Quantum Computation and Quantum Information - Selected Solutions

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1 Chapter 1: Introduction and Overview

2 Chapter 2: Introduction to Quantum Mechanics

Exercise 2.1: Show that $(1,-1),(1,2)$, and $(2,1)$ are linearly dependent.

Solution: Vectors $|v_1\rangle, |v_2\rangle, \dots, |v_n\rangle$ are linearly dependent in a vector space V with $\dim(V) = m$ if $n > m$. Because we have three two-dimensional vectors, they must be linearly dependent. This could also be directly demonstrated by representing the vectors as an augmented matrix

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{pmatrix} \tag{1}$$

and applying Gaussian elimination to show that a nontrivial solution for a_1, a_2, a_3 of the form

$$a_1 |v_1\rangle + a_2 |v_2\rangle + a_3 |v_3\rangle = 0$$

exists.

Exercise 2.2: Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle, |1\rangle$, and the output basis $|0\rangle, |1\rangle$. Find input and output basis which give rise to a different matrix representation of A.

Solution: Because we have a two-dimensional vector space, we can define $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We are being asked to find the transformation matrix A such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. These conditions give us:

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies a_{00} = 0 \ \&\ a_{10} = 1 \tag{2}$$

and

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies a_{01} = 1 \ \&\ a_{11} = 0. \tag{3}$$

So we have learned the matrix representation of the transformation A is given as:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

Taking inspiration from page 63, define $|0\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|1\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. To find the linear transformation A that satisfies the conditions $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$, the same method is carried out as before; requiring that

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \implies \frac{a_{00}}{\sqrt{2}} + \frac{a_{01}}{\sqrt{2}} = \frac{1}{2} \ \&\ \frac{a_{10}}{\sqrt{2}} + \frac{a_{11}}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \tag{5}$$

and

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \implies \frac{a_{00}}{\sqrt{2}} - \frac{a_{01}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \ \&\ \frac{a_{10}}{\sqrt{2}} - \frac{a_{11}}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \tag{6}$$

Now this system of equations must be solved for $a_{00}, a_{01}, a_{10}, a_{11}$ using whatever method you'd like. Carrying this out results in finding that the transformation matrix A is now represented as:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

Exercise 2.3: Suppose A is a linear operator from vector space V to vector space W, and B is a linear operator from vector space W to vector space X. Let $|v_i\rangle, |w_j\rangle$, and $|x_k\rangle$ be bases for the vector spaces V, W, and X, respectively. Show that the matrix representation for the linear transformation BA is the matrix product of the matrix representations for B and A, with respect to the appropriate bases.

Solution: We are given that $A : V \to W$ and $B : W \to X$ and want to show that

$$BA|v_i\rangle = \sum_k (BA)_{ki}|x_k\rangle \tag{8}$$

where $(BA)_{ki}$, the matrix product of B and A, is defined as

$$(BA)_{ki} := \sum_j B_{kj}A_{ji}. \tag{9}$$

To do this, we first start with the matrix representation of the operator BA on $|v_i\rangle$ and use the transformations that have been defined for us:

$$\begin{aligned} BA|v_i\rangle &= B\left(\sum_j A_{ji}|w_j\rangle\right) \\ &= \sum_j A_{ji}B|w_j\rangle \\ &= \sum_j A_{ji}\sum_k B_{kj}|x_k\rangle. \\ &= \sum_k\sum_j B_{kj}A_{ji}|x_k\rangle \\ &\equiv \sum_k (BA)_{ki}|x_k\rangle \end{aligned} \tag{10}$$

Exercise 2.4: Show that the identity operator on a vector space V has a matrix representation which is one along the diagonal and zero everywhere else, if the matrix representation is taken with respect to the same input and output bases. This matrix is known as the *identity matrix*.

Solution: We are looking for the elements of the transformation matrix I where $I : V \to V$ such that $IV = V$. We are therefore looking for the form of I such that

$$\sum_i I_{ij}|v_i\rangle = |v_j\rangle \tag{11}$$

where $|v_i\rangle = |v_j\rangle$. This therefore requires that $I_{ij} = 0\ \forall\ j \neq i$ and $I_{ij} = 1\ \forall\ j = i$. The only matrix that satisfies such a property is diagonal with 1's. Note that such a matrix is called the Kronecker-Delta function, δ_{ij}.

