DECOMPOSING A FACTORIAL INTO LARGE FACTORS

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ABSTRACT. Let $t(N)$ denote the largest number such that $N!$ can be expressed as the product of N numbers greater than or equal to $t(N)$. The bound $t(N)/N = 1/e - o(1)$ was apparently established in unpublished work of Erdős, Selfridge, and Straus; but the proof is lost. Here we obtain the more precise asymptotic

$$\frac{t(N)}{N} = \frac{1}{e} - \frac{c_0}{\log N} + O\left(\frac{1}{\log^{1+c} N}\right)$$

for an explicit constant $c_0 = 0.30441901\ldots$ and some absolute constant $c > 0$, answering a question of Erdős and Graham. For the upper bound, a further lower order term in the asymptotic expansion is also obtained. With numerical assistance, we also establish several conjectures of Guy and Selfridge concerning effective estimates of this quantity, for instance establishing $t(N) \geq N/3$ for $N \geq 43632$, with the threshold shown to be best possible.

1. INTRODUCTION

Given a natural number M, define a *factorization* of M to be a finite multiset $\mathcal{B}$ such that the product

$$\prod \mathcal{B} := \prod_{a \in \mathcal{B}} a$$

(where the elements are counted with multiplicity) is equal to M; more generally, define a *subfactorization* of M to be a finite multiset $\mathcal{B}$ such that $\prod \mathcal{B}$ divides M. Given a threshold t, we say that a multiset $\mathcal{B}$ is *t-admissible* if $a \geq t$ for all $a \in \mathcal{B}$. For a given natural number N, we then define $t(N)$ to be the largest t for which there exists a t-admissible factorization $\mathcal{B}$ of $N!$ of cardinality $|\mathcal{B}| = N$.

Example 1.1. The multiset

$$\{3, 3, 3, 3, 4, 4, 5, 7, 8\}$$

is a 3-admissible factorization of

$$\prod \{3, 3, 3, 3, 4, 4, 5, 7, 8\} = 3^4 \times 4^2 \times 5 \times 7 \times 8 = 9!$$

of cardinality

$$|\{3, 3, 3, 3, 4, 4, 5, 7, 8\}| = 9,$$

hence $t(9) \geq 3$. One can check that no 4-admissible factorization of 9! of this cardinality exists, hence $t(9) = 3$.

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It is easy to see that $t(N)$ is non-decreasing in N, (any cardinality N factorization of $N!$ can be extended to a cardinality $N+1$ factorization of $(N+1)!$ by adding $N+1$ to the multiset). The first few elements of the sequence $t(N)$ are

$$1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, \dots$$

(OEIS A034258). The values of $t(N)$ for $N \leq 79$ were computed in [10], and the values for $N \leq 200$ can be extracted from OEIS A034259, which describes the inverse sequence to t. As part of our work, we extend this sequence to $N \leq 10^4$; see [17] and Figure 5.

When the factorial $N!$ is replaced with an arbitrary number the problem of determining $t(N)$ essentially becomes the bin covering problem, which is known to be NP-hard; see e.g., [2]. However, as we shall see in this paper, the special structure of the factorial (and in particular, the profusion of factors at the "tiny primes" $2, 3$) make it more tractable to estimate $t(N)$ with high precision than in the general case.

Remark 1.2. One can equivalently define $t(N)$ as the greatest t for which there exists a t-admissible *subfactorization* of $N!$ of cardinality *at least* N. This is because every such subfactorization can be converted into a t-admissible factorization of cardinality exactly N by first deleting elements from the subfactorization to make the cardinality N, and then multiplying one of the elements of the subfactorization by a natural number to upgrade the subfactorization to a factorization. This "relaxed" formulation of the problem turns out to be more convenient for both theoretical analysis of $t(N)$ and numerical computations.

By combining the obvious lower bound

$$\prod \mathcal{B} \geq t^{|\mathcal{B}|} \tag{1.1}$$

for any t-admissible multiset $\mathcal{B}$ with Stirling's formula (2.4), we obtain the trivial upper bound

$$\frac{t(N)}{N} \leq \frac{(N!)^{1/N}}{N} = \frac{1}{e} + O\left(\frac{\log N}{N}\right) \tag{1.2}$$

for $N \geq 2$; see Figure 1. In [9, p.75] it was reported that an unpublished work of Erdős, Selfridge, and Straus established the asymptotic

$$\frac{t(N)}{N} = \frac{1}{e} + o(1) \tag{1.3}$$

(first conjectured in [7]) and asked if one could show the bound

$$\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c}{\log N} \tag{1.4}$$

for some constant $c > 0$ (problem #391 in `https://www.erdosproblems.com`; see also [10, Section B22, p. 122–123]); it was also noted that similar results were obtained in [1] if one restricted the a_i to be prime powers. However, as later reported in [8], Erdős "believed that Straus had written up our proof [of (1.3)]. Unfortunately Straus suddenly died and no trace was ever found of his notes. Furthermore, we never could reconstruct our proof, so our assertion now can be called only a conjecture". In [10] the lower bound $\frac{t(N)}{N} \geq \frac{1}{4}$ was established for sufficiently large N, by rearranging powers of 2 and 3 in the obvious factorization $1 \times 2 \times \cdots \times N$ of $N!$. A variant lower bound of the asymptotic shape $\frac{t(N)}{N} \geq \frac{3}{16} - o(1)$ obtained by rearranging only powers of 2, and which is superior for medium values of N, can also be found in [10]. The following conjectures in [10] were also made:

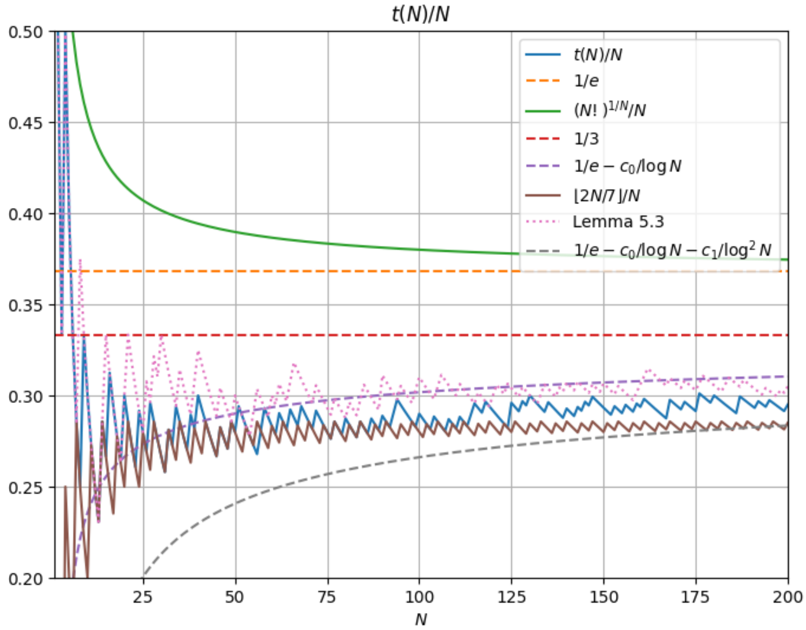

FIGURE 1. The function $t(N)/N$ (blue) for $N \leq 200$, using the data from OEIS A034258, as well as the trivial upper bound $(N!)^{1/N}/N$ (green), the improved upper bound from Lemma 5.1 (pink), which is asymptotic to (1.5) (purple), and the function $\lfloor 2N/7 \rfloor/N$ (brown), which we show to be a lower bound for $N \neq 56$. Theorem 1.3 implies that $t(N)/N$ is asymptotic to (1.5) (purple), which in turn converges to $1/e$ (orange). The threshold $1/3$ (red) is permanently crossed at $N = 43632$.

(1) One has $t(N) \leq N/e$ for $N \neq 1, 2, 4$.
(2) One has $t(N) \geq \lfloor 2N/7 \rfloor$ for $N \neq 56$.
(3) One has $t(N) \geq N/3$ for $N \geq 3 \times 10^5$. (It was also asked if the threshold 3×10^5 could be lowered.)

In this paper we answer all of these questions.

Theorem 1.3 (Main theorem)**.** *Let N be a natural number.*

(i) *If $N \neq 1, 2, 4$, then $t(N) \leq N/e$.*
(ii) *If $N \neq 56$, then $t(N) \geq \lfloor 2N/7 \rfloor$.*
(iii) *If $N \geq 43632$, then $t(N) \geq N/3$. The threshold 43632 is best possible.*
(iv) *For large N, one has*

$$\frac{t(N)}{N} = \frac{1}{e} - \frac{c_0}{\log N} + O\left(\frac{1}{\log^{1+c} N}\right) \tag{1.5}$$

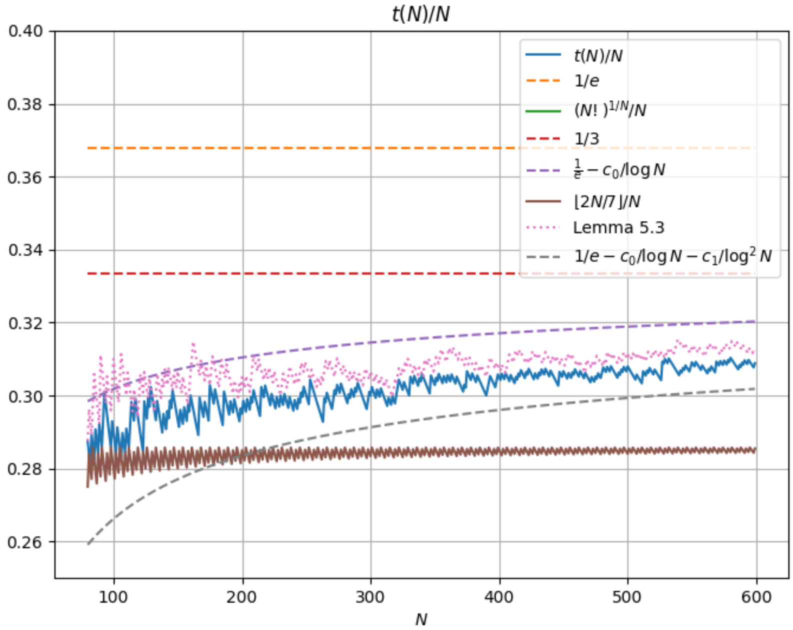

FIGURE 2. A continuation of Figure 1 to the region $80 \leq N \leq 599$.

for some constant $c > 0$*, where* c_0 *is the explicit constant*

$$c_0 := \frac{1}{e} \int_0^1 f_e(x)\, dx \\ = 0.30441901\ldots \tag{1.6}$$

and for any $\alpha > 0$*,* $f_\alpha : (0, \infty) \to \mathbb{R}$ *denotes the piecewise smooth function*

$$f_\alpha(x) := \left\lfloor \frac{1}{x} \right\rfloor \log \frac{\lceil 1/\alpha x \rceil}{1/\alpha x}. \tag{1.7}$$

In particular, (1.3) *and* (1.4) *hold. In fact the upper bound can be sharpened to*

$$\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0}{\log N} - \frac{c_1 + o(1)}{\log^2 N} \tag{1.8}$$

for an explicit constant $c_1 = 0.75554808\ldots$*; see Proposition 5.2.*

For future reference, we observe the simple bounds

$$\begin{aligned} 0 \leq f_\alpha(x) &\leq \frac{1}{x} \log \frac{1/\alpha x + 1}{1/\alpha x} \\ &= \frac{1}{x} \log(1 + \alpha x) \\ &\leq \alpha \end{aligned} \tag{1.9}$$

for all $x > 0$; in particular, f_α is a bounded function. It however has an oscillating singularity at $x = 0$; see Figure 3.

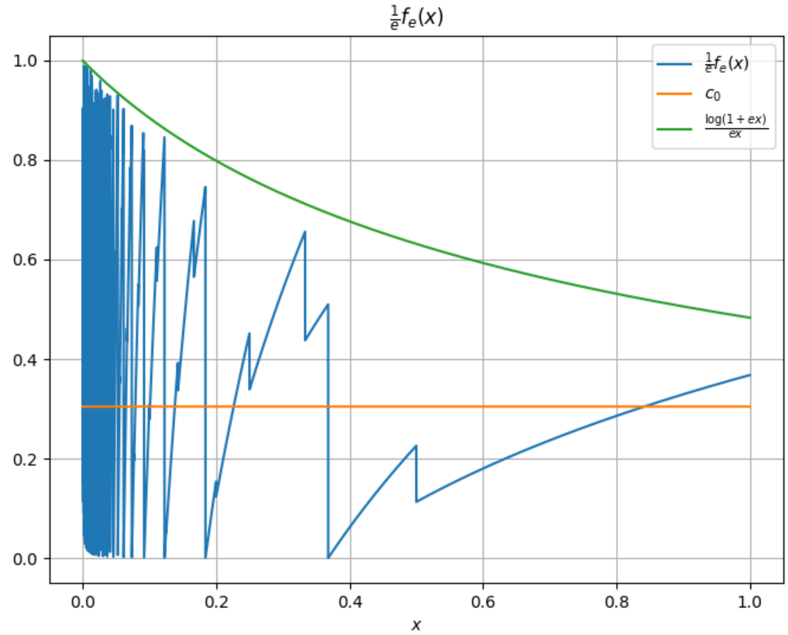

FIGURE 3. The piecewise continuous function $x \mapsto \frac{1}{e} f_e(x)$, together with its mean value $c_0 = 0.30441901\dots$ and the upper bound $\frac{\log(1+ex)}{ex}$. The function exhibits an oscillatory singularity at $x = 0$ similar to $\sin \frac{1}{x}$ (but it is always nonnegative and bounded). Informally, the function f_e quantifies the difficulty that large primes in the factorization of $N!$ have in becoming slightly larger than N/e after multiplying by a natural number.

In Appendix C we give some details on the numerical computation of the constant c_0.

Remark 1.4. In a previous version [16] of this manuscript, the weaker bounds

$$\frac{1}{e} - \frac{O(1)}{\log N} \le \frac{t(N)}{N} \le \frac{1}{e} - \frac{c_0 + o(1)}{\log N}$$

were established, which were enough to recover (1.3), (1.4), and Theorem 1.3(i).

As one might expect, the proof of Theorem 1.3 proceeds by a combination of both theoretical analysis and numerical calculations. Our main tools to obtain upper and lower bounds on $t(N)$ can be summarized as follows:

- In Section 3, we discuss *greedy algorithms* to construct subfactorizations, that provide quickly computable, though suboptimal, lower bounds on $t(N)$ for small and medium values;
- In Section 4, we present a *linear programming* (or *integer programming*) method that provides quite accurate upper and lower bounds on $t(N)$ for small and medium values of N;

- In Section 6, we introduce an *accounting equation* linking the "t-excess" of a subfactorization with its "p-surpluses" at various primes, which provides an reasonable upper bound on $t(N)$ for all N, and is discussed in more detail in Section 6;
- In Section 7, we extend the *rearrangement approach* from [11] to give a computer-assisted proof that Theorem 1.3(iii) holds for sufficiently large N.
- In Section 8, we give *modified approximate factorization* strategy, which provides lower bounds on $t(N)$, that become asymptotically quite efficient.

The final approach is significantly more complicated than the other four, but gives the most efficient lower bounds in the asymptotic limit $N \to \infty$. The key idea is to start with an approximate factorization

$$N! \approx \left(\prod_{j \in I} j \right)^A$$

for some relatively small natural number A (e.g., $A = \lfloor \log^2 N \rfloor$) and a suitable set I of natural numbers greater than or equal to t; there is some freedom to select parameters here, and we will take I to be the natural numbers in $(t, t(1+\sigma)]$ that are 3-rough (coprime to 6), where t is the target lower bound for $t(N)$ we wish to establish, and $\sigma := \frac{3N}{tA}$ is chosen to bring the number of terms in the approximate factorization close to N. With this choice of I, this product contains approximately the right number of copies of p for medium-sized primes p; but it has the "wrong" number of copies of large primes, and is also constructed to avoid the "tiny" primes $p = 2, 3$. One then performs a number of alterations to this approximate factorization to correct for the "surpluses" or "deficits" at various primes $p > 3$, using the supply of available tiny primes $p = 2, 3$ as a sort of "liquidity pool" to efficiently reallocate primes in the factorization. A key point will be that the incommensurability of $\log 2$ and $\log 3$ (i.e., the irrationality of $\log 3 / \log 2$) means that the 3-smooth numbers (numbers of the form $2^n 3^m$) are asymptotically dense (in logarithmic scale), allowing for other factors to be exchanged for 3-smooth factors with little loss[1].

1.1. **Author contributions and data.** This project was initially concieved as a single-author manuscript by Terence Tao, but since the release of the initial preprint [16], grew to become a collaborative project organized via the Github repository [17], which also contains the supporting code and data for the project. The contributions of the individual authors, according to the CRediT categories[2], are as follows:

- Boris Alexeev: ...
- Evan Conway: ...
- Terence Tao: Conceptualization, Formal Analysis, Methodology, Project Administration, Visualization, Writing – original draft, Writing – review & editing.
- Markus Uhr: Initial conception and implementation of linear programming, numerical analysis.
- Kevin Ventullo: Software

[1]The weaker results alluded to in Remark 1.4 only used the prime 2 as a supply of "liquidity", and thus encountered inefficiencies due to the inability to "make change" when approximating another factor by a power of two.

[2]`https://credit.niso.org/`

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list here all contributors to the project who did not wish to be listed as co-authors.

