NOTES ON UPPER AND LOWER BOUNDING $t(N)$

TERENCE TAO

1. BASICS

The symbol p will always denote a prime. The primes $2, 3$ will play a special role here and will be referred to as *tiny primes*.

We use $v_p(a/b) = v_p(a) - v_p(b)$ to denote the p-adic valuation of a positive natural number a/b, that is to say the number of times p divides the numerator a, minus the number of times p divides the denominator b. For instance, $v_2(32/27) = 5$ and $v_3(32/27) = -3$. If one applies a logarithm to the fundamental theorem of arithmetic, one obtains the identity

$$\sum_p v_p(r) \log p = \log r \tag{1.1}$$

for any positive rational r.

Asymptotically, it is known that

$$\frac{1}{e} - \frac{O(1)}{\log N} \leq \frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$$

where

$$\begin{aligned} c_0 &:= \frac{1}{e} \int_0^1 \left\lfloor \frac{1}{x} \right\rfloor \log \left(ex \left\lceil \frac{1}{ex} \right\rceil \right) \, dx \\ &= \frac{1}{e} \int_1^\infty \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e} \, \frac{dy}{y^2} \\ &= 0.3044190 \dots \end{aligned} \tag{1.2}$$

We recall Legendre's formula

$$v_p(N!) = \sum_{j=1}^\infty \left\lfloor \frac{N}{p^j} \right\rfloor = \frac{N - s_p(N)}{p-1}. \tag{1.3}$$

To bound the factorial, we have the explicit Stirling approximation [6]

$$N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N+1} \leq \log N! \leq N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N}, \tag{1.4}$$

valid for all natural numbers N.

In addition to the usual asymptotic notation, we use $O_{\leq}(X)$ to denote any quantity whose magnitude is bounded by at most X (note the absence of an additional constant factor).

Lemma 1.1 (Integration by parts)**.** *Let* $(y, x]$ *be a half-open interval in* $(0, +\infty)$*. Suppose that one has a function* $a\colon \mathbb{N} \to \mathbb{R}$ *and a continuous function* $f : (y, x] \to \mathbb{R}$ *such that*

$$\sum_{y<n\leq z} a_n = \int_z^y f(t)\, dt + C + O_{\leq}(A)$$

for all $y \leq z \leq x$*, and some* $C \in \mathbb{R}$*,* $A > 0$*. Then, for any function* $b : (y, x] \to \mathbb{R}$ *of bounded total variation, one has*

$$\sum_{y<n\leq x} b(n)a_n = \int_x^y b(t)f(t)\, du + O_{\leq}(A(|b(y^+)| + |b(x)| + \|b\|_{\mathrm{TV}})) \tag{1.5}$$

Proof. If, for every natural number $y < n \leq x$, one modifies b to be equal to the constant $b(n)$ in a small neighborhood of n, then one does not affect the left-hand side of (1.5) or increase the total variation of b, while only modifying the integral in (1.5) by an arbitrarily small amount. Hence, by the usual limiting argument, we may assume without loss of generality that b is locally constant at each such n. If we define the function $g\colon (y, x] \to \mathbb{R}$ by

$$g(z) := \sum_{y<n\leq z} a_n - \int_z^y f(u)\, du - C$$

then g has jump discontinuities at the natural numbers, but is otherwise continuously differentiable, and is also bounded uniformly in magnitude by A. We can then compute the Riemann–Stieltjes integral

$$\int_{(y,x]} b\, dg = \sum_{y<n\leq x} b(n)a_n - \int_y^x f(t)b(t)\, dt.$$

Since the discontinuities of g and b do not coincide, we may integrate by parts to obtain

$$\int_{(y,x]} b\, dg = b(x)g(x) - b(y^+)g(y^+) - \int_{(y,x]} g db.$$

The left-hand side is $O_{\leq}(A(|b(y^+)| + |b(x)| + \|b\|_{\mathrm{TV}}))$, and the claim follows. □

We will use this lemma to control primes. One form of the prime number theorem with classical error term is that

$$\pi(x) := \sum_{p\leq x} 1 = \int_2^x \frac{dt}{\log t} + O(\frac{x}{\log^{10} x})$$

(say).

In [2, Theorem 2] the bound

$$\psi(x) = x + O_{\leq}(0.94\sqrt{x})$$

was demonstrated for $11 \leq x \leq 10^{19}$. For $10^{19} \leq x \leq 10^{21} \approx e^{48.35}$, the slightly weaker bounds

$$\psi(x) = x + O_{\leq}(\frac{\sqrt{x}}{8\pi}\log x(\log x - 3))$$

were obtained in [1, (7.3)]. On the other hand, in [1, Theorem 1, Table 1] the bound

$$\psi(x) = x + O_{\leq}(1.11742 \times 10^{-8} x)$$

was shown for $x \geq e^{45} \approx 10^{19.54}$. We may simply combine the latter two bounds to conclude that

$$\psi(x) = x + O_{\leq}(\frac{\sqrt{x}}{8\pi} \log x(\log x - 3) + 1.11742 \times 10^{-8} x)$$

whenever $x \geq 10^{19}$.

2. Criteria for upper bounding $t(N)$

We have the trivial upper bound $t(N) \leq (N!)^{1/N}$. This can be improved to $t(N) \leq N/e$ for $N \neq 1, 2, 4$, answering a conjecture of Guy and Selfridge [5]; see [7]. This was derived from the following slightly stronger criterion, which asymptotically gives $\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0+o(1)}{\log N}$:

Lemma 2.1 (Upper bound criterion)**.** [7, Lemma 2.1] *Suppose that* $1 \leq t \leq N$ *are such that*

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor \log \left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil \right) > \log N! - N \log t \tag{2.1}$$

Then $t(N) < t$.

A surprisingly sharp upper bound comes from linear programming.

Lemma 2.2 (Linear programming bound)**.** *Let* N *be an natural number and* $1 \leq t \leq N/2$. *Suppose for each prime* $p \leq N$, *one has a non-negative real number* w_p *which is weakly non-decreasing in* p *(thus* $w_p \leq w_{p'}$ *when* $p \leq p'$*), and such that*

$$\sum_p w_p \nu_p(j) \geq 1 \tag{2.2}$$

for all $t \leq j \leq N$, *and such that*

$$\sum_p w_p \nu_p(N!) < N. \tag{2.3}$$

Then $t(N) < t$.

Proof. We first observe that the bound (2.2) in fact holds for all $j \geq t$, not just for $t \leq j \leq N$. Indeed, if this were not the case, consider the first $j \geq t$ where (2.2) fails. Take a prime p dividing j and replace it by a prime in the interval $[p/2, p)$ which exists by Bertrand's postulate (or remove p entirely, if $p = 2$); this creates a new j' in $[j/2, j)$ which is still at least t. By the weakly decerasing hypothesis on w_p, we have

$$\sum_p w_p \nu_p(j) \geq \sum_p w_p \nu_p(j')$$

and hence by the minimality of j we have

$$\sum_p w_p \nu_p(j) > 1,$$

a contradiction.

Now suppose for contradiction that $t(N) \geq t$, thus we have a factorization $N! = \prod_{j \geq t} j^{m_j}$ for some natural numbers m_j summing to N. Taking p-valuations, we conclude that

$$\sum_{j \geq t} m_j \nu_p(j) \leq \nu_p(N!)$$

for all $p \leq N$. Multiplying by w_p and summing, we conclude from (2.2) that

$$N = \sum_{j \geq t} m_j \leq \sum_p w_p \nu_p(N!),$$

contradicting (2.3). □

This bound is sharp for all $N \leq 600$, with the exception of $N = 155$, where it gives the upper bound $t(155) \leq 46$. A more precise integer program gives $t(155) = 45$.

Remark 2.3. A variant of the linear programming method also gives good lower bound constructions. Specifically, one can use linear programming to find non-negative real numbers m_j for $t \leq j \leq N$ that maximize the quantity $\sum_{t \leq j \leq N} m_j$ subject to the constraints

$$\sum_{t \leq j \leq N} m_j \nu_p(j) \leq \nu_p(N!).$$

The expression $\prod_{t \leq j \leq N} j^{\lfloor m_j \rfloor}$ will then be a subfactorization of $N!$ into $\sum_{t \leq j \leq N} \lfloor m_j \rfloor$ factors j, each of which is at least t. If $\sum_{t \leq j \leq N} \lfloor m_j \rfloor \geq N$, this demonstrates that $t(N) \geq t$. Numerically, this procedure attains the exact value of $t(N)$ for all $N \leq 600$; for instance for $N = 155$, it shows that $t(155) \geq 45$.

