NOTES ON UPPER AND LOWER BOUNDING $t(N)$

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1. Basics

Let $\mathcal{B} = \{b_1, \dots, b_M\}$ be a finite multiset of natural numbers (thus each natural number may appear in $\mathcal{B}$ multiple times); the ordering of elements in the multiset will not be of relevance to us. The *cardinality* $|\mathcal{B}| = M$ of the multiset is the number of elements counting multiplicity; for example,

$$|\{2, 2, 3\}| = 3.$$

The *product* $\prod \mathcal{B}$ of the finite multiset is defined by $\prod \mathcal{B} := \prod_{b \in \mathcal{B}} b$, where we count for multiplicity; for example

$$\prod\{2, 2, 3\} = 12.$$

The tuple $\mathcal{B}$ is a *factorization* of a natural number M if $\mathcal{B} = M$, and a *subfactorization* if $\mathcal{B}|M$. For example, $\{2, 2, 3\}$ is a factorization of 12 and a subfactorization of 24.

We use $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ to denote the p-adic valuation of a positive natural number a/b, that is to say the number of times p divides the numerator a, minus the number of times p divides the denominator b. For instance, $\nu_2(32/27) = 5$ and $\nu_3(32/27) = -3$. By the fundamental theorem of arithmetic, we see that a finite multiset $\mathcal{B}$ is a factorization of M if and only if

$$\nu_p(M/\prod \mathcal{B}) = 0$$

for all primes p, and a subfactorization if and only if

$$\nu_p(M/\prod \mathcal{B}) \geq 0$$

for all primes p. We refer to $\nu_p(M/\mathcal{B})$ as the *p-surplus* of $\mathcal{B}$ (as an attempted factorization) of M at prime p, and $-\nu_p(M/\prod \mathcal{B}) = \nu_p(\prod \mathcal{B}/M)$ as the *p-deficit*, and say that the factorization is p-balanced if $\nu_p(M/\prod \mathcal{B}) = 0$. Thus a subfactorization (resp. factorization) occurs when one has non-negative surpluses (resp. balance) at all primes p.

Example 1.1. Suppose one wishes to factorize $5! = 2^3 \times 3 \times 5$. The attempted factorization $\mathcal{B} := \{3, 4, 5, 5\}$ has a 2-surplus of $\nu_2(5!/\prod \mathcal{B}) = 1$, is in balance at 3, and has a 5-deficit of $\nu_2(\prod \mathcal{B}/5!) = 1$, so it is not a factorization or subfactorization of $5!$. However, if one replaces one of the copies of 5 in $\mathcal{B}$ with a 2, this erases both the 2-surplus and the 5-deficit, and creates a factorization $\{2, 3, 4, 5\}$ of $5!$.

If one applies a logarithm to the fundamental theorem of arithmetic, one obtains the identity

$$\sum_p \nu_p(r) \log p = \log r \tag{1.1}$$

for any positive rational r.

A finite multiset $\mathcal{B}$ is said to be *t-admissible* for some $t > 0$ if $b \geq t$ for all $b \in \mathcal{B}$. We define $t(N)$ denotes the largest quantity such that there exists a $t(N)$-admissible factorization of $N!$ of cardinality N. Clearly, $t(N)$ is also the largest quantity such that there exists a $t(N)$-admissible subfactorization of $N!$ of cardinality at least N, since when starting from such a subfactorization, we may delete elements and then distribute any surpluses at any primes arbitrarily to create a factorization of cardinality exactly N.

Example 1.2. The finite multiset $\{2, 2, 3, 3, 4, 4, 5, 7\}$ is a 2-admissible subfactorization of $8! = 2^7 \times 3^2 \times 5 \times 7$, having a 2-surplus of 1. If one deletes a copy of 2 to make the cardinality exactly 8, one now has a surplus of 2 at 2; one can distribute these two powers of 2 to the remaining element of 2 to obtain a factorization $\{3, 3, 4, 4, 5, 7, 8\}$ that is still 2-admissible, and is in fact now 3-admissible.

