

Variational Autoencoder Mathematics

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Introduction

The **Variational Autoencoder** (aka **VAE**) is a generative model. This means that it is a model which produces new unseen data. Unlike the normal Autoencoder, VAE focuses on understanding the distribution of a smaller representation of the data. This lower-dimensional representation of the data is known as “latent vector \mathbf{z} ”.

The dimension of the latent vector \mathbf{z} is a hyperparameter which we choose along with the architecture of the Network. Keep in mind that we don’t want \mathbf{z} to be too large. It should be a relatively small vector, so that an information bottleneck is created. One other reason for \mathbf{z} being small, is that we want to be able to sample easily new vectors, without having to take into consideration many features.

With that said, the question arises: How can we pick the values of \mathbf{z} which will make sense, that is, which will generate a new data point from the distribution of our original data?

Here is the beauty of the **Variational Autoencoder**: We will learn the distribution of \mathbf{z} . That is, for every component of \mathbf{z} , we will learn a mean and a standard deviation.

Suppose \mathbf{z} has k components:

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}$$

Then, the mean and standard deviation vectors are defined as:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_k \end{bmatrix}$$

Our goal is to learn the $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ vectors in order to be able to sample \mathbf{z} as follows

$$\mathbf{z} = \boldsymbol{\mu} + \epsilon \odot \boldsymbol{\sigma}$$

where $\epsilon \sim N(0, 1)$ is a gaussian with mean 0 and standard deviation 1.

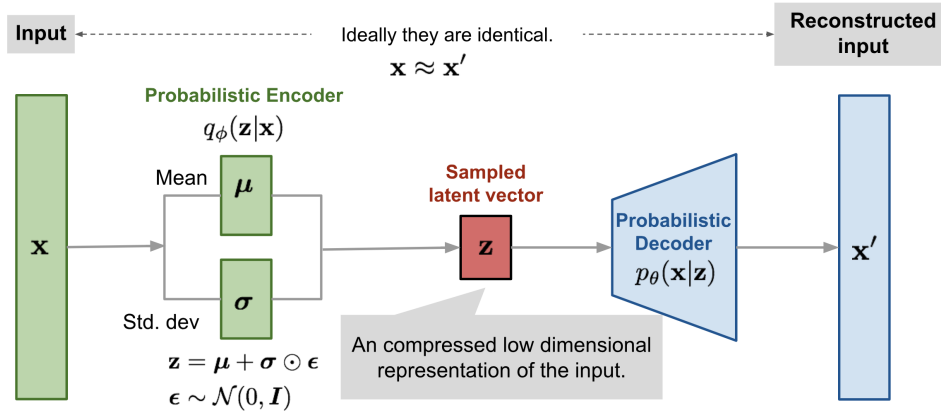


Figure 1: This picture demonstrates the architecture of a Variational Autoencoder. The input \mathbf{x} gets fed in a Probabilistic Encoder $q_\phi(z|x)$, which in turns connects with the μ and σ layers. Note that usually there is a encoder Network before the mean and std layers, but here in the figure it is ommitted. Then, they sample \mathbf{z} which in turn is fed to the Probabilistic Decoder $p_\theta(x|z)$. The result is then fed to an output layer which represents the reconstructed input data. The original picture can be found [here](#).

Brief Explanation of architecture

The architecture of a **VAE** is briefly portrayed in Figure 1. Let's take a closer look in each part:

1. The encoder part consists of a Probabilistic Encoder $q_\phi(z|x)$. Given some parameters ϕ (which are parameters of the model), $q_\phi(z|x)$ models the probability of obtaining the latent vector \mathbf{z} given input data \mathbf{x} . Afterwards, it connects to the μ and σ layers, as there might a whole encoder network before those.
2. The latent vector \mathbf{z} .
3. The decoder part which consists of a Probabilistic Decoder $p_\theta(x|z)$. As with the probabilistic encoder, given some parameters θ which are parameters of the model, we want to learn the probability of obtaining a data point \mathbf{x} given a latent vector \mathbf{z} .
4. The reconstructed input \hat{x} .

Loss function

The loss function of the VAE is:

$$L(\theta, \phi, x) = -E_{z \sim Q_\phi(z|x)} [\log(P(x|z))] + D_{KL}[Q_\phi(z|x) \parallel P(z)]$$

It may seem daunting at first, but if we break it down into pieces then it gets much simpler.

KL-Divergence and multivariate Normal Distribution

Let's start by explaining what the second term of the loss function is. The **Kullback Leiber Divergence**, also known as **Relative Entropy**, is a measure of similarity between two probability distributions. It is denoted by $D_{KL}(\cdot \parallel \cdot)$, its unit of measure it called **nat** and it can computed by the formula (for continuous probability distributions P, Q):

$$D_{KL}(P \parallel Q) = \int P(x) \log \left(\frac{P(x)}{Q(x)} \right) dx \quad (1)$$

Of course, this implies that $D_{KL}(P \parallel Q) \neq D_{KL}(Q \parallel P)$.