2.1. **Asymptotic analysis of upper bound.**

We refine the upper bound in [7] slightly.

Proposition 2.4. *For large N, one has*

$$\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0}{\log N} + O\left(\frac{1}{\log^2 N}\right).$$

Proof. We apply Lemma 2.1 with

$$t := \frac{1}{e} - \frac{c_0}{\log N} + \frac{C_0}{\log^2 N}$$

with C_0 a large absolute constant to be chosen later. From the Stirling approximation one sees that

$$\log N! - N \log t \geq e c_0 \frac{N}{\log N} + (C_0 - O(1)) \frac{N}{\log^2 N}$$

so it will suffice to establish the upper bound

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor \log\left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil\right) \leq e c_0 \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right).$$

For N large enough, we have $\frac{t}{\lfloor\sqrt{t}\rfloor} \leq \frac{N}{\log N}$, so it suffices to show that

$$\sum_{\frac{N}{\log N} \leq p \leq N} \left\lfloor \frac{N}{p} \right\rfloor \log\left(\frac{p}{t}\left\lceil\frac{t}{p}\right\rceil\right) \leq ec_0\frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right).$$

The summand is a piecewise monotone function of p, with $O(\log N)$ pieces, and bounded in size by $O(N)$. A routine application of the prime number theorem (with classical error term) and summation by parts then allows one to express the left-hand side as

$$\int_{N/\log N}^{N} \left\lfloor\frac{N}{x}\right\rfloor \log\left(\frac{x}{t}\left\lceil\frac{t}{x}\right\rceil\right)\frac{dx}{\log x} + O\left(\frac{N}{\log^2 N}\right)$$

(in fact the error term can be made much stronger than this). We use the approximation

$$\frac{1}{\log x} = \frac{1}{\log N} + O\left(\frac{\log(N/x)}{\log^2 N}\right).$$

To control the error term, we observe from Taylor expansion that

$$\log\left(\frac{x}{t}\left\lceil\frac{t}{x}\right\rceil\right) \ll \frac{\lceil\frac{t}{x}\rceil - \frac{t}{x}}{t/x} \ll \frac{x}{t} \ll \frac{x}{N} \tag{2.4}$$

and the contribution of the error term is

$$\ll \int_{N/\log N}^{N} \frac{N}{x}\frac{x}{N}\frac{\log(N/x)}{\log^2 N}\, dx \ll \frac{N}{\log^2 N}$$

which is acceptable. As for the main term, we can rescale it to

$$\frac{et}{\log N}\int_{N/et\log N}^{N/et} \left\lfloor\frac{N/et}{x}\right\rfloor \log\left(ex\left\lceil\frac{1}{ex}\right\rceil\right)dx.$$

Since $N/et = 1 + O(1/\log N)$, we see that the integrand here is within $O(1/\log N)$ of $\lfloor\frac{1}{x}\rfloor \log\left(ex\lceil\frac{1}{ex}\rceil\right)$ unless $\frac{1}{x}$ is within $O(1/\log N)$ of an integer, which one can calculate to occur on a set of measure zero. A variant of (2.4) shows that both integrandd are bounded by $O(1)$ for all $x \in [0, N/et]$, so by the triangle inequality the above expression can be rewritten as

$$\frac{N}{\log N}\int_0^1 \left\lfloor\frac{1}{x}\right\rfloor \log\left(ex\left\lceil\frac{1}{ex}\right\rceil\right)dx + O\left(\frac{N}{\log^2 N}\right),$$

and the claim follows from (1.2). □

3. A general factorization algorithm

In this section we present and then analyze an algorithm that, when given parameters $1 \leq t \leq N$, will attempt to construct a factorization $N! = \prod \mathcal{B}$ of $N!$ by a finite multiset $\mathcal{B}$ of N elements that are all at least t. The algorithm will not always succeed, but when it does, it will certify that $t(N) \geq t$.

3.1. **Notational preliminaries.** We begin with some key definitions.

Let $\mathcal{B} = \{b_1, \dots, b_M\}$ be a finite multiset of natural numbers (thus each natural number may appear in $\mathcal{B}$ multiple times); the ordering of elements in the multiset will not be of relevance to us. The *cardinality* $|\mathcal{B}| = M$ of the multiset is the number of elements counting multiplicity; for example,

$$|\{2, 2, 3\}| = 3.$$

The *product* $\prod \mathcal{B}$ of the finite multiset is defined by $\prod \mathcal{B} := \prod_{b \in \mathcal{B}} b$, where we count for multiplicity; for example

$$\prod \{2, 2, 3\} = 12.$$

The tuple $\mathcal{B}$ is a *factorization* of a natural number M if $\mathcal{B} = M$, and a *subfactorization* if $\mathcal{B}|M$. For example, $\{2, 2, 3\}$ is a factorization of 12 and a subfactorization of 24.

By the fundamental theorem of arithmetic (or (1.1)), we see that a finite multiset $\mathcal{B}$ is a factorization of M if and only if

$$\nu_p(M/\prod \mathcal{B}) = 0$$

for all primes p, and a subfactorization if and only if

$$\nu_p(M/\prod \mathcal{B}) \geq 0$$

for all primes p. We refer to $\nu_p(M/\prod \mathcal{B})$ as the *p-surplus* of $\mathcal{B}$ (as an attempted factorization of M) at prime p, and $-\nu_p(M/\prod \mathcal{B}) = \nu_p(\prod \mathcal{B}/M)$ as the *p-deficit*, and say that the factorization is *p-balanced* if $\nu_p(M/\prod \mathcal{B}) = 0$. Thus a subfactorization (resp. factorization) occurs when one has non-negative surpluses (resp. balance) at all primes p.

Example 3.1. Suppose one wishes to factorize $5! = 2^3 \times 3 \times 5$. The attempted factorization $\mathcal{B} := \{3, 4, 5, 5\}$ has a 2-surplus of $\nu_2(5!/\prod \mathcal{B}) = 1$, is in balance at 3, and has a 5-deficit of $\nu_2(\prod \mathcal{B}/5!) = 1$, so it is not a factorization or subfactorization of $5!$. However, if one replaces one of the copies of 5 in $\mathcal{B}$ with a 2, this erases both the 2-surplus and the 5-deficit, and creates a factorization $\{2, 3, 4, 5\}$ of $5!$.

A finite multiset $\mathcal{B}$ is said to be *t-admissible* for some $t > 0$ if $b \geq t$ for all $b \in \mathcal{B}$. Then $t(N)$ is largest quantity such that there exists a $t(N)$-admissible factorization of $N!$ of cardinality N.

Call a natural number 3-*smooth* if it is of the form $2^n 3^m$ for some natural numbers n, m. Given a positive real number x, we use $\lceil x \rceil^{\langle 2,3 \rangle}$ to denote the smallest 3-smooth number greater than or equal to x. For instance, $\lceil 5 \rceil^{\langle 2,3 \rangle} = 6$ and $\lceil 10 \rceil^{\langle 2,3 \rangle} = 12$.

3.2. **Description of algorithm.** We now describe an algorithm that, for given $1 \leq t \leq N$, either successfully demonstrates that $t(N) \geq t$, or halts with an error.

(0) Select natural numbers A, K such that $K^2(1 + \frac{3N/t}{A}) < t$. There is some freedom to select parameters here, but roughly speaking one would like to have $1 \lll A \lll K \lll \sqrt{t}$.

