NOTES ON UPPER AND LOWER BOUNDING $t(N)$

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1. Basics

$t(N)$ denotes the largest quantity such that $N!$ can be factored into N factors, each of which is at most $t(N)$.

$\nu_p(N)$ denotes the p-adic valuation of N, i.e., the exponent of the largest power of p dividing N.

We recall Legendre's formula

$$\nu_p(N) = \sum_{j=1}^{\infty} \left\lfloor \frac{N}{p^j} \right\rfloor = \frac{N - s_p(N)}{p-1}. \tag{1.1}$$

Asymptotically, it is known that

$$\frac{1}{e} - \frac{O(1)}{\log N} \leq \frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$$

where

$$\begin{aligned} c_0 &:= \frac{1}{e}\int_0^1 \left\lfloor \frac{1}{x} \right\rfloor \log\left(ex \left\lceil \frac{1}{ex} \right\rceil \right)\, dx \\ &= \frac{1}{e}\int_1^\infty \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e}\, \frac{dy}{y^2} \\ &= 0.3044\dots \end{aligned}$$

To bound the factorial, we have the explicit Stirling approximation [4]

$$N\log N - N + \log\sqrt{2\pi N} + \frac{1}{12N+1} \leq \log N! \leq N\log N - N + \log\sqrt{2\pi N} + \frac{1}{12N}, \tag{1.2}$$

valid for all natural numbers N.

To estimate the prime counting function, we have the following good asymptotics up to a large height.

Theorem 1.1 (Buthe's bounds)**.** [1] *For any* $2 \leq x \leq 10^{19}$, *we have*

$$\mathrm{li}(x) - \frac{\sqrt{x}}{\log x}\left(1.95 + \frac{3.9}{\log x} + \frac{19.5}{\log^2 x}\right) \leq \pi(x) < \mathrm{li}(x)$$

and

$$\mathrm{li}(x) - \frac{\sqrt{x}}{\log x} \leq \pi^*(x) < \mathrm{li}(x) + \frac{\sqrt{x}}{\log x}.$$

For $x > 10^{19}$ we have the bounds of Dusart [2]. One such bound is

$$|\psi(x) - x| \leq 59.18 \frac{x}{\log^4 x}.$$

For $a_+, a_- \in [0, +\infty]$, we define the asymmetric norm $|x|_{a_+,a_-}$ of a real number x by the formula

$$|x|_{a_+,a_-} := \max(a_+ x, -a_- x),$$

thus this is $a_+|x|$ when x is positive and $a_-|x|$ when x is negative. This function is Lipschitz with constant $\max(a_+, a_-)$.

2. Powers of 2 and 3

For every natural number B, let δ_B denote the smallest gap in the set $\{\{b\frac{\log 3}{\log 2}\} : 0 \leq b < B\} \cup \{1\}$, where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x. Clearly this quantity is nonincreasing in B. Numerical calculations show that

$$\delta_B \leq \frac{2.7}{B} \text{ for } 1 \leq B \leq 1900$$

see Figure 1. By the monotonicity of δ_B, we thus have

$$\delta_B \leq \frac{2.7}{\min(B, 1900)} \tag{2.1}$$

for all $B \geq 1$.

For larger values of B, $B\delta_B$ appears to be unbounded (presumably due to fluctuations in the continued fraction expansion of $\frac{\log 3}{\log 2}$). However, one has the following decay bound.

Proposition 2.1. *One has $\delta_B \ll B^{-c}$ for some absolute constant $c > 0$.*

Proof. Let $N > 1$ be a parameter to be chosen later. By Dirichlet's theorem, one can write $\frac{\log 3}{\log 2} = \frac{a}{q} + \frac{\varepsilon}{q}$ for some $1 \leq q \leq N$ with a, q coprime, and some $|\varepsilon| \leq 1/N$. From Baker's theorem we also have the lower bound $|\varepsilon| \gg q^{-O(1)}$. Writing $(mq+r)\frac{\log 3}{\log 2} = \frac{ar}{q} + (m + \frac{r}{q})\varepsilon \bmod 1$ for $r = 0, \dots, q-1$ and $0 \leq m \leq \frac{1}{q\varepsilon}$, we see that the gaps between these numbers on the unit circle do not exceed $O(1/N)$. Since all of the $mq + r$ are of the size $O(N^{O(1)})$, we conclude that $\delta_B \ll 1/N$ for some $B = O(N^{O(1)})$, giving the claim. □

We can use these quantities to efficiently approximate a given number by a number of the form $2^m 3^n$.

