6c_3t + 2c_2$$

Plugging in the boundary conditions, we can factor out the coefficients and place them in a vector such that $\mathbf{T}\vec{C} = \vec{X}$, or:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ t_f^5 & t_f^4 & t_f^3 & t_f^2 & t_f & 1 \\ 5t_f^4 & 4t_f^3 & 3t_f^2 & 2t_f & 1 & 0 \\ 20t_f^3 & 12t_f^2 & 6t_f & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_f \\ 0 \\ 0 \end{bmatrix}$$

Clearly, the first three rows of the matrix imply that the three lower-index coefficients are zero, i.e. $c_0 = c_1 = c_2 = 0$, so we can reduce the 6x6 matrix to a 3x3 matrix:

$$\begin{bmatrix} t_f^5 & t_f^4 & t_f^3 \\ 5t_f^4 & 4t_f^3 & 3t_f^2 \\ 20t_f^3 & 12t_f^2 & 6t_f \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_f \\ 0 \\ 0 \end{bmatrix}$$

We can obtain the coefficients by inverting this 3x3 matrix and multiplying by the vector on the right-hand side. To invert a 3x3 matrix, we first construct its matrix of minors, obtain its cofactor matrix by negating all values in even positions on odd rows and odd positions in even rows, transpose the cofactor matrix to obtain the adjoint, and divide this by the determinant. The matrix of minors here is obtained by finding the 2x2 determinant of the values that do not share a row or column with the position of interest. For example, the first entry at row 1, column 1, has the value $24t_f^4 - 36t_f^4 = -12t_f^4$. The matrix of minors is then:

$$\begin{bmatrix} -12t_f^4 & -30t_f^5 & -20t_f^6 \\ -6t_f^5 & -14t_f^6 & -8t_f^7 \\ -t_f^6 & -2t_f^7 & -t_f^8 \end{bmatrix}$$

The cofactor matrix is obtained by negating positions 2, 4, 6, and 8:

$$\begin{bmatrix} -12t_f^4 & 30t_f^5 & -20t_f^6 \\ 6t_f^5 & -14t_f^6 & 8t_f^7 \\ -t_f^6 & 2t_f^7 & -t_f^8 \end{bmatrix}$$

The adjoint matrix is its transpose:

$$\begin{bmatrix} -12t_f^4 & 6t_f^5 & -t_f^6 \\ 30t_f^5 & -14t_f^6 & 2t_f^7 \\ -20t_f^6 & 8t_f^7 & -t_f^8 \end{bmatrix}$$

Finally, the determinant of the initial matrix is $\det T = t_f^5(24t_f^4 - 36t_f^4) - t_f^4(30t_f^5 - 60t_f^5) + t_f^3(60t_f^6 - 80t_f^6) = -2t_f^9$, so the inverse of the matrix is:

$$\begin{bmatrix} 6t_f^{-5} & -3t_f^{-4} & 0.5t_f^{-3} \\ -15t_f^{-4} & 7t_f^{-3} & -t_f^{-2} \\ 10t_f^{-3} & -4t_f^{-2} & 0.5t_f^{-1} \end{bmatrix}$$

Thus, our complete polynomial is:

$$x(t) = 6t_f^{-5}x_ft^5 - 15t_f^{-4}x_ft^4 + 10t_f^{-3}x_ft^3$$

b) Figure 2 shows the position, velocity, and acceleration from time 0 to the final time, here set to be 0.5 s. The final distance is set to 10.

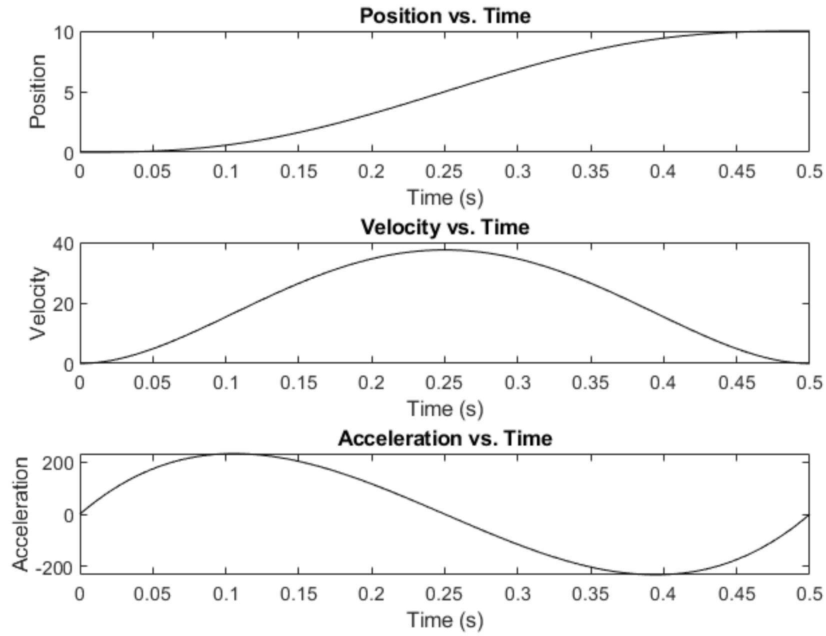

Figure 2: The position, velocity, and acceleration profiles for the function which minimizes the cumulative squared jerk over the interval from 0 to 0.5 s moving a distance of 10.

