NOTES ON UPPER AND LOWER BOUNDING $t(N)$

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1. Basics

$t(N)$ denotes the largest quantity such that $N!$ can be factored into N factors, each of which is at most $t(N)$.

$v_p(N)$ denotes the p-adic valuation of N, i.e., the exponent of the largest power of p dividing N.

We recall Legendre's formula

$$v_p(N) = \sum_{j=1}^{\infty} \lfloor \frac{N}{p^j} \rfloor = \frac{N - s_p(N)}{p-1}. \tag{1.1}$$

Asymptotically, it is known that

$$\frac{1}{e} - \frac{O(1)}{\log N} \leq \frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$$

where

$$\begin{aligned} c_0 &:= \frac{1}{e} \int_0^1 \left\lfloor \frac{1}{x} \right\rfloor \log\left(ex \left\lceil \frac{1}{ex} \right\rceil \right)\, dx \\ &= \frac{1}{e} \int_1^\infty \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e} \frac{dy}{y^2} \\ &= 0.3044 \dots \end{aligned}$$

To bound the factorial, we have the explicit Stirling approximation [4]

$$N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N+1} \leq \log N! \leq N \log N - N + \log \sqrt{2\pi N} + \frac{1}{12N}, \tag{1.2}$$

valid for all natural numbers N.

To estimate the prime counting function, we have the following good asymptotics up to a large height.

Theorem 1.1 (Buthe's bounds)**.** [1] *For any* $2 \leq x \leq 10^{19}$*, we have*

$$\mathrm{li}(x) - \frac{\sqrt{x}}{\log x}\left(1.95 + \frac{3.9}{\log x} + \frac{19.5}{\log^2 x}\right) \leq \pi(x) < \mathrm{li}(x)$$

and

$$\mathrm{li}(x) - \frac{\sqrt{x}}{\log x} \leq \pi^*(x) < \mathrm{li}(x) + \sqrt{x}\log x.$$

For $x > 10^{19}$ we have the bounds of Dusart [2]. One such bound is

$$|\psi(x) - x| \leq 59.18\frac{x}{\log^4 x}.$$

2. Criteria for lower bounding $t(N)$

Suppose we are trying to factorize $N!$ into factors of size at least t. A candidate tuple $\vec{b} = (b_1, \dots, b_{N'})$ is said to be *admissible* if $b_j \geq t$ for all $j = 1, \dots, N'$. The *undershoot* $S_p^-(\vec{b})$ and *overshoot* of a tuple at a prime p are defined by the formulae

$$S_p^-(\vec{b}) := (\nu_p(N) - \sum_{i=1}^{N'} \nu_p(b_i))_+$$

$$S_p^+(\vec{b}) := (\sum_{i=1}^{N'} \nu_p(b_i) - \nu_p(N!))_+.$$

By the fundamental theorem of arithmetic, we obtain a perfect factorization $N! = b_1 \dots b_{N'}$ if the undershoots and overshoots all vanish. The *excess* $E(\vec{b})$ is defined to be the quantity

$$E(\vec{b}) := \sum_{i=1}^{N'} \log \frac{b_i}{t}. \tag{2.1}$$

This quantity is non-negative for admissible tuples. Intuitively, the smaller the excess, the more efficient the candidate factorization.

Proposition 2.1. *Let $2 \leq t \leq N$ with $t = N/e^{1+\delta}$. Suppose one can find an admissible tuple $\vec{b}$ obeying the inequality*

$$E(\vec{b}) + \sum_p (S_p^-(\vec{b}) \log p + S_p^+(\vec{b}) \log 2) \leq \delta N \tag{2.2}$$

as well as the side condition

$$\sum_{p>2} S_p^+(\vec{b}) \left\lceil \frac{\log p}{\log 2} \right\rceil < S_2^-(\vec{b}). \tag{2.3}$$

Then $t(N) \geq t$.

For the purposes of the Guy–Selfridge conjecture $t(N) \geq N/3$, we may take $\delta = \log \frac{3}{e} \approx 0.098$.

In [5, Proposition 3.1] the slightly weaker criterion

$$E(\vec{b}) + \sum_p (S_p^-(\vec{b}) \log p + S_p^+(\vec{b}) \log 2) + |N' - N| \log N \leq \delta N$$

was given in place of (2.2), (2.3), but this formulation seems slightly superior numerically (we no longer need to maintain direct control on N').

Proof. Since there is a non-zero undershoot at $p = 2$, there is no overshoot at this prime. Suppose that we have an overshoot at an odd prime $p > 2$, thus one of the b_i has a factor of p. If we replace this factor of p by the larger quantity $2^{\lceil \frac{\log p}{\log 2} \rceil}$, then $S_p^+(\vec{b})$ goes down by one, $E(\vec{b})$ goes up by at most $\log 2$, and both sides of (2.3) decrease by the same amount. Thus neither (2.2) nor (2.3) can be destroyed by this process. Iterating, we may assume without loss of generality that there are no overshoots, thus

$$W(\vec{b}) + \sum_p (\nu_p(N!) - \sum_{i=1}^{N'} \nu_p(b_i)) \log p \le \delta N.$$

From the Stirling approximation we have

$$\delta N \le \log N! - N \log t.$$

From the fundamental theorem of arithmetic we have

$$\sum_p \nu_p(b_i) \log p = \log b_i; \quad \sum_p \nu_p(N!) \log p = \log N!.$$

Using this and (2.1), we may rearrange the previous inequality as

$$N \log t \le N' \log t.$$

If we then delete all but N of the terms in the tuple $(b_1, \dots, b_{N'})$, and then distribute all undershoots amongst these surviving terms arbitrarily, we obtain a factorization $N! = a_1 \dots a_N$ with all $a_i \ge t$, so that $t(N) \ge t$ as required. □

In view of this proposition, we no longer need to keep direct track of the number of terms in the factorization; as long as we keep the excess small, and have not too much undershoot or overshoot, taking particular care to avoid undershoot at large primes, and to keep some undershoot at 2 for the side condition, we get a lower bound on $t(N)$. For $t = N/3$, the right-hand side $\log N! - N \log t$ is roughly $N \log \frac{3}{e} \approx 0.098N$ by the Stirling approximation.

3. Criteria for upper bounding $t(N)$

We have the trivial upper bound $t(N) \le (N!)^{1/N}$. This can be improved to $t(N) \le N/e$ for $N \ne 1, 2, 4$, answering a conjecture of Guy and Selfridge [3]; see [5]. This was derived from the following slightly stronger criterion, which asymptotically gives $\frac{t(N)}{N} \le \frac{1}{e} - \frac{c_0+o(1)}{\log N}$:

Lemma 3.1 (Upper bound criterion)**.** [5, Lemma 2.1] *Suppose that* $1 \le t \le N$ *are such that*

$$\sum_{p > \frac{t}{\lfloor \sqrt{t} \rfloor}} \left\lfloor \frac{N}{p} \right\rfloor \log \left(\frac{p}{t} \left\lceil \frac{t}{p} \right\rceil \right) > \log N! - N \log t \tag{3.1}$$

Then $t(N) < t$.

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