Exercise 2.5: Verify that $(\cdot,\cdot)$ just defined is an inner product on $\mathbf{C}^n$.

To first demonstrate its linearity, we compute

$$\begin{aligned} \left((y_1,\ldots,y_n), \sum_i \lambda_i (z_1,\ldots,z_n)_i\right) &\equiv \sum_j y_j^* \sum_i \lambda_i z_{ij} \\ = \sum_i \lambda_i \sum_j y_j^* zij &\equiv \sum_i \lambda_i((y_1,\ldots,y_n),(z_1,\ldots,z_n)_i). \end{aligned} \tag{12}$$

To demonstrate that this inner product is equal to its complex conjugate, we must show

$$((y_1,\ldots,y_n),(z_1,\ldots,z_n)) = ((y_1,\ldots,y_n),(z_1,\ldots,z_n))^*. \tag{13}$$

This follows naturally from the definition, since

$$\begin{aligned} ((y_1,\ldots,y_n),(z_1,\ldots,z_n)) &\equiv \sum_i y_i^* z_i \\ = \sum_i (y_i z_i^*)^* &= ((z_1,\ldots,z_n),(y_1,\ldots,y_n))^* \end{aligned} \tag{14}$$

To demonstrate the positivity of this inner product, we compute:

$$((v_1, \ldots, v_n), (v_1, \ldots, v_n)) \equiv \sum_i v_i^* v_i \tag{15}$$

and note that $v_i^* v_i$ is the modulus of v_i squared, $|v_i|^2$, which is always positive.

Exercise 2.7: Verify that $|w\rangle = (1, 1)$ and $|v\rangle = (1, -1)$ are orthogonal. What are the normalized form of these vectors?

Solution: The inner product on $\mathbb{R}^n$ is defined as

$$(v, w) := \sum_i v_i w_i \tag{16}$$

so when $|w\rangle = (1, 1)$ and $|v\rangle = (1, -1)$, we compute that $(w, v) = (v, w) = 1 - 1 = 0$.

The normalized forms of these vectors will be given as $\frac{|v\rangle}{\sqrt{(|v\rangle, |v\rangle)}}$ so we compute the normalized form of $|w\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and the normalized form of $|v\rangle = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$.

Exercise 2.9: The Pauli matrices can be considered as operators with respect to an orthonormal basis $|0\rangle, |1\rangle$ for a two-dimensional Hilbert space. Express each of the Pauli operators in the outer product notation.

Solution: Since the outer product notation of an operator is given as:

$$\sum_{ij} \langle w_j| A |v_i\rangle |w_j\rangle \langle v_i| \tag{17}$$

and $\langle w_j| A |v_i\rangle = A_{ij}$ since we have defined $A |v_j\rangle = \sum_i A_{ij} |w_i\rangle$, we have the outer product notation of an operator A given as:

$$\sum_{ij} A_{ji} |w_j\rangle \langle v_i| \tag{18}$$

where $i, j \in \{0, 1\}$ since we are working in the $|0\rangle, |1\rangle$ basis. Fully expanded, we have:

$$A = a_{00} |0\rangle \langle 0| + a_{01} |0\rangle \langle 1| + a_{10} |1\rangle \langle 0| + a_{11} |1\rangle \langle 1| . \tag{19}$$

So for each Pauli matrix, I, X, Y, Z, we have the outer product notations:

$$\begin{aligned} I &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ X &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ Y &= -i|0\rangle\langle 1| + i|1\rangle\langle 0| \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \end{aligned} \tag{20}$$

Exercise 2.11: Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y, Z.

Solution: To first find the eigenvalues of the X Pauli operator, find the roots of the characteristic function $c(\lambda) = \det|A - \lambda I|$:

$$0 = \det \left| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| \implies 0 = \lambda^2 - 1 \implies \lambda = \pm 1. \tag{21}$$

To solve for the eigenvectors of the Pauli X matrix, we must find the null space of $(X - \lambda_1 I) |v_1\rangle$ and $(X - \lambda_2 I) |v_2\rangle$:

$$0 = (X - \lambda_1 I) |v_1\rangle = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{22}$$

$$0 = (X - \lambda_2 I) |v_2\rangle = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{23}$$

