Quantum Field Theory for the Gifted Amateur - Selected Solutions

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1 Chapter 1: Lagrangians

Exercise (1.5): For a three-dimensional elastic medium, the potential energy is

$$V = \frac{\mathcal{T}}{2} \int d^3x \, (\nabla \psi)^2 \, ,$$

and the kinetic energy is

$$T = \frac{\rho}{2} \int d^3x \left(\frac{\partial \psi}{\partial t} \right)^2 .$$

Use these results and show the functional derivative approach to show that ψ obeys the wave equation:

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2},$$

where v is the velocity of the wave.

Solution: Beginning by constructing the Lagrangian for this system, one finds that:

$$L = \int d^3x \frac{\mathcal{T}}{2} (\nabla \psi)^2 - \frac{\rho}{2} \left(\frac{\partial \psi}{\partial t} \right)^2 \tag{1}$$

which means that the Lagrangian density is:

$$\mathcal{L} = \frac{\mathcal{T}}{2} (\nabla \psi)^2 - \frac{\rho}{2} \left(\frac{\partial \psi}{\partial t} \right)^2 . \tag{2}$$

Applying Hamilton's principle of least action to this system:

$$\frac{\delta S}{\delta \psi} = 0 \implies \frac{\partial \mathcal{L}}{\partial \psi} - \partial_b \frac{\partial \mathcal{L}}{\partial (\partial_b \psi)} = 0 = \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial t)} - \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial x^i)} = 0 \tag{3}$$

Where $x^1 = x, x^2 = y$, and $x^3 = z$. Because $\mathcal{L}$ has no dependence on a ψ term, this functional derivative becomes:

$$-\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial t)} = \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial x^i)} . \tag{4}$$

Taking these partial derivative of the Lagrangian density results in:

$$\begin{aligned} -