(1) Let I denote the elements of the interval[1] $(t, t+3N/A]$ that are coprime to 6. Let $\mathcal{B}^{(1)}$ be the elements of I, each occurring with multiplicity A. This multiset is t-admissible, and $\prod \mathcal{B}^{(1)}$ is not divisible by tiny primes $2, 3$. (It will have approximately the right number of primes for $3 < p \leq t/K$, though it may have quite different prime factorization at primes $p > t/K$.)
(2) Remove any element from $\mathcal{B}^{(1)}$ that contains a prime factor p with $p > t/K$, and call this new multiset $\mathcal{B}^{(2)}$. It remains t-admissible with no tiny prime factors, though it tends to acquire a p-surplus in the range $3 < p \leq K$.
(3) For each $p > t/K$, add in $v_p(N!)$ copies of the number $p\lceil t/p\rceil$ to $\mathcal{B}^{(2)}$, and call this new multiset $\mathcal{B}^{(3)}$. Now $\mathcal{B}^{(3)}$ is t-admissible and in balance at all primes $p > t/K$, but will typically be in a slight deficit at primes $3 < p \leq t/K$, particularly in the range $3 < p \leq K$. (It will now also contain a few tiny prime factors, but will generally still have a large surplus at those primes.)
(4) For each prime $3 < p \leq t/K$ at which there is a surplus $v_p(N!/\prod \mathcal{B}) > 0$, replace $v_p(N!/\prod \mathcal{B})$ copies of p in the prime factorizations of elements of $\mathcal{B}^{(3)}$ with $\lceil p\rceil^{\langle 2,3\rangle}$ instead, and call this new multiset $\mathcal{B}^{(4)}$. Thus $\mathcal{B}^{(4)}$ has no surplus at primes $3 < p \leq t/K$ (and is still t-admissible and in balance for $p > t/K$).
(5) For the primes $3 < p \leq t/K$ at which there is a deficit $v_p(\prod \mathcal{B}/N!) > 0$, multiply all these primes together, and use the greedy algorithm to group them into factors $x_1, \dots, x_M$ in the range $(\sqrt{t/K}, t/K]$, together with possibly one exceptional factor x_* in the range $(1, t/K]$. For each of these factors x_i or x_*, add the quantity $x_i\lceil t/x_i\rceil^{\langle 2,3\rangle}$ or $x_*\lceil t/x_*\rceil^{\langle 2,3\rangle}$ to $\mathcal{B}^{(4)}$, and call this new multiset $\mathcal{B}^{(5)}$.
(6) By construction, $\mathcal{B}^{(5)}$ is t-admissible and will be in balance at all primes $p > 3$, and is thus $N!/\prod \mathcal{B}^{(5)}$ is of the form $2^n 3^m$ for some integers n, m. If at least one of n, m is negative, then `HALT` the algorithm with an error. Otherwise, select a 3-smooth number $2^{n_1}3^{m_1}$ greater than equal to t with $n_1/m_1 \leq n/m$ (which one can interpret as $n_1 m \leq n m_1$ in case some of the denominators here vanish), and similarly select a 3-smooth number $2^{n_2}3^{m_2}$ greater than or equal to t with $n_2/m_2 \geq n/m$. (It is reasonable to select the smallest such 3-smooth numbers in both cases, although this is not absolutely necessary for the algorithm to be successful.) By construction, we can express (n, m) as a positive linear combination $\alpha_1(n_1, m_1) + \alpha_2(n_2, m_2)$ of (n_1, m_1) and (n_2, m_2). Add $\lfloor \alpha_1 \rfloor$ copies of $2^{n_1}3^{m_1}$ and $\lfloor \alpha_2 \rfloor$ copies of $2^{n_2}3^{m_2}$ to $\mathcal{B}^{(5)}$, and call this tuple $\mathcal{B}^{(6)}$. (This will largely eliminate the surplus at 2 and 3.)
(7) If the multiset $\mathcal{B}^{(6)}$ has cardinality less than N, `HALT` the algorithm with an error. Otherwise, delete elements from $\mathcal{B}^{(6)}$ to bring the cardinality to N, and arbitrarily distribute any surplus primes to one of the remaining elements, and call the resulting multiset $\mathcal{B}^{(7)}$. By construction, $\mathcal{B}^{(7)}$ is a t-admissible factorization of $N!$ into N numbers, demonstrating that $t(N) \geq t$.

It will be convenient to divide the set of primes into the following ranges:

- *Tiny primes* $p = 2, 3$.
- *Small primes* $3 < p \leq K$.

[1] Numerically, it would be slightly better to use the closed interval $[t, t+3N/A]$ instead of the half-open interval $(t, t+3N/A]$, but we will consistently aim to use half-open intervals here to be compatible with standard notation for the prime counting function $\pi(x)$.

- *Borderline small primes* $K < p \leq K(1 + \frac{3N/t}{A})$.
- *Medium primes* $K(1 + \frac{3N/t}{A}) < p \leq t/K$.
- *Large primes* $p > t/K$.

The expected p-surpluses or p-deficits at various stages of this process are summarized in Table 1.

	Tiny p	Small p	Borderline p	Medium p	Large p
$\mathcal{B}^{(1)}$	Max. surplus	Near balance	Near balance	Near balance	???
$\mathcal{B}^{(2)}$	Max. surplus	Med. surplus	Med. surplus?	Near balance	Max. surplus
$\mathcal{B}^{(3)}$	Lg. surplus	Sm. surplus?	Med. surplus?	Near balance	Balance
$\mathcal{B}^{(4)}$	Lg. surplus	Balance?	Balance?	Balance/sm. deficit	Balance
$\mathcal{B}^{(5)}$	Lg. surplus	Balance	Balance	Balance	Balance
$\mathcal{B}^{(6)}$	Sm. surplus	Balance	Balance	Balance	Balance
$\mathcal{B}^{(7)}$	Balance	Balance	Balance	Balance	Balance

TABLE 1. Evolution of the surpluses and deficits of the multisets $\mathcal{B}^{(i)}$, $i = 1, \dots, 7$; we describe the size of these surpluses and deficits informally as "small", "medium", "large", or "maximal". For entries with a question mark, we allow the possibility of a tiny deficit. For the entry marked ???, all behavior from large surpluses to large deficits are possible.

3.3. **Analysis of Step 7.** We now analyze the above algorithm, starting from the final Step 7 and working backwards to Step 1, to establish sufficient conditions for the algorithm to successfully demonstrate that $t(N) \geq t$.

It will be convenient to introduce the following notation. For $a_+, a_- \in [0, +\infty]$, we define the asymmetric norm $|x|_{a_+,a_-}$ of a real number x by the formula

$$|x|_{a_+,a_-} := \begin{cases} a_+|x| & x \geq 0 \\ a_-|x| & x \leq 0. \end{cases}$$

If a_+, a_- are finite, this function is Lipschitz with constant $\max(a_+, a_-)$. One can think of a_+ as the "cost" of making x positive, and a_- as the "cost" of making x negative.

We now begin the analysis of Step 9. This procedure will terminate successfully as long as the length $|\mathcal{B}^{(6)}|$ of the tuple is at least N. To ensure this, we introduce the *t-excess* of a multiset $\mathcal{B}$ by the formula

$$\mathrm{E}_t(\mathcal{B}) := \prod_{b \in \mathcal{B}} \log \frac{b}{t} = \log \prod \mathcal{B} - |\mathcal{B}| \log t.$$

Thus, to ensure the success of this step, it suffices to establish the inequality

$$\mathrm{E}_t(\mathcal{B}^{(6)}) \leq \log \prod \mathcal{B}^{(6)} - N \log t.$$

From (1.1) we have

$$\log \prod \mathcal{B}^{(6)} = \log N! - \sum_p \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(6)}} \right) \log p,$$

so we can rewrite the previous condition (using the fact that $\mathcal{B}^{(6)}$ is a subfactorization of $N!$) as

$$\mathrm{E}_t(\mathcal{B}^{(6)}) + \sum_p \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(6)}}\right)\right|_{\log p,\infty} \leq \log N! - N\log t.$$

If we assume that $t = N/e^{1+\delta}$ for some $\delta > 0$, we can use the Stirling approximation (1.4) to reduce to the sufficient condition

$$\mathrm{E}_t(\mathcal{B}^{(6)}) + \sum_p \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(6)}}\right)\right|_{\log p,\infty} \leq \delta N + \log\sqrt{2\pi N}. \tag{3.1}$$

3.4. **Analysis of Step 6.** Now we analyze Step 6. For any $L \geq 1$, let κ_L be the least quantity such that

$$x \leq \lceil x \rceil^{\langle 2,3\rangle} \leq \exp(\kappa_L)x \tag{3.2}$$

holds for all $x \geq L$. Just from considering the powers of two, we have the trivial upper bound

$$\kappa_L \leq \log 2. \tag{3.3}$$

We shall obtain better estimates on this quantity in Section 4. For now we use this quantity to help achieve efficient subfactorizations of 3-smooth numbers, as follows.

