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L-functions of $Q(\sqrt{-N})$, which are closely related to the field $K(2^n)$, either. The precise results will be appeared.

REMARK 1. It is conjectured by A. Weil that the main part of the Hasse zeta function of an abelian curve with conductor N corresponds to a cusp form of weight two with respect to $\Gamma_0(N)$.

REMARK 2. Behaviours of prime spots in the extension $K(\ell^n)/Q$ are informed by Hasse zeta function (Shimura [3], [4]).

References

[1] E. Hecke: Zur Theorie der Elliptischen Modulfunktionen, Math. Ann. 97 (1926), 210-242 (= Werke, 428-460).

[2] A.P. Ogg: Abelan Curves of small conductor, J. Reine Angew. Math. 226 (1967), 204-215.

[3] G. Shimura: The zeta-function of an algebraic variety and automorphic functions, AMS conference on algebraic geometry, Woods Hole (1964).

[4] G. Shimura: A reciprocity law in non-solvable extensions, J. Reine Angew. Math. 221 (1966), 209-220.

Tables of Hecke operators (1)

by Hideo Wada

§ 1. Notations and method

Let q be a prime number such that $q \equiv 1 \pmod 4$ and S(q) be the set of cusp forms f such that:

$$f\left(\frac{az+b}{cz+d}\right)=\chi(a)(cz+d)^2 f(z), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q),$$

where $\chi(a)=\left(\frac{a}{q}\right)$=Legendre symbol.

Then S(q) is a finite dimensional vector field over the complex number field C. Let m be an integer which is relatively prime to q. The Hecke operator T(m) is a linear transformation of S(q) and we have the following well known formulas:

$$T(m_1 \cdot m_2)=T(m_1)\cdot T(m_2), \text{ when } (m_1, m_2)=1 \qquad (1)$$

$$T(p^k)=T(p)\cdot T(p^{k-1})-\chi(p)\cdot p\cdot T(p^{k-2}),$$

where p is a prime number (2)

Furthermore it is known that if we choose suitable basis of S(q) (independently of m), each T(m) can be represented in the following form

$$T(m)=\begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{pmatrix}$$

We want to compute x_1, x_2, $\cdots$, x_n. These values depend of course on m, but because of the above two formulas, we have only to know these values for prime numbers p (different from q) for the purpose of computing them for general m. We can compute the trace of T(m) using the Eichler-Serberg formula. When we get traces of T(p), $T(p^2)$, $\cdots$, $T(p^n)$, we can compute traces of $T(p)^2$, $T(p)^3$, $\cdots$, $T(p)^n$ using the Shimura formulas. The trace of $T(p)^k$ is $x_1{}^k+x_2{}^k+\cdots+x_n{}^k$. So we can compute fundamental symmetric functions of x_1, x_2, $\cdots$, x_n by means of the Newton Formulas. In this way we can get the polynomial:

$$F(x)=(x-x_1)(x-x_2)\cdots(x-x_n).$$

§ 2. Examples

When $q=401$, then $\dim_C S(q)$=the trace of $T(1)=32$. As $\chi(2)=1$, $T(2)$ can be expressed as follows:

$$T(2)=\begin{pmatrix} x_1 & & & & & & 0 \\ & x_1 & & & & & \\ & & x_2 & & & & \\ & & & x_2 & & & \\ & & & & \ddots & & \\ & & & & & x_{16} & \\ 0 & & & & & & x_{16} \end{pmatrix}$$

Therefore from traces of $T(2)$, $T(2^2)$, $\cdots$, $T(2^{16})$, we can get

$$F(x)=(x^{16}-x^{15}-22x^{14}+20x^{13}+192x^{12}-154x^{11}-854x^{10}+585x^9+2064x^8$$
$$-1169x^7-2653x^6+1197x^5+1623x^4-532x^3-351x^2+38x+12)^2.$$