So we have found that $|v_1\rangle = |0\rangle + |1\rangle$ and $|v_2\rangle = |0\rangle - |1\rangle$. To provide the diagonal representation of X, we must first normalize $|v_1\rangle$ and $|v_2\rangle$. Reference Exercise 2.7 to find that the normalized forms are:

$$\begin{aligned} |v_1\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \equiv |+\rangle \\ |v_2\rangle &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \equiv |-\rangle . \end{aligned} \tag{24}$$

The diagonal representation of X is therefore:

$$X = \lambda_1|+\rangle\langle+| + \lambda_2|-\rangle\langle-| = |+\rangle\langle+| - |-\rangle\langle-|. \tag{25}$$

Exercise 2.16: Show that any projector P satisfies the equation $P^2 = P$.

Solution: We compute that

$$P^2 = \sum_i |i\rangle\langle i| \sum_j |j\rangle\langle j| = \sum_{i,j} |i\rangle \langle i|j\rangle \langle j| = \sum_{i,j} |i\rangle \delta_{ij} \langle j| = \sum_i |i\rangle\langle i|. \tag{26}$$

Exercise 2.17: Show that a normal matrix is Hermitian if and only if it has real eigenvalues.

Solution: By spectral decomposition, we know that normal operators are diagonalizable. A diagonalizable operator is one that satisfies the property:

$$A = \sum_i \lambda_i |i\rangle\langle i| \tag{27}$$

and since the operator is also Hermitian, we know that

$$\sum_i \lambda_i |i\rangle\langle i| = \left(\sum_i \lambda_i |i\rangle\langle i|\right)^\dagger. \tag{28}$$

From the result of Exercise 2.14, we know that

$$\left(\sum_i \lambda_i |i\rangle\langle i|\right)^\dagger = \sum_i a_i^* A_i^\dagger \tag{29}$$

so we have that

$$\sum_i \lambda_i |i\rangle\langle i| = \sum_i \lambda_i^* \left(|i\rangle\langle i|\right)^\dagger. \tag{30}$$

From the result of Exercise 2.13, we know that

$$\left(|i\rangle\langle i|\right)^\dagger = |i\rangle\langle i| \tag{31}$$

so this tells us that

$$\sum_i \lambda_i |i\rangle\langle i| = \sum_i \lambda_i^* |i\rangle\langle i| \implies \lambda = \lambda^* \tag{32}$$

so A must have real eigenvalues if it is normal and Hermitian.

Exercise 2.18: Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Solution: Again by spectral decomposition, we know that all normal matrices (unitary matrices are normal) are diagonalizable so we may write that

$$\left(\sum_i \lambda_i |i\rangle\langle i|\right)\left(\sum_j \lambda_j |j\rangle\langle j|\right)^\dagger = \left(\sum_i \lambda_i |i\rangle\langle i|\right)\left(\sum_j \lambda_j^* |j\rangle\langle j|\right) = I \tag{33}$$

using the result of Exercise 2.14. We then rearrange the terms in this summation to find that

$$\begin{aligned}\left(\sum_i \lambda_i |i\rangle\langle i|\right)\left(\sum_i \lambda_i^* |i\rangle\langle i|\right) &= \sum_{ij} \lambda_i \lambda_j^* \,|i\rangle\, |i\rangle\langle j|\, \langle j| = \sum_{ij} \lambda_i \lambda_j^* \,|i\rangle\, \delta_{ij}\, \langle j| \\ &= \sum_i \lambda_i \lambda_i^* \,|i\rangle\, \langle i| \equiv \sum_i |\lambda_i|^2 \,|i\rangle\, \langle i| = I\end{aligned} \tag{34}$$

and this tells us that $|\lambda_i|^2 = 1 \implies |\lambda_i| = 1$.

Exercise 2.26: Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle\,|1\rangle$, and using the Kronecker product.

Solution: We compute

$$|\psi\rangle^{\otimes 2} = \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}{2} = \begin{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \tag{35}$$

$$|\psi\rangle^{\otimes 3} = |\psi\rangle \otimes |\psi\rangle^{\otimes 2} = \frac{|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle + |0\rangle|1\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|0\rangle|1\rangle + |1\rangle|1\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{2\sqrt{2}} \tag{36}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1/2\sqrt{2} \\ 1/2\sqrt{2} \\ 1/2\sqrt{2} \\ 1/2\sqrt{2} \\ 1/2\sqrt{2} \\ 1/2\sqrt{2} \\ 1/2\sqrt{2} \\ 1/2\sqrt{2} \end{pmatrix} \tag{37}$$

Exercise 2.27: Calculate the matrix representation of the tensor products of the Pauli operators $X \otimes Z, I \otimes X$, and $X \otimes I$. Is the tensor product commutative?