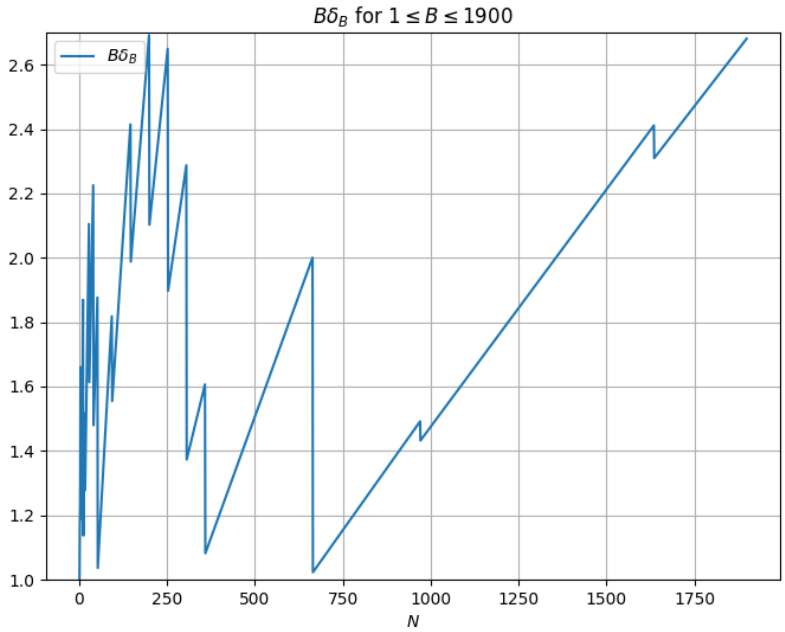

FIGURE 1. $B\delta_B$.

Lemma 2.2. *Let $t \geq 1$. Then there exist natural numbers n, m such that*

$$t \leq 2^n 3^m \leq \exp(\delta_{\lceil \log t/\log 3 \rceil} \log 2) t.$$

In particular, if $N \leq 3^{1900} \approx 10^{906.5}$, one has

$$t \leq 2^n 3^m \leq \exp((2.7 \log 3 \log 2)/\log t) t \leq \exp(2.06/\log t) t,$$

while for $t > 3^{1900}$, one has

$$t \leq 2^n 3^m \leq t \exp(\frac{2.7 \log 2}{1900}) \leq \exp(1/1000) t$$

and asymptotically one has

$$t \leq 2^n 3^m \leq \exp(O(\log^{-c} t)) t$$

for some absolute constant $c > 0$.

Proof. Set $B := \lceil \log s/\log 3 \rceil$. By definition of δ_B, one can find $0 \leq m \leq B-1$ and an integer n such that

$$\frac{\log s}{\log 2} \leq n + m \frac{\log 3}{\log 2} \leq \frac{\log s}{\log 2} + \delta_B.$$

Since $m \leq \log s/\log 3$, we see that n must be non-negative. Multiplying by $\log 2$ and exponentiating, we obtain the claim. □

3. Criteria for lower bounding $t(N)$

Suppose we are trying to factorize $N!$ into factors of size at least t. A candidate tuple $\vec{b} = (b_1, \dots, b_{N'})$ is said to be *admissible* if $b_j \geq t$ for all $j = 1, \dots, N'$. The *p-saving* $S_p(\vec{b})$ of the tuple is defined by the formula

$$S_p(\vec{b}) := \nu_p(N) - \sum_{i=1}^{N'} \nu_p(b_i).$$

A tuple is *debt-free* if $S_p(\vec{b}) \geq 0$ for all p.