Approximate values of x_1, x_2, $\cdots$, x_{16} are

$x_1=-2.62663649449576933816249807 15038$
$x_2=-2.27030574428055468615436890 30746$
$x_3=-1.74246715754726488589010736 48570$
$x_4=-1.65853078636371798464032416 60856$
$x_5=-1.32321906182068346731145909 09668$
$x_6=-1.15605018839646380011019459 05867$
$x_7=-0.43565002104630961045857626 69852$
$x_8=-0.16012542835457496517019121 58036$
$x_9=\ 0.23481102547782444096979076 12897$
$x_{10}=\ 0.90724085554191663442356270 82861$
$x_{11}=\ 0.90850959472820352313384051 86619$

$x_{12}= 1.4382109972742404068745725733928$
$x_{13}= 1.7840617150544319416728468179825$
$x_{14}= 2.1363629500000262718493486524642$
$x_{15}= 2.3864879358612055447718458797536$
$x_{16}= 2.5772998083674899742019117581683$

As $\chi(3)=-1$, $T(3)$ can be expressed as follows:

$$T(3)=\begin{pmatrix} y_1 & & & & & & 0 \\ & -y_1 & & & & & \\ & & y_2 & & & & \\ & & & -y_2 & & & \\ & & & & \ddots & & \\ & & & & & y_{16} & \\ 0 & & & & & & -y_{16} \end{pmatrix}$$

It would take some 200000 years to compute directly the trace of $T(3^{32})$ using an electronic computer of today. But we can compute $T(3)$ within 30 minutes in the following way.

Put $z_i=y_i^2+3$, then

$$T(9)=\begin{pmatrix} z_1 & & & & & & 0 \\ & z_1 & & & & & \\ & & z_2 & & & & \\ & & & z_2 & & & \\ & & & & \ddots & & \\ & & & & & z_{16} & \\ 0 & & & & & & z_{16} \end{pmatrix}$$

because $T(9)=T(3)\cdot T(3)+3E$. From special cases of the formulas (1), (2):

$$T(2^k\cdot 3)=T(2^k)\cdot T(3),$$
$$T(2^k)=T(2)\cdot T(2^{k-1})-2\cdot T(2^{k-2}),$$

we get the linear equations

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{16} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_{16}^{(2)} \\ \vdots & \vdots & & \vdots \\ x_1^{(15)} & x_2^{(15)} & \cdots & x_{16}^{(15)} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{16} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{tr}\, T(9) \\ \mathrm{tr}\, T(2\cdot 9) \\ \mathrm{tr}\, T(2^2\cdot 9) \\ \vdots \\ \mathrm{tr}\, T(2^{15}\cdot 9) \end{pmatrix}$$

where $x_i^{(k)}=x_i x_i^{(k-1)}-2x_i^{(k-2)}$. Solving these equations we get approximate values of z_i:

z_1=−2.8019922974854530471398070132012
z_2= 1.0110964155279577911482055139930
z_3=−0.6103592792973946504800740905501
z_4=−5.5676680806925427944877962965686
z_5= 2.9105314471853195806033039723338
z_6=−3.3550370064589151418478983423360
z_7= 0.4376373215351353986257700386329
z_8= 1.9113526421006485080168690592373
z_9=−3.7688655069113914542021813627062
z_{10}= 1.9873913549634117327456582894462
z_{11}=−8.2618397810843319015739546290236
z_{12}= 0.0409345346094928050268921203491
z_{13}=−1.8261585325346331224572627972309
z_{14}= 2.4352664502166506719833234742289
z_{15}=−0.6840259066785417290457901445926
z_{16}=−6.8582637749954126469152577677536

From these values we can compute approximate values of coefficients of the polynomial

$$F(x)=(x^2-y_1^{\,2})(x^2-y_2^{\,2})\cdots(x^2-y_{16}^{\,2})$$
$$=x^{32}+a_1x^{30}+a_2x^{28}+\cdots+a_{15}x^2+a_{16}.$$