2. Notation and basic estimates

We use the usual asymptotic notation $X = O(Y)$, $X \ll Y$, or $Y \gg X$ to denote an inequality of the form $|X| \leq CY$ for some absolute constant C. We also write $X \asymp Y$ for $X \ll Y \ll X$. For effective estimates, we will use the more precise notation $O_{\leq}(Y)$ to denote any quantity whose magnitude is bounded by exactly at most Y. We also use $O_{\leq}(Y)^+$ to denote a quantity of size $O_{\leq}(Y)$ that is also non-negative, that is to say it lies in the interval $[0, Y]$. We also use $o(X)$ to denote any quantity bounded in magnitude by $c(N)X$, for some $c(N)$ that goes to zero as $N \to \infty$.

If S is a statement, we use 1_S to denote its indicator, thus $1_S = 1$ when S is true and $1_S = 0$ when S is false. If x is a real number, we use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x, and $\lceil x \rceil$ to be the least integer greater than or equal to x.

Throughout this paper, the symbol p (or p_0, p_1, etc.) is always understood to be restricted to be prime. We use (a, b) to denote the greatest common divisor of a and b, $a|b$ to denote the assertion that a divides b, and $\pi(x) = \sum_{p\leq x} 1$ to denote the usual prime counting function.

We use $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ to denote the p-adic valuation of a positive natural number a/b, that is to say the number of times p divides the numerator a, minus the number of times p divides the denominator b. For instance, $\nu_2(32/27) = 5$ and $\nu_3(32/27) = -3$. If one applies a logarithm to the fundamental theorem of arithmetic, one obtains the identity

$$\sum_p \nu_p(r) \log p = \log r \tag{2.1}$$

for any positive rational r.

For a natural number n, we can write

$$\nu_p(n) = \sum_{j=1}^{\infty} 1_{p^j|n}. \tag{2.2}$$

Upon taking partial sums, we recover Legendre's formula

$$\nu_p(N!) = \sum_{j=1}^{\infty} \left\lfloor \frac{N}{p^j} \right\rfloor = \frac{N - s_p(N)}{p-1} \tag{2.3}$$

where $s_p(N)$ is the sum of the digits of N in the base p expansion.

Given a putative factorization $\mathcal{B}$ of $N!$, we refer to the quantity $\nu_p\left(\frac{N!}{\prod \mathcal{B}}\right)$ as the *p-surplus* of $\mathcal{B}$ with respect to the target $N!$, and similarly refer to the negative $-\nu_p\left(\frac{N!}{\prod \mathcal{B}}\right) = \nu_p\left(\frac{\prod \mathcal{B}}{N!}\right)$ of this surplus as the *p-deficit*, with the multiset being *p-balanced* if the p-surplus (or p-deficit) is zero. Thus, a factorization of $N!$ is achieved if and only if one is balanced at every prime p, whereas a subfactorization is achieved if one is either in balance or surplus at every prime p.

To bound the factorial, we have the explicit Stirling approximation [14]

$$\log N! = N \log N - N + \log \sqrt{2\pi N} + O_{\leq}^{+}\left(\frac{1}{12N}\right), \tag{2.4}$$

valid for all natural numbers N.

2.1. Approximation by 3-smooth numbers.

The primes $2, 3$ will play a special role[3] in this paper and will be referred to as *tiny primes*. Call a natural number 3-*smooth* if it is the product of tiny primes, i.e., it is of the form $2^n 3^m$ for some natural numbers n, m, and 3-*rough* if it is not divisible by any tiny prime, that is to say it is coprime to 6. Given a positive real number x, we use $\lceil x \rceil^{\langle 2,3 \rangle}$ to denote the smallest 3-smooth number greater than or equal to x. For instance, $\lceil 5 \rceil^{\langle 2,3 \rangle} = 6$ and $\lceil 10 \rceil^{\langle 2,3 \rangle} = 12$.

It will be convenient to introduce a variant of this quantity that is close to a power[4] of 12. If $1 \leq L \leq x$ is an additional real parameter, we define

$$\lceil x \rceil_L^{\langle 2,3 \rangle} := 12^a \lceil x/12^a \rceil^{\langle 2,3 \rangle} \tag{2.5}$$

for any real $x \geq L \geq 1$, where $a := \lfloor \frac{x/L}{\log 12} \rfloor$ is the largest integer such that $12^a \leq x/L$.

For any $L \geq 1$, let κ_L be the least quantity such that

$$x \leq \lceil x \rceil^{\langle 2,3 \rangle} \leq \exp(\kappa_L) x \tag{2.6}$$

holds for all $x \geq L$; see Figure 4. In Appendix A we establish the following facts:

Lemma 2.1 (Approximation by 3-smooth numbers)**.**

(i) *We have* $\kappa_{4.5} = \log \frac{4}{3} = 0.28768\ldots$ *and* $\kappa_{40.5} = \log \frac{32}{27} = 0.16989\ldots$.
(ii) *For large L, one has* $\kappa_L \ll \log^{-c} L$ *for some absolute constant* $c > 0$.
(iii) *If* $1 \leq L \leq x$ *are real numbers, then*

$$x \leq \lceil x \rceil_L^{\langle 2,3 \rangle} \leq \exp(\kappa_L) x \tag{2.7}$$

[3] One could also run analogous arguments with other sets of tiny primes; for instance, the initial version [16] of this paper only utilized the prime 2 in this fashion.

[4] The significance of the base 12 is that the 3-smooth portion $2^{\nu_2(N!)}3^{\nu_3(N!)}$ of $N!$, which serves as our "liquidity pool", is approximately $2^N 3^{N/2} = \sqrt{12}^N$; see (2.3) below. This makes $\log \sqrt{12}$ a natural "unit of currency" in which to conduct various factor exchanges, with various integer linear combinations of $\log 2$ and $\log 3$ usable as "small change" to approximate quantities that are not integer multiples of $\log \sqrt{12} = \log 2 + \frac{1}{2} \log 3$.

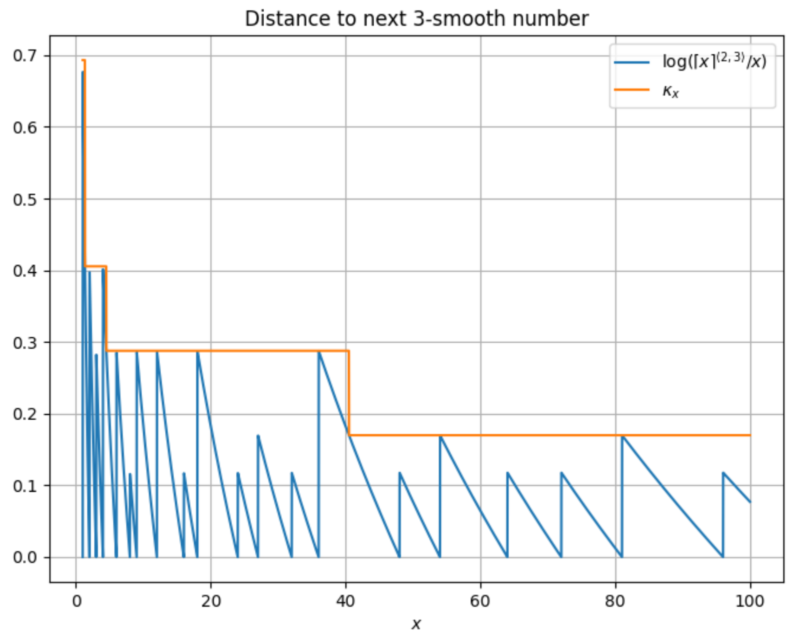

FIGURE 4. The function $\log \frac{\lceil x\rceil^{\langle 2,3\rangle}}{x}$, compared against κ_x.

and for any $0 \leq \gamma < 1$ *we have*

$$\frac{\nu_2(\lceil x\rceil_L^{\langle 2,3\rangle}) - 2\gamma\nu_3(\lceil x\rceil_L^{\langle 2,3\rangle})}{1-\gamma} \leq \frac{\log x + \kappa_{L,\gamma}^{(2)}}{\log\sqrt{12}} \tag{2.8}$$

and

$$\frac{2\nu_3(\lceil x\rceil_L^{\langle 2,3\rangle}) - \gamma\nu_2(\lceil x\rceil_L^{\langle 2,3\rangle})}{1-\gamma} \leq \frac{\log x + \kappa_{L,\gamma}^{(3)}}{\log\sqrt{12}} \tag{2.9}$$

where

$$\kappa_{L,\gamma}^{(2)} := \left(\frac{\log\sqrt{12}}{(1-\gamma)\log 2} - 1\right)\log(12L) + \frac{\kappa_L \log\sqrt{12}}{(1-\gamma)\log 2} \tag{2.10}$$

$$\kappa_{L,\gamma}^{(3)} := \left(\frac{\log\sqrt{12}}{(1-\gamma)\log\sqrt{3}} - 1\right)\log(12L) + \frac{\kappa_L \log\sqrt{12}}{(1-\gamma)\log\sqrt{3}}. \tag{2.11}$$

We remark that when x is a power of 12, the left-hand sides of (2.8), (2.9) are both equal to $\frac{\log x}{\log\sqrt{12}}$; thus the estimates (2.8), (2.9) are quite efficient asymptotically.

2.2. **Sums over primes.** We recall the effective prime number theorem from [6, Corollary 5.2], which asserts that

$$\pi(x) \geq \frac{x}{\log x} + \frac{x}{\log^2 x} \tag{2.12}$$

for $x \geq 599$ and

$$\pi(x) \leq \frac{x}{\log x} + \frac{1.2762x}{\log^2 x} \tag{2.13}$$

for $x > 1$.

We will also need to control sums of somewhat oscillatory functions over primes, for which the bounds in (2.12), (2.13) are of insufficient strength. Let $y < x$ be real numbers. Given a function $b\colon (y,x] \to \mathbb{R}$, its *total variation* $\|b\|_{\mathrm{TV}(y,x]}$ is defined as the supremum of the quantities $\sum_{j=0}^{J-1} |b(x_{j+1}) - b(x_j)|$ for $y < x_0 \leq \cdots \leq x_J \leq x$, and the *augmented total variation* $\|b\|_{\mathrm{TV}^*(y,x]}$ is defined as

$$\|b\|_{\mathrm{TV}^*(y,x]} := |b(y^+)| + |b(x)| + \|b\|_{\mathrm{TV}(y,x]},$$

$b(y^+) := \lim_{t\to y^+} b(t)$ denotes the right limit of b at y (which exists if b is of finite total variation). Equivalently, $\|b\|_{\mathrm{TV}^*(y,x]}$ is the total variation of b if extended by zero outside of $(y,x]$. The indicator function $1_{(y,x]}$ clearly has an augumented total variation of 2.

We will use this augmented total variation to control sums over primes. More precisely, in Appendix B we will show

Lemma 2.2 (Effective bounds for oscillatory sums over primes)**.** *Let* $1423 \leq y \leq x$*, and let* $b : (y,x] \to \mathbb{R}$ *be of bounded total variation. Then we have the bound*

$$\sum_{y<p\leq x} b(p)\log p = \int_y^x \left(1 - \frac{2}{\sqrt{t}}\right) b(t)\, dt + O_{\leq}(\|b\|_{\mathrm{TV}^*(y,x]} E(x)) \tag{2.14}$$

where the error function $E(x)$ *is defined as*

$$E(x) := 0.95\sqrt{x} + 3.83 \times 10^{-9} x. \tag{2.15}$$

In particular one has

$$\pi(x) - \pi(y) = \int_y^x \left(1 - \frac{2}{\sqrt{t}}\right) \frac{dt}{\log t} + O_{\leq}\left(2\frac{E(x)}{\log y}\right). \tag{2.16}$$

If b is non-negative, one also has the upper bound

$$\sum_{y<p\leq x} b(p) \leq \frac{1}{\log y}\int_y^x b(t)\, dt + \|b\|_{\mathrm{TV}^*(y,x]}\frac{E(x)}{\log y} \tag{2.17}$$

and the lower bound

$$\sum_{y<p\leq x} b(p) \leq \frac{1 - \frac{2}{\sqrt{y}}}{\log x}\int_y^x b(t)\, dt - \|b\|_{\mathrm{TV}^*(y,x]}\frac{E(x)}{\log x}. \tag{2.18}$$

Thus for instance

$$\pi(x) - \pi(y) \leq \frac{x-y}{\log y} + 2\frac{E(x)}{\log y} \tag{2.19}$$

and

$$\pi(x) - \pi(y) \geq \left(1 - \frac{2}{\sqrt{y}}\right)\frac{x-y}{\log x} - 2\frac{E(x)}{\log x}. \tag{2.20}$$

One can also replace all occurrences of $E(x)$ *here by the classical error term* $O(x\exp(-c\sqrt{\log x}))$ *for some absolute constant* $c > 0$ *(in which case the* $\frac{2}{\sqrt{t}}$ *type terms can be absorbed into the error term).*

We remark that the accuracy in (2.14), (2.16) in particular is on par with what would be provided by the Riemann hypothesis, as long as x is not too large (e.g., $x \leq 10^{18}$). The other estimates in this lemma are not quite as precise, but still adequate for our applications. The error term $E(x)$ can be improved somewhat for large x (see (B.3)), but this simplified version will suffice for our analysis (in particular, the contribution of the second term in (2.15) will be negligible for our applications). We make the easy remark that $E(x)$ is non-decreasing in x, while $E(x)/x$ is non-increasing.

3. Greedy algorithms

The following simple greedy algorithm gives reasonably good performance to obtain large t-admissible subfactorizations $\mathcal{B}$ of $N!$ for a given choice of t and N:

(0) Initialize $\mathcal{B}$ to be the empty multiset.
(1) If $\mathcal{B}$ is not a factorization, locate the largest prime p which is currently in surplus: $v_p(N!/\prod \mathcal{B}) > 0$.
(2) If $N!/\prod \mathcal{B}$ contains a multiple of p that is greater than or equal to t, locate the smallest such multiple, add it to $\mathcal{B}$, and return to Step 1. Otherwise, `HALT` the algorithm.

This procedure clearly halts in finite time to produce a t-admissible subfactoriation of $N!$. For instance, applying this procedure with $N = 9$, $t = 3$ produces the 3-admissible subfactorization

$$\{7\times 1, 5\times 1, 3\times 1, 3\times 1, 3\times 1, 3\times 1, 2\times 2, 2\times 2, 2\times 2\}$$

which recovers the bound $t(9) \geq 3$ from Example 1.1 (though with a slightly different subfactorization, in which the 8 is replaced by 4).

This procedure is efficient for small N, for instance attaining the exact value of $t(N)$ for all $N \leq 79$, though it begins to degrade for larger N; see Figure 7. The performance is also respectable (though not optimal) for medium N; for instance, when $N = 3\times 10^5$ and $t = N/3$, it locates a t-admissible subfactorization of $N!$ of cardinality $N + 372$, which is close to the linear programming limit of $N + 455$ established in the next section.

discuss modifications to the algorithm to make it perform both faster and more accurately

In order to get this algorithm to validate all $8\times 10^4 \leq N \leq 10^{11}$ on commodity hardware in a reasonable amount of time, two major modifications were implemented.

First, when trying to improve an inequality $t(N) \geq \lambda N$ for all N in some range, one can avoid running the algorithm for every single N in that range by instead proving a stronger inequality on a sparse subset. Namely, if one can show that $t(N_0) \geq (\lambda+\varepsilon)N_0$, it follows that $t(N) \geq \lambda N$ for all $N \in \left[N_0, (1+\frac{\varepsilon}{\lambda})N_0\right)$. If one can fix ε, this reduces a brute force check of every value in a range of length L to checking just $\log_{1+\frac{\varepsilon}{\lambda}}(L)$ values. From the estimates in Section 1, one would expect to be able to take $\varepsilon = \frac{1}{e} - \lambda$ asymptotically; in practice the algorithm uses slightly smaller values.

The other major modification is related to Step (2) in the above algorithm, where one is searching for the least value of c such that $cp \geq t$ and c can be constructed from the remaining factors. In practice, the algorithm pre-computes and store information about all such candidates; absent any further heuristics, this amounts to storing information about all integers less than N, which becomes prohibitively expensive for large N. The key observation is that any such c must satisfy a further arithmetic condition, namely that $\frac{cL(c)}{S(c)} < t$. By only storing information about c which satisfy this condition, the memory footprint is reduced by several orders of magnitude.

By using the greedy method, Theorem 1.3(ii) can be verified for $N \leq 3 \times 10^5$, and Theorem 1.3(iii) can be verified for $8 \times 10^4 \leq N \leq 10^{11}$. Thus, to resolve these claims, it remains to only establish Theorem 1.3(iii) in the regime $43632 \leq N < 8 \times 10^4$ and $N > 10^{11}$, and also to show that this claim fails for $N = 43631$.

would be nice to have some data to plot on the greedy algorithm performance for the range $10^4 \leq N \leq 10^{11}$

4. Linear programming

A t-admissible subfactorization of $N!$ can also be viewed as a product

$$\prod_{j \geq t} j^{m_j}$$

for some non-negative integers m_j that obey the linear constraints

$$\sum_{j \geq t} m_j \nu_p(j) \leq \nu_p(N!) \tag{4.1}$$

for all primes p. Thus, we have an alternative description of $t(N)$:

Proposition 4.1. *For any $N \geq 1$, $t(N)$ is the largest quantity t for which there is a solution to the infinite integer program* (4.1)*, where m_j are constrained to be non-negative integers with*

$$\sum_j m_j \leq N.$$

This proposition suggests the possibility of using linear programming (or integer programming) methods to provide upper and lower bounds on $t(N)$. An immediate issue for computational purposes is that this program involves an infinite number of variables m_j. However, observe that any j not dividing $N!$ cannot be used; and furthermore if j contains a strictly smaller factor that is at least t, then it could be replaced by that factor while still generating a

t-admissible subfactorization of N. As a consequence of these two facts, j can be restricted to a finite set $J_{t,N}$ of the numbers $t \leq j \leq \max(N, t^2)$ dividing $N!$ with no proper factors greater than or equal to t. This makes the integer program finite (though somewhat large), and in particular computable for small N, such as $N \leq 10^4$. By relaxing the integer program to a linear program, one can also obtain upper and lower bounds as follows:

- If one uses linear programming to maximize the quantity $\sum_{j \in J_{t,N}} m_j$ for non-negative *reals* m_j subject to the constraints (8.1), and $\sum_{j \in J_{t,N}} \lfloor m_j \rfloor \geq N$, then $\prod_{j \in J_{t,N}} j^{\lfloor m_j \rfloor}$ represents a t-admissible subfactorization of $N!$ that witnesses $t(N) \geq t$.
- If the dual linear program of finding non-negative reals w_p for each prime $p \leq N$ that satisfy the dual constraints

$$\sum_p w_p \nu_p(j) \geq 1 \tag{4.2}$$

for all $j \in J_{t,N}$ as well as

$$\sum_p w_p \nu_p(N!) < N, \tag{4.3}$$

then by multiplying (4.1) by w_p and summing over $p \leq N$, we see that these constraints force $\sum_{j \in J_{t,N}} m_j < N$, and hence $t(N) < t$.

