RELATIVE MONADS ON SYMMETRIC MULTICATEGORIES

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Definition 0.1. A *relative monad* $(T, i, ^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each $A \in \operatorname{ob} \mathbb{C}$ an object TA and map $i_A : JA \to TA$, and
- for each $f : JA \to TB$ a map $f^* : TA \to TB$

such that we have

$$f = f^* i,$$
$$(f^* g)^* = f^* g^*,$$
$$i^* = 1$$

for all $g : JA \to TB$, $f : JB \to TC$.

T has the structure of a functor from $\mathbb{D}$ to $\mathbb{C}$, with action on maps given by $Tf := (if)^*$. Indeed, a relative monad along the identity $1_{\mathbb{C}}$ is equivalent to an ordinary monad, with multiplication $m_X : TTX \to TX$ defined by

$$m_X := (1_{TX})^*.$$

In what follows we abbreviate 'relative monad' to 'RM'.

1. Strength

In this section, we define a notion of RM suitable for the multicategorical setting. This notion of *strong RM* recovers the usual notion of strong monad on a monoidal category when the multicategory is representable and the RM is along the identity. We go on to derive, when T is a strong RM, the following chain of implications:

$$T \text{ is an idempotent strong RM}$$
$$\Downarrow$$
$$T \text{ is a commutative RM}$$
$$\Downarrow$$
$$T \text{ is a symmetric multi-RM}$$
$$\Downarrow$$
$$T \text{ lifts to an RM along } \mathrm{CMon}(\mathbb{C}) \to \mathrm{CMon}(\mathbb{C}).$$

Definition 1.1. A *multicategory* $\mathbb{C}$ comprises

- a class of *objects* $\operatorname{ob} \mathbb{C}$,
- for all n and objects $X_1, ..., X_n, Y$ a class of *n-ary maps* $\mathbb{C}(X_1, ..., X_n; Y)$; an element of which is denoted by $f : X_1, ..., X_n \to Y$,

Date: September 12, 2023.

- for each object A an *identity* map $1_A \in \mathbb{C}(A; A)$,
- composition

$$\mathbb{C}(X_1, ..., X_n; Y) \times \mathbb{C}(W_{1,1}, ..., W_{1,m_1}; X_1) \times ... \times \mathbb{C}(W_{n,1}, ..., W_{n,m_n}; X_n)$$
$$\to \mathbb{C}(W_{1,1}, ..., W_{n,m_n}; Y)$$
$$(f, g_1, ..., g_n) \mapsto f \circ (g_1, ..., g_n)$$

  for all arities $n, m_1, ..., m_n$ and all objects $Y, X_1, ..., X_n, W_{1,1}, ..., W_{n,m_n}$ in $\mathbb{C}$,

where the identities and composition satisfy associativity and identity axioms.

We furthermore call $\mathbb{C}$ a *symmetric multicategory* if for all n we have actions of the symmetric group S_n on the class of n-ary maps

$$(-)_\sigma : \mathbb{C}(X_1, ..., X_n; Y) \to \mathbb{C}(X_{\sigma(1)}, ..., X_{\sigma(n)}; Y)$$
$$f \mapsto f_\sigma$$

compatible with the composition.

Note that any multicategory can be restricted to a category by considering only the unary maps. We can also define for $f : X_1, ..., X_n \to Y$, $g : W_1, ..., W_m \to X_j$ the single-index composite $f \circ_j g$ by

$$f \circ (1, ..., 1, g, 1, ..., 1) : X_1, ..., X_{j-1}, W_1, ..., W_m, X_{j+1}, ..., X_n \to Y.$$

