A First Course in General Relativity - Selected Solutions

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1 Chapter 2: Vector Analysis in Special Relativity

Exercise 2.3: Prove Eq. (2.5)

Solution: By convention, Latin indices are not summed over 0 so if we are to interchange them with Greek indices as a dummy index, we must perform the following:

$$\Lambda^{\bar{\alpha}}_{\beta}\Delta x^{\beta} = \Lambda^{\bar{\alpha}}_{0}\Delta x^{0} + \Lambda^{\bar{\alpha}}_{i}\Delta x^{i} \tag{1}$$

since $\Lambda^{\bar{\alpha}}_{\beta}\Delta x^{\beta}$ implies a sum over all of the positive real numbers where $\Lambda^{\bar{\alpha}}_{i}\Delta x^{i}$ is a sum over positive real numbers not including zero.

Exercise 2.7a: Prove Eq. (2.10) for all α, β

Solution: To verify that $(\vec{e}_\alpha)^\beta = \delta^\beta_\alpha$, consider an arbitrary basis vector, $\vec{e}_\alpha$, meaning that the elements in its list are all zero except for the single entry at the αth component. This can be written as:

$$\vec{e}_\alpha = (..., 0, 0, 1, 0, ...) \tag{2}$$

Where the index of each value in the list can be traced with respect to α

$$(\alpha - n, ..., \alpha - 2, \alpha - 1, \alpha, \alpha + 1, ..., \alpha + n) \tag{3}$$

Then $(\vec{e}_\alpha)^\beta$ indicates the βth component of the basis vector $\vec{e}_\alpha$. By the definition of a basis vector, we know that all entries in $\vec{e}_\alpha$ are zero except the one at the αth component. So if we choose β to be any non-α index, the result must be 0:

$$(\vec{e}_\alpha)^{\alpha-1} = 0 \tag{4}$$

It's for this reason that we can define the βth component of the $\vec{e}_\alpha$ basis vector to be equal to the Kronecker delta, meaning that $(\vec{e}_\alpha)^\beta = 1$ only when $\alpha = \beta$.

Exercise 2.29: Prove, using component expressions, Eqs. (2.24) and (2.26), that

$$\frac{d}{d\tau}(\vec{U}\cdot\vec{U}) = 2\vec{U}\cdot\frac{d\vec{U}}{d\tau} \tag{5}$$

Solution: By (2.26):

$$\begin{aligned}\vec{U}\cdot\vec{U} &= -U^0U^0 + U^1U^1 + U^2U^2 + U^3U^3 \\ &= -(U^0)^2 + (U^1)^2 + (U^2)^2 + (U^3)^2\end{aligned} \tag{6}$$

and by (2.24):

$$\vec{U}\cdot\vec{U} = \vec{U}^2 \implies \frac{d}{d\tau}(\vec{U}\cdot\vec{U}) = \frac{d}{d\tau}(\vec{U}^2) = 2\vec{U}\cdot\frac{d\vec{U}}{d\tau} \tag{7}$$

2 Chapter 3: Tensor Analysis in Special Relativity

Exercise (3.1a): Given an arbitrary set of numbers $\{M_{\alpha\beta}; \alpha = 0, ..., 3; \beta = 0, ..., 3\}$ and two arbitrary vector components $\{A^\mu, \mu = 0, ..., 3\}$ and $\{B^\nu, \nu = 0, ..., 3\}$, show that the two expressions

$$M_{\alpha\beta}A^\alpha B^\beta \tag{8}$$

and

$$M_{\alpha\alpha}A^\alpha B^\alpha \tag{9}$$

are not equivalent.

Solution:

$$M_{\alpha\alpha}A^\alpha B^\alpha = M_{00}A^0B^0 + M_{11}A^1B^1 + M_1A^1B^1 + M_{11}A^1B^1 \tag{10}$$

where

$$M_{\alpha\beta}A^\alpha B^\beta = B^\beta(M_{0\beta}A^0 + M_{1\beta}A^1 + M_{2\beta}A^2 + M_{3\beta}A^3) \tag{11}$$

So $M_{\alpha\alpha}A^\alpha B^\alpha$ only contains the diagonal terms of $M_{\alpha\beta}A^\alpha B^\beta$.