By the duality of linear programming, these upper bounds and lower bounds would match if the constraint $\sum_{j \in J_{t,N}} \lfloor m_j \rfloor \leq N$ in the lower bound were replaced with the linear constraint $\sum_{j \in J_{t,N}} m_j \leq N$.

In practice, the linear programming method is extremely accurate in upper and lower bounding $t(N)$; for instance, they give the exact value of $t(N)$ for all $N \leq 600$, with the sole exception of $N = 155$, where the linear programming bounds instead give $45 \leq t(155) \leq 46$. In this case, the (slower) integer programming method can be deployed to verify that $t(155) = 45$.

However, the large size of $J_{t,N}$ renders a direct application of the linear programming approach computationally expensive once N exceeds 10^3 or so. For the purpose of establishing lower bounds, one can of course reduce this set arbitrarily; we have found a good choice to be the integers between t and N. For the dual problem, we are also able to make such a reduction, under the additional hypotheses that the weights w_p are non-decreasing:

Lemma 4.2 (Linear programming bound)**.** *Let N be an natural number and* $1 \leq t \leq N/2$. *Suppose for each prime $p \leq N$, one has a non-negative real number w_p which is weakly non-decreasing in p (thus $w_p \leq w_{p'}$ when $p \leq p'$), and such that* (4.2) *holds for all $t \leq j \leq N$, and such that* (4.3) *also holds.*

Proof. By the previous remarks, it will suffice to show that the bound (4.2) in fact holds for all $j \geq t$, not just for $t \leq j \leq N$. Indeed, if this were not the case, consider the first $j \geq t$ where (4.2) fails. Take a prime p dividing j and replace it by a prime in the interval $[p/2, p)$ which exists by Bertrand's postulate (or remove p entirely, if $p = 2$); this creates a new j' in $[j/2, j)$

which is still at least t. By the weakly decerasing hypothesis on w_p, we have

$$\sum_p w_p \nu_p(j) \geq \sum_p w_p \nu_p(j')$$

and hence by the minimality of j we have

$$\sum_p w_p \nu_p(j) > 1,$$

a contradiction. □

We have found empirically that using linear programming to maximize the left-hand side of (4.3) subject to (4.2) for $t \leq j \leq N$ tends to generate weights w_p that are in fact weakly decreasing, so that a rigorous upper bound $t(N) < t$ can be established by this method without needing to expand the linear program to all $j \in J_{t,N}$. With this technique, the linear programming method can now be run to cover all $N \leq 10^4$ with the exception of $N = 155, 765, 1528, 1618, 1619, 2574, 2935, 3265, 5122, 5680, 9633$, but in these cases $t(N)$ can be computed exactly by integer programming; see [17] and Figure 5.

One can also use this method to accurately bound subfactorizations of $N!$ for larger N, although the runtime becomes slow. For instance, with $N = 3 \times 10^5$ and $t = N/3 = 10^5$, Lemma 4.2 can be used to show that any t-admissible factorization has cardinality at most $N + 455$, while the lower bound linear program produces a t-admissible factorization of exactly this cardinality. This demonstrates Theorem 1.3(ii), (iii) for this value of N. The linear programming method can also establish Theorem 1.3(iii) in the range $43632 \leq N \leq 8 \times 10^4$, but show that this conjecture fails for $N = 43631$; it also holds for $N = 41006$, but fails for all smaller N except for $N = 1, 2, 3, 4, 5, 6, 9$. When combined with the greedy algorithm computations, this resolves Theorem 1.3(ii), (iii) except in the asymptotic range $N > 10^{11}$, where it suffices to establish the lower bound $t(N) \geq N/3$.

more discussion here

5. Some upper bounds

It is easy to check using (2.1) that the weights $w_p := \frac{\log p}{\log t}$ will obey the requirements (4.2), (4.3) as long as $0 > \log N! - N \log t$. This recovers the trivial upper bound (1.2). By adjusting these weights at large primes, one can improve this bound as follows:

Lemma 5.1 (Upper bound criterion)**.** *Suppose that* $1 \leq t \leq N$ *are such that*

$$\sum_{\frac{t}{\lfloor\sqrt{t}\rfloor} < p \leq N} f_{N/t}(p/N) > \log N! - N \log t, \tag{5.1}$$

where $f_{N/t}$ *was defined in* (1.7)*. Then* $t(N) < t$*.*

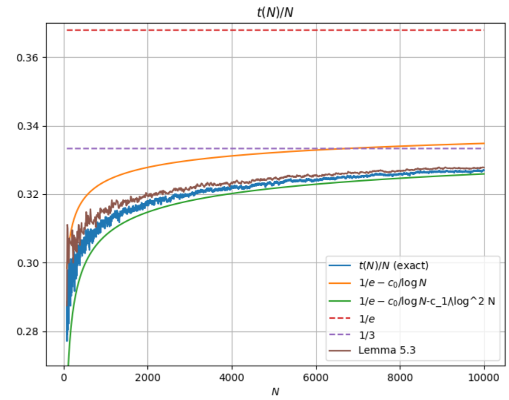

FIGURE 5. $t(N)/N$ for $80 \leq N \leq 10^4$, obtained via linear programming in most cases (and integer programming in some exceptional cases). The upper bound from Lemma 5.1 is surprisingly sharp, as is the refined asymptotic $1/e - c_0/\log N - c_1/\log^2 N$, though the cruder asymptotics $1/e$ or $1/e - c_0/\log N$ are significantly poorer approximations.

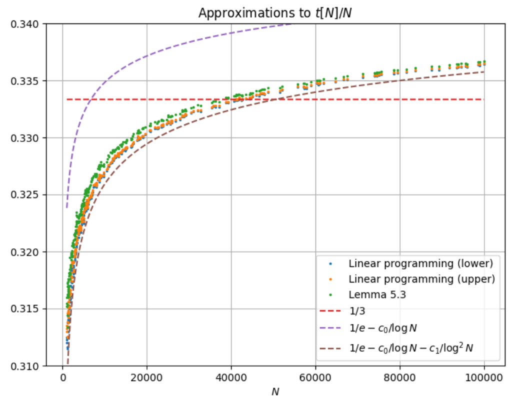

FIGURE 6. Upper and lower bounds $t(N)/N$ obtained by linear programming for some randomly sampled $10^3 \leq N \leq 10^5$. The refined asymptotic $1/e - c_0/\log N - c_1/\log^2 N$ is now a slight underestimate, hinting at further terms in the asymptotic expansion.

Proof. We introduce the weights

$$w_p := \begin{cases} \frac{\log p}{\log t} & p \leq \frac{t}{\lfloor \sqrt{t} \rfloor} \\ \frac{\log p}{\log t} - \frac{\log \frac{\lceil t/p \rceil}{t/p}}{\log t} = 1 - \frac{\log \lceil t/p \rceil}{\log t} & p > \frac{t}{\lfloor \sqrt{t} \rfloor}. \end{cases}$$

Clearly the w_p are non-negative. It will suffice to verify the conditions (4.2), (4.3). If $j \in J_{t,N}$ contains no prime factor $p > \frac{t}{\lfloor\sqrt{t}\rfloor}$, then from (2.1) we have

$$\sum_p w_p \nu_p(j) = \frac{\sum_p \nu_p(j)\log p}{\log t} = \frac{\log j}{\log t} \geq 1.$$

If $j \in J_{t,N}$ is of the form $j = mp_1$ where $p_1 > \frac{t}{\lfloor\sqrt{t}\rfloor}$ and m contains no prime factor exceeding $\frac{t}{\lfloor\sqrt{t}\rfloor}$, then $m \geq \lceil t/p_1 \rceil$, and we have

$$\begin{aligned}\sum_p w_p \nu_p(j) &= \frac{\sum_p \nu_p(j)\log p}{\log t} - \frac{\log \frac{\lceil t/p_1\rceil}{t/p_1}}{\log t} \\ &= \frac{\log(mp_1)}{\log t} - \frac{\log \frac{m}{t/p_1}}{\log t} \\ &= 1.\end{aligned}$$

Finally, if $j \in J_{t,N}$ is divisible by two primes $p_1, p_2 > t\lfloor\sqrt{t}\rfloor$ (possibly equal), then

$$\begin{aligned}\sum_p w_p \nu_p(j) &\geq 1 - \frac{\log\lceil t/p_1\rceil}{\log t} + 1 - \frac{\log\lceil t/p_1\rceil}{\log t} \\ &\geq 1 - \frac{\log\sqrt{t}}{\log t} + 1 - \frac{\log\sqrt{t}}{\log t} \\ &= 1.\end{aligned}$$

Thus we have verified (4.2) for all $j \in J_{t,N}$. Finally, from (2.1), (2.3), (5.1) we have

$$\begin{aligned}\sum_p w_p \nu_p(N!) &= \frac{\sum_p \nu_p(N!)\log p}{\log t} - \sum_{p > \frac{t}{\lfloor\sqrt{t}\rfloor}} \frac{\nu_p(N!)\log\frac{\lceil t/p\rceil}{t/p}}{\log t} \\ &\geq \frac{\log N!}{\log t} - \frac{\sum_{p>\frac{t}{\lfloor\sqrt{t}\rfloor}} \lfloor\frac{N}{p}\rfloor \log\frac{\lceil t/p\rceil}{t/p}}{\log t} \\ &= \frac{\log N!}{\log t} - \frac{\sum_{p>\frac{t}{\lfloor\sqrt{t}\rfloor}} f_{N/t}(p/N)}{\log t} \\ &< N,\end{aligned}$$

giving (4.3). The claim follows. ☐

In practice, Lemma 5.1 gives quite good upper bounds on N, especially when N is large, although for medium N the linear programming method is superior: see Figure 1, Figure 2, Figure 7.

We can now prove the upper bound portion of Theorem 1.3(iv):

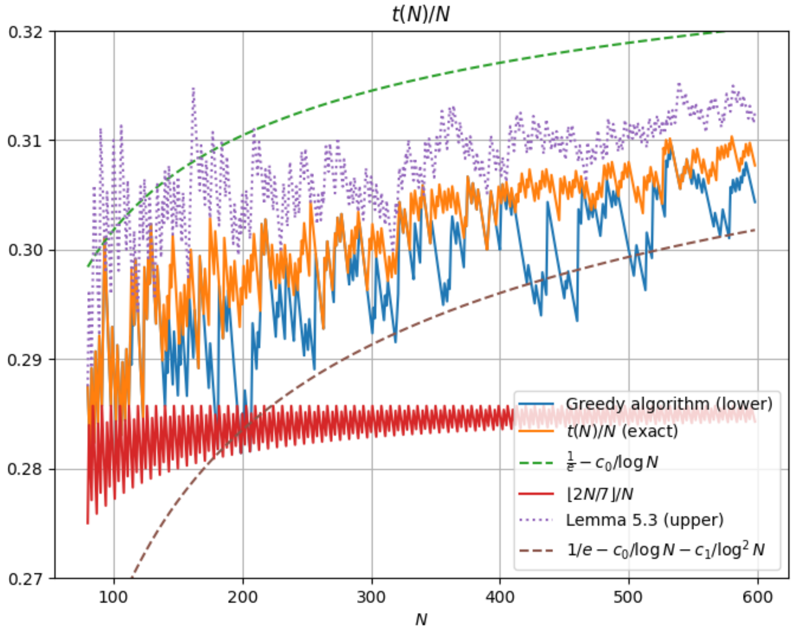

FIGURE 7. An enlarged version of Figure 2, displaying the lower bound from the greedy algorithm and the upper bound from Lemma 5.1. The linear programming upper and lower bounds are exact in this region, except for $N = 155$ in which the upper bound is off by one.

Proposition 5.2. *For large N, one has*

$$\frac{t(N)}{N} \leq \frac{1}{e} - \frac{c_0}{\log N} - \frac{c_1 + o(1)}{\log^2 N}$$

where

$$c_1' := \frac{1}{e}\int_0^1 f_e(x) \log \frac{1}{x}\, dx = 0.3702015\ldots \tag{5.2}$$

$$c_1'' := \sum_{k=1}^{\infty} \frac{1}{k} \log\left(\frac{e}{k}\left\lceil \frac{k}{e} \right\rceil\right) \approx 1.6796 \tag{5.3}$$

$$c_1 := c_1' + c_0 c_1'' - e c_0^2/2 \approx 0.75554808. \tag{5.4}$$

We discuss the numerical evaluation of these constants in Appendix C.

Numerically, this bound is a reasonably good approximation for medium-sized N, see Figure 5, Figure 6, although it may be possible to improve the approximation further with additional terms. Based on these numerics it seems natural to conjecture that one in fact has

$$\frac{t(N)}{N} = \frac{1}{e} - \frac{c_0}{\log N} - \frac{c_1 + o(1)}{\log^2 N}$$

as $N \to \infty$.

Proof. We apply Lemma 5.1 with
$$t := \frac{1}{e} - \frac{c_0}{\log N} - \frac{c_1 - \varepsilon}{\log^2 N}$$
for a given small constant $\varepsilon > 0$. From Taylor expansion of the logarithm and the Stirling approximation (2.4) one sees that
$$\log N! - N \log t = ec_0 \frac{N}{\log N} + (ec_1 - \frac{1}{2}e^2 c_0^2 - e\varepsilon + o(1)) \frac{N}{\log^2 N}$$
so it will suffice to establish the lower bound
$$\sum_{\frac{t}{\lfloor\sqrt{t}\rfloor} < p \leq N} f_{N/t}(p/N) \geq ec_0 \frac{N}{\log N} + (ec_1 - \frac{1}{2}e^2 c_0^2 - e\varepsilon + o(1)) \frac{N}{\log^2 N} \tag{5.5}$$
for N sufficiently large depending on ε.

For N large enough, we have $\frac{t}{\lfloor\sqrt{t}\rfloor} \leq \frac{N}{\log^3 N}$. On the interval $[1/\log^3 N, 1]$, the piecewise smooth function $f_{N/t}$ is bounded by $O(1)$ thanks to (1.9), and has a total variation of $O(\log^3 N)$; the same is then true for the rescaled function $x \mapsto f_{N/t}(x/N)$ on $[N/\log^3 N, 1]$. By Lemma 2.2 (with classical error term), the left-hand side of (5.5) is at least
$$\int_{N/\log^3 N}^{N} f_{N/t}(x/N) \frac{dx}{\log x} + O\left(N \exp(-c\sqrt{\log N})\right)$$
for some $c > 0$. Performing a change of variable, we reduce to showing that
$$\int_{1/\log^3 N}^{1} f_{N/t}(x) \frac{\log N}{\log(Nx)}\, dx \geq ec_0 + \frac{ec_1 - \frac{1}{2}e^2 c_0^2 - e\varepsilon + o(1)}{\log N}.$$
By Taylor expansion, we have
$$\frac{\log N}{\log(Nx)} = 1 + \frac{\log \frac{1}{x}}{\log N} + o\left(\frac{1}{\log N}\right)$$
and from dominated convergence we have
$$\int_{1/\log^3 N}^{1} f_{N/t}(x) \log \frac{1}{x}\, dx = ec_1' + o(1)$$
and hence by definition of c_1, we reduce to showing that
$$\int_{1/\log^3 N}^{1} f_{N/t}(x)\, dx \geq ec_0 + \frac{ec_0 c_1'' - e^2 c_0^2 - e\varepsilon + o(1)}{\log N}.$$
By performing a rescaling by $N/et = 1 + \frac{ec_0 + o(1)}{\log N}$, the left-hand side may be written as
$$\left(1 - \frac{ec_0 + o(1)}{\log N}\right) \int_{N/et \log^3 N}^{N/et} \left\lfloor \frac{N/et}{x} \right\rfloor \log\left(ex \left\lceil \frac{1}{ex} \right\rceil\right)$$
so it will suffice to show that
$$\int_{N/et \log^3 N}^{N/et} \left\lfloor \frac{N/et}{x} \right\rfloor \log\left(ex \left\lceil \frac{1}{ex} \right\rceil\right)\, dx \geq ec_0 + \frac{ec_0 c_1'' - e\varepsilon + o(1)}{\log N}.$$

From (1.6), (1.9) we have

$$\int_{1/\log^2 N}^1 \left\lfloor \frac{1}{x} \right\rfloor \log \left(ex \left\lceil \frac{1}{ex} \right\rceil \right) = ec_0 - \frac{o(1)}{\log N}$$

it suffices to show that

$$\int_{1/log^2 N}^{N/et} \left(\left\lfloor \frac{N/et}{x} \right\rfloor - \left\lfloor \frac{1}{x} \right\rfloor \right) \log \left(ex \left\lceil \frac{1}{ex} \right\rceil \right)\, dx \geq \frac{ec_0 c'' - e\varepsilon + o(1)}{\log N}.$$

Let K be sufficiently large depending on ε, then for N sufficiently large depending on K we can lower bound the left-hand side by

$$\sum_{k=1}^K \int_{1/k}^{N/etk} \log \left(ex \left\lceil \frac{1}{ex} \right\rceil \right)\, dx;$$

since $\frac{N}{etk} = \frac{1}{k} + \frac{ec_0}{k \log N}$, we can lower bound this (using the irrationality of e) by

$$\frac{ec_0 + o(1)}{\log N} \sum_{k=1}^K \frac{1}{k} \log \left(\frac{e}{k} \left\lceil \frac{k}{e} \right\rceil \right)$$

for sufficiently large N. Since the sum here can be made arbitrarily close to c_0'' by increasing K, we obtain the claim. □

We can now establish Theorem 1.3(i):

Proposition 5.3. *One has $t(N)/N < 1/e$ for $N \neq 1, 2, 4$.*

Proof. From existing data on $t(N)$ (or the linear programming method) one can verify this claim for $N < 80$ (see Figure 1), so we assume that $N \geq 80$.