A useful measure of the efficiency of a t-admissible finite multiset $\mathcal{B}$ is the *t-excess*

$$E_t(\mathcal{B}) := \sum_{i=1}^{N'} \log \frac{b_i}{t} = \log \prod \mathcal{B} - |\mathcal{B}| \log t.$$

Example 1.3. The 3-excess of $\{3, 3, 4, 4, 5, 7, 8\}$ is

$$E_t(\{3, 3, 4, 4, 5, 7, 8\}) = 2 \log \frac{4}{3} + \log \frac{5}{3} + \log \frac{7}{3} + \log \frac{8}{3} = 2.914 \dots .$$

The t-excess clearly non-negative when $\mathcal{B}$ is t-admissible. Combining this with (1.1), we obtain the basic *balance identity*

$$E_t(\mathcal{B}) + \sum_p \nu_p(N!/\prod \mathcal{B}) \log p = \log N! - |\mathcal{B}| \log t. \tag{1.2}$$

In particular, when one has a subfactorization, the gap between $\log N!$ and $|\mathcal{B}| \log t$ must be somehow distributed between the t-excess $E_t(\mathcal{B})$ and the p-surpluses $\nu_p(N!/\prod \mathcal{B})$.

Example 1.4. The 3-admissible finite multiset $\{3, 3, 4, 4, 5, 7, 8\}$ is a factorization of 8! of cardinality 7, and the gap

$$\log 8! - 7 \log 3 = 2.914 \dots$$

is entirely absorbed by the 3-excess of the multiset. If one replaces the element 8 of this multiset with 4, this reduces the excess to

$$E_t(\{3, 3, 4, 4, 5, 7, 8\}) = 3 \log \frac{4}{3} + \log \frac{5}{3} + \log \frac{7}{3} = 2.221 \dots ,$$

but creates a 2-surplus of 1 that contributes $\log 2 = 0.693 \dots$ to (1.2), restoring balance.

From (1.2), we have the following equivalent definition of $t(N)$:

Lemma 1.5 (Equivalent form of $t(N)$)**.** *$t(N)$ is the supremum of all t for which there exists a t-admissible subfactorization $\mathcal{B}$ of $N!$ with*

$$E_t(\mathcal{B}) + \sum_p \nu_p(N!/\mathcal{B}) \log p \leq \log N! - N \log t.$$

The advantage of this formulation is that one no longer needs to directly track the cardinality $|\mathcal{B}|$ of the t-admissible subfactorization $\mathcal{B}$. The formulation highlights the need to locate subfactorizations in which both the t-excess and the p-surpluses are kept as low as possible.

We recall Legendre's formula

$$\nu_p(N!) = \sum_{j=1}^{\infty} \left\lfloor \frac{N}{p^j} \right\rfloor = \frac{N - s_p(N)}{p-1}. \tag{1.3}$$

Asymptotically, it is known that

$$\frac{1}{e} - \frac{O(1)}{\log N} \leq \frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$$

where

$$\begin{aligned} c_0 &:= \frac{1}{e} \int_0^1 \left\lfloor \frac{1}{x} \right\rfloor \log \left(ex \left\lceil \frac{1}{ex} \right\rceil \right) \, dx \\ &= \frac{1}{e} \int_1^\infty \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e} \, \frac{dy}{y^2} \\ &= 0.3044\ldots \end{aligned}$$

To bound the factorial, we have the explicit Stirling approximation [4]

$$N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N+1} \leq \log N! \leq N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N}, \tag{1.4}$$

valid for all natural numbers N.

To estimate the prime counting function, we have the following good asymptotics up to a large height.

Theorem 1.6 (Buthe's bounds)**.** [1] *For any* $2 \leq x \leq 10^{19}$, *we have*

$$\mathrm{li}(x) - \frac{\sqrt{x}}{\log x} \left(1.95 + \frac{3.9}{\log x} + \frac{19.5}{\log^2 x} \right) \leq \pi(x) < \mathrm{li}(x)$$

and

$$\mathrm{li}(x) - \frac{\sqrt{x}}{\log x} \leq \pi^*(x) < \mathrm{li}(x) + \frac{\sqrt{x}}{\log x}.$$

For $x > 10^{19}$ we have the bounds of Dusart [2]. One such bound is

$$|\psi(x) - x| \leq 59.18 \frac{x}{\log^4 x}.$$

2. Criteria for upper bounding $t(N)$

We have the trivial upper bound $t(N) \leq (N!)^{1/N}$. This can be improved to $t(N) \leq N/e$ for $N \neq 1, 2, 4$, answering a conjecture of Guy and Selfridge [3]; see [5]. This was derived from the following slightly stronger criterion, which asymptotically gives $\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0+o(1)}{\log N}$:

Lemma 2.1 (Upper bound criterion)**.** [5, Lemma 2.1] *Suppose that $1 \leq t \leq N$ are such that*

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor \log \left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil \right) > \log N! - N \log t \tag{2.1}$$

Then $t(N) < t$.