Exercise (3.1b): Show that $A^\alpha B^\beta \eta_{\alpha\beta} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$

Solution:

Because

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

Any component of $A^\alpha B^\beta \eta_{\alpha\beta}$ where $\alpha \neq \beta$ means multiplying $A^\alpha B^\beta$ by an off-diagonal component of $\eta_{\alpha\beta}$, which are all 0. Treating A^α and B^β as row/column matrices and carrying out their multiplication with $\eta_{\alpha\beta}$ will result in $-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$.

Exercise (3.3a): Prove, by writing out all of the terms, the validity of the following:

$$\tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha) \tag{13}$$

Solution:

Since one-forms act on vector arguments, the scalar values associated with A^α may be pulled out of the expression like so:

$$\begin{aligned} \tilde{p}(A^\alpha \vec{e}_\alpha) = \tilde{p}(A^0\vec{e}^0 + A^1\vec{e}^1 + A^2\vec{e}^2 + A^3\vec{e}^3) &= \tilde{p}(A^0\vec{e}_0) + \tilde{p}(A^1\vec{e}_1) + \tilde{p}(A^2\vec{e}_2) + \tilde{p}(A^3\vec{e}_3) \\ &= A^0\tilde{p}(\vec{e}_0) + A^1\tilde{p}(\vec{e}_1) + = A^2\tilde{p}(\vec{e}_2) + A^3\tilde{p}(\vec{e}_3) = A^\alpha\tilde{p}(\vec{e}_\alpha) \end{aligned} \tag{14}$$

Exercise 3.5: Justify each step leading from Eqs. (3.10a) to (3.10d).

Solution: To establish the frame-independence of $A^{\bar{\alpha}}p_{\bar{\alpha}}$:

$$\begin{aligned} A^{\bar{\alpha}}p_{\bar{\alpha}} &= A^{\bar{\alpha}}\tilde{p}(\vec{e}_{\bar{\alpha}}), \\ \vec{e}_{\bar{\alpha}} &= \Lambda^\mu_{\bar{\alpha}}\vec{e}_\mu, \\ A^{\bar{\alpha}} &= \Lambda^{\bar{\alpha}}_\beta A^\beta \\ \implies A^{\bar{\alpha}}\tilde{p}(\vec{e}_{\bar{\alpha}}) = \Lambda^{\bar{\alpha}}_\beta A^\beta \tilde{p}(\Lambda^\mu_{\bar{\alpha}}\vec{e}_\mu) = \Lambda^{\bar{\alpha}}_\beta \Lambda^\mu_{\bar{\alpha}} A^\beta \tilde{p}(\vec{e}_\mu) &= \Lambda^{\bar{\alpha}}_\beta \Lambda^\mu_{\bar{\alpha}} A^\beta p_\mu \end{aligned} \tag{15}$$

and by Eq. (2.18):

$$\Lambda^{\bar{\alpha}}_\beta \Lambda^\mu_{\bar{\alpha}} A^\beta p_\mu = \delta^\mu_\beta A^\beta p_\mu = A^\beta p_\beta \implies \boxed{A^{\bar{\alpha}}p_{\bar{\alpha}} = A^\beta p_\beta} \tag{16}$$

which should not be a surprising result given that the one-form of a vector produces a scalar and scalars are invariant quantities under Lorentz transformations.

Exercise (3.10a): Given a frame $\mathcal{O}$ whose coordinates are $\{x^\alpha\}$, show that:

$$\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta \tag{17}$$

Solution: Taking the partial with respect to x^β means holding all terms in x constant except for the β index. When $\alpha \neq \beta$, you are taking a partial of a fixed value (a constant), meaning your derivative will be equal to zero. When you differentiate the β index with respect to β, you will always get 1.

Exercise (3.10b): For any two frames, we have Eq. (3.18):

$$\frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda^\beta_{\bar{\alpha}}. \tag{18}$$

Show that (a) and the chain rule imply

$$\Lambda^{\beta}_{\bar{\alpha}}\Lambda^{\bar{\alpha}}_{\mu} = \delta^{\beta}_{\mu} \tag{19}$$

Solution:

$$\begin{aligned}
\Lambda^{\beta}_{\bar{\alpha}} &= \frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}}, \\
\Lambda^{\bar{\alpha}}_{\mu} &= \frac{\partial x^{\bar{\alpha}}}{\partial x^{\mu}} \\
\implies \Lambda^{\beta}_{\bar{\alpha}}\Lambda^{\bar{\alpha}}_{\mu} &= \frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}}\frac{\partial x^{\bar{\alpha}}}{\partial x^{\mu}} = \frac{\partial x^{\beta}}{\partial x^{\mu}} = \delta^{\beta}_{\mu}
\end{aligned} \tag{20}$$

Exercise (3.11a): Use the notation $\partial\phi/\partial x^{\alpha} = \phi_{,\alpha}$ to rewrite Eqs. (3.14), (3.15), and (3.18).

