1 General Relativity Review

1.1 Defining the topological space

In relativity and related theories, the space of physical events is represented by a Lorentzian manifold $\mathcal{M}$. Unlike a Riemannian manifold, which requires positive semi-definiteness, a Lorentzian manifold relaxes this and allows inner products (distances) to be negative, to preserve the casual structure of spacetime (**read up on more of this**). A Lorentzian manifold also has a tangent space that is locally homeomorphic to flat spacetime $\eta_{\mu\nu}$. The tangent space is the collection of possible directions you can move in to move tangentially to a point, so this allows you to define differential objects like gradients on the manifold. The Lorentzian manifold being locally flat at each point preserves the local accuracy of special relativity (when at small scale, spacetime will always appear flat).

The manifold is endowed with a metric $g_{\mu\nu}$ that gives a smooth assignment to each point p on $\mathcal{M}$ (metric varies smoothly across manifold, which is crucial for the continuity and differentiability of the manifold) and allows lengths and angles to be calculated on $T_p\mathcal{M}$, the tangent space, at each point. The metric $g_{\mu\nu}$ captures the geometry of space near a point.

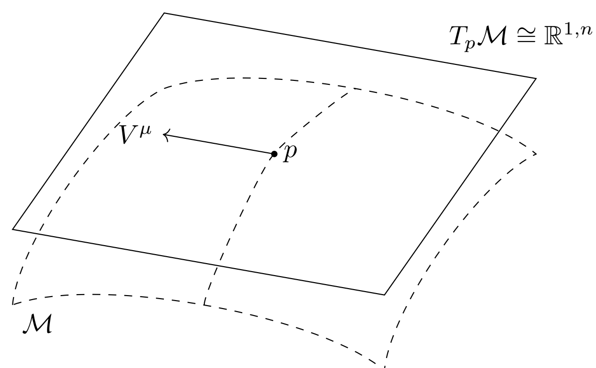

Figure 1: The tangent space at a point p on a manifold $\mathcal{M}$. This space is locally flat and is endowed with the flat spacetime metric $\eta_{\mu\nu}$. A contravariant vector V^μ is drawn on this space.

So the finding the tangent space at a point gives an operation for going from the full manifold to flat spacetime any point on that manifold:

$$(\mathcal{M}, g_{\mu\nu}) \xrightarrow{T_p\mathcal{M}} (\mathbb{R}^{1,n}, \eta_{\mu\nu}). \tag{1}$$

Important note! (I think this is true...): contravariant tensors (the ones you're used to dealing with... those with indices upstairs) are defined on these *tangent spaces*, while covariant tensors are defined on *cotangent spaces*. This seems true because the tangent space is the set of all possible ways you could put a tangent line through a point and you know that the derivative at a point will give you a tangent vector at that point... AND when we take derivatives in GR, we use $\partial/\partial x^\mu$.

Every Lorentzian manifold comes with a **Levi-Civita connection**, which is a specific type of affine connection that is associated with a Lorentzian manifold. An **affine connection** is a geometrical object that connects tangent spaces at points on a manifold so that derivatives can be taken on the manifold.

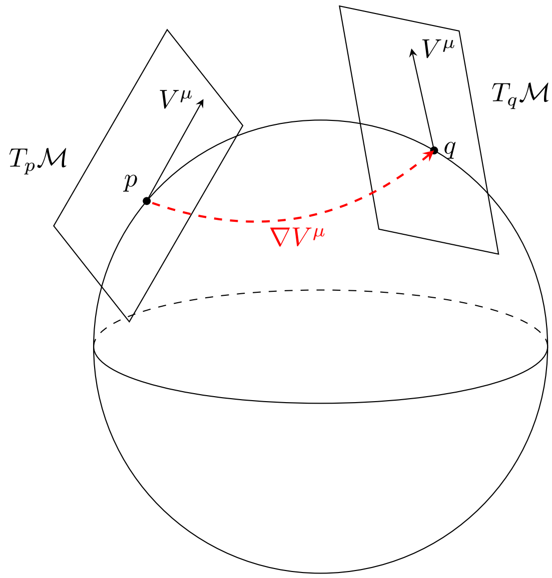

Figure 2: Showing the affine connection from point p to q on a spherical manifold. The tangent spaces and a vector in them for each point are also shown.