The result is as follows:

a_1=70.999999999999999999999999975720
a_2=2275.9999999999999999999998546942
a_3=43638.999999999999999999991512986
a_4=558856.99999999999999999977729244
a_5=5055253.9999999999999999965378066
a_6=33307003.999999999999999964500937
a_7=162429516.99999999999999974701186
a_8=589806265.99999999999999871371174
a_9=1590734164.9999999999999952884164
a_{10}=3152518982.9999999999999876057610

a_{11}=4498023797.99999999999997695788840
a_{12}=4468102724.99999999999997068501 26
a_{13}=2926805999.99999999999997588432 03
a_{14}=1151599999.99999999999998841 10551
a_{15}=225599999.99999999999997353882 78
a_{16}=12799999.99999999999999844142656

As a_i are integers, a_i must be next values:

$a_1=71$	$a_9=1590734165$
$a_2=2276$	$a_{10}=3152518983$
$a_3=43639$	$a_{11}=4498023798$
$a_4=558857$	$a_{12}=4468102725$
$a_5=5055254$	$a_{13}=2926806000$
$a_6=33307004$	$a_{14}=1151600000$
$a_7=162429517$	$a_{15}=225600000$
$a_8=589806266$	$a_{16}=12800000$

In the same way when $\chi(2)=\chi(3)=1$ or -1, we can compute T(3) from T(2).

§ 3. Tables

For computing next tables, The auther used electronic computer TOSBAC-3000 installed in the Department of Mathematics College of Tsudajuku.

Table of dimensions

q	dim $S(q)$	$\chi(2)$	$\chi(3)$	q	dim $S(q)$	$\chi(2)$	$\chi(3)$
5	0	−	−	233	18	+	−
13	0	−	+	241	18	+	+
17	0	+	−	257	20	+	−
29	2	−	−	269	22	−	−
37	2	−	+	277	22	−	+
41	2	+	−	281	22	+	−
53	4	−	−	293	24	−	−
61	4	−	+	313	24	+	+
73	4	+	+	317	26	−	−
89	6	+	−	337	26	+	+
97	6	+	+	349	28	−	+
101	8	−	−	353	28	+	−
109	8	−	+	373	30	−	+
113	8	+	−	389	32	−	−
137	10	+	−	397	32	−	+
149	12	−	−	401	32	+	−
157	12	−	+	409	32	+	+
173	14	−	−	421	34	−	+
181	14	−	+	433	34	+	+
193	14	+	+	449	36	+	−
197	16	−	−	457	36	−	+
229	18	−	+	461	38	−	−

Table of T(2)

29	x^2+5
37	x^2+4
41	$(x+1)^2$
53	x^4+6x^2+7
61	x^4+8x^2+13
73	$(x^2+x-1)^2$
89	$(x^3+x^2-3x-1)^2$
97	$(x^3-3x-1)^2$
101	$x^8+13x^6+51x^4+67x^2+20$
109	$x^8+15x^6+75x^4+146x^2+87$
113	$(x^4+x^3-5x^2-4x+3)^2$
137	$(x^5+2x^4-5x^3-8x^2+5x+3)^2$
149	$x^{12}+20x^{10}+148x^8+499x^6+766x^4+465x^2+61$
157	$x^{12}+18x^{10}+120x^8+369x^6+539x^4+344x^2+71$
173	$x^{14}+20x^{12}+151x^{10}+542x^8+972x^6+833x^4+276x^2+13$
181	$x^{14}+23x^{12}+210x^{10}+974x^8+2441x^6+3234x^4+2030x^2+435$
193	$(x^7-x^6-8x^5+8x^4+15x^3-13x^2-4x+3)^2$
197	$x^{16}+24x^{14}+228x^{12}+1095x^{10}+2834x^8+3942x^6+2795x^4+925x^2+112$
229	$x^{18}+29x^{16}+349x^{14}+2267x^{12}+8673x^{10}+20038x^8+27557x^6+21285x^4$ $+8100x^2+1125$
233	$(x^9+2x^8-11x^7-20x^6+38x^5+60x^4-43x^3-53x^2+15x+9)^2$
241	$(x^9-11x^7+38x^5-2x^4-45x^3+3x^2+13x+1)^2$
257	$(x^{10}+2x^9-13x^8-24x^7+56x^6+91x^5-89x^4-115x^3+38x^2+30x-9)^2$
269	?