Lemma 3.2. *Let $L \geq 1$. Let $t > 3L$ and let 2^n3^m be a 3-smooth number with $n, m > 0$ obeying the conditions*

$$\frac{\log(3L)+\kappa_L}{\log t - \log(3L)} \leq \frac{n\log 2}{m\log 3} \leq \frac{\log t - \log(2L)}{\log(2L)+\kappa_L}. \tag{3.4}$$

Then one can find a t-admissible subfactorization $\mathcal{B}$ of 2^n3^m such that

$$\mathrm{E}_t(\mathcal{B}) \leq \kappa_L \frac{n\log 2 + m\log 3}{\log t} \tag{3.5}$$

and

$$|\nu_2(2^n3^m/\mathcal{B})|_{\log 2,\infty} + |\nu_3(2^n3^m/\mathcal{B})|_{\log 3,\infty} \leq 2(\log t + \kappa_L). \tag{3.6}$$

In practice, $\log t$ will be significantly larger than $\log(2L)$ or $\log(3L)$, and so the hypothesis (3.4) will be quite mild, as long as n and m are both reasonably large.

Proof. Let $2^{n_0}, 3^{m_0}$ be the largest powers of 2 and 3 less than or equal to t/L respectively, thus

$$L \leq \frac{t}{2^{n_0}} \leq 2L \tag{3.7}$$

and

$$L \leq \frac{t}{3^{m_0}} \leq 3L. \tag{3.8}$$

From (3.2), the 3-smooth numbers $\lceil t/2^{n_0}\rceil^{\langle 2,3\rangle} = 2^{n_1}3^{m_1}$, $\lceil t/3^{m_0}\rceil^{\langle 2,3\rangle} = 2^{n_2}3^{m_2}$ obey the estimates

$$\frac{t}{2^{n_0}} \leq 2^{n_1}3^{m_1} \leq e^{\kappa_L}\frac{t}{2^{n_0}} \tag{3.9}$$

and

$$\frac{t}{3^{m_0}} \leq 2^{n_2}3^{m_2} \leq e^{\kappa_L}\frac{t}{3^{m_0}}, \tag{3.10}$$

or equivalently

$$t \leq 2^{n_0+n_1}3^{m_1}, 2^{n_2}3^{m_0+m_2} \leq e^{\kappa_L}t. \tag{3.11}$$

We can use (3.7), (3.9) to bound

$$\frac{n_0+n_1}{m_1} \geq \frac{n_0}{\log(e^{\kappa_L}\frac{t}{2^{n_0}})/\log 3} \geq \frac{(\log t - \log(2L))/\log 2}{(\log(2L)+\kappa_L)/\log 3}$$

(with the convention that this bound is vacuously true for $m_1 = 0$). Similarly, from (3.8), (3.10) we have

$$\frac{n_2}{m_0+m_2} \leq \frac{\log(e^{\kappa}\frac{t}{3^{m_0}})/\log 2}{m_0} \leq \frac{(\log(3L)+\kappa)/\log 2}{(\log t - \log(3L))/\log 3}$$

and hence by (3.4)

$$\frac{n_2}{m_0+m_2} \leq \frac{n}{m} \leq \frac{n_0+n_1}{m_1}. \tag{3.12}$$

Thus we can write (n, m) as a non-negative linear combination

$$(n, m) = \alpha_1(n_0+n_1, m_1) + \alpha_2(n_2, m_0+m_2)$$

for some real $\alpha_1, \alpha_2 \geq 0$. We now take our subfactorization $\mathcal{B}$ to consist of $\lfloor \alpha_1 \rfloor$ copies of the 3-smooth number $2^{n_0+n_1}3^{m_1}$ and $\lfloor \alpha_2 \rfloor$ copies of the 3-smooth number $2^{n_2}3^{m_0+m_2}$. By (3.11), each term $2^{n'}3^{m'}$ here is admissible and contributes a t-excess of at most κ_L, which is in turn bounded by $\kappa_L \frac{n' \log 2 + m' \log 3}{\log t}$. Adding these bounds together, we obtain (3.5).

The expression $2^n 3^m / \prod \mathcal{B}$ contains at most $n_0+n_1+n_2$ factors of 2 and at most $m_0+m_2+m_1$ factors of 3, hence

$$\nu_2(2^n 3^m / \prod \mathcal{B}) \log 2 + \nu_3(2^n 3^m / \prod \mathcal{B}) \log 3 \leq \log 2^{n_0+n_1}3^{m_1} + \log 2^{n_2}3^{m_0+m_2},$$

and the bound (3.6) follows from (3.11). $\square$

We now use this lemma to analyze Step 6 as follows.

Proposition 3.3. *Let $L \geq 1$. Let $3L < t = N/e^{1+\delta}$ for some $\delta > 0$, and let $1 \leq K \leq t$ and $A \geq 1$. Suppose that the algorithm in Section 3.2 with the indicated parameters reaches the end of Step 5 with a multiset $\mathcal{B}^{(5)}$ obeying the following hypotheses:*

(i) *(Small excess and surplus at non-tiny primes)*

$$\mathrm{E}_t(\mathcal{B}^{(5)}) + \sum_{p>3} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(5)}}\right)\right|_{\log p, \infty} \leq \delta N + \log\sqrt{2\pi} - \frac{3}{2}\log N - (\kappa_L \log\sqrt{12})\frac{N}{\log t}. \tag{3.13}$$

(ii) *(Large surpluses at tiny primes) The surpluses $\nu_2(N!/\prod \mathcal{B}^{(5)})$, $\nu_3(N!/\prod \mathcal{B}^{(5)})$ lie in the sector $\Gamma_{t,L} \subset \mathbb{R}^2$, defined to be the set of pairs (n, m) with $n, m > 0$ and*

$$\frac{\log(3L)+\kappa_L}{\log t - \log(3L)} \leq \frac{n}{m} \leq \frac{\log t - \log(2L)}{\log(2L)+\kappa_L}.$$

Then $t(N) \geq t$.

Proof. Write $n := \nu_2(N!/\prod \mathcal{B}^{(5)})$ and $m := \nu_3(N!/\prod \mathcal{B}^{(5)})$. From (1.3) we have $n \leq N$ and $m \leq N/2$, hence

$$n \log 2 + m \log 3 \leq N \log \sqrt{12}.$$

Applying Lemma 3.2, we can find a subfactorization $\mathcal{B}'$ of $2^n 3^m$ with an excess of at most $(\kappa_L \log \sqrt{12})\frac{N}{\log t}$, and with

$$\left| \nu_2 \left(\frac{2^n 3^m}{\prod \mathcal{B}'} \right) \right|_{\log 2, \infty} + \left| \nu_3 \left(\frac{2^n 3^m}{\prod \mathcal{B}'} \right) \right|_{\log 3, \infty} \leq 2(\log t + \kappa_L) \leq 2 \log N$$

where we have used (3.3) and the fact that $\log t \leq \log N - 1$. Then $\mathcal{B}^{(6)} = \mathcal{B}^{(5)} \cup \mathcal{B}'$ is another t-admissible multiset, and from (3.13), we obtain the previous sufficient condition (3.1). □

3.5. **Analysis of Step 5.**

Proposition 3.4. *Let* $1 \leq K \leq t \leq N$, $A \geq 1$, *and* $L \geq 1$ *be parameters such that* $9L < t = N/e^{1+\delta}$ *for some* $\delta > 0$. *Suppose that the algorithm in Section 3.2 with the indicated parameters reaches the end of Step 4 to produce a multiset* $\mathcal{B}^{(4)}$ *obeying the following hypotheses.*

(i) *(Small excess and surplus at small/medium primes)*

$$\begin{aligned} &\mathrm{E}_t(\mathcal{B}^{(4)}) + \sum_{3<p\leq t/K} \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(4)}} \right) \right|_{\kappa_K \min(\frac{\log p}{\log \sqrt{t/K}},1),\infty} \\ &\leq \delta N - \frac{3}{2} \log N - \kappa_L (\log \sqrt{12}) \frac{N}{\log t}. \end{aligned} \tag{3.14}$$

(ii) *(Large surpluses at tiny primes) Whenever* n_{**}, m_{**} *are natural numbers obeying the bounds*

$$n_{**} \log 2 + m_{**} \log 3 \leq \sum_{3<p\leq t/K} \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(4)}} \right) \right|_{\frac{\log \sqrt{tK}+\kappa_K}{\log \sqrt{t/K}} \log p, \infty} + \log t + \kappa_L,$$

then one has

$$\left(\nu_2 \left(\frac{N!}{\prod \mathcal{B}^{(4)}} \right) - n_{**}, \nu_3 \left(\frac{N!}{\prod \mathcal{B}^{(4)}} \right) - m_{**} \right) \in \Gamma_{t,L}.$$

Then $t(N) \geq t$.