A surprisingly sharp upper bound comes from linear programming.

Lemma 2.2 (Linear programming bound)**.** *Let N be an natural number and $1 \leq t \leq N/2$. Suppose for each prime $p \leq N$, one has a non-negative real number w_p which is weakly non-decreasing in p (thus $w_p \leq w_{p'}$ when $p \leq p'$), and such that*

$$\sum_p w_p \nu_p(j) \geq 1 \tag{2.2}$$

for all $t \leq j \leq N$, and such that

$$\sum_p w_p \nu_p(N!) < N. \tag{2.3}$$

Then $t(N) < t$.

Proof. We first observe that the bound (2.2) in fact holds for all $j \geq t$, not just for $t \leq j \leq N$. Indeed, if this were not the case, consider the first $j \geq t$ where (2.2) fails. Take a prime p dividing j and replace it by a prime inthe interval $[p/2, p)$ which exists by Bertrand's postulate (or remove p entirely, if $p = 2$); this creates a new j' in $[j/2, j)$ which is still at least t. By the weakly decerasing hypothesis on w_p, we have

$$\sum_p w_p \nu_p(j) \geq \sum_p w_p \nu_p(j')$$

and hence by the minimality of j we have

$$\sum_p w_p \nu_p(j) > 1,$$

a contradiction.

Now suppose for contradiction that $t(N) \geq t$, thus we have a factorization $N! = \prod_{j \geq t} j^{m_j}$ for some natural numbers m_j summing to N. Taking p-valuations, we conclude that

$$\sum_{j \geq t} m_j \nu_p(j) \leq \nu_p(N!)$$

for all $p \leq N$. Multiplying by w_p and summing, we conclude from (2.2) that

$$N = \sum_{j \geq t} m_j \leq \sum_p w_p \nu_p(N!),$$

contradicting (2.3). □

3. Powers of 2 and 3

We now begin the study of constructions that can establish lower bounds of the form $t(N) \geq t$ for some

$$1 \leq t \leq N. \tag{3.1}$$

It will be convenient to parameterize

$$t = \frac{N}{e^{1+\delta}} \tag{3.2}$$

where we shall assume

$$\delta > 0; \tag{3.3}$$

for instance, if $t = N/3$, we will have $\delta = \log \frac{3}{3} \approx 0.098$. We also a parameter $L \geq 1$ for which

$$9L \leq t \tag{3.4}$$

and divide the primes into three categories:

- The *tiny primes* $p = 2, 3$;
- The *small primes* $3 < p \leq \sqrt{t/L}$;
- The *large primes* $p > \sqrt{t}/L$.

For any $B \geq 1$, define a ***B*-smooth number** to be a number whose prime factors are all at most B. Here we will be primarily interested in the cases $B = 2, 3$.

A 2-smooth number is just a power of two; and for any $t \geq 1$, there exists a 2-smooth number 2^n in the interval $[t, 2t]$; indeed one can take $n = \lfloor \log t / \log 2 \rfloor$. For 3-smooth numbers $2^n 3^m$ - that is to say, products of tiny primes - one can do better. For any $L \geq 1$, let κ_L be the least quantity such that for any real nmber $t \geq L$, there exists a 3-smooth number $2^n 3^m$ such that

$$t \leq 2^n 3^m \leq \exp(\kappa_L) t.$$

Thus for instance $\kappa_1 = \log 2$ thanks to the aforementioned fact about 2-smooth numbers, and it is clear that κ_L is non-decreasing in L. We have the following explicit bounds that noticeably improve upon $\log 2 = 0.69314\dots$:

Lemma 3.1. *If n_1, n_2, m_1, m_2 are positive integers such that*

$$1 \leq \frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}$$

then

$$\kappa_{\min(2^{n_1+n_2}, 3^{m_1+m_2})/6} \leq \log \max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right).$$

Thus, for instance, setting $n_1 = 3$, $m_1 = 2$, $n_2 = 2$, $m_2 = 1$, we have

$$\kappa_{4.5} \leq \log \frac{2^2}{3} = 0.28768\dots,$$

setting $n_1 = 3$, $m_1 = 2$, $n_2 = 5$, $m_2 = 3$, we have

$$\kappa_{40.5} \leq \log \frac{2^5}{3^3} = 0.16989\dots$$

and setting $n_1 = 11$, $m_1 = 7$, $n_2 = 8$, $m_2 = 5$, we have

$$\kappa_{2^{18}/3} \leq \log \frac{3^7}{2^{11}} = 0.06566\dots$$

($2^{18}/3 = 87381.33\dots$).

Proof. If $\min(2^{n_1+n_2}, 3^{m_1+m_2})/6 \leq t \leq 2^{n_2-1}3^{m_1-1}$, then we have

$$t \leq 2^{n_2-1}3^{m_1-1} \leq \max(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}})t, \tag{3.5}$$

so we are done in this case. Now suppose that $t > 2^{n_2-1}3^{m_1-1}$. Let 2^n3^m be the smallest 3-smooth number that is at least t, then we must have $n \geq n_2$ or $m \geq m_1$ (or both). Thus at least one of $\frac{2^{n_1}}{3^{m_1}}2^n3^m$ and $\frac{3^{m_2}}{2^{n_2}}2^n3^m$ is an integer, and is thus at most t by construction. This gives (3.5), and the claim follows. □

Asymptotically, we have logarithmic-type decay:

Lemma 3.2 (Baker bound)**.** *We have*

$$\kappa_L \ll \log^{-c} L$$

for all $L \geq 2$ and some absolute constant $c > 0$.

Proof. From the classical theory of continued fractions, we can find rational approximants

$$\frac{p_{2j}}{q_{2j}} \leq \frac{\log 3}{\log 2} \leq \frac{p_{2j+1}}{q_{2j+1}} \tag{3.6}$$

to the irrational number $\log 3/\log 2$, where the convergents p_j/q_j obey the recursions

$$p_j = b_jp_{j-1} + p_{j-2}; \quad q_j = b_jq_{j-1} + q_{j-2}$$

with $p_{-1} = 1, q = -1 = 0, p_0 = b_0, q_0 = 1$, and $[b_0; b_1, b_2, \dots] = [1; 1, 1, 2, \dots]$ is the continued fraction expansion of $\frac{\log 3}{\log 2}$. Furthermore, $p_{2j+1}q_{2j} - p_{2j}q_{2j+1} = 1$, and hence

$$\frac{\log 3}{\log 2} - \frac{p_{2j}}{q_{2j}} = \frac{1}{q_{2j}q_{2j+1}}. \tag{3.7}$$

By Baker's theorem, $\frac{\log 3}{\log 2}$ is a Diophantine number, giving a bound of the form

$$q_{2j+1} \ll q_{2j}^{O(1)} \tag{3.8}$$

and a similar argument gives

$$q_{2j+2} \ll q_{2j+1}^{O(1)}. \tag{3.9}$$

We can rewrite (3.6) as

$$1 \leq \frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}}$$

and routine Taylor expansion using (3.14) gives the upper bounds

$$\frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}} \leq \exp\left(O\left(\frac{1}{q_{2j}}\right)\right).$$

If one then sets j to be the largest natural number for which (**??**) holds with t replaced by L, the claim then follows from (3.8), (3.9), and Lemma 3.1. $\square$

We can now obtain efficient t-admissible subfactorizations of $2^n 3^m$ when n, m are somewhat comparable.