Applying Lemma 5.1 and (2.4), it suffices to show that

$$\sum_{p \geq \frac{N/e}{\lfloor \sqrt{N/e} \rfloor}} f_e(p/N) > \frac{1}{2} \log(2\pi N) + \frac{1}{12N}. \tag{5.6}$$

This may be easily verified numerically in the range $80 \leq N \leq 5000$ (see Figure 8). We will discard the $\lfloor \sqrt{N/e} \rfloor$ denominator, and reduce to showing

$$\sum_{N/e < p \leq N} f_e(p/N) > \frac{1}{2} \log(2\pi N) + \frac{1}{12N} \tag{5.7}$$

for $N > 5000$. On $[1/e, 1]$, one can compute

$$\|f_e\|_{\mathrm{TV}^*(1/e,1]} = 4 - 2\log 2$$

so by Lemma 2.2 (noting that $5000/e > 1423$) we have

$$\sum_{N/e < p \leq N} f_e(p/N) \log p \geq N \int_{1/e}^1 \left(1 - \frac{2}{\sqrt{Nx}} \right) f_e(x)\, dx - (4 - 2\log 2) E(N).$$

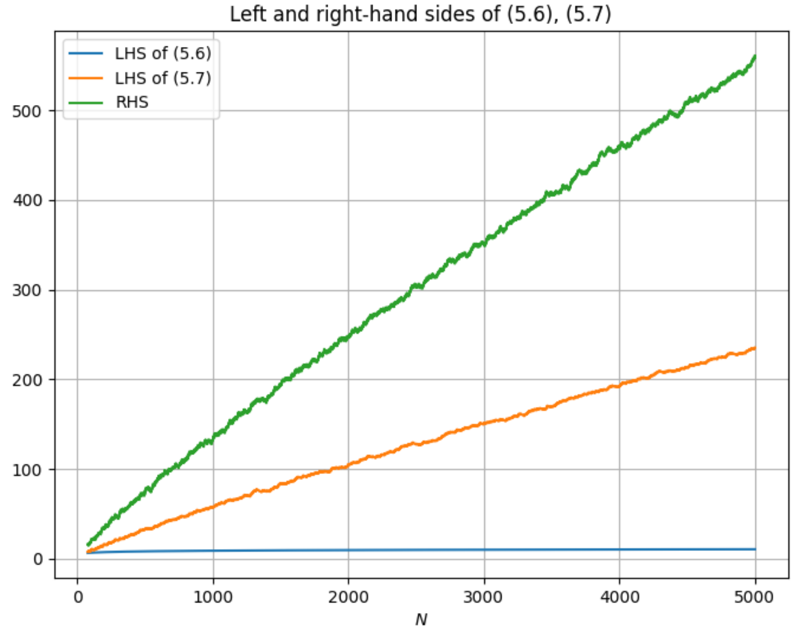

FIGURE 8. A plot of the left and right-hand sides of (5.6), (5.7) for $80 \leq N < 5000$.

By upper bounding $\log p$ by $\log N$ and lower bounding $\left(1 - \frac{2}{\sqrt{Nx}}\right)$ by $1 - \frac{2}{\sqrt{N/e}}$, it suffices to show that

$$\left(1 - \frac{2}{\sqrt{N/e}}\right) \int_{1/e}^{1} f_e(x)\, dx \geq (4 - 2\log 2)\frac{E(N)}{N} + \frac{\log N \log(2\pi N)}{2N} + \frac{\log N}{12N^2}.$$

The right-hand side is increasing in N and the left-hand side is decreasing for $N \geq 5000$, so it suffices to verify this claim for $N = 5000$; but this is a routine calculation (with plenty of room to spare; cf., Figure 8). $\square$

6. THE ACCOUNTING EQUATION

Given a t-admissible multiset $\mathcal{B}$ (which we view as an approximate factorization of $N!$), we can apply the fundamental theorem of arithmetic (2.1) to the rational number $N!/\prod \mathcal{B}$ and rearrange to obtain the *accounting equation*

$$\mathcal{E}_t(\mathcal{B}) + \sum_p \nu_p\left(\frac{N!}{\prod \mathcal{B}}\right) \log p = \log N! - |\mathcal{B}| \log t \tag{6.1}$$

where we define the *t-excess* $\mathcal{E}_t(\mathcal{B})$ of the multiset $\mathcal{B}$ by the formula

$$\mathcal{E}_t(\mathcal{B}) := \sum_{a \in \mathcal{B}} \log \frac{a}{t}. \tag{6.2}$$

Example 6.1. Suppose one wishes to factorize $5! = 2^3 \times 3 \times 5$. The attempted 3-admissible factorization $\mathcal{B} := \{3, 4, 5, 5\}$ has a 2-surplus of $\nu_2(5!/\prod \mathcal{B}) = 1$, is in balance at 3, and has a 5-deficit of $\nu_5(\prod \mathcal{B}/5!) = 1$, so it is not a factorization or subfactorization of 5!. The 3-excess of this multiset is

$$\mathcal{E}_3(\mathcal{B}) = \log\frac{3}{3} + \log\frac{4}{3} + \log\frac{5}{3} + \log\frac{5}{3} = 1.3093\ldots$$

and the accounting equation (6.1) becomes

$$1.3093\cdots + \log 2 - \log 5 = 0.3930\cdots = \log 5! - 4\log 3.$$

If one replaces one of the copies of 5 in $\mathcal{B}$ with a 2, this erases both the 2-surplus and the 5-deficit, and creates a factorization $\mathcal{B}' = \{2, 3, 4, 5\}$ of 5!; the 3-excess now drops to

$$\mathcal{E}_3(\mathcal{B}) = \log\frac{2}{3} + \log\frac{3}{3} + \log\frac{4}{3} + \log\frac{5}{3} = 0.3930\ldots,$$

bringing the accounting equation back into balance.

In view of Remark 1.2, one can now equivalently describe $t(N)$ as follows:

Lemma 6.2 (Equivalent description of $t(N)$)**.** *$t(N)$ is the largest quantity t for which there exists a t-admissible subfactorization of $N!$ with*

$$\mathcal{E}_t(\mathcal{B}) + \sum_p \nu_p\left(\frac{N!}{\prod \mathcal{B}}\right)\log p \le \log N! - N\log t.$$

One can view $\log N! - N\log t$ as an available "budget" that one can "spend" on some combination of t-excess and p-surpluses. For t of the form $t = N/e^{1+\delta}$ for some $\delta > 0$, the budget can be computed using the Stirling approximation (2.4) to be $\delta N + O(\log N)$. The non-negativity of the t-excess and p-surpluses recovers the trivial upper bound (1.2); but note that any prime $p > \frac{t}{\lfloor\sqrt{t}\rfloor}$ must inevitably contribute at least $\log\frac{\lceil t/p\rceil}{t/p}$ to the t-excess if it is to appear in the multiset $\mathcal{B}$. By pursuing this line of reasoning, one can obtain an alternate proof of Lemma 5.1; see [16, Lemma 2.1].

7. Rearranging the standard factorization

In this section we describe an approach to establishing lower bounds on $t(N)$ by starting with the standard factorization $\{1, \ldots, N\}$, dividing out some small prime factors from some of the terms, and then redistributing them to other terms. This approach was introduced in [10] to give lower bounds of the shape $\frac{t(N)}{N} \ge \frac{3}{16} + o(1)$ (by redistributing powers of two only) and $\frac{t(N)}{N} \ge \frac{1}{4} + o(1)$. With computer assistance, we are also able to show that $\frac{t(N)}{N} \ge \frac{1}{3} + o(1)$ for sufficiently large N, in a simpler fashion than the method used to prove Theorem 1.3(iv) in the next section.

details needed here

8. Modified approximate factorizations

In this section we present and then analyze an algorithm that starts with an *approximate* factorization $\mathcal{B}^{(0)}$ of $N!$, which is t-admissible but omits all tiny primes, and is approximately in balance in small and medium primes, and attempts to "repair" this factorization to establish a lower bound of the form $t(N) \geq t$.

To describe the criterion for the algorithm to succeed, it will be convenient to introduce the following notation. For $a_+, a_- \in [0, +\infty]$, we define the asymmetric norm $|x|_{a_+,a_-}$ of a real number x by the formula

$$|x|_{a_+,a_-} := \begin{cases} a_+|x| & x \geq 0 \\ a_-|x| & x \leq 0, \end{cases}$$

with the usual convention $+\infty \times 0 = 0$. If a_+, a_- are finite, this function is Lipschitz with constant $\max(a_+, a_-)$. One can think of a_+ as the "cost" of making x positive, and a_- as the "cost" of making x negative.

The analysis of the algorithm is now captured by the following proposition.

Proposition 8.1 (Repairing an approximate factorization)**.** *Let N, K be natural numbers, and let $1 \leq t \leq N$ be an additional parameter obeying the conditions*

$$\frac{t}{K} \geq \sqrt{N}; \quad \frac{t}{K^2} \geq K \geq 5. \tag{8.1}$$

We also assume that there are additional parameters $\kappa_* > 0$ and $0 \leq \gamma_2, \gamma_3 < 1$, such that there exist 3-smooth numbers

$$t \leq 2^{n_2}3^{m_2}, 2^{n_3}3^{m_3} \leq e^{\kappa_*}t \tag{8.2}$$

such that

$$2m_2 \leq \gamma_2 n_2; \quad n_3 \leq 2\gamma_3 m_3. \tag{8.3}$$

We define the "norm" of a pair n, m of real numbers by the formula

$$\|(n,m)\|_\gamma := \max\left(\frac{n - 2\gamma_2 m}{1-\gamma_2}, \frac{2m - \gamma_3 n}{1-\gamma_3}\right).$$

Let $\mathcal{B}^{(0)}$ be a t-admissible multiset of natural numbers, with all elements of $\mathcal{B}^{(0)}$ at most $(t/K)^2$, and suppose that one has the inequalities

$$\sum_{i=1}^{8} \delta_i \leq \delta \tag{8.4}$$

and

$$\sum_{i=1}^{7} \alpha_i \leq 1 \tag{8.5}$$

where

$$\delta_1 := \frac{1}{N}\mathcal{E}_t(\mathcal{B}^{(1)}) \tag{8.6}$$

$$\delta_2 := \frac{1}{N}\sum_{t/K<p\le N} f_{N/t}(p/N) \tag{8.7}$$

$$\delta_3 := \frac{\kappa_{4.5}}{N}\sum_{3<p_1\le t/K}\left|\nu_{p_1}\left(\frac{N!}{\prod\mathcal{B}^{(0)}}\right)\right| \tag{8.8}$$

$$\delta_4 := \kappa_{4.5}\sum_{K<p_1\le t/K} A_{p_1} \tag{8.9}$$

$$\delta_5 := \kappa_{4.5}\sum_{3<p_1\le K} |A_{p_1}-B_{p_1}|_{\frac{\log p_1}{\log(t/K^2)},1} \tag{8.10}$$

$$\delta_6 := \frac{\kappa_{4.5}}{N} \tag{8.11}$$

$$\delta_7 := \frac{\kappa_*}{\log t}\left(\log\sqrt{12}-B_2\log 2-B_3\log 3\right) \tag{8.12}$$

$$\delta_8 := \frac{2(\log t+\kappa_*)}{N} \tag{8.13}$$

$$\delta := \frac{1}{N}\log N! - \log t \tag{8.14}$$

$$\alpha_1 := \frac{1}{N}\left\|\left(\nu_2\left(\prod\mathcal{B}^{(0)}\right),\nu_3\left(\prod\mathcal{B}^{(0)}\right)\right)\right\|_\gamma \tag{8.15}$$

$$\alpha_2 := \|(B_2,B_3)\|_\gamma \tag{8.16}$$

$$\alpha_3 := \frac{\log\frac{t}{K}+\kappa_{**}}{N\log\sqrt{12}}\sum_{3<p_1\le t/K}\left|\nu_{p_1}\left(\frac{N!}{\prod\mathcal{B}^{(0)}}\right)\right| \tag{8.17}$$

$$\alpha_4 := \frac{1}{\log\sqrt{12}}\sum_{K<p_1\le t/K}\left(\log\frac{t}{p_1}+\kappa_{**}\right)A_{p_1} \tag{8.18}$$

$$\alpha_5 := \frac{1}{\log\sqrt{12}}\sum_{3<p_1\le K}\left|A_{p_1}-B_{p_1}\right|_{\frac{\log p_1}{\log(t/K^2)}(\log K^2+\kappa_{**}),\log p_1+\kappa_{**}} \tag{8.19}$$

$$\alpha_6 := \frac{\log t+\kappa_{**}}{N\log\sqrt{12}} \tag{8.20}$$

$$\alpha_7 := \max\left(\frac{\log(2N)}{(1-\gamma_2)N\log 2},\frac{\log(3N)}{(1-\gamma_3)N\log\sqrt{3}}\right) \tag{8.21}$$

$$\kappa_{**} := \max(\kappa^{(2)}_{4.5,\gamma_2},\kappa^{(3)}_{4.5,\gamma_3}) \tag{8.22}$$

$$A_{p_1} := \frac{1}{N}\sum_m \nu_{p_1}(m)|\{a\in\mathcal{B}^{(0)} : a = mp \textit{ for a prime } p>t/K\}| \tag{8.23}$$

$$B_{p_1} := \frac{1}{N}\sum_{m\le K}\nu_{p_1}(m)\sum_{\frac{t}{m}\le p<\frac{t}{m-1}}\left\lfloor\frac{N}{p}\right\rfloor, \tag{8.24}$$

with the convention that the upper bound $p < \frac{t}{m-1}$ in (8.24) *is vacuous when $m = 1$. Then $t(N) \geq t$.*

Note that the quantities δ_2, δ essentially appeared previously in Lemma 5.1. Informally, δ and the condition (8.4) represent the "budget" of what one can "spend" on various imperfections of the approximate factorization $\mathcal{B}^{(0)}$ until we become unable to "afford" to "buy" a t-admissible subfactorization of cardinality at least N. The quantity δ_2 represents the "fixed expense" coming from large primes that is unavoidable; but the other expenditures $\delta_i, i \neq 2$ can be made small by suitable exploitation of the pool of tiny primes present in the factorization of $N!$ (which multiply out to roughly $\sqrt{12}^N$). The secondary constraint (8.5) can roughly speaking be viewed as a requirement that one does not completely exhaust this pool.

In practice, the parameter K will be quite small compared to N, and the quantities $\gamma_2, \gamma_3, \kappa_*$ will also be somewhat smaller than 1.

The rest of this section will be devoted to the proof of this proposition. It will be convenient to divide the primes into four classes:

- *Tiny primes* $p = 2, 3$.
- *Small primes* $3 < p \leq K$.
- *Medium primes* $K < p \leq t/K$.
- *Large primes* $p > t/K$.

Initially, the multiset $\mathcal{B}^{(0)}$ may have the "wrong" number of factors at large primes. We fix this by applying the following modifications to $\mathcal{B}^{(0)}$:

(a) Remove all elements of $\mathcal{B}^{(0)}$ that are divisible by a large prime $p > t/K$ from the multiset.
(b) For each large prime $p > t/K$, add $v_p(N!)$ copies of $p\lceil t/p \rceil$ to the multiset.

We let $\mathcal{B}^{(1)}$ be the multiset formed after completing both Step (a) and Step (b). We make two simple observations:

(A) Since the elements of $\mathcal{B}^{(0)}$ are at most $(t/K)^2$, all the elements removed in Step (a) are of the form mp where $m \leq t/K$.
(B) For each large prime p considered in Step (b), one has $v_p(N!) = \lfloor N/p \rfloor$ by (2.3) and (8.1), while $\lceil t/p \rceil \leq K \leq t/K$ (again by (8.1)).