Proof. By (3.14), $\mathcal{B}^{(4)}$ is a subfactorization of $N!$, and by construction it is in balance at all primes $p > t/K$. Consider all the p-surplus primes in the small, borderline small, and medium range $3 < p \leq t/K$, thus each such prime is considered with multiplicity $\nu_p(N!/\prod \mathcal{B}^{(4)})$. Using the greedy algorithm, one can factor the product of all these primes into M factors $c_1, \dots, c_M$ in the interval $(\sqrt{t/K}, t/K]$, times at most one exceptional factor c_* in $(1, \sqrt{t/K}]$, for some M. If we let M' denote the number of factors in $c_1, \dots, c_M$ that are not divisible by a prime larger than $\sqrt{t/K}$, we have the bound

$$\left(\sqrt{t/K}\right)^{M'} \leq \prod_{3<p\leq \sqrt{t/K}} \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)$$

and hence on taking logarithms

$$M' \leq \sum_{3<p\leq \sqrt{t/K}} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)\right|_{\frac{\log p}{\log \sqrt{t/K}},\infty}.$$

Restoring the factors divisible by primes $p > \sqrt{t/K}$, we conclude that

$$M \leq \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)\right|_{\min(\frac{\log p}{\log \sqrt{t/K}},1),\infty}. \tag{3.15}$$

For each of the M factors c_i, we introduce the 3-smooth number $\lceil t/c_i \rceil^{\langle 2,3\rangle} = 2^{n_i}3^{m_i}$, which by (3.2) lies in the interval $[t/c_i, e^{\kappa_K} t/c_i]$; similarly, for the exceptional factor c_* we introduce a 3-smooth number $\lceil t/c_* \rceil^{\langle 2,3\rangle} = 2^{n_*}3^{m_*}$ in the interval $[t/c_*, e^{\kappa_K} t/c_*]$. If we then adjoin the 3-smooth numbers $\lceil t/c_i \rceil^{\langle 2,3\rangle} c_i = 2^{n_i}3^{m_i} c_i$ for $i = 1, \dots, M$ as well as $\lceil t/c_* \rceil^{\langle 2,3\rangle} c_* = 2^{n_*}3^{m_*} c_*$ to the t-admissible multiset $\mathcal{B}^{(4)}$ to create a new t-admissible multiset $\mathcal{B}^{(5)}$. The quantity $\log \lceil t/c_* \rceil^{\langle 2,3\rangle} = n_i \log 2 + m_i \log 3$ is bounded by $\log \sqrt{tK} + \kappa_K$, and the quantity $\log \lceil t/c_* \rceil^{\langle 2,3\rangle} = n_* \log 2 + m_* \log 3$ is similarly bounded by $\log t + \kappa$, hence if we denote $n_{**} := n_1 + \dots + n_M + n_*$ and $m_{**} := m_1 + \dots + m_M + m_*$, we have

$$n_{**} \log 2 + m_{**} \log 3 \leq \frac{\log \sqrt{tK} + \kappa_K}{\log \sqrt{t/K}} \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)\right|_{\log p,\infty} + \log t + \kappa_K.$$

Each of the new factors in $\mathcal{B}^{(5)}$ contributes an excess of at most κ_K, so the total excess of $\mathcal{B}^{(5)}$ is at most

$$\mathrm{E}_t(\mathcal{B}^{(4)}) + \kappa_K M + \kappa_K$$

which by (3.15) is bounded by

$$\mathrm{E}_t(\mathcal{B}^{(4)}) + \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)\right|_{\kappa_K \min(\frac{\log p}{\log \sqrt{t/K}},1),\infty} + \kappa_K.$$

We conclude that $\mathcal{B}^{(5)}$ obeys the hypotheses of Proposition 3.3 (using (3.3) to bound κ_K by $\log \sqrt{2\pi}$), and the claim follows. □

3.6. **Analysis of Step 4.**

Proposition 3.5. *Let $L \geq 1$. Let $9L < t = N/e^{1+\delta}$ for some $\delta > 0$, and suppose that the algorithm reaches the end of Step 3 to produce a multiset $\mathcal{B}^{(3)}$ obeying the following hypotheses:*

(i) *(Small excess and surplus at small/medium primes) One has*

$$\begin{aligned} &\mathrm{E}_t(\mathcal{B}^{(3)}) + \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \right|_{\kappa_K \min(\frac{\log p}{\log \sqrt{t/K}},1),\kappa_p} \\ &\leq \delta N - \frac{3}{2}\log N - \kappa_L(\log\sqrt{12})\frac{N}{\log t}. \end{aligned} \tag{3.16}$$

(ii) *(Large surpluses at tiny primes) Whenever n_{**}, m_{**} are natural numbers obeying the bounds*

$$\begin{aligned} n_{**}\log 2 + m_{**}\log 3 \leq \sum_{3<p\leq t/K} &\left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \right|_{\frac{\log\sqrt{tK}+\kappa_K}{\log\sqrt{t/K}}\log p,\log p+\kappa_p} \\ &+ \log t + \kappa_K, \end{aligned} \tag{3.17}$$

then one has

$$\left(\nu_2\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) - n_{**}, \nu_3\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) - m_{**} \right) \in \Gamma_{t,L}. \tag{3.18}$$

Then $t(N) \geq t$.

Proof. Suppose there is a non-tiny prime $p > 3$ with a positive p-deficit $|\nu_p(N!/\prod \mathcal{B}^{(3)})|_{0,1} > 0$. Since $\mathcal{B}^{(3)}$ is in balance at all large primes, we have $3 < p \leq t/K$. We locate an element of $\mathcal{B}^{(3)}$ that contains p as a factor, and replaces it with $\lceil p \rceil^{\langle 2,3\rangle} = 2^{n_p}3^{m_p}$, which increases that factor by at most $\exp(\kappa_p)$ thanks to (3.2). This procedure reduces the p-deficit by one, adds at most κ_p to the t-excess, and decrements $\nu_2(N!/\prod \mathcal{B}^{(3)})$, $\nu_3(N!/\prod \mathcal{B}^{(3)})$ by n_p, m_p respectively. Since $n_p \log 2 + m_p \log 3 \leq \log p + \kappa_p$, if we apply this procedure to clear all deficits at non-tiny primes, the resulting multiset $\mathcal{B}^{(4)}$ has a t-excess of

$$\mathrm{E}_t(\mathcal{B}^{(4)}) \leq \mathrm{E}_t(\mathcal{B}^{(3)}) + \sum_{p>3} |\nu_p(N!/\prod \mathcal{B})|_{0,\kappa_p}$$

and we have

$$\nu_2(N!/\prod \mathcal{B}^{(4)}) = \nu_2(N!/\prod \mathcal{B}^{(3)}) - n', \quad \nu_3(N!/\prod \mathcal{B}^{(4)}) = \nu_3(N!/\prod \mathcal{B}^{(3)}) - m'$$

with

$$n'\log 2 + m'\log 3 \leq \sum_{p>3} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \right|_{0,\log p+\kappa_p}.$$

The hypotheses of Proposition 3.4 are now satisfied, and we are done. □

3.7. **Analysis of Steps 1,2,3.** To apply Proposition 3.5, we now compute the various statistics of $\mathcal{B}^{(3)}$ produced by Steps 1-3.