c) We can modify the above scenario for a case where we desire an "out and back" motion that returns to the initial starting point at time t_f. This can be best accomplished by decomposing the problem into two segments that are antisymmetric. The first segment is an outward motion from 0 to x_f from time 0 to $\frac{t_f}{2}$. The second segment follows the reverse trajectory and can be constructed by time-reversing the first segment. Clearly, the initial setup of this problem is the same as the case above with t_f set to $\frac{t_f}{2}$. Therefore, we can use the same functional form of the position, since our goal is still to minimize the cumulative squared jerk, but our boundary conditions will change. Thus, we have:

$$x(t) = c_5t^5 + c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$$

with the constraints that $t_0 = 0, x(0) = v(0) = a(0) = 0$, $x(\frac{t_f}{2}) = x_f$, and $v(\frac{t_f}{2}) = 0$. For simplicity, we also set t_f in the above solutions to be equal to $\frac{t_f}{2}$, and then replace this later. Due to the antisymmetry, we still require the velocity at the midpoint to be zero, as a non-zero velocity would imply that the arm travelled too far beforehand (hence a negative velocity) or will travel too far afterwards (a positive velocity), and is thus not the ideal motion, especially with the time-reversal. The boundary condition on acceleration is less clear, however. While a positive acceleration is unlikely to give a more optimal solution, since this would drive the velocity to increase in the positive direction, a negative acceleration could potentially be more optimal than requiring the acceleration to be zero at the midpoint. Thus, our main task here is to minimize the cumulative squared jerk by finding the optimal acceleration at the midpoint. As before, we have the same expressions for our velocity and acceleration, and the first three equations still imply that the lower-index coefficients are zero due to the initial conditions. We then end up with an almost identical system of three equations:

$$\begin{bmatrix} t_f^5 & t_f^4 & t_f^3 \\ 5t_f^4 & 4t_f^3 & 3t_f^2 \\ 20t_f^3 & 12t_f^2 & 6t_f \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_f \\ 0 \\ a_{min} \end{bmatrix}$$

As our matrix is the same, we can use the same inverse we found earlier, so the coefficients we obtain are from the product:

$$\begin{bmatrix} 6t_f^{-5} & -3t_f^{-4} & 0.5t_f^{-3} \\ -15t_f^{-4} & 7t_f^{-3} & -t_f^{-2} \\ 10t_f^{-3} & -4t_f^{-2} & 0.5t_f^{-1} \end{bmatrix} \begin{bmatrix} x_f \\ 0 \\ a_{min} \end{bmatrix} = \begin{bmatrix} c_5 \\ c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6t_f^{-5}x_f + 0.5t_f^{-3}a_{min} \\ -15t_f^{-4}x_f - t_f^{-2}a_{min} \\ 10t_f^{-3}x_f + 0.5t_f^{-1}a_{min} \end{bmatrix}$$

We can then represent our position function as:

$$x(t) = (6t_f^{-5}x_f + 0.5t_f^{-3}a_{min})t^5 + (-15t_f^{-4}x_f - t_f^{-2}a_{min})t^4 + (10t_f^{-3}x_f + 0.5t_f^{-1}a_{min})t^3$$
$$x(t) = a_{min}(0.5t_f^{-3}t^5 - t_f^{-2}t^4 + 0.5t_f^{-1}t^3) + x_f(6t_f^{-5}t^5 - 15t_f^{-4}t^4 + 10t_f^{-3}t^3)$$

In order to minimize our cumulative squared jerk, we use the same cost function as for the above solution, that is $H(x(t)) = \int_0^{t_f}(\dddot{x}(t))^2dt$. Using the above form of $x(t)$, we can construct the jerk as:

$$J(t) = \dddot{x}(t) = a_{min}(30t_f^{-3}t^2 - 24t_f^{-2}t + 3t_f^{-1}) + x_f(360t_f^{-5}t^2 - 360t_f^{-4}t + 60t_f^{-3})$$

Our problem is then to solve the equation:

$$\frac{d}{da_{min}}\int_0^{t_f}(a_{min}(30t_f^{-3}t^2 - 24t_f^{-2}t + 3t_f^{-1}) + x_f(360t_f^{-5}t^2 - 360t_f^{-4}t + 60t_f^{-3}))^2dt = 0$$
$$\int_0^{t_f}\frac{d}{da_{min}}(a_{min}(30t_f^{-3}t^2 - 24t_f^{-2}t + 3t_f^{-1}) + x_f(360t_f^{-5}t^2 - 360t_f^{-4}t + 60t_f^{-3}))^2dt = 0$$
$$2\int_0^{t_f}(a_{min}(30t_f^{-3}t^2 - 24t_f^{-2}t + 3t_f^{-1}) + x_f(360t_f^{-5}t^2 - 360t_f^{-4}t + 60t_f^{-3}))(30t_f^{-3}t^2 - 24t_f^{-2}t + 3t_f^{-1})dt = 0$$
$$\int_0^{t_f}a_{min}(900t_f^{-6}t^4 - 2*720t_f^{-5}t^3 + 2*90t_f^{-4}t^2 + 576t_f^{-4}t^2 - 2*72t_f^{-3}t + 9t_f^{-2})+$$
$$x_f(10800t_f^{-8}t^4 - 8640t_f^{-7}t^3 + 1080t_f^{-6}t^2 - 10800t_f^{-7}t^3 + 8640t_f^{-6}t^2 - 1080t_f^{-5}t + 1800t_f^{-6}t^2 - 1440t_f^{-5}t + 180t_f^{-4})dt = 0$$