Solution: We compute

$$X \otimes Z = \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{38}$$

$$I \otimes X = \begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{39}$$

$$X \otimes I = \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{40}$$

By looking at the last two results, we may conclude that the tensor product is not commutative.

Exercise 2.29: Show that the tensor product of two unitary matrices is also unitary.

Solution: We compute using the result of Exercise 2.28 and Eq. 2.27 that

$$C^\dagger C \equiv (A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B) = A^\dagger A \otimes B^\dagger B = I \otimes I = I. \tag{41}$$

Exercise 2.30: Show that the tensor product of two Hermitian operators is Hermitian.

Solution: Assume that A and B are Hermitian (that $A^\dagger = A$ and $B^\dagger = B$). The using the result of Exercise 2.28, we compute

$$C^\dagger \equiv (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B \equiv C \implies C^\dagger = C. \tag{42}$$

Exercise 2.32: Show that the tensor product of two projectors is also a projector.

Solution: Let P_1 and P_2 be projectors. From Exercise 2.16, we know that $P^2 = P$. We then compute

$$(P_1 \otimes P_2)^2 = (P_1 \otimes P_2)(P_1 \otimes P_2) = P_1^2 \otimes P_2^2 \equiv P_1 \otimes P_2 \tag{43}$$

which tells us that $P_1 \otimes P_2$ is a projector since it satisfies $P^2 = P$.

Exercise 2.33: The Hadamard operator on one qubit maybe written as

$$H = \frac{1}{\sqrt{2}}\Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\Big]. \tag{44}$$

Show explicitly that the Hadamard transform on n qubits, $H^{\otimes n}$, may be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y| . \tag{45}$$

Write out an explicit matrix representation for $H^{\otimes 2}$.

Solution: We must expand the expression for the Hadamard operator, resulting in

$$H = \frac{1}{\sqrt{2}} \Big[|0\rangle \langle 0| + |1\rangle \langle 0| + |0\rangle \langle 1| - |1\rangle \langle 1| \Big] \tag{46}$$

and notice that this can be compactly written as

$$\frac{1}{\sqrt{2}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y| \tag{47}$$

where $x, y \in \{0, 1\}$. When we tensor up the Hadamard operator over n qubits, we then have that

$$H^{\otimes n} = \underbrace{\frac{1}{\sqrt{2}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y| \otimes \frac{1}{\sqrt{2}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y| \otimes \cdots \otimes \frac{1}{\sqrt{2}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y|}_{n \text{ times}} \tag{48}$$

and this becomes

$$= \frac{1}{(\sqrt{2})^n} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y| \tag{49}$$

where now x, y are length n binary strings.

We then compute using the Kronecker product that

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} . \tag{50}$$

Exercise 2.36: Show that all of the Pauli matrices except for I have trace zero.

Solution: Using the definition of trace,

$$\mathrm{tr}(A) \equiv \sum_i A_{ii} \tag{51}$$

and the definitions of the Pauli matrices, we compute that

$$\begin{aligned} \mathrm{tr}(I) &= 1 + 1 = 2 \\ \mathrm{tr}(X) &= 0 + 0 = 0 \\ \mathrm{tr}(Y) &= 0 + 0 = 0 \\ \mathrm{tr}(Z) &= 1 - 1 = 0. \end{aligned} \tag{52}$$

Exercise 2.38: If A and B are two linear operators, show that

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B) \tag{53}$$

and if z is an arbitrary complex number show that

$$\mathrm{tr}(zA) = z\mathrm{tr}(A). \tag{54}$$

Solution: We compute that

$$\mathrm{tr}(A) + \mathrm{tr}(B) = \sum_i A_{ii} + \sum_i B_{ii} = \sum_i (A + B)_{ii} \equiv \mathrm{tr}(A + B) \tag{55}$$

and

$$\mathrm{tr}(zA) = \sum_i zA_{ii} = z \sum_i A_{ii} \equiv z\mathrm{tr}(A). \tag{56}$$