We begin with the analysis of $\mathcal{B}^{(1)}$, constructed in Step 1 of the algorithm. To count elements coprime to 6, we use the following lemma:

Lemma 3.6. *For any interval $(a, b]$ with $0 \leq a \leq b$, the number of natural numbers in the interval that are coprime to 6 is $\frac{b-a}{3} + O_\leq(4/3)$.*

Proof. By the triangle inequality, it suffices to show that the number of natural numbers coprime to 6 in $[0, a]$, minus $a/3$, is $O_{\leq}(2/3)$. The claim is easily verified for $0 \leq a \leq 6$, and the quantity in question is 6-periodic in a, giving the claim. $\square$

The excess of $\mathcal{B}^{(1)}$ is clearly given by

$$\mathrm{E}_t(\mathcal{B}^{(1)}) = A \sum_{n \in I} \log \frac{n}{t}.$$

By the fundamental theorem of calculus, this is

$$A \int_0^{3t/A} |I \cap (t, t+h]| \, \frac{dh}{t+h}.$$

Bounding $\frac{1}{t+h}$ by $\frac{1}{t}$ and applying Lemma 3.6, we conclude that

$$\mathrm{E}_t(\mathcal{B}^{(1)}) \leq A \int_0^{3N/A} \left(\frac{h}{3} + \frac{4}{3}\right) \frac{dh}{t} = \frac{3N^2}{2tA} + 4. \tag{3.19}$$

Next, we compute p-valuations $v_p(\mathcal{B}^{(1)})$. By construction, this quantity vanishes at tiny primes $p = 2, 3$. For $p > 3$, we can use Lemma 3.6 again to conclude

$$\begin{aligned}
v_p(\mathcal{B}^{(1)}) &= A \sum_{1 \leq j \leq \frac{\log N}{\log p}} |I \cap p^j \mathbb{Z}| \\
&= A \sum_{1 \leq j \leq \frac{\log N}{\log p}} \left(\frac{N}{p^j A} + O_{\leq}(4/3)\right) \\
&= \frac{N}{p-1} + O_{\leq}\left(\frac{3}{p-1}\right) + O_{\leq}\left(\frac{4A}{3} \left\lceil \frac{\log N}{\log p} \right\rceil\right) \\
&= \frac{N}{p-1} + +O_{\leq}\left(\frac{4A+3}{3} \left\lceil \frac{\log N}{\log p} \right\rceil\right)
\end{aligned}$$

since $\frac{3}{p-1} \leq \frac{3}{4}$. Meanwhile, from (1.3) one has

$$v_p(N!) = \frac{N}{p-1} + O_{\leq}\left(\left\lceil \frac{\log N}{\log p} \right\rceil\right)$$

and thus

$$v_p(N!/\mathcal{B}^{(1)}) = O_{\leq}\left(\frac{4A+6}{3} \left\lceil \frac{\log N}{\log p} \right\rceil\right). \tag{3.20}$$

Now we pass to $\mathcal{B}^{(2)}$ by performing Step 2 of the algorithm. Removing elements from a t-admissible multiset cannot increase the t-excess, so from (3.19) we have

$$\mathrm{E}_t(\mathcal{B}^{(2)}) \leq \frac{3N^2}{2tA} + 4. \tag{3.21}$$

The elements removed are of the form pm with $m \leq K(1 + \frac{3N/t}{A})$ coprime to 6, and p in the interval $(\frac{t}{\min(m,K)}, \frac{t}{m} + \frac{3N}{mA}]$. We conclude that

$$v_p(N!/\mathcal{B}^{(2)}) = v_p(N!/\mathcal{B}^{(1)}) \tag{3.22}$$

for medium primes $K(1+\frac{3N/t}{A}) < p \leq t/K$. For small and borderline small primes $3 < p \leq K(1+\frac{3}{A})$ one has

$$\nu_p(N!/\mathcal{B}^{(2)}) = \nu_p(N!/\mathcal{B}^{(1)}) + A \sum_{\substack{m \leq K(1+\frac{3N/t}{A}) \\ (m,6)=1}} \nu_p(m) \left(\pi\left(\frac{t}{m} + \frac{3N}{Am}\right) - \pi\left(\frac{t}{\min(m,K)}\right)\right). \tag{3.23}$$

Finally, by construction we are in balance

$$\nu_p(N!/\mathcal{B}^{(2)}) = 0$$

for large primes $p > t/K$, while for tiny primes $p = 2, 3$ we have

$$\nu_p(N!/\mathcal{B}^{(2)}) = \nu_p(N!)$$

We now pass to $\mathcal{B}^{(3)}$ by performing Step 3 of the algorithm. In other words, we add $\nu_p(N!)$ copies of $p\lceil t/p \rceil$ for each prime large $p > t/K$. The t-excess is now given by

$$\mathrm{E}_t(\mathcal{B}^{(3)}) = \mathrm{E}_t(\mathcal{B}^{(2)}) + \sum_{p > t/K} \nu_p(N!) \log \frac{\lceil t/p \rceil}{t/p}. \tag{3.24}$$

By construction one has balance (3.25) at primes large $p > t/K$,

$$\nu_p(N!/\mathcal{B}^{(3)}) = 0 \tag{3.25}$$

and no modification at borderline small or medium primes $K < p \leq t/K$,

$$\nu_p(N!/\mathcal{B}^{(3)}) = 0 \tag{3.26}$$

but now the p-surplus or p-deficit at small primes $3 < p \leq K$ is modified:

$$\nu_p(N!/\mathcal{B}^{(3)}) = \nu_p(N!/\mathcal{B}^{(2)}) - \sum_{p' > t/K} \nu_{p'}(N!) \nu_p(\lceil t/p' \rceil). \tag{3.27}$$

Similarly, at tiny primes $p = 2, 3$ we have

$$\nu_p(N!/\mathcal{B}^{(3)}) = \nu_p(N!) - \sum_{p' > t/K} \nu_{p'}(N!) \nu_p(\lceil t/p' \rceil). \tag{3.28}$$

4. Powers of 2 and 3

We now obtain good bounds on the quantity κ_L. Clearly κ_L is a non-increasing function of L with $\kappa_1 = \log 2$. The following lemma gives improved control on κ_L for large L:

Lemma 4.1. *If n_1, n_2, m_1, m_2 are natural numbers such that $n_1 + n_2, m_1 + m_2 \geq 1$ and*

$$1 \leq \frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}$$

then

$$\kappa_{\min(2^{n_1+n_2}, 3^{m_1+m_2})/6} \leq \log \max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right).$$

Proof. If $\min(2^{n_1+n_2}, 3^{m_1+m_2})/6 \leq t \leq 2^{n_2-1}3^{m_1-1}$, then we have

$$t \leq 2^{n_2-1}3^{m_1-1} \leq \max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right) t, \tag{4.1}$$

so we are done in this case. Now suppose that $t > 2^{n_2-1}3^{m_1-1}$. If we write $\lceil t \rceil^{\langle 2,3\rangle} = 2^n 3^m$ be the smallest 3-smooth number that is at least t, then we must have $n \geq n_2$ or $m \geq m_1$ (or both). Thus at least one of $\frac{2^{n_1}}{3^{m_1}} 2^n 3^m$ and $\frac{3^{m_2}}{3^{n_2}} 2^n 3^m$ is an integer, and is thus at most t by construction. This gives (4.1), and the claim follows. $\square$

Some efficient choices of parameters for this lemma are given in Table 2. For instance, $\kappa_{4.5} \leq 0.28768\dots$ and $\kappa_{40.5} \leq 0.16989\dots$.

n_1	m_1	n_2	m_2	$\min(2^{n_1+n_2}, 3^{m_1+m_2})/6$	$\log\max(3^{m_1}/2^{n_1}, 2^{n_2}/3^{m_2})$
1	1	**1**	**0**	$1/2 = 0.5$	$\log 2 = 0.69314\dots$
1	**1**	2	1	$2^2/3 = 1.33\dots$	$\log(3/2) = 0.40546\dots$
3	2	**2**	**1**	$3^2/2 = 4.5$	$\log(2^2/3) = 0.28768\dots$
3	2	**5**	**3**	$3^4/2 = 40.5$	$\log(2^5/3^3) = 0.16989\dots$
3	**2**	8	5	$2^{10}/3 = 341.33\dots$	$\log(3^2/2^3) = 0.11778\dots$
11	**7**	8	5	$2^{18}/3 = 87381.33\dots$	$\log(3^7/2^{11}) = 0.06566\dots$
19	12	**8**	**5**	$3^{17}/2 \approx 6.4 \times 10^7$	$\log(2^8/3^5) = 0.05211\dots$
19	12	**27**	**17**	$3^{29}/2 \approx 3.4 \times 10^{13}$	$\log(2^{27}/3^{17}) = 0.03856\dots$
19	12	**46**	**29**	$3^{41}/2 \approx 1.8 \times 10^{19}$	$\log(2^{46}/3^{29}) = 0.02501\dots$

TABLE 2. Efficient parameter choices for Lemma 4.1. The parameters used to attain the minimum or maximum are indicated in **boldface**. Note how the number of rows in each group matches the terms $1, 1, 2, 2, 3, \dots$ in the continued fraction expansion.

