DECOMPOSING A FACTORIAL INTO LARGE FACTORS

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ABSTRACT. Let $t(N)$ denote the largest number such that $N!$ can be expressed as the product of N numbers greater than or equal to $t(N)$. The bound $t(N)/N = 1/e - o(1)$ was apparently established in unpublished work of Erdős, Selfridge, and Straus; but the proof is lost. Here we obtain the more precise asymptotic

$$\frac{t(N)}{N} = \frac{1}{e} - \frac{c_0}{\log N} + O\left(\frac{1}{\log^{1+c} N}\right)$$

for an explicit constant $c_0 = 0.3044190\dots$ and some absolute constant $c > 0$, answering a question of Erdős and Graham. With numerical assistance, we also establish several conjectures of Guy and Selfridge concerning effective estimates of this quantity, for instance establishing $t(N) \geq N/3$ for $N \geq 43632$, with the threshold shown to be best possible.

1. INTRODUCTION

Given a natural number M, define a *factorization* of M to be a finite multiset $\mathcal{B}$ such that the product

$$\prod \mathcal{B} := \prod_{a \in \mathcal{B}} a$$

(where the product is counted with multiplicity) is equal to M; more generally, define a *sub-factorization* of M to be a finite multiset $\mathcal{B}$ such that $\prod \mathcal{B}$ divides M. Given a threshold t, we say that a multiset $\mathcal{B}$ is *t-admissible* if $a \geq t$ for all $a \in \mathcal{B}$. For a given natural number N, we then define $t(N)$ to be the largest t for which there exists a t-admissible factorization $\mathcal{B}$ of $N!$ of cardinality $|\mathcal{B}| = N$.

Example 1.1. The multiset

$$\{3, 3, 3, 3, 4, 4, 5, 7, 8\}$$

is a 3-admissible factorization of $9!$ of cardinality 9, hence $t(9) \geq 3$. One can check that no 4-admissible factorization of $9!$ of this cardinality exists, hence $t(9) = 3$.

The first few elements of this sequence are

$$1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, \dots$$

and the values of $t(N)$ for $N \leq 79$ were computed in [9] and also found at OEIS A034258. It is easy to see that $t(N)$ is non-decreasing in N (basically because any cardinality N factorization of $N!$ can be extended to a cardinality $N + 1$ factorization of $(N + 1)!$ by adding $N + 1$ to the multiset). The values for $N \leq 200$ can also be recovered from the entries of the inverse sequence of t at OEIS A034259.

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When the factorial $N!$ is replaced with an arbitrary number, this problem is essentially the bin covering problem, which is known to be NP-hard; see e.g., [2]. However, as we shall see in this paper, the special structure of the factorial (and in particular, the profusion of factors at the "tiny primes" $2, 3$) make it more tractable than the general case.

Remark 1.2. One can equivalently define $t(N)$ as the greatest t for which there exists a t-admissible *subfactorization* of $N!$ of cardinality *at least* N. This is because every such subfactorization can be converted into a t-admissible factorization of cardinality exactly N by first deleting elements from the subfactorization to make the cardinality N, and then multiplying one of the elements of the subfactorization by a natural number to upgrade the subfactorization to a factorization. This "relaxed" formulation of the problem turns out to be more convenient for both theoretical analysis of $t(N)$ and numerical computations.

By combining the obvious lower bound

$$\prod \mathcal{B} \geq t^{|\mathcal{B}|} \tag{1.1}$$

for any t-admissible tuple with Stirling's formula (2.6), we obtain the trivial upper bound

$$\frac{t(N)}{N} \leq \frac{(N!)^{1/N}}{N} = \frac{1}{e} + O\left(\frac{\log N}{N}\right) \tag{1.2}$$

for $N \geq 2$; see Figure 1. In [8, p.75] it was reported that an unpublished work of Erdős, Selfridge, and Straus established the asymptotic

$$\frac{t(N)}{N} = \frac{1}{e} + o(1) \tag{1.3}$$

(first conjectured in [6]) and asked if one could show the bound

$$\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c}{\log N} \tag{1.4}$$

for some constant $c > 0$ (problem #391 in `https://www.erdosproblems.com`; see also [9, Section B22, p. 122–123]); it was also noted that similar results were obtained in [1] if one restricted the a_i to be prime powers. However, as later reported in [7], Erdős "believed that Straus had written up our proof [of (1.3)]. Unfortunately Straus suddenly died and no trace was ever found of his notes. Furthermore, we never could reconstruct our proof, so our assertion now can be called only a conjecture". In [9] the lower bound $\frac{t(N)}{N} \geq \frac{1}{4}$ was established for sufficiently large N, by rearranging powers of 2 and 3 in the obvious factorization $1\times 2\times\cdots\times N$ of $N!$. A variant lower bound of the asymptotic shape $\frac{t(N)}{N} \geq \frac{3}{16} - o(1)$ obtained by rearranging only powers of 2, and which is superior for medium values of N, can also be found in [9]. The following conjectures in [9] were also made:

(1) One has $t(N) \leq N/e$ for $N \neq 1, 2, 4$.
(2) One has $t(N) \geq \lfloor 2N/7 \rfloor$ for $N \neq 56$.
(3) One has $t(N) \geq N/3$ for $N \geq 3 \times 10^5$. (It was also asked if the threshold 3×10^5 could be lowered.)

In this paper we answer all of these questions.

Theorem 1.3 (Main theorem)**.** *Let N be a natural number.*

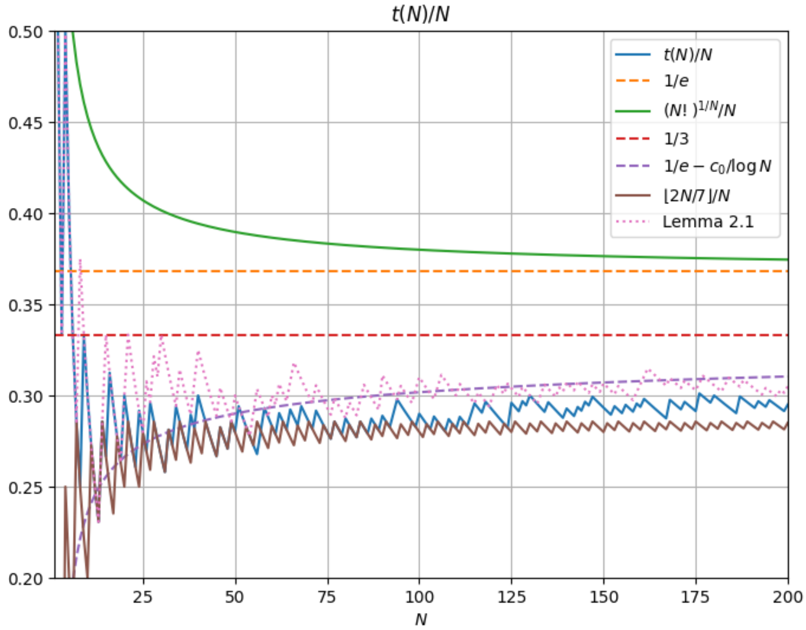

FIGURE 1. The function $t(N)/N$ (blue) for $N \leq 200$, using the data from OEIS A034258, as well as the trivial upper bound $(N!)^{1/N}/N$ (green), the improved upper bound from Lemma 5.3 (pink), which is asymptotic to (1.5) (purple), and the function $\lfloor 2N/7 \rfloor/N$ (brown), which is a lower bound for $N \neq 56$ [9]. Theorem 1.3 implies that $t(N)/N$ is asymptotic to (1.5) (purple), which in turn converges to $1/e$ (orange). The threshold $1/3$ (red) is permanently crossed at $N = 43632$. **TODO: relabel image to reflect new lemma numbering**

(i) *If* $N \neq 1, 2, 4$, *then* $t(N) \leq N/e$.
(ii) *If* $N \neq 56$, *then* $t(N) \geq \lfloor 2N/7 \rfloor$.
(iii) *If* $N \geq 43632$, *then* $t(N) \geq N/3$. *The threshold* 43632 *is best possible.*
(iv) *For large* N, *one has*

$$\frac{t(N)}{N} = \frac{1}{e} - \frac{c_0}{\log N} + O\left(\frac{1}{\log^{1+c} N}\right) \tag{1.5}$$

for some constant $c > 0$, *where* c_0 *is the explicit quantity*

$$\begin{aligned} c_0 &:= \frac{1}{e} \int_0^1 \left\lfloor \frac{1}{x} \right\rfloor \log\left(ex \left\lceil \frac{1}{ex} \right\rceil \right) dx \\ &= 0.3044190\ldots \end{aligned} \tag{1.6}$$

(see Figure 3). In particular, (1.3) *and* (1.4) *hold.*

In Appendix D we give some details on the numerical computation of the constant c_0.

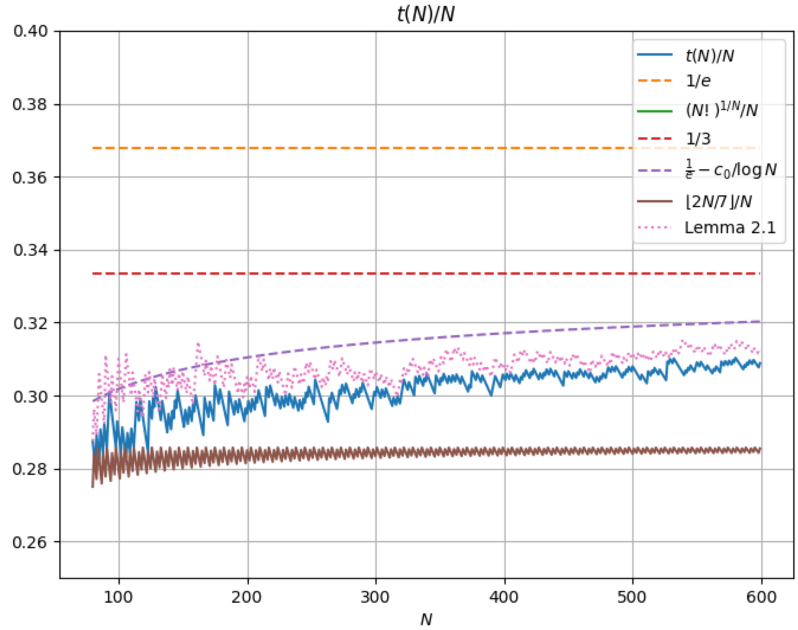

FIGURE 2. A continuation of Figure 1 to the region $80 \leq N \leq 599$. **TODO: relabel image to reflect new lemma numbering**

Remark 1.4. In a previous version [14] of this manuscript, the weaker bounds

$$\frac{1}{e} - \frac{O(1)}{\log N} \leq \frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$$

were established, which were enough to recover (1.3), (1.4), and Theorem 1.3(i).

As one might expect, the proof of Theorem 1.3 proceeds by a combination of both theoretical analysis and numerical calculations. Our main tools to obtain upper and lower bounds on $t(N)$ can be summarized as follows:

- In Section 4, we discuss *greedy algorithms* to construct subfactorizations, that provide quickly computable, though suboptimal, lower bounds on $t(N)$ for small and medium values;
- In Section 3, we present a *linear programming* (or *integer programming*) method that provides quite accurate upper and lower bounds on $t(N)$ for small and medium values of N;
- In Section 5, we introduce an *accounting identity* linking the "t-excess" of a subfactorization with its "p-surpluses" at various primes, which provides an reasonable upper bound on $t(N)$ for all N, and is discussed in more detail in Section 5;
- In Section 5.1, we give *modified approximate factorization* strategy, which provides lower bounds on $t(N)$, that become asymptotically quite efficient.

The final approach is significantly more complicated than the other three, but is the only one which gives efficient lower bounds in the asymptotic limit $N \to \infty$. The key idea is to start

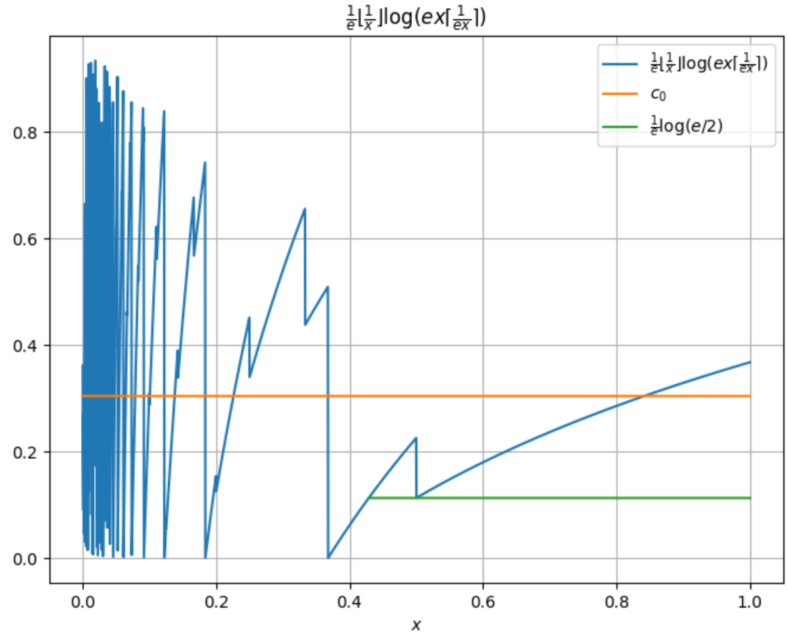

FIGURE 3. The piecewise continuous function $x \mapsto \frac{1}{e}\left\lfloor\frac{1}{x}\right\rfloor \log(ex\left\lceil\frac{1}{ex}\right\rceil)$, together with its mean value $c_0 = 0.3044190\dots$. The function exhibits an oscillatory singularity at $x = 0$ similar to $\sin\frac{1}{x}$ (but it is always nonnegative and bounded). We also display the (crude) lower bound of $\frac{1}{e}\log(e/2)$ for $x \geq \frac{1}{\sqrt{2e}} = 0.4288\dots$. Informally, this function quantifies the difficulty that large primes in the factorization of $N!$ have in becoming slightly larger than N/e after multiplying by a natural number.

with an approximate factorization

$$N! \approx \left(\prod_{j\in I} j\right)^A$$

for some small natural number A (e.g., $A = \lfloor\log^2 N\rfloor$) and a suitable set I of natural numbers greater than or equal to t; there is some freedom to select parameters here, and we will take I to be the natural numbers in $(t, t(1+\sigma)]$ that are coprime to 6, where t is the target lower bound for $t(N)$ we wish to establish, and $\sigma := \frac{3N}{tA}$. With a suitable choice of I, this product contains approximately the right number of copies of p for medium-sized primes p; but it has the "wrong" number of copies of large primes, and is also constructed to avoid the "tiny" primes $p = 2, 3$. One then performs a number of alterations to this approximate factorization to correct for the "surpluses" or "deficits" at various primes $p > 3$, using the supply of available tiny primes $p = 2, 3$ as a sort of "liquidity pool" to efficiently reallocate primes in the factorization. A key point will be that the incommensurability of $\log 2$ and $\log 3$ (i.e., the irrationality of $\log 3/\log 2$) means that the 3-smooth numbers (numbers of the form 2^n3^m)

are asymptotically dense (in logarithmic scale), allowing for other factors to be exchanged for 3-smooth factors with little loss[1].