277 $x^{22}+34x^{20}+495x^{18}+4045x^{16}+20447x^{14}+66497x^{12}+140432x^{10}$
$+190102x^8+158753x^6+75363x^4+16786x^2+871$

281 $(x^{11}+2x^{10}-13x^9-23x^8+60x^7+86x^6-123x^5-122x^4+107x^3+56x^2-32x$
$-3)^2$

293 ?

313 $(x^{12}+x^{11}-16x^{10}-14x^9+93x^8+71x^7-238x^6-158x^5+256x^4+148x^3$
$-78x^2-41x-3)^2$

317 ?

337 $(x^{13}+x^{12}-17x^{11}-15x^{10}+108x^9+79x^8-322x^7-180x^6+461x^5+172x^4$
$-295x^3-47x^2+63x-9)^2$

349 ?

353 $(x^{14}+2x^{13}-19x^{12}-35x^{11}+138x^{10}+225x^9-483x^8-654x^7+839x^6$
$+870x^5-629x^4-490x^3+109x^2+98x+12)^2$

373 ?

389 ?

397 ?

401 $(x^{16}-x^{15}-22x^{14}+20x^{13}+192x^{12}-154x^{11}-854x^{10}+585x^9+2064x^8$
$-1169x^7-2653x^6+1197x^5+1623x^4-532x^3-351x^2+38x+12)^2$

409 $(x^{16}+x^{15}-21x^{14}-18x^{13}+175x^{12}+124x^{11}-738x^{10}-409x^9+1665x^8$
$+653x^7-1957x^6-428x^5+1078x^4+29x^3-227x^2+32x+4)^2$

421 ?

433 $(x^{17}-24x^{15}+232x^{13}-2x^{12}-1162x^{11}+28x^{10}+3233x^9-145x^8-4979x^7$
$+328x^6+3973x^5-264x^4-1403x^3-12x^2+177x+27)^2$

449 $(x^{18}+x^{17}-25x^{16}-24x^{15}+254x^{14}+231x^{13}-1356x^{12}-1143x^{11}$
$+4114x^{10}+3110x^9-7167x^8-4640x^7+6843x^6+3544x^5-3139x^4-1155x^3$
$+496x^2+104x+3)^2$

Table of T(3)

29	x^2+5
37	$(x+1)^2$
41	x^2+8
53	x^4+10x^2+7
61	$(x^2+2x-2)^2$
73	$(x^2-x-1)^2$
89	$x^6+17x^4+83x^2+125$
97	$(x^3-3x-1)^2$
101	$x^8+15x^6+68x^4+103x^2+20$
109	$(x^4-7x^2+3x+6)^2$
113	$x^8+19x^6+122x^4+297x^2+194$
137	$x^{10}+23x^8+188x^6+670x^4+989x^2+436$
149	$x^{12}+25x^{10}+231x^8+961x^6+1733x^4+1065x^2+61$
157	$(x^6+3x^5-7x^4-19x^3+10x^2+19x-10)^2$
173	$x^{14}+27x^{12}+274x^{10}+1302x^8+2958x^6+2963x^4+1187x^2+117$
181	$(x^7+x^6-13x^5-10x^4+41x^3+25x^2-26x-4)^2$
193	$(x^7+x^6-10x^5-10x^4+25x^3+16x^2-20x+1)^2$
197	$x^{16}+33x^{14}+431x^{12}+2855x^{10}+10171x^8+18702x^6+14559x^4+1577x^2+28$
229	$(x^9-x^8-17x^7+12x^6+89x^5-40x^4-143x^3+39x^2+64x-4)^2$
233	$x^{18}+38x^{16}+601x^{14}+5159x^{12}+26274x^{10}+81337x^8+149882x^6+151865x^4+68528x^2+5789$
241	$(x^9-x^8-13x^7+11x^6+52x^5-30x^4-74x^3+13x^2+36x+8)^2$
257	$x^{20}+41x^{18}+709x^{16}+6772x^{14}+39275x^{12}+142977x^{10}+325042x^8+441046x^6+319484x^4+93696x^2+4096$