Evaluating the integral with respect to t yields the equation (evaluated between 0 and t_f):

$$a_{min}(180t_f^{-6}t^5 - 360t_f^{-5}t^4 + 60t_f^{-4}t^3 + 192t_f^{-4}t^3 - 72t_f^{-3}t^2 + 9t_f^{-2}t)+$$
$$x_f(2160t_f^{-8}t^5 - 2160t_f^{-7}t^4 + 360t_f^{-6}t^3 - 2700t_f^{-7}t^4 + 2880t_f^{-6}t^3 - 540t_f^{-5}t^2 + 600t_f^{-6}t^3 - 720t_f^{-5}t^2 + 180t_f^{-4}t) = 0$$

Clearly, evaluating the above at 0 leads to 0, as every term contains t to some power. We can then plug in t_f, simplify, move the x_f term to the right, and divide through by the coefficients of a_{min} to obtain an expression for our acceleration.

$$a_{min} = -\frac{x_f(2160t_f^{-3} - 2160t_f^{-3} + 360t_f^{-3} - 2700t_f^{-3} + 2880t_f^{-3} - 540t_f^{-3} + 600t_f^{-3} - 720t_f^{-3} + 180t_f^{-3})}{(180t_f^{-1} - 360t_f^{-1} + 60t_f^{-1} + 192t_f^{-1} - 72t_f^{-1} + 9t_f^{-1})}$$
$$a_{min} = -\frac{60x_ft_f^{-3}}{9t_f^{-1}} = -\frac{20x_f}{3t_f^2}$$

d) Figure 3 displays the results for the above solution with a total time of 0.5 seconds (i.e. t_f was set to 0.25 s in the above equations) and a total distance of 10 cm. In order to obtain these results, the position, velocity, and acceleration vectors were flipped using the fliplr function in Matlab and concatenated with the forward vectors (removing the duplicate at $\frac{t_f}{2}$). The second half of the velocity profile was also negated due to the change in direction and slope of $x(t)$. The acceleration, however, still captured the correct behavior with respect to both the concavity of $x(t)$ and slope of $v(t)$, so it did not need to be negated. As expected, the optimal acceleration at the boundary is a large negative number, rather than zero. We can see that this satisfies our other boundary conditions at the midpoint as well, as required.

Question 3

a) As discussed previously, the problem of minimizing some specific property of an unknown function can be solved using the calculus of variations. In the above case, we considered a solution that maximized the "smoothness," which we equated to minimizing the cumulative squared jerk, the third time derivative of position, over the interval. Here, we instead seek a function that will minimize the cumulative squared acceleration. That is, our problem can be stated as:

$$\arg\min_{x(t)}\int_{t_0}^{t_f}(\ddot{x}(t))^2dt$$

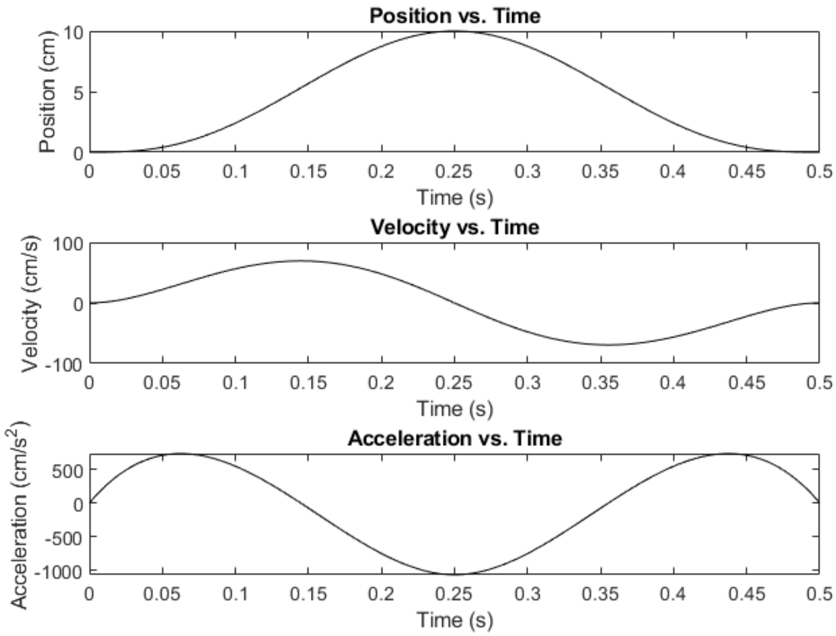

Figure 3: The position, velocity, and acceleration profiles for the function which minimizes the cumulative squared jerk over the interval from 0 to 0.25 s moving a distance of 10 cm and traveling back to the original position following a time-reversed trajectory.