1.1. **Author contributions and data.** This project was initially concieved as a single-author manuscript by Terence Tao, but since the release of the initial preprint [14], grew to become a collaborative project organized via the Github repository [15], which also contains the supporting code and data for the project. The contributions of the individual authors, according to the CRediT categories at `https://credit.niso.org/`, are as follows:

authors should be arranged in alphabetical order of surname.

- ...
- Terence Tao: Conceptualization, Formal Analysis, Methodology, Project Administration, Visualization, Writing – original draft, Writing – review & editing.

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list here all contributors to the project who did not wish to be listed as co-authors.

2. Notation and basic estimates

If S is a statement, we use 1_S to denote its indicator, thus $1_S = 1$ when S is true and $1_S = 0$ when S is false. If x is a real number, we use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x, and $\lceil x \rceil$ to be the least integer greater than or equal to x.

Throughout this paper, the symbol p (or p', p_1, p_2, etc.) is always understood to be restricted to be prime. The primes $2, 3$ will play a special role in this paper and will be referred to as *tiny primes*. Call a natural number 3-*smooth* if it is the product of tiny primes, i.e., it is of the form $2^n 3^m$ for some natural numbers n, m. Given a positive real number x, we use $\lceil x \rceil^{\langle 2,3 \rangle}$ to denote the smallest 3-smooth number greater than or equal to x. For instance, $\lceil 5 \rceil^{\langle 2,3 \rangle} = 6$ and $\lceil 10 \rceil^{\langle 2,3 \rangle} = 12$. For any $L \geq 1$, let κ_L be the least quantity such that

$$x \leq \lceil x \rceil^{\langle 2,3 \rangle} \leq \exp(\kappa_L) x \tag{2.1}$$

holds for all $x \geq L$. Just from considering the powers of two, we have the trivial upper bound

$$\kappa_L \leq \log 2. \tag{2.2}$$

In fact κ_L decays to zero as L goes to infinity, due to the incommensurability of $\log 2$ and $\log 3$; we quantify this decay in Appendix A.

In practice, $\lceil x \rceil^{\langle 2,3 \rangle}$ will only be slightly larger than x; we quantify this in Appendix A.

[1]The weaker results alluded to in Remark 1.4 only used the prime 2 as a supply of "liquidity", and thus encountered inefficiencies due to the inability to "make change" when approximating another factor by a power of two.

We use (a,b) to denote the greatest common divisor of a and b, $a|b$ to denote the assertion that a divides b, and $\pi(x) = \sum_{p \leq x} 1$ to denote the usual prime counting function. The effective and asymptotic estimates over primes that we will use are summarized in Appendix C.

We use $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ to denote the p-adic valuation of a positive natural number a/b, that is to say the number of times p divides the numerator a, minus the number of times p divides the denominator b. For instance, $\nu_2(32/27) = 5$ and $\nu_3(32/27) = -3$. If one applies a logarithm to the fundamental theorem of arithmetic, one obtains the identity

$$\sum_p \nu_p(r) \log p = \log r \tag{2.3}$$

for any positive rational r.

For a natural number n, we can write

$$\nu_p(n) = \sum_{j=1}^{\infty} 1_{p^j | n}. \tag{2.4}$$

Upon taking partial sums, we recover Legendre's formula

$$\nu_p(N!) = \sum_{j=1}^{\infty} \left\lfloor \frac{N}{p^j} \right\rfloor = \frac{N - s_p(N)}{p-1} \tag{2.5}$$

where $s_p(N)$ is the sum of the digits of N in the base p expansion.

Given a putative factorization $\mathcal{B}$ of $N!$, we refer to the quantity $\nu_p\left(\frac{N!}{\prod \mathcal{B}}\right)$ as the *p-surplus* of $\mathcal{B}$ with respect to the target $N!$; if it is negative, we refer to $-\nu_p\left(\frac{N!}{\prod \mathcal{B}}\right)$ as the *p-deficit*, with the multiset being *p-balanced* if the p-surplus (or p-deficit) is zero. Thus, a factorization of $N!$ is achieved if and only if one is balanced at every prime p, whereas a subfactorization is achieved if one is either in balance or surplus at every prime p.

We use the usual asymptotic notation $X = O(Y)$, $X \ll Y$, or $Y \gg X$ to denote an inequality of the form $|X| \leq CY$ for some absolute constant C. We also write $X \asymp Y$ for $X \ll Y \ll X$. For effective estimates, we will use the more precise notation $O_{\leq}(Y)$ to denote any quantity whose magnitude is bounded by exactly at most Y.

To bound the factorial, we have the explicit Stirling approximation [12]

$$N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N+1} \leq \log N! \leq N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N}, \tag{2.6}$$

valid for all natural numbers N.

3. Linear programming

A surprisingly sharp upper bound on $t(N)$ comes from linear programming.

Lemma 3.1 (Linear programming bound)**.** *Let N be an natural number and $1 \leq t \leq N/2$. Suppose for each prime $p \leq N$, one has a non-negative real number w_p which is weakly non-decreasing in p (thus $w_p \leq w_{p'}$ when $p \leq p'$), and such that*

$$\sum_p w_p \nu_p(j) \geq 1 \tag{3.1}$$

for all $t \leq j \leq N$, and such that

$$\sum_p w_p \nu_p(N!) < N. \tag{3.2}$$

Then $t(N) < t$.

Proof. We first observe that the bound (3.1) in fact holds for all $j \geq t$, not just for $t \leq j \leq N$. Indeed, if this were not the case, consider the first $j \geq t$ where (3.1) fails. Take a prime p dividing j and replace it by a prime in the interval $[p/2, p)$ which exists by Bertrand's postulate (or remove p entirely, if $p = 2$); this creates a new j' in $[j/2, j)$ which is still at least t. By the weakly decerasing hypothesis on w_p, we have

$$\sum_p w_p \nu_p(j) \geq \sum_p w_p \nu_p(j')$$

and hence by the minimality of j we have

$$\sum_p w_p \nu_p(j) > 1,$$

a contradiction.

Now suppose for contradiction that $t(N) \geq t$, thus we have a factorization $N! = \prod_{j \geq t} j^{m_j}$ for some natural numbers m_j summing to N. Taking p-valuations, we conclude that

$$\sum_{j \geq t} m_j \nu_p(j) \leq \nu_p(N!)$$

for all $p \leq N$. Multiplying by w_p and summing, we conclude from (3.1) that

$$N = \sum_{j \geq t} m_j \leq \sum_p w_p \nu_p(N!),$$

contradicting (3.2). □

This bound is sharp for all $N \leq 600$, with the exception of $N = 155$, where it gives the upper bound $t(155) \leq 46$. A more precise integer program (discussed below) gives $t(155) = 45$.

A variant of the linear programming method also gives good lower bound constructions. Specifically, one can use linear programming to find non-negative real numbers m_j for $t \leq j \leq N$ that maximize the quantity $\sum_{t \leq j \leq N} m_j$ subject to the constraints

$$\sum_{t \leq j \leq N} m_j \nu_p(j) \leq \nu_p(N!).$$

The expression $\prod_{t \leq j \leq N} j^{\lfloor m_j \rfloor}$ will then be a subfactorization of $N!$ into $\sum_{t \leq j \leq N} \lfloor m_j \rfloor$ factors j, each of which is at least t. If $\sum_{t \leq j \leq N} \lfloor m_j \rfloor \geq N$, this demonstrates that $t(N) \geq t$. Numerically, this procedure attains the exact value of $t(N)$ for all $N \leq 600$; for instance for $N = 155$, it shows that $t(155) \geq 45$.

discuss integer programming, need to restrict j to a finite set of "useful" integers

These methods also give quite precise upper and lower bounds for larger values of N, but with quite slow runtime. For instance, with $N = 3 \times 10^5$ and $t = N/3 = 10^5$, the upper bound method can be used to show that any t-admissible factorization has cardinality at most $N + 455$, while the lower bound method produces a t-admissible factorization of exactly this cardinality.

more discussion here

By using the greedy method, Theorem 1.3(ii) can be verified for $N \leq 3 \times 10^5$, and Theorem 1.3(iii) can be verified for $8 \times 10^4 \leq N \leq$???. The linear programming method can also estabalish Theorem 1.3(iii) in the range $43632 \leq N \leq 8 \times 10^4$. Thus, to resolve these claims, it remains to only establish Theorem 1.3(iii) in the regime $N >$???.

4. Greedy algorithms

The following simple greedy algorithm gives reasonably good performance to obtain large t-admissible subfactorizations $\mathcal{B}$ of $N!$ for a given choice of t and N:

(0) Initialize $\mathcal{B}$ to be the empty multiset.
(1) If $\mathcal{B}$ is not a factorization, locate the largest prime p which is currently in surplus: $\nu_p(N!/\prod \mathcal{B}) > 0$.
(2) If $N!/\prod \mathcal{B}$ contains a multiple of p that is greater than or equal to t, locate the smallest such multiple, add it to $\mathcal{B}$, and return to Step 1. Otherwise, `HALT` the algorithm.

This procedure clearly halts in finite time to produce a t-admissible subfactoriation of $N!$. For instance, applying this procedure with $N = 9$, $t = 3$ produces the 3-admissible subfactorization

$$\{7 \times 1, 5 \times 1, 3 \times 1, 3 \times 1, 3 \times 1, 3 \times 1, 2 \times 2, 2 \times 2, 2 \times 2\}$$

which recovers the bound $t(9) \geq 3$ from Example 1.1 (though with a slightly different subfactorization, in which the 8 is replaced by 4).

This procedure is efficient for small N, for instance attaining the exact value of $t(N)$ for all $N \leq 79$, though it begins to degrade for larger N; see Figure 4. The performance is also respectable (though not optimal) for medium N; for instance, when $N = 3\times10^5$ and $t = N/3$, it locates a t-admissible subfactorization of $N!$ of cardinality $N + 372$, which is close to the linear programming limit of $N + 455$.

discuss modifications to the algorithm to make it perform both faster and more accurately

5. The accounting identity

Given a t-admissible multiset $\mathcal{B}$ (which we view as an approximate factorization of $N!$), we can apply (2.3) to the $r := N!/\prod \mathcal{B}$ and rearrange to obtain the *accounting identity*

$$\mathcal{E}_t(\mathcal{B}) + \sum_p \nu_p\left(\frac{N!}{\prod \mathcal{B}}\right)\log p = \log N! - |\mathcal{B}|\log t \tag{5.1}$$

where we define the *t-excess* $\mathcal{E}_t(\mathcal{B})$ of the multiset $\mathcal{B}$ by the formula

$$\mathcal{E}_t(\mathcal{B}) := \sum_{a\in\mathcal{B}} \log\frac{a}{t}. \tag{5.2}$$

Example 5.1. Suppose one wishes to factorize $5! = 2^3 \times 3 \times 5$. The attempted 3-admissible factorization $\mathcal{B} := \{3,4,5,5\}$ has a 2-surplus of $\nu_2(5!/\prod \mathcal{B}) = 1$, is in balance at 3, and has a 5-deficit of $\nu_2(\prod \mathcal{B}/5!) = 1$, so it is not a factorization or subfactorization of $5!$. The 3-excess of this multiset is

$$\mathcal{E}_3(\mathcal{B}) = \log\frac{3}{3} + \log\frac{4}{3} + \log\frac{5}{3} + \log\frac{5}{3} = 1.3093\ldots$$

and the accounting identity (5.1) become

$$1.3093\cdots + \log 2 - \log 5 = 0.3930\cdots = \log 5! - 4\log 3.$$

If one replaces one of the copies of 5 in $\mathcal{B}$ with a 2, this erases both the 2-surplus and the 5-deficit, and creates a factorization $\mathcal{B}' = \{2,3,4,5\}$ of $5!$; the 3-excess now drops to

$$\mathcal{E}_3(\mathcal{B}) = \log\frac{2}{3} + \log\frac{3}{3} + \log\frac{4}{3} + \log\frac{5}{3} = 0.3930\ldots,$$

bringing the accounting identity back into balance.

In view of Remark 1.2, one can now equivalently describe $t(N)$ as follows:

Lemma 5.2 (Equivalent description of $t(N)$)**.** *$t(N)$ is the largest quantity t for which there exists a t-admissible subfactorization of $N!$ with*

$$\mathcal{E}_t(\mathcal{B}) + \sum_p \nu_p\left(\frac{N!}{\prod \mathcal{B}}\right)\log p \leq \log N! - N\log t.$$

One can view $\log N! - N\log t$ as an available "budget" that one can "spend" on some combination of t-excess and p-surpluses. For t of the form $t = N/e^{1+\delta}$ for some $\delta > 0$, the budget can be computed using the Stirling approximation (2.6) to be $\delta N + O(\log N)$. The non-negativity of the t-excess and p-surpluses recovers the trivial upper bound (1.2); but one can improve upon this bound by observing that large prime factors of $N!$ inevitably generate a noticeable t-excess, as follows.

Lemma 5.3 (Upper bound criterion)**.** *Suppose that $1 \leq t \leq N$ are such that*

$$\sum_{p > \frac{t}{\lfloor\sqrt{t}\rfloor}} \left\lfloor\frac{N}{p}\right\rfloor \log\left(\frac{p}{t}\left\lceil\frac{t}{p}\right\rceil\right) > \log N! - N\log t \tag{5.3}$$

Then $t(N) < t$.

