

NOTES ON UPPER AND LOWER BOUNDING $t(N)$

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1. BASICS

Let $\mathcal{B} = \{b_1, \dots, b_M\}$ be a finite multiset of natural numbers (thus each natural number may appear in \mathcal{B} multiple times); the ordering of elements in the multiset will not be of relevance to us. The *cardinality* $|\mathcal{B}| = M$ of the multiset is the number of elements counting multiplicity; for example,

$$|\{2, 2, 3\}| = 3.$$

The *product* $\prod \mathcal{B}$ of the finite multiset is defined by $\prod \mathcal{B} := \prod_{b \in \mathcal{B}} b$, where we count for multiplicity; for example

$$\prod \{2, 2, 3\} = 12.$$

The tuple \mathcal{B} is a *factorization* of a natural number M if $\prod \mathcal{B} = M$, and a *subfactorization* if $\mathcal{B} | M$. For example, $\{2, 2, 3\}$ is a factorization of 12 and a subfactorization of 24.

We use $v_p(a/b) = v_p(a) - v_p(b)$ to denote the p -adic valuation of a positive natural number a/b , that is to say the number of times p divides the numerator a , minus the number of times p divides the denominator b . For instance, $v_2(32/27) = 5$ and $v_3(32/27) = -3$. By the fundamental theorem of arithmetic, we see that a finite multiset \mathcal{B} is a factorization of M if and only if

$$v_p(M / \prod \mathcal{B}) = 0$$

for all primes p , and a subfactorization if and only if

$$v_p(M / \prod \mathcal{B}) \geq 0$$

for all primes p . We refer to $v_p(M / \prod \mathcal{B})$ as the p -*surplus* of \mathcal{B} (as an attempted factorization) of M at prime p , and $-v_p(M / \prod \mathcal{B}) = v_p(\prod \mathcal{B} / M)$ as the p -*deficit*, and say that the factorization is p -balanced if $v_p(M / \prod \mathcal{B}) = 0$. Thus a subfactorization (resp. factorization) occurs when one has non-negative surpluses (resp. balance) at all primes p .

Example 1.1. Suppose one wishes to factorize $5! = 2^3 \times 3 \times 5$. The attempted factorization $\mathcal{B} := \{3, 4, 5, 5\}$ has a 2-surplus of $v_2(5! / \prod \mathcal{B}) = 1$, is in balance at 3, and has a 5-deficit of $v_5(\prod \mathcal{B} / 5!) = 1$, so it is not a factorization or subfactorization of $5!$. However, if one replaces one of the copies of 5 in \mathcal{B} with a 2, this erases both the 2-surplus and the 5-deficit, and creates a factorization $\{2, 3, 4, 5\}$ of $5!$.

If one applies a logarithm to the fundamental theorem of arithmetic, one obtains the identity

$$\sum_p v_p(r) \log p = \log r \tag{1.1}$$

for any positive rational r .

A finite multiset \mathcal{B} is said to be t -admissible for some $t > 0$ if $b \geq t$ for all $b \in \mathcal{B}$. We define $t(N)$ denotes the largest quantity such that there exists a $t(N)$ -admissible factorization of $N!$ of cardinality N . Clearly, $t(N)$ is also the largest quantity such that there exists a $t(N)$ -admissible subfactorization of $N!$ of cardinality at least N , since when starting from such a subfactorization, we may delete elements and then distribute any surpluses at any primes arbitrarily to create a factorization of cardinality exactly N .

Example 1.2. The finite multiset $\{2, 2, 3, 3, 4, 4, 5, 7\}$ is a 2-admissible subfactorization of $8! = 2^7 \times 3^2 \times 5 \times 7$, having a 2-surplus of 1. If one deletes a copy of 2 to make the cardinality exactly 8, one now has a surplus of 2 at 2; one can distribute these two powers of 2 to the remaining element of 2 to obtain a factorization $\{3, 3, 4, 4, 5, 7, 8\}$ that is still 2-admissible, and is in fact now 3-admissible.

