Quantum Computation and Quantum Information - Selected Solutions

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1 Chapter 1: Introduction and Overview

2 Chapter 2: Introduction to Quantum Mechanics

Exercise 2.1: Show that $(1,-1),(1,2)$, and $(2,1)$ are linearly dependent.

Solution: Vectors $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$ are linearly dependent in a vector space V with $\dim(V) = m$ if $n > m$. Because we have three two-dimensional vectors, they must be linearly dependent. This could also be directly demonstrated by representing the vectors as an augmented matrix

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{pmatrix} \tag{1}$$

and applying Gaussian elimination to show that a nontrivial solution for a_1, a_2, a_3 of the form

$$a_1 |v_1\rangle + a_2 |v_2\rangle + a_3 |v_3\rangle = 0$$

exists.

Exercise 2.2: Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle, |1\rangle$, and the output basis $|0\rangle, |1\rangle$. Find input and output basis which give rise to a different matrix representation of A.

Solution: Because we have a two-dimensional vector space, we can define $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We are being asked to find the transformation matrix A such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. These conditions give us:

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies a_{00} = 0 \ \& \ a_{10} = 1 \tag{2}$$

and

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies a_{01} = 1 \ \& \ a_{11} = 0. \tag{3}$$

So we have learned the matrix representation of the transformation A is given as:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

Taking inspiration from page 63, define $|0\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|1\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. To find the linear transformation A that satisfies the conditions $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$, the same method is carried out as before; requiring that

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \implies \frac{a_{00}}{\sqrt{2}} + \frac{a_{01}}{\sqrt{2}} = \frac{1}{2} \ \& \ \frac{a_{10}}{\sqrt{2}} + \frac{a_{11}}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \tag{5}$$

and

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \implies \frac{a_{00}}{\sqrt{2}} - \frac{a_{01}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \ \& \ \frac{a_{10}}{\sqrt{2}} - \frac{a_{11}}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \tag{6}$$

Now this system of equations must be solved for $a_{00}, a_{01}, a_{10}, a_{11}$ using whatever method you'd like. Carrying this out results in finding that the transformation matrix A is now represented as:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

Exercise 2.3: Suppose A is a linear operator from vector space V to vector space W, and B is a linear operator from vector space W to vector space X. Let $|v_i\rangle, |w_j\rangle$, and $|x_k\rangle$ be bases for the vector spaces V, W, and X, respectively. Show that the matrix representation for the linear transformation BA is the matrix product of the matrix representations for B and A, with respect to the appropriate bases.

Solution: We are given that $A : V \to W$ and $B : W \to X$ and want to show that

$$BA|v_i\rangle = \sum_k (BA)_{ki} |x_k\rangle \tag{8}$$

where $(BA)_{ki}$, the matrix product of B and A, is defined as

$$(BA)_{ki} := \sum_j B_{kj} A_{ji}. \tag{9}$$

To do this, we first start with the matrix representation of the operator BA on $|v_i\rangle$ and use the transformations that have been defined for us:

$$\begin{aligned} BA|v_i\rangle &= B\left(\sum_j A_{ji}|w_j\rangle\right) \\ &= \sum_j A_{ji} B|w_j\rangle \\ &= \sum_j A_{ji} \sum_k B_{kj}|x_k\rangle. \\ &= \sum_k \sum_j B_{kj} A_{ji}|x_k\rangle \\ &\equiv \sum_k (BA)_{ki}|x_k\rangle \end{aligned} \tag{10}$$

Exercise 2.4: Show that the identity operator on a vector space V has a matrix representation which is one along the diagonal and zero everywhere else, if the matrix representation is taken with respect to the same input and output bases. This matrix is known as the *identity matrix*.

Solution: We are looking for the elements of the transformation matrix I where $I : V \to V$ such that $IV = V$. We are therefore looking for the form of I such that

$$\sum_i I_{ij}|v_i\rangle = |v_j\rangle \tag{11}$$

where $|v_i\rangle = |v_j\rangle$. This therefore requires that $I_{ij} = 0 \; \forall \; j \neq i$ and $I_{ij} = 1 \; \forall \; j = i$. The only matrix that satisfies such a property is diagonal with 1's. Note that such a matrix is called the Kronecker-Delta function, δ_{ij}.

Exercise 2.5: Verify that $(\cdot, \cdot)$ just defined is an inner product on $\mathbf{C}^n$.

To first demonstrate its linearity, we compute

$$\begin{aligned} \left((y_1, \ldots, y_n), \sum_i \lambda_i (z_1, \ldots, z_n)_i\right) &\equiv \sum_j y_j^* \sum_i \lambda_i z_{ij} \\ = \sum_i \lambda_i \sum_j y_j^* zij &\equiv \sum_i \lambda_i ((y_1, \ldots, y_n), (z_1, \ldots, z_n)_i). \end{aligned} \tag{12}$$

To demonstrate that this inner product is equal to its complex conjugate, we must show

$$((y_1, \ldots, y_n), (z_1, \ldots, z_n)) = ((y_1, \ldots, y_n), (z_1, \ldots, z_n))^*. \tag{13}$$

This follows naturally from the definition, since

$$\begin{aligned} ((y_1, \ldots, y_n), (z_1, \ldots, z_n)) &\equiv \sum_i y_i^* z_i \\ = \sum_i (y_i z_i^*)^* &= ((z_1, \ldots, z_n), (y_1, \ldots, y_n))^* \end{aligned} \tag{14}$$

To demonstrate the positivity of this inner product, we compute:

$$((v_1, \ldots, v_n), (v_1, \ldots, v_n)) \equiv \sum_i v_i^* v_i \tag{15}$$

and note that $v_i^* v_i$ is the modulus of v_i squared, $|v_i|^2$, which is always positive.

Exercise 2.7: Verify that $|w\rangle = (1,1)$ and $|v\rangle = (1,-1)$ are orthogonal. What are the normalized form of these vectors?

Solution: The inner product on $\mathbb{R}^n$ is defined as

$$(v, w) := \sum_i v_i w_i \tag{16}$$

so when $|w\rangle = (1,1)$ and $|v\rangle = (1,-1)$, we compute that $(w,v) = (v,w) = 1 - 1 = 0$.

The normalized forms of these vectors will be given as $\frac{|v\rangle}{\sqrt{(|v\rangle, |v\rangle)}}$ so we compute the normalized form of $|w\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and the normalized form of $|v\rangle = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$.

Exercise