Proof. Suppose for contradiction that $t(N) \geq t$, then we can find a t-admissible factorization $\mathcal{B}$ of $N!$. The accounting identity then gives

$$\sum_{a \in \mathcal{B}} \log \frac{a}{t} = \mathcal{E}_t(\mathcal{B}) = \log N! - N \log t. \tag{5.4}$$

Let $f_t(p) := \log(\frac{p}{t}\lceil \frac{t}{p} \rceil)$. We claim that

$$\log \frac{a}{t} \geq f_t(p_{a,1}) + \cdots + f_t(p_{a,k_a}) \tag{5.5}$$

for all $a \in \mathcal{B}$, where $p_{a,1}, \dots, p_{a,k_a}$ are the primes greater than $\frac{t}{\lfloor \sqrt{t} \rfloor}$ that divide a (counting multiplicity). For $k_a = 0$ this is clear since $a \geq t$. For $k_a = 1$, we can write $a = d_a p_{a,1}$ where $p_{a,1} > \frac{t}{\sqrt{t}+1}$ and $d_a \geq \lceil \frac{t}{p_{a,1}} \rceil$, so that

$$\log \frac{a}{t} = \log \left(\frac{p_{a,1}}{t} d_a \right) \geq f_t(p_{a,1}),$$

again giving (5.5). For $k_a \geq 2$, we have $a \geq p_{a,1} \dots p_{a,k}$, hence

$$\begin{aligned}
\log \frac{a}{t} - \sum_{j=1}^{k_a} f_t(p_{a,j}) &\geq \sum_{j=1}^{k_a} (\log p_{a,j} - f_t(p_{a,j})) - \log t \\
&= \sum_{j=1}^{k_a} \left(\log t - \log \left\lceil \frac{t}{p_{a,j}} \right\rceil \right) - \log t \\
&\geq \sum_{j=1}^{k_a} \left(\log t - \log \sqrt{t} \right) - \log t \\
&\geq 0
\end{aligned}$$

which again gives (5.5). Summing (5.5) over all $a \in \mathcal{B}$ and inserting into (5.4), we conclude that

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \nu_p(N!) f_t(p) \leq \log N! - N \log t.$$

By (2.5), this contradicts (5.3), giving the claim. □

In practice, Lemma 5.3 gives reasonable upper bounds on N, especially when N is large, although for medium N the linear programming method is superior: see Figure 1, Figure 2, Figure 4

We can now prove the upper bound portion of Theorem 1.3(iv):

Proposition 5.4. *For large N, one has*

$$\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0}{\log N} + O\left(\frac{1}{\log^2 N} \right).$$

Proof. We apply Lemma 5.3 with

$$t := \frac{1}{e} - \frac{c_0}{\log N} + \frac{C_0}{\log^2 N}$$

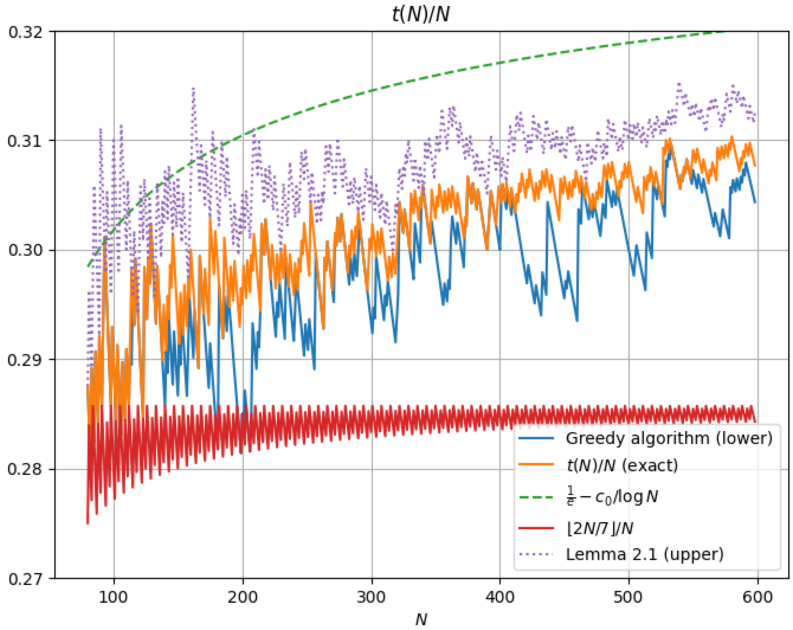

FIGURE 4. An enlarged version of Figure 2, displaying the lower bound from the greedy algorithm and the upper bound from Lemma 5.3. The linear programming upper and lower bounds are exact in this region, except for $N = 155$ in which the upper bound is off by one. **TODO: relabel image to reflect new lemma numbering**

with C_0 a large absolute constant to be chosen later. From Taylore expansion and the Stirling approximation one sees that

$$\log N! - N \log t \geq ec_0 \frac{N}{\log N} + (C_0 - O(1)) \frac{N}{\log^2 N}$$

so it will suffice to establish the upper bound

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor \log \left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil \right) \leq ec_0 \frac{N}{\log N} + O\left(\frac{N}{\log^2 N} \right).$$

For N large enough, we have $\frac{t}{\lfloor \sqrt{t} \rfloor} \leq \frac{N}{\log N}$, so it suffices to show that

$$\sum_{\frac{N}{\log N} \leq p \leq N} \left\lfloor \frac{N}{p} \right\rfloor \log \left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil \right) \leq ec_0 \frac{N}{\log N} + O\left(\frac{N}{\log^2 N} \right).$$

The summand is a piecewise monotone function of p, with $O(\log N)$ pieces, and bounded in size by $O(N)$, so the total variation is $O(N \log N)$. By Lemma C.2, the left-hand side is then

$$\int_{N/\log N}^{N} \left\lfloor \frac{N}{x} \right\rfloor \log \left(\frac{x}{t} \left\lceil \frac{t}{x} \right\rceil \right) \frac{dx}{\log x} + O\left(N \exp(-c\sqrt{\log N}) \right)$$

for some $c > 0$. Thus it remains to show that

$$\int_{N/\log N}^{N} \left\lfloor \frac{N}{x} \right\rfloor \log\left(\frac{x}{t}\left\lceil \frac{t}{x} \right\rceil\right) \frac{dx}{\log x} \leq ec_0 \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right).$$

We use the approximation

$$\frac{1}{\log x} = \frac{1}{\log N} + O\left(\frac{\log(N/x)}{\log^2 N}\right).$$

To control the error term, we observe from Taylor expansion that

$$\log\left(\frac{x}{t}\left\lceil \frac{t}{x} \right\rceil\right) \ll \frac{\lceil \frac{t}{x} \rceil - \frac{t}{x}}{t/x} \ll \frac{x}{t} \ll \frac{x}{N} \tag{5.6}$$

and the contribution of the error term is

$$\ll \int_{N/\log N}^{N} \frac{N}{x} \frac{x}{N} \frac{\log(N/x)}{\log^2 N}\, dx \ll \frac{N}{\log^2 N}$$

which is acceptable. As for the main term, we can rescale it to

$$\frac{et}{\log N} \int_{N/et \log N}^{N/et} \left\lfloor \frac{N/et}{x} \right\rfloor \log\left(ex\left\lceil \frac{1}{ex} \right\rceil\right) dx.$$

Since $N/et = 1 + O(1/\log N)$, we see that the integrand here is within $O(1/\log N)$ of $\lfloor \frac{1}{x} \rfloor \log\left(ex\lceil \frac{1}{ex} \rceil\right)$ unless $\frac{1}{x}$ is within $O(1/\log N)$ of an integer, which one can calculate to occur on a set of measure $O(1/\log N)$. A variant of (5.6) shows that both integrands are bounded by $O(1)$ for all $x \in [0, N/et]$, so by the triangle inequality the above expression can be rewritten as

$$\frac{N}{\log N} \int_0^1 \left\lfloor \frac{1}{x} \right\rfloor \log\left(ex\left\lceil \frac{1}{ex} \right\rceil\right) dx + O\left(\frac{N}{\log^2 N}\right),$$

and the claim follows from (1.6). □

We can now establish Theorem 1.3(i):

Proposition 5.5. *One has $t(N)/N < 1/e$ for $N \neq 1, 2, 4$.*

Proof. From the linear programming method one can verify this claim for $N < 599$ (see Figure 2), so we assume that $N \geq 599$, so that the prime number theorem bound (C.1) becomes available.

Applying Lemma 5.3, (2.6), it suffices to show that

$$\sum_{p \geq \frac{N/e}{\lfloor \sqrt{N/e} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor f_{N/e}(p) > \frac{1}{2}\log(2\pi N) + \frac{1}{12N} \tag{5.7}$$

where $f_{N/e}(p) = \log(\frac{ep}{N}\lceil \frac{N}{ep} \rceil)$ is as in the proof of the proposition. As the left-hand side is $\asymp N/\log N$, while the right-hand side is $\asymp \log N$, there is significant room to spare here, and we can use somewhat lossy arguments.

For $N/\sqrt{2e} < p \leq N$ one can obtain the lower bound $\lfloor \frac{N}{p} \rfloor f_{N/e}(p) \geq \log(e/2)$ (see Figure 3), and (since $\frac{N/e}{\lfloor \sqrt{N/e} \rfloor} \leq N/2e$ in the regime $N \geq 599$), so we may crudely bound the left-hand side of (5.7) from below by

$$\left(\pi(N) - \pi(N/\sqrt{2e}) \right) \log(e/2).$$

Applying (C.1), (C.2), we reduce to showing that

$$\begin{aligned} &\left(\frac{N}{\log N} \left(1 + \frac{1}{\log N} \right) - \frac{N/\sqrt{2e}}{\log(N/\sqrt{2e})} \left(1 + \frac{1.2762}{\log(N/\sqrt{2e})} \right) \right) \log(e/2) \\ &\quad > \frac{1}{2} \log(2\pi N) + \frac{1}{12N} \end{aligned} \tag{5.8}$$

for $N \geq 599$. This can be numerically verified for $N = 599$ (see Figure 5), so by the fundamental theorem of calculus it suffices to show that the derivative of the left-hand side is at least that of the right hand side for (real) $N \geq 599$. Computing this derivative, dividing by $\log(e/2)$, and discarding some terms with a favorable sign, we reduce to showing that

$$\frac{1}{\log N} - \frac{2}{\log^3 N} - \frac{1}{\sqrt{2e}\log(N/\sqrt{2e})} - \frac{0.2762}{\sqrt{2e}\log^2(N/\sqrt{2e})} \geq \frac{1}{2\log(e/2)N}$$

for $N \geq 599$. But in this range we have the crude lower bounds $\log N \geq \log(N/\sqrt{2e}) \geq 5$, $\sqrt{2e}\log(N/\sqrt{2e}) \geq 2\log N$, and $2\log(e/2)N \geq 50\log N$, and the claim then follows (with room to spare) by estimating all terms here by constant multiples of $\frac{1}{\log N}$. □

5.1. **Modified approximate factorizations.** In this section we present and then analyze an algorithm that, when given parameters $1 \leq t \leq N$, will attempt to construct a t-admissible subfactorization $\prod \mathcal{B}$ of $N!$ that obeys the criterion in Lemma 5.2. The algorithm will not always succeed, but when it does, it will certify that $t(N) \geq t$.

5.2. **Description of algorithm.** In addition to the given parameters $1 \leq t \leq N$, we require additional natural number parameters A, K

$$K^2(1+\sigma) < t, \tag{5.9}$$

where

$$\sigma := \frac{3N/t}{A}. \tag{5.10}$$

There is some freedom to select parameters here, but roughly speaking one would like to have $1 \lll A \lll K \lll \sqrt{t}$.

With such parameters in hand, we can consider the following algorithm.

(1) Let I denote the elements of the interval[2] $(t, t(1+\sigma)]$ that are coprime to 6. Let $\mathcal{B}^{(1)}$ be the elements of I, each occurring with multiplicity A. This multiset is t-admissible,

[2]Numerically, it would be slightly better to use the closed interval $[t, t(1+\sigma)]$ instead of the half-open interval $(t, t(1+\sigma)]$, but we will consistently aim to use half-open intervals here to be compatible with standard notation for the prime counting function $\pi(x)$.

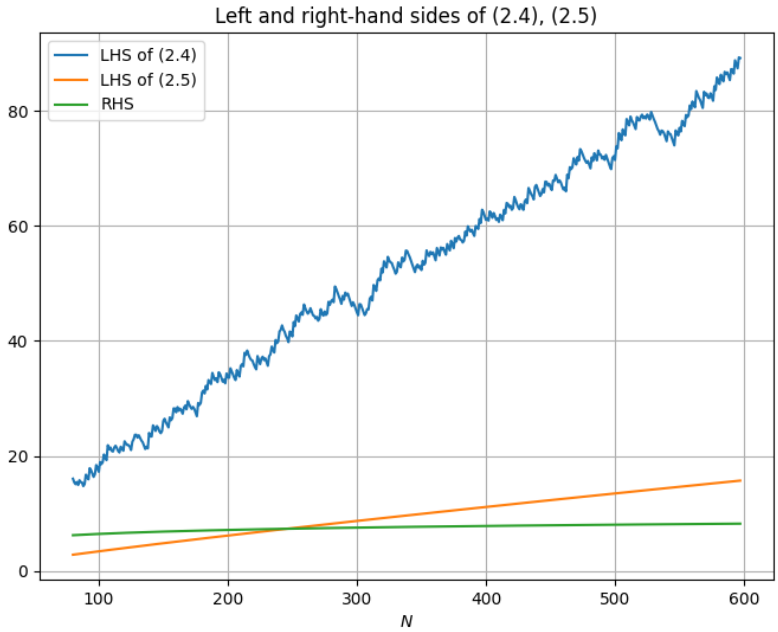

FIGURE 5. A plot of the left and right-hand sides of (5.7), (5.8) for $80 \leq N < 599$. For $N \geq 599$, the effective prime number theorem from (C.2), (C.1) rigorously establishes the left-hand side of (5.8) as a (crude) lower bound for the left-hand side of (5.7).

and $\prod \mathcal{B}^{(1)}$ is not divisible by tiny primes $2, 3$. (It will have approximately the right number of primes for $3 < p \leq t/K$, though it may have quite different prime factorization at primes $p > t/K$.)

(2) Remove any element from $\mathcal{B}^{(1)}$ that contains a prime factor p with $p > t/K$, and call this new multiset $\mathcal{B}^{(2)}$. It remains t-admissible with no tiny prime factors, though it tends to acquire a p-surplus in the range $3 < p \leq K$.

(3) For each $p > t/K$, add in $\nu_p(N!)$ copies of the number $p\lceil t/p \rceil$ to $\mathcal{B}^{(2)}$, and call this new multiset $\mathcal{B}^{(3)}$. Now $\mathcal{B}^{(3)}$ is t-admissible and in balance at all primes $p > t/K$, but will typically be in a slight deficit at primes $3 < p \leq t/K$, particularly in the range $3 < p \leq K$. (It will now also contain a few tiny prime factors, but will generally still have a large surplus at those primes.)