This affine connection (just referred to as *connection*) is used to define the covariant derivative of a vector V^μ, defined as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda \tag{2}$$

where the $\partial_\nu V^\mu$ term captures how the vector itself changes from p to q (as usual) and the $\Gamma^\mu_{\nu\lambda} V^\lambda$ term, called **Christoffel symbols**, captures how the shape of the manifold (V^μ's basis vectors) changes along the path. The Christoffel symbols are the coefficients of the connection. Since the Christoffel symbols capture how the manifold changes from one point to another, it shouldn't be surprising that their definition contains first derivatives of the metric;

$$.\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\sigma\mu} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial g_{\sigma\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\sigma} \right). \tag{3}$$

In a flat spacetime, the metric would never change from one point to another, meaning $\Gamma^\mu_{\nu\lambda} V^\lambda = 0$, which means our covariant derivative would become

$$\nabla_\nu V^\mu = \partial_\nu V^\mu, \tag{4}$$

which is the regular definition of the gradient operator (so the covariant derivative is just the generalization of the gradient operator to curved spacetimes). If the covariant derivative of a vector along a path is zero, this means that the vector has not changed its orientation over the path. In this case, we say it has been parallel transported across the manifold.

Note that the shortest path on a manifold, called its **geodesic**, is described by the geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \tag{5}$$

where the x^μ terms are the coordinates on the manifold and τ is the proper time; the time measured by the observer moving along the geodesic.

To measure the **curvature of the manifold**, we must know how the metric, which, again, describes how the geometry near a point changes, itself changes over the manifold. Knowing how the geometry of spacetime near all points on a manifold change and how this change in geometry itself changes from point to point. The curvature of a manifold is described by the **Riemann curvature tensor**, defined as

$$R^\lambda_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\sigma\mu} - \partial_\nu \Gamma^\lambda_{\sigma\mu} + \Gamma^\lambda_{\rho\mu} \Gamma^\rho_{\sigma\nu} - \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\sigma\mu}. \tag{6}$$

With the Riemann tensor, we define another tensorial object of two lower rank by contracting (summing over the indices of) the Riemann tensor:

$$R_{\mu\nu} := R^{\lambda}_{\mu\lambda\nu}. \tag{7}$$

This new quantity, called the **Ricci tensor**, still contains information about the curvature of your manifold from the Riemann tensor but now in a more compact form. The Ricci tensor therefore also describes curvature of a manifold but in a more compact form.

We then define another quantity, called the **Ricci scalar** (or often referred to as the scalar curvature), as the trace of the Ricci tensor with respect to the inverse metric:

$$R := g^{\mu\nu} R_{\mu\nu}. \tag{8}$$

Since the metric defines the geometry of your manifold near a point and the Ricci tensor is a compact measure for the curvature of the manifold, this Ricci scalar describes the **average curvature at a point**. It should be no surprise that a measure for the average curvature at a point depends on some notion for what the curvature of a manifold looks like and what the geometry around a single point looks like.

The Einstein field equations are defined as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{9}$$

1.2 Einstein-Hilbert action

The Einstein-Hilbert action is the action that correctly yields the Einstein field equations when the action principle is applied to it. (I assume the field equations came first and there was some amount of guessing that went into determining which action resulted in them after the variation is taken). This is defined as

$$S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x R\sqrt{-g} \tag{10}$$

where $\sqrt{-g} := \det(g_{\mu\nu})$. Recall that the action tells you all the possible configurations/paths of your system ($g_{\mu\nu}$ here... all possible spacetime geometries at a point). Taking $\delta S = 0$ tells you the path that objects will take over the possibilities of configurations/paths. It makes sense that the Einstein-Hilbert action would include a dependence on R, since this tells you about average geometry around a point and you are eventually looking to minimize spcaetime paths, so you'd naturally want to take the geometry of your spacetime into account. Also recall that the determinant of a tensor tells you how volumes change under transformation of that tensor.

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \delta(R\sqrt{-g}) = 0 \implies G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}. \tag{11}$$

The implies symbol is skipping over a whole lot of derivation.

2 Modified theories of gravity

Lovelock's theorem tells us that general relativity is the **only** possible theory of gravity that satisfies the following assumptions:

1. The theory be four-dimensional ($D = 4$).
2. Only second order derivatives allowed in the theory's equations of motion (EFEs).
3. That there be no additional fields; $g_{\mu\nu}$ is the fundamental field in GR describing the geometry of spacetime near a point.
4. The theory of gravity be <u>local</u>; the spacetime geometry at a point is only due to the value of $g_{\mu\nu}$ and its derivatives at *that point*.
5. The theory be <u>covariant</u>; that the underlying theory does not change under coordinate transformations.
6. The equations of motion come from the theory's action.