A useful measure of the efficiency of a t -admissible finite multiset \mathcal{B} is the t -excess

$$E_t(\mathcal{B}) := \sum_{i=1}^{N'} \log \frac{b_i}{t} = \log \prod \mathcal{B} - |\mathcal{B}| \log t.$$

Example 1.3. The 3-excess of $\{3, 3, 4, 4, 5, 7, 8\}$ is

$$E_t(\{3, 3, 4, 4, 5, 7, 8\}) = 2 \log \frac{4}{3} + \log \frac{5}{3} + \log \frac{7}{3} + \log \frac{8}{3} = 2.914 \dots$$

The t -excess clearly non-negative when \mathcal{B} is t -admissible. Combining this with (1.1), we obtain the basic *balance identity*

$$E_t(\mathcal{B}) + \sum_p v_p(N! / \prod \mathcal{B}) \log p = \log N! - |\mathcal{B}| \log t. \quad (1.2)$$

In particular, when one has a subfactorization, the gap between $\log N!$ and $|\mathcal{B}| \log t$ must be somehow distributed between the t -excess $E_t(\mathcal{B})$ and the p -surpluses $v_p(N! / \prod \mathcal{B})$.

Example 1.4. The 3-admissible finite multiset $\{3, 3, 4, 4, 5, 7, 8\}$ is a factorization of $8!$ of cardinality 7, and the gap

$$\log 8! - 7 \log 3 = 2.914 \dots$$

is entirely absorbed by the 3-excess of the multiset. If one replaces the element 8 of this multiset with 4, this reduces the excess to

$$E_t(\{3, 3, 4, 4, 5, 7, 8\}) = 3 \log \frac{4}{3} + \log \frac{5}{3} + \log \frac{7}{3} = 2.221 \dots,$$

but creates a 2-surplus of 1 that contributes $\log 2 = 0.693 \dots$ to (1.2), restoring balance.

From (1.2), we have the following equivalent definition of $t(N)$:

Lemma 1.5 (Equivalent form of $t(N)$). $t(N)$ is the supremum of all t for which there exists a t -admissible subfactorization \mathcal{B} of $N!$ with

$$E_t(\mathcal{B}) + \sum_p v_p(N! / \mathcal{B}) \log p \leq \log N! - N \log t.$$

The advantage of this formulation is that one no longer needs to directly track the cardinality $|\mathcal{B}|$ of the t -admissible subfactorization \mathcal{B} . The formulation highlights the need to locate subfactorizations in which both the t -excess and the p -surpluses are kept as low as possible.

We recall Legendre's formula

$$\nu_p(N!) = \sum_{j=1}^{\infty} \left\lfloor \frac{N}{p^j} \right\rfloor = \frac{N - s_p(N)}{p - 1}. \quad (1.3)$$

Asymptotically, it is known that

$$\frac{1}{e} - \frac{O(1)}{\log N} \leq \frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$$

where

$$\begin{aligned} c_0 &:= \frac{1}{e} \int_0^1 \left\lfloor \frac{1}{x} \right\rfloor \log \left(ex \left\lceil \frac{1}{ex} \right\rceil \right) dx \\ &= \frac{1}{e} \int_1^{\infty} \lfloor y \rfloor \log \frac{[y/e]}{y/e} \frac{dy}{y^2} \\ &= 0.3044 \dots \end{aligned}$$

To bound the factorial, we have the explicit Stirling approximation [4]

$$N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N + 1} \leq \log N! \leq N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N}, \quad (1.4)$$

valid for all natural numbers N .

To estimate the prime counting function, we have the following good asymptotics up to a large height.

Theorem 1.6 (Buthe's bounds). [1] *For any $2 \leq x \leq 10^{19}$, we have*

$$\text{li}(x) - \frac{\sqrt{x}}{\log x} \left(1.95 + \frac{3.9}{\log x} + \frac{19.5}{\log^2 x} \right) \leq \pi(x) < \text{li}(x)$$

and

$$\text{li}(x) - \frac{\sqrt{x}}{\log x} \leq \pi^*(x) < \text{li}(x) + \frac{\sqrt{x}}{\log x}.$$

For $x > 10^{19}$ we have the bounds of Dusart [2]. One such bound is

$$|\psi(x) - x| \leq 59.18 \frac{x}{\log^4 x}.$$

2. CRITERIA FOR UPPER BOUNDING $t(N)$

We have the trivial upper bound $t(N) \leq (N!)^{1/N}$. This can be improved to $t(N) \leq N/e$ for $N \neq 1, 2, 4$, answering a conjecture of Guy and Selfridge [3]; see [5]. This was derived from the following slightly stronger criterion, which asymptotically gives $\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$:

Lemma 2.1 (Upper bound criterion). [5, Lemma 2.1] *Suppose that $1 \leq t \leq N$ are such that*

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor \log \left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil \right) > \log N! - N \log t \quad (2.1)$$

Then $t(N) < t$.