(4) For each prime $3 < p \leq t/K$ at which there is a surplus $\nu_p(N!/\prod \mathcal{B}) > 0$, replace $\nu_p(N!/\prod \mathcal{B})$ copies of p in the prime factorizations of elements of $\mathcal{B}^{(3)}$ with $\lceil p \rceil^{\langle 2,3 \rangle}$ instead, and call this new multiset $\mathcal{B}^{(4)}$. Thus $\mathcal{B}^{(4)}$ has no surplus at primes $3 < p \leq t/K$ (and is still t-admissible and in balance for $p > t/K$).

(5) For the primes $3 < p \leq t/K$ at which there is a deficit $\nu_p(\prod \mathcal{B}/N!) > 0$, multiply all these primes together, and use the greedy algorithm to group them into factors $x_1, \dots, x_M$ in the range $(\sqrt{t/K}, t/K]$, together with possibly one exceptional factor x_* in the range $(1, t/K]$. For each of these factors x_i or x_*, add the quantity $x_i \lceil t/x_i \rceil^{\langle 2,3 \rangle}$ or $x_* \lceil t/x_* \rceil^{\langle 2,3 \rangle}$ to $\mathcal{B}^{(4)}$, and call this new multiset $\mathcal{B}^{(5)}$.

(6) By construction, $\mathcal{B}^{(5)}$ is t-admissible and will be in balance at all primes $p > 3$, and is thus $N!/\prod \mathcal{B}^{(5)}$ is of the form $2^n 3^m$ for some integers n, m. If at least one of n, m is

negative, then HALT the algorithm with an error. Otherwise, select a 3-smooth number $2^{n_1}3^{m_1}$ greater than equal to t with $n_1/m_1 \leq n/m$ (which one can interpret as $n_1 m \leq nm_1$ in case some of the denominators here vanish), and similarly select a 3-smooth number $2^{n_2}3^{m_2}$ greater than or equal to t with $n_2/m_2 \geq n/m$. (It is reasonable to select the smallest such 3-smooth numbers in both cases, although this is not absolutely necessary for the algorithm to be successful.) By construction, we can express (n,m) as a positive linear combination $\alpha_1(n_1,m_1) + \alpha_2(n_2,m_2)$ of (n_1,m_1) and (n_2,m_2). Add $\lfloor \alpha_1 \rfloor$ copies of $2^{n_1}3^{m_1}$ and $\lfloor \alpha_2 \rfloor$ copies of $2^{n_2}3^{m_2}$ to $\mathcal{B}^{(5)}$, and call this tuple $\mathcal{B}^{(6)}$. (This will largely eliminate the surplus at 2 and 3.)

(7) If the criterion in Lemma 5.2 is obeyed by $\mathcal{B}^{(6)}$, then we have successfully established[3] that $t(N) \geq t$. Otherwise, HALT the algorithm with an error.

To analyze this algorithm, it will be convenient to divide the set of primes into the following ranges:

- *Tiny primes* $p = 2, 3$.
- *Small primes* $3 < p \leq K$.
- *Borderline small primes* $K < p \leq K(1+\sigma)$.
- *Medium primes* $K(1+\sigma) < p \leq t/K$.
- *Large primes* $p > t/K$.

The expected p-surpluses or p-deficits at various stages of this process are summarized in Table 1.

	Tiny p	Small p	Borderline p	Medium p	Large p
$\mathcal{B}^{(1)}$	Max. surplus	Near balance	Near balance	Near balance	???
$\mathcal{B}^{(2)}$	Max. surplus	Med. surplus	Med. surplus?	Near balance	Max. surplus
$\mathcal{B}^{(3)}$	Lg. surplus	Sm. surplus?	Med. surplus?	Near balance	Balance
$\mathcal{B}^{(4)}$	Lg. surplus	Balance?	Balance?	Balance/sm. deficit	Balance
$\mathcal{B}^{(5)}$	Lg. surplus	Balance	Balance	Balance	Balance
$\mathcal{B}^{(6)}$	Sm. surplus	Balance	Balance	Balance	Balance
$\mathcal{B}^{(7)}$	Balance	Balance	Balance	Balance	Balance

TABLE 1. Evolution of the surpluses and deficits of the multisets $\mathcal{B}^{(i)}$, $i = 1, \dots, 6$; we describe the size of these surpluses and deficits informally as "small", "medium", "large", or "maximal". For entries with a question mark, we allow the possibility of a tiny deficit. For the entry marked ???, all behavior from large surpluses to large deficits are possible. The final step $\mathcal{B}^{(7)}$ is an optional one, if one wishes to convert the subfactorization $\mathcal{B}^{(6)}$ to an exact factorization.

5.3. **Analysis of Step 7.** We now analyze the above algorithm, starting from the final Step 7 and working backwards to Step 1, to establish sufficient conditions for the algorithm to successfully demonstrate that $t(N) \geq t$.

[3] If desired, one could implement the proof of Lemma 5.2 as a final component of this algorithm, that is to say one removes elements from $\mathcal{B}^{(6)}$ to make the cardinality exactly N, and then distributes any surplus primes arbitrarily to create a t-admissible factorization of $N!$ of cardinality exactly N.

It will be convenient to introduce the following notation. For $a_+, a_- \in [0, +\infty]$, we define the asymmetric norm $|x|_{a_+,a_-}$ of a real number x by the formula

$$|x|_{a_+,a_-} := \begin{cases} a_+|x| & x \geq 0 \\ a_-|x| & x \leq 0, \end{cases}$$

with the usual convention $+\infty \times 0 = 0$. If a_+, a_- are finite, this function is Lipschitz with constant $\max(a_+, a_-)$. One can think of a_+ as the "cost" of making x positive, and a_- as the "cost" of making x negative. We can then rewrite the termination condition of Lemma 5.2 (using the fact that $\mathcal{B}^{(6)}$ is a subfactorization of $N!$) as

$$\mathcal{E}_t(\mathcal{B}^{(6)}) + \sum_p \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(6)}} \right) \right|_{\log p, \infty} \leq \log N! - N \log t.$$

If we assume that $t = N/e^{1+\delta}$ for some $\delta > 0$, we can use the Stirling approximation (2.6) to reduce to the sufficient condition

$$\mathcal{E}_t(\mathcal{B}^{(6)}) + \sum_p \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(6)}} \right) \right|_{\log p, \infty} \leq \delta N + \log \sqrt{2\pi N}. \tag{5.11}$$

5.4. **Analysis of Step 6.** Now we analyze Step 6, using the quantity κ_L introduced in (2.1). **TODO: draw picture of n_0, m_0, etc.**

Lemma 5.6. *Let $L \geq 1$. Let $t > 3L$ and let $2^n 3^m$ be a 3-smooth number with n, m positive and in the sector*

$$\frac{\log(3L) + \kappa_L}{\log t - \log(3L)} \leq \frac{n \log 2}{m \log 3} \leq \frac{\log t - \log(2L)}{\log(2L) + \kappa_L}. \tag{5.12}$$

Then one can find a t-admissible subfactorization $\mathcal{B}$ of $2^n 3^m$ such that

$$\mathcal{E}_t(\mathcal{B}) \leq \kappa_L \frac{n \log 2 + m \log 3}{\log t} \tag{5.13}$$

and

$$\sum_{p_0 = 2,3} |\nu_{p_0}(2^n 3^m / \mathcal{B})|_{\log p_0, \infty} \leq 2(\log t + \kappa_L). \tag{5.14}$$

In practice, $\log t$ will be significantly larger than $\log(2L)$ or $\log(3L)$, and so the hypothesis of containment in the sector $\Gamma_{t,L}$, as long as n and m are both reasonably large.

Proof. Let $2^{n_0}, 3^{m_0}$ be the largest powers of 2 and 3 less than or equal to t/L respectively, thus

$$L \leq \frac{t}{2^{n_0}} \leq 2L \tag{5.15}$$

and

$$L \leq \frac{t}{3^{m_0}} \leq 3L. \tag{5.16}$$

From (2.1), the 3-smooth numbers $2^{n_1} 3^{m_1} := \lceil t/2^{n_0} \rceil^{\langle 2,3 \rangle}$, $2^{n_2} 3^{m_2} := \lceil t/3^{m_0} \rceil^{\langle 2,3 \rangle}$ obey the estimates

$$\frac{t}{2^{n_0}} \leq 2^{n_1} 3^{m_1} \leq e^{\kappa_L} \frac{t}{2^{n_0}} \tag{5.17}$$

and

$$\frac{t}{3^{m_0}} \leq 2^{n_2}3^{m_2} \leq e^{\kappa_L}\frac{t}{3^{m_0}}, \tag{5.18}$$

or equivalently

$$t \leq 2^{n_0+n_1}3^{m_1}, 2^{n_2}3^{m_0+m_2} \leq e^{\kappa_L}t. \tag{5.19}$$

We can use (5.15), (5.17) to bound

$$\begin{aligned}\frac{n_0+n_1}{m_1} &\geq \frac{n_0}{\log(e^{\kappa_L}\frac{t}{2^{n_0}})/\log 3} \\ &\geq \frac{(\log t - \log(2L))/\log 2}{(\log(2L)+\kappa_L)/\log 3}\end{aligned}$$

(with the convention that this bound is vacuously true for $m_1 = 0$). Similarly, from (5.16), (5.18) we have

$$\begin{aligned}\frac{n_2}{m_0+m_2} &\leq \frac{\log(e^{\kappa}\frac{t}{3^{m_0}})/\log 2}{m_0} \\ &\leq \frac{(\log(3L)+\kappa)/\log 2}{(\log t - \log(3L))/\log 3}\end{aligned}$$

and hence by (5.12)

$$\frac{n_2}{m_0+m_2} \leq \frac{n}{m} \leq \frac{n_0+n_1}{m_1}. \tag{5.20}$$

Thus we can write (n, m) as a non-negative linear combination

$$(n, m) = \alpha_1(n_0+n_1, m_1) + \alpha_2(n_2, m_0+m_2)$$

for some real $\alpha_1, \alpha_2 \geq 0$. We now take our subfactorization $\mathcal{B}$ to consist of $\lfloor \alpha_1 \rfloor$ copies of the 3-smooth number $2^{n_0+n_1}3^{m_1}$ and $\lfloor \alpha_2 \rfloor$ copies of the 3-smooth number $2^{n_2}3^{m_0+m_2}$. By (5.19), each term $2^{n'}3^{m'}$ here is admissible and contributes a t-excess of at most κ_L, which is in turn bounded by $\kappa_L \frac{n'\log 2 + m'\log 3}{\log t}$. Adding these bounds together, we obtain (5.13).

As a subfactorization of 2^n3^m, the multiset $\mathcal{B}$ has a 2-surplus of at most $n_0+n_1+n_2$ and a 3-surplus of at most $m_0+m_2+m_1$, hence

$$\sum_{p_0=2,3} \nu_{p_0}\left(\frac{2^n3^m}{\prod \mathcal{B}}\right)\log p_0 \leq \log 2^{n_0+n_1}3^{m_1} + \log 2^{n_2}3^{m_0+m_2},$$

and the bound (5.14) follows from (5.19). □

We now use this lemma to analyze Step 6 as follows.

Proposition 5.7. *Let $L \geq 1$. Let $3L < t = N/e^{1+\delta}$ for some $\delta > 0$, and let $1 \leq K \leq t$ and $A \geq 1$. Suppose that the algorithm in Section 5.2 with the indicated parameters reaches the end of Step 5 with a multiset $\mathcal{B}^{(5)}$ obeying the following hypotheses:*

(i) *(Small excess and surplus at non-tiny primes)*

$$\mathcal{E}_t(\mathcal{B}^{(5)}) + \sum_{p>3}\left|\nu_p\left(\frac{N!}{\prod \mathcal{B}^{(5)}}\right)\right|_{\log p,\infty} \leq \delta N + \log\sqrt{2\pi} - \frac{3}{2}\log N - (\kappa_L \log\sqrt{12})\frac{N}{\log t}. \tag{5.21}$$

(ii) *(Large surpluses at tiny primes) One has*

$$\sum_{p=2,3} \nu_p \left(\prod \mathcal{B}^{(5)}\right) \log p < \min(Q_{N,t,L}, Q'_{N,t,L}) \tag{5.22}$$

where

$$Q_{N,T,L} := \nu_2(N!) \log 2 - \frac{\log(3L) + \kappa_L}{\log t - \log(3L)} \nu_3(N!) \log 3 \tag{5.23}$$

and

$$Q'_{N,T,L} := \nu_3(N!) \log 3 - \frac{\log(2L) + \kappa_L}{\log t - \log(2L)} \nu_2(N!) \log 2. \tag{5.24}$$

Then $t(N) \geq t$.

