Limiti notevoli

Esponenziali e logaritmici

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to-\infty}\left(1+\frac{a}{x}\right)^x = e^a$$

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{nx} = e^{na}$$

$$\lim_{x\to-\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$$

$$\lim_{x\to0}(1+ax)^{\frac{1}{x}} = e^a$$

$$\lim_{x\to0}\log_a(1+x)^{\frac{1}{x}} = \frac{1}{\ln a}$$

$$\lim_{x\to0}\frac{\log_a(1+x)}{x} = \log_a e = \frac{1}{\ln a}$$

$$\lim_{x\to0}\frac{a^x-1}{x} = \ln a$$

$$\lim_{x\to0}\frac{(1+x)^a-1}{x} = a$$

$$\lim_{x\to0}\frac{(1+x)^a-1}{ax} = 1$$

$$\lim_{x\to0} x^r\cdot\log_a x = 0 \qquad \forall a\in\mathbb{R}^+-\{1\},\ \forall r\in\mathbb{R}^+$$

$$\lim_{x\to0}\frac{\log_a x}{x^r} = 0 \qquad \forall a\in\mathbb{R}^+-\{1\},\ \forall r\in\mathbb{R}^+$$

$$\lim_{x\to+\infty} x^r a^x = \lim_{x\to+\infty} a^x \qquad \forall a\in\mathbb{R}^+-\{1\},\ \forall r\in\mathbb{R}^+$$

$$\lim_{x\to-\infty} |x|^r\cdot a^x = \lim_{x\to-\infty} a^x \qquad \forall a\in\mathbb{R}^+-\{1\},\ \forall r\in\mathbb{R}^+$$

$$\lim_{x\to+\infty}\frac{e^x}{x^r} = \lim_{x\to+\infty} a^x \qquad \forall r\in\mathbb{R}^+$$

$$\lim_{x\to+\infty}\frac{x^r}{e^x} = \lim_{x\to+\infty} a^x \qquad \forall r\in\mathbb{R}^+$$

$$\lim_{x\to-\infty} e^x\cdot x^r = 0 \qquad \forall r\in\mathbb{R}^+$$

Trigonometrici

$$\lim_{x\to0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to0}\frac{\sin ax}{bx} = \frac{a}{b}$$

$$\lim_{x\to0}\frac{\tan x}{x} = 1$$

$$\lim_{x\to0}\frac{\tan ax}{bx} = \frac{a}{b}$$

$$\lim_{x\to0}\frac{1-\cos x}{x} = 0$$

$$\lim_{x\to0}\frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to0}\frac{\arcsin x}{x} = 1$$

$$\lim_{x\to0}\frac{\arcsin ax}{bx} = \frac{a}{b}$$

$$\lim_{x\to0}\frac{\arctan x}{x} = 1$$

$$\lim_{x\to0}\frac{\arctan ax}{bx} = \frac{a}{b}$$

$$\lim_{x\to0}\frac{\sinh x}{x} = 1$$

$$\lim_{x\to0}\frac{\operatorname{arcsinh} x}{x} = 1$$

$$\lim_{x\to0}\frac{\tanh x}{x} = 1$$

$$\lim_{x\to0}\frac{arctanhx}{x} = 1$$

$$\lim_{x\to0}\frac{x-\sin x}{x^3} = \frac{1}{6}$$

$$\lim_{x\to0}\frac{x-\arctan x}{x^3} = \frac{1}{3}$$

Regole di derivazione

Siano $f(x)$ e $g(x)$ funzioni reali di variabile reale x derivabili, e sia D l'operazione di derivazione rispetto a x:

$$\mathrm{D}[f(x)] = f'(x) \qquad \mathrm{D}[g(x)] = g'(x)$$

- **Regola della somma (linearità):**

  $$\mathrm{D}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x)$$

- **Regola del prodotto**

  $$\mathrm{D}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- **Regola del quoziente**

  $$\mathrm{D}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

- **Regola della funzione reciproca**

  $$\mathrm{D}\left[\frac{1}{f(x)}\right] = -\frac{f'(x)}{f^2(x)}$$

- **Regola della funzione inversa**

  $$\mathrm{D}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

- **Regola della catena**

  $$\mathrm{D}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

- **Regola della potenza**

  $$\mathrm{D}\left[f(x)^{g(x)}\right] = f(x)^{g(x)}\left[g'(x)\ln(f(x)) + \frac{g(x)f'(x)}{f(x)}\right]$$

Derivate fondamentali

Ognuna di queste funzioni, se non altrimenti specificato, è derivabile in tutto il suo campo di esistenza.