Exercise 2.40: Verify the commutation relations

$$[X, Y] = 2iZ;\ [Y, Z] = 2iX;\ [Z, X] = 2iY. \tag{57}$$

Solution: We compute that

$$[X, Y] \equiv XY - YX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \equiv 2iZ \tag{58}$$

$$[Y, Z] = YZ - ZY = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \equiv 2iX \tag{59}$$

$$[Z, X] = ZX - XZ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2iY \tag{60}$$

Exercise 2.51: Verify that the Hadamard gate H is unitary.
Solution: H is unitary if $H^\dagger H = I$ so we compute

$$H^\dagger H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv I. \tag{61}$$

Exercise 2.53: What are the eigenvalues and eigenvectors of H?
Solution: We compute that

$$\det(H - \lambda I) = 0 \implies \det\left(\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{pmatrix}\right) = \lambda^2 - 1 = 0 \tag{62}$$

which tells us that $\lambda = \pm 1$. And we compute that

$$H - \lambda_1 I = \begin{pmatrix} \frac{1}{\sqrt{2}} - 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - 1 \end{pmatrix}\begin{pmatrix} v_1 1 \\ v_1 2 \end{pmatrix} = \vec{0} \implies |v_1\rangle = \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{63}$$

$$H - \lambda_2 I = \begin{pmatrix} \frac{1}{\sqrt{2}} + 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} + 1 \end{pmatrix}\begin{pmatrix} v_1 1 \\ v_1 2 \end{pmatrix} = \vec{0} \implies |v_1\rangle = \begin{pmatrix} -1 + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{64}$$

Exercise 2.71: Let ρ be a density operator. Show that $\mathrm{tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.
Solution: We first compute $\mathrm{tr}(\rho)$. We find:

$$\mathrm{tr}(\rho) = \mathrm{tr}\left(\sum_i p_i |i\rangle\langle i|\right) = \sum_i p_i \mathrm{tr}(|i\rangle\langle i|) = \sum_i p_i = 1. \tag{65}$$

We then compute

$$\rho^2 = \sum_i p_i|i\rangle\langle i| \sum_j p_j|j\rangle\langle j| = \sum_{i,j} p_i p_j \,|i\rangle\,\langle i|j\rangle\,\langle j| = \sum_{i,j} p_i p_j \delta_{ij} |i\rangle\langle j| = \sum_i p_i^2 |i\rangle\langle j| \tag{66}$$

which allows us to compute $\mathrm{tr}(\rho^2)$:

$$\mathrm{tr}\left(\sum_i p_i^2 |i\rangle\langle j|\right) = \sum_i p_i^2 \mathrm{tr}(|i\rangle\langle j|) = \sum_i p_i^2. \tag{67}$$

Recall that $p_i \leq 1\ \forall\ i$ since p_i represents the probabilistic weight of each state. This tells us that

$$\sum_i p_i^2 \leq \sum_i p_i = 1 \implies \sum_i p_i^2 \leq 1 \tag{68}$$

Exercise 2.75: For each of the four Bell states, find the reduced density operator for each qubit.

Solution: The Bell states, defined in section 1.3.6 of the text, are given as:

$$
\begin{aligned}
|\beta_{00}\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
|\beta_{01}\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
|\beta_{10}\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
|\beta_{11}\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}
\end{aligned} \tag{69}
$$

Beginning with $|\beta_{00}\rangle$, we first determine the total density operator for the two qubit system:

$$
\rho^{AB} = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2} \tag{70}
$$

We then obtain the reduced density operator for qubit A using the partial trace:

$$
\begin{aligned}
\rho^A &= \mathrm{tr}_B(\rho^{AB}) = |a_1\rangle\langle a_2|\mathrm{tr}(|b_1\rangle\langle b_2|) \\
&= \frac{\mathrm{tr}_B(|00\rangle\langle 00|) + \mathrm{tr}_B(|00\rangle\langle 11|) + \mathrm{tr}_B(|11\rangle\langle 00|) + \mathrm{tr}_B(|11\rangle\langle 11|)}{2} \\
&= \frac{\mathrm{tr}_B((|0\rangle\otimes|0\rangle)(\langle 0|\otimes\langle 0|)) + \mathrm{tr}_B((|0\rangle\otimes|0\rangle)(\langle 1|\otimes\langle 1|)) + \mathrm{tr}_B((|1\rangle\otimes|1\rangle)(\langle 0|\otimes\langle 0|)) + \mathrm{tr}_B((|1\rangle\otimes|1\rangle)(\langle 1|\otimes\langle 1|))}{2} \\
&= \frac{\mathrm{tr}_B(|0\rangle\langle 0|\otimes|0\rangle\langle 0|) + \mathrm{tr}_B(|0\rangle\langle 1|\otimes|0\rangle\langle 1|) + \mathrm{tr}_B(|1\rangle\langle 0|\otimes|1\rangle\langle 0|) + \mathrm{tr}_B(|1\rangle\langle 1|\otimes|1\rangle\langle 1|)}{2} \\
&= \frac{|0\rangle\langle 0|\langle 0|0\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \equiv \frac{I}{2}.
\end{aligned} \tag{71}
$$

The same result will be found for the reduced density operator for qubit B.

Now for $|\beta_{01}\rangle$:

$$
\rho^{AB} = \left(\frac{|01\rangle + |10\rangle}{2}\right)\left(\frac{\langle 01| + \langle 10|}{2}\right) = \frac{|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|}{2} \tag{72}
$$

We find determine the reduced density matrix for qubit A as before:

$$
\begin{aligned}
\rho^A &= \frac{\mathrm{tr}_B(|01\rangle\langle 01|) + \mathrm{tr}_B(|01\rangle\langle 10|) + \mathrm{tr}_B(|10\rangle\langle 01|) + \mathrm{tr}_B(|10\rangle\langle 10|)}{2} \\
&= \frac{\mathrm{tr}_B(|0\rangle\langle 0|\otimes|1\rangle\langle 1|) + \mathrm{tr}_B(|0\rangle\langle 1|\otimes|1\rangle\langle 0|) + \mathrm{tr}_B(|1\rangle\langle 0|\otimes|0\rangle\langle 1|) + \mathrm{tr}_B(|1\rangle\langle 1|\otimes|0\rangle\langle 0|)}{2} \\
&= \frac{|0\rangle\langle 0|\langle 1|1\rangle + |0\rangle\langle 1|\langle 0|1\rangle + |1\rangle\langle 0|\langle 1|0\rangle + |1\rangle\langle 1|\langle 0|0\rangle}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}.
\end{aligned} \tag{73}
$$

Again, the same will be true for the B qubit.

Moving on to $|\beta_{10}\rangle$:

$$
\rho^{AB} = \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| - \langle 11|}{\sqrt{2}}\right) = \frac{|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|}{2} \tag{74}
$$

Then finding the reduced density matrix for qubit A:

$$
\begin{aligned}
\rho^A &= \frac{\mathrm{tr}_B(|00\rangle\langle 00|) - \mathrm{tr}_B(|00\rangle\langle 11|) + \mathrm{tr}_B(|11\rangle\langle 00|) + \mathrm{tr}_B(|11\rangle\langle 11|)}{2} \\
&= \frac{|0\rangle\langle 0|\langle 0|0\rangle - |0\rangle\langle 1|\langle 1|0\rangle - |1\rangle\langle 0|\langle 0|1\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}.
\end{aligned} \tag{75}
$$

The result will again be the same for the B qubit since the only negative terms will always go to zero.

Finally for $|\beta_{11}\rangle$:

$$\rho^{AB} = \left(\frac{|01\rangle - |10\rangle}{2}\right)\left(\frac{\langle 01| - \langle 10|}{2}\right) = \frac{|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|}{2} \tag{76}$$

Computing the reduced density matrix for qubit A:

$$\begin{aligned} \rho^A &= \frac{\mathrm{tr}_B(|01\rangle\langle 01|) - \mathrm{tr}_B(|01\rangle\langle 10|) - \mathrm{tr}_B(|10\rangle\langle 01|) + \mathrm{tr}_B(|10\rangle\langle 10|)}{2} \\ &= \frac{|0\rangle\langle 0|\langle 1|1\rangle - |0\rangle\langle 1|\langle 0|1\rangle - |1\rangle\langle 0|\langle 1|0\rangle + |1\rangle\langle 1|\langle 0|0\rangle}{2} = \frac{I}{2}. \end{aligned} \tag{77}$$

For the same reason as before, the result will be the same for qubit B.