A surprisingly sharp upper bound comes from linear programming.

Lemma 2.2 (Linear programming bound). *Let N be an natural number and $1 \leq t \leq N/2$. Suppose for each prime $p \leq N$, one has a non-negative real number w_p which is weakly non-decreasing in p (thus $w_p \leq w_{p'}$ when $p \leq p'$), and such that*

$$\sum_p w_p v_p(j) \geq 1 \quad (2.2)$$

for all $t \leq j \leq N$, and such that

$$\sum_p w_p v_p(N!) < N. \quad (2.3)$$

Then $t(N) < t$.

Proof. We first observe that the bound (2.2) in fact holds for all $j \geq t$, not just for $t \leq j \leq N$. Indeed, if this were not the case, consider the first $j \geq t$ where (2.2) fails. Take a prime p dividing j and replace it by a prime in the interval $[p/2, p)$ which exists by Bertrand's postulate (or remove p entirely, if $p = 2$); this creates a new j' in $[j/2, j)$ which is still at least t . By the weakly decreasing hypothesis on w_p , we have

$$\sum_p w_p v_p(j) \geq \sum_p w_p v_p(j')$$

and hence by the minimality of j we have

$$\sum_p w_p v_p(j) > 1,$$

a contradiction.

Now suppose for contradiction that $t(N) \geq t$, thus we have a factorization $N! = \prod_{j \geq t} j^{m_j}$ for some natural numbers m_j summing to N . Taking p -valuations, we conclude that

$$\sum_{j \geq t} m_j v_p(j) \leq v_p(N!)$$

for all $p \leq N$. Multiplying by w_p and summing, we conclude from (2.2) that

$$N = \sum_{j \geq t} m_j \leq \sum_p w_p v_p(N!),$$

contradicting (2.3). □

3. POWERS OF 2 AND 3

We now begin the study of constructions that can establish lower bounds of the form $t(N) \geq t$ for some

$$1 \leq t \leq N. \quad (3.1)$$

It will be convenient to parameterize

$$t = \frac{N}{e^{1+\delta}} \quad (3.2)$$

where we shall assume

$$\delta > 0; \quad (3.3)$$

for instance, if $t = N/3$, we will have $\delta = \log \frac{3}{e} \approx 0.098$. We also a parameter $L \geq 1$ for which

$$9L \leq t \quad (3.4)$$

and divide the primes into three categories:

- The *tiny primes* $p = 2, 3$;
- The *small primes* $3 < p \leq \sqrt{t/L}$;
- The *large primes* $p > \sqrt{t/L}$.

For any $B \geq 1$, define a *B-smooth number* to be a number whose prime factors are all at most B . Here we will be primarily interested in the cases $B = 2, 3$.

A 2-smooth number is just a power of two; and for any $t \geq 1$, there exists a 2-smooth number 2^n in the interval $[t, 2t]$; indeed one can take $n = \lfloor \log t / \log 2 \rfloor$. For 3-smooth numbers $2^n 3^m$ - that is to say, products of tiny primes - one can do better. For any $L \geq 1$, let κ_L be the least quantity such that for any real number $t \geq L$, there exists a 3-smooth number $2^n 3^m$ such that

$$t \leq 2^n 3^m \leq \exp(\kappa_L)t.$$

Thus for instance $\kappa_1 = \log 2$ thanks to the aforementioned fact about 2-smooth numbers, and it is clear that κ_L is non-decreasing in L . We have the following explicit bounds that noticeably improve upon $\log 2 = 0.69314 \dots$:

Lemma 3.1. *If n_1, n_2, m_1, m_2 are positive integers such that*

$$1 \leq \frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}$$

then

$$\kappa_{\min(2^{n_1+n_2}, 3^{m_1+m_2})/6} \leq \log \max \left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}} \right).$$

Thus, for instance, setting $n_1 = 3$, $m_1 = 2$, $n_2 = 2$, $m_2 = 1$, we have

$$\kappa_{4.5} \leq \log \frac{2^2}{3} = 0.28768 \dots,$$

setting $n_1 = 3$, $m_1 = 2$, $n_2 = 5$, $m_2 = 3$, we have

$$\kappa_{40.5} \leq \log \frac{2^5}{3^3} = 0.16989 \dots$$

and setting $n_1 = 11$, $m_1 = 7$, $n_2 = 8$, $m_2 = 5$, we have

$$\kappa_{2^{18}/3} \leq \log \frac{3^7}{2^{11}} = 0.06566 \dots$$

($2^{18}/3 = 87381.33 \dots$).

Proof. If $\min(2^{n_1+n_2}, 3^{m_1+m_2})/6 \leq t \leq 2^{n_2-1}3^{m_1-1}$, then we have

$$t \leq 2^{n_2-1}3^{m_1-1} \leq \max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right)t, \quad (3.5)$$

so we are done in this case. Now suppose that $t > 2^{n_2-1}3^{m_1-1}$. Let $2^n 3^m$ be the smallest 3-smooth number that is at least t , then we must have $n \geq n_2$ or $m \geq m_1$ (or both). Thus at least one of $\frac{2^{n_1}}{3^{m_1}} 2^n 3^m$ and $\frac{3^{m_2}}{2^{n_2}} 2^n 3^m$ is an integer, and is thus at most t by construction. This gives (3.5), and the claim follows. \square

Asymptotically, we have logarithmic-type decay:

Lemma 3.2 (Baker bound). *We have*

$$\kappa_L \ll \log^{-c} L$$

for all $L \geq 2$ and some absolute constant $c > 0$.

Proof. From the classical theory of continued fractions, we can find rational approximants

$$\frac{p_{2j}}{q_{2j}} \leq \frac{\log 3}{\log 2} \leq \frac{p_{2j+1}}{q_{2j+1}} \quad (3.6)$$

to the irrational number $\log 3 / \log 2$, where the convergents p_j/q_j obey the recursions

$$p_j = b_j p_{j-1} + p_{j-2}; \quad q_j = b_j q_{j-1} + q_{j-2}$$

with $p_{-1} = 1, q_{-1} = -1, p_0 = b_0, q_0 = 1$, and $[b_0; b_1, b_2, \dots] = [1; 1, 1, 2, \dots]$ is the continued fraction expansion of $\frac{\log 3}{\log 2}$. Furthermore, $p_{2j+1}q_{2j} - p_{2j}q_{2j+1} = 1$, and hence

$$\frac{\log 3}{\log 2} - \frac{p_{2j}}{q_{2j}} = \frac{1}{q_{2j}q_{2j+1}}. \quad (3.7)$$

By Baker's theorem, $\frac{\log 3}{\log 2}$ is a Diophantine number, giving a bound of the form

$$q_{2j+1} \ll q_{2j}^{O(1)} \quad (3.8)$$

and a similar argument gives

$$q_{2j+2} \ll q_{2j+1}^{O(1)}. \quad (3.9)$$

We can rewrite (3.6) as

$$1 \leq \frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}}$$

and routine Taylor expansion using (3.14) gives the upper bounds

$$\frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}} \leq \exp \left(O \left(\frac{1}{q_{2j}} \right) \right).$$

If one then sets j to be the largest natural number for which (??) holds with t replaced by L , the claim then follows from (3.8), (3.9), and Lemma 3.1. \square

We can now obtain efficient t -admissible subfactorizations of $2^n 3^m$ when n, m are somewhat comparable.