Lemma 3.3. *Let $L \geq 1$. Let $t > 3L$ and let $2^n 3^m$ be a 3-smooth number obeying the conditions*

$$\frac{\log(3L) + \kappa}{\log t - \log(3L)} \leq \frac{n \log 2}{m \log 3} \leq \frac{\log t - \log(2L)}{\log(2L) + \kappa}. \tag{3.10}$$

Then one can find a t-admissible subfactorization $\mathcal{B}$ of $2^n 3^m$ such that

$$E_t(\mathcal{B}) \leq \frac{\kappa_L}{\log t}(n \log 2 + m \log 3) \tag{3.11}$$

and

$$|\nu_2(2^n 3^m / \mathcal{B})|_{\log 2, \infty} + |\nu_3(2^n 3^m / \mathcal{B})|_{\log 3, \infty} \leq 2(\log t + \kappa_L). \tag{3.12}$$

Proof. Let $2^{n_0}, 3^{m_0}$ be the largest powers of 2 and 3 less than t/L respectively. By definition of κ_L, we can find 3-smooth numbers $2^{n_1}3^{m_1}$, $2^{n_2}3^{m_2}$ such that

$$\frac{t}{2^{n_0}} \leq 2^{n_1} 3^{m_1} \leq e^{\kappa} \frac{t}{2^{n_0}} \tag{3.13}$$

and

$$\frac{t}{3^{m_0}} \leq 2^{n_2} 3^{m_2} \leq e^{\kappa} \frac{t}{3^{m_0}}, \tag{3.14}$$

or equivalently

$$t \leq 2^{n_0+n_1} 3^{m_1}, 2^{n_2} 3^{m_0+m_2} \leq e^{\kappa} t. \tag{3.15}$$

We can bound

$$\begin{aligned} \frac{n_0 + n_1}{m_1} &\geq \frac{n_0}{\log(e^{\kappa} \frac{t}{2^{n_0}}) / \log 3} \\ &\geq \frac{(\log t - \log(2L)) / \log 2}{(\log(3L) + \kappa) / \log 3} \end{aligned}$$

(with the convention that this bound is vacuously true for $m_1 = 0$) and similarly

$$\begin{aligned} \frac{n_2}{m_0 + m_2} &\leq \frac{\log(e^{\kappa} \frac{t}{3^{m_0}}) / \log 2}{m_0} \\ &\leq \frac{(\log(2L) + \kappa) / \log 2}{(\log t - \log(3L)) / \log 3} \end{aligned}$$

and hence by (3.10)

$$\frac{n_2}{m_0 + m_2} \leq \frac{n}{m} \leq \frac{n_0 + n_1}{m_1}. \tag{3.16}$$

Thus we can write (n, m) as a non-negative linear combination

$$(n, m) = \alpha_1(n_0 + n_1, m_1) + \alpha_2(n_2, m_0 + m_2)$$

for some real $\alpha_1, \alpha_2 \geq 0$. We now take our subfactorization $\mathcal{B}$ to consist of $\lfloor \alpha_1$ copies of the 3-smooth number $2^{n_0+n_1}3^{m_1}$ and $\lfloor \alpha_2 \rfloor$ copies of the 3-smooth number $2^{n_2}3^{m_0+m_2}$. By (3.15), each term $2^{n'}3^{m'}$ here is admissible and contributes an excess of at most κ, which is in turn bounded by $\frac{\kappa}{\log t}(n' \log 2 + m' \log 3)$. Adding these bounds together, we obtain (3.11).

The expression $2^n3^m / \prod \mathcal{B}$ contains at most $n_0 + n_1 + n_2$ factors of 2 and at most $m_0 + m_2 + m_1$ factors of 3, hence

$$\nu_2(2^n3^m / \prod \mathcal{B}) \log 2 + \nu_3(2^n3^m / \prod \mathcal{B}) \log 3 \leq \log 2^{n_0+n_1}3^{m_1} + \log 2^{n_2}3^{m_0+m_2},$$

and the bound (3.12) follows. □

4. Criteria for lower bounding $t(N)$

Lemma 1.5 gives an initial criterion for lower bounding $t(N)$. We now perform various manipulations on tuples to replace this criterion with a more tractable one. For $a_+, a_- \in [0, +\infty]$, we define the asymmetric norm $|x|_{a_+,a_-}$ of a real number x by the formula

$$|x|_{a_+,a_-} := \max(a_+x, -a_-x),$$

thus this is $a_+|x|$ when x is positive and $a_-|x|$ when x is negative. If a_+, a_- are finite, this function is Lipschitz with constant $\max(a_+, a_-)$. One can think of a_+ as the "cost" of making x positive, and a_- as the "cost" of making x negative. One can then reformulate Lemma 1.5 as follows.