1.1. **Strong relative monads.** We seek to generalise Kock's notion of a strong monad on a monoidal category. A strong monad structure on a monoidal category is given by a map

$$t_{X,Y} : X \otimes TY \to T(X \otimes Y)$$

satisfying some axioms. To define a suitable notion of strong RM in the multicategorical setting, we extend an RM's extension maps $\mathbb{C}(JX, TY) \xrightarrow{(-)^*} \mathbb{C}(TX, TY)$ to general n-ary hom-categories

$$\mathbb{C}(B_1, ..., X, ..., B_n; TY) \xrightarrow{(-)^j} \mathbb{C}(B_1, ..., TX, ..., B_n; TY),$$

which we call *strengthenings*. To use this to construct the map t in the ordinary and representable case, we begin with the unit

$$i : X \otimes Y \to T(X \otimes Y).$$

Passing to the underlying multicategory, this corresponds to a map

$$i : X, Y \to T(X \otimes Y).$$

We can strengthen this map in the second argument to obtain

$$i^2 : X, TY \to T(X \otimes Y).$$

Now passing back to the original monoidal category we have found a strength map $X \otimes TY \to T(X \otimes Y)$, and one can check that this satisfies the strength axioms. This derivation justifies the use of the terminology 'strength' to refer to the maps

$$\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^j} \mathbb{C}(B_1, ..., TX, ..., B_n; TY)$$

below.

Definition 1.2. A *strong RM* $(T, i, {}^t)$ along a map of multicategories $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each $A \in \operatorname{ob}\mathbb{C}$ an object TA and map $i_A : JA \to TA$, and
- for each arity n, $1 \leq j \leq n$ and $f : A_1, ..., A_{j-1}, JX, A_{j+1}, ..., A_n \to TY$ a map $f^j : A_1, ..., TX, ..., A_n \to TY$, where $(-)^j$ is natural in all arguments,

such that we have

$$\begin{aligned} f &= f^j \circ_j i, \\ (f^j \circ_j g)^{j+k-1} &= f^j \circ_j g^k, \\ i^1 &= 1 \end{aligned}$$

for all $g : A_1, ..., JX, ..., A_m \to TY$, $f : B_1, ..., JY, ..., B_n \to TC$.

We see that a strong RM restricts to an RM along the map between the categories of unary maps $J : \mathbb{D} \to \mathbb{C}$. Thus, on unary maps, T has a functor structure given by $Tf := (i \circ f)^1$.

For the next result, consider maps of the form $JX_1, ..., JX_n \to TY$, i.e. maps which can be strengthened in any index. In this case we can extend our notation from strengthenings in only one argument $f \mapsto f^j$ to strengthenings in any subset of the domain $f \mapsto f^S$ for $S \subseteq [n]$. Here we introduce the notation $- \circ_S g_j$ to mean 'compose with the map g_j at index j for all $j \in S$'.

Proposition 1.3. *Let T be a strong RM. Then for each n, subset $S \subseteq [n]$ and $JX_1, ..., JX_n \to TY$ we have a map $f^S : X_1, ..., X_n \to TY$, where*

$$X_j = \begin{cases} TX_j & j \in S \\ JX_j & j \notin S \end{cases}$$

such that $(-)^S$ is natural in all arguments, and such that we have

$$\begin{aligned} f &= f^S \circ_S i, \\ (f^{S_1} \circ_j g)^{S_2+j-1} &= f^{S_1} \circ_j g^{S_2}, \end{aligned}$$

for all $g : JX_1, ..., JX_m \to TY_j$, $f : JY_1, ..., JY_n \to TZ$ when $j \in S_1$.

Proof. The action $(-)^S$ is defined by applying the strengths $(-)^j$ for $j \in S$ from left to right. We must now prove the two equalities. To show that $f = f^S \circ_S i$, we apply the equality $f^j \circ_j i = f$ in turn for each of the elements of S. To show that $(f^{S_1} \circ_j g)^{S_2+j-1} = f^{S_1} \circ_j g^{S_2}$, let $S_1 = U \sqcup U'$ where $U = S_1 \cap [j]$. Then

$$\begin{aligned} (f^{S_1} \circ_j g)^{S_2+j-1} &= (f^U \circ_j g)^{(S_2+j-1)\sqcup(U'+m-1)} \\ &= (f^U \circ_j g^{S_2})^{(U'+m-1)} \\ &= f^{S_1} \circ_j g^{S_2}. \end{aligned}$$

∎

Definition 1.4. A multifunctor $F : \mathbb{D} \to \mathbb{C}$ between multicategories comprises for each hom-category $\mathbb{D}(A_1, ..., A_n; B)$ a function

$$\mathbb{D}(A_1, ..., A_n; B) \to \mathbb{C}(FA_1, ..., FA_n; FB) : f \mapsto Ff,$$

such that the following two equalities hold:

- $F1_A = 1_{FA}$, and
- $F(f \circ_j g) = Ff \circ_j Fg$.