From (2.5) we have

$$Q_{N,T,L} \geq N \log 2 - \log N - \frac{\log(3L) + \kappa_L}{\log t - \log(3L)} \frac{N}{2} \log 3$$

and

$$Q'_{N,T,L} \geq \frac{N}{2} \log 3 - \log N - \frac{\log(2L) + \kappa_L}{\log t - \log(2L)} N \log 2;$$

since $\frac{1}{2} \log 3 < \log 2$, we may replace $\min(Q_{N,T,L}, Q'_{N,T,L})$ in the above expression by the simpler

$$\frac{N}{2} \log 3 - \log N - \frac{\log(3L) + \kappa_L}{\log t - \log(3L)} N \log 2$$

Proof. Write $n := \nu_2(N!/\prod \mathcal{B}^{(5)})$ and $m := \nu_3(N!/\prod \mathcal{B}^{(5)})$. From (2.5) we have $n \leq N$ and $m \leq N/2$, hence

$$n \log 2 + m \log 3 \leq N \log \sqrt{12}.$$

From (5.22) we have

$$(\nu_2(N!) - n) \log 2 \leq Q_{N,T,L}$$

and hence

$$n > \frac{\log(3L) + \kappa_L}{\log t - \log(3L)} \frac{\nu_3(N!) \log 3}{\log 2}$$

and similarly

$$m > \frac{\log(2L) + \kappa_L}{\log t - \log(2L)} \frac{\nu_2(N!) \log 2}{\log 3}.$$

In particular, n, m are positive. Since we also have $n \leq \nu_2(N!)$ and $m \leq \nu_3(N!)$, the condition (5.6) holds. Applying Lemma 5.6, we can find a subfactorization $\mathcal{B}'$ of $2^n 3^m$ with an excess of at most $(\kappa_L \log \sqrt{12}) \frac{N}{\log t}$, and with

$$\sum_{p_0=2,3} \left| \nu_{p_0} \left(\frac{2^n 3^m}{\prod \mathcal{B}'} \right) \right|_{\log p_0, \infty} \leq 2(\log t + \kappa_L) \leq 2 \log N$$

where we have used (2.2) and the fact that $\log t \leq \log N - 1$. Then $\mathcal{B}^{(6)} = \mathcal{B}^{(5)} \cup \mathcal{B}'$ is another t-admissible multiset, and from (5.21), we obtain the previous sufficient condition (5.11). $\square$

5.5. **Analysis of Step 5.**

Proposition 5.8. *Let $1 \leq K \leq t \leq N$, $A \geq 1$, and $L \geq 1$ be parameters such that $9L < t = N/e^{1+\delta}$ for some $\delta > 0$, and (5.9) holds. Suppose that the algorithm in Section 5.2 with the indicated parameters reaches the end of Step 4 to produce a multiset $\mathcal{B}^{(4)}$ obeying the following hypotheses.*

(i) *(Small excess and surplus at small/medium primes)*

$$\begin{aligned} &\mathcal{E}_t(\mathcal{B}^{(4)}) + \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)\right|_{\kappa_K \min(\frac{\log p}{\log \sqrt{t/K}},1),\infty} \\ &\quad \leq \delta N - \frac{3}{2}\log N - \kappa_L(\log\sqrt{12})\frac{N}{\log t}. \end{aligned} \tag{5.25}$$

(ii) *(Large surpluses at tiny primes) We have*

$$\begin{aligned} &\sum_{p_0=2,3} \nu_{p_0}\left(\prod \mathcal{B}^{(4)}\right)\log p_0 \\ &\quad + \sum_{3<p\leq t/K} \left|\nu_p\left(\frac{N!}{\prod\mathcal{B}^{(4)}}\right)\right|_{\frac{\log\sqrt{tK}+\kappa_K}{\log\sqrt{t/K}}\log p,\infty} + \log t + \kappa_L \\ &\quad \leq \frac{N}{2}\log 3 - \log N - \frac{\log(3L)+\kappa_L}{\log t - \log(3L)} N\log 2. \end{aligned}$$

Then $t(N) \geq t$.

Proof. By (5.25), $\mathcal{B}^{(4)}$ is a subfactorization of $N!$, and by construction it is in balance at all large primes $p > t/K$. Consider all the p-surplus primes in the small, borderline small, and medium range $3 < p \leq t/K$, thus each such prime is considered with multiplicity $\nu_p(N!/\prod\mathcal{B}^{(4)})$. Using the greedy algorithm, one can factor the product of all these primes into M factors $c_1,\dots,c_M$ in the interval $(\sqrt{t/K}, t/K]$, times at most one exceptional factor c_* in $(1,\sqrt{t/K}]$, for some M. If we let M' denote the number of factors in $c_1,\dots,c_M$ that are not divisible by a prime larger than $\sqrt{t/K}$, we have the bound

$$\left(\sqrt{t/K}\right)^{M'} \leq \prod_{3<p\leq\sqrt{t/K}} \nu_p\left(\frac{N!}{\prod\mathcal{B}^{(4)}}\right)$$

and hence on taking logarithms

$$M' \leq \sum_{3<p\leq\sqrt{t/K}} \left|\nu_p\left(\frac{N!}{\prod\mathcal{B}^{(4)}}\right)\right|_{\frac{\log p}{\log\sqrt{t/K}},\infty}.$$

Restoring the factors divisible by primes $p > \sqrt{t/K}$, we conclude that

$$M \leq \sum_{3<p\leq t/K} \left|\nu_p\left(\frac{N!}{\prod\mathcal{B}^{(4)}}\right)\right|_{\min(\frac{\log p}{\log\sqrt{t/K}},1),\infty}. \tag{5.26}$$

For each of the M factors c_i, we introduce the 3-smooth number $\lceil t/c_i \rceil^{\langle 2,3\rangle} = 2^{n_i}3^{m_i}$, which by (2.1) lies in the interval $[t/c_i, e^{\kappa_K} t/c_i]$; similarly, for the exceptional factor c_* we introduce a 3-smooth number $\lceil t/c_* \rceil^{\langle 2,3\rangle} = 2^{n_*}3^{m_*}$ in the interval $[t/c_*, e^{\kappa_K} t/c_*]$. If we then adjoin the 3-smooth numbers $\lceil t/c_i \rceil^{\langle 2,3\rangle} c_i = 2^{n_i}3^{m_i}c_i$ for $i = 1, \dots, M$ as well as $\lceil t/c_* \rceil^{\langle 2,3\rangle} c_* = 2^{n_*}3^{m_*}c_*$ to the t-admissible multiset $\mathcal{B}^{(4)}$ to create a new t-admissible multiset $\mathcal{B}^{(5)}$. The quantity $\log\lceil t/c_* \rceil^{\langle 2,3\rangle} = n_i \log 2 + m_i \log 3$ is bounded by $\log\sqrt{tK} + \kappa_K$, and the quantity $\log\lceil t/c_* \rceil^{\langle 2,3\rangle} = n_* \log 2 + m_* \log 3$ is similarly bounded by $\log t + \kappa$, hence if we denote $n_{**} := n_1 + \dots + n_M + n_*$ and $m_{**} := m_1 + \dots + m_M + m_*$, we have

$$\sum_{p_0=2,3} \nu_{p_0}(\prod \mathcal{B}^{(5)}) \log p_0 \leq \sum_{p_0=2,3} \nu_{p_0}(\prod \mathcal{B}^{(4)}) \log p_0 + (\log\sqrt{tK}+\kappa_K) \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)\right|_{\min(\frac{\log p}{\log\sqrt{t/K}},1),\infty} + \log$$

Each of the new factors in $\mathcal{B}^{(5)}$ contributes an excess of at most κ_K, so the total excess of $\mathcal{B}^{(5)}$ is at most

$$\mathcal{E}_t(\mathcal{B}^{(4)}) + \kappa_K M + \kappa_K$$

which by (5.26) is bounded by

$$\mathcal{E}_t(\mathcal{B}^{(4)}) + \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(4)}}\right)\right|_{\kappa_K \min(\frac{\log p}{\log\sqrt{t/K}},1),\infty} + \kappa_K.$$

We conclude that $\mathcal{B}^{(5)}$ obeys the hypotheses of Proposition 5.7 (using (2.2) to bound κ_K by $\log\sqrt{2\pi}$), and the claim follows. □

5.6. **Analysis of Step 4.**

Proposition 5.9. *Let $L \geq 1$. Let $9L < t = N/e^{1+\delta}$ for some $\delta > 0$ be such that* (5.9) *holds, and suppose that the algorithm reaches the end of Step 3 to produce a multiset $\mathcal{B}^{(3)}$ obeying the following hypotheses:*

(i) *(Small excess and surplus at small/medium primes) One has*

$$\mathcal{E}_t(\mathcal{B}^{(3)}) + \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right)\right|_{\kappa_K \min(\frac{\log p}{\log\sqrt{t/K}},1),\kappa_p} \leq \delta N - \frac{3}{2}\log N - \kappa_L(\log\sqrt{12})\frac{N}{\log t}. \tag{5.27}$$

(ii) *One has*

$$\sum_{p_0=2,3} \nu_{p_0}\left(\prod \mathcal{B}^{(3)}\right) \log p_0 + \sum_{3<p\leq t/K} \left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right)\right|_{(\log\sqrt{tK}+\kappa_K)\min(\frac{\log p}{\log\sqrt{t/K}},1),\log p+\kappa_p} + \log t + \kappa_K \leq \frac{N}{2}\log 3 - \log N - \frac{\log(3L)+\kappa_L}{\log t - \log(3L)} N \log 2 \tag{5.28}$$

Then $t(N) \geq t$.

Proof. Suppose there is a non-tiny prime $p > 3$ with a positive p-deficit $|\nu_p(N!/\prod \mathcal{B}^{(3)})|_{0,1} > 0$. Since $\mathcal{B}^{(3)}$ is in balance at all large primes, we have $3 < p \leq t/K$. We locate an element of $\mathcal{B}^{(3)}$ that contains p as a factor, and replaces it with $\lceil p \rceil^{\langle 2,3 \rangle} = 2^{n_p} 3^{m_p}$, which increases that factor by at most $\exp(\kappa_p)$ thanks to (2.1). This procedure reduces the p-deficit by one, adds at most κ_p to the t-excess, and increments $\sum_{p'=2,3} \nu_{p'}(N!/\prod \mathcal{B}^{(3)}) \log p'$ by $n_p \log 2 + m_p \log 3$. Since $n_p \log 2 + m_p \log 3 \leq \log p + \kappa_p$, if we apply this procedure to clear all deficits at non-tiny primes, the resulting multiset $\mathcal{B}^{(4)}$ has a t-excess of

$$\mathcal{E}_t(\mathcal{B}^{(4)}) \leq \mathcal{E}_t(\mathcal{B}^{(3)}) + \sum_{p>3} |\nu_p(N!/\prod \mathcal{B})|_{0,\kappa_p}$$

and we have

$$\sum_{p'=2,3} \nu_p(\prod \mathcal{B}^{(4)}) \log p \leq \sum_{p'=2,3} \nu_p(\prod \mathcal{B}^{(3)}) \log p + \sum_{p>3} \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(3)}} \right) \right|_{0, \log p + \kappa_p}.$$

The hypotheses of Proposition 5.8 are now satisfied, and we are done. ☐

To simplify the criteria here, we introduce the quantities

$$X_1 := \sum_{3<p\leq K} \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(3)}} \right) \right|_{\frac{\log p}{\log \sqrt{t/K}}, 0} + \sum_{K<p\leq t/K} \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(3)}} \right) \right| \tag{5.29}$$

$$X_2 := \sum_{3<p\leq K} \left| \nu_p \left(\frac{N!}{\prod \mathcal{B}^{(3)}} \right) \right|_{0,1}. \tag{5.30}$$

Since $\kappa_K \min(\frac{\log p}{\log \sqrt{t/K}}, 1), \kappa_p$ are both bounded by κ_K for $p \geq K$, and bounded by $\kappa_K \frac{\log p}{\log \sqrt{t/K}}$, κ_5 respectively for $3 < p \leq K$, we can replace (5.27) with

$$\mathcal{E}_t(\mathcal{B}^{(3)}) + \kappa_K X_1 + \kappa_5 X_2 \leq \delta N - \frac{3}{2} \log N - \kappa_L (\log \sqrt{12}) \frac{N}{\log t}. \tag{5.31}$$

Similarly, since $\log p + \kappa_p$ is bounded by $\log \sqrt{tK} + \kappa_K$ for $K < p \leq t/K$, and $\log p + \kappa_p$ is bounded by $\log K + \kappa_5$ for $3 < p \leq K$, we can replace (5.28) with

$$\begin{aligned} &\sum_{p_0=2,3} \nu_{p_0} \left(\prod \mathcal{B}^{(3)} \right) \log p_0 + (\log \sqrt{tK} + \kappa_K)(X_1 + 2) + (\log K + \kappa_5) X_2 \\ &\quad \leq \frac{N}{2} \log 3 - \log N - \frac{\log(3L) + \kappa_L}{\log t - \log(3L)} N \log 2 \end{aligned} \tag{5.32}$$

5.7. **Analysis of Steps 1,2,3.** To apply Proposition 5.9, we now compute the various statistics of $\mathcal{B}^{(3)}$ produced by Steps 1-3.

We begin with the analysis of $\mathcal{B}^{(1)}$, constructed in Step 1 of the algorithm. To count elements coprime to 6, we use the following lemma:

Lemma 5.10. *For any interval $(a,b]$ with $0 \leq a \leq b$, the number of natural numbers in the interval that are coprime to 6 is $\frac{b-a}{3} + O_{\leq}(4/3)$.*

TODO: display the sawtooth function used in the proof

Proof. By the triangle inequality, it suffices to show that the number of natural numbers coprime to 6 in $[0,a]$, minus $a/3$, is $O_{\leq}(2/3)$. The claim is easily verified for $0 \leq a \leq 6$, and the quantity in question is 6-periodic in a, giving the claim. □

The excess of $\mathcal{B}^{(1)}$ can be computed as

$$\mathcal{E}_t(\mathcal{B}^{(1)}) = A \sum_{n \in I} \log \frac{n}{t}.$$

By the fundamental theorem of calculus, this is

$$A \int_0^{3t/A} |I \cap (t, t+h]| \, \frac{dh}{t+h}.$$

Bounding $\frac{1}{t+h}$ by $\frac{1}{t}$ and applying Lemma 5.10, we conclude that

$$\mathcal{E}_t(\mathcal{B}^{(1)}) \leq A \int_0^{3N/A} \left(\frac{h}{3} + \frac{4}{3}\right) \frac{dh}{t} = \frac{3N^2}{2tA} + 4. \tag{5.33}$$

Next, we compute p-valuations $v_p(\mathcal{B}^{(1)})$. By construction, this quantity vanishes at tiny primes $p_0 = 2, 3$, thus

$$\sum_{p_0 = 2,3} v_{p_0}\left(\prod \mathcal{B}^{(1)}\right) \log p_0 = 0.$$

For $p > 3$, we can use Lemma 5.10 again to conclude

$$\begin{aligned}
v_p(\mathcal{B}^{(1)}) &= A \sum_{1 \leq j \leq \frac{\log N}{\log p}} |I \cap p^j \mathbb{Z}| \\
&= A \sum_{1 \leq j \leq \frac{\log N}{\log p}} \left(\frac{N}{p^j A} + O_{\leq}(4/3)\right) \\
&= \frac{N}{p-1} + O_{\leq}\left(\frac{1}{p-1}\right) + O_{\leq}\left(\frac{4A}{3} \left\lceil \frac{\log N}{\log p} \right\rceil\right) \\
&= \frac{N}{p-1} + O_{\leq}\left(\frac{4A + 0.75}{3} \left\lceil \frac{\log N}{\log p} \right\rceil\right)
\end{aligned}$$

since $\frac{1}{p-1} \leq \frac{0.75}{3}$. Meanwhile, from (2.5) one has

$$v_p(N!) = \frac{N}{p-1} + O_{\leq}\left(\left\lceil \frac{\log N}{\log p} \right\rceil\right)$$

and thus

$$v_p(N!/\mathcal{B}^{(1)}) = O_{\leq}\left(\frac{4A + 3.75}{3} \left\lceil \frac{\log N}{\log p} \right\rceil\right). \tag{5.34}$$