As before, the calculus of variations considers perturbations to the function $x(t) \rightarrow x(t) + \varepsilon\eta(t)$, where $\eta(t)$ is any smooth, bounded function, and ε is a parameter describing the strength of the perturbation. We seek a function that corresponds to a minimum in this perturbation space, i.e. that any perturbation of any form will cause an increase in our cost function (the cumulative squared acceleration) for any perturbation size. Thus, we consider the derivative of the cost with respect to the perturbation parameter, evaluated at $\varepsilon = 0$, and set this derivative equal to zero to find our extremum. Our problem is then to solve:

$$\frac{d}{d\varepsilon}\int_{t_0}^{t_f}(\ddot{x}(t) + \varepsilon\ddot{\eta}(t))^2 dt = \int_{t_0}^{t_f}\frac{d}{d\varepsilon}(\ddot{x}(t) + \varepsilon\ddot{\eta}(t))^2 dt = 2\int_{t_0}^{t_f}\ddot{x}(t)\ddot{\eta}(t)dt = 0$$

Rather than consider any functions with smooth, continuous second derivatives in the above integral, we would instead rather consider the functions themselves, as requiring their second derivatives to be smooth and continous is restrictive and unnecessary. We then must restate the above integral in terms of the function $\eta(t)$ rather than its second derivative. To do so, we utilize integration by parts, which states that $\int_a^b u dv = uv\big|_a^b - \int_a^b v du$. We use $\ddot{\eta}(t)$ as our dv and $\ddot{x}(t)$ as our u. We also specify boundary conditions for the perturbation function such that η and its first derivative are zero at both boundaries, which is necessary since otherwise this would violate our constraints on our position function. This forces the uv term to be zero, as we are evaluating $\dot{\eta}\ddot{x}$. We thus obtain:

$$\int_{t_0}^{t_f}\ddot{x}(t)\ddot{\eta}(t)dt = -\int_{t_0}^{t_f}\dddot{x}(t)\dot{\eta}(t)dt$$

We can repeat this procedure once more to obtain a final expression in terms of the perturbation function. Using the same reasoning and boundary conditions, we then obtain:

$$\int_{t_0}^{t_f}\ddot{x}(t)\ddot{\eta}(t)dt = -\int_{t_0}^{t_f}\dddot{x}(t)\dot{\eta}(t)dt = \int_{t_0}^{t_f}\ddddot{x}(t)\eta(t)dt$$

We then employ the aforementioned fundamental theorem of the calculus of variations, which states that if an integral of a product of two functions is zero over an interval for any arbitrary smooth and continuous form of one function, the other function must be zero over the integral. Since $\eta(t)$ is an arbitrary function satisfying these criteria, the fourth derivative of $x(t)$ must be zero, implying that $x(t)$ is a third-order polynomial, that is:

$$x(t) = c_3t^3 + c_2t^2 + c_1t + c_0$$

b) As before, we can solve for the coefficients of this polynomial given the proper boundary conditions. Here, we utilize the bounds on position and velocity as above, so $t_0 = x(0) = v(0) = v(t_f) = 0$ and $x(t_f) = x_f$. We once again two equations for both position and velocity, which we can summarize in the following matrix equation:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ t_f^3 & t_f^2 & t_f & 1 \\ 3t_f^2 & 2t_f & 1 & 0 \end{bmatrix} \begin{bmatrix} c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_f \\ 0 \end{bmatrix}$$

Once again, we can reduce this by inspection, which reveals that the lower-index coefficients, c_0 and c_1, are zero.

$$\begin{bmatrix} t_f^3 & t_f^2 \\ 3t_f^2 & 2t_f \end{bmatrix} \begin{bmatrix} c_3 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_f \\ 0 \end{bmatrix}$$

We can invert the matrix quite easily, as the inverse of a 2x2 matrix can be obtained by switching the diagonal entries and negating the off-diagonal entries, and dividing the result by the determinant. The determinant of the above matrix is $\det(T) = 2t_f^4 - 3t_f^4 = -t_f^4$, so the inverse is:

$$\begin{bmatrix} -2t_f^{-3} & t_f^{-2} \\ 3t_f^{-2} & -t_f^{-1} \end{bmatrix}$$

Multiplying this matrix by our boundary condition vector returns the coefficients c_3 and c_2, so our completed polynomial is:

