RELATIVE MONADS ON SYMMETRIC MULTICATEGORIES

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Definition 0.1. A *relative monad* $(T, i, ^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each $A \in \mathrm{ob}\,\mathbb{C}$ an object TA and map $i_A : JA \to TA$, and
- for each $f : JA \to TB$ a map $f^* : TA \to TB$

such that we have

$$\begin{aligned} f &= f^* i, \\ (f^* g)^* &= f^* g^*, \\ i^* &= 1 \end{aligned}$$

for all $g : JA \to TB$, $f : JB \to TC$.

$$\bullet \xrightarrow{\qquad f \qquad} \bullet \qquad = \qquad \bullet \xrightarrow{i} \bullet \xrightarrow{f^*} \bullet$$

$$(\bullet \xrightarrow{g} \bullet \xrightarrow{f^*} \bullet)^* \qquad = \qquad \bullet \xrightarrow{g^*} \bullet \xrightarrow{f^*} \bullet$$

$$\bullet \xrightarrow{\qquad i^* \qquad} \bullet \qquad = \qquad \bullet \overset{1}{=\!=\!=\!=} \bullet$$

T has the structure of a functor from $\mathbb{D}$ to $\mathbb{C}$, with action on maps given by

$$Tf := (if)^*.$$

$$T(\bullet \xrightarrow{\qquad f \qquad} \bullet) \qquad = \qquad (\bullet \xrightarrow{f} \bullet \xrightarrow{i} \bullet)^*$$

Indeed, a relative monad along the identity $1_{\mathbb{C}}$ is equivalent to an ordinary monad, with multiplication $m_X : TTX \to TX$ defined by

$$m_X := (1_{TX})^*.$$

In the rest of this document we abbreviate 'relative monad' to 'RM'.

1. Strength

Definition 1.1. A (multicategorical) *strong RM* $(T, i, ^t)$ along a multifunctor $J : \mathbb{D} \to \mathbb{C}$ between multicategories comprises

- for each $A \in \mathrm{ob}\,\mathbb{C}$ an object TA and map $i_A : JA \to TA$, and
- for each arity n, $1 \leq j \leq n$ and $f : A_1, ..., A_{j-1}, JX, A_{j+1}, ..., A_n \to TY$ a map $f^j : A_1, ..., TX, ..., A_n \to TY$, where $(-)^j$ is natural in all arguments,

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such that we have

$$\begin{aligned} f &= f^j \circ_j i, \\ (f^j \circ_j g)^{j+k-1} &= f^j \circ_j g^k, \\ i^1 &= 1 \end{aligned}$$

for all $g : A_1, ..., JX, ..., A_m \to TY,\ f : B_1, ..., JY, ..., B_n \to TC$.

Note that such a strong RM is also a (unary) RM, with $(-)^*$ given by $(-)^1$.

Definition 1.2. A (monoidal) *strong RM* (T, i, t) along a lax monoidal functor $(J, \phi) : \mathbb{D} \to \mathbb{C}$ between monoidal categories comprises

- a relative monad $(T, i, {}^*)$ along J, and
- a natural family of maps $t_{X,Y} : JX \times TY \to T(X \times Y)$

such that the following two diagrams commute:

$$\begin{tikzcd} I \times TX \arrow[r, "\phi.\times 1"] \arrow[dr, "\lambda"'] & JI \times TX \arrow[r, "t"] & T(I \times X) \arrow[dl, "T\lambda"] \\ & TX & \end{tikzcd}$$

$$\begin{tikzcd} (JX \times JY) \times TZ \arrow[r, "\phi\times 1"] \arrow[d, "\alpha"'] & J(X \times Y) \times TZ \arrow[r, "t"] & T((X \times Y) \times Z) \arrow[d, "T\alpha"] \\ JX \times (JY \times TZ) \arrow[r, "1\times t"'] & JX \times T(Y \times Z) \arrow[r, "t"'] & T(X \times (Y \times Z)) \end{tikzcd}$$

(expressing coherence with the monoidal structure), and the following diagrams also commute:

$$\begin{tikzcd} JX \times JY \arrow[r, "\phi"] \arrow[d, "1\times i"'] & J(X \times Y) \arrow[d, "i"] \\ JX \times TY \arrow[r, "t"'] & T(X \times Y) \end{tikzcd}$$

$$\begin{tikzcd} JX \times JY \arrow[r, "\phi"] \arrow[d, "1\times k"'] & J(X \times Y) \arrow[d, "l"'] \\ JX \times TY' \arrow[r, "t"'] & T(X \times Y') \end{tikzcd} \Longrightarrow \begin{tikzcd} JX \times TY \arrow[r, "t"] \arrow[d, "1\times k^*"] & T(X \times Y) \arrow[d, "l^*"] \\ JX \times TY' \arrow[r, "t"'] & T(X \times Y') \end{tikzcd}$$

(expressing coherence with the monad structure).

