Violation of the CHSH-Bell Inequality with Photon Entanglement via Spontaneous Parametric Down Conversion (SPDC)

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We analyze the polarization-entanglement between two photons produced via spontaneous parametric down-conversion (SPDC) in two type-1 beta barium borate crystals, and demonstrate the violation of the CHSH-Bell Inequality leading one to the conclusion of a quantum mechanical over a hidden-variable theory. The evidence of entanglement lies within the violation of the CHSH-Bell inequality (the statement that $S \leq 2$), wherein we find experimentally that $S = 2.346 \pm 0.0483$ beating the inequality to seven standard deviations and confirming that the correlation between photon polarization's cannot be accounted for by a hidden variable theory.

I. INTRODUCTION

> "To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements."[1]
> -John Bell

The main question addressed in this paper is whether the state of correlated photons is best described via a quantum mechanical or local hidden variable theory. Simply put, what structure of statistics best describes the prepared state of our polarized photons? In short, are our photons entangled?

Quantum mechanics implications of non-locality was deeply unnerving to Einstein, Podolsky, and Rosen who said as much in their well known paper discussing the possible incompleteness of the theory [2]. These feelings led to an argument that physical states may have "local hidden variables" associated with them which would allow particles to behave in a deterministic way (restoring causality and locality).

The result of this tension would ultimately lead to John Bell proving in his 1964 paper titled "On the Einstein Podolsky Rosen Paradox"[3] that quantum physics was fundamentally incompatible with a local hidden variable theory. One way in which to demonstrate this fundamental difference between non-locality and a hidden variable theory is to show the polarization-entanglement of two photons via spontaneous parametric down conversion, a process by which a photon of higher energy is converted into a pair of photons of lower energy (while of course preserving conservation of energy, momentum, etc.).

The procedure we will be following will be modeled after Kwiat *et. al.*[4]. The goal of this experiment was to demonstrate the entanglement of a pair of photons produced via SPDC- by taking data on the coincidence count rates on opposite sides of an ejection cone produced by an incoming beam of blue photons ($\lambda = 410$ nm) impinging on nonlinear crystals. These blue photons hit two different orientations of these nonlinear crystals, each converting one incoming blue photon to two photons of double the frequency.

The orientation of the crystals is as follows, the optical axis of one of the crystals is orientated 90 degrees with respect to the other so that one is in the direction of horizontally polarized light and the other vertical. Then the polarization of the incoming blue beam is orientated to be at an angle of 45 degrees with respect to the horizontal and vertical basis such that there is an equal probability that a photon in the beam will be either down converted in the first or second crystal.

II. THEORY

A. The Local Hidden Variable and CHSH-Bell Inequality

The derivation of the CHSH-Bell inequality we will be considering will be one which first appeared in Bell [3], though I will be relying on one which appears in Zhou [5] for its simplicity and greater applicability to our situation.

The specific case we will be considering will be two particles, which have been prepared in a state that move in two distinguishable directions towards measuring devices at location 1 and 2. The devices can measure properties of the state via adjustable detector settings $\hat{a}$ and $\hat{b}$. Suppose that there is in the initial state of the two particles can be completely described in terms of a "hidden variable" λ which has a probability distribution $\rho(\lambda)$. Let A and B be the measurement outcomes at location 1 and 2 so that A,B $\in \{\pm 1\}$. Assuming a general form for the measurements, we can construct A and B as

$$A = A(\hat{a}, \hat{b}, \lambda) \qquad B = B(\hat{a}, \hat{b}, \lambda) \tag{1}$$

but to preserve locality in the sense of EPR [2] we infact can reduce Equation (1) to Equation (2)

$$A = A(\hat{a}, \lambda) \qquad B = B(\hat{b}, \lambda) \tag{2}$$

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Now, one can imagine hidden variables being contained within the measuring devices but this will be resolved if we assume the devices are "far" away enough from one another so as to ensure they don't interfere due to locality, and we can further solve this issue by averaging over the hidden variable λ.