Lemma 3.3. *Let $L \geq 1$. Let $t > 3L$ and let $2^n 3^m$ be a 3-smooth number obeying the conditions*

$$\frac{\log(3L) + \kappa}{\log t - \log(3L)} \leq \frac{n \log 2}{m \log 3} \leq \frac{\log t - \log(2L)}{\log(2L) + \kappa}. \quad (3.10)$$

Then one can find a t -admissible subfactorization B of $2^n 3^m$ such that

$$E_t(B) \leq \frac{\kappa_L}{\log t} (n \log 2 + m \log 3) \quad (3.11)$$

and

$$|v_2(2^n 3^m / B)|_{\log 2, \infty} + |v_3(2^n 3^m / B)|_{\log 3, \infty} \leq 2(\log t + \kappa_L). \quad (3.12)$$

Proof. Let $2^{n_0}, 3^{m_0}$ be the largest powers of 2 and 3 less than t/L respectively. By definition of κ_L , we can find 3-smooth numbers $2^{n_1} 3^{m_1}, 2^{n_2} 3^{m_2}$ such that

$$\frac{t}{2^{n_0}} \leq 2^{n_1} 3^{m_1} \leq e^{\kappa} \frac{t}{2^{n_0}} \quad (3.13)$$

and

$$\frac{t}{3^{m_0}} \leq 2^{n_2} 3^{m_2} \leq e^{\kappa} \frac{t}{3^{m_0}}, \quad (3.14)$$

or equivalently

$$t \leq 2^{n_0+n_1} 3^{m_1}, 2^{n_2} 3^{m_0+m_2} \leq e^{\kappa} t. \quad (3.15)$$

We can bound

$$\begin{aligned} \frac{n_0 + n_1}{m_1} &\geq \frac{n_0}{\log(e^{\kappa} \frac{t}{2^{n_0}}) / \log 3} \\ &\geq \frac{(\log t - \log(2L)) / \log 2}{(\log(3L) + \kappa) / \log 3} \end{aligned}$$

(with the convention that this bound is vacuously true for $m_1 = 0$) and similarly

$$\begin{aligned} \frac{n_2}{m_0 + m_2} &\leq \frac{\log(e^{\kappa} \frac{t}{3^{m_0}}) / \log 2}{m_0} \\ &\leq \frac{(\log(2L) + \kappa) / \log 2}{(\log t - \log(3L)) / \log 3} \end{aligned}$$

and hence by (3.10)

$$\frac{n_2}{m_0 + m_2} \leq \frac{n}{m} \leq \frac{n_0 + n_1}{m_1}. \quad (3.16)$$

Thus we can write (n, m) as a non-negative linear combination

$$(n, m) = \alpha_1(n_0 + n_1, m_1) + \alpha_2(n_2, m_0 + m_2)$$

for some real $\alpha_1, \alpha_2 \geq 0$. We now take our subfactorization \mathcal{B} to consist of $\lfloor \alpha_1 \rfloor$ copies of the 3-smooth number $2^{n_0+n_1}3^{m_1}$ and $\lfloor \alpha_2 \rfloor$ copies of the 3-smooth number $2^{n_2}3^{m_0+m_2}$. By (3.15), each term $2^{n'}3^{m'}$ here is admissible and contributes an excess of at most κ , which is in turn bounded by $\frac{\kappa}{\log t}(n' \log 2 + m' \log 3)$. Adding these bounds together, we obtain (3.11).

The expression $2^{n_0}3^{m_0}/\prod \mathcal{B}$ contains at most $n_0 + n_1 + n_2$ factors of 2 and at most $m_0 + m_2 + m_1$ factors of 3, hence

$$v_2(2^{n_0}3^{m_0}/\prod \mathcal{B}) \log 2 + v_3(2^{n_0}3^{m_0}/\prod \mathcal{B}) \log 3 \leq \log 2^{n_0+n_1}3^{m_1} + \log 2^{n_2}3^{m_0+m_2},$$

and the bound (3.12) follows. \square

4. CRITERIA FOR LOWER BOUNDING $t(N)$

Lemma 1.5 gives an initial criterion for lower bounding $t(N)$. We now perform various manipulations on tuples to replace this criterion with a more tractable one. For $a_+, a_- \in [0, +\infty]$, we define the asymmetric norm $|x|_{a_+, a_-}$ of a real number x by the formula

$$|x|_{a_+, a_-} := \max(a_+x, -a_-x),$$

thus this is $a_+|x|$ when x is positive and $a_-|x|$ when x is negative. If a_+, a_- are finite, this function is Lipschitz with constant $\max(a_+, a_-)$. One can think of a_+ as the “cost” of making x positive, and a_- as the “cost” of making x negative. One can then reformulate Lemma 1.5 as follows.