Solution:
Eq. (3.14):

$$\frac{\partial\phi}{\partial t}U^t + \frac{\partial\phi}{\partial t}U^x + \frac{\partial\phi}{\partial t}U^y + \frac{\partial\phi}{\partial t}U^z \implies \phi_{,t}U^t + \phi_{,x}U^x + \phi_{,y}U^y + \phi_{,z}U^z \tag{21}$$

Eq. (3.15):

$$\tilde{d}\phi \underset{\mathcal{O}}{\rightarrow} \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \implies \tilde{d}\phi \underset{\mathcal{O}}{\rightarrow} (\phi_{,t}, \phi_{,x}, \phi_{,y}, \phi_{,z}) \tag{22}$$

Eq. (3.18):

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} = \Lambda^{\beta}_{\bar{\alpha}} \implies x^{\beta}_{,\bar{\alpha}} = \Lambda^{\beta}_{\bar{\alpha}} \tag{23}$$

Exercise (3.13): Prove, by geometric or algebraic arguments, that $\tilde{d}f$ is normal to surfaces of constant f.

Solution: If we consider any point $P = (t_0, x_0, y_0, z_0)$ along a parameterized level curve $f(\tau) = \phi(t(\tau), x(\tau), y(\tau), z(\tau) = c$, then we can take the gradient to be as follows:

$$\tilde{d}f = \partial\phi_{,t}\Big|_P \frac{dt}{d\tau}\Big|_{\tau_0} + \partial\phi_{,x}\Big|_P \frac{dx}{d\tau}\Big|_{\tau_0} + \partial\phi_{,y}\Big|_P \frac{dy}{d\tau}\Big|_{\tau_0} + \partial\phi_{,z}\Big|_P \frac{dz}{d\tau}\Big|_{\tau_0} = 0 \tag{24}$$

Since this is also the definition of the dot product between two vectors:

$$\left\langle \partial\phi_{,t}\Big|_P, \partial\phi_{,x}\Big|_P, \partial\phi_{,y}\Big|_P, \partial\phi_{,z}\Big|_P \right\rangle \cdot \left\langle \frac{dt}{d\tau}\Big|_{\tau_0}, \frac{dx}{d\tau}\Big|_{\tau_0}, \frac{dy}{d\tau}\Big|_{\tau_0}, \frac{dz}{d\tau}\Big|_{\tau_0} \right\rangle = 0 \tag{25}$$

whose product is equal to zero, we can say that $\tilde{d}f$ and f are normal to each other at every point along a level curve.

Exercise (3.14): Let $\tilde{p} \rightarrow_{\mathcal{O}} (1, 1, 0, 0)$ and $\tilde{q} \rightarrow_{\mathcal{O}} (-1, 0, 1, 0)$ be two one-forms. Prove, by trying two vectors $\vec{A}$ and $\vec{B}$ as arguments, that $\tilde{p} \otimes q \neq \tilde{\tilde{q}} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.

Solution: Since a one-form supplied with a vector argument: $\tilde{p}(\vec{A}) = p^{\alpha}A^{\alpha}$, we can perform the following operations to show $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$:

$$\begin{aligned}
\tilde{p} \otimes \tilde{q} = \tilde{p}(\vec{A})\tilde{q}(\vec{B}) = (A^0 + A^1)(-B^0 + B^2) \\
\tilde{q} \otimes \tilde{p} = \tilde{q}(\vec{A})\tilde{p}(\vec{B}) = (-A^0 + A^2)(B^0 + B^1)
\end{aligned} \tag{26}$$

Exercise (3.16a): Prove that $\mathbf{h}_{(s)}$ defined by

$$\mathbf{h}_{(s)}(\vec{A}, \vec{B}) = \frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) + \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) \tag{27}$$

is a symmetric tensor.