From this, we see that $\mathcal{B}^{(1)}$ is automatically t-admissible, and in balance at any large prime $p > t/K$:

$$v_p\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right) = 0.$$

For medium primes $K < p_1 \leq t/K$, one can have some increase in the p_1-surplus coming from Step (a), which is described by (8.23):

$$\nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right) = \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right) + NA_{p_1}.$$

For small or tiny primes $p \leq K$, one also has some possible decrease in the p_1-surplus coming from Step (b), which is described by (8.24):

$$\nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right) = \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right) + N(A_{p_1} - B_{p_1}).$$

In particular, we have from (8.15), (8.16) and the triangle inequality that

$$\frac{1}{N}\left\|\left(\nu_2\left(\prod \mathcal{B}^{(1)}\right), \nu_3\left(\prod \mathcal{B}^{(1)}\right)\right)\right\|_\gamma \leq \alpha_1 + \alpha_2. \tag{8.25}$$

Each element removed in Step (a) reduces the t-excess, while each element $p\lceil t/p\rceil$ added in Step (b) increases the t-excess by $\log \frac{\lceil t/p\rceil}{t/p}$, so each large prime $t/K < p \leq N$ contributes a net of $\lfloor \frac{N}{p} \rfloor \log \frac{\lceil t/p\rceil}{t/p} = f_{N/t}(p/N)$ to the t-excess. Thus by (8.6), (8.7) we have

$$\frac{1}{N}\mathcal{E}_t(\mathcal{B}^{(1)}) \leq \delta_1 + \delta_2. \tag{8.26}$$

Now we bring the multiset $\mathcal{B}^{(1)}$ into balance at small and medium primes $3 < p \leq t/K$. We make the following observations:

(C) If an element in $\mathcal{B}^{(1)}$ is divisible by some small or medium prime $3 < p \leq t/K$, and one replaces p by $\lceil p\rceil_{4.5}^{\langle 2,3\rangle}$ in the factorization of that element, then the p-deficit decreases by one, while (by Lemma 2.1) the t-excess increases by at most $\kappa_{4.5}$, and the quantity $\|(\nu_2(\prod \mathcal{B}^{(1)}), \nu_3(\prod \mathcal{B}^{(1)}))\|_\gamma$ increases by at most $\frac{\log p + \kappa_{**}}{\log \sqrt{12}}$. All other p_1-surpluses or p_1-deficits for $p_1 \neq 2, 3, p$ remain unaffected.

(D) If one adds an element of the form $m\lceil t/m\rceil_{4.5}^{\langle 2,3\rangle}$ to $\mathcal{B}^{(1)}$ for some $m \leq t/K$ that is the product of small or medium primes $3 < p \leq t/K$, then the p-surpluses at small or medium primes p decrease by $\nu_p(m)$, while (by Lemma 2.1) the t-excess increases by at most $\kappa_{4.5}$, and the quantity $\|(\nu_2(\frac{N!}{\prod \mathcal{B}^{(1)}}), \nu_3(\frac{N!}{\prod \mathcal{B}^{(1)}}))\|_\gamma$ increases by at most $\frac{\log(t/m) + \kappa_{**}}{\log \sqrt{12}}$. The p-surpluses or p-deficits at medium or large primes remain unaffected.

With these observations in mind, we perform the following modifications to the multiset $\mathcal{B}^{(1)}$.

(c) If there is a p_1-deficit $\nu_{p_1}(\prod \mathcal{B}^{(1)}/N!) > 0$ at some small or medium prime $3 < p_1 \leq t/K$, then we perform the replacement of p_1 in one of the elements of $\mathcal{B}^{(1)}$ with $\lceil p_1\rceil_{4.5}^{\langle 2,3\rangle}$ as per observation (C), repeated $\nu_{p_1}(\prod \mathcal{B}^{(1)}/N!)$ times, in order to eliminate all such deficits.

(d) If there is a p-surplus $\nu_p(\prod N!/\mathcal{B}^{(1)}) > 0$ at some medium prime $K < p \leq t/K$, we add the element $p\lceil t/p\rceil_{4.5}^{\langle 2,3\rangle}$ to $\mathcal{B}^{(1)}$ as per observation (D), $\nu_p(\prod N!/\mathcal{B}^{(1)})$ times, in order to eliminate all such surpluses at medium primes.

(d') If there are p-surpluses $\nu_p(\prod N!/\mathcal{B}^{(1)}) > 0$ at some small primes $3 < p \leq K$, we multiply all these primes together, then apply the greedy algorithm to factor them into products m in the range $t/K^2 < m \leq t/K$, plus at most one exceptional product in the range $1 < m \leq t/K$. For each of these m, add $m\lceil t/m \rceil_{4.5}^{\langle 2,3\rangle}$ to $\mathcal{B}^{(1)}$ as per observation (D), to eliminate all such surpluses at small primes.

Call the multiset formed from $\mathcal{B}^{(1)}$ formed as the outcome of applying Steps (c), (d), (d') as $\mathcal{B}^{(2)}$. The product of all the primes arising in Step (d') has logarithm equal to

$$\sum_{3<p_1\leq K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right)\right|_{\log p_1,0} = \sum_{3<p_1\leq K} \left| \nu_{p}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right)\right|_{\log p_1,0}$$

and hence the number of non-exceptional m arising in (d') is at most

$$\sum_{3<p_1\leq K} \left| \nu_{p}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right)\right|_{\frac{\log p_1}{\log(t/K^2)},0}.$$

The total excess of $\mathcal{B}^{(2)}$ is increased in Step (c) by at most

$$\kappa_{4.5} \sum_{3<p_1\leq t/K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right)\right|_{0,1} = \kappa_{4.5} \sum_{3<p_1\leq t/K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right) + N(A_{p_1} - B_{p_1})\right|_{0,1},$$

in Step (d) by at most

$$\kappa_{4.5} \sum_{K<p_1\leq t/K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right)\right|_{1,0} = \kappa_{4.5} \sum_{K<p_1\leq t/K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right) + NA_{p_1}\right|_{1,0},$$

and in Step (c) by at most

$$\kappa_{4.5}\left(1 + \sum_{3<p_1\leq K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right)\right|_{\frac{\log p_1}{\log(t/K^2)},0}\right).$$

From the triangle inequality and (8.26), (8.8), (8.9), (8.10), (8.11), we then have

$$\frac{1}{N}\mathcal{E}_t(\mathcal{B}^{(2)}) \leq \sum_{i=1}^{6} \delta_i. \tag{8.27}$$

Similarly, the quantity $\frac{1}{N}\|(\nu_2(\prod \mathcal{B}^{(1)}), \nu_3(\prod \mathcal{B}^{(1)}))\|_\gamma$ is increased in Step (c) by at most

$$\frac{1}{N\log\sqrt{12}} \sum_{3<p_1\leq t/K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right) + N(A_{p_1} - B_{p_1})\right|_{0,\log p_1+\kappa_{**}},$$

in Step (d) by at most

$$\frac{1}{N\log\sqrt{12}} \sum_{K<p_1\leq t/K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right) + NA_{p_1}\right|_{\log(t/p_1)+\kappa_{**},0},$$

and in Step (d') by at most the sum of

$$\frac{1}{N\log\sqrt{12}} \sum_{3<p_1\leq K} \left| \nu_{p_1}\left(\frac{N!}{\prod \mathcal{B}^{(0)}}\right) + N(A_{p_1} - B_{p_1})\right|_{\log(K^2)+\kappa_{**},0}$$

and

$$\frac{1}{N \log \sqrt{12}} \left(\log t + \kappa_{**}\right)$$

so by (8.25), (8.17), (8.18), (8.19), (8.20), and the triangle inequality we have

$$\frac{1}{N} \|(\nu_2(\prod \mathcal{B}^{(2)}), \nu_3(\prod \mathcal{B}^{(3)}))\|_\gamma \leq \sum_{i=1}^{6} \alpha_i. \tag{8.28}$$

By construction, the multiset $\mathcal{B}^{(2)}$ is t-admissible, and in balance at all small, medium, and large primes $p > 3$; thus $N!/\prod \mathcal{B}^{(2)} = 2^n 3^m$ for some integers n, m. From (8.28), (8.5), (2.3), (8.21) we have

$$\begin{aligned}
n - 2\gamma_2 m &= \nu_2(N!) - 2\gamma_2\nu_3(N!) - \left(\nu_2\left(\prod \mathcal{B}^{(2)}\right) - 2\gamma_2\nu_3\left(\prod \mathcal{B}^{(2)}\right)\right)\\
&\geq \nu_2(N!) - 2\gamma_2\nu_3(N!) - N(1-\gamma_2)\sum_{i=1}^{6}\alpha_i\\
&> N - \frac{\log N}{\log 2} - 1 - \gamma_2 N - N(1-\gamma_2)(1-\alpha_7)\\
&= N(1-\gamma_2)\alpha_7 - \frac{\log(2N)}{\log 2}\\
&\geq 0
\end{aligned}$$

and similarly

$$\begin{aligned}
2m - \gamma_3 n &= 2\nu_3(N!) - \gamma_3\nu_2(N!) - \left(2\nu_3\left(\prod \mathcal{B}^{(2)}\right) - \gamma_3\nu_2\left(\prod \mathcal{B}^{(2)}\right)\right)\\
&\geq 2\nu_3(N!) - \gamma_3\nu_2(N!) - N(1-\gamma_3)\sum_{i=1}^{6}\alpha_i\\
&> N - \frac{\log N}{\log\sqrt{3}} - 2 - \gamma_3 N - N(1-\gamma_3)(1-\alpha_7)\\
&= N(1-\gamma_3)\alpha_7 - \frac{\log(3N)}{\log\sqrt{3}}\\
&\geq 0.
\end{aligned}$$

From (8.3) and Cramer's rule we conclude that that $(n, 2m)$ lies in the non-negative linear span of $(n_2, 2m_2)$, $(n_3, 2m_3)$, thus

$$(n, 2m) = \beta_2(n_2, 2m_2) + \beta_3(n_3, 2m_3) \tag{8.29}$$

for some reals $\beta_2, \beta_3 \geq 0$. We now create the multiset $\mathcal{B}^{(3)}$ by adding $\lfloor \beta_2 \rfloor$ copies of $2^{n_2}3^{m_2}$ and $\lfloor \beta_3 \rfloor$ copies of $2^{n_3}3^{m_3}$ to $\mathcal{B}^{(2)}$. By (8.2), this multiset remains t-admissible, and each element added increases the t-excess by at most κ_*. The number of such elements can be upper bounded using (8.29), (2.3) as

$$\begin{aligned}
\lfloor \beta_2 \rfloor + \lfloor \beta_3 \rfloor &\le \beta_2 + \beta_3 \\
&\le \frac{1}{\log t}\big(\beta_2(n_2 \log 2 + m_2 \log 3) + \beta_3(n_3 \log 2 + m_3 \log 3)\big) \\
&= \frac{1}{\log t}(n \log 2 + m \log 3) \\
&\le \frac{1}{\log t}((\nu_2(N!) - N B_2) \log 2 + (\nu_3(N!) - N B_3) \log 3) \\
&\le \frac{1}{\log t}\left(N \log 2 + \frac{N}{2} \log 3 - N B_2 \log 2 - N B_3 \log 3\right) \\
&= \frac{N \log \sqrt{12}}{\log t} - \frac{N(B_2 \log 2 + B_3 \log 3)}{\log t}.
\end{aligned}$$

By (8.27), (8.12), we thus have

$$\frac{1}{N}\mathcal{E}_t(\mathcal{B}^{(3)}) \le \sum_{i=1}^{7} \delta_i. \tag{8.30}$$

Meanwhile by construction we see that $\mathcal{B}^{(3)}$ is a subfactorization of $N!$ that is in balance at all non-tiny primes, with tiny prime surpluses bounded by

$$\nu_2\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \le n_2 + n_3; \quad \nu_3\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \le m_2 + m_3.$$

and thus by (8.2), (8.13), we thus have

$$\frac{1}{N}\sum_p \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \log p \le \frac{\log 2^{n_2} 3^{m_2} + \log 2^{n_3} 3^{m_3}}{N} \le \delta_8$$

and thus by (8.30), (8.4) we have

$$\mathcal{E}_t(\mathcal{B}^{(3)}) + \sum_p \nu_p\left(\frac{N!}{\prod \mathcal{B}^{(3)}}\right) \log p \le \log N! - N \log t.$$

Applying Lemma 6.2, we conclude that $t(N) \ge t$ as claimed.

9. Estimating terms

In order to use Proposition 8.1 for a given choice of N, t, we need to find a t-admissible multiset $\mathcal{B}^{(0)}$ and parameters $K, \kappa_*, \gamma_2, \gamma_3$ obeying (8.1) as well as good upper bounds on the quantities δ_i, $i = 1, \dots, 8$ and α_i, $i = 1, \dots, 7$, that can either be evaluated asymptotically or numerically. Many of theaw terms will be straightforward to estimate; we discuss only the more difficult ones gwew.

We introduce a further natural number parameter A and define

$$\sigma := \frac{3N}{At}. \tag{9.1}$$

We let $\mathcal{B}^{(0)}$ be the multiset of 3-rough elements of the interval $(t, t(1+\sigma)]$, with each element repeated precisely A times. This is clearly t-admissible. It has no presence at tiny primes, so

$$\alpha_1 = 0. \tag{9.2}$$

We will also introduce an auxiliary parameter L to assist us with the estimates. The influence of the parameters A, K, L on the other parameters δ_i, α_i (and $\gamma_2, \gamma_3, \kappa_{**}$) can be roughly summarized as follows:

- $\gamma_2, \gamma_3 \asymp \log L/\log N$; assuming this quantity is small enough, we have $\kappa_{**} \asymp 1$.
- $\delta_1 \asymp 1/A$.
- $\delta_2 \asymp 1/\log N$.
- $\delta_3 \asymp A/K \log N$ and $\alpha_3 \asymp A/K$.
- $\delta_5 \asymp \log^{O(1)} K/\log^2 N$ and $\alpha_2, \alpha_5 \asymp \log^{O(1)} K/\log N$.
- $\delta_7 \asymp \kappa_L/\log N$.
- $\delta_4, \delta_6, \delta_8, \alpha_4, \alpha_6, \alpha_7$ will be lower order terms.

We will quantify these relationships more precisely below, but they already suggest that one should take A to only be moderately large (e.g., of logarithmic size), that K should only be slightly larger than A, and that L should be significantly smaller than N.

We use the notation $\sum^*$ to denote summation restricted to 3-rough numbers, thus for instance $\sum^*_{a<k\leq b} 1$ denotes the number of 3-rough numbers in $(a, b]$. We have a simple estimate for such counts:

Lemma 9.1. *For any interval $(a, b]$ with $0 \leq a \leq b$ one has $\sum^*_{a<k\leq b} 1 = \frac{b-a}{3} + O_{\leq}(4/3)$.*

Proof. By the triangle inequality, it suffices to show that $\sum^*_{0<k\leq x} 1 - \frac{x}{3} = O_{\leq}(2/3)$ for all $x \geq 0$. This is easily verified for $0 \leq x \leq 6$, and the left-hand side is 6-periodic in x, giving the claim; see Figure 9. □

This lets us estimate δ_1:

Lemma 9.2. *We have*

$$\delta_1 \leq \frac{3N}{2tA} + \frac{4}{N}.$$

Proof. By definition, we have

$$\mathcal{E}_t(\mathcal{B}^{(1)}) = A \sum^*_{t<n\leq t(1+\sigma)} \log \frac{n}{t}.$$

By the fundamental theorem of calculus, this is

$$A \int_0^{t\sigma} \sum^*_{t<n\leq t+h} 1 \, \frac{dh}{t+h}.$$

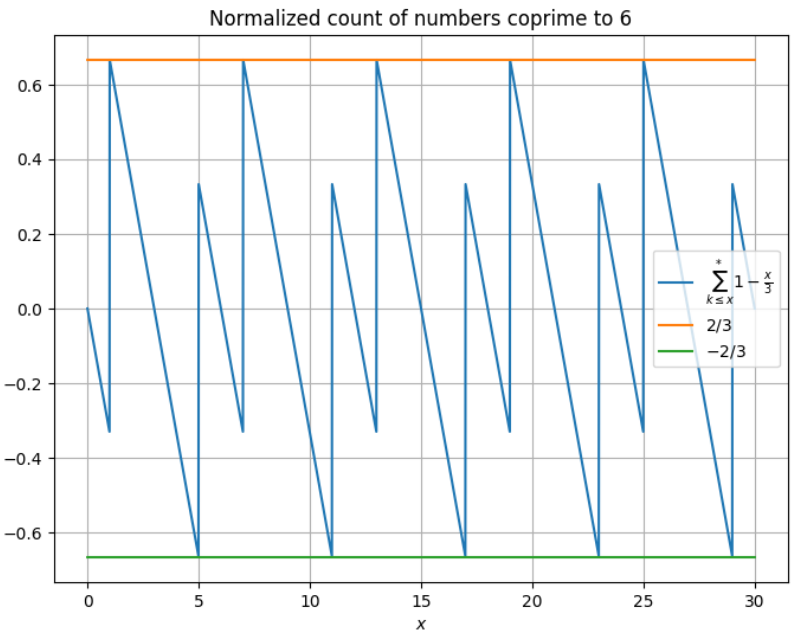

FIGURE 9. The function $\sum^*_{k\le x} 1 - \frac{x}{3}$.