Proposition 4.1 (Reformulated balance criterion)**.** *Let $1 \leq t \leq N$, and suppose that one has a t-admissible tuple $\mathcal{B}$ obeying the following hypothesis:*

(i) *(Small excess and surplus at all primes)*

$$E_t(\mathcal{B}) + \sum_p |\nu_p(N!/ \prod \mathcal{B})|_{\log p, \infty} \leq \log N! - N \log t. \tag{4.1}$$

Then $t(N) \geq t$.

Indeed, the infinite penalty for making $\nu_p(N!/\mathcal{B})$ in (4.1) ensures that $\mathcal{B}$ is a subfactorization of $N!$.

We will reduce this infinite penalty term later, but let us work on other aspects of the criterion (4.1) first. In practice we will apply this criterion with $t := N/e^{1+\delta}$ for some $\delta > 0$; for instance, if we wish to set $t = N/3$, then $\delta = \log \frac{e}{3} \approx 0.098$. From (1.4) we may then replace $\log N! - N \log t = \log N! - N \log N + N + \delta N$ by the slightly smaller quantity

$$\delta N + \log \sqrt{2\pi N}.$$

The $\log \sqrt{2\pi N}$ is a lower order term, and we shall use it only to clean up some other lower order terms.

Using Lemma 3.3, we can leave a large surplus at tiny primes and still get good bounds:

Proposition 4.2 (Criterion with tiny-prime surplus)**.** *Let $L \geq 1$. Let $3L < t = N/e^{1+\delta}$ for some $\delta > 0$, and suppose that one has a t-admissible tuple $\mathcal{B}$ obeying the following hypotheses:*

(i) *(Small excess and surplus at non-tiny primes)*

$$E_t(\mathcal{B}) + \sum_{p>3} |\nu_p(N!/\prod \mathcal{B})|_{\log p,\infty} \leq \delta N + \kappa_L - \frac{3}{2}\log N - \kappa_L(\log\sqrt{12})\frac{N}{\log t}. \tag{4.2}$$

(ii) *(Large surpluses at tiny primes) The surpluses $\nu_2(N!/\prod \mathcal{B})$, $\nu_3(N!/\prod \mathcal{B})$ are positive and obey the bounds*

$$\frac{\log(3L)+\kappa_L}{\log t - \log(3L)} \leq \frac{\nu_2(N!/\prod \mathcal{B})\log 2}{\nu_3(N!/\prod \mathcal{B})\log 3} \leq \frac{\log t - \log(2L)}{\log(2L)+\kappa_L}.$$

Then $t(N) \geq t$.

Proof. Write $n := \nu_2(N!/\prod \mathcal{B})$ and $m := \nu_3(N!/\prod \mathcal{B})$. From (1.3) we have $n \leq N$ and $m \leq N/2$, hence

$$n\log 2 + m\log 3 \leq N\log\sqrt{12}.$$

Applying Lemma 3.3, we can find a subfactorization $\mathcal{B}'$ of 2^n3^m with an excess of at most $(\kappa_L\log\sqrt{12})\frac{N}{\log t}$, and with

$$|\nu_2(2^n3^m/\prod \mathcal{B}')|_{\log 2,\infty} + |\nu_3(2^n3^m/\prod \mathcal{B}')|_{\log 3,\infty} \leq 2(\log t + \kappa_L) \leq 2\log N - 2 + 2\kappa_L.$$

If we let $\mathcal{B}''$ be the concatenation of $\mathcal{B}$ and $\mathcal{B}'$, then $\mathcal{B}''$ is another t-admissible tuple, and from (4.2) and the observation that $-2+3\kappa_L \leq \log\sqrt{2\pi}$, we see that

$$E_t(\mathcal{B}'') + \sum_p |\nu_p(N!/\prod \mathcal{B}'')|_{\log p,\infty} \leq \delta N + \log\sqrt{2\pi N},$$

and the claim now follows from Proposition 4.1. □

The criterion (4.2) will still be somewhat expensive at small primes $3 < p \leq \sqrt{t/L}$. We can improve the situation as follows.