Every multifunctor restricts to an ordinary functor between the categories of unary maps in $\mathbb{D}$ and those in $\mathbb{C}$.

Proposition 1.5. *Let T be a strong RM along a multifunctor $J : \mathbb{D} \to \mathbb{C}$. Then T is a multifunctor.*

Proof. We can define the action of T on morphisms by $(i \circ J-)^{[n]}$. To show that $T1_A = 1_{TA}$, we have

$$T1_A := (i_A \circ J1_A)^1 = (i_A \circ 1_{JA})^1 = i_A^1 = 1_{TA}.$$

To show that $T(f \circ_j g) = Tf \circ_j Tg$, we have

$$\begin{aligned} T(f \circ_j g) &:= (i \circ J(f \circ_j g))^{[n+m-1]} = (i \circ Jf \circ_j Jg)^{[n+m-1]} \\ &= ((i \circ Jf)^{[j-1]} \circ_j Jg)^{j \dots (n+m-1)} \\ &= ((i \circ Jf)^{[j]} \circ_j (i \circ Jg))^{j \dots (n+m-1)} \\ &= ((i \circ Jf)^{[j]} \circ_j (i \circ Jg)^{[m]})^{(j+m-1) \dots (n+m-1)} \\ &= (i \circ Jf)^{[n]} \circ_j (i \circ Jg)^{[m]} \\ &= Tf \circ_j Tg. \end{aligned}$$

Hence a strong RM is a multifunctor. □

1.2. **Idempotent strong relative monads.**

Definition 1.6. Let T be a strong RM. We say T is *idempotent* if the strengthenings are inverse to precomposition with the unit: the maps

$$\begin{tikzcd} \mathbb{C}(\dots, JX, \dots; TY) \arrow[r, bend left, "(-)^j"] & \mathbb{C}(\dots, TX, \dots; TY) \arrow[l, bend left, "- \circ_j i"] \end{tikzcd}$$

are inverses for all n and all objects $A_1, \dots, X, \dots, A_n; Y$. That is, as well as the equality $f^j \circ_j i = f$ (which holds for all strong RMs), we also have $(g \circ_j i)^j = g$.

1.3. **Commutative relative monads.** When we defined the subset strengths $(-)^S$ we had to choose an order in which to apply the individual strengths. Commutativity says that any choice of order gives the same result.

Definition 1.7. Let T be a strong RM. We say T is a *commutative RM* if for all $f : A_1, \dots, JX, \dots, JY, \dots, A_n \to TZ$ and $1 \leq j < k \leq n$ we have

$$f^{kj} = f^{jk} : \dots, TX, \dots, TY, \dots \to TZ.$$

Note that being able to commute any two strengths lets us reorder the application of n strengths in any way we choose. This lets us manipulate the subset strengths more freely, as the following proposition shows.

Proposition 1.8. *Let T be a commutative RM, let $f : JX_1, \dots, JX_n \to TY$ be a map, let $S \subseteq [n]$, let $g_j : JZ_{j1}, \dots, JZ_{jm_j} \to TX_j$ for $j \in S$, and let $S_j \subseteq [m_j]$. Then we have*

$$(f^S \circ_S g_j)^{\bigcup (S_j + k_j)} = f^S \circ_S g_j^{S_j},$$

where the k_j are the required index shifts so that the strengths line up.

Proof. Since T is commutative, we can rearrange the indices of S so that any of them is rightmost. Thus if we start from $(f^S \circ_S g_j)^{\bigcup(S_j+k_j)}$, for each $j \in S$ in turn, we can

- shuffle S so that j is rightmost, then
- apply the axioms of a strength to bring the indices of S_j inside the parentheses.