Proposition 4.1 (Reformulated balance criterion). *Let $1 \leq t \leq N$, and suppose that one has a t -admissible finite multiset \mathcal{B} obeying the following hypothesis:*

(i) *(Small excess and surplus at all primes)*

$$E_t(\mathcal{B}) + \sum_p |v_p(N!/\prod \mathcal{B})|_{\log p, \infty} \leq \log N! - N \log t. \quad (4.1)$$

Then $t(N) \geq t$.

Indeed, the infinite penalty for making $v_p(N!/\mathcal{B})$ in (4.1) ensures that \mathcal{B} is a subfactorization of $N!$.

We will reduce this infinite penalty term later, but let us work on other aspects of the criterion (4.1) first. In practice we will apply this criterion with $t := N/e^{1+\delta}$ for some $\delta > 0$; for instance, if we wish to set $t = N/3$, then $\delta = \log \frac{e}{3} \approx 0.098$. From (1.4) we may then replace $\log N! - N \log t = \log N! - N \log N + N + \delta N$ by the slightly smaller quantity

$$\delta N + \log \sqrt{2\pi N}.$$

The $\log \sqrt{2\pi N}$ is a lower order term, and we shall use it only to clean up some other lower order terms.

Using Lemma 3.3, we can leave a large surplus at tiny primes and still get good bounds:

Proposition 4.2 (Criterion with tiny-prime surplus). *Let $L \geq 1$. Let $3L < t = N/e^{1+\delta}$ for some $\delta > 0$, and suppose that one has a t -admissible finite multiset \mathcal{B} obeying the following hypotheses:*

(i) *(Small excess and surplus at non-tiny primes)*

$$E_t(\mathcal{B}) + \sum_{p>3} |v_p(N!/\prod \mathcal{B})|_{\log p, \infty} \leq \delta N + \kappa_L - \frac{3}{2} \log N - \kappa_L (\log \sqrt{12}) \frac{N}{\log t}. \quad (4.2)$$

(ii) *(Large surpluses at tiny primes) The surpluses $v_2(N!/\prod \mathcal{B})$, $v_3(N!/\prod \mathcal{B})$ are positive and obey the bounds*

$$\frac{\log(3L) + \kappa_L}{\log t - \log(3L)} \leq \frac{v_2(N!/\prod \mathcal{B}) \log 2}{v_3(N!/\prod \mathcal{B}) \log 3} \leq \frac{\log t - \log(2L)}{\log(2L) + \kappa_L}.$$

Then $t(N) \geq t$.

Proof. Write $n := v_2(N!/\prod \mathcal{B})$ and $m := v_3(N!/\prod \mathcal{B})$. From (1.3) we have $n \leq N$ and $m \leq N/2$, hence

$$n \log 2 + m \log 3 \leq N \log \sqrt{12}.$$

Applying Lemma 3.3, we can find a subfactorization \mathcal{B}' of $2^n 3^m$ with an excess of at most $(\kappa_L \log \sqrt{12}) \frac{N}{\log t}$, and with

$$|v_2(2^n 3^m / \prod \mathcal{B}')|_{\log 2, \infty} + |v_3(2^n 3^m / \prod \mathcal{B}')|_{\log 3, \infty} \leq 2(\log t + \kappa_L) \leq 2 \log N - 2 + 2\kappa_L.$$

If we let \mathcal{B}'' be the concatenation of \mathcal{B} and \mathcal{B}' , then \mathcal{B}'' is another t -admissible tuple, and from (4.2) and the observation that $-2 + 3\kappa_L \leq \log \sqrt{2\pi}$, we see that

$$E_t(\mathcal{B}'') + \sum_p |v_p(N!/\prod \mathcal{B}'')|_{\log p, \infty} \leq \delta N + \log \sqrt{2\pi N},$$

and the claim now follows from Proposition 4.1. \square

The criterion (4.2) will still be somewhat expensive at small primes $3 < p \leq \sqrt{t/L}$. We can improve the situation as follows.