Proposition 4.3 (Improved criterion with tiny-prime surplus)**.** *Let $L \geq 1$. Let $9L < t = N/e^{1+\delta}$ for some $\delta > 0$, and suppose that one has a t-admissible tuple $\mathcal{B}$ obeying the following hypotheses.*

(i) *(Small excess and surplus at non-tiny primes)*

$$\begin{aligned} E_t(\mathcal{B}) &+ \sum_{3<p\leq\sqrt{t/L}} |\nu_p(N!/\prod \mathcal{B})|_{\frac{\kappa_L\log p}{\log\sqrt{t/L}},\infty} + \sum_{p>\sqrt{t/L}} |\nu_p(N!/\prod \mathcal{B})|_{\log p,\infty} \\ &\leq \delta N - \frac{3}{2}\log N - \kappa_L(\log\sqrt{12})\frac{N}{\log t}. \end{aligned} \tag{4.3}$$

(ii) *(Large surpluses at tiny primes) Whenever n_{**}, m_{**} are natural numbers obeying the bounds*

$$n_{**}\log 2 + m_{**}\log 3 \leq \sum_{3<p\leq\sqrt{t/L}} |v_p(N!/\prod\mathcal{B})|_{\frac{\log\sqrt{tL}+\kappa}{\log\sqrt{t/L}}\log p,\infty} + \log t + \kappa,$$

then $v_2(N!/\prod\mathcal{B}) > n_{**}$, $v_3(N!/\prod\mathcal{B}) > m_{**}$, and furthermore

$$\frac{\log(3L)+\kappa_L}{\log t - \log(3L)} \leq \frac{(v_2(N!/\prod\mathcal{B}) - n_{**})\log 2}{(v_3(N!/\prod\mathcal{B}) - m_{**})\log 3} \leq \frac{\log t - \log(2L)}{\log(2L)+\kappa_L}.$$

Then $t(N) \geq t$.

Proof. By (4.3), $\mathcal{B}$ is a subfactorization of $N!$. Consider all the p-surplus primes in the range $3 < p \leq \sqrt{t/L}$, thus each such prime is considered with multiplicity $v_p(N!/\prod\mathcal{B})$. Denoting their product as B, we have

$$\log B = \sum_{3<p\leq\sqrt{t/L}} |v_p(N!/\prod\mathcal{B})|_{\log p,\infty}.$$

Using the greedy algorithm, one can factor B into M factors $c_1,\dots,c_M$ in the interval $[\sqrt{t/L}, \frac{t/L}{]}$, times one exceptional factor c_* in $[1,\sqrt{t/L}]$, for some M. We have the bound

$$(\sqrt{t/L})^M \leq B$$

and hence

$$M \leq \sum_{3<p\leq\sqrt{t/L}} |v_p(N!/\prod\mathcal{B})|_{\frac{\log p}{\log\sqrt{t/L}},\infty}.$$

For each of the M factors c_i, we may use the definition of κ_L and find 3-smooth number $2^{n_i}3^{m_i}$ in the interval $[t/c_i, e^{\kappa_L}t/c_i]$, and similarly for the exceptional factor c_* we may find a 3-smooth number $2^{n_*}3^{m_*}$ in the interval $[t/c_*, e^{\kappa_L}t/c_*]$. If we then adjoin the 3-smooth numbers $2^{n_i}3^{m_i}c_i$ for $i=1,\dots,M$ as well as $2^{n_*}3^{m_*}c_*$ to the tuple $\mathcal{B}$ to create a new tuple $\mathcal{B}'$. This tuple is still t-admissible, but now has no p-surplus (or p-deficit) at any prime $3 < p \leq \sqrt{t/L}$. The quantity $n_i\log 2 + m_i\log 3$ is bounded by $\log\sqrt{tL}+\kappa_L$, and the quantity $n\log 2 + m\log 3$ is similarly bounded by $\log t+\kappa$, hence if we denote $n_{**} := n_1+\dots+n_M+n_*$ and $m_{**} := m_1+\dots+m_M+m_*$, we have

$$n_{**}\log 2 + m_{**}\log 3 \leq \frac{\log\sqrt{tL}+\kappa_L}{\log\sqrt{t/L}} \sum_{3<p\leq\sqrt{t/L}} |v_p(N!/\prod\mathcal{B})|_{\log p,\infty} + \log t + \kappa_L.$$