From the fundamental theorem of arithmetic we have

$$\log b_i = \sum_p \nu_p(b_i) \log p$$

and

$$\log N! = \sum_p \nu_p(N!) \log p$$

and hence we obtain the identity

$$\log N! - \sum_{i=1}^{N'} \log b_i = \sum_p S_p(\vec{b}) \log p. \tag{3.1}$$

By the fundamental theorem of arithmetic, we obtain a perfect factorization $N! = b_1 \dots b_{N'}$ if all the p-savings vanish. The *excess* $E(\vec{b})$ is defined to be the quantity

$$E(\vec{b}) := \sum_{i=1}^{N'} \log \frac{b_i}{t}. \tag{3.2}$$

This quantity is non-negative for admissible tuples. Intuitively, the smaller the excess, the more efficient the candidate factorization. By combining (3.2) with (3.1) we obtain the identity

$$E(\vec{b}) + \sum_p S_p(\vec{b}) \log p = \log N! - N' \log t. \tag{3.3}$$

We conclude

Proposition 3.1. *Let $2 \leq t \leq N$ with $t = N/e^{1+\delta}$. Suppose one can find an admissible tuple $\vec{b}$ such that*

$$E(\vec{b}) + \sum_p |S_p(\vec{b})|_{\log p, \infty} \leq \delta N + \log \sqrt{2\pi N}. \tag{3.4}$$

Then $t(N) \geq t$.

For the purposes of the Guy–Selfridge conjecture $t(N) \geq N/3$, we may take $\delta = \log \frac{3}{e} \approx 0.098$.

In [5, Proposition 3.1] the variant criterion

$$E(\vec{b}) + \sum_p |S_p(\vec{b})|_{\log p, \log 2} + |N' - N| \log N \leq \delta N$$

was given in place of (3.4), but the condition (3.4) seems slightly superior numerically (we no longer need to maintain direct control on N').

Proof. From (3.4), the tuple must be debt-free. Applying (1.2), we have

$$E(\vec{b}) + \sum_p S_p(\vec{b}) \log p < \log N! - N \log t,$$

and hence by (3.3) we have $N' \geq N$. If we then delete all but N of the terms in the tuple $(b_1, \dots, b_{N'})$, and then distribute all p-savings amongst these surviving terms arbitrarily, we obtain a factorization $N! = a_1 \dots a_N$ with all $a_i \geq t$, so that $t(N) \geq t$ as required. □

In view of this proposition, we no longer need to keep direct track of the number of terms in the factorization; as long as we keep the excess small, and have not too much p-saving, particularly at large primes, while staying debt-free, we get a lower bound on $t(N)$.

For very large N, a promising strategy to improve the criterion is to initially allow for a large 2-saving and 3-saving, and then "spend" those available primes later on well chosen factors of the form $2^m 3^n$. Here is a more precise formulation.

Proposition 3.2 (Second criterion)**.** *Let $2 \leq t \leq N$ with $t = N/e^{1+\delta}$. Suppose we can find pairs (m_1, n_1), (m_2, n_2) of natural numbers with*

$$t \leq 2^{m_i} 3^{n_i} \leq e^{\varepsilon} t \leq N \tag{3.5}$$

for $i = 1, 2$ and some $\varepsilon > 0$. Suppose we also have an admissible tuple $\vec{b}$ obeying the following axioms:

(i) *The vector*

$$(S_2^-(\vec{b}), S_3^-(\vec{b})) \tag{3.6}$$

in $\mathbb{R}^2$ is a non-negative linear combination of (m_1, n_1) and (m_2, n_2).

(ii) *We have*

$$\begin{aligned} E(\vec{b}) &+ \sum_{p>3} |S_p(\vec{b})|_{\log p, \infty} \\ &+ \frac{3}{2} \log N + \frac{\varepsilon \log 12}{2} \frac{N}{\log t} \leq \delta N + \log \sqrt{2\pi}. \end{aligned} \tag{3.7}$$

Then $t(N) \geq t$.

In practice, the $\frac{3}{2} \log N$ term is negligible. The point here is that this version of the criterion largely frees up the need to track the undershoot at 2 and 3, other than to verify the (quite mild) condition (i). The quantity ε can be easily bounded by $\log 2$ in most cases, but one expects (based on the irrationality of $\log 3 / \log 2$) that one can do better than this; and this quantity can be bounded numerically quite easily even for rather large N.