Now, let's suppose that both P, Q are multivariate normal distributions with means μ_1, μ_2 and **covariance** matrices Σ_1, Σ_2 :

$$P(x) = N(x; \mu_1, \Sigma_1) = \frac{1}{\sqrt{(2\pi)^k |\Sigma_1|}} e^{-\frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)}$$
$$Q(x) = N(x; \mu_2, \Sigma_2) = \frac{1}{\sqrt{(2\pi)^k |\Sigma_2|}} e^{-\frac{1}{2}(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)}$$

where k is the magnitude (length) of vector x .

Hence

$$\begin{aligned} \log(P(x)) &= \log \left(\frac{1}{\sqrt{(2\pi)^k |\Sigma_1|}} e^{-\frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)} \right) \\ &= \log \left(\frac{1}{\sqrt{(2\pi)^k |\Sigma_1|}} \right) + \log \left(e^{-\frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)} \right) \\ &= -\frac{k}{2} \log(2\pi) - \frac{1}{2} \log(|\Sigma_1|) - \frac{1}{2} (x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \end{aligned}$$

Following the exact same steps, we also get that

$$\log(Q(x)) = -\frac{k}{2} \log(2\pi) - \frac{1}{2} \log(|\Sigma_2|) - \frac{1}{2} (x - \mu_2)^T \Sigma_2^{-1} (x - \mu_2)$$

With the help of the above equalities, expanding (1) yields:

$$\begin{aligned} D_{KL}(P \parallel Q) &= \int P(x) [\log(P(x)) - \log(Q(x))] dx \\ &= \int P(x) \left[\frac{1}{2} \log\left(\frac{|\Sigma_2|}{|\Sigma_1|}\right) - \frac{1}{2} (x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \right. \\ &\quad \left. + \frac{1}{2} (x - \mu_2)^T \Sigma_2^{-1} (x - \mu_2) \right] dx \end{aligned}$$

We can rewrite the above term as an Expectation over P :

$$\begin{aligned} D_{KL}(P \parallel Q) &= E_P \left[\frac{1}{2} \log\left(\frac{|\Sigma_2|}{|\Sigma_1|}\right) - \frac{1}{2} (x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \right. \\ &\quad \left. + \frac{1}{2} (x - \mu_2)^T \Sigma_2^{-1} (x - \mu_2) \right] \end{aligned}$$

Since the logarithmic term is independent of x , we can move it outside the expectation. This leaves us with

$$\begin{aligned} D_{KL}(P \parallel Q) &= \frac{1}{2} \log\left(\frac{|\Sigma_2|}{|\Sigma_1|}\right) \\ &\quad - \frac{1}{2} E_P [(x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1)] \\ &\quad + \frac{1}{2} E_P [(x - \mu_2)^T \Sigma_2^{-1} (x - \mu_2)] \end{aligned} \tag{2}$$

Let's now try to simplify the 2nd and 3rd terms of the above expression.

First, we have to recall the [trace](#) function and some of its properties. The trace of a square matrix A , denoted as $tr(A)$, is the sum of the elements along the main diagonal of A . The [properties](#) of the trace function which we will need are:

1. [Trace of scalar](#): Considering the scalar as a 1×1 matrix, gives: $x = tr(x)$
2. [Trace of Expectation](#): From 1: $E[x] = E[tr(x)] \Rightarrow tr(E[x]) = E[tr(x)]$
3. [Cyclic Property](#): $tr(ABC) = tr(CAB)$

Having these properties in mind, we are now ready to simplify the expectation terms computed before during the simplification of the KL Divergence.

- Term 2. Note that the matrix multiplications inside the expectation reduce to a scalar value.

$$\begin{aligned} E_P \left[(x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \right] &\stackrel{(1)}{=} E_P \left[\text{tr} \left((x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \right) \right] \\ &\stackrel{(3)}{=} E_P \left[\text{tr} \left((x - \mu_1)(x - \mu_1)^T \Sigma_1^{-1} \right) \right] \\ &\stackrel{(2)}{=} \text{tr} \left(E_P \left[(x - \mu_1)(x - \mu_1)^T \Sigma_1^{-1} \right] \right) \end{aligned}$$

Σ_1^{-1} is independent from the expectation over P , so it can be moved outside, thus giving:

$$E_P \left[(x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \right] = \text{tr} \left(E_P \left[(x - \mu_1)(x - \mu_1)^T \right] \Sigma_1^{-1} \right)$$