Now we pass to $\mathcal{B}^{(2)}$ by performing Step 2 of the algorithm. Removing elements from a t-admissible multiset cannot increase the t-excess, so from (5.33) we have

$$\mathcal{E}_t(\mathcal{B}^{(2)}) \leq \frac{3N^2}{2tA} + 4. \tag{5.35}$$

The elements removed are of the form pm with $m \leq K(1+\nu)$ coprime to 6, and p in the interval $(\frac{t}{\min(m,K)}, \frac{t(1+\sigma)}{m}]$. We conclude that

$$\nu_p(N!/\mathcal{B}^{(2)}) = \nu_p(N!/\mathcal{B}^{(1)}) \tag{5.36}$$

for medium primes $K(1+\sigma) < p \leq t/K$. For small and borderline small primes $3 < p \leq K(1+\sigma)$ one has

$$\nu_p(N!/\mathcal{B}^{(2)}) = \nu_p(N!/\mathcal{B}^{(1)}) + A \sum_{\substack{m \leq K(1+\sigma) \\ (m,6)=1}} \nu_p(m) \left(\pi\left(\frac{t(1+\sigma)}{m}\right) - \pi\left(\frac{t}{\min(m,K)}\right) \right). \tag{5.37}$$

Finally, for tiny primes $p_0 = 2, 3$ we again have the maximal surplus:

$$\sum_{p_0=2,3} \nu_{p_0}\left(\prod \mathcal{B}^{(2)}\right) \log p_0 = 0.$$

We now pass to $\mathcal{B}^{(3)}$ by performing Step 3 of the algorithm. In other words, we add $\nu_p(N!)$ copies of $p\lceil t/p \rceil$ for each large prime $p > t/K$. The t-excess is now given by

$$\mathcal{E}_t(\mathcal{B}^{(3)}) = \mathcal{E}_t(\mathcal{B}^{(2)}) + \sum_{p > t/K} \nu_p(N!) \log \frac{\lceil t/p \rceil}{t/p}$$

and hence by (5.35) and (2.5)

$$\mathcal{E}_t(\mathcal{B}^{(3)}) \leq \frac{3N^2}{2tA} + 4 + \sum_{p > t/K} \left\lceil \frac{N}{p} \right\rceil \log \frac{\lceil t/p \rceil}{t/p}. \tag{5.38}$$

By construction one has balance (5.39) at large primes $p > t/K$,

$$\nu_p(N!/\mathcal{B}^{(3)}) = 0 \tag{5.39}$$

and no modification at borderline small or medium primes $K < p \leq t/K$,

$$\nu_p(N!/\mathcal{B}^{(3)}) = \nu_p(N!/\mathcal{B}^{(2)}) \tag{5.40}$$

but now the surplus or deficit at small primes $3 < p_1 \leq K$ is modified:

$$\nu_{p_1}(N!/\mathcal{B}^{(3)}) = \nu_{p_1}(N!/\mathcal{B}^{(2)}) - \sum_{p > t/K} \left\lceil \frac{N}{p} \right\rceil \nu_{p_1}(\lceil t/p \rceil). \tag{5.41}$$

Similarly, at tiny primes $p_0 = 2, 3$ we have

$$\nu_{p_0}(\mathcal{B}^{(3)}) = \sum_{p > t/K} \left\lfloor \frac{N}{p} \right\rfloor \nu_p(\lceil t/p \rceil). \tag{5.42}$$

At medium primes, $K(1+\sigma) < p \leq t/K$, we see from (5.40), (5.34) that

$$\left| \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \right| \leq \frac{4A + 3.75}{3} \left\lceil \frac{\log N}{\log p} \right\rceil.$$

For borderline primes $K \leq p < K(1+\sigma)$, we have from (5.40), (5.37) that

$$\left|\nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right)\right| \leq \frac{4A+3.75}{3}\left\lceil\frac{\log N}{\log p}\right\rceil + A \sum_{\substack{m\leq K(1+\sigma)\\(m,6)=1}} \nu_p(m)\left(\pi\left(\frac{t(1+\sigma)}{m}\right) - \pi\left(\frac{t}{\min(m,K)}\right)\right).$$

The only m which contributes here is $m = p$, thus we may simplify to

$$\begin{aligned}\left|\nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right)\right| &\leq \frac{4A+3.75}{3}\left\lceil\frac{\log N}{\log p}\right\rceil + A\left(\pi\left(\frac{t(1+\sigma)}{p}\right) - \pi\left(\frac{t}{K}\right)\right) \\ &\leq \frac{4A+3.75}{3}\left\lceil\frac{\log N}{\log p}\right\rceil + A\left(\pi\left(\frac{t(1+\sigma)}{K}\right) - \pi\left(\frac{t}{K}\right)\right).\end{aligned}$$

For the small primes $3 < p_1 \leq K$, we see from (5.34), (5.37), (5.41) that we have the upper bound

$$\nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \leq \frac{4A+3.75}{3}\left\lceil\frac{\log N}{\log p_1}\right\rceil + Y_{p_1} + Z_{p_1}$$

and the lower bound

$$\nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \leq -\frac{4A+3.75}{3}\left\lceil\frac{\log N}{\log p_1}\right\rceil + Y_{p_1}$$

where Y_{p_1} is the quantity

$$\begin{aligned}Y_{p_1} := A &\sum_{\substack{m\leq K\\(m,6)=1}} \nu_p(m)\left(\pi\left(\frac{t(1+\sigma)}{m}\right) - \pi\left(\frac{t}{m}\right)\right) \\ &- \sum_{p>t/K}\left\lfloor\frac{N}{p}\right\rfloor \nu_{p_1}(\lceil t/p\rceil)\end{aligned} \tag{5.43}$$

and Z_{p_1} is the (non-negative) error term

$$Z_{p_1} := A \sum_{\substack{K<m\leq K(1+\sigma)\\(m,6)=1}} \nu_{p_1}(m)\left(\pi\left(\frac{t(1+\sigma)}{m}\right) - \pi\left(\frac{t}{K}\right)\right). \tag{5.44}$$

An important phenomenon for us will be that Y_{p_1} is usually positive (so that $\mathcal{B}^{(3)}$ typically enjoys a (modest) surplus at small primes rather than a deficit); as such, $|Y_{p_1}|_{0,1}$ will enjoy better estimates than $|Y_{p_1}|_{1,0}$. From the triangle inequality we now have $X_1 \leq X_1'$ and $X_2 \leq X_2'$, where

$$\begin{aligned}X_1' := \frac{4A+3.75}{3} &\sum_{3<p\leq t/K}\left\lceil\frac{\log N}{\log p}\right\rceil \\ &+ A\left(\pi(K(1+\sigma)) - \pi(K)\right)\left(\pi\left(\frac{t(1+\sigma)}{K}\right) - \pi\left(\frac{t}{K}\right)\right) \\ &+ \sum_{3<p_1\leq K}\frac{\log p_1}{\log\sqrt{t/K}}(|Y_{p_1}|_{1,0} + Z_{p_1})\end{aligned} \tag{5.45}$$

and

$$X_2' := \frac{4A+3.75}{3}\sum_{3<p\leq K}\left\lceil\frac{\log N}{\log p}\right\rceil + \sum_{3<p_1\leq K}|Y_{p_1}|_{0,1}. \tag{5.46}$$

To summarize the previous discussion, we have

Proposition 5.11 (Criterion for $t(N) \geq t$)**.** *Let $L \geq 1$. Let $9L < t = N/e^{1+\delta}$ for some $\delta > 0$ be such that* (5.9) *holds. Let X_1', X_2' be defined by* (5.45), (5.46). *If one has*

$$\begin{aligned} &\frac{3N^2}{2tA} + 4 + \sum_{p > t/K} \left\lfloor \frac{N}{p} \right\rfloor \log \frac{\lceil t/p \rceil}{t/p} \\ &\quad + \kappa_K X_1' + \kappa_5 X_2' \\ &\quad \leq \delta N - \frac{3}{2} \log N - \kappa_L (\log \sqrt{12}) \frac{N}{\log t} \end{aligned} \tag{5.47}$$

and

$$\begin{aligned} &\sum_{p_0 = 2,3} \sum_{p > t/K} \left\lfloor \frac{N}{p} \right\rfloor \nu_{p_0}(\lceil t/p \rceil) \log p_0 \\ &\quad + (\log \sqrt{tK} + \kappa_K)(X_1' + 2) + (\log K + \kappa_5) X_2' \\ &\quad \leq \frac{N}{2} \log 3 - \log N - \frac{\log(3L) + \kappa_L}{\log t - \log(3L)} N \log 2 \end{aligned} \tag{5.48}$$

then $t(N) \geq t$.

5.8. **Estimation of relevant quantities.** Suppose N, A, K, L are as in Proposition 5.11. We now use effective prime number theorems to control the various quantities appearing in that proposition.

The quantity

$$\sum_{p > t/K} \left\lceil \frac{N}{p} \right\rceil \log \frac{\lceil t/p \rceil}{t/p}$$

appearing in (5.47) can be bounded by

$$\frac{1}{\log(t/K)} \sum_{t/K < p \leq N} f_{N/t}(p/N) \log p$$

where for $\alpha > 0$, $f_\alpha \colon (t/NK, 1] \to \mathbb{R}$ is the piecewise smooth function

$$f_\alpha(x) := \left\lfloor \frac{1}{x} \right\rfloor \log \frac{\lceil 1/\alpha x \rceil}{1/\alpha x}.$$

Applying Lemma C.2 and a change of variables, and discarding the negative term arising from $-\frac{2}{\sqrt{x}}$, we thus have

$$\begin{aligned} \sum_{p > t/K} \left\lceil \frac{N}{p} \right\rceil \log \frac{\lceil t/p \rceil}{t/p} \leq \frac{N}{\log(t/K)} \int_{t/NK}^1 f_{N/t}(x)\, dx \\ + \frac{E(N)}{\log(t/K)} \left(f_{N/t}\left(\frac{t}{NK} + \right) + f_{N/t}(1) + \| f_{N/t} \|_{\mathrm{TV}(t/NK, 1]} \right). \end{aligned}$$

Now we consider the expression

$$\sum_{3 < p \leq t/K} \left\lceil \frac{\log N}{\log p} \right\rceil$$

appearing in (5.45). The quantity $\left\lceil \frac{\log N}{\log p} \right\rceil$ equals 1 for $p > \sqrt{t}$ and is at most $\frac{\log N}{\log 5} + 1$ for $3 < p \leq \sqrt{t}$, so we have

$$\sum_{3<p\leq t/K} \left\lceil \frac{\log N}{\log p} \right\rceil \leq \pi(t/K) + \frac{\log N}{\log 5}\pi(\sqrt{t}). \tag{5.49}$$

One can bound this using for instance the bound (C.2).

As for the quantity $\sum_{3<p\leq K} \left\lceil \frac{\log N}{\log p} \right\rceil$ appearing in (5.46), we crudely bound it by

$$\sum_{3<p\leq K} \left\lceil \frac{\log N}{\log p} \right\rceil \leq \left(\frac{\log N}{\log 5} + 1 \right) (\pi(K) - \pi(3)). \tag{5.50}$$

To estimate the second term in (5.45), we can invoke (C.5) to bound

$$\pi\left(\frac{t(1+\sigma)}{K}\right) - \pi\left(\frac{t}{K}\right) \leq \frac{\sigma t}{K \log(t/K)} + \frac{2E\left(\frac{t(1+\sigma)}{K}\right)}{\log(t/K)}$$

assuming that $t/K \geq 1423$.

In a similar vein, we can bound the quantity Z_{p_1} defined in (5.44) by

$$Z_{p_1} \leq A \sum_{\substack{K<m<K(1+\sigma) \\ (m,6)=1}} \nu_{p_1}(m) \left(\frac{\sigma t}{K \log(t/K)} + \frac{2E\left(\frac{t(1+\sigma)}{K}\right)}{\log(t/K)} \right) \tag{5.51}$$

again assuming that $t/K \geq 1423$.

Now we consider the quantity

$$\sum_{p>t/K} \lfloor \frac{N}{p} \rfloor \nu_{p_*}(\lceil t/p \rceil)$$

for a prime p_*; this appears in in (5.48) when p_* is a tiny prime, and also in (5.43) when p_* is a small prime. We can rearrange this expression as

$$\sum_{m\leq K} \nu_{p_*}(m) \sum_{t/m\leq p<t/(m-1)} \lfloor \frac{N}{p} \rfloor.$$

Applying Lemma C.2, we can write this as

$$\sum_{m\leq K} \nu_{p_*}(m) \left(\int_{t/m}^{t/(m-1)} (1 - \frac{2}{\sqrt{x}}) \lfloor \frac{N}{x} \rfloor \frac{dx}{\log x} + O_{\leq}\left(\frac{2E(t/(m-1))(mN/t)}{\log(t/m)} \right) \right).$$

Thus we have the upper bound

$$\sum_{p>t/K} \lfloor \frac{N}{p} \rfloor \nu_{p_*}(\lceil t/p \rceil) \leq \sum_{m\leq K} \nu_{p_*}(m) \left(\int_{t/m}^{t/(m-1)} \lfloor \frac{N}{x} \rfloor \frac{dx}{\log(t/m)} + \frac{2tE(t/(m-1))}{mN \log(t/m)} \right) \tag{5.52}$$

and the lower bound

$$\sum_{p>t/K} \lfloor \frac{N}{p} \rfloor \nu_{p_*}(\lceil t/p \rceil) \geq \sum_{m \leq K} \nu_{p_*}(m) \left(\left(1 - \frac{2}{\sqrt{t/(m-1)}}\right) \int_{t/m}^{t/(m-1)} \lfloor \frac{N}{x} \rfloor \frac{dx}{\log(t/(m-1))} - \frac{2tE(t/(m-1))}{mN\log(t/m)} \right)$$

6. The asymptotic regime

With the above estimates, we can now establish the lower bound in Theorem 1.3(iv). Thus we aim to show that $t(N) \geq t$ for sufficiently large N, where

$$t := \frac{N}{e} - \frac{c_0 N}{\log N} + \frac{N}{\log^{1+c_1} N} \tag{6.1}$$

and $0 < c_1 < 1$ is a small absolute constant. With this choice of parameters, one has

$$\delta = \frac{ec_0}{\log N} + \frac{1}{\log^{1+c_1} N} + O\left(\frac{1}{\log^2 N}\right).$$

Let N be sufficiently large. We introduce parameters

$$A := \lfloor \log^2 N \rfloor$$

and

$$K := \lfloor \log^3 N \rfloor$$

and

$$L := N^{0.1},$$

so from (**??**) one has

$$\sigma = \frac{3N}{tA} \asymp \frac{1}{\log^2 N}.$$

The conditions $K^2(1+\sigma) < t$ and $t > 9L$ are easily verified for N large enough.