Proof. By hypothesis, the vector (3.6) can be written as $s_1(m_1, n_1)+s_2(m_2, n_2)$ for some positive reals s_1, s_2. Splitting into integer and fractional parts, we can thus write (3.6) as the sum of $\lfloor s_1 \rfloor$ copies of (m_1, n_1), $\lfloor s_2 \rfloor$ copies of (m_2, n_2), and a vector with coefficients at most $(m_1 + m_2, n_1 + n_2)$. If we then add s_1 copies of $2^{m_1}3^{n_1}$ and s_2 copies of $2^{m_2}3^{n_2}$ to the admissible tuple, then it remains admissible and debt-free; but now $S_2(\vec{b})$, $S_3(\vec{b})$ are reduced to at most $m_1 + m_2$, $n_1 + n_2$ respectively. Also, each $2^{m_i}3^{n_i}$ contributes an excess of at most ε, which in turn is at most $\frac{\varepsilon}{\log t} \log 2^{m_i}3^{n_i}$; hence the total additional excess produced here is at most $\frac{\varepsilon}{\log t} \log 2^{S_2(\vec{b})}3^{S_3(\vec{b})}$. From (1.1) we have $S_p(\vec{b}) \leq \frac{N}{p-1}$, hence the additional excess is at most

$$\frac{\varepsilon}{\log t} \log 2^N 3^{N/2} \leq \frac{\varepsilon \log 12}{2} \frac{N}{\log t}.$$

The new value of $S_2(\vec{b}) \log 2 + S_3(\vec{b}) \log 3$ is at most

$$\begin{aligned} (m_1 + m_2) \log 2 + (n_1 + n_2) \log 3 &= \log 2^{m_1}3^{n_1} + \log 2^{m_2}3^{n_2} \\ &\leq 2 \log N \qquad = \frac{3}{2} \log N + \log \sqrt{N}. \end{aligned}$$

From (3.7) we conclude that the new admissible tuple obeys (3.4), and the claim now follows from the previous proposition. □

We can now allow for some debt at various primes p, as well as handle savings at small primes more efficiently.

Proposition 3.3 (Third criterion). *Let $2 \leq t \leq N$ with $t = N/e^{1+\delta}$. Suppose we can find pairs (m_1, n_1), (m_2, n_2) of natural numbers obeying (**??**), and an admissible tuple $\vec{b}$ obeying the following axioms:*

(i) *The vector*
$$(S_2(\vec{b}) - u, S_3(\vec{b})) \tag{3.8}$$
in $\mathbb{R}^2$ is a non-negative linear combination of (m_1, n_1) and (m_2, n_2), whenever
$$0 \leq u \leq \sum_{3<p\leq\sqrt{t}} |S_p(\vec{b})|_{\frac{2\log p \lceil (\log t)/2\log 2 \rceil}{\log t}, \lceil \frac{\log p}{\log 2} \rceil} + \sum_{p>\sqrt{t}} |S_p(\vec{b})|_{0, \lceil \frac{\log p}{\log 2} \rceil} + \left\lceil \frac{\log t}{\log 2} \right\rceil.$$

(ii) *We have*
$$\begin{aligned} E(\vec{b}) &+ \sum_{3<p\leq\sqrt{t}} |S_p(\vec{b})|_{\frac{2(\log p)\log 2}{\log t}, \log 2} + \sum_{p>\sqrt{t}} |S_p(\vec{b})|_{\log p, \log 2} \\ &+ \frac{3}{2} \log N + \frac{\varepsilon \log 12}{2} \frac{N}{\log t} \leq \delta N. \end{aligned} \tag{3.9}$$

Then $t(N) \geq t$.

The point here is that the "cost" of excessive savings at primes $3 < p \leq \sqrt{t}$ has been significantly reduced. The condition (i) has become more complicated, but is easy to satisfy in practice.

Proof. Suppose we have a p-debt $S_p(\vec{b}) < 0$ at some some prime $p > 3$. Then one of the elements of the tuple $\vec{b}$ is divisible by p. If we replace p by $2^{\lceil \log p/\log 2\rceil}$ in that element, then we keep the tuple admissible, increasing the excess by at most $\log 2$, while decreasing $S_2(\vec{b})$ by $\lceil \log p/\log 2\rceil$, increasing $S_p(\vec{b})$ by one (and thus decreasing $|S_p(\vec{b})|_{0,\lceil \frac{\log p}{\log 2}\rceil}$ or $|S_p(\vec{b})|_{0,\lceil \frac{\log p}{\log 2}\rceil}$ by $\lceil \frac{\log p}{\log 2}\rceil$), and not affecting any of the other p-savings. Thus, by iterating this procedure, we may assume that the tuple is debt-free.