Bounding $\frac{1}{t+h}$ by $\frac{1}{t}$ and applying Lemma 9.1, (9.1), we conclude that

$$\mathcal{E}_t(\mathcal{B}^{(1)}) \le A\int_0^{3N/A}\left(\frac{h}{3}+\frac{4}{3}\right)\frac{dh}{t} = \frac{3N^2}{2tA}+4.$$

and the claim follows. □

To construct $\gamma_2, \gamma_3, \kappa_*, n_2, m_2, n_3, m_3$, we introduce another parameter $L \ge 1$ and assume that

$$t > 3L. \tag{9.3}$$

We define n_2, n_3, m_2, n_3 by setting

$$2^{n_2}3^{m_2} := 2^{n_0}\lceil t/2^{n_0}\rceil^{\langle 2,3\rangle}; \quad 2^{n_3}3^{m_3} := 3^{m_0}\lceil t/3^{m_0}\rceil^{\langle 2,3\rangle}$$

where $2^{n_0}, 3^{m_0}$ are the largest powers of $2, 3$ respectively that are at most t/L. By construction and (2.6), (8.2) holds with

$$\kappa_* = \kappa_L. \tag{9.4}$$

We have

$$2m_2 \le \frac{\log\lceil t/2^{n_0}\rceil^{\langle 2,3\rangle}}{\log\sqrt{3}} \le \frac{\log(2L)+\kappa_L}{\log\sqrt{3}}$$

and

$$n_2 \ge n_0 \ge \frac{\log t - \log(2L)}{\log 2};$$

similarly

$$n_3 \le \frac{\log(3L)+\kappa_L}{\log 2}$$

and

$$2m_3 \geq \frac{\log t - \log(3L)}{\log \sqrt{3}}.$$

We conclude that (8.3) holds with

$$\begin{aligned}\gamma_2 &:= \frac{\log 2}{\log \sqrt{3}} \frac{\log(2L) + \kappa_L}{\log t - \log(2L)} \\ \gamma_3 &:= \frac{\log \sqrt{3}}{\log 2} \frac{\log(3L) + \kappa_L}{\log t - \log(3L)};\end{aligned} \tag{9.5}$$

one can of course also take larger values of γ_2, γ_3 if desired. This lets us compute the quantity κ_{**} defined in (8.22).

To estimate δ_3, α_3 we use

Lemma 9.3. *For every $3 < p \leq t/K$, one has*

$$\nu_p\left(\frac{N!}{\prod \mathcal{B}^{(1)}}\right) = O_{\leq}\left(\frac{4A+3}{3}\left\lceil \frac{\log N}{\log p}\right\rceil\right). \tag{9.6}$$

Proof. One has

$$\begin{aligned}\nu_p(\prod \mathcal{B}^{(1)}) &= A \sum^*_{t<n\leq t(1+\sigma)} \nu_p(n) \\ &= A \sum_{1\leq j\leq \frac{\log N}{\log p}} \sum^*_{t/p^j<n\leq t(1+\sigma)/p^j} 1 \\ &= A \sum_{1\leq j\leq \frac{\log N}{\log p}} \left(\frac{N}{p^j A} + O_{\leq}(4/3)\right) \\ &= \frac{N}{p-1} - O^+_{\leq}\left(\frac{1}{p-1}\right) + O_{\leq}\left(\frac{4A}{3}\left\lceil \frac{\log N}{\log p}\right\rceil\right) \\ &= \frac{N}{p-1} - O^+_{\leq}\left(\left\lceil \frac{\log N}{\log p}\right\rceil\right) + O_{\leq}\left(\frac{4A}{3}\left\lceil \frac{\log N}{\log p}\right\rceil\right).\end{aligned}$$

Meanwhile, from (2.3) one has

$$\nu_p(N!) = \frac{N}{p-1} - O^+_{\leq}\left(\left\lceil \frac{\log N}{\log p}\right\rceil\right)$$

and the claim follows. □

Corollary 9.4. *One has*

$$\delta_3 \leq \frac{(4A+3)\kappa_{4.5}}{3N}\left(\pi\left(\frac{t}{K}\right) + \frac{\log N}{\log 5}\pi\left(\sqrt{N}\right)\right)$$

and

$$\alpha_3 \leq \frac{(4A+3)\left(\log \frac{t}{K} + \kappa_{**}\right)}{3N \log \sqrt{12}}\left(\pi\left(\frac{t}{K}\right) + \frac{\log N}{\log 5}\pi\left(\sqrt{N}\right)\right).$$

Proof. This is immediate from Lemma 9.3 and (8.17), (8.8) after noting that $\lfloor \frac{\log N}{\log p} \rfloor \leq 1 + \frac{\log N}{\log 5} 1_{p \leq \sqrt{N}}$ for $3 < p \leq t/K$. $\square$

The main quantities left to estimate are the quantities $\delta_4, \delta_5, \alpha_4, \alpha_5$ that involve A_{p_1}. By construction of $\mathcal{B}^{(0)}$, we have

$$A_{p_1} = \frac{1}{N} \sum_m^* \nu_{p_1}(m) \sum_{\frac{t}{K}, \frac{t}{m} < p \leq \frac{t(1+\sigma)}{m}} A.$$

In particular, for $p > K(1+\sigma)$ the quantity A_{p_1} vanishes entirely:

$$A_{p_1} = 0. \tag{9.7}$$

For the remaining primes $3 < p \leq K(1+\sigma)$ one has

$$A_{p_1} = \frac{A}{N} \sum_{m \leq K(1+\sigma)}^* \nu_{p_1}(m) \left(\pi\left(\frac{t(1+\sigma)}{m}\right) - \pi\left(\frac{t}{\min(m,K)}\right)\right). \tag{9.8}$$

In practice, these expressions can be adequately controlled by Lemma 2.2, as can the quantities B_{p_1}.

10. The asymptotic regime

With the above estimates, we can now establish the lower bound in Theorem 1.3(iv). Thus we aim to show that $t(N) \geq t$ for sufficiently large N, where

$$t := \frac{N}{e} - \frac{c_0 N}{\log N} + \frac{N}{\log^{1+c_1} N} \asymp N \tag{10.1}$$

and $0 < c_1 < 1$ is a small absolute constant. We use the construction of the previous section with the parameters

$$A := \lfloor \log^2 N \rfloor \tag{10.2}$$

$$K := \lfloor \log^3 N \rfloor \tag{10.3}$$

$$L := N^{0.1}, \tag{10.4}$$

so from (9.1) one has

$$\sigma = \frac{3N}{tA} \asymp \frac{1}{A} \asymp \frac{1}{\log^2 N}. \tag{10.5}$$

The conditions (8.1), (9.3) are easily verified for N large enough.

By (9.4), (10.4), and Lemma 2.1(ii) we have

$$\kappa_* \ll \log^{-c} N$$

for some absolute constant $c > 0$. From (9.5), (10.1), (10.4) we have

$$\gamma_2 = \frac{1}{10}\frac{\log 2}{\log \sqrt{3}} + O\left(\frac{1}{\log N}\right), \quad \gamma_3 = \frac{1}{10}\frac{\log \sqrt{3}}{\log 2} + O\left(\frac{1}{\log N}\right)$$

and hence by (8.22), (2.10), (2.11) we have for sufficienty large N that

$$\kappa_{**} \ll 1.$$

By Proposition 8.1, it thus suffices to establish the inequalities (8.4), (8.5). Several of the quantities $\delta, \delta_i, \alpha_i$ can now be immediately estimated using (9.2), (9.2), Corollary 9.4, (2.4), and the prime number theorem:

$$\begin{aligned}
\delta_1 &\ll \frac{1}{A} \asymp \frac{1}{\log^2 N} \\
\delta_3 &\ll \frac{A}{K \log N} \asymp \frac{1}{\log^2 N} \\
\delta_6 &\ll \frac{1}{N} \\
\delta_7 &\ll \frac{\kappa_*}{\log N} \ll \frac{1}{\log^{1+c} N} \\
\delta_8 &\ll \frac{\log N}{N} \\
\delta &= \frac{ec_0}{\log N} + \frac{e}{\log^{1+c_1} N} + O\left(\frac{1}{\log^2 N}\right)
\end{aligned}$$

$$\begin{aligned}
\alpha_1 &= 0 \\
\alpha_3 &\ll \frac{A}{K} \asymp \frac{1}{\log N} \\
\alpha_6, \alpha_7 &\ll \frac{\log N}{N}
\end{aligned}$$

On the interval $(t/NK, 1]$, the function $f_{N/t}$ is piecewise monotone with $O(K)$ pieces, and bounded by 1, so its augmented total variation norm is $O(K)$. Applying (8.7) and Lemma 2.2 (with classical error term), we have

$$\begin{aligned}
\delta_2 &\leq \frac{1}{\log(t/K)} \int_{t/NK}^{1} f_{N/t}(x)\, dx + O\left(\frac{1}{\log^2 N}\right) \\
&\leq \frac{1}{\log N} \int_{1/eK}^{N/et} f_{N/t}(etx/N)\, dx + O\left(\frac{1}{\log^2 N}\right)
\end{aligned}$$

where we have used (1.9) to manage error terms. Similarly to the proof of Proposition 5.2, the function $f_{N/t}(etx/N)$ differs from $f_e(x)$ outside of an exceptional set of measure $O(1/\log N)$, and hence by (1.6) (and (1.9)) we have

$$\delta_2 \leq \frac{ec_0}{\log N} + O\left(\frac{1}{\log^2 N}\right).$$

To finish the verification of the conditions (8.4), (8.5), it will suffice to show that

$$\delta_4, \delta_5 \ll \frac{(\log\log N)^{O(1)}}{\log^2 N} \tag{10.6}$$

and

$$\alpha_2, \alpha_4, \alpha_5 \ll \frac{(\log\log N)^{O(1)}}{\log N}. \tag{10.7}$$

By Mertens' theorem (or Lemma 2.2) and (8.9), (8.10), (8.16), (8.18), (8.19), (10.5), it suffices to show that

$$A_{p_1}, B_{p_1} \ll \frac{(\log\log N)^{O(1)}}{p_1 \log N} \tag{10.8}$$

for all $p_1 \leq K(1+\sigma)$ (recalling from (9.7) that A_{p_1} vanishes for any larger p_1), as well as the variant

$$|A_{p_1} - B_{p_1}|_{0,1} \ll \frac{(\log\log N)^{O(1)}}{p_1 \log^2 N} \tag{10.9}$$

for $3 < p_1 \leq K$.

For (10.8) we use (9.8), (8.24), and the crude bound

$$\nu_{p_1}(m) \ll 1_{p_1|m} \log\log N \tag{10.10}$$

for $m \leq K(1+\sigma)$, and reduce to showing that

$$\frac{A}{N} \sum_{m \leq K(1+\sigma)} 1_{p_1|m} \left(\pi\left(\frac{t(1+\sigma)}{m}\right) - \pi\left(\frac{t}{\min(m,K)}\right)\right). \ll \frac{(\log\log N)^{O(1)}}{p_1 \log N}$$

and

$$\frac{1}{N} \sum_{m \leq K} 1_{p_1|m} \sum_{\frac{t}{m} \leq p < \frac{t}{m-1}} \left\lfloor \frac{N}{p} \right\rfloor \ll \frac{(\log\log N)^{O(1)}}{p_1 \log N}.$$

But from the Brun–Titchmarsh inequality (or Lemma 2.2) and (10.5) one has

$$\pi\left(\frac{t(1+\sigma)}{m}\right) - \pi\left(\frac{t}{\min(m,K)}\right) \ll \frac{t\sigma}{m \log N} \ll \frac{N}{Am \log N}$$

and

$$\sum_{\frac{t}{m} \leq p < \frac{t}{m-1}} \left\lfloor \frac{N}{p} \right\rfloor \ll \frac{tm}{m^2 \log N} \ll \frac{N}{m \log N}$$

and the claim then follows from summing the harmonic series.

It remains to show (10.9). For $3 < p_1 \leq K$, we see from (9.8), (10.5), (10.10) and Lemma 2.2 (with classical error term) that

$$\begin{aligned} A_{p_1} &\geq \frac{1}{N} \sideset{}{^*}\sum_{m \leq K(1+\sigma)} \nu_{p_1}(m) \left(\frac{At\sigma}{m \log N} + O\left(\frac{(\log\log N)^{O(1)} At\sigma}{m \log^2 N} \right)\right) \\ &= \frac{1}{\log N} \sideset{}{^*}\sum_{m \leq K(1+\sigma)} \nu_{p_1}(m) \frac{3}{m} + O\left(\frac{(\log\log N)^{O(1)}}{\log^2 N} \right) \\ &= \frac{1}{\log N} \sideset{}{^*}\sum_{m \leq K} \nu_{p_1}(m) \frac{3}{m} + O\left(\frac{(\log\log N)^{O(1)}}{\log^2 N} \right) \end{aligned}$$

and similarly from (8.24), (10.10), and Lemma 2.2 (again with classical error term)

$$\begin{aligned} B_{p_1} &\leq \frac{1}{N}\sum_{m\leq K}\nu_{p_1}(m)\sum_{\frac{t}{m}\leq p<\frac{t}{m-1}}\frac{N}{p} \\ &\leq \frac{1}{N}\sum_{m\leq K}\nu_{p_1}(m)\left(\frac{N}{\log(t/m)}\int_{t/m}^{t/(m-1)}\frac{dx}{x}+O\left(\frac{N}{\log^{10}N}\right)\right) \\ &\leq \frac{1}{\log N}\sum_{m\leq K}\nu_{p_1}(m)\log\frac{m}{m-1}+O\left(\frac{(\log\log N)^{O(1)}}{\log^2 N}\right) \end{aligned}$$

so it will suffice to establish the inequality

$$\sum_{m\leq K}\nu_{p_1}(m)\log\frac{m}{m-1}\leq \sum_{m\leq K}^{*}\nu_{p_1}(m)\frac{3}{m} \tag{10.11}$$

for all $p_1 > 3$.

Writing $\nu_{p_1}(m) = \sum_{j\geq 1} 1_{p_1^j|m}$, it suffices to show that

$$\sum_{m\leq K; p^j|m}\frac{3}{m}1_{(m,6)=1}-\log\frac{m}{m-1}\geq 0.$$

Making the change of variables $m = p_1^j n$, it suffices to show that

$$\sum_{n\leq K'}\frac{3}{n}1_{(n,6)=1}-p_1^j\log\frac{p_1^j n}{p_1^j n-1}\geq 0$$

for any $K' > 0$. Using the bound

$$\log\frac{p_1^j n}{p_1^j n-1}=\int_{p_1^j n-1}^{p_1^j n}\frac{dx}{x}\leq\frac{1}{p_1^j n-1}$$

and $p^j \geq 5$, we have

$$p_1^j\log\frac{p_1^j n}{p_1^j n-1}\leq\frac{1}{n-0.2}$$

and so it suffices to show that

$$\sum_{n\leq K'}^{*}\frac{3}{n}1_{(n,6)=1}-\frac{1}{n-0.2}\geq 0. \tag{10.12}$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{n-0.2}-\frac{1}{n}=\psi(0.8)-\psi(1)=0.353473\dots,$$

where ψ here denotes the digamma function rather than the von Mangoldt summatory function, it will suffice to show that

$$\sum_{n\leq K'}\frac{3}{n}1_{(n,6)=1}-\frac{1}{n}\geq 0.4. \tag{10.13}$$

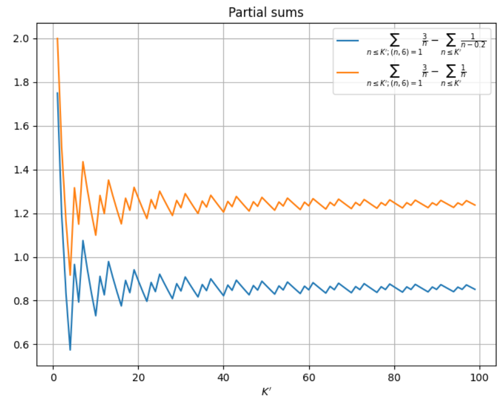

FIGURE 10. A plot of (10.12), (10.13).

This can be numerically verified for $K' \leq 100$, with substantial room to spare for K' large; see Figure 10. On a block $6a - 1 \leq n \leq 6a + 4$ with $a > 1$, the sum is positive:

$$\begin{aligned} \sideset{}{^*}\sum_{6a-1 \leq n \leq 6a+4} \frac{3}{n} - \frac{1}{n} &= \left(\frac{1}{6a-1} - \frac{1}{6a}\right) + \left(\frac{1}{6a-1} - \frac{1}{6a+2}\right) \\ &\quad + \left(\frac{1}{6a+1} - \frac{1}{6a+3}\right) + \left(\frac{1}{6a+1} - \frac{1}{6a+4}\right) \\ &\quad > 0. \end{aligned}$$

The inequality for $K' > 100$ is then easily verified from the $K' \leq 100$ data and the triangle inequality.

11. GUY–SELFRIDGE CONJECTURE

We now establish the Guy–Selfridge conjecture $t(N) \geq N/3$ in the range

$$N \geq N_0 := 10^{11}.$$

We will apply Proposition 8.1 with the construction in Section 9 and the choice of parameters

$$\begin{aligned} t &:= N/3 \\ A &:= 189 \\ K &:= 293 \\ L &:= 4.5; \end{aligned}$$

the choice of A and K was obtained after some numerical experimentation. In particular, by (9.1) we have

$$\sigma = \frac{3N}{At} = \frac{9}{189} = 0.047619\dots$$

One can readily check the required conditions (8.1), (9.3) for $N \geq N_0$, so it remains to verify the hypotheses (8.4), (8.5) of Proposition 8.1 in this range. Some of the quantities in these hypotheses involve sums over large ranges, such as $(t/K, N]$; but one can use Lemma 2.2 to obtain adequate upper or lower bounds on such quantities, leaving one with sums over short ranges such as $p \leq K$ or $p \leq K(1+\sigma)$. As such, all of the bounds needed can be quickly computed even for very large N with simple computer code[5].

Many of the bounds we will use will be monotone decreasing in N, so that they only need to be tested at the left endpoint $N = N_0$. However, this is not the case for all of the bounds, as some involve subtracting one monotone quantity from another. For those estimates, we will initially only establish bounds in two extreme cases, $N = N_0$ and $N \geq 10^{70}$, and discuss how to cover the intervening ranges $N_0 < N < 10^{70}$ at the end of the section.