$$x(t) = -2t_f^{-3} x_f t^3 + 3t_f^{-2} x_f t^2$$

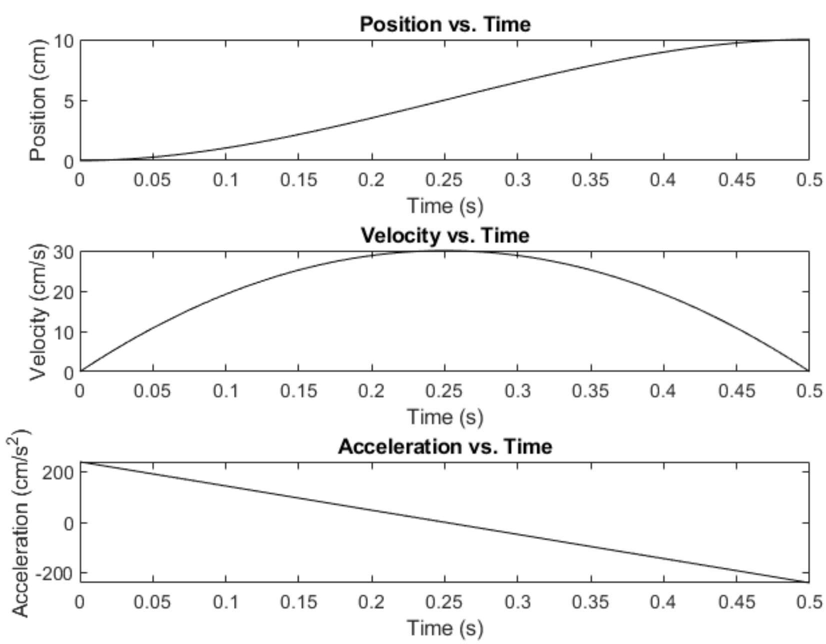

Figure 4: The position, velocity, and acceleration profiles for the function which minimizes the cumulative squared acceleration over the interval from 0 to 0.5 s moving a distance of 10 cm.

c) Figure 4 shows the position, velocity and acceleration profiles using boundary conditions $t_f = 0.5$ s and $x_f = 10$ cm for the trajectory which minimizes the cumulative squared acceleration. We can see that the acceleration profile is linear with respect to time, unlike in the previous cases which sought to minimze the cumulative squared jerk instead.

Question 4

a) The following derivation follows the same procedure as above. Rather than minimizing the cumulative squared acceleration, we instead seek to minimize the cumulative squared snap, the fourth time derivative of position.

$$\arg\min_{x(t)} \int_{t_0}^{t_f} (\ddddot{x}(t))^2 dt$$

Using the calculus of variations, this problem becomes to solving:

$$\frac{d}{d\varepsilon}\int_{t_0}^{t_f} (\ddddot{x}(t) + \varepsilon\ddddot{\eta}(t))^2 dt = \int_{t_0}^{t_f} \frac{d}{d\varepsilon}(\ddddot{x}(t) + \varepsilon\ddddot{\eta}(t))^2 dt = 2\int_{t_0}^{t_f} \ddddot{x}(t)\ddddot{\eta}(t) dt = 0$$

As before, we can transform the final equality to one in terms of the perturbation function itself rather than its derivatives using integration by parts. Here, we use this method four times, so the fourth derivative of position becomes the eighth derivative of position, similarly to how the second derivative became the fourth for acceleration or the third became the sixth for jerk. This implies that our position function is a seventh-order polynomial:

$$x(t) = c_7t^7 + c_6t^6 + c_5t^5 + c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$$

b) Solving for the coefficients of this equation is more difficult than in previous cases, as it involves inverting a 4x4 matrix after simplification, but the process is essentially identical. We begin by finding equations for velocity, acceleration, and jerk, and use the boundary conditions at time zero and time t_f to construct a system of eight equations which can be summarized in the following matrix equation:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ t_f^7 & t_f^6 & t_f^5 & t_f^4 & t_f^3 & t_f^2 & t_f & 1 \\ 7t_f^6 & 6t_f^5 & 5t_f^4 & 4t_f^3 & 3t_f^2 & 2t_f & 1 & 0 \\ 42t_f^5 & 30t_f^4 & 20t_f^3 & 12t_f^2 & 6t_f & 2 & 0 & 0 \\ 210t_f^4 & 120t_f^3 & 60t_f^2 & 24t_f & 6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_7 \\ c_6 \\ c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_f \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As before, we see that the four lower-index coefficients are zero, and we can simplify this system to a set of four equations with $c_0 = c_1 = c_2 = c_3 = 0$.

$$\begin{bmatrix} t_f^7 & t_f^6 & t_f^5 & t_f^4 \\ 7t_f^6 & 6t_f^5 & 5t_f^4 & 4t_f^3 \\ 42t_f^5 & 30t_f^4 & 20t_f^3 & 12t_f^2 \\ 210t_f^4 & 120t_f^3 & 60t_f^2 & 24t_f \end{bmatrix} \begin{bmatrix} c_7 \\ c_6 \\ c_5 \\ c_4 \end{bmatrix} = \begin{bmatrix} x_f \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, we must invert the 4x4 matrix on the left-hand side and multiply the inverse by the boundary condition vector on the right-hand side to obtain our coefficients. As in the case for the 3x3 matrix, we can invert the matrix by constructing its matrix of minors, negating every other entry to form its cofactor matrix, transposing to obtain the adjoint, and dividing by the determinant. The most time-consuming step is the matrix of minors, which involves finding 16 3x3 determinants. Rather than do this by hand, we note that the powers of t_f can be computed separately from the coefficients, and a simple Matlab script is used to compute the 3x3 determinants of the coefficients by constructing temporary matrices with the correct rows and columns removed. By inspection, it can also be seen that each entry contains only one power of t_f, and these powers follow a pattern, increasing by 1 along each row and column. Therefore, the matrix of minors can be decomposed into the following pairwise product of two matrices:

$$\begin{bmatrix} t_f^9 & t_f^{10} & t_f^{11} & t_f^{12} \\ t_f^{10} & t_f^{11} & t_f^{12} & t_f^{13} \\ t_f^{11} & t_f^{12} & t_f^{13} & t_f^{14} \\ t_f^{12} & t_f^{13} & t_f^{14} & t_f^{15} \end{bmatrix} \begin{bmatrix} -240 & -840 & -1008 & -420 \\ -120 & -408 & -468 & -180 \\ -24 & -78 & -84 & -30 \\ -2 & -6 & -6 & -2 \end{bmatrix}$$