$$\bar{A}(\hat{a},\lambda) = \langle A(\hat{a},\lambda)\rangle_{inst} \qquad \bar{B}(\hat{b},\lambda) = \left\langle B(\hat{b},\lambda)\right\rangle_{inst} \tag{3}$$

where $\bar{A},\bar{B} \in [-1,1]$. Now the inter-particle correlation is measured by the expected value of the product $\bar{A}\bar{B}$ which we can find via the following structure

$$E(\hat{a},\hat{b}) = \int \bar{A}(\hat{a},\lambda)\bar{B}(\hat{b},\lambda)\rho(\lambda)d\lambda \tag{4}$$

if we then allow for $\hat{a}'$ and $\hat{b}'$ to be different settings of the devices we then can find the following relationship (substituting $\bar{A}(\hat{a}',\lambda)$ for $\bar{A}'$)

$$E(\hat{a},\hat{b}) - E(\hat{a}',\hat{b}') = \int \left(\bar{A}\bar{B} - \bar{A}'\bar{B}'\right)\rho(\lambda)d\lambda \tag{5}$$

which we can put into the form seen in Equation (6)

$$\int \left(\bar{A}\bar{B}\left(1 \pm \bar{A}'\bar{B}'\right)\right)\rho(\lambda)d\lambda - \int \left(\bar{A}\bar{B}'\left(1 \pm \bar{A}'\bar{B}\right)\right)\rho(\lambda)d\lambda \tag{6}$$

Then, using the fact that $|\hat{A}| \leq 1$ and $|\hat{B}| \leq 1$ we can simplify Equation (6) further

$$|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')| \leq 2 \pm \left(E(\hat{a}',\hat{b}') + E(\hat{a}',\hat{b})\right) \tag{7}$$

and then after rearranging we can define a new object S as

$$S \equiv |E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')| + [E(\hat{a}',\hat{b}') + E(\hat{a}',\hat{b})] \leq 2 \tag{8}$$

where Equation (8) is the general result due to Clauser, Horne, Shimony and Holt [6]. Now, to arrive at Bell's inequality we set $\hat{a}' = \hat{b}'$, and assume that $E(\hat{b}',\hat{b}') = -1$ which will produce the following inequality

$$|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')| \leq 1 + E(\hat{b},\hat{b}') \tag{9}$$

which is the form of the equation found in J. Bell's paper "On the Einstein Podolsky Rosen Paradox"[3].

B. Entangled States of Interest

Entanglement is the property of a multi-particle quantum state whereby the wave function which describes the system as a whole cannot be factored into a product of states which individually describe one of the particles, summed up neatly in Equation (10).

$$|\psi_{1,2}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \tag{10}$$

Using the quED Entanglement Demonstrator, we can create the following entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|H_1\rangle|H_2\rangle + e^{i\phi}|V_1\rangle|V_2\rangle\right) \tag{11}$$

and by choosing an orientation for our precompensating crystal we can then further specify either of the following two Bell states: the plus state $|\Psi_+\rangle$ given by Equation (12),

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left(|H_1\rangle|H_2\rangle + |V_1\rangle|V_2\rangle\right) \tag{12}$$

or the minus state $|\Psi_-\rangle$ given by Equation (13). Using the minus Bell state we show anti-correlations in coincidence rates of the diagonal basis and using the plus Bell state we are able to show correlations in the coincidence counts for the diagonal basis.

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|H_1\rangle|H_2\rangle - |V_1\rangle|V_2\rangle\right) \tag{13}$$

once a state has been specified, we can then investigate the quality of the correlations for the different polarization's.