We now bound some of the terms appearing in Proposition 8.1. From Lemma 2.1 we have

$$\kappa_{4.5} = \log\frac{4}{3} = 0.28768\dots.$$

From (9.5) one can take

$$\gamma_2 := \frac{\log 2}{\log\sqrt{3}}\frac{\log(2L)+\kappa_L}{\log(N_0/3)-\log(2L)} = 0.1423165\dots$$

and

$$\gamma_3 := \frac{\log\sqrt{3}}{\log 2}\frac{\log(3L)+\kappa_L}{\log(N_0/3)-\log(3L)} = 0.1059116\dots$$

for all $N \geq N_0$; by (8.22) and some calculation we then have

$$\kappa_{**} \leq 6.830101\dots.$$

From (2.4) one has

$$\delta \geq \log N - \log t = \log\frac{3}{e} = 0.0986122\dots$$

for all $N \geq N_0$.

We will use this lower bound as our unit of reference for all other δ_i quantities, bounding them by suitable multiples of δ. For instance, from (9.2) one has

$$\delta_1 \leq \frac{9}{2A} + \frac{4}{N_0} \leq 0.023810\delta$$

for all $N \geq N_0$.

From (8.7) and Lemma 2.2, and the monotonicity of $E(N)/N$, one has

$$\begin{aligned}
\delta_2 &\leq \frac{\int_{1/3K}^1 f_3(x)\,dx}{\log(t/K)} + \frac{\|f_3\|_{\mathrm{TV}((1/3K,1])}}{\log(t/K)}\frac{E(N)}{N} \\
&\leq \frac{0.919785}{\log(N_0/3K)} + \frac{1159.795}{\log(N_0/3K)}\frac{E(N_0)}{N_0} \\
&\leq 0.504735\delta
\end{aligned}$$

[5] `https://github.com/teorth/erdos-guy-selfridge/blob/main/src/python/interval_computations.py`

for all[6] $N \geq N_0$. For $N \geq 10^{70}$ we may replace N_0 by 10^{70} and obtain the significantly better bound

$$\delta_2 \leq 0.060410\delta.$$

From Corollary 9.4 and (2.13) one has

$$\begin{aligned}
\delta_3 &\leq \frac{(4A+3)\kappa_{4.5}}{3}\left(\frac{1}{3K\log\frac{N}{3K}} + \frac{1.2762}{3K\log^2\frac{N}{3K}} + \frac{\log N}{\sqrt{N}\log\sqrt{N}\log 5} + \frac{1.2762\log N}{\sqrt{N}\log^2\sqrt{N}\log 5}\right)\\
&\leq \frac{(4A+3)\kappa_{4.5}}{3}\left(\frac{1}{3K\log\frac{N_0}{3K}} + \frac{1.2762}{3K\log^2\frac{N_0}{3K}} + \frac{\log N_0}{\sqrt{N_0}\log\sqrt{N_0}\log 5} + \frac{1.2762\log N_0}{\sqrt{N_0}\log^2\sqrt{N_0}\log 5}\right)\\
&\leq 0.051574\delta
\end{aligned}$$

for all $N \geq N_0$.

We skip $\delta_4, \delta_5, \delta_7$ for now. From (8.11) we have

$$\delta_6 \leq \frac{\kappa_{4.5}}{N_0} \leq 3 \times 10^{-11}\delta$$

for all $N \geq N_0$, and from (8.13) we have

$$\delta_8 \leq \frac{2(\log(N_0/3) + \kappa_{4.5})}{N_0} \leq 6 \times 10^{-10}\delta$$

for all $N \geq N_0$, so these two terms are negligible in the analysis.

From (9.2) we have

$$\alpha_1 = 0.$$

We skip $\alpha_2, \alpha_4, \alpha_5$ for now. From Corollary 9.4 and (2.13) one has

$$\begin{aligned}
\alpha_3 \leq{}& \frac{4A+3}{3\log\sqrt{12}}\left(\log\frac{N}{3K} + \kappa_{**}\right)\\
&\times\left(\frac{1}{3K\log\frac{N}{3K}} + \frac{1.2762}{3K\log^2\frac{N}{3K}} + \frac{\log N}{\sqrt{N}\log\sqrt{N}\log 5} + \frac{1.2762\log N}{\sqrt{N}\log^2\sqrt{N}\log 5}\right).
\end{aligned}$$

Expanding out the product, one can check that all terms are non-increasing in N; so we may substitute N_0 for N in the right-hand side, which after some calculation gives

$$\alpha_3 \leq 0.361121$$

for all $N \geq N_0$. From (8.20) we have

$$\begin{aligned}
\alpha_6 &\leq \frac{\log(N_0/3) + \kappa_{**}}{N_0\log\sqrt{12}}\\
&\leq 3 \times 10^{-10}
\end{aligned}$$

[6]Despite the seemingly large numerator, the second term is in fact negligible in the regime $N \geq N_0$, due to the square root type decay in $E(N)/N$.

for all $N \geq N_0$, and similarly from (8.21) we have

$$\alpha_7 \leq \max\left(\frac{\log(2N_0)}{(1-\gamma_2)N_0\log 2}, \frac{\log(3N_0)}{(1-\gamma_3)N_0\log\sqrt{3}}\right)$$
$$\leq 6\times 10^{-10}$$

for all $N \geq N_0$. so the contribution of these two terms are negligible.

Conveniently[7], the choice of parameters A, K ensure that there are no primes in the range

$$293 = K < p \leq K(1+\sigma) = K(1+\sigma) = 306.952\ldots$$

and thus

$$\delta_4 = \alpha_4 = 0$$

for all $N \geq N_0$.

The remaining terms $\delta_5, \delta_7, \alpha_2, \alpha_5$ to estimate involve the quantities A_{p_1}, B_{p_1} defined in (8.23), (8.24), and require a bit more care. For B_{p_1}, we can split the expression as

$$B_{p_1} = \sum_{m\leq K} \nu_{p_1}(m) \sum_{k:a_{k,m}<b_{k,m}} \frac{k}{N}(\pi(Nb_{k,m}) - \pi(Na_{k,m}))$$

where

$$a_{k,m} := \max\left(\frac{1}{3m}-, \frac{1}{k}\right); \quad b_{k,m} := \max\left(\frac{1}{3(m-1)}-, \frac{1}{k-1}\right)$$

where the $-$ denotes the subtraction of an infinitesimal quantity to reflect the restriction to the range $\frac{t}{m} \leq p < \frac{t}{m-1}$ rather than $\frac{t}{m} < p \leq \frac{t}{m-1}$. Using Lemma 2.2 (and a limiting argument), we can upper bound this quantity by

$$B_{p_1} \leq \sum_{m\leq K} \nu_{p_1}(m) \sum_{k:a_{k,m}<b_{k,m}} \frac{k}{\log(Na_{k,m})}\left(a_{k,m} - b_{k,m} + 2\frac{E(N_0 b_{k,m})}{N_0}\right)$$

and lower bound it by

$$B_{p_1} \geq \sum_{m\leq K} \nu_{p_1}(m) \sum_{k:a_{k,m}<b_{k,m}} \frac{k}{\log(Nb_{k,m})}\left(\left(1 - \frac{2}{\sqrt{a_{k,m}N}}\right)(a_{k,m} - b_{k,m}) - 2\frac{E(N_0 b_{k,m})}{N_0 b_{k,m}}\right).$$

We caution here that while the upper bound for B_{p_1} is monotone decreasing in N, the lower bound does not have a favorable monotonicity property, particularly as it will be used when *subtracting* copies of B_{p_1} rather than *adding* then.

From the monotonicity of the upper bound, one can use (8.16) to calculate that

$$\alpha_2 \leq 0.270233$$

for all $N \geq N_0$. For (8.12), subtraction is involved, and one must proceed with more caution. For $N = N_0$, one has

$$\delta_7 \leq 0.113636.$$

[7]Even if this were not the case, the quantities δ_4, α_4 should be viewed as lower order terms, and are far smaller than several of the other δ_i or α_i for typical choices of parameters.

For $N \geq 10^{70}$, we simply discard the negative terms here and obtain the bound

$$\delta_7 \leq \frac{\kappa_{4.5} \log \sqrt{12}}{\log(10^{70}/3)} \leq 0.02265\delta$$

As for the A_{p_1}, we know from (9.7) that this vanishes unless $3 < p_1 \leq K(1+\sigma)$. From (9.8) and Lemma 2.2 one has the upper bound

$$A_{p_1} \leq \sum^{*}_{m \leq K(1+\sigma)} \frac{A\nu_{p_1}(m)}{\log(N/3\min(m,K))} \left(\frac{1+\sigma}{3m} - \frac{1}{3\min(m,K)} + \frac{2E(N_0(1+\sigma)/3m)}{N_0} \right)$$

and the lower bound

$$A_{p_1} \geq \sum^{*}_{m \leq K(1+\sigma)} \frac{A\nu_{p_1}(m)}{\log(N(1+\sigma)/3m)} \times \left(\left(1 - \frac{2}{\sqrt{N_0(1+3\sigma)/3m}}\right) \left(\frac{1+\sigma}{3m} - \frac{1}{3\min(m,K)} \right) - \frac{2E(N_0(1+\sigma)/3m)}{N_0} \right).$$

Again, the upper bound is monotone decreasing in N, but the lower bound does not have a favorable monotonicity. At $N = N_0$, one can calculate using these bounds and (8.10), (8.19) to obtain

$$\delta_5 \leq 0.062824\delta$$
$$\alpha_5 \leq 0.315286$$

which, when combined with the previous bounds, gives

$$\sum_{i=1}^{8} \delta_i \leq 0.9742\delta$$

and

$$\sum_{i=1}^{7} \alpha_i \leq 0.9467$$

at $N = N_0$, thus verifying (8.4), (8.5) in those cases.

For $N \geq 10^{70}$, we use the triangle inequality to crudely upper bound

$$\delta_5 \leq \kappa_{4.5} \sum_{3 < p_1 \leq K} \frac{\log p_1}{\log(t/K^2)} A_{p_1} + B_{p_1}$$

and

$$\alpha_5 \leq \frac{1}{\log \sqrt{12}} \sum_{3 < p_1 \leq K} \frac{(\log K^2 + \kappa_{**}) \log p_1}{\log(t/K^2)} A_{p_1} + (\log p_1 + \kappa_{**}) B_{p_1}.$$

The bounds available for the right-hand side are now monotone in N, and one can calculate that

$$\delta_5 \leq 0.077304\delta$$
$$\alpha_5 \leq 0.184984$$

for $N \geq 10^{70}$. This is better than the previous bound for α_5. For δ_5, the bound is slightly worse, but this is more than compensated for by the improved bounds on δ_2, δ_7, and (8.4), (8.5) can be verified here with significant room to spare.

This completes the proof of Theorem 1.3(iii) (and hence Theorem 1.3(ii)) in the cases $N = N_0$ and $N \geq 10^{70}$. It remains to cover the intermediate range $N_0 < N \leq 10^{70}$. Here we adopt the perspective of interval arithmetic. If N is constrained to a given interval, such as $[10^{11}, 5 \times 10^{11}]$, we can use the worst-case upper and lower bounds for A_{p_1}, B_{p_1} to obtain conservative upper bounds on the most delicate quantities $\delta_5, \delta_7, \alpha_2, \alpha_5$, thus potentially verifying the conditions (8.4), (8.5) simultaneously for all N in such an interval. As it turns out, there is enough room to spare in theese estimates, particularly for large N, that this strategy works using only a small number of intervals; specifically, by considering N in the intervals

$$[10^{11}, 5\times10^{11}]; \quad [5\times10^{11}, 10^{14}]; \quad [10^14, 10^{20}]; \quad [10^{20}, 10^{70}]$$

one can check that such bounds are sufficient to verify (8.4), (8.5) in these cases. This now verifies Theorem 1.3(ii), (iii) for all $N \geq 10^{11}$. (In fact, with more effort, this verification can be pushed down to $N \geq 6 \times 10^{10}$ using the same choice of parameters A, K, L.)

Appendix A. Distance to the next 3-smooth number

We now establish the various claims in Lemma 2.1. We begin with part (iii). The claim (2.7) is immediate from (2.6), (2.5). Now prove (2.8), (2.9). If we write $\lceil x/12^a \rceil^{\langle 2,3\rangle} = 2^b 3^c$, then by (2.6) we have

$$b \log 2 + c \log 3 \leq \log x - a \log 12 + \kappa_L,$$

while from definition of a we have

$$\log x - a \log 12 \leq \log(12L). \tag{A.1}$$

We now compute

$$\begin{aligned}
\frac{\nu_2(\lceil x \rceil_L^{\langle 2,3\rangle}) - 2\gamma \nu_3(\lceil x \rceil_L^{\langle 2,3\rangle})}{1-\gamma} &= \frac{2a + b - 2\gamma(a+c)}{1-\gamma} \\
&\leq 2a + \frac{\log x - a\log 12 + \kappa_L}{(1-\gamma)\log 2} \\
&= \frac{\log x}{\log\sqrt{12}} + \left(\frac{1}{(1-\gamma)\log 2} - \frac{1}{\log\sqrt{12}}\right)(\log x - a \log 12) \\
&\quad + \frac{\kappa_L}{(1-\gamma)\log 2}
\end{aligned}$$

giving (2.8) from (A.1); similarly, we have

$$\begin{aligned}\frac{2\nu_3(\lceil x\rceil_L^{\langle 2,3\rangle}) - \gamma\nu_2(\lceil x\rceil_L^{\langle 2,3\rangle})}{1-\gamma} &= \frac{2(a+c)-\gamma(2a+b)}{1-\gamma}\\ &\leq 2a + \frac{2(\log x - a\log 12 + \kappa_L)}{(1-\gamma)\log 3}\\ &= \frac{\log x}{\log\sqrt{12}} + \left(\frac{2}{(1-\gamma)\log 3} - \frac{1}{\log\sqrt{12}}\right)(\log x - a\log 12)\\ &\quad + \frac{\kappa_L}{(1-\gamma)\log\sqrt{3}}\end{aligned}$$

giving (2.9) from (A.1).

To prove parts (i) and (ii) of Lemma 2.1, we establish the following lemma to upper bound κ_L.

Lemma A.1. *If n_1, n_2, m_1, m_2 are natural numbers such that $n_1+n_2, m_1+m_2 \geq 1$ and*

$$\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}} \geq 1$$

then

$$\kappa_{\min(2^{n_1+n_2}, 3^{m_1+m_2})/6} \leq \log\max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right).$$

Proof. If $\min(2^{n_1+n_2}, 3^{m_1+m_2})/6 \leq t \leq 2^{n_2-1}3^{m_1-1}$, then we have

$$t \leq 2^{n_2-1}3^{m_1-1} \leq \max\left(\frac{3^{m_1}}{2^{n_1}}, \frac{2^{n_2}}{3^{m_2}}\right)t, \tag{A.2}$$

so we are done in this case. Now suppose that $t > 2^{n_2-1}3^{m_1-1}$. If we write $\lceil t\rceil^{\langle 2,3\rangle} = 2^n3^m$ be the smallest 3-smooth number that is at least t, then we must have $n \geq n_2$ or $m \geq m_1$ (or both). Thus at least one of $\frac{2^{n_1}}{3^{m_1}}2^n3^m$ and $\frac{3^{m_2}}{3^{n_2}}2^n3^m$ is an integer, and is thus at most t by construction. This gives (A.2), and the claim follows. □

Some efficient choices of parameters for this lemma are given in Table 1. For instance, $\kappa_{4.5} \leq \log\frac{4}{3} = 0.28768\ldots$ and $\kappa_{40.5} \leq \log\frac{32}{27} = 0.16989\ldots$. In fact, since $\lceil 4.5+\varepsilon\rceil^{\langle 2,3\rangle} = 6$ and $\lceil 40.5+\varepsilon\rceil^{\langle 2,3\rangle} = 48$ for all sufficiently small $\varepsilon > 0$, we see that these bounds are sharp (And similarly for the other entries in Table 1); this establishes part (i).

Remark A.2. It should be unsurprising that the continued fraction convergents $1/1, 2/1, 3/2, 8/5, 19/12, \ldots$ to

$$\frac{\log 3}{\log 2} = 1.5849\cdots = [1; 1, 1, 2, 2, 3, 1, \ldots]$$

are often excellent choices for n_1/m_1 or n_2/m_2, although other approximants such as $5/3$ or $11/7$ are also usable.

n_1	m_1	n_2	m_2	$\min(2^{n_1+n_2}, 3^{m_1+m_2})/6$	$\log\max(3^{m_1}/2^{n_1}, 2^{n_2}/3^{m_2})$
1	1	**1**	**0**	$1/2 = 0.5$	$\log 2 = 0.69314\dots$
1	**1**	2	1	$2^2/3 = 1.33\dots$	$\log(3/2) = 0.40546\dots$
3	2	**2**	**1**	$3^2/2 = 4.5$	$\log(2^2/3) = 0.28768\dots$
3	2	**5**	**3**	$3^4/2 = 40.5$	$\log(2^5/3^3) = 0.16989\dots$
3	**2**	8	5	$2^{10}/3 = 341.33\dots$	$\log(3^2/2^3) = 0.11778\dots$
11	**7**	8	5	$2^{18}/3 = 87381.33\dots$	$\log(3^7/2^{11}) = 0.06566\dots$
19	12	**8**	**5**	$3^{17}/2 \approx 6.4 \times 10^7$	$\log(2^8/3^5) = 0.05211\dots$
19	12	**27**	**17**	$3^{29}/2 \approx 3.4 \times 10^{13}$	$\log(2^{27}/3^{17}) = 0.03856\dots$
19	12	**46**	**29**	$3^{41}/2 \approx 1.8 \times 10^{19}$	$\log(2^{46}/3^{29}) = 0.02501\dots$

TABLE 1. Efficient parameter choices for Lemma A.1. The parameters used to attain the minimum or maximum are indicated in **boldface**. Note how the number of rows in each group matches the terms $1, 1, 2, 2, 3, \dots$ in the continued fraction expansion.