Any modified theory of gravity must therefore bend one of these rules.

3 3+1 Decomposition

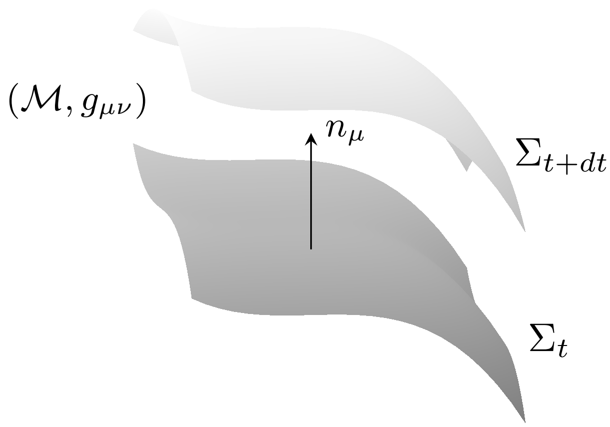

Figure 3: The foliation of a manifold $\mathcal{M}$, which is endowed with a metric $g_{\mu\nu}$ into two hypersurfaces. The normal vector n_μ to the Σ_t surface is shown. Note that $(\mathcal{M}, g_{\mu\nu})$ indicates the Lorentzian manifold which is endowed with the metric $g_{\mu\nu}$.

3.1 Defining lapse function and shift vector

We wish to develop a *general method* for slicing a spacetime manifold $\mathcal{M}$ into spatial hypersurfaces so that their evolution can be tracked over a coordinate time t. We first imagine creating two slices, Σ_t and Σ_{t+dt} and define the normal vector to any surface as

$$n_\mu = -\alpha \nabla_\mu t \tag{12}$$

where α is added as a normalization constant, t is added to correctly order the hypersurfaces in time, ∇_μ is the covariant derivative, added because the normal vector to a plane will be the partial derivatives of the vector or function describing the plane (**need better reasoning here**), and the negative sign is introduced since we are using the metric signature $(-,+,+,+)$. This means we have

$$n_\mu = (-\alpha, 0, 0, 0) \tag{13}$$

and the dot product of the normal vector with itself is

$$n_\mu n^\mu \equiv g^{\mu\nu} n_\mu n_\nu = -1. \tag{14}$$

With two hypersurfaces, Σ_t and Σ_{t+dt}, we would first like to know how the change in coordinate time corresponds to a change in proper time for an observer on the surfaces. (again, really need image here) The proper time separation will be $n^\mu d\tau$ and we want to know how this is related to dt. To do this, we take the dot product of the separation vector $n^\mu d\tau$ and the gradient of coordinate time (describing the rate of change of coordinate time along the direction of $n^\mu d\tau$):

$$dt = (\nabla_\mu t)(n^\mu d\tau). \tag{15}$$

Using Eq. (12), we have

$$dt = \left(-\frac{n_\mu}{\alpha}\right)(n^\mu d\tau) = \frac{d\tau}{\alpha} \implies \boxed{\alpha dt = d\tau}. \tag{16}$$

So we have learned that the normalization constant α of the normal vectors tells us how much proper time τ (measured by an observer on these hypersurfaces) changes as coordinate time advances between the hypersurfaces. For this reason, α is called a **lapse function**.

We would similarly like to know how spatial coordinates change for an observer on the hypersurface as coordinate time advances. First note that we can introduce a factor β^i into the definition of n^μ's spatial components without alerting our normalization condition $n^\mu n_\mu = -1$ (**I don't understand this at all**) so we can say

$$\boxed{n^\mu = \frac{1}{\alpha}(-1, \beta^i)}. \tag{17}$$

To determine how a change in coordinate time corresponds to a change in spatial coordinates on the hypersurface, we again play the same game of taking the dot product of the change in the spatial separation vector $\nabla_\mu x^i$ along the change in the observer's proper time vector $n^\mu d\tau$ (again remember that n^μ gives direction to the change in proper time $d\tau$). This results in

$$dx^i = (\nabla_\mu x^i)(n^\mu d\tau) \tag{18}$$

and we can say that $\nabla_\mu x^i = \partial_\mu x^i$ since x^i are just coordinates (**I don't understand why the Christoffel symbols vanish...**). We can then say that $\partial_\mu x^i = \delta^i_\mu$ (since $\partial_y x = 0$ but $\partial_x x = 1$), leaving us with

$$dx^i = \delta^i_\mu n^\mu d\tau = n^i d\tau = \frac{\beta^i}{\alpha} d\tau \tag{19}$$

then using Eq. (16), we have

$$dx^i = \frac{\beta^i}{\alpha}\alpha dt \implies \boxed{dx^i = \beta^i d\tau}. \tag{20}$$