Having done this for each $j \in S$, we obtain $f^S \circ_S g_j^{S_j}$ as required. □

Having defined idempotent strong RM and commutative RMs, we now prove the implication between them.

Theorem 1.9. *If T is an idempotent strong RM, then T is commutative.*

Proof. Suppose T is idempotent and let $f : A_1, ..., JX, ..., JY, ..., A_n \to TZ$. Then

$$f^{kj} = (f^j \circ_j i)^{kj} = (f^{jk} \circ_j i)^j = f^{jk},$$

and so T is commutative. □

1.4. **Multi relative monads.**

Definition 1.10. Let T be an RM. We say T is a *multi-RM* if

- T is a multifunctor, and
- the multifunctoriality of T is compatible with the monad structure,

which is to say that we have

- $i \circ Jf = Tf \circ (i, ..., i)$ for any $f : X_1, ..., X_n \to Y$, and
- whenever $h \circ Jf = Tf' \circ (g_1, ..., g_n)$ we also have $h^* \circ Tf = Tf' \circ (g_1^*, ..., g_n^*)$:

$$\begin{tikzcd}
TX_1, ..., TX_n \arrow[r, "g_1^*, ..., g_n^*"] \arrow[d, "Tf"'] & TX'_1, ..., TX'_n \arrow[d, "Tf'"] \\
TY \arrow[r, "h^*"'] & TY'
\end{tikzcd}$$

We further say that T is a *symmetric multi-RM* if we have $(Tf)_\sigma = T(f_\sigma)$ for all n-ary f and $\sigma \in S_n$.

Theorem 1.11. *Let T be a commutative RM along a symmetric multifunctor J : $\mathbb{D} \to \mathbb{C}$. Then T is a symmetric multi-RM.*

Proof. Suppose T is commutative. Since T is strong, T is a multifunctor. We have two conditions to check to show T is a multi-RM. For the first, we simply have

$$i \circ Jf = (i \circ Jf)^{[n]} \circ (i, ..., i) = Tf \circ (i, ..., i).$$

Note that this holds for any strong RM, not necessarily commutative. For the second condition, suppose $h \circ Jf = Tf' \circ (g_1, ..., g_n)$. Then

$$\begin{aligned}
h^* \circ Tf &= h^* \circ (i \circ Jf)^{[n]} = (h^* \circ i \circ Jf)^{[n]} \\
&= (h \circ Jf)^{[n]} = (Tf' \circ (g_1, ..., g_n))^{[n]} \\
&= ((i \circ f')^{[n]} \circ (g_1, ..., g_n))^{[n]} \\
&\overset{\dagger}{=} (i \circ Jf')^{[n]} \circ (g_1^*, ..., g_n^*) \\
&= Tf' \circ (g_1^*, ..., g_n^*),
\end{aligned}$$

where the step marked † holds by Proposition 1.8 and the commutativity of T. To show that T is furthermore symmetric, we have

$$\begin{aligned}(Tf)_\sigma &:= ((i \circ Jf)^{[n]})_\sigma = ((i \circ Jf)_\sigma)^{\sigma(1)\dots\sigma(n)} \\ &= ((i \circ Jf)_\sigma)^{[n]} = (i \circ Jf_\sigma)^{[n]} \\ &= T(f_\sigma).\end{aligned}$$

Hence indeed T is a symmetric multimonad. □

1.5. **Commutative monoids in $\mathbb{C}$.**

Definition 1.12. Let C be a symmetric multicategory. The category $\mathrm{CMon}(\mathbb{C})$ of commutative monoids in $\mathbb{C}$ comprises

- commutative monoid objects (M, m) consisting of an object $M \in \mathbb{C}$ and n-ary maps
$$m_n : M, ..., M \to M$$
for each n, such that
  - $m_n \circ_k m_p = m_{n+p-1}$ for all $1 \leq k \leq n$, and
  - $(m_n)_\sigma = m_n$ for all $\sigma \in S_n$.
- monoid morphisms $f : (M, m) \to (M', m')$ comprising a map $f : M \to M'$ such that
$$\begin{tikzcd} M, ..., M \arrow[r, "m_n"] \arrow[d, "{f,...,f}"'] & M \arrow[d, "f"] \\ M', ..., M' \arrow[r, "m'_n"'] & M' \end{tikzcd}$$
commutes for all n.