By Proposition 5.11, it suffices to verify the criteria (5.47), (5.48). From Lemma A.3 we have

$$\delta N - \frac{3}{2}\log N - \kappa_L(\log\sqrt{12})\frac{N}{\log t} \geq ec_0 \frac{N}{\log N} + \frac{N}{2\log^{1+c_1} N}$$

if c_1 is small enough and N is large enough, while from the choice of t, L one has

$$\frac{N}{2}\log 3 - \log N - \frac{\log(3L) + \kappa_L}{\log t - \log(3L)} N \log 2 \gg N.$$

By repeating the proof of Proposition 5.4, we see that

$$\sum_{p>t/K} \nu_p(N!) \log \frac{\lceil t/p \rceil}{t/p} = ec_0 \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right)$$

Thus, it will suffice to establish the bounds

$$\begin{aligned}&\frac{3N^2}{2tA}+4\\&\quad+\kappa_K X_1'+\kappa_5 X_2'\\&\quad\ll\frac{N(\log\log N)^3}{\log^2 N}\end{aligned}\tag{6.2}$$

and

$$\begin{aligned}&\sum_{p_0=2,3}\sum_{p>t/K}\left\lfloor\frac{N}{p}\right\rfloor \nu_{p_0}(\lceil t/p\rceil)\log p_0\\&\quad+(\log\sqrt{tK}+\kappa_K)(X_1'+2)+(\log K+\kappa_5)X_2'\\&\ll\frac{N(\log\log N)^3}{\log N}.\end{aligned}\tag{6.3}$$

From the choice of parameters t, A, we see that the $\frac{3N^2}{2tA}+4$ term in (6.2) is acceptable. For a tiny prime $p_0=2,3$, we have from (5.52) that

$$\sum_{p>t/K}\lfloor\frac{N}{p}\rfloor\nu_{p_0}(\lceil t/p\rceil)\ll\sum_{m\le K}\nu_{p_0}(m)\frac{N}{m\log N}.$$

Using the bound

$$\begin{aligned}\sum_{m\le K(1+\sigma)}\frac{\nu_p(m)}{m}&=\sum_{j=1}^{\infty}\sum_{m\le K(1+\sigma):\,p^j|m}\frac{1}{m}\\&\ll\sum_{j=1}^{\infty}\frac{\log K}{p^j}\\&\ll\frac{\log\log N}{p}\end{aligned}\tag{6.4}$$

we thus see that the first term on the left-hand side of (**??**) is acceptable. It will now suffice to show that

$$X_1', X_2'\ll\frac{N(\log\log N)^3}{\log^2 N}.\tag{6.5}$$

From (5.49) and the prime number theorem we have

$$\frac{4A+3.75}{3}\sum_{3<p\le t/K}\left\lceil\frac{\log N}{\log p}\right\rceil\ll A\frac{t/K}{\log N}\ll\frac{N}{\log^2 N}$$

so the contribution of the first term in (5.45) is acceptable. Next, from the Brun–Titchmarsh inequality (or (C.5)) we have

$$A\left(\pi(K(1+\sigma))-\pi(K)\right)\left(\pi\left(\frac{t(1+\sigma)}{K}\right)-\pi\left(\frac{t}{K}\right)\right)\ll A\frac{K\sigma}{\log K}\frac{t\sigma/K}{\log N}\ll\frac{N}{\log^3 N}$$

so the contribution of the second term in (5.45) is also acceptable. From (5.51) one has

$$Z_p \ll A \sum_{K<m<K(1+\sigma)} \nu_p(m) \frac{\sigma N}{K \log N} \ll A \frac{\sigma^2 N \log K}{p \log N} \ll \frac{N \log\log N}{p \log^3 N}$$

so the contribution of these terms to (5.45) is also acceptable. From (5.50) we have

$$\frac{4A + 3.75}{3} \sum_{3<p\leq K} \left\lceil \frac{\log N}{\log p} \right\rceil \ll A(\log N) \frac{K}{\log N} \ll \log^6 N$$

which is certainly acceptable for (5.46). By Mertens' theorem, it will now suffice to show that

$$|Y_{p_1}|_{1,0} \leq \frac{N(\log\log N)^2}{p_1 \log N} \tag{6.6}$$

and

$$|Y_{p_1}|_{0,1} \ll \frac{N(\log\log N)^2}{p_1 \log^2 N} \tag{6.7}$$

for all small primes p.

For the first bound (6.6), we crudely discard the second term of (5.43) and use the Brun–Titchmarsh inequality to obtain

$$\begin{aligned} |Y_{p_1}|_{1,0} &\ll A \sum_{m\leq K} \nu_{p_1}(m) \left(\pi\left(\frac{t(1+\sigma)}{m} \right) - \pi\left(\frac{t}{m} \right) \right) \\ &\ll \frac{At\sigma}{\log N} \sum_{m\leq K} \frac{\nu_{p_1}(m)}{m} \\ &\ll \frac{N \log\log N}{p_1 \log N} \end{aligned}$$

which is acceptable. For the second bound (6.7), we argue more carefully. From Lemma C.2 we have

$$\begin{aligned} &A \sum_{m\leq K:(m,6)=1} \nu_{p_1}(m) \left(\pi\left(\frac{t(1+\sigma)}{m} \right) - \pi\left(\frac{t}{m} \right) \right) \\ &\qquad = A \sum_{m\leq K:(m,6)=1} \nu_{p_1}(m)(1 + O(\frac{\log\log N}{\log N})) \frac{t\sigma m}{\log N} \\ &\qquad = \frac{N}{\log N} \sum_{m\leq K:(m,6)=1} \frac{3\nu_{p_1}(m)}{m} + O\left(\frac{N(\log\log N)^2}{p_1 \log^2 N} \right) \end{aligned}$$

where we again used (6.4) to control the error. From the upper bound (5.52) we also have

$$\begin{aligned} \sum_{p>t/K} \left\lfloor \frac{N}{p} \right\rfloor \nu_{p_1}(\lceil t/p \rceil) &\leq \sum_{m\leq K} \nu_{p_1}(m)(1 + O(\frac{\log\log N}{\log N})) \int_{t/m}^{t/(m-1)} \frac{N}{x} \frac{dx}{\log N} \\ &= \frac{N}{\log N} \sum_{m\leq K} \nu_{p_1}(m) \log \frac{m}{m-1} O\left(\frac{N(\log\log N)^2}{p_1 \log^2 N} \right) \end{aligned}$$

and thus

$$|Y_{p_1}|_{0,1} \ll \frac{N}{\log N}|\sum_{m\le K:(m,6)=1} \frac{3\nu_{p_1}(m)}{m} - \sum_{m\le K} \nu_{p_1}(m)\log\frac{m}{m-1}|_{0,1} + \frac{N(\log\log N)^2}{p_1\log^2 N}.$$

The claim now follows from the following numerical inequality.

Lemma 6.1 (Key inequality)**.** *For $p \ge 5$ and $K > 0$, we have*

$$0 \le \sum_{m\le K;(m,6)=1} \nu_p(m)\frac{3}{m} - \sum_{m\le K} \nu_p(m)\log\frac{m}{m-1} \le \frac{2}{p-1}.$$

But this can be easily verified; see Appendix B. The proof of (6.1) is now complete.

7. Guy–Selfridge conjecture

We now establish the Guy–Selfridge conjecture $t(N) \ge N/3$ in the range

$$N \ge ???.$$

We will apply Proposition 5.9 with the choice of parameters

$$\begin{aligned} t &:= N/3 \\ A &:= ??? \\ K &:= 342 \\ L &:= 342. \end{aligned}$$

Clearly $\delta = \log\frac{3}{e} = 0.09861\ldots$, and

$$\sigma = \frac{9}{A}.$$

From Lemma A.1, we have

$$\kappa_K = \kappa_L \le \log\frac{9}{8} = 0.11778\ldots \tag{7.1}$$

Thus the right-hand side of (5.31) is at least

$$N\log\frac{3}{e} - \frac{3}{2}\log N - (\log\frac{9}{8})(\log\sqrt{12})\frac{N}{\log(N/3)}.$$

Direct numerical calculation (cf. Figure 6) reveals that

$$\int_{1/3K}^{1} f(x)\,dx \le 0.9201$$

$$f(\frac{1}{3K}+) + f(1) + \|f\|_{\mathrm{TV}(1/3K,1]} \le 2044$$

and thus

$$\mathcal{E}_t(\mathcal{B}^{(3)}) \le \frac{N}{20} + 4 + \frac{N}{\log(N/3K)}\left(0.9201 + 2044\frac{E(N)}{N}\right).$$

The 2044 factor may seem large, but for large N the quantity $\frac{E(N)}{N}$ is so small that this term is in fact negligible.

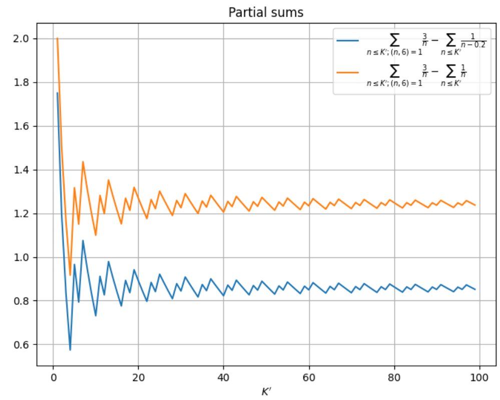

FIGURE 6. A plot of $f(x)$. The integral $c_1 = \int_{1/K}^1 f(x)\, dx \approx 0.9200$ is slightly larger than $ec_0 \approx 0.8244$.

We can directly compute $\pi(K) - \pi(3) = 66$.

APPENDIX A. POWERS OF 2 AND 3

We now obtain good bounds on the quantity κ_L introduced in (2.1). Clearly κ_L is a non-increasing function of L with $\kappa_1 = \log 2$. The following lemma gives improved control on κ_L for large L:

Lemma A.1. *If n_1, n_2, m_1, m_2 are natural numbers such that $n_1 + n_2, m_1 + m_2 \ge 1$ and*

$$1 \le \frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}$$

then

$$\kappa_{\min(2^{n_1+n_2}, 3^{m_1+m_2})/6} \le \log \max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right).$$

Proof. If $\min(2^{n_1+n_2}, 3^{m_1+m_2})/6 \le t \le 2^{n_2-1}3^{m_1-1}$, then we have

$$t \le 2^{n_2-1}3^{m_1-1} \le \max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right) t, \tag{A.1}$$

so we are done in this case. Now suppose that $t > 2^{n_2-1}3^{m_1-1}$. If we write $\lceil t \rceil^{\langle 2,3 \rangle} = 2^n 3^m$ be the smallest 3-smooth number that is at least t, then we must have $n \ge n_2$ or $m \ge m_1$ (or both). Thus at least one of $\frac{2^{n_1}}{3^{m_1}} 2^n 3^m$ and $\frac{3^{m_2}}{2^{n_2}} 2^n 3^m$ is an integer, and is thus at most t by construction. This gives (A.1), and the claim follows. $\square$

Some efficient choices of parameters for this lemma are given in Table 2. For instance, $\kappa_{4.5} \le 0.28768\dots$ and $\kappa_{40.5} \le 0.16989\dots$.

Remark A.2. It should be unsurprising that the continued fraction convergents $1/1, 2/1, 3/2, 8/5, 19/12, \dots$ to

$$\frac{\log 3}{\log 2} = 1.5849\dots = [1; 1, 1, 2, 2, 3, 1, \dots]$$

n_1	m_1	n_2	m_2	$\min(2^{n_1+n_2}, 3^{m_1+m_2})/6$	$\log\max(3^{m_1}/2^{n_1}, 2^{n_2}/3^{m_2})$
1	1	**1**	**0**	$1/2 = 0.5$	$\log 2 = 0.69314\dots$
1	**1**	2	1	$2^2/3 = 1.33\dots$	$\log(3/2) = 0.40546\dots$
3	2	**2**	**1**	$3^2/2 = 4.5$	$\log(2^2/3) = 0.28768\dots$
3	2	**5**	**3**	$3^4/2 = 40.5$	$\log(2^5/3^3) = 0.16989\dots$
3	**2**	8	5	$2^{10}/3 = 341.33\dots$	$\log(3^2/2^3) = 0.11778\dots$
11	**7**	8	5	$2^{18}/3 = 87381.33\dots$	$\log(3^7/2^{11}) = 0.06566\dots$
19	12	**8**	**5**	$3^{17}/2 \approx 6.4 \times 10^7$	$\log(2^8/3^5) = 0.05211\dots$
19	12	**27**	**17**	$3^{29}/2 \approx 3.4 \times 10^{13}$	$\log(2^{27}/3^{17}) = 0.03856\dots$
19	12	**46**	**29**	$3^{41}/2 \approx 1.8 \times 10^{19}$	$\log(2^{46}/3^{29}) = 0.02501\dots$

TABLE 2. Efficient parameter choices for Lemma A.1. The parameters used to attain the minimum or maximum are indicated in **boldface**. Note how the number of rows in each group matches the terms $1, 1, 2, 2, 3, \dots$ in the continued fraction expansion.

are often excellent choices for n_1/m_1 or n_2/m_2, although other approximants such as $5/3$ or $11/7$ are also usable.

Asymptotically, we have logarithmic-type decay:

Lemma A.3 (Baker bound)**.** *We have*

$$\kappa_L \ll \log^{-c} L$$

for all $L \geq 2$ and some absolute constant $c > 0$.