Solution: From Eq. (3.27), we know a tensor $\mathbf{f}$ is symmetric if:

$$\mathbf{f}(\vec{A}, \vec{B}) = \mathbf{f}(\vec{B}, \vec{A}) \forall \vec{A}, \vec{B} \tag{28}$$

So if $\mathbf{h}$ is to be symmetric, then:

$$\frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) - \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) = 0 \tag{29}$$

Let $\vec{A} = (A_0, A_1, A_2, A_3)$ and $\vec{B} = (B_0, B_1, B_2, B_3)$ then:

$$\frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) = \frac{1}{2}(A_0 B_0 + A_1 B_1 + A_2 B_2 + A_3 B_3) \tag{30}$$

and

$$\frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) = \frac{1}{2}(B_0 A_0 + B_1 A_0 + B_2 A_2 + B_3 A_3) \tag{31}$$

and since multiplication is commutative, Eq. 21 must be true, meaning that $\mathbf{h}_{(s)}(\vec{A}, \vec{B})$ is a symmetric tensor.

3 Chapter 5: Preface to Curvature

Exercise (5.3a): Show that the coordinate transformation $(x, y) \to (\xi, \eta)$ with $\xi = x$ and $\eta = 1$ violates Eq. (5.6).

Solution: For a transformation to be reasonable, it must assign all coordinates in the source (x, y) to distinct coordinates in the target (ξ, η). This property will be satisfied if the Jacobian is non-zero, which is the definition given by Eq. (5.6). So to show this transformation is not reasonable, it must be shown to violate Eq. (5.6):

$$\det \begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \tag{32}$$

So this transformation of coordinates is not reasonable.

Exercise (5.7): Calculate all elements of the transformation matrices $\Lambda^{\alpha'}_{\beta}$ and $\Lambda^{\nu'}_{\mu}$ for the transformation from Cartesian (x, y) - the unprimed indices - to polar (r, θ) - the primed indices.

Solution: Since, by Eq. (5.8):

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} \tag{33}$$

We can directly compute the transformation from Cartesian into Polar components by computing the terms of this matrix (knowing that $\xi(x, y) = r = \sqrt{x^2 + y^2}$ and $\eta(x, y) = \theta = \arctan(y/x)$ in polar coordinates). This results in:

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \tag{34}$$

Since $\Lambda^{\nu'}_{\mu}$ is defined as:

$$\Lambda^{\mu}_{\nu'} = \begin{pmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{pmatrix} = \begin{pmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{pmatrix} \tag{35}$$

in Eq. (5.13), the matrix is the following:

$$\Lambda^{\mu}_{\nu'} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \tag{36}$$

Exercise (5.8a): (Use the result of Exer.7.) Let $f = x^2 + y^2 + 2xy$ and in Cartesian Coordinates $\vec{V} \to (x^2 + 3y, y^2 + 3x)$, $\vec{W} \to (1, 1)$. Compute f as a function of r and θ, and find the components of $\vec{V}$ and $\vec{W}$ on the polar basis, expressing them as functions of r and θ.

Solution: Expressing f as a polar function is as simple as making the substitutions $x = r\cos\theta$ and $y = \sin\theta$, arriving at $f = r^2 + 2r^2\cos\theta\sin\theta$. To express $\vec{V}$ and $\vec{W}$ as polar functions, the same process can be applied. This results in $\vec{V} = (r^2\cos^2\theta + 3r\sin\theta, r^2\sin^2\theta + 3r\cos\theta)$ and $\vec{W} = (1, 1)$. To express $\vec{V}$ and $\vec{W}$ in a polar *basis*, though, you must use the transformations found in the previous problem:

$$V^{\alpha'} = \Lambda^{\alpha'}_{\beta} V^{\beta} \implies \vec{V} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix} = \begin{pmatrix} r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta \\ r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta - \sin^2\theta) \end{pmatrix} \tag{37}$$

$$\vec{W} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ (\cos\theta - \sin\theta)/r \end{pmatrix} \tag{38}$$

Exercise (5.8b): Find the components of $\tilde{d}f$ in Cartesian Coordinates and obtain them in polars (i) by direct calculation in polars, and (ii) by transforming components from Cartesian.

