A First Course in Loop Quantum Gravity - Selected Solutions

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1 Chapter 2: Special Relativity and Electromagnetism

Exercise (2.4): Show that $\epsilon^{\sigma\alpha\beta\gamma}\partial_\gamma F_{\alpha\beta} = 0$ is equivalent to half of the Maxwell equations.

Solution: With the knowledge that the Levi-Civita symbol, $\epsilon^{\sigma\alpha\beta\gamma}$, is equal to one only when there is an even permutation in the $\sigma\alpha\beta\gamma$ indices, this becomes:

$$\epsilon^{\sigma\alpha\beta\gamma} = \begin{cases} 1 & \text{if } \sigma\alpha\beta\gamma \text{ is an even permutation} \\ 0 & \text{repeated indices} \\ -1 & \text{else} \end{cases} \tag{1}$$

in four dimensions where $0 \to t$, $1 \to x$, $2 \to y$, and $3 \to z$. We can begin to arrive half of the Maxwell equations by plugging values into the σ index of $\epsilon^{\sigma\alpha\beta\gamma}$ since this restrict the values of the other indices that do not result in zero. Starting with $\sigma = 0$:

$$\begin{aligned} &\epsilon^{0\alpha\beta\gamma}\partial_\gamma F_{\alpha\beta} = 0 \\ \implies &\epsilon^{0123}\partial_3 F_{12} + \epsilon^{0231}\partial_1 F_{23} + \epsilon^{0312}\partial_2 F_{31} + \epsilon^{0321}\partial_1 F_{32} + \epsilon^{0213}\partial_3 F_{21} + \epsilon^{0132}\partial_2 F_{13} = 0 \\ \implies &\partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} - \partial_1 F_{32} - \partial_3 F_{21} - \partial_2 F_{13} = 0 \implies \end{aligned} \tag{2}$$

And since $F_{\alpha\beta}$ is defined as the field equation below:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{3}$$

Eq. (2) becomes:

$$\begin{aligned} &2\partial_1 B_1 + 2\partial_2 B_2 + 2\partial_3 B_3 = 0 \\ \implies &\partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \\ \implies &\nabla \cdot \vec{B} = 0 \end{aligned} \tag{4}$$