Proposition 4.3 (Improved criterion with tiny-prime surplus). *Let $L \geq 1$. Let $9L < t = N/e^{1+\delta}$ for some $\delta > 0$, and suppose that one has a t -admissible finite multiset \mathcal{B} obeying the following hypotheses.*

(i) *(Small excess and surplus at non-tiny primes)*

$$\begin{aligned} E_t(\mathcal{B}) + \sum_{3 < p \leq \sqrt{t/L}} |v_p(N!/\prod \mathcal{B})|_{\frac{\kappa_L \log p}{\log \sqrt{t/L}}, \infty} + \sum_{p > \sqrt{t/L}} |v_p(N!/\prod \mathcal{B})|_{\log p, \infty} \\ \leq \delta N - \frac{3}{2} \log N - \kappa_L (\log \sqrt{12}) \frac{N}{\log t}. \end{aligned} \quad (4.3)$$

(ii) (*Large surpluses at tiny primes*) Whenever n_{**}, m_{**} are natural numbers obeying the bounds

$$n_{**} \log 2 + m_{**} \log 3 \leq \sum_{3 < p \leq \sqrt{t/L}} |v_p(N! / \prod B)|_{\frac{\log \sqrt{tL} + \kappa}{\log \sqrt{t/L}} \log p, \infty} + \log t + \kappa,$$

then $v_2(N! / \prod B) > n_{**}$, $v_3(N! / \prod B) > m_{**}$, and furthermore

$$\frac{\log(3L) + \kappa_L}{\log t - \log(3L)} \leq \frac{(v_2(N! / \prod B) - n_{**}) \log 2}{(v_3(N! / \prod B) - m_{**}) \log 3} \leq \frac{\log t - \log(2L)}{\log(2L) + \kappa_L}.$$

Then $t(N) \geq t$.

Proof. By (4.3), B is a subfactorization of $N!$. Consider all the p -surplus primes in the range $3 < p \leq \sqrt{t/L}$, thus each such prime is considered with multiplicity $v_p(N! / \prod B)$. Denoting their product as B , we have

$$\log B = \sum_{3 < p \leq \sqrt{t/L}} |v_p(N! / \prod B)|_{\log p, \infty}.$$

Using the greedy algorithm, one can factor B into M factors c_1, \dots, c_M in the interval $[\sqrt{t/L}, \frac{t/L}{\sqrt{t/L}}]$, times one exceptional factor c_* in $[1, \sqrt{t/L}]$, for some M . We have the bound

$$(\sqrt{t/L})^M \leq B$$

and hence

$$M \leq \sum_{3 < p \leq \sqrt{t/L}} |v_p(N! / \prod B)|_{\frac{\log p}{\log \sqrt{t/L}}, \infty}.$$

For each of the M factors c_i , we may use the definition of κ_L and find 3-smooth number $2^{n_i} 3^{m_i}$ in the interval $[t/c_i, e^{\kappa_L} t/c_i]$, and similarly for the exceptional factor c_* we may find a 3-smooth number $2^{n_*} 3^{m_*}$ in the interval $[t/c_*, e^{\kappa_L} t/c_*]$. If we then adjoin the 3-smooth numbers $2^{n_i} 3^{m_i} c_i$ for $i = 1, \dots, M$ as well as $2^{n_*} 3^{m_*} c_*$ to the tuple B to create a new tuple B' . This tuple is still t -admissible, but now has no p -surplus (or p -deficit) at any prime $3 < p \leq \sqrt{t/L}$. The quantity $n_i \log 2 + m_i \log 3$ is bounded by $\log \sqrt{tL} + \kappa_L$, and the quantity $n \log 2 + m \log 3$ is similarly bounded by $\log t + \kappa$, hence if we denote $n_{**} := n_1 + \dots + n_M + n_*$ and $m_{**} := m_1 + \dots + m_M + m_*$, we have

$$n_{**} \log 2 + m_{**} \log 3 \leq \frac{\log \sqrt{tL} + \kappa_L}{\log \sqrt{t/L}} \sum_{3 < p \leq \sqrt{t/L}} |v_p(N! / \prod B)|_{\log p, \infty} + \log t + \kappa.$$