Solution: (i) To compute by direct calculation in polar: $\tilde{d}f = (\partial f/\partial r, \partial f/\partial \theta)$ we can use the definition of f in polar that was derived in part (a):

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left(r^2 + 2r^2\cos\theta\sin\theta\right) = 2r + 4r\cos\theta\sin\theta \tag{39}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta}\left(r^2 + 2r^2\cos\theta\sin\theta\right) = 2r^2\cos(2\theta) \tag{40}$$

(ii) To compute $\tilde{d}f$ by transforming components from Cartesian,

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial \phi}{\partial y} \implies \frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} \tag{41}$$

$$\frac{\partial f}{\partial r} = \cos\theta(2x+2y) + \sin\theta(2x+2y) = (\cos\theta + \sin\theta)(2r\cos\theta + 2r\sin\theta) = 2r + 4r\sin\theta\cos\theta \tag{42}$$

Similarly:

$$\frac{\partial f}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y} \tag{43}$$

$$\frac{\partial f}{\partial \theta} = (-r\sin\theta)(2x+2y) + (r\cos\theta)(2x+2y) = 2r^2\cos\theta \tag{44}$$

It should be noted that the expressions from (ii) match those derived from (i).

Exercise (5.8c): (i) Use the metric tensor in polar coordinates to find the polar components of the one-forms $\tilde{V}$ and $\tilde{W}$ associated with $\vec{V}$ and $\vec{W}$. (ii) Obtain the polar components of $\tilde{V}$ and $\tilde{W}$ by transformation of their Cartesian components.

Solution: (i) By Eq. (5.31), the metric tensor in polar coordinates is:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{45}$$

The metric in polar coordinates can be used to find the polar components of the one-forms by:

$$\tilde{W}_\alpha = g_{\alpha\beta}W^\beta = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}\begin{pmatrix} \cos\theta + \sin\theta \\ (\cos\theta - \sin\theta)/r \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ r(\cos\theta - \sin\theta) \end{pmatrix} \tag{46}$$

The same can be done for $\vec{V}$ as computed in polar form from part (a) of this problem.

(ii) Using the transformation matrix $\Lambda^\alpha_{\beta'}$ to obtain $\tilde{V}$ and $\tilde{W}$:

$$\Lambda^\alpha_{\beta'}\tilde{W}^\alpha = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ r(\cos\theta - \sin\theta) \end{pmatrix} \tag{47}$$

And the same process can be employed to solve for $\tilde{V}$. Note that $\tilde{V}$ and $\tilde{W}$ is just $\vec{V}$ and $\vec{W}$ in Cartesian coordinates since the metric tensor in Cartesian coordinates is the identity matrix.

Exercise (5.11a): For the vector field $\vec{V}$ whose Cartesian components are (x^2+3y, y^2+3x), compute $V^\alpha_{,\beta}$ in Cartesian.

Solution: Since $V^\alpha_{,\beta} \equiv \partial V^\alpha/\partial x^\beta$:

$$V^\alpha_{,\beta} = \begin{pmatrix} \partial V^1/\partial x & \partial V^1/\partial y \\ \partial V^2/\partial x & \partial V^2/\partial y \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \tag{48}$$

Exercise (5.11b): Compute the transformation $\Lambda^{\mu'}_\alpha \Lambda^\beta_{\nu'} V^\alpha_{,\beta}$ to polars.

This computation is a straightforward usage of the transformation matrices $\Lambda^{\mu'}_\alpha$ and $\Lambda^\beta_{\nu'}$ from Cartesian to polar coordinates derived in Exercise 5.7 and the polar form of $V^\alpha_{,\beta}$ found in the previous part to this problem. The order of multiplication for these matrices should be noted, however, since computing $\Lambda^{\mu'}_\alpha \Lambda^\beta_{\nu'} V^\alpha_{,\beta}$ would leave $V^\alpha_{,\beta}$ unchanged. Computing $\Lambda^{\mu'}_\alpha V^\alpha_{,\beta} \Lambda^\beta_{\nu'}$ results in:

$$\begin{aligned} \Lambda^{\mu'}_\alpha V^\alpha_{,\beta}\Lambda^\beta_{\nu'} &= \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 2(r\cos^3\theta + 3\cos\theta\sin\theta + r\sin^3\theta) - r(\cos\theta - \sin\theta) & (-3\sin\theta + \cos\theta)(-3 + 2r\sin\theta) \\ \frac{(\cos\theta - \sin\theta)(3\cos\theta + 3\sin\theta - r\sin(2\theta))}{r} & (-3 + r\cos\theta + r\sin\theta)\sin(2\theta) \end{pmatrix} \end{aligned} \tag{49}$$

Exercise (5.11c): Compute the components $V^{\mu'}_{;\nu'}$ directly in polars using the Christoffel symbols.