Performing the pairwise multiplication and inverting the negative entries (i.e. entries where the sum of the row and column subscripts is odd) gives us the cofactor matrix:

$$\begin{bmatrix} -240t_f^9 & 840t_f^{10} & -1008t_f^{11} & 420t_f^{12} \\ 120t_f^{10} & -408t_f^{11} & 468t_f^{12} & -180t_f^{13} \\ -24t_f^{11} & 78t_f^{12} & -84t_f^{13} & 30t_f^{14} \\ 2t_f^{12} & -6t_f^{13} & 6t_f^{14} & -2t_f^{15} \end{bmatrix}$$

We transpose this matrix to obtain the adjoint:

$$\begin{bmatrix} -240t_f^9 & 120t_f^{10} & -24t_f^{11} & 2t_f^{12} \\ 840t_f^{10} & -408t_f^{11} & 78t_f^{12} & -6t_f^{13} \\ -1008t_f^{11} & 468t_f^{12} & -84t_f^{13} & 6t_f^{14} \\ 420t_f^{12} & -180t_f^{13} & 30t_f^{14} & -2t_f^{15} \end{bmatrix}$$

The determinant of the original matrix can be obtained by taking each entry of the first row of the cofactor matrix, multiplying it by its corresponding entry in the initial matrix, and summing the results. Therefore, the determinant $\det T = t_f^7(-240t_f^9) + t_f^6(840t_f^{10}) + t_f^5(-1008t_f^{11}) + t_f^4(420t_f^{12}) = 12t_f^{16}$. We can then construct the inverse by dividing each entry in our adjoint matrix by this value, yielding:

$$\begin{bmatrix} -20t_f^{-7} & 10t_f^{-6} & -2t_f^{-5} & \frac{1}{6}t_f^{-4} \\ 70t_f^{-6} & -34t_f^{-5} & 6.5t_f^{-4} & -\frac{1}{2}t_f^{-3} \\ -84t_f^{-5} & 39t_f^{-4} & -7t_f^{-3} & \frac{1}{2}t_f^{-2} \\ 35t_f^{-4} & -15t_f^{-3} & 2.5t_f^{-2} & -\frac{1}{6}t_f^{-1} \end{bmatrix}$$

Using this matrix and multiplying our boundary condition vector, we obtain the following equation for the position to minimize snap:

$$x(t) = -20t_f^{-7}x_f t^7 + 70t_f^{-6}x_f t^6 - 84t_f^{-5}x_f t^5 + 35t_f^{-4}x_f t^4$$

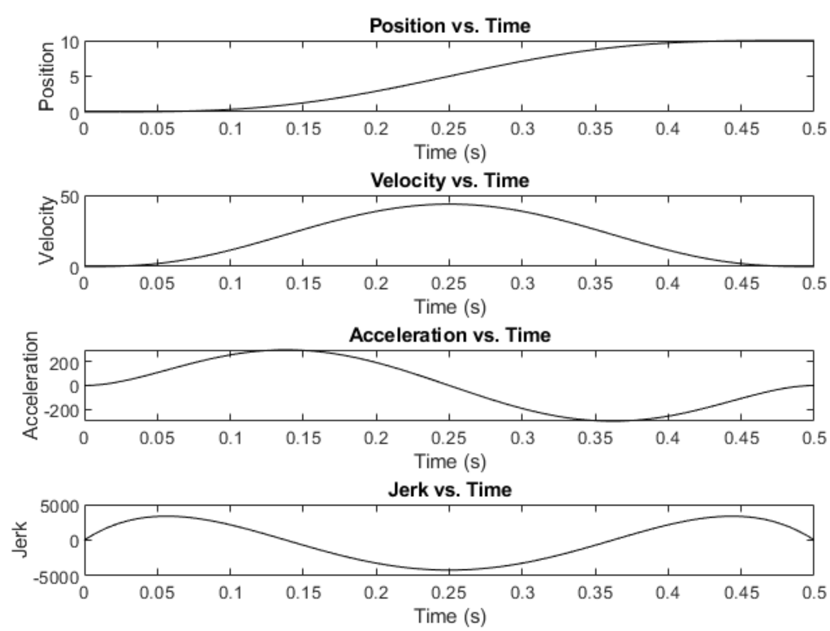

Figure 5: The position, velocity, and acceleration profiles for the function which minimizes the cumulative squared snap over the interval from 0 to 0.5 s moving a distance of 10 cm.

c) Figure 5 displays the position, velocity, acceleration, and jerk profiles which minimize the cumulative squared snap, the fourth derivative of position with respect to time.