By hypothesis, we now see that B' has no p -deficit at 2 or 3 either, so B' is still a subfactorization of $N!$. Each of the new factors in B' contributes an excess of at most κ_L , so the total excess of B' is at most

$$E_t(B) + \kappa_L M + \kappa_L$$

which is in turn bounded by

$$E_t(B) + \sum_{3 < p \leq \sqrt{t/L}} |v_p(N! / \prod B)|_{\frac{\kappa_L \log p}{\log \sqrt{t/L}}, \infty} + \kappa_L.$$

We conclude that B' obeys the hypotheses of Equation (4.2), and the claim follows. \square

Finally, we relax the subfactorization condition by permitting some p -deficit at various non-tiny primes $p > 3$.

Proposition 4.4 (Improved criterion with tiny-prime surplus, and some deficit). *Let $L \geq 1$. Let $9L < t = N/e^{1+\delta}$ for some $\delta > 0$, and suppose that one has a t -admissible finite multiset \mathcal{B} obeying the following hypotheses:*

(i) (Small excess and surplus at small and large primes) *One has*

$$\begin{aligned} E_t(\mathcal{B}) + \sum_{3 < p \leq \sqrt{t/L}} |v_p(N!/\prod \mathcal{B})|_{\frac{\kappa_L \log p}{\log \sqrt{t/L}}, \kappa_p} + \sum_{p > \sqrt{t/L}} |v_p(N!/\prod \mathcal{B})|_{\log p, \kappa_p} \\ \leq \delta N - \frac{3}{2} \log N - \kappa_L (\log \sqrt{12}) \frac{N}{\log t}. \end{aligned} \quad (4.4)$$

(ii) (Large surpluses at tiny primes) *Whenever n_{**}, m_{**} are natural numbers obeying the bounds*

$$\begin{aligned} n_{**} \log 2 + m_{**} \log 3 \leq \sum_{3 < p \leq \sqrt{t/L}} |v_p(N!/\prod \mathcal{B})|_{\frac{\log \sqrt{t/L} + \kappa_L}{\log \sqrt{t/L}} \log p, \log p + \kappa_p} \\ + \sum_{p > \sqrt{t/L}} |v_p(N!/\prod \mathcal{B})|_{0, \log p + \kappa_p} + \log t + \kappa, \end{aligned}$$

*then $v_2(N!/\prod \mathcal{B}) > n_{**}$, $v_3(N!/\prod \mathcal{B}) > m_{**}$, and furthermore*

$$\frac{\log(3L) + \kappa_L}{\log t - \log(3L)} \leq \frac{(v_2(N!/\prod \mathcal{B}) - n_{**}) \log 2}{(v_3(N!/\prod \mathcal{B}) - m_{**}) \log 3} \leq \frac{\log t - \log(2L)}{\log(2L) + \kappa_L}.$$

Then $t(N) \geq t$.

Proof. Consider all the primes with a positive deficit, that is to say the primes p with a multiplicity of $|v_p(N!/\prod \mathcal{B})|_{0,1}$. If p is one of these primes, we select an element of the tuple that contains p as a factor, and replace it with the least 3-smooth number $2^{n_p} 3^{m_p}$ larger than p , thus increasing this element by a factor of at most $\exp(\kappa_p)$; meanwhile, $v_2(N!/\prod \mathcal{B}) \log 2 + v_3(N!/\prod \mathcal{B}) \log 3$ goes down by at most $\log p + \kappa_p$. Performing this for all the primes in deficit, we can clear this deficit at the cost of raising the excess of \mathcal{B} by at most $\sum_{p > 3} \kappa_p$, and decreasing $v_2(N!/\prod \mathcal{B})$, $v_3(N!/\prod \mathcal{B})$ by some n, m with $n \log 2 + m \log 3 \leq \sum_p |v_p(N!/\prod \mathcal{B})|_{0, \log p + \kappa_p}$. The hypotheses of Proposition 4.3 are now satisfied, and we are done. \square

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