β therefore measures the rate at which a normal observer would notice the spatial coordinates change (if an observer at Σ_t could see the hypersurface and all of its coordinates evolve to Σ_{t+dt}, they would see a spacetime difference of $(t + dt, x^i - \beta^i dt)$ ... **shaky on this... not totally clear. Not clear on the next part either**). This is why β is called the **shift vector**. We can then define a time vector according to a coordinate observer (one outside of $\mathcal{M}$... the person evolving the simulation) as

$$t^\mu = \alpha n^\mu + \beta^\mu = (1, 0, 0, 0) \tag{21}$$

3.2 Decomposing tensors

We now wish to split tensors into spatial and temporal components. Given a vector A^μ, we want to decompose it into a component aligned with the normal vector (along the direction of proper time flow) and components that are aligned with the spatial slices. We can always decompose a tensor into components that are parallel and perpendicular to n^μ:

$$A^\mu = A^\mu_\perp + A^\mu_\parallel \tag{22}$$

where $A^\mu_\perp$ would be the normal component and $A^\mu_\parallel$ would be aligned with the spatial slice. First define the magnitude of the vector parallel to n^μ as

$$\phi := -n_\nu A^\nu \tag{23}$$

(negative sign added because n_ν is timelike, so for future-pointing vectors, n^ν will be positive). Because the parallel component of A^μ will point in the same direction as n^μ, we have that

$$A^\mu_\parallel = \phi n^\mu = (-n_\nu A^\nu) n^\mu = -n_\nu n^\mu A^\nu. \tag{24}$$

Now to compute $A^\mu_\perp$, we rearrange the definition given in Eq. (22) and apply what we have found in Eq. (24):

$$A^\mu_\perp = A^\mu - A^\mu_\perp = A^\mu + n_\nu n^\mu A^\nu \tag{25}$$

and use the trick $A^\mu = \delta^\mu_\nu A^\nu$ so we have

$$A^\mu_\perp = \delta^\mu_\nu A^\nu + n_\nu n^\mu A^\nu = (\delta^\mu_\nu + n_\nu n^\mu) A^\nu. \tag{26}$$

We have therefore found that, given a spacetime vector A^ν, the operator $\delta^\mu_\nu + n_\nu n^\mu$ will return only the components of A^ν that are aligned with the spatial slices. For this reason, we define a projection operator γ^μ_ν as

$$\gamma^\mu_\nu := \delta^\mu_\nu + n_\nu n^\mu. \tag{27}$$

As an informative example, we will decompose a rank 2 tensor into its spatial and normal components using the method developed above. The object we would like to get spatial projections of is $\nabla_\mu n_\nu$; how the normal vectors change with respect to each direction.

Motivated by our previous derivation, we will use a similar trick by letting

$$\nabla_\mu n_\nu = \delta^\lambda_\mu \delta^\sigma_\lambda \nabla_\lambda n_\sigma \tag{28}$$

and we rearrange Eq. (27) as $\delta^\beta_\alpha = \gamma^\beta_\alpha - n_\alpha n^\beta$, leaving us with

$$\nabla_\mu n_\nu = (\gamma^\lambda_\mu - n_\mu n^\lambda)(\gamma^\sigma_\nu - n_\nu n^\sigma)\nabla_\lambda n_\sigma \tag{29}$$

and you can multiply this out to get an idea of what each component will mean individually. In Thomas Baumgarte's lecture, he eventually defines an acceleration from one of these terms. I will skip this now but may come back to finish it with textbooks.

The following definition is made from one of the terms in the previous expression:

$$\boxed{K_{\mu\nu} := -\gamma^\lambda_\mu \gamma^\sigma_\nu \nabla_\lambda n_\sigma}. \tag{30}$$

This is called the **extrinsic curvature**: it is the spatial projection of the gradient of the normal vector (how the normal vector changes along multiple directions). This captures how each slice Σ_t is curved in its embedding within the manifold $\mathcal{M}$. (So I think intrinsic curvature will be captured by a projection of the metric along all points of the manifold and this extrinsic curvature is how curved the *borders* of Σ_t along $\mathcal{M}$ are).