d) Schematically, the position versus time plots in Questions 2, 3, and 4 all look qualitatively similar. However, upon closer inspection, it can be seen that they differ slightly as they correspond to different odd-order polynomials. In all cases, $x(t)$ increases slowly with the steepest slope in the center, after which it slows back down to the final position. The velocity profiles, too, look somewhat similar, but larger differences are apparent, especially between the minimal cumulative squared acceleration and the other two cases. In the former case, there are no inflections in the velocity profile, but these are present in the other two, consistent with the velocity profile corresponding to a quadratic function in the former case, and higher-order even polynomials in the other cases. In all cases, however, the velocity is positive throughout the duration of the motion, as expected since there is no backward motion in these scenarios. The changes in the acceleration profile are most apparent, transforming from a linear function of time in the cumulative squared acceleration case to cubic and fifth-order polynomials in the other scenarios. Even with these quantitative differences, however, like the position and velocity functions, the acceleration consistently follows similar qualitative patterns, beginning to increase at time zero and remaining positive during the first half of the motion, crossing through zero at the midpoint, and remaining negative during the latter half. The ranges of the velocity and acceleration are comparable amongst all three cases as well, with maximal velocity between $30 - 50cm/s$ and total acceleration falling within the range between $-250 - 250cm^2/s$.

The position and velocity profiles in parts **2b** and (the first 0.25 s of) **2d** are also qualitatively similar. Like all the other profiles, $x(t)$ follows a similar trajectory with the largest rate of increase at the midpoint of the displacement. The velocity follows a similar bell-shaped profile in both cases, but interestingly, it appears that the peak velocity occurs slightly after the midpoint of the trajectory (i.e. slightly after 0.125 s), rather than directly at the center. The peak velocity is also larger than in all other cases, consistent with the faster average rate to the maximal displacement of $40cm/s$, as opposed to $20cm/s$ in all other cases. The acceleration profile shows the largest difference, primarily due to the removed constraint at the final displacement. The peak negative acceleration is nearly 4 times larger than that seen in other cases, and occurs at the point where all other profiles are constrained to be zero. Even the positive acceleration peaks are near 2 times as large as the other cases, which is consistent with the doubled average velocity as well.

The appendix below contains codes used for each of the above problems. For Question 1, the Twitch.m script first constructs and plots the impulse-response function to a stimulus exhibited by a slow-twitch muscle. The following for loop convolves this response with spike trains with varying inter-spike intervals that change the summed response. These are then plotted in subsequent subfigures.

```
                K = .17;
tau1 = 50;
tau2 = 15;
t = 1:.5:1500;
h = K*(exp(-t/tau1)-exp(-t/tau2));

subplot(4,1,1);
plot(t,h,'k');
title('Slow-twitch Muscle Response');
xlabel('Time (ms)');
ylabel('Force (N)');
xlim([0 1500]);
ylim([0 0.3]);

tsep = [200,50,20];

for i = 1:3
    impulse_train = zeros(1,4*tsep(i)+tsep(i)/2);
    impulse_train(tsep(i):tsep(i):(4*tsep(i)+tsep(i)/2)) = 1;
    tsep(i):tsep(i):(4*tsep(i)+tsep(i)/2)
    force = conv(impulse_train,h);
    subplot(4,1,i+1);
    plot(force,'k');
    xlabel('Time (ms)');
    ylabel('Force (N)');
    title([num2str(tsep(i)),'ms interval']);
    xlim([0 1500]);
    ylim([0 0.3]);
end
```

For Question 2, the Jerk.m script creates the forward and inverse matrices used to obtain the boundary conditions from the coefficients and vice versa. These two matrices are checked to make sure they are inverses, and then the polynomial coefficients are extracted from the inverse. The position, velocity, and acceleration profiles are constructed and plotted versus time.

```
tf = 0.5;
xf = 10;

A=[tf^5 tf^4 tf^3;5*tf^4 4*tf^3 3*tf^2;20*tf^3 12*tf^2 6*tf];
B=[6*tf^-5 -3*tf^-4 0.5*tf^-3;-15*tf^-4 7*tf^-3 -1*tf^-2;10*tf^-3 -4*tf^-2 0.5*
   tf^-1];

A*B;

C = B*[xf;0;0];
t=0:0.001:tf;

x = C(1)*t.^5+C(2)*t.^4+C(3)*t.^3;
v = 5*C(1)*t.^4 + 4*C(2)*t.^3 + 3*C(3)*t.^2;
a = 20*C(1)*t.^3 + 12*C(2)*t.^2 + 6*C(3)*t;

figure;
subplot(3,1,1);
```

```
plot(t,x,'k');
xlabel('Time (s)');
ylabel('Position');
title('Position vs. Time');
subplot(3,1,2);
plot(t,v,'k');
xlabel('Time (s)');
ylabel('Velocity');
title('Velocity vs. Time');
subplot(3,1,3);
plot(t,a,'k');
xlabel('Time (s)');
ylabel('Acceleration');
title('Acceleration vs. Time');
```