III. EXPERIMENTAL SETUP

The setup for the quED Demonstrator device is surprisingly simple, it begins with the blue laser ($\lambda = 410$ nm) which due to its highly divergent nature is then focused through a telescope consisting of an aspheric lens and a negative spherical lens. The beam then travels through a pinhole to further culminate the beam, and is then adjusted via a half-wave-plate. The beam is then passed through the pre-compensating birefringent crystal (used as described earlier to select either the plus or minus state). At this point the beam is prepared to pass through the two non-linear crystals, which are located at the focus of the Gaussian beam. When the beam exits it produces a cone of red entangled photons ($\lambda = 820$ nm), where the entangled photons are those which are on opposite sides of the cone.

We then collect the entangled photons using two single mode fibers, but not before passing the entangled photons through a polarizing filter used to determine the entanglement in the two different bases we will be testing. In order to best exploit the CHSH-Bell inequality we will be using the angles $\alpha = -\pi/4$, $\beta = -\pi/8$, $\alpha' = \pi/4$ and $\beta' = \pi/8$ which then means that $\alpha_\perp = \pi/4$, $\beta_\perp = 3\pi/8$, $\alpha'_\perp = \pi/2$ and $\beta'_\perp = 5\pi/8$. Then the predicted value of S we would expect would be $2\sqrt{2}$, if we achieve this value, and the standard deviation of our measurements are small then we can be reasonably sure we have violated the CHSH-Bell inequality.

To investigate the quality of the correlations for the different polarization's we can use a quantity called the visibility (V) defined in Equation (14).

$$V = \frac{C_{max} - C_{min}}{C_{max} + C_{min}} \tag{14}$$

Where C_{max} and C_{min} correspond to the maximum or minimum coincidence count rate. We can then find the error of the visibility via applying a Gaussian error propagation rule seen in Equation (15).

$$\Delta V = \frac{2}{(C_{max} + C_{min})^2} \cdot \sqrt{C_{min}^2 C_{max} + C_{max}^2 C_{min}} \tag{15}$$

Now, when comparing the statistical predictions between a hidden variable and quantum mechanical theory a well know inequality to violate which will thus demonstrate the entanglement of a state is the CHSH inequality, attributed to Clauser, Horne, Shimony and Holt.[6] The idea behind the CHSH-Bell inequality seen in equation (16), is that in a hidden variable theory the absolute value of a particular combination between two particles is bounded by 2, where the α's and β's correspond to the local measurements of the observers, where in our case this is found via polarization state of photons either going down one arm of a fiber optic to a detector or another.

$$S = (E_{\alpha,\beta} + E_{\alpha',\beta'} - E_{\alpha,\beta'} + E_{\alpha',\beta'}) \leq 2 \tag{16}$$

What we aim to show is that $S \geq 2$, where the uncertainty in our calculation for S is much less than the difference with 2. If one then defines the coincidence count rate obtained from the combination of polarizer settings α and β as $C_{\alpha,\beta}$ then the normalized expectation value $E_{\alpha,\beta}$ of correlations is given by equation (17).

$$E_{\alpha,\beta} = \frac{C_{\alpha,\beta} - C_{\alpha,\beta_\perp} - C_{\alpha_\perp,\beta} + C_{\alpha_\perp,\beta_\perp}}{C_{\alpha,\beta} + C_{\alpha,\beta_\perp} + C_{\alpha_\perp,\beta} + C_{\alpha_\perp,\beta_\perp}} \tag{17}$$

Where $\perp$ indicates polarization orientations perpendicular to either α or β. In other-words, entanglement can be demonstrated by showing that there is no basis which will separate the two states.

$$\Delta E = 2 \cdot \frac{(C_{\alpha,\beta} + C_{\alpha_\perp,\beta_\perp})(C_{\alpha_\perp,\beta} + C_{\alpha,\beta_\perp})}{(C_{\alpha,\beta} + C_{\alpha_\perp,\beta} + C_{\alpha,\beta_\perp} + C_{\alpha_\perp,\beta_\perp})^2} \cdot \sqrt{\frac{1}{C_{\alpha,\beta} + C_{\alpha_\perp,\beta_\perp}} + \frac{1}{C_{\alpha_\perp,\beta} + C_{\alpha,\beta_\perp}}} \tag{18}$$