As you would expect, the spatial covariant derivative of a spatial vector $A^\mu_\perp$ is taken by spatially-projecting the derivative

$$D_\mu A^\nu_\perp := \gamma^\lambda_\mu \gamma^\nu_\sigma \nabla_\lambda A^\mu_\perp. \tag{31}$$

If you want to take spatial covariant derivatives of a spacetime vector, the following operation is used:

$$D_\mu A^\nu_\perp = \gamma^\lambda_\mu \gamma^\nu_\sigma \nabla_\lambda A^\mu_\perp. \tag{32}$$

This covariant derivative would return zero if the vector was parallel transported (meaning if was moved along the curve without changing its direction relative to that curve), meaning $\nabla_\mu A^\nu = 0$ in 4D and $D_\mu A^\nu_\perp = 0$ on the spatial slices.

Lie Derivative: I don't know where its form comes from (how it's derived) but it tells you what part of changes to the input tensor are *not* due to simple coordinate transformations after it is dragged along some curve.

Contracted Biancchi identity:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{33}$$

(divergence of Einstein tensor is zero). Using the field equations, this also tells us that

$$\nabla_\mu T^{\mu\nu} = 0, \tag{34}$$

which is the conservation of energy.

Recipe for decomposing a theory into its spatial and normal (temporal) components:

1. Decompose the theory's variables
   (a) In GR, the fundamental quantity is the metric so we must reduce $g_{\mu\nu}$ into a component which only contains information about spatial curvature and another component that only contains information about time in the theory
2. Decompose the equations (which will give you constraints on the theory's quantities (like momentum and energy)
   (a) In GR, this is the action of decomposing the EFEs

3.3 Decomposing the metric

To decompose the metric, use the same trick as before where we say

$$g_{\mu\nu} = \delta_\mu^\lambda \delta_\nu^\sigma g_{\lambda\sigma} \tag{35}$$

and

$$\delta_\nu^\mu = \gamma_\nu^\mu - n_\nu n^\mu \tag{36}$$

so we have

$$\begin{aligned} g_{\mu\nu} &= (\gamma_\mu n^\lambda - n_\mu^\lambda)(\gamma_\nu^\sigma - n_\nu n^\sigma) g_{\lambda\sigma} \\ &= \gamma_\mu^\lambda \gamma_\nu^\sigma g_{\lambda\sigma} - \gamma_\mu n^\lambda n_\nu n^\sigma g_{\lambda\sigma} - \gamma_\nu^\sigma n_\mu^\lambda g_{\lambda\sigma} + n_\mu n^\lambda n_\nu n^\sigma g_{\lambda\sigma} \end{aligned} \tag{37}$$

and

$$\gamma_\mu^\lambda n_\nu n^\sigma g_{\lambda\sigma} = \gamma_\nu^\lambda n_\mu n_\sigma \tag{38}$$

and since γ_ν^λ describes a spatial slice while the n-vectors describe vectors normal to those spatial slices, there will be no components that the two variables can project onto eachother, so the whole sum goes to zero. This is true for both cross-terms in the previous expansion, leaving us with

$$\begin{aligned} g_{\mu\nu} &= \gamma_\mu^\lambda \gamma_\nu^\sigma g_{\lambda\sigma} + n_\mu n^\lambda n_\nu n^\sigma g_{\lambda\sigma} \\ &= \gamma_\mu^\lambda \gamma_{\nu\lambda} + n_\mu n^\lambda n_\nu n_\lambda \\ &= \gamma_{\mu\nu} - n_\mu n_\nu \\ &\implies \boxed{\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu} \end{aligned} \tag{39}$$

where $\gamma_{\mu\nu}$ is defined as the <u>induced spatial metric</u>; given a full spacetime metric $g_{\mu\nu}$, one uses Eq. (39) to derive a purely spatial metric on a hypersurface Σ_t.