We have a forgetful functor $U : \mathrm{CMon}(\mathbb{C}) \to \mathbb{C}$ with $U(M, m) = M$ and $Uf = f$.

Proposition 1.13. *If $J : \mathbb{D} \to \mathbb{C}$ is a symmetric multifunctor between symmetric multicategories, then J lifts to a functor $\tilde{J} : \mathrm{CMon}(\mathbb{D}) \to \mathrm{CMon}(\mathbb{C})$.*

Proof. The map $\tilde{J}$ sends an object (M, m) to (JM, Jm); we see that this is a commutative monoid object since

$$\begin{aligned}Jm_n \circ_k Jm_p = J(m_n \circ_k m_p) = Jm_{n+p-1} \\ (Jm_n)_\sigma = J(m_n)_\sigma = Jm_n\end{aligned}$$

by the symmetric multifunctoriality of J. On morphisms we have $\tilde{J}f = Jf$; we need to check that if $f : (M, m) \to (M', m')$ is a monoid morphism, then so is Jf. Indeed, we have

$$Jm'_n \circ (Jf, ..., Jf) = J(m'_n \circ (f, ..., f)) = J(f \circ m_n) = Jf \circ Jm_n,$$

as required. Functoriality follows from the functor structure of J. So indeed if J is a symmetric multifunctor then it lifts to $\tilde{J} : \mathrm{CMon}(\mathbb{D}) \to \mathrm{CMon}(\mathbb{C})$. □

Theorem 1.14. *Let $(T, i, ^*)$ be a symmetric multi-RM along the symmetric multifunctor $J : \mathbb{D} \to \mathbb{C}$. Then T lifts to a monad $(\tilde{T}, i, ^*)$ along $\tilde{J} : \mathrm{CMon}(\mathbb{D}) \to$*

CMon($\mathbb{C}$) *such that*

$$
\begin{aligned}
U\tilde{T} &= TU,\\
U(i) &= i,\\
U(f^*) &= f^*.
\end{aligned}
$$

Proof. Suppose T is a symmetric multimonad along $J : \mathbb{D} \to \mathbb{C}$. Let $\tilde{T}(M, m) = (TM, Tm)$; this is a commutative monoid object due to the symmetric multifunctor structure on T, as above in Proposition 1.13.

The map $i : JM \to TM$ lifts to a monoid morphism $i : (JM, Jm) \to (TM, Tm)$ because the diagram

$$
\begin{tikzcd}
JM, ..., JM \arrow[r, "Jm_n"] \arrow[d, "{i,...,i}"'] & JM \arrow[d, "i"] \\
TM, ..., TM \arrow[r, "Tm_n"'] & TM
\end{tikzcd}
$$

commutes for all n, being one of the axioms of a multimonad.

Given a monoid morphism $f : (JM, Jm) to (TM', Tm')$ we have that

$$
\begin{tikzcd}
JM, ..., JM \arrow[r, "Jm_n"] \arrow[d, "{f,...,f}"'] & JM \arrow[d, "f"] \\
TM', ..., TM \arrow[r, "Tm'_n"'] & TM'
\end{tikzcd}
$$

commutes for all n. Since T is a multimonad, we therefore also have that

$$
\begin{tikzcd}
TM, ..., TM \arrow[r, "Tm_n"] \arrow[d, "{f^*,...,f^*}"'] & TM \arrow[d, "f^*"] \\
TM', ..., TM \arrow[r, "Tm'_n"'] & TM'
\end{tikzcd}
$$

commutes for all n, and so f^* is also a monoid morphism. Hence T indeed lifts to the required monad $(\tilde{T}, i, ^*)$ on CMon($\mathbb{C}$). $\square$

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