Now consider the positive p-savings $S_p(\vec{b}) > 0$ coming from primes $3 < p \leq \sqrt{t}$, which multiply to an expression B with

$$\log B = \sum_{3<p\leq\sqrt{t}} S_p(\vec{b}) \log p.$$

By the greedy algorithm, one can factor B into M expressions in the interval $(\sqrt{t}, t]$, plus at most one further factor bounded by t, where M obeys the bound

$$\left(\sqrt{t}\right)^M \leq B$$

and hence on taking logarithms and rearranging

$$M \leq \sum_{3<p\leq\sqrt{t}} S_p(\vec{b}) \frac{2\log p}{\log t}.$$

For each of the M factors, one can make it be larger than or equal to t (but less than $2t$) by inserting at most $\lceil (\log \sqrt{t})/\log 2\rceil = \lceil (\log t)/2\log 2\rceil$ factors of two; and the final factor can also be similarly adjusted using at most $\lceil \log t/\log 2\rceil$ factors of two. Each such adjustment increases the excess by at most $\log 2$; since $\log 2 \leq \log\sqrt{2\pi}$, we can upper bound the net increase in the excess by

$$\sum_{3<p\leq\sqrt{t}} |S_p(\vec{b})|_{\frac{2(\log p)\log 2}{\log t},0} + \log\sqrt{2\pi},$$

while the 2-saving $S_2(\vec{b})$ has been reduced by at most

$$\sum_{3<p\leq\sqrt{t}} |S_p(\vec{b})|_{\frac{2\log p\lceil(\log t)/2\log 2\rceil}{\log t},0} + \left\lceil \frac{\log t}{\log 2}\right\rceil.$$

Performing these adjustments to remove all p-savings at primes $3 < p \leq K$, we obtain the current criterion from the previous one. □

4. Criteria for upper bounding $t(N)$

We have the trivial upper bound $t(N) \leq (N!)^{1/N}$. This can be improved to $t(N) \leq N/e$ for $N \neq 1, 2, 4$, answering a conjecture of Guy and Selfridge [3]; see [5]. This was derived from the following slightly stronger criterion, which asymptotically gives $\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0+o(1)}{\log N}$:

Lemma 4.1 (Upper bound criterion)**.** [5, Lemma 2.1] *Suppose that* $1 \leq t \leq N$ *are such that*

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor \log\left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil \right) > \log N! - N \log t \tag{4.1}$$

Then $t(N) < t$.

A surprisingly sharp upper bound comes from linear programming.

Lemma 4.2 (Linear programming bound)**.** *Let* N *be an natural number and* $1 \leq t \leq N/2$. *Suppose for each prime* $p \leq N$*, one has a non-negative real number* w_p *which is weakly non-decreasing in p (thus* $w_p \leq w_{p'}$ *when* $p \leq p'$*), and such that*

$$\sum_p w_p \nu_p(j) \geq 1 \tag{4.2}$$

for all $t \leq j \leq N$*, and such that*

$$\sum_p w_p \nu_p(N!) < N. \tag{4.3}$$

Then $t(N) < t$.

Proof. We first observe that the bound (4.2) in fact holds for all $j \geq t$, not just for $t \leq j \leq N$. Indeed, if this were not the case, consider the first $j \geq t$ where (4.2) fails. Take a prime p dividing j and replace it by a prime inthe interval $[p/2, p)$ which exists by Bertrand's postulate (or remove p entirely, if $p = 2$); this creates a new j' in $[j/2, j)$ which is still at least t. By the weakly decerasing hypothesis on w_p, we have

$$\sum_p w_p \nu_p(j) \geq \sum_p w_p \nu_p(j')$$

and hence by the minimality of j we have

$$\sum_p w_p \nu_p(j) > 1,$$

a contradiction.

Now suppose for contradiction that $t(N) \geq t$, thus we have a factorization $N! = \prod_{j \geq t} j^{m_j}$ for some natural numbers m_j summing to N. Taking p-valuations, we conclude that

$$\sum_{j \geq t} m_j \nu_p(j) \leq \nu_p(N!)$$

for all $p \leq N$. Multiplying by w_p and summing, we conclude from (4.2) that

$$N = \sum_{j \geq t} m_j \leq \sum_p w_p \nu_p(N!),$$

contradicting (4.3). $\square$

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