By hypothesis, we now see that $\mathcal{B}'$ has no p-deficit at 2 or 3 either, so $\mathcal{B}'$ is still a subfactorization of $N!$. Each of the new factors in $\mathcal{B}'$ contributes an excess of at most κ_L, so the total excess of $\mathcal{B}'$ is at most

$$E_t(\mathcal{B}) + \kappa_L M + \kappa_L$$

which is in turn bounded by

$$E_t(\mathcal{B}) + \sum_{3<p\leq\sqrt{t/L}} |v_p(N!/\prod\mathcal{B})|_{\frac{\kappa_L\log p}{\log\sqrt{t/L}},\infty} + \kappa_L.$$

We conclude that $\mathcal{B}'$ obeys the hypotheses of Equation (4.2), and the claim follows. □

Finally, we relax the subfactorization condition by permitting some p-deficit at various non-tiny primes $p > 3$.

Proposition 4.4 (Improved criterion with tiny-prime surplus, and some deficit)**.** *Let $L \geq 1$. Let $9L < t = N/e^{1+\delta}$ for some $\delta > 0$, and suppose that one has a t-admissible tuple $\mathcal{B}$ with the property that whenever n_{**}, m_{**} are natural numbers obeying the bounds*

$$n_{**} \log 2 + m_{**} \log 3 \leq \sum_{3<p\leq\sqrt{t/L}} |\nu_p(N!/\prod \mathcal{B})|_{\frac{\log\sqrt{tL}+\kappa_L}{\log\sqrt{t/L}} \log p, \log p+\kappa_p}$$
$$+ \sum_{p>\sqrt{t/L}} |\nu_p(N!/\prod \mathcal{B})|_{0,\log p+\kappa_p} + \log t + \kappa,$$

then $\nu_2(N!/\prod \mathcal{B}) > n_{**}$, $\nu_3(N!/\prod \mathcal{B}) > m_{**}$, and furthermore

$$\frac{\log(3L)+\kappa_L}{\log t - \log(3L)} \leq \frac{(\nu_2(N!/\prod \mathcal{B}) - n_{**})\log 2}{(\nu_3(N!/\prod \mathcal{B}) - m_{**})\log 3} \leq \frac{\log t - \log(2L)}{\log(2L)+\kappa_L},$$

and furthermore suppose that

$$E_t(\mathcal{B}) + \sum_{3<p\leq\sqrt{t/L}} |\nu_p(N!/\prod \mathcal{B})|_{\frac{\kappa_L \log p}{\log\sqrt{t/L}}, \kappa_p} + \sum_{p>\sqrt{t/L}} |\nu_p(N!/\prod \mathcal{B})|_{\log p, \kappa_p}$$
$$\leq \delta N - \frac{3}{2}\log N - \kappa_L(\log\sqrt{12})\frac{N}{\log t}. \tag{4.4}$$

Then $t(N) \geq t$.

Proof. Consider all the primes with a positive deficit, that is to say the primes p with a multiplicity of $|\nu_p(N!/\prod \mathcal{B})|_{0,1}$. If p is one of these primes, we select an element of the tuple that contains p as a factor, and replace it with the least 3-smooth number $2^{n_p}3^{n_p}$ larger than p, thus increasing this element by a factor of at most $\exp(\kappa_p)$; meanwhile, $\nu_2(N!/\prod \mathcal{B})\log 2 + \nu_3(N!/\prod \mathcal{B})\log 3$ goes down by at most $\log p + \kappa p$. Performing this for all the primes in deficit, we can clear this deficit at the cost of raising the excess of $\mathcal{B}$ by at most $\sum_{p>3}\kappa_p$, and decreasing $\nu_2(N!/\prod \mathcal{B})$, $\nu_3(N!/\prod \mathcal{B})$ by some n, m with $n\log 2 + m\log 3 \leq \sum_p |\nu_p(N!/\prod \mathcal{B})|_{0,\log p+\kappa_p}$. The hypotheses of Proposition 4.3 are now satisfied, and we are done. □

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