We can now use this spatial metric to raise and lower indices just as the full spacetime metric can. Given a vector β^ν,

$$\beta_\mu = g_{\mu\nu}\beta^\nu = (\gamma_{\mu\nu} - n_\mu n_\nu)\beta^\nu \tag{40}$$

and if β^ν is fully spatial, $n_\mu n_\nu \beta^\nu = 0$ since they are always orthogonal, meaning

$$\beta^\mu = \gamma_{\mu\nu}\beta^\nu. \tag{41}$$

Additionally, since we found

$$n_\mu \beta^\mu = 0 \tag{42}$$

and said previously that $n_\mu = (-\alpha, 0, 0, 0)$ (since the normal vectors only point in the direction of forward time), the only non-zero component we have is

$$n_t \beta^t = -\alpha\beta^t = 0 \implies \beta^t = 0, \tag{43}$$

which tells us that <u>any contravariant spatial vector like β^μ must have vanishing normal components</u>. Note that it could be the case that covariant spatial vectors have normal components since

$$n^\mu \beta_\mu = n^t \beta_t + n^i \beta_i = 0. \tag{44}$$

This equality must hold but it does not prevent the scenario where $n^t\beta_t = -n^i\beta_i$.

We then write

$$\begin{aligned} g^{\mu\nu} &= \gamma^{\mu\nu} - n^\mu n^\nu \\ &= \begin{pmatrix} n^t n^t & n^t n^i \\ n^i n^t & n^i n^j \end{pmatrix} = \begin{pmatrix} \alpha^{-2} & -\beta^i \alpha^{-2} \\ -\beta^i \alpha^{-2} & -\alpha^{-2}\gamma^{ij}\beta^i\beta^j \end{pmatrix} \end{aligned} \tag{45}$$

using the fact that $n^\mu = \frac{1}{\alpha}(-1, \beta^i)$. We then invert this matrix to find (I'm trusting the computation done in the lecture here... haven't done it myself):

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_j\beta^j & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}. \tag{46}$$

We can then define the spacetime interval with the decomposed metric:

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= (-\alpha^2 + \beta_j\beta^j)dt^2 + \beta_i dt dx^i + \beta_i dx^i dt + \gamma_{ij}dx^i dx^j \\ &= -\alpha^2 dt^2 + \gamma_{ij}\beta^i\beta^j dt^2 + 2\gamma_{ij}\beta_j dt dx^i + \gamma_{ij}dx^i dx^j \\ &= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \end{aligned} \tag{47}$$

So we have successfully decomposed the metric into spatial and normal components, with the induced spatial metric $\gamma_{\mu\nu}$ being derived from a given spacetime metric $g_{\mu\nu}$.

As an example, consider the Schwarzschild line element in isotropic coordinates

$$ds^2 = \left(\frac{1-\frac{M}{2r}}{1-\frac{M}{2r}}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega). \tag{48}$$

To decompose this metric, since there are no cross-terms ($dtdx^i$), we have that $\beta = 0$ and we have that

$$\alpha = \left(\frac{1-\frac{M}{2r}}{1-\frac{M}{2r}}\right) \tag{49}$$

and

$$\gamma_{ij} = \left(1+\frac{M}{2r}\right)^4 \eta_{ij} \tag{50}$$

by inspection.

4 Decomposing the field equations

Step 2 of decomposing a theory: decomposing the equations. This is what is done below:

We must first decompose the variables that show up in the field equations (Riemann tensor, Ricci tensor, Ricci scalar). **NOTE:** This is the derivation-heavy part and I'm skipping all of the details right now (which *are* important for my purposes and I will eventually go over them). This part will mainly be repreating the results and explaining the steps for applying the decomposition to the Einstein field equations.

First we define the Lie derivative along a normal vector of the spatial metric to be

$$\mathcal{L}_n\gamma_{\mu\nu} = n^\lambda\nabla_\lambda\gamma_{\mu\nu} + \gamma_{\lambda\nu}\nabla_\mu n^\lambda + \gamma_{\mu\lambda}\nabla_\lambda n^\lambda \tag{51}$$

and this somehow (don't understand how yet) leads to

$$\mathcal{L}_n\gamma_{ij} = -2K_{ij}. \tag{52}$$

We also have the pure time derivative of the spatial metric

$$\partial_t\gamma_{ij} = 2 - \alpha K_{ij} + {}^{(3)}\nabla_i\beta_j + {}^{(3)}\nabla_j\beta_i \tag{53}$$

where ${}^{(3)}\nabla_i$ is the purely spatial covariant derivative, defined as

$${}^{(3)}\nabla_i V^j = \partial_i V^j + V^k\,{}^{(3)}\Gamma^j_{ki} \tag{54}$$

where ${}^{(3)}\Gamma^j_{ki}$ are the purely spatial Christoffel symbols, defined exactly as before in Eq. (3) with $g_{\mu\nu}$ being replaced by Γ_{ij} in the expression.