Remark 4.2. It should be unsurprising that the continued fraction convergents $1/1, 2/1, 3/2, 8/5, 19/12, \dots$ to

$$\frac{\log 3}{\log 2} = 1.5849\dots = [1; 1, 1, 2, 2, 3, 1, \dots]$$

are often excellent choices for n_1/m_1 or n_2/m_2, although other approximants such as $5/3$ or $11/7$ are also usable.

Asymptotically, we have logarithmic-type decay:

Lemma 4.3 (Baker bound)**.** *We have*

$$\kappa_L \ll \log^{-c} L$$

for all $L \geq 2$ and some absolute constant $c > 0$.

Proof. From the classical theory of continued fractions, we can find rational approximants

$$\frac{p_{2j}}{q_{2j}} \leq \frac{\log 3}{\log 2} \leq \frac{p_{2j+1}}{q_{2j+1}} \tag{4.2}$$

to the irrational number $\log 3/\log 2$, where the convergents p_j/q_j obey the recursions

$$p_j = b_j p_{j-1} + p_{j-2}; \quad q_j = b_j q_{j-1} + q_{j-2}$$

with $p_{-1} = 1, q = -1 = 0, p_0 = b_0, q_0 = 1$, and

$$[b_0; b_1, b_2, \dots] = [1; 1, 1, 2, 2, 3, 1 \dots]$$

is the continued fraction expansion of $\frac{\log 3}{\log 2}$. Furthermore, $p_{2j+1}q_{2j} - p_{2j}q_{2j+1} = 1$, and hence

$$\frac{\log 3}{\log 2} - \frac{p_{2j}}{q_{2j}} = \frac{1}{q_{2j}q_{2j+1}}. \tag{4.3}$$

By Baker's theorem, $\frac{\log 3}{\log 2}$ is a Diophantine number, giving a bound of the form

$$q_{2j+1} \ll q_{2j}^{O(1)} \tag{4.4}$$

and a similar argument (using $p_{2j+2}q_{2j+1} - p_{2j+1}q_{2j+2} = -1$) gives

$$q_{2j+2} \ll q_{2j+1}^{O(1)}. \tag{4.5}$$

We can rewrite (4.2) as

$$1 \leq \frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}}$$

and routine Taylor expansion using (4.3) gives the upper bounds

$$\frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}} \leq \exp\left(O\left(\frac{1}{q_{2j}}\right)\right).$$

From Lemma 4.1 we obtain

$$\kappa_{\min(2^{p_{2j}+p_{2j+1}}, 3^{q_{2j}+q_{2j+1}})/6} \ll \frac{1}{q_{2j}}.$$

The claim then follows from (4.4), (4.5) after optimizing in j.

□

It seems reasonable to conjecture that c can be taken to be arbitrarily close to 1, but this is essentially equivalent to the open problem of determining that irrationality measure of $\log 3/\log 2$ is equal to 2.

5. Asymptotic evaluation of $t(N)$

In this section we establish the lower bound

$$\frac{t}{N} \geq \frac{1}{e} - \frac{c_0}{\log N} + \frac{1}{\log^{1+c_1} N} \tag{5.1}$$

for some small absolute constant $0 < c_1 < 1$, if N is sufficiently large. With this choice of parameters, one has

$$\delta = \frac{ec_0}{\log N} + \frac{1}{\log^{1+c_1} N} + O\left(\frac{1}{\log^2 N}\right).$$

Let N be sufficiently large. We introduce parameters

$$A := \lfloor \log^2 N \rfloor$$

and

$$K := \lfloor \log^3 N \rfloor$$

and

$$L := N^{0.1}.$$

We apply the algorithm from Section 3.2, using the first option for Step 3, and invoke Proposition 3.5. With these parameters, we see from Lemma 4.3 that the right-hand side of (3.16) is at least

$$ec_0 \frac{N}{\log N} + \frac{N}{2\log^{1+c_1} N}$$

if c_1 is small enough and N is large enough.

Now we work on the left-hand side of (3.16). From (3.21) one has

$$\mathrm{E}_t(\mathcal{B}^{(2)}) \ll \frac{t}{A} + 1 \ll \frac{N}{\log^2 N}.$$

By repeating the proof of Proposition 2.4, we see that

$$\sum_{p > t/K} \nu_p(N!) \log \frac{\lceil t/p \rceil}{t/p} = ec_0 \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right)$$

and hence by (3.24)

$$\mathrm{E}_t(\mathcal{B}^{(2)}) = ec_0 \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right).$$

Thus, to establish (3.16), and bounding $\kappa_K, \kappa_p = O(1)$ and $\min(\frac{\log p}{\log \sqrt{t/K}}, 1) = O(\log p/\log N)$, it will suffice to show that

$$\sum_{3 < p \leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \right|_{\log p/\log N, 1} \ll \frac{N(\log\log N)^2}{\log^2 N}. \tag{5.2}$$

We begin by considering medium primes $K(1 + \frac{3N/t}{A}) < p \leq t/K$. In this range, we have from (3.22), (3.20), (3.26) that

$$\nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) = \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right) \ll A \frac{\log N}{\log p}; \tag{5.3}$$

bounding $\log p/\log N$ crudely by $O(1)$, we then have

$$\sum_{K(1+\frac{3N/t}{A}) < p \leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \right|_{\log p/\log N, 1} \ll A \sum_{K(1+\frac{3N/t}{A}) < p \leq t/K} \frac{\log N}{\log p} \ll \frac{AN}{K \log N} \ll \frac{N}{\log^2 N}$$

using the prime number theorem to bound the sum over primes (dividing into the regions $p \leq \sqrt{N}$ and $p > \sqrt{N}$ if desired). Thus the contribution of these primes to (5.2) is acceptable.

Now we deal with the (technical) contribution of the borderline small $K < p \leq K(1 + \frac{3N/t}{A})$. We still have (3.26), but the second term in the right-hand side of (3.23) has one non-trivial term, namely when $m = p$. By the Brun–Titchmarsh theorem, the contribution of this term is $O(\frac{N}{K \log N})$. Hence in this range we have the crude bound

$$\left| v_p \left(\frac{N!}{\prod \mathcal{B}^{(3)}} \right) \right| \ll \frac{N}{K \log N} \tag{5.4}$$

and hence the contribution of this case is $O(N/A \log N) = O(N/\log^3 N)$, which is acceptable.