But the term $E_P \left[(x - \mu_1)(x - \mu_1)^T \right]$ is equal to the [Covariance Matrix](#) Σ_1 , thus yielding

$$E_P \left[(x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \right] = \text{tr} \left(\Sigma_1 \Sigma_1^{-1} \right) = \text{tr}(I_k) = k \quad (3)$$

- Term 3. Again, the term inside the expectation reduces to a scalar.

$$E_P \left[(x - \mu_2)^T \Sigma_2^{-1} (x - \mu_2) \right]$$

Add and subtract μ_1 :

$$\begin{aligned} &= E_P \left[\left[(x - \mu_1) + (\mu_1 - \mu_2) \right]^T \Sigma_2^{-1} \left[(x - \mu_1) + (\mu_1 - \mu_2) \right] \right] \\ &= E_P \left[\left[(x - \mu_1)^T + (\mu_1 - \mu_2)^T \right] \Sigma_2^{-1} \left[(x - \mu_1) + (\mu_1 - \mu_2) \right] \right] \end{aligned}$$

$$(A^T + B^T)C(A + B) = A^T C A + A^T C B + B^T C A + B^T B :$$

$$\begin{aligned} &= E_P \left[(x - \mu_1)^T \Sigma_2^{-1} (x - \mu_1) + (x - \mu_1)^T \Sigma_2^{-1} (\mu_1 - \mu_2) + \right. \\ &\quad \left. (\mu_1 - \mu_2)^T \Sigma_2^{-1} (x - \mu_1) + (\mu_1 - \mu_2)^T \Sigma_2^{-1} (\mu_1 - \mu_2) \right] \end{aligned}$$

Taking the expectation over each term individually:

$$\begin{aligned}
E_P \left[(x - \mu_2)^T \Sigma_2^{-1} (x - \mu_2) \right] &= E_P \left[(x - \mu_1)^T \Sigma_2^{-1} (x - \mu_1) \right] + \\
&E_P \left[(x - \mu_1)^T \Sigma_2^{-1} (\mu_1 - \mu_2) \right] + \\
&E_P \left[(\mu_1 - \mu_2)^T \Sigma_2^{-1} (x - \mu_1) \right] + \\
&E_P \left[(\mu_1 - \mu_2)^T \Sigma_2^{-1} (\mu_1 - \mu_2) \right]
\end{aligned}$$

- The first sub-term has the same derivation as the first term from before:

$$E_P \left[(x - \mu_1)^T \Sigma_2^{-1} (x - \mu_1) \right] = \text{tr}(\Sigma_1 \Sigma_2^{-1})$$

- The second and third sub-terms are equal to 0 due to the expectation over $(x - \mu_1)^T$. Specifically, the factor $\Sigma_2^{-1}(\mu_1 - \mu_2)$ can be moved out of the expectation as it is a constant:

$$\begin{aligned}
E_P \left[(x - \mu_1)^T \Sigma_2^{-1} (\mu_1 - \mu_2) \right] &= E_P \left[(x - \mu_1)^T \right] \Sigma_2^{-1} (\mu_1 - \mu_2) \\
&= 0_k \Sigma_2^{-1} (\mu_1 - \mu_2) = 0 \\
E_P \left[(\mu_1 - \mu_2)^T \Sigma_2^{-1} (x - \mu_1) \right] &= (\mu_1 - \mu_2)^T \Sigma_2^{-1} E_P \left[(x - \mu_1)^T \right] \\
&= \Sigma_2^{-1} (\mu_1 - \mu_2) 0_k = 0
\end{aligned}$$

- The fourth sub-term is the expectation of a constant, so it is equal to the constant itself:

$$E_P \left[(\mu_1 - \mu_2)^T \Sigma_2^{-1} (\mu_1 - \mu_2) \right] = (\mu_1 - \mu_2)^T \Sigma_2^{-1} (\mu_1 - \mu_2)$$

These simplifications leave us with:

$$\begin{aligned}
E_P \left[(x - \mu_2)^T \Sigma_2^{-1} (x - \mu_2) \right] &= \text{tr}(\Sigma_1 \Sigma_2^{-1}) + \\
&(\mu_1 - \mu_2)^T \Sigma_2^{-1} (\mu_1 - \mu_2)
\end{aligned} \tag{4}$$

Finally, formula (2) for the KL-Divergence can be ultimately simplified using equations (3), (4) to:

$$D_{KL}(P \parallel Q) = \frac{1}{2} \left[\log \left(\frac{|\Sigma_2|}{|\Sigma_1|} \right) - k + \text{tr}(\Sigma_1 \Sigma_2^{-1}) + (\mu_1 - \mu_2)^T \Sigma_2^{-1} (\mu_1 - \mu_2) \right] \tag{5}$$

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