Given a representable multicategory $\mathbb{C}$, we have objects I, $X \times Y$ and maps $u : - \to I$, $\theta : X, Y \to X \times Y$ such that the induced maps

$$-\circ_j u : \mathbb{C}(..., I, ...; Y) \to \mathbb{C}(..., -, ...; Y),\ -\circ_j \theta : \mathbb{C}(..., X\times Y, ...; Z) \to \mathbb{C}(..., X, Y, ...; Z)$$

are isomorphisms. The maps λ, ρ, α may in this setting be defined by the equations

$$\begin{aligned} \lambda \circ \theta \circ_1 u &= 1, \\ \rho \circ \theta \circ_2 u &= 1, \\ \alpha \circ \theta \circ_1 \theta &= \theta \circ_2 \theta. \end{aligned}$$

Proposition 1.3. *Suppose that $\mathbb{D}$ and $\mathbb{C}$ are representable multicategories. Then a multicategorical RM is a monoidal RM, with strength map defined by*

$$t \circ \theta := (i \circ J\theta)^2.$$

Proof. We have four properties to check. For the λ condition, we need

$$T\lambda \circ t \circ (\phi. \times 1) = \lambda.$$

We precompose with $\theta \circ_1 u$, a bijection on 1-cells, and obtain:

$$\begin{aligned}
T\lambda \circ t \circ (\phi. \times 1) \circ \theta \circ_1 u &= T\lambda \circ t \circ \theta \circ_1 \phi. \circ_1 u \\
&= T\lambda \circ t \circ \theta \circ_1 Ju \\
&= T\lambda \circ (i \circ J\theta)^2 \circ_1 Ju \\
&= (i \circ J\lambda)^1 \circ (i \circ J\theta)^2 \circ_1 Ju \\
&= ((i \circ J\lambda)^1 \circ i \circ J\theta)^2 \circ_1 Ju \\
&= (i \circ J\lambda \circ J\theta)^2 \circ_1 Ju \\
&= (i \circ J\lambda \circ J\theta \circ_1 Ju)^1 \\
&= (i \circ J(\lambda \circ \theta \circ_1 u))^1 \\
&= (i \circ J1)^1 \\
&= i^1 = 1 \\
&= \lambda \circ \theta \circ_1 u,
\end{aligned}$$

as required.

Next, for the α condition, we need

$$T\alpha \circ t \circ (\phi \times 1) = t \circ (1 \times t) \circ \alpha.$$

This time we precompose with $\theta \circ_1 \theta$ and obtain:

$$\begin{aligned}
T\alpha \circ t \circ (\phi \times 1) \circ \theta \circ_1 \theta &= T\alpha \circ t \circ \theta \circ_1 \phi \circ_1 \theta \\
&= T\alpha \circ t \circ \theta \circ_1 J\theta \\
&= T\alpha \circ (i \circ J\theta)^2 \circ_1 J\theta \\
&= (i \circ J\alpha)^1 \circ (i \circ J\theta)^2 \circ_1 J\theta \\
&= ((i \circ J\alpha)^1 \circ i \circ J\theta)^2 \circ_1 J\theta \\
&= (i \circ J\alpha \circ J\theta)^2 \circ_1 J\theta \\
&= (i \circ J\alpha \circ J\theta \circ_1 J\theta)^3 \\
&= (i \circ J(\alpha \circ \theta \circ_1 \theta))^3 \\
&= (i \circ J(\theta \circ_2 \theta))^3 \\
&= (i \circ J\theta \circ_2 J\theta)^3 \\
&= ((i \circ J\theta)^2 \circ_2 (i \circ J\theta))^3 \\
&= (i \circ J\theta)^2 \circ_2 (i \circ J\theta)^2 \\
&= t \circ \theta \circ_2 (t \circ \theta) \\
&= t \circ (1 \times t) \circ \theta \circ_2 \theta \\
&= t \circ (1 \times t) \circ \alpha \circ \theta \circ_1 \theta,
\end{aligned}$$

as required.

We move on to the two monad structure conditions. For the unit condition, we need

$$t \circ (1 \times i) = i \circ \phi.$$

Precomposing with θ, we obtain:

$$\begin{aligned} t \circ (1 \times i) \circ \theta &= t \circ \theta \circ_2 i \\ &= (i \circ J\theta)^2 \circ_2 i \\ &= i \circ J\theta \\ &= icirc\phi \circ \theta, \end{aligned}$$

as required.

Finally, for the extension condition, given

$$t \circ (1 \times k) = l \circ \phi,$$

we need to show that

$$t \circ (1 \times k^*) = l^* \circ t.$$

Precomposing with θ we obtain:

$$\begin{aligned} t \circ (1 \times k^*) \circ \theta &= t \circ \theta \circ_2 k^1 \\ &= (i \circ J\theta)^2 \circ_2 k^1 \\ &= ((i \circ J\theta)^2 \circ_2 k)^2 \\ &= (t \circ (1 \times k^*) \circ \theta)^2 \\ &= (l \circ \phi \circ \theta)^2 \\ &= (l \circ J\theta)^2 \\ &= (l^1 \circ i \circ J\theta)^2 \\ &= l^1 \circ (i \circ J\theta)^2 \\ &= l^* \circ t, \end{aligned}$$

as required. Hence a multicategorical RM between representable multicategories is a monoidal RM. □

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