Proof. From the classical theory of continued fractions, we can find rational approximants

$$\frac{p_{2j}}{q_{2j}} \leq \frac{\log 3}{\log 2} \leq \frac{p_{2j+1}}{q_{2j+1}} \tag{A.2}$$

to the irrational number $\log 3/\log 2$, where the convergents p_j/q_j obey the recursions

$$p_j = b_j p_{j-1} + p_{j-2}; \quad q_j = b_j q_{j-1} + q_{j-2}$$

with $p_{-1} = 1, q = -1 = 0, p_0 = b_0, q_0 = 1$, and

$$[b_0; b_1, b_2, \dots] = [1; 1, 1, 2, 2, 3, 1 \dots]$$

is the continued fraction expansion of $\frac{\log 3}{\log 2}$. Furthermore, $p_{2j+1}q_{2j} - p_{2j}q_{2j+1} = 1$, and hence

$$\frac{\log 3}{\log 2} - \frac{p_{2j}}{q_{2j}} = \frac{1}{q_{2j}q_{2j+1}}. \tag{A.3}$$

By Baker's theorem, $\frac{\log 3}{\log 2}$ is a Diophantine number, giving a bound of the form

$$q_{2j+1} \ll q_{2j}^{O(1)} \tag{A.4}$$

and a similar argument (using $p_{2j+2}q_{2j+1} - p_{2j+1}q_{2j+2} = -1$) gives

$$q_{2j+2} \ll q_{2j+1}^{O(1)}. \tag{A.5}$$

We can rewrite (A.2) as

$$1 \leq \frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}}$$

and routine Taylor expansion using (A.3) gives the upper bounds

$$\frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}} \leq \exp\left(O\left(\frac{1}{q_{2j}}\right)\right).$$

From Lemma A.1 we obtain

$$\kappa_{\min(2^{p_{2j}+p_{2j+1}}, 3^{q_{2j}+q_{2j+1}})/6} \ll \frac{1}{q_{2j}}.$$

The claim then follows from (A.4), (A.5) after optimizing in j.

□

It seems reasonable to conjecture that c can be taken to be arbitrarily close to 1, but this is essentially equivalent to the open problem of determining that irrationality measure of $\log 3/\log 2$ is equal to 2.

Appendix B. Key inequality

We now prove Lemma 6.1. Writing $v_p(m) = \sum_{j\geq 1} 1_{p^j|m}$, it suffices to show that

$$0 \leq \sum_{m\leq K;(m,6)=1,p^j|m} \frac{3}{m} - \sum_{m\leq K,p^j|m} \log\frac{m}{m-1} \leq \frac{2}{p^j}$$

for all j. Making the change of variables $m = p^j n$, it suffices to show that

$$0 \leq \sum_{n\leq K'} \frac{3}{n} 1_{(n,6)=1} - p^j \log\frac{p^j n}{p^j n - 1} \leq 2$$

for any $K' > 0$. Using the bound

$$\log\frac{p^j n}{p^j n-1} = \int_{p^j n-1}^{p^j n} \frac{dx}{x} \in [\frac{1}{p^j n}, \frac{1}{p^j n-1}]$$

and $p^j \geq 5$, we have

$$\frac{1}{n} \leq p^j \log\frac{p^j n}{p^j n-1} \leq \frac{1}{n-0.2}$$

and so it suffices to show that

$$0 \leq \sum_{n\leq K'} \frac{3}{n} 1_{(n,6)=1} - \frac{1}{n-0.2} \leq \sum_{n\leq K'} \frac{3}{n} 1_{(n,6)=1} - \frac{1}{n} \geq 2. \tag{B.1}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n-0.2} - \frac{1}{n} = \psi(0.8) - \psi(1) = 0.353473,$$

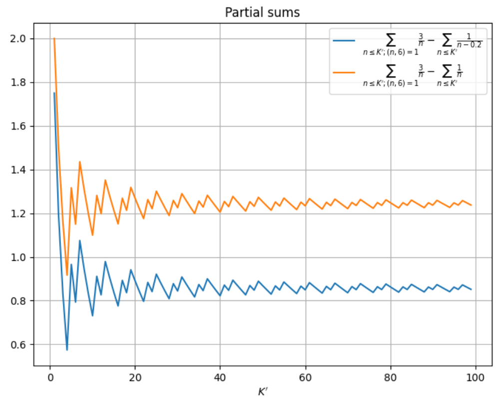

FIGURE 7. A plot of (B.1).

where ψ here denotes the digamma function rather than the von Mangoldt summatory function, it will suffice to show that

$$0.4 \leq \sum_{n \leq K'} \frac{3}{n} 1_{(n,6)=1} - \frac{1}{n} \geq 2. \tag{B.2}$$

This can be numerically verified for $K' \leq 100$, with substantial room to spare for K' large; see Figure 7. On a block $6a - 1 \leq n \leq 6a + 4$, the sum is positive:

$$\begin{aligned}\sum_{6a-1 \leq n \leq 6a+4} \frac{3}{n} 1_{(n,6)=1} - \frac{1}{n} &= \left(\frac{1}{6a-1} - \frac{1}{6a}\right) + \left(\frac{1}{6a-1} - \frac{1}{6a+2}\right) \\ &\quad + \left(\frac{1}{6a+1} - \frac{1}{6a+3}\right) + \left(\frac{1}{6a+1} - \frac{1}{6a+4}\right) \\ &> 0.\end{aligned}$$

Similarly, on a block $6a - 4 \leq n \leq 6a + 1$, the sum is negative:

$$\begin{aligned}\sum_{6a-4 \leq n \leq 6a+1} \frac{3}{n} 1_{(n,6)=1} - \frac{1}{n} &= \left(\frac{1}{6a+1} - \frac{1}{6a}\right) + \left(\frac{1}{6a+1} - \frac{1}{6a-2}\right) \\ &\quad + \left(\frac{1}{6a-1} - \frac{1}{6a-3}\right) + \left(\frac{1}{6a-1} - \frac{1}{6a-4}\right) \\ &< 0.\end{aligned}$$

Thus the sum in (B.2) is increasing for $K' = 4$ (6) and decreasing for $K' = 1$ (6), and the inequality for $K' > 100$ is then easily verified from the $K' \leq 100$ data and the triangle inequality

From this and the triangle inequality one can easily establish (B.1) in the remaining ranges $K' \geq 98$.

APPENDIX C. ESTIMATING SUMS OVER PRIMES

In this section we collect some estimates on sums over primes from the literature that we will use in this paper.

We recall the effective prime number theorem from [5, Corollary 5.2], which asserts that

$$\pi(x) \geq \frac{x}{\log x} + \frac{x}{\log^2 x} \tag{C.1}$$

for $x \geq 599$ and

$$\pi(x) \leq \frac{x}{\log x} + \frac{1.2762x}{\log^2 x} \tag{C.2}$$

for $x > 1$.

Lemma C.1 (Integration by parts). *Let $(y, x]$ be a half-open interval in $(0, +\infty)$. Suppose that one has a function $a \colon \mathbb{N} \to \mathbb{R}$ and a continuous function $f : (y, x] \to \mathbb{R}$ such that*

$$\sum_{y<n\leq z} a_n = \int_z^y f(t)\, dt + C + O_{\leq}(A)$$

for all $y \leq z \leq x$, and some $C \in \mathbb{R}$, $A > 0$. Then, for any function $b : (y, x] \to \mathbb{R}$ of bounded total variation, one has

$$\sum_{y<n\leq x} b(n)a_n = \int_x^y b(t)f(t)\, dt + O_{\leq}(A(|b(y^+)| + |b(x)| + \|b\|_{\mathrm{TV}(y,x]})), \tag{C.3}$$

where $b(y^+) := \lim_{t\to y^+} b(t)$ denotes the right limit of b at y, and the total variation $\|b\|_{\mathrm{TV}(y,x]}$ is defined as the supremum of the quantities $\sum_{j=0}^{J-1} |b(x_{j+1}) - b(x_j)|$ for $y < x_0 \leq \cdots \leq x_J \leq x$.

Proof. If, for every natural number $y < n \leq x$, one modifies b to be equal to the constant $b(n)$ in a small neighborhood of n, then one does not affect the left-hand side of (C.3) or increase the total variation of b, while only modifying the integral in (C.3) by an arbitrarily small amount. Hence, by the usual limiting argument, we may assume without loss of generality that b is locally constant at each such n. If we define the function $g : (y, x] \to \mathbb{R}$ by

$$g(z) := \sum_{y<n\leq z} a_n - \int_z^y f(u)\, du - C$$

then g has jump discontinuities at the natural numbers, but is otherwise continuously differentiable, and is also bounded uniformly in magnitude by A. We can then compute the Riemann–Stieltjes integral

$$\int_{(y,x]} b\, dg = \sum_{y<n\leq x} b(n)a_n - \int_y^x f(t)b(t)\, dt.$$

Since the discontinuities of g and b do not coincide, we may integrate by parts to obtain

$$\int_{(y,x]} b\, dg = b(x)g(x) - b(y^+)g(y^+) - \int_{(y,x]} g db.$$

The left-hand side is $O_{\leq}(A(|b(y^+)| + |b(x)| + \|b\|_{\mathrm{TV}(y,x]}))$, and the claim follows. ∎

By combining this lemma with effective prime number estimates, we obtain

Lemma C.2 (Effective prime number theorem)**.** *Under the above hypotheses with* $1423 \leq y \leq x$*, one has*

$$\sum_{y<p\leq x} b(p)\log p = \int_y^x b(t)(1-\frac{2}{\sqrt{t}})\,dt + O_{\leq}\left((|b(y^+)| + |b(x)| + \|b\|_{\mathrm{TV}})E(x)\right)$$

where

$$E(x) := 0.95\sqrt{x} + \frac{\sqrt{x}}{8\pi}\log x(\log x - 3)1_{x\geq 10^{19}} + \min(\varepsilon_0, \varepsilon_1(x), \varepsilon_2(x))x1_{x\geq e^{45}}$$

and

$$\varepsilon_0 := 1.11742\times 10^{-8}$$
$$\varepsilon_1(x) := 9.39(\log^{1.515} x)\exp(-0.8274\sqrt{\log x})$$
$$\varepsilon_2(x) := 0.026(\log^{1.801} x)\exp(-0.1853(\log^{3/5} x)(\log\log x)^{-1/5})$$

for some absolute constant $c > 0$*.*

Applying the above lemma to $b(t) = 1/\log t$, we conclude in particular that

$$\pi(x) - \pi(y) = \int_y^x (1-\frac{2}{\sqrt{t}})\frac{dt}{\log t} + O_{\leq}(\frac{2E(x)}{\log y}) \tag{C.4}$$

for $1423 \leq y \leq x$. In particular, we have the slightly crude upper bound

$$\pi(x) - \pi(y) \leq \frac{x-y}{\log y} + \frac{2E(x)}{\log y} \tag{C.5}$$

in this range.

Proof. Observe that E is monotone non-decreasing. Thus by Lemma C.1, it will suffice to show that

$$\sum_{p\leq x}\log p = x - \sqrt{x} + O_{\leq}(E(x)) = \int_0^x (1-\frac{2}{\sqrt{t}})\,dt + O_{\leq}(E(x))$$

for all $x \geq 1423$.

For $1423 \leq x \leq 10^{19}$, this follows from [4, Theorem 2]. In the range For $10^{19} \leq x \leq 10^{21} \approx e^{48.35}$, we use the bound

$$\psi(x) = x + O_{\leq}(\frac{\sqrt{x}}{8\pi}\log x(\log x - 3))$$

which was established for $5000 \leq x \leq 10^{21}$ in [3, (7.3)], where $\psi(x) := \sum_{n\leq x}\Lambda(n)$ is the usual von Mangoldt summatory function. To use this, we apply [3, (6.10), (6.11)] to conclude that

$$\sum_{p\leq x}\log p = \psi(x) - \psi(\sqrt{x}) + O_{\leq}(1.03883(x^{1/3} + x^{1/5} + 2(\log x)x^{1/13})).$$

From [13, Theorems 10,12] we have

$$\psi(\sqrt{x}) = \sqrt{x} + O_{\leq}(0.18\sqrt{x}).$$

Since

$$0.18\sqrt{x} + 1.03883(x^{1/3} + x^{1/5} + 2(\log x)x^{1/13}) \leq 0.95\sqrt{x}$$

for $x \geq 10^{19}$, the claim follows.

Finally, in the range $x \geq 10^{21}$, we see from [3, Theorem 1, Table 1] that one has the bound

$$\psi(x) = x + O_{\leq}(\varepsilon_0)$$

for $x \geq e^{45} \approx 10^{19.54}$, while from [11, Theorems 1.1, 1.4] one has

$$\psi(x) = x + O_{\leq}(\varepsilon_1(x))$$

and

$$\psi(x) = x + O_{\leq}(\varepsilon_2(x))$$

for all $x \geq 2$. The claim then follows by repeating the previous arguments. □

Remark C.3. Assuming the Riemann hypothesis, the final term in the definition of $E(x)$ may be deleted, since [3, (7.3)] then holds for all $x \geq 5000$.

Appendix D. Computation of c_0

In this appendix we give some details regarding the rigorous numerical estimation of the constant c_0 defined in (1.6). As one might imagine from an inspection of Figure 3, direct application of numerical quadrature converges quite slowly due to the oscillatory singularity. To resolve the singularity, we can perform a change of variables $x = 1/y$ to express c_0 as an improper integral:

$$c_0 = \frac{1}{e}\int_1^\infty \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e}\, \frac{dy}{y^2}. \tag{D.1}$$

The integrand is piecewise smooth and the integral can be computed explicitly on any interval $[a, b]$ of the form

$$[a, b] \subset [n, n+1] \cap [(m-1)e, me]$$

for some non-negative integers n, m as

$$\int_a^b \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e}\, \frac{dy}{y^2} = n(\frac{\log(b/m)}{b} - \frac{\log(a/m)a}{)}.$$

This formula permits one to evaluate $\int_1^b \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e} \frac{dy}{y^2}$ exactly for any finite b. To control the tail, we see from the crude bounds $0 \leq \lfloor y \rfloor \leq y$ and

$$0 \leq \log \frac{\lceil y/e \rceil}{y/e} \leq \log\left(1 + \frac{e}{y}\right) \leq \frac{e}{y}$$

that

$$0 \leq \int_b^\infty \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e}\, \frac{dy}{y^2} \leq \frac{e}{b} \tag{D.2}$$

which allows for rigorous upper and lower bounds on the improper integral. For instance, this procedure gives

$$0.304419004 \leq c_0 \leq 0.304419017.$$

Heuristically, the tail integral (D.2) should be approximately $e/2b$ due to the equidistribution properties of the fractional part of y/e. Using this heuristic approximation, one obtains the prediction

$$c_0 \approx 0.30441901087.$$

It should be possible to obtain this level of precision more rigorously (using interval arithmetic to preclude any possibility of roundoff error), but we have not attempted to do so.

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