With the three-dimensional Christoffel symbols defined, we can proceed exactly as before when working with the full metric by first defining a purely spatial (three dimensional) Riemann curvature tensor. There are three unique ways to express the spatial Riemann curvature tensor, though, since we can now decompose the full four-dimensional object

1. Twice along the spatial dimensions
2. Twice along the normal direction (flow of coordinate time)
3. Once along the spatial dimensions and once along the normal direction

and each of these three decompositions of the four dimensional Riemann tensor will result in three different expressions, each of which will introduce **constraint equations** on the spatial hypersurfaces when they are used in the field equations. There are only three unique ways to decompose $R_{\mu\nu\lambda\sigma}$ because if we decomposed twice along the spatial dimensions and once along the normal direction, you would always get zero.

Again, I'm only showing results here. I will *need* to work the derivations out for myself once I have the books. From this process, we get:

1. Purely spatial decompositions result in:
$$\gamma_i^\mu \gamma_j^\nu \gamma_k^\lambda \gamma_l^\sigma R_{\mu\nu\lambda\sigma} = {}^{(3)}R_{ijkl} + K_{ij}K_{jl} - K_{il}K_{jk}. \tag{55}$$
This is called **Gauss' equation**
2. purely normal decompositions result in:
$$\gamma_i^\mu \gamma_j^\nu n^\lambda n^\sigma R_{\mu\nu\lambda\sigma} = \mathcal{L}_n K_{ij} + \frac{1}{\alpha}\, {}^{(3)}\nabla_i\, {}^{(3)}\nabla_j \alpha + K_i^k K_{kj}. \tag{56}$$
This is called **Ricci's equation**. Notice that the left side only contains derivatives along the normal direction. **there is something wrong with the indices here. figure it out once the books come**
3. Normal-spatial decomposition:
$$\gamma_j^\mu \gamma_j^\nu \gamma_k^\lambda n^\sigma R_{\mu\nu\lambda\sigma} = {}^{(3)}\nabla_j K_{ik} - {}^{(3)}\nabla_i K_{jk} \tag{57}$$
These are called the **Codazzi-Mainardi equations**

Now with these equations, we have projections of the Riemann curvature tensor and now can build the Ricci tensor and Ricci scalar out of it by contracting each of the three expressions' indices twice (**again, work this out once you have the books**).

After this is done, you have successfully decomposed all of the variables that show up in the field equations so you are set to fully decompose them. Since there are three unique ways to decompose the fundamental quantity, $R_{\mu\nu\lambda\sigma}$, you will have three unique field equations, each of which will result in a constraint on your variables. Carrying out this process results in:

1. Spatial-spatial:
$$\boxed{\partial_t K_{ij} - \alpha(R_{ij} - 2K_{ik}K_j^k + KK_{ij}) = {}^{(3)}\nabla_i\, {}^{(3)}\nabla_j \alpha - 4\pi M_{ij} + \mathcal{L}_\beta K_{ij}} \tag{58}$$
where M_{ij} are the matter terms, defined as
$$M_{ij} := 2S_{ij} - \gamma_{ij}(S - \rho) \tag{59}$$
(see below for the definition of ρ) where we define
$$S_{ij} := \gamma_i^\mu \gamma_j^\nu T_{\mu\nu}, \tag{60}$$
which is the stress observed by a normal observer (one living on the hyperplanes).
2. Normal-normal:
$$\boxed{{}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho} \tag{61}$$
This called the **Hamiltonian constraint** where
$$\rho := n^\mu n_\nu T_{\mu\nu} \tag{62}$$
and K^2 is called the mean curvature, defined as $K^2 := (\gamma^{ij}K_{ij})$.

3. Normal-spatial:

$$\boxed{{}^{(3)}\nabla_i(K^{ij} - \gamma^{ij}K) = 8\pi J^i} \tag{63}$$

These are called the **momentum constraint**, where

$$J^i := -\gamma^{i\mu}n^\nu T_{\mu\nu} \tag{64}$$

These three boxed equations along with our previous condition

$$\boxed{\partial_t\gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta\gamma_{ij}} \tag{65}$$

are called the **ADM equations**, which are the full reformulation of the Einstein field equations in the 3+1 decomposition.