Problem 71: Ordered Fractions

Consider a proper reduced fraction $\frac{n}{d} < \frac{3}{7}$ with $d \leq 1000000$.

Of all such fractions, the closest to $\frac{3}{7}$ will minimize:

$$\frac{3}{7} - \frac{n}{d} = \frac{3d - 7n}{7d} \tag{1}$$

To find this number, we will minimize its numerator and maximize its denominator.

The smallest possible numerator is 1, so set

$$3d - 7n = 1 \tag{2}$$

Then:

$$\begin{aligned} n &= \frac{3d - 1}{7} \\ &= \frac{3(d+2) - 7}{7} \\ &= \frac{3(d+2)}{7} - 1 \end{aligned} \tag{3}$$

For n to be an integer, this requires that $d + 2$ is divisible by 7.

A unit change in the numerator is more harmful than a unit change in the denominator. So maximize the denominator **subject** to the above condition.

To maximize the denominator, $7d$, choose the largest value of $d \leq 1000000$ such that $7 \mid (d + 2)$. This is $d + 2 = 999999 = 7 \times 142857$.

This gives:

$$d = 999997 \tag{4}$$

and our answer:

$$\begin{aligned} n &= \frac{3(d+2)}{7} - 1 \\ &= 3 \times 142857 - 1 \\ &= 428570 \end{aligned} \tag{5}$$

We can also be sure that $\frac{n}{d}$ is a reduced fraction. How?

Well, recall from Equation 2 that $3d - 7n = 1$. This means that $\gcd(n, d) = 1$, since any divisor of both d and n also divides $3d - 7n$ and hence must be a divisor of 1. But the only positive integer that divides 1 is 1 itself.