Solution: Since $\alpha, \beta \in \{x, y\}$ in Cartesian coordinates, there will be four components to compute: $V^r_{;r}, V^\theta_{;r}, V^r_{;\theta}, V^\theta_{;\theta}$. Beginning with $V^r_{;r}$:

$$V^r_{;r} = \frac{\partial V^r}{\partial r} + V^\mu \Gamma^r_{\mu r} \ \& \ \Gamma^\mu_{rr} = \forall \mu \implies V^r_{;r} = \frac{\partial V^r}{\partial r} + V^\theta \Gamma^r_{\theta r}$$
$$\Gamma^r_{\theta r} : \frac{\partial \vec{e_\theta}}{\partial r} = -\sin\theta \vec{e_x} + \cos\theta \vec{e_y} = \frac{1}{r}\vec{e_\theta} \implies \Gamma^r_{\theta r} = 0 \tag{50}$$
$$\implies V^r_{;r} = \frac{\partial V^r}{\partial r} = \frac{\partial}{\partial r}\left(r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta\right) = 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta$$

$$V^\theta_{;r} = \frac{\partial V^\theta}{\partial r} + V^\mu \Gamma^\theta_{\mu r} = \frac{\partial V^\theta}{\partial r} + \frac{1}{r}V^\theta =$$
$$\frac{\partial}{\partial r}\left(r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta - \sin^2\theta)\right) + \frac{1}{r}\left(r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta - \sin^2\theta)\right) \tag{51}$$
$$\implies V^\theta_{;r} = \frac{(\cos\theta - \sin\theta)(3\cos\theta + 3\sin\theta) - r\sin(2\theta)}{r}$$

$$V^r_{;\theta} = \frac{\partial V^r}{\partial \theta} + V^\mu \Gamma^r_{\mu\theta} = \frac{\partial V^r}{\partial \theta} - rV^\theta$$
$$\implies V^r_{;\theta} = -r(\cos\theta - \sin\theta)(-3\cos\theta + 3\sin\theta) + r\sin(2\theta) \tag{52}$$

$$V^\theta_{;\theta} = \frac{\partial V^\theta}{\partial \theta} + \frac{1}{r}V^r = \sin(2\theta)(-3 + r\cos\theta + r\sin\theta)) \tag{53}$$

Exercise (5.11d): Compute the divergence $V^\alpha_{,\alpha}$ using results from part (a).
Solution:

$$V^\alpha_{,\alpha} = \frac{\partial V^\alpha}{\partial x^\alpha} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} = 2(x+y) = 2r(\cos\theta + \sin\theta) \tag{54}$$

Exercise (5.11e): Compute the divergence $V^{\mu'}_{;\mu'}$ using results from either part (b) or (c).
Solution:

$$V^{\mu'}_{;\mu'} = V^r_{;r} + V^\theta_{;\theta} = \frac{\partial V^r}{\partial r} + \Gamma^r_{rr}V^r + \Gamma^r_{\theta r}V^\theta + \frac{\partial V^r}{\partial r} + \Gamma^\theta_{\theta\theta}V^\theta + \Gamma^\theta_{r\theta}V^r$$
$$= \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta} + \frac{1}{r}V^r = 2r(\cos\theta + \sin\theta) \tag{55}$$

Exercise (5.11f): Compute the divergence $V^{\mu'}_{;\mu'}$ using Eq. (5.55) directly.

$$V^{\mu'}_{;\mu'} = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial \theta}V^\theta = 2r(\cos\theta + \sin\theta) \tag{56}$$

(This and the majority of the results given for Exercise 5.11 were computed in Mathematica)

Exercise (5.12a): For the one-form field $\tilde{p}$ whose Cartesian coordinates are $(x^2 + 3y, y^2 + 3x)$, compute $p_{\alpha,\beta}$ in Cartesian.
Solution:

$$p_{\alpha,\beta} = \begin{pmatrix} p_{rr} & p_{r\theta} \\ p_{\theta r} & p_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix} \tag{57}$$

Exercise (5.12b): Compute the transformation $\Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'}p_{\alpha,\beta}$ to polars.
Solution:

$$\Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'}p_{\alpha,\beta} = (\Lambda^\alpha_{\mu'})^T p_{\alpha,\beta}\Lambda^\beta_{\nu'} = \left(2r(\cos^3\theta - 3\cos\theta\sin\theta + r^2\sin^3\theta) \quad\right) \tag{58}$$