Finally, we establish (ii). From the classical theory of continued fractions, we can find rational approximants

$$\frac{p_{2j}}{q_{2j}} \le \frac{\log 3}{\log 2} \le \frac{p_{2j+1}}{q_{2j+1}} \tag{A.3}$$

to the irrational number $\log 3/\log 2$, where the convergents p_j/q_j obey the recursions

$$p_j = b_j p_{j-1} + p_{j-2}; \quad q_j = b_j q_{j-1} + q_{j-2}$$

with $p_{-1} = 1, q = -1 = 0, p_0 = b_0, q_0 = 1$, and

$$[b_0; b_1, b_2, \dots] = [1; 1, 1, 2, 2, 3, 1\dots]$$

is the continued fraction expansion of $\frac{\log 3}{\log 2}$. Furthermore, $p_{2j+1}q_{2j} - p_{2j}q_{2j+1} = 1$, and hence

$$\frac{\log 3}{\log 2} - \frac{p_{2j}}{q_{2j}} = \frac{1}{q_{2j}q_{2j+1}}. \tag{A.4}$$

By Baker's theorem (see, e.g., [3]), $\frac{\log 3}{\log 2}$ is a Diophantine number, giving a bound of the form

$$q_{2j+1} \ll q_{2j}^{O(1)} \tag{A.5}$$

and a similar argument (using $p_{2j+2}q_{2j+1} - p_{2j+1}q_{2j+2} = -1$) gives

$$q_{2j+2} \ll q_{2j+1}^{O(1)}. \tag{A.6}$$

We can rewrite (A.3) as

$$\frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}} \ge 1$$

and routine Taylor expansion using (A.4) gives the upper bounds

$$\frac{3^{q_{2j}}}{2^{p_{2j}}}, \frac{2^{p_{2j+1}}}{3^{q_{2j+1}}} \le \exp\left(O\left(\frac{1}{q_{2j}}\right)\right).$$

From Lemma A.1 we obtain

$$\kappa_{\min(2^{p_{2j}+p_{2j+1}}, 3^{q_{2j}+q_{2j+1}})/6} \ll \frac{1}{q_{2j}}.$$

The claim then follows from (A.5), (A.6) (and the obvious fact that κ is monotone non-increasing after optimizing in j.

Remark A.3. It seems reasonable to conjecture that c can be taken to be arbitrarily close to 1, but this is essentially equivalent to the open problem of determining that irrationality measure of $\log 3/\log 2$ is equal to 2.

Appendix B. Estimating sums over primes

In this appendix we establish Lemma 2.2. The key tool is

Lemma B.1 (Integration by parts)**.** *Let $(y, x]$ be a half-open interval in $(0, +\infty)$. Suppose that one has a function $a \colon \mathbb{N} \to \mathbb{R}$ and a continuous function $f : (y, x] \to \mathbb{R}$ such that*

$$\sum_{y<n\leq z} a_n = \int_z^y f(t)\, dt + C + O_{\leq}(A)$$

for all $y \leq z \leq x$, and some $C \in \mathbb{R}$, $A > 0$. Then, for any function $b : (y, x] \to \mathbb{R}$ of bounded total variation, one has

$$\sum_{y<n\leq x} b(n) a_n = \int_x^y b(t) f(t)\, dt + O_{\leq}(A\|b\|_{\mathrm{TV}^*(y,x]}). \tag{B.1}$$

Proof. If, for every natural number $y < n \leq x$, one modifies b to be equal to the constant $b(n)$ in a small neighborhood of n, then one does not affect the left-hand side of (B.1) or increase the total variation of b, while only modifying the integral in (B.1) by an arbitrarily small amount. Hence, by the usual limiting argument, we may assume without loss of generality that b is locally constant at each such n. If we define the function $g \colon (y, x] \to \mathbb{R}$ by

$$g(z) := \sum_{y<n\leq z} a_n - \int_z^y f(u)\, du - C$$

then g has jump discontinuities at the natural numbers, but is otherwise continuously differentiable, and is also bounded uniformly in magnitude by A. We can then compute the Riemann–Stieltjes integral

$$\int_{(y,x]} b\, dg = \sum_{y<n\leq x} b(n) a_n - \int_y^x f(t) b(t)\, dt.$$

Since the discontinuities of g and b do not coincide, we may integrate by parts to obtain

$$\int_{(y,x]} b\, dg = b(x) g(x) - b(y^+) g(y^+) - \int_{(y,x]} g db.$$

The left-hand side is $O_{\leq}(A\|b\|_{\mathrm{TV}^*(y,x]})$, and the claim follows. ☐

We now prove (2.14). In fact we prove the sharper estimate

$$\sum_{y<p\leq x} b(p) \log p = \int_y^x b(t) \left(1 - \frac{2}{\sqrt{t}} \right) dt + O_{\leq} \left(\|b\|_{\mathrm{TV}^*((y,x])} \tilde{E}(x) \right) \tag{B.2}$$

where

$$\tilde{E}(x) := 0.95\sqrt{x} + \min(\max(\varepsilon_0, \varepsilon_1(x)), \varepsilon_2(x), \varepsilon_3(x))1_{x\geq 10^{19}} \tag{B.3}$$

and

$$\begin{aligned}
\varepsilon_0(x) &:= \frac{\sqrt{x}}{8\pi}\log x(\log x - 3)\\
\varepsilon_1(x) &:= 1.12494\times 10^{-10}\\
\varepsilon_2(x) &:= 9.39(\log^{1.515} x)\exp(-0.8274\sqrt{\log x})\\
\varepsilon_3(x) &:= 0.026(\log^{1.801} x)\exp(-0.1853(\log^{3/5} x)(\log\log x)^{-1/5})
\end{aligned}$$

From using the ε_2 term, it is clear that

$$\tilde{E}(x) \ll x\exp(-c\sqrt{\log x})$$

for some absolute constant $c > 0$; and by using the $\varepsilon_0, \varepsilon_1$ term and routine calculations one can show that

$$\tilde{E}(x) \leq E(x)$$

for all $x \geq 1423$.

Observe that $\tilde{E}$ is monotone non-decreasing. Thus by Lemma B.1, to show (B.2) will suffice to show that

$$\sum_{p\leq x}\log p = x - \sqrt{x} + O_{\leq}(\tilde{E}(x)) = \int_0^x \left(1 - \frac{2}{\sqrt{t}}\right)dt + O_{\leq}(\tilde{E}(x))$$

for all $x \geq 1423$.

For $1423 \leq x \leq 10^{19}$, this claim follows from [5, Theorem 2]. For $x > 10^{19}$, we apply [4, (6.10), (6.11)] to conclude that

$$\sum_{p\leq x}\log p = \psi(x) - \psi(\sqrt{x}) + O_{\leq}(1.03883(x^{1/3} + x^{1/5} + 2(\log x)x^{1/13})),$$

where $\psi(x) := \sum_{n\leq x}\Lambda(n)$ is the usual von Mangoldt summatory function. From [15, Theorems 10,12] we have

$$\psi(\sqrt{x}) = \sqrt{x} + O_{\leq}(0.18\sqrt{x}).$$

Since

$$0.18\sqrt{x} + 1.03883(x^{1/3} + x^{1/5} + 2(\log x)x^{1/13}) \leq 0.95\sqrt{x}$$

in this range of x, it suffices to show that

$$\psi(x) = x + O_{\leq}(\min(\max(\varepsilon_0(x), \varepsilon_1(x)), \varepsilon_2(x), \varepsilon_3(x)))$$

for $x > 10^{19}$. The claims for $i = 2, 3$ follow from [12, Theorems 1.1, 1.4]. In [4, Theorem 2, (7.3)], the bound

$$\psi(x) = x + O_{\leq}(\varepsilon_0(x))$$

is established whenever $x \geq 5000$ and $4.92\frac{x}{\sqrt{\log x}} \leq T$, where T is a height up to which the Riemann hypothesis has been established. Using the value $T = 3\times 10^{12}$ from [13], we can therefore cover the range $10^{19} < x < e^{55}$ (in fact we could go up to $e^{58.33} \approx 2.1\times 10^{25}$). For $x \geq e^{55}$, we can use [4, Table 2] (the value $T = 2.445\times 10^{12}$ used there following from [13]).

Remark B.2. Assuming the Riemann hypothesis, the $\varepsilon_1, \varepsilon_2, \varepsilon_3$ terms in the definition of $\tilde{E}(x)$ may be deleted, since [4, (7.3)] then holds for all $x \geq 5000$.

The claim (2.16) now follows from (2.14) by setting $b(t) := \frac{1}{\log t}$. For non-negative b, the claims (2.17), (2.18) follow from (2.14) and the pointwise bounds

$$\frac{b(p)\log p}{\log x} \leq b(p) \leq \frac{b(p)\log p}{\log y}$$

and

$$1 - \frac{2}{\sqrt{y}} \leq 1 - \frac{2}{\sqrt{t}} \leq 1.$$

Finally, (2.19), (2.20) come from specializing (2.17), (2.18) to the case of an indicator function $b = 1_{(y,x]}$.

Appendix C. Computation of c_0 and related quantities

In this appendix we give some details regarding the numerical estimation of the constants c_0, c_1', c_1'', c_1 defined in (1.6), (5.2), (5.3), (5.4).

We begin with c_0. As one might imagine from an inspection of Figure 3, direct application of numerical quadrature converges quite slowly due to the oscillatory singularity. To resolve the singularity, we can perform a change of variables $x = 1/y$ to express c_0 as an improper integral:

$$c_0 = \frac{1}{e}\int_1^\infty \lfloor y \rfloor \log \frac{\lceil y/e \rceil}{y/e} \frac{dy}{y^2}. \tag{C.1}$$

Next, observe[8] that

$$\begin{aligned}
\frac{1}{e}\int_e^\infty y \log \frac{\lceil y/e \rceil}{y/e} \frac{dy}{y^2} &= \sum_{k=1}^\infty \int_{ke}^{(k+1)e} y \log \frac{k+1}{y/e} \frac{dy}{y^2} \\
&= \frac{1}{e}\sum_{k=1}^\infty \int_k^{k+1} (\log(k+1) - \log y) \frac{dy}{y} \\
&= \frac{1}{2e}\sum_{k=1}^\infty \log^2\left(1 + \frac{1}{k}\right) \\
&= 0.1797439053\ldots;
\end{aligned}$$

The value here was computed in interval arithmetic by subtracting off the asymptotically similar sum $\frac{1}{2e}\sum_{k=1}^\infty \frac{1}{k^2} = \frac{1}{2e}\frac{\pi^2}{6}$, summing the resulting partial sum up to $k = 10^5$, bounding the tail of the sum rigorously.

$$\frac{1}{e}\int_1^e \lfloor y \rfloor \log \frac{e}{y} \frac{dy}{y^2} = \frac{2}{e^2} - \frac{\log 2}{2e} = 0.143173268\ldots$$

[8]We thank an anonymous commentor on the blog of one of the authors for this suggestion.

and hence

$$c_0 = \frac{1}{2e}\sum_{k=1}^{\infty}\log^2\left(1+\frac{1}{k}\right) + \frac{2}{e^2} - \frac{\log 2}{2e} - \frac{1}{e}\int_e^\infty \{y\}\log\frac{\lceil y/e\rceil}{y/e}\,\frac{dy}{y^2}$$

where $\{x\} := x - \lfloor x\rfloor$. The integrand here lies between 0 and $1/y^3$, so the integral for $y \geq T$ lies between 0 and $1/2T^2$. Truncating to say $T = 10^5$ and performing the integral exactly, one can evaluate

$$\frac{1}{e}\int_e^\infty \{y\}\log\frac{\lceil y/e\rceil}{y/e}\,\frac{dy}{y^2} = 0.018498162\dots$$

so that

$$c_0 = 0.30441901\dots.$$

A similar calculation (which we omit) reveals that

$$\begin{aligned} c_1' &= \sum_{k=1}^{\infty}\frac{1+\log(k+1)}{2e}\log^2\left(1+\frac{1}{k}\right) - \frac{1}{3e}\log^3\left(1+\frac{1}{k}\right)\\ &\quad + \frac{6}{e^2} - \frac{\log^2 2 + \log 2 + 3}{2e}\\ &\quad - \frac{1}{e}\int_e^\infty \{y\}(\log y)\log\frac{\lceil y/e\rceil}{y/e}\,\frac{dy}{y^2}\\ &\approx 0.3702051\dots. \end{aligned}$$

Computing the sum c_1'' to reasonable accuracy requires some further analysis. From the crude bound

$$0 \leq \frac{1}{k}\log\left(\frac{e}{k}\left\lceil\frac{k}{e}\right\rceil\right) \leq \frac{e}{k^2}$$

and the integral test, one has the simple tail bound

$$0 \leq \sum_{k=K+1}^{\infty}\frac{1}{k}\log\left(\frac{e}{k}\left\lceil\frac{k}{e}\right\rceil\right) \leq \frac{e}{K}$$

but the convergence rate here is slow. To accelerate the convergence, we write $\lceil\frac{k}{e}\rceil = \frac{k}{e} + \{\frac{k}{e}\}$ and use the more precise Taylor approximation

$$\frac{e\left\{\frac{k}{e}\right\}}{k^2} - \frac{e^2\left\{\frac{k}{e}\right\}^2}{2k^3} \leq \frac{1}{k}\log\left(\frac{e}{k}\left\lceil\frac{k}{e}\right\rceil\right) \leq \frac{e\left\{\frac{k}{e}\right\}}{k^2}.$$

Bounding $\{k/e\}$ by one, we have the tail bound

$$0 \leq \sum_{k=K+1}^{\infty}\frac{e^2\left\{\frac{k}{e}\right\}^2}{2k^3} \leq \frac{e^2}{4K^2}$$

so the main task is then to control the simplified tail

$$\sum_{k=K+1}^{\infty}\frac{e\left\{\frac{k}{e}\right\}}{k^2}.$$

From the integral test one has

$$\frac{e}{2(K+1)} \leq \sum_{k=K+1}^{\infty} \frac{\frac{e}{2}}{k^2} \leq \frac{e}{2K}$$

so one can instead look at the normalized tail

$$\sum_{k=K+1}^{\infty} e \frac{\left\{\frac{k}{e}\right\} - \frac{1}{2}}{k^2}.$$

The Erdős–Turán inequality states that, for any absolutely convergent non-negative weights c_k, any interval $I \subset [0,1]$ of length $|I|$, and any real numbers ξ_k, and any $N \geq 1$, one has

$$|\sum_k c_k(1_I(\xi_k \bmod 1) - |I|)| \leq \frac{1}{N+1}\sum_k c_k + \sum_{n=1}^{N}\left(\frac{2}{\pi n} + \frac{2}{N+1}\right)|\sum_k c_k e^{2\pi i n \xi_k}|;$$

see the inequality[9] after [18, Theorem 20]. Applying this for $I = [0,h]$ and then averaging in h from 0 to 1, we conclude that

$$\left|\sum_k c_k \left(\{\xi_k\} - \frac{1}{2}\right)\right| \leq \frac{1}{N+1}\sum_k c_k + \sum_{n=1}^{N}\left(\frac{2}{\pi n} + \frac{2}{N+1}\right)|\sum_k c_k e^{2\pi i n \xi_k}|.$$

In particular, we have

$$\left|\sum_{k=K+1}^{\infty} e \frac{\left\{\frac{k}{e}\right\} - \frac{1}{2}}{k^2}\right| \leq \frac{1}{N+1}\sum_{k=K+1}^{\infty}\frac{e}{k^2} + \sum_{n=1}^{N}\left(\frac{2e}{\pi n} + \frac{2e}{N+1}\right)\left|\sum_{k=K+1}^{\infty}\frac{e^{2\pi i n k/e}}{k^2}\right|.$$

To estimate the exponential sum

$$S_{n,K} := \sum_{k=K+1}^{\infty}\frac{e^{2\pi i n k/e}}{k^2}$$

observe from shifting k by one that

$$S_{n,K} = e^{2\pi i n/e}\sum_{k=K}^{\infty}\frac{e^{2\pi i n k/e}}{(k+1)^2} = e^{2\pi i n/e} S_{n,K} + O_{\leq}\left(\frac{1}{(K+1)^2} + \sum_{k=K+1}^{2}\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$

and hence on summing the telescoping series

$$|S_{n,K}| \leq \frac{2}{|e^{2\pi i n/e} - 1|(K+1)^2} = \frac{1}{(K+1)^2 \sin(\pi n/e)}.$$

Because the irrationality measure of e is 2, this will give error terms of the shape $O(\log K/K^2)$ if one sets $N \approx K/\log K$. Setting for instance $K = 10^6$, $N = 10^5$, an interval arithmetic computation then gives

$$c_1'' = 1.679578996\ldots$$

and thus by (5.4)

$$c_1 = 0.7554808\ldots.$$

[9]In the cited reference, only the special case in which c_k is a uniform probability distribution function on $\{1,\ldots,M\}$ is discussed, but it is easy to see that the argument in fact works for arbitrary absolutely convergent non-negative weights c_k.

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