Now we consider the small primes $3 < p \leq K$. From (**??**), (3.23), (3.20) and the Brun–Titchmarsh inequality we have the crude upper bound

$$\begin{aligned} v_p(N!/\mathcal{B}^{(3)}) &\leq v_p(N!/\mathcal{B}^{(2)}) \\ &\leq v_p(N!/\mathcal{B}^{(1)}) + A \sum_{m \ll K} v_p(m) \left(\pi \left(\frac{t}{m} + \frac{3N}{Am} \right) - \pi \left(\frac{t}{m} \right) \right) \\ &\ll A \frac{\log N}{\log p} + A \sum_{m \ll K} v_p(m) \frac{N}{Am \log N} \\ &\ll \log^3 N + \sum_{j \leq \log\log N} \sum_{m \ll K/p^j} \frac{N}{p^j m \log N} \\ &\ll \frac{N \log\log N}{p \log N}. \end{aligned} \tag{5.5}$$

We can thus control the net contribution of the surpluses at small primes by

$$\begin{aligned} \sum_{3 < p \leq K} \left| v_p \left(\frac{N!}{\prod \mathcal{B}^{(3)}} \right) \right|_{\log p/\log N, 0} &\ll \sum_{3 < p \leq K} \frac{N \log\log N}{p \log N} \frac{\log p}{\log N} \\ &\ll \frac{(N \log\log N)^2}{\log^2 N} \end{aligned}$$

by Mertens' theorem. Hence this contribution is acceptable. It remains to control the deficits at small primes:

$$\sum_{3 < p \leq K} \left| v_p \left(\frac{N!}{\prod \mathcal{B}^{(3)}} \right) \right|_{0,1} \ll \frac{N (\log\log N)^2}{\log^2 N}. \tag{5.6}$$

Here the situation is more delicate. From (3.23), (3.20) and the prime number theorem (with classical error term) we have

$$\begin{aligned} v_p(N!/\mathcal{B}^{(2)}) &\geq -O(A \frac{\log N}{\log p}) + A \sum_{m \leq K; (m,6)=1} v_p(m) \left(\pi \left(\frac{t}{m} + \frac{3N}{Am} \right) - \ pi \left(\frac{t}{m} \right) \right) \\ &\geq -O(\log^3 N) + \left(1 + O\left(\frac{\log\log N}{\log N} \right) \right) \sum_{m \leq K; (m,6)=1} v_p(m) \frac{3N}{m \log N} \\ &= \frac{N}{\log N} \sum_{m \leq K; (m,6)=1} v_p(m) \frac{3}{m} + O\left(\frac{N (\log\log N)^2}{p \log^2 N} \right). \end{aligned}$$

Next, we control the final term in (3.27) using (1.3) and the prime number theorem:

$$\begin{aligned}\sum_{p'>t/K} \nu_{p'}(N!)\nu_p(\lceil t/p'\rceil) &\leq \sum_{p'>t/K} \frac{N}{p'}\nu_p(\lceil t/p'\rceil)\\ &= \sum_{m\leq K} \nu_p(m) \sum_{t/m\leq p'<t/(m-1)} \frac{N}{p'}\\ &= \left(1+O\left(\frac{\log\log N}{\log N}\right)\right)\sum_{m\leq K}\nu_p(m)\int_{t/m}^{t/(m-1)} \frac{N}{x}\frac{dx}{\log N}\\ &= \frac{N}{\log N}\sum_{m\leq K}\nu_p(m)\log\frac{m}{m-1} + O\left(\frac{N(\log\log N)^2}{p\log^2 N}\right).\end{aligned}$$

We now have a crucial inequality:

Lemma 5.1 (Key inequality). *We have*

$$\sum_{m\leq K;(m,6)=1} \nu_p(m)\frac{3}{m} \geq \sum_{m\leq K}\nu_p(m)\log\frac{m}{m-1}.$$

Proof. Writing $\nu_p(m) = \sum_{j\geq 1} 1_{p^j|m}$, it suffices to show that

$$\sum_{m\leq K;(m,6)=1,p^j|m} \frac{3}{m} \geq \sum_{m\leq K,p^j|m} \log\frac{m}{m-1}$$

for all j. Making the change of variables $m = p^j n$, it suffices to show that

$$\sum_{n\leq K'} \frac{3}{n}1_{(n,6)=1} - p^j\log\frac{p^j n}{p^j n-1} \geq 0$$

for any $K' > 0$. From the Taylor expansion

$$p^j\log\frac{p^j n}{p^j n-1} = \frac{1}{n}+\frac{1}{2p^j n^2}+\frac{1}{3p^{2j}n^3}+\dots$$

and the fact that $p^j \geq 5$, it suffices to show that

$$\sum_{n\leq K'} \frac{3}{n}1_{(n,6)=1} - 5\log\frac{5n}{5n-1} \geq 0.$$

Using the bound

$$\log\frac{5n}{5n-1} = \int_{5n-1}^{5n}\frac{dx}{x} \leq \frac{1}{5n-1}$$

it suffices to show that

$$\sum_{n\leq K'} \frac{3}{n}1_{(n,6)=1} - \frac{1}{n-0.2} \geq 0. \tag{5.7}$$

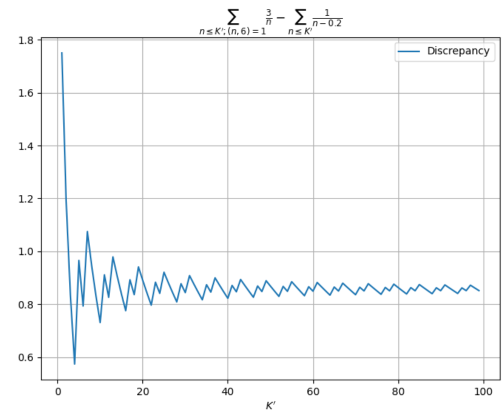

FIGURE 1. A plot of (5.7)

This can be numerically verified for $K' \leq 98$, with the value exceeding 0.8 for at $K = 98$; see Figure 1. On a block $6a - 1 \leq n \leq 6a + 4$, one can compute

$$\sum_{6a-1\leq n\leq 6a+4} \frac{3}{n} 1_{(n,6)=1} - \frac{1}{n-0.2} = \frac{3}{6a-1} + \frac{3}{6a+1} - \frac{1}{6a-1.2} - \frac{1}{6a-0.2} - \frac{1}{6a+0.8} - \frac{1}{6a+1.8} - \frac{1}{6a+2.8}$$

Since $\sum_{a\geq 16} \frac{1}{5(6a-1)(6a-1.2)} < 0.1$ (say), we thus see that

$$\sum_{99\leq n\leq 6a+4} \frac{3}{n} 1_{(n,6)=1} - \frac{1}{n-0.2} \geq -0.1$$

for any $a \geq 16$, and from this and the triangle inequality one can easily establish (5.7) in the remaining ranges $K \geq 98$. $\square$

From this inequality and (3.27), we

$$\nu_p(N!/\mathcal{B}^{(3)}) \geq -O\left(\frac{N(\log\log N)^2}{p\log^2 N}\right) \tag{5.8}$$

and thus by Mertens' theorem we obtain (5.6). This completes the proof of (3.16).

To finish the proof of (5.1), it suffices by Proposition 3.5 to establish the condition (3.18) whenever n_{**}, m_{**} obey (3.17). The left-hand side of (3.17) can be bounded by

$$\ll \sum_{3<p\leq t/K} \left|\nu_p\left(\frac{N!}{\mathcal{B}^{(3)}}\right)\right| \log p + \log N.$$

We claim that this expression is $O(N(\log\log N)^2/\log N)$, which implies

$$n_{**}, m_{**} \ll \frac{N(\log\log N)^2}{\log N}.$$

For the contribution of medium primes $K(1+\frac{3N/t}{A}) < p \leq t/K$ this follows readily from (5.3); the contribution of borderline small primes $K < p \leq K(1+\frac{3N/t}{A})$ follows readily from (5.4); and the contribution of small primes $3 < p \leq K$ follows readily from (5.5), (5.8), and Mertens' theorem. Also, from (3.28), (1.3) we see for a tiny prime $p = 2, 3$ that

$$\begin{aligned} \nu_p(N!/\mathcal{B}^{(3)}) &= \frac{N}{p-1} + O(\log N) - O\left(\sum_{p'>t/K} \frac{N}{p'} \nu_p(\lceil t/p' \rceil)\right) \\ &= \frac{N}{p-1} + O(\log N) - O\left(\sum_{p'>t/K} \frac{N}{p'} \log\log N\right) \\ &= \frac{N}{p-1} + O\left(\frac{N(\log\log N)^2}{\log N}\right) \end{aligned}$$

(with room to spare), and so

$$\begin{aligned} \nu_2(N!/\mathcal{B}^{(3)}) - n_{**} &= N + O\left(\frac{N(\log\log N)^2}{\log N}\right) \\ \nu_3(N!/\mathcal{B}^{(3)}) - m_{**} &= \frac{N}{2} + O\left(\frac{N(\log\log N)^2}{\log N}\right). \end{aligned}$$

By choice of L, this implies (3.18) for N large enough. The proof of (5.1) is now complete.

6. Guy–Selfridge conjecture for $N > 10^{19}$

7. Guy–Selfridge conjecture for medium values of N

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