Exercise (5.12c): Compute the components $p_{\mu';\nu'}$ directly in polars using the Christoffel symbols, Eq. (4.44), in Eq. (5.62).
Solution:

$$p_{r;r} = p_{r,r} - p_\mu \Gamma^\mu_{\alpha\beta} = \frac{\partial p_r}{\partial r} - p_r\Gamma^r_{rr} - p_\theta\Gamma^\theta_{rr} \implies p_{r;r} = \frac{\partial p_r}{\partial r} \tag{59}$$

Where p_r is the r-component of the one-form in a polar basis.

$$p_r = r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta \implies p_{r;r} = \frac{\partial p_r}{\partial r} = 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta \tag{60}$$

$$p_{r;\theta} = \frac{\partial p_r}{\partial \theta} - p_r \Gamma^r_{r\theta} - p_\theta \Gamma^\theta_{r\theta} = \frac{\partial p_r}{\partial \theta} - \frac{1}{r} p_\theta \tag{61}$$

Exercise (5.14): For the tensor whose polar coordinates are $(A^{rr} = r^2, A^{r\theta} = r\sin\theta, A^{\theta r} = r\cos\theta, A^{\theta\theta} = \tan\theta)$, compute in Eq. (5.65) in polars for all possible indices:

$$\begin{aligned} \nabla_r A^{rr} = \frac{\partial A^{rr}}{\partial r} + A^{\alpha r}\Lambda^r_{\alpha r} + A^{r\alpha}\Lambda^r_{\alpha r} = \frac{\partial A^{rr}}{\partial r} + A^{rr}\Lambda^r_{rr} + A^{\theta r}\Lambda^r_{\theta r} + A^{rr}\Lambda^r_{rr} + A^{r\theta}\Lambda^r_{\theta r} \\ \implies \nabla_r A^{rr} = \frac{\partial A^{rr}}{\partial r} = \frac{\partial}{\partial r}\left(r^2\right) = 2r \end{aligned} \tag{62}$$

$$\begin{aligned} \nabla_\theta A^{rr} = \frac{\partial A^{rr}}{\partial \theta} + A^{rr}\Lambda^r_{r\theta} + A^{\theta r}\Lambda^r_{\theta\theta} + A^{rr}\Lambda^r_{r\theta} + A^{r\theta}\Lambda^r_{\theta\theta} \\ \implies \nabla_\theta A^{rr} = -r\left(A^{\theta r} + A^{r\theta}\right) + \frac{\partial A^{rr}}{\partial \theta} = -r(r\cos\theta + r\sin\theta) + \frac{\partial}{\partial \theta}\left(r^2\right) = -r^2(\cos\theta + \sin\theta) \end{aligned} \tag{63}$$

$$\begin{aligned} \nabla_\theta A^{r\theta} = \frac{\partial A^{r\theta}}{\partial \theta} + A^{r\theta}\Gamma^r_{r\theta} + A^{\theta\theta}\Gamma^r_{\theta\theta} + A^{rr}\Gamma^\theta_{r\theta} + A^{r\theta}\Gamma^\theta_{\theta\theta} = \frac{\partial A^{r\theta}}{\partial \theta} - r(A^{\theta\theta}) + \frac{1}{r}(A^{rr}) \\ \implies \nabla_\theta A^{r\theta} = r(\cos\theta - \tan\theta - 1) \end{aligned} \tag{64}$$

And the five remaining computations for all possible indices $(\nabla_r A^{r\theta}, \nabla_r A^{\theta r}, \nabla_\theta A^{\theta r}, \nabla_r A^{\theta\theta}, \nabla_\theta A^{\theta\theta})$ can be computed in exactly the same manner.

Exercise (5.16): Fill in all the missing steps leading from Eq. (5.74) to Eq. (5.75).

Solution: Starting with Eq. (5.72):

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^\nu_{\alpha\mu} g_{\nu\beta} - \Gamma^\nu_{\beta\mu} g_{\alpha\nu} \tag{65}$$

And using the fact that $g_{\alpha'\mu';\beta'} = 0$:

$$g_{\alpha'\beta',\mu'} = \Gamma^{\nu'}_{\alpha'\mu'} g_{\nu'\beta'} + \Gamma^{\nu'}_{\beta'\mu'} g_{\alpha'\nu'} \implies g_{\alpha\beta,\mu} = \Gamma^\nu_{\alpha\mu} g_{\nu\beta} + \Gamma^\nu_{\beta\mu} g_{\alpha\nu} \tag{66}$$