Funzioni polinomiali

- $\mathrm{D}(a) = 0 \qquad (a \text{ costante})$
- $\mathrm{D}(x) = 1$
- $\mathrm{D}(ax) = a \qquad (a \text{ costante})$
- $\mathrm{D}(x^2) = 2x$
- $\mathrm{D}(x^3) = 3x^2$

  Più in generale si ha:
- $\mathrm{D}(x^n) = nx^{n-1} \qquad (n \in \mathbb{N})$

Potenze, radici e valore assoluto

- $\mathrm{D}(x^\alpha) = \alpha x^{\alpha-1} \qquad (\alpha \in \mathbb{R})$
- $\mathrm{D}(\sqrt[2]{x}) = \dfrac{1}{2\sqrt[2]{x}}$
- $\mathrm{D}(\sqrt[n]{x^m}) = \dfrac{m}{n} \cdot \sqrt[n]{x^{m-n}} \qquad (x > 0)$
- $\mathrm{D}(|x|) = \dfrac{|x|}{x} = \dfrac{x}{|x|}$

Funzioni esponenziali e logaritmiche

- $\mathrm{D}(\log_b x) = \dfrac{\log_b e}{x} = \dfrac{1}{x \cdot \ln b}$
- $\mathrm{D}(\ln x) = \dfrac{1}{x}$
- $\mathrm{D}(e^x) = e^x$
- $\mathrm{D}(a^x) = a^x \cdot \ln a$
- $\mathrm{D}(x^x) = x^x(1 + \ln x)$

Funzioni trigonometriche

- $\mathrm{D}(\sin x) = \cos x$
- $\mathrm{D}(\cos x) = -\sin x$
- $\mathrm{D}(\tan x) = 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$
- $\mathrm{D}(\cot x) = -(1 + \cot^2 x) = -\dfrac{1}{\sin^2 x}$
- $\mathrm{D}(\sec x) = \sec x \tan x$
- $\mathrm{D}(\csc x) = -\csc x \cot x$
- $\mathrm{D}(\arcsin x) = \dfrac{1}{\sqrt{1 - x^2}}$
- $\mathrm{D}(\arccos x) = -\dfrac{1}{\sqrt{1 - x^2}}$
- $\mathrm{D}(\arctan x) = \dfrac{1}{1 + x^2}$
- $\mathrm{D}(\operatorname{arccot} x) = -\dfrac{1}{1 + x^2}$
- $\mathrm{D}(\operatorname{arcsec} x) = \dfrac{1}{|x|\sqrt{x^2 - 1}}$
- $\mathrm{D}(\operatorname{arccsc} x) = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$

Derivate di funzioni composte

- $\mathrm{D}(|f(x)|) = f'(x)\dfrac{f(x)}{|f(x)|} = f'(x)\dfrac{|f(x)|}{f(x)}$
- $\mathrm{D}\left([f(x)]^n\right) = n \cdot f(x)^{n-1} \cdot f'(x)$
- $\mathrm{D}(\ln f(x)) = \mathrm{D}(\ln |f(x)|) = \dfrac{f'(x)}{f(x)}$
- $\mathrm{D}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$
- $\mathrm{D}\left(a^{f(x)}\right) = a^{f(x)} \cdot f'(x) \cdot \ln a$
- $\mathrm{D}(\sin f(x)) = \cos f(x) \cdot f'(x)$
- $\mathrm{D}(\cos f(x)) = -\sin f(x) \cdot f'(x)$
- $\mathrm{D}(\tan f(x)) = \dfrac{f'(x)}{\cos^2 f(x)}$
- $\mathrm{D}(\arcsin f(x)) = \dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$

- $\mathrm{D}(\arccos f(x)) = \dfrac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
- $\mathrm{D}(\arctan f(x)) = \dfrac{f'(x)}{1+[f(x)]^2}$
- $\mathrm{D}\left(f(x)^{g(x)}\right) = f(x)^{g(x)} \cdot \left[g'(x) \cdot \ln f(x) + g(x) \cdot \dfrac{f'(x)}{f(x)}\right]$
- $\mathrm{D}\left(x^{f(x)}\right) = x^{f(x)} \cdot \left[f'(x) \cdot \ln x + \dfrac{f(x)}{x}\right]$

Regole di integrazione

Integrazione di funzioni generiche

- **Costante**

$$\int af(x)\ \mathrm{d}x = a\int f(x)\ \mathrm{d}x$$

- **Somma**

$$\int [f(x)+g(x)]\ \mathrm{d}x = \int f(x)\ \mathrm{d}x + \int g(x)\ \mathrm{d}x$$

- **Integrazione per parti**

$$\int f(x)g'(x)\ \mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\ \mathrm{d}x$$

Funzioni razionali

- $\displaystyle\int \mathrm{d}x = x + c$
- $\displaystyle\int x^a\ \mathrm{d}x = \frac{x^{a+1}}{a+1} + c \qquad (a \neq -1)$
- $\displaystyle\int \frac{1}{x}\ \mathrm{d}x = \ln|x| + c$
- $\displaystyle\int \sqrt{x}\ \mathrm{d}x = \frac{2}{3}\sqrt{x^3} + c$
- $\displaystyle\int \frac{f'(x)}{1+[f(x)]^2}\ \mathrm{d}x = \arctan f(x) + c$