The OutandBack.m script perfoms essentially the same calculations, but also adds in the derived acceleration to yield the minimum cumulative squared jerk. It should also be noted that the tf parameter in this function really corresponds to tf/2, the midpoint time at which the furthest position is reached. The script also time-reverses the profiles for position, velocity, and acceleration, and concatenates these time-reversed profiles with the forward trajectories to obtain the out and back motion.

```
tf = 0.25;
xf = 10;

A=[tf^5 tf^4 tf^3;5*tf^4 4*tf^3 3*tf^2;20*tf^3 12*tf^2 6*tf];
B=[6*tf^-5 -3*tf^-4 0.5*tf^-3;-15*tf^-4 7*tf^-3 -1*tf^-2;10*tf^-3 -4*tf^-2 0.5*
    tf^-1];

amin=-20*xf/(3*tf^2);
A*B;

C = B*[xf;0;amin];
t=0:0.001:tf;

x = C(1)*t.^5+C(2)*t.^4+C(3)*t.^3;
v = 5*C(1)*t.^4 + 4*C(2)*t.^3 + 3*C(3)*t.^2;
a = 20*C(1)*t.^3 + 12*C(2)*t.^2 + 6*C(3)*t;

x2=fliplr(x);
v2=-1*fliplr(v);
a2=fliplr(a);

xtot=[x,x2];
vtot=[v,v2];
atot=[a,a2];
ttot=0:0.001:2*tf;
l = length(ttot);
xtot(floor(l/2))=[];
vtot(floor(l/2))=[];
atot(floor(l/2))=[];

figure;
subplot(3,1,1);
plot(ttot,xtot,'k');
xlabel('Time (s)');
ylabel('Position (cm)');
title('Position vs. Time');
subplot(3,1,2);
```

```
plot(ttot,vtot,'k');
xlabel('Time (s)');
ylabel('Velocity (cm/s)');
title('Velocity vs. Time');
subplot(3,1,3);
plot(ttot,atot,'k');
xlabel('Time (s)');
ylabel('Acceleration (cm/s^2)');
title('Acceleration vs. Time');
```

For Question 3, the Acc.m script performs the same calculations and plots as those for Jerk.m, but instead uses the equations that minimize the cumulative squared acceleration.

```
tf = 0.5;
xf = 10;
t = 0:0.001:tf;

A = [tf^3 tf^2;3*tf^2 2*tf];
B = [-2*tf^-3 tf^-2;3*tf^-2 -1*tf^-1];

A*B;

C=B*[xf;0];

x=C(1)*t.^3+C(2)*t.^2;
v=3*C(1)*t.^2+2*C(2)*t;
a=6*C(1)*t+2*C(2);

figure;
subplot(3,1,1);
plot(t,x,'k');
xlabel('Time (s)');
ylabel('Position (cm)');
title('Position vs. Time');
subplot(3,1,2);
plot(t,v,'k');
xlabel('Time (s)');
ylabel('Velocity (cm/s)');
title('Velocity vs. Time');
subplot(3,1,3);
plot(t,a,'k');
xlabel('Time (s)');
ylabel('Acceleration (cm/s^2)');
title('Acceleration vs. Time');
```

For Question 4, the Snap.m script once again extracts the polynomial coefficients, position, velocity, and acceleration profiles, as well as the jerk profile, for the polynomial which minimizes the cumulative squared snap.

```
tf = 0.5;
xf = 10;

A=[tf^7 tf^6 tf^5 tf^4;7*tf^6 6*tf^5 5*tf^4 4*tf^3;42*tf^5 30*tf^4 20*tf^3 12*tf
    ^2;210*tf^4 120*tf^3 60*tf^2 24*tf];
B=[-20*tf^-7 10*tf^-6 -2*tf^-5 (1/6)*tf^-4;70*tf^-6 -34*tf^-5 6.5*tf^-4 -0.5*tf
    ^-3;-84*tf^-5 39*tf^-4 -7*tf^-3 0.5*tf^-2;35*tf^-4 -15*tf^-3 2.5*tf^-2 (-1/6)
    *tf^-1];

A*B;
```

```
C = B*[xf;0;0;0];
t=0:0.001:tf;

x = C(1)*t.^7+C(2)*t.^6+C(3)*t.^5+C(4)*t.^4;
v = 7*C(1)*t.^6+6*C(2)*t.^5+5*C(3)*t.^4+4*C(4)*t.^3;
a = 42*C(1)*t.^5+30*C(2)*t.^4+20*C(3)*t.^3+12*C(4)*t.^2;
J = 210*C(1)*t.^4+120*C(2)*t.^3+60*C(3)*t.^2+24*C(4)*t;

figure;
subplot(4,1,1);
plot(t,x,'k');
xlabel('Time (s)');
ylabel('Position');
title('Position vs. Time');
subplot(4,1,2);
plot(t,v,'k');
xlabel('Time (s)');
ylabel('Velocity');
title('Velocity vs. Time');
subplot(4,1,3);
plot(t,a,'k');
xlabel('Time (s)');
ylabel('Acceleration');
title('Acceleration vs. Time');
subplot(4,1,4);
plot(t,J,'k');
xlabel('Time (s)');
ylabel('Jerk');
title('Jerk vs. Time');
```

The final script, Cofactors.m, was used to obtain the numerical portions of the cofactors during the inversion of the 4x4 matrix in Question 4. The script iterates through each position of the matrix, creating a temporary matrix without the rows and columns of the position of interest, finds its determinant, and places this determinant in a cofactor matrix, d.

```
A=[1 1 1 1;7 6 5 4;42 30 20 12;210 120 60 24];

num = 1:4;
d = zeros(4);
for i=1:4
    for j=1:4
        Atemp = A;
        Atemp(i,:)=[];
        Atemp(:,j)=[];
        d(i,j) = det(Atemp);
    end
end
```