And since α, β, μ are dummy indices whose order can be rearranged in the previous expression, the following form can be arrived at by switching the β and μ indices:

$$g_{\alpha\mu,\beta} = \Gamma^\nu_{\alpha\beta} g_{\nu\mu} + \Gamma^\nu_{\mu\beta} g_{\alpha\nu} \tag{67}$$

And the following expression can be arrived at by switching α with β in Eq. (67) and multiplying the whole expression by a negative sign:

$$g_{\beta\mu,\alpha} = \Gamma^\nu_{\beta\alpha} g_{\nu\mu} + \Gamma^\nu_{\mu\alpha} g_{\beta\nu} \implies -g_{\beta\mu,\alpha} = -\Gamma^\nu_{\beta\alpha} g_{\nu\mu} - \Gamma^\nu_{\mu\alpha} g_{\beta\nu} \tag{68}$$

We can now consider the addition of the three terms, $g_{\alpha\beta,\mu}, g_{\alpha\mu,\beta}, -g_{\beta\mu,\alpha}$:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \Gamma^\nu_{\alpha\mu} g_{\nu\beta} + \Gamma^\nu_{\beta\mu} g_{\alpha\nu} + \Gamma^\nu_{\alpha\beta} g_{\nu\mu} + \Gamma^\nu_{\mu\beta} g_{\alpha\nu} - \Gamma^\nu_{\beta\alpha} g_{\nu\mu} - \Gamma^\nu_{\mu\alpha} g_{\beta\nu} \tag{69}$$

And, using the fact that the indices of the metric can be interchanged ($g_{\beta\nu} = g_{\nu\beta}$), we arrive at:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \left(\Gamma^\nu_{\alpha\mu} - \Gamma^\nu_{\mu\alpha}\right) g_{\nu\beta} + \left(\Gamma^\nu_{\alpha\beta} - \Gamma^\nu_{\beta\alpha}\right) g_{\nu\mu} + \left(\Gamma^\nu_{\beta\mu} + \Gamma^\nu_{\mu\beta}\right) g_{\alpha\nu} \tag{70}$$

Since the lower indices of the Christoffel symbols may be interchanged, this leaves us with:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2\Gamma^\nu_{\beta\mu} g_{\alpha\nu} \tag{71}$$

Using the fact that inverting the metric just turns its covariant indices into contravariant indices $(1/g_{\alpha\beta} = g^{\alpha\beta})$:

$$\Gamma^\nu_{\beta\mu} = \frac{1}{2} g^{\alpha\nu}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right) \tag{72}$$

It's important to remind the reader of the notation being used here to understand the meaning of this result. Recall that $\phi_{,\alpha} \equiv \frac{\partial \phi}{\partial x^\alpha}$ so the previous expression becomes:

$$\Gamma^\nu_{\beta\mu} = \frac{1}{2} g^{\alpha\nu}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \frac{\partial g_{\beta\mu}}{\partial x^\alpha}\right) \tag{73}$$

Meaning the Christoffel symbols can be written in terms of derivatives of the metric.

Exercise (5.22): Show that if $U^\alpha \nabla_\alpha V^\beta = W^\beta$, then $U^\alpha \nabla_\alpha V_\beta = W_\beta$

Solution: Recall the notation that $\nabla_\alpha V^\beta \equiv V^\beta_{;\alpha}$ from Eq. (5.51). This turns the expression into:

$$U^\alpha V^\beta_{;\alpha} = W^\beta \tag{74}$$

We can then multiply both sides of the expression by the metric $g_{\mu\beta}$:

$$U^\alpha g_{\mu\beta} V^\beta_{;\alpha} = g_{\mu\beta} W^\beta \tag{75}$$

From Eq. (5.68), $V_{\alpha;\beta} = g_{\alpha\mu} V^\alpha_{;\beta}$ so we can transform the left hand side of this expression to be:

$$U^\alpha V_{\mu;\alpha} = g_{\mu\beta} W^\beta \tag{76}$$

And we can finally use $V_\alpha = g_{\alpha\mu} V^\mu$ from Eq. (5.69) to simply the right side of the expression into:

$$U^\alpha V_{\mu;\alpha} = W_\mu \tag{77}$$

And since μ is just a dummy index, it can be changed for β, resulted in the desired expression:

$$U^\alpha \nabla_\alpha V_\beta = W_\beta \tag{78}$$