269 ?

277 $(x^{11}-21x^9+152x^7-13x^6-445x^5+102x^4+450x^3-147x^2-56x+4)^2$

281 $x^{22}+50x^{20}+1081x^{18}+13285x^{16}+102611x^{14}+519821x^{12}+1748638x^{10}$
$+3865180x^8+5420534x^6+4487135x^4+1884424x^2+265883$

293 ?

313 $(x^{12}-21x^{10}-5x^9+158x^8+63x^7-509x^6-241x^5+648x^4+272x^3-250x^2$
$-81x-4)^2$

317 ?

337 $(x^{13}-x^{12}-22x^{11}+16x^{10}+182x^9-91x^8-697x^7+221x^6+1217x^5-215x^4$
$-808x^3+61x^2+175x)^2$

349 ?

353 $x^{28}+57x^{26}+1429x^{24}+20814x^{22}+196024x^{20}+1256811x^{18}$
$+5621908x^{16}+17671372x^{14}+38764478x^{12}+58122566x^{10}$
$+57447036x^8+35475661x^6+12722068x^4+2334034x^2+160344$

373 ?

389 ?

397 ?

401 $x^{32}+71x^{30}+2276x^{28}+43639x^{26}+558857x^{24}+5055254x^{22}$
$+33307004x^{20}+162429517x^{18}+589806266x^{16}+1590734165x^{14}$
$+3152518983x^{12}+4498023798x^{10}+4468102725x^8+2926806000x^6$
$+1151600000x^4+225600000x^2+12800000$

409 $(x^{16}+x^{15}-27x^{14}-27x^{13}+284x^{12}+272x^{11}-1487x^{10}-1296x^9+4094x^8$
$+2998x^7-5757x^6-3006x^5+3774x^4+907x^3-964x^2+32)^2$

421 ?

433 $(x^{17}+2x^{16}-29x^{15}-58x^{14}+327x^{13}+666x^{12}-1798x^{11}-3845x^{10}$
$+4859x^9+11637x^8-5274x^7-17447x^6-326x^5+11020x^4+3120x^3$
$-1660x^2-416x+104)^2$

On zeta functions associated with prehomogeneous vector spaces.

Takuro SHINTANI

§0. This note is a summary of a joint work [2] of the author with Mikio Sato.

§1. In this section we briefly describe results of the theory of prehomogeneous vector spaces initiated by M. Sato (for details see [1] and [4]).

1° Let G be a complex linear algebraic group and ρ be a rational representation of G on an n-dimensional vector space V. Then G naturally operates on V. We call a triple (G, ρ, V) a prehomogeneous vector space when there exists a proper algebraic subset S in V such that $V - S$ is a single G-orbit. We call S the set of singular points in V.

2° Now we assume that S is an irreducible hypersurface. Then there exists a prime homogeneous polynomial P on V such that

$$S = \{ x \in V;\ P(x) = 0 \}.$$

There exists a rational character χ of G such that

$$P(\rho(g)\cdot x) = \chi(g)P(x) \qquad (\forall g \in G,\ \forall x \in V).$$

We put d = degree of P.

3° Assume further that G is reductive. Denote by V^* the dual vector space of V and by ρ^* the representation of G on