3 Chapter 8: Quantum Noise and Quantum Operations

Exercise 8.1: Pure states evolve under unitary transformations as $|\psi\rangle \to U|\psi\rangle$. Show that, equivalently, we may write $\rho \to \mathcal{E}(\rho) \equiv U\rho U^\dagger$, for $\rho = |\psi\rangle\langle\psi|$.

Solution: Recall that $\langle\psi| \equiv |\psi\rangle^\dagger$. Since we are given that $|\psi\rangle \to U|\psi\rangle$ we can compute:

$$|\psi\rangle\langle\psi| \equiv |\psi\rangle\,|\psi\rangle^\dagger \to U|\psi\rangle\left(U^\dagger\,|\psi\rangle^\dagger\right) \equiv U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger. \tag{78}$$

Exercise 8.2: Recall from Section 2.2.3 (on page 84) that a quantum measurement with outcomes labeled m is described by a set of measurement operators M_m such that $\sum_m M_m^\dagger M_m = I$. Let the state of the system immediately before the measurement be ρ. Show that for $\mathcal{E}(\rho) = M_m\rho M_m^\dagger$, the state of the system immediately after the measurement is

$$\frac{\mathcal{E}(\rho)}{\mathrm{tr}(\mathcal{E}(\rho))}.$$

Also show that the probability of obtaining this measurement result is $p(m) = \mathrm{tr}(\mathcal{E}(\rho)$.

Solution: Recall from Section 2.4.1 that the density operator can be defined as

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{79}$$

Where we know $|\psi\rangle$ will transform as

$$\frac{M_m\,|\psi_i\rangle}{\sqrt{\langle\psi_i|\,M_m^\dagger M_m\,|\psi_i\rangle}} \tag{80}$$

and $\langle\psi_i|$ will transform as

$$\frac{\langle\psi_i|\,M_m^\dagger}{\sqrt{\langle\psi_i|\,M_m M_m^\dagger\,|\psi_i\rangle}} \tag{81}$$

after measurement from Section 2.2.3. Then we know the whole density operator will transform as

$$\sum_i p_i \frac{M_m\,|\psi_i\rangle}{\sqrt{\langle\psi_i|\,M_m^\dagger M_m\,|\psi_i\rangle}}\frac{\langle\psi_i|\,M_m^\dagger}{\sqrt{\langle\psi_i|\,M_m M_m^\dagger\,|\psi_i\rangle}} \tag{82}$$

and we know the denominator of this expression is $\mathrm{tr}(M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger)$ from Section 2.1.8. We therefore have

$$\sum_i p_i \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\mathrm{tr}(M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger)} = \frac{M_m\rho M_m^\dagger}{\mathrm{tr}(M_m\rho M_m^\dagger)} \equiv \frac{\mathcal{E}(\rho)}{\mathrm{tr}(\mathcal{E}(\rho))}. \tag{83}$$

Exercise 8.4: Suppose we have a single qubit principle system, interacting with a single qubit environment through the transform

$$U = P_0 \otimes I + P_1 \otimes X, \tag{84}$$

where X is the usual Pauli matrix (acting on the environment), and $P_0 \equiv |0\rangle\langle 0|$, $P_1 \equiv |1\rangle\langle 1|$ are projectors (acting on the system). Give the quantum operation for this process, in the operator-sum representation, assuming the environment starts in the state $|0\rangle$.

Solution: Using the operator-sum representation with the fact that $|e_0\rangle = |0\rangle$, we compute

$$\begin{aligned} E_k &= \langle e_k| U |e_0\rangle = \langle 0| (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) |0\rangle = \langle 1| (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) |0\rangle \\ &= \langle 0| \otimes I |0\rangle + \langle 1| \otimes X |0\rangle = \langle 0| \otimes |0\rangle + \langle 1| \otimes |1\rangle = |0\rangle\langle 0| + |1\rangle\langle 1|. \end{aligned} \tag{85}$$

Since we have found that $E_k = P_k$, we have found that

$$\mathcal{E}(\rho) = E_0 \rho E^\dagger \rho + E_1 \rho E_1^\dagger = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|. \tag{86}$$