- $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + c$
- $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + c$
- $\displaystyle\int \frac{1}{a+bx^2}\,\mathrm{d}x = \frac{\arctan\frac{x\sqrt{b}}{\sqrt{a}}}{\sqrt{ab}} + c$
- $\displaystyle\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + c \qquad (b^2-4ac>0)$
- $\displaystyle\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x = \frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + c \qquad (b^2-4ac<0)$
- $\displaystyle\int \frac{x+c}{(x+b)^2+a^2}\,\mathrm{d}x = \frac{1}{2}\ln\left(x^2+2bx+a^2+b^2\right) + \frac{c-b}{a}\arctan\left(\frac{x+b}{a}\right) + c$

Logaritmi

- $\displaystyle\int \ln x\,\mathrm{d}x = x\ln x - x + c$
- $\displaystyle\int \log_b x\,\mathrm{d}x = x\log_b x - x\log_b e + c$

Funzioni esponenziali

- $\displaystyle\int e^x\,\mathrm{d}x = e^x + c$
- $\displaystyle\int e^{ax}\,\mathrm{d}x = \frac{e^{ax}}{a} + c$
- $\displaystyle\int f'(x)\cdot e^{f(x)}\,\mathrm{d}x = e^{f(x)} + c$
- $\displaystyle\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + c$
- $\displaystyle\int a^{f(x)}\cdot f'(x)\,\mathrm{d}x = \frac{a^{f(x)}}{\ln a} + c$

Funzioni irrazionali

- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + c$
- $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arccos x + c$
- $\displaystyle\int \frac{1}{|x|\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{arcsec} x + c$

- $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \operatorname{arcsinh} x + c$
- $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{arccosh} x + c$
- $\displaystyle\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$
- $\displaystyle\int \sqrt{a^2+x^2}\,\mathrm{d}x = \frac{a^2}{2}\operatorname{arcsinh}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2} + c$

Funzioni trigonometriche

- $\displaystyle\int \cos x\,\mathrm{d}x = \sin x + c$
- $\displaystyle\int \sin x\,\mathrm{d}x = -\cos x + c$
- $\displaystyle\int f'(x)\cdot\cos f(x)\,\mathrm{d}x = \sin f(x) + c$
- $\displaystyle\int f'(x)\cdot\sin f(x)\,\mathrm{d}x = -\cos f(x) + c$
- $\displaystyle\int \tan x\,\mathrm{d}x = -\ln|\cos x| + c$
- $\displaystyle\int \csc x\,\mathrm{d}x = -\ln|\csc x + \cot x| + c$
- $\displaystyle\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + c$
- $\displaystyle\int \cot x\,\mathrm{d}x = \ln|\sin x| + c$
- $\displaystyle\int \sec^2 x\,\mathrm{d}x = \tan x + c$
- $\displaystyle\int \csc^2 x\,\mathrm{d}x = -\cot x + c$
- $\displaystyle\int \sin^2 x\,\mathrm{d}x = \frac{1}{2}(x - \sin x\cos x) + c$
- $\displaystyle\int \cos^2 x\,\mathrm{d}x = \frac{1}{2}(x + \sin x\cos x) + c$
- $\displaystyle\int \cos(ax)\,\mathrm{d}x = \frac{1}{a}\sin(ax) + c$
- $\displaystyle\int \sin(ax)\,\mathrm{d}x = -\frac{1}{a}\cos(ax) + c$

Funzioni iperboliche

- $\int \sinh x \ \mathrm{d}x = \cosh x + c$
- $\int \cosh x \ \mathrm{d}x = \sinh x + c$
- $\int \tanh x \ \mathrm{d}x = \ln(\cosh x) + c$
- $\int \operatorname{csch} x \ \mathrm{d}x = \ln\left(\left|\tanh \frac{x}{2}\right|\right) + c$
- $\int \operatorname{sech} x \ \mathrm{d}x = \arctan(\sinh x) + c$
- $\int \coth x \ \mathrm{d}x = \ln(|\sinh x|) + c$
- $\int \operatorname{arccosh} x \ \mathrm{d}x = x \operatorname{arccosh} x - \sqrt{x^2 - 1} + c$
- $\int \operatorname{arcsinh} x \ \mathrm{d}x = x \operatorname{arcsinh} x - \sqrt{x^2 + 1} + c$
- $\int \operatorname{arctanh} x \ \mathrm{d}x = x \operatorname{arctanh} x + \frac{\log(1 - x^2)}{2} + c$

Formule parametriche (funzioni razionali in seno/coseno)

Posto

$t = \tan\left(\frac{x}{2}\right)$	$\mathrm{d}x = \frac{2}{1 + t^2} \ \mathrm{d}t$

si ha

$\sin x = \frac{2t}{1 + t^2}$	$\cos x = \frac{1 - t^2}{1 + t^2}$	$\tan x = \frac{2t}{1 - t^2}$

Sostituzioni interessanti

$x = \frac{e^t - e^{-t}}{2} = \sinh(t)$	$\mathrm{d}x = \frac{e^t + e^{-t}}{2} = \cosh(t)$	$\cosh^2(t) - \sinh^2(t) = 1$