Now, one can compute the uncertainty in E using Equation (18) which then can be used to calculate the uncertainty ΔS Equation (19).

$$\Delta S = \sqrt{\Delta E_{\alpha,\beta}^2 + \Delta E_{\alpha',\beta}^2 + \Delta E_{\alpha,\beta'}^2 + \Delta E_{\alpha',\beta'}^2} \tag{19}$$

By using the formulation for the normalized expectation value of correlations, we can plug these values into Equation (16) to test whether we have violated the CHSH-Bell inequality. Now, when thinking about the degree to which the CHSH-Bell inequality has been violated one can consider the number n_Δ. This is the number of standard deviations which add to the gap between our calculated value of S and the hidden variable bound of 2 and defined in Equation (20).

$$n_\Delta \equiv \frac{S-2}{\Delta S} \tag{20}$$

IV. RESULTS

In Figure 1 we see a very high degree of polarization for the photons coming from our apparatus, in our case the Bell state was set to the minus state Equation (13). As one would expect, the coincidence rate follows a sinusoidal pattern when the visibility of our apparatus is high enough. In this run we calculated a visibility of 95.97% in the HV basis and 90.74% in the AD basis.

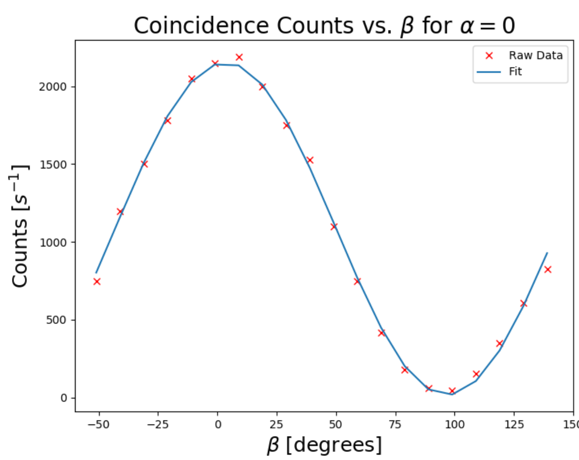

FIG. 1. Measurement of Polarization Entanglement for $\alpha = 0$, while the angle of the β polarizer was varied

In Figure 2 we see that the rate can be considered effectively constant, indicative of the fact that the individual photons are nearly unpolarized while as we know from Figure 1 that together they correlations are quantum mechanical. We see a similar situation for Figure 3 in that there is a strong correlation for the AD basis as well. Again we can infer a sinusoidal pattern indicative

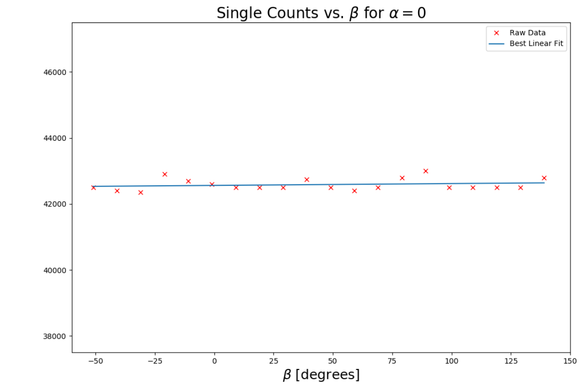

FIG. 2. Rate at the Second Detector for $\alpha = 0$

of a quantum mechanical correlation between the polarization of the state of the paired photons. Similarly, to

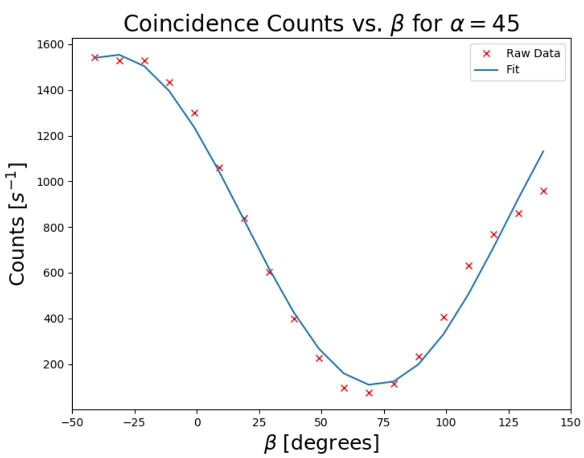

FIG. 3. Measurement of Polarization Entanglement for $\alpha = 45$, while the angle of the β polarizer was varied

the HV basis we see in Figure 4 that there is little change in the coincidence count rate for the individual photons. We can be reasonably sure from the above plots that our photons are infact entangled, though to verify this we will need to explicitly violate the CHSH-Bell inequality. To do this, we have used the angles outlined in the experimental set-up section of this paper and have found the corresponding coincidence count rates at those specific angles. The results of these measurements are summarized in Table I. Using the values from Table I we can then calculate the normalized expectation values using Equation (8), those values have been tabulated in Table II.

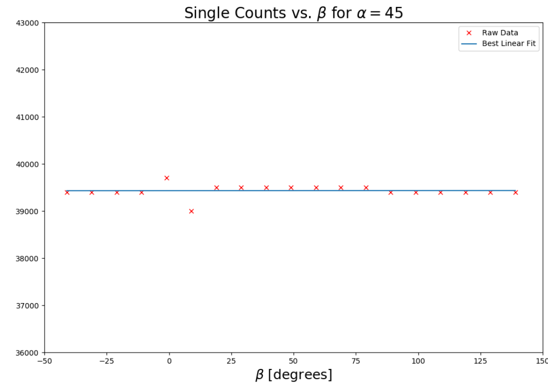

FIG. 4. Rate at the Second Detector for $\alpha = 45$

α	β	C.C.'s (1/s)
-45	-22.5	113
-45	22.5	463
-45	67.5	438.7
-45	112.5	86
0	-22.5	458.7
0	22.5	557.4
0	67.5	156.4
0	112.5	140.7
45	-22.5	484
45	22.5	220.4
45	67.5	43.7
45	112.5	421
90	-22.5	110.4
90	22.5	46.4
90	67.5	296
90	112.5	401.4

TABLE I. Coincidence Rates for Difference Choices of Angles

$E(-45,-22.5)$	$E(-45,22.5)$	$E(0,-22.5)$	$E(0,22.5)$
-0.70969	0.48529	0.47781	0.67355
$\Delta E(-45,-22.5)$	$\Delta E(-45,22.5)$	$\Delta E(0,-22.5)$	$\Delta E(0,22.5)$
0.0214	0.0253	0.0275	0.0218

TABLE II. Normalized Expectation Value (E_{α_i,β_j}) of Correlations Between the Measurements and the Errors ($\Delta E_{\alpha_i,\beta_j}$) on the Individual Correlation Coefficients

V. CONCLUSION

From Table II we can use Equation (16), (19) & (20) to find $S = 2.346$, $\Delta S = 0.0483$ and $n_\Delta = 7.17$ thereby demonstrating a violation of the CHSH-Bell's inequality to 7 standard deviations which implies that the state prepared using SPDC fundamentally cannon be described by a hidden variable theory.

VI. SUMMARY

Having found experimentally $S = 2.346 \pm 0.0483$, we can answer the question posed at the beginning of the paper, namely what kind of theory best describes the states prepared using spontaneous parametric down conversion. The answer of this question is elegantly answered by some simple counting arguments which leads to one of two cases, the violation or agreement of an inequality, the result of this experiment demonstrates conclusively the violation of the CHSH-Bell inequality and with it the contravention of a hidden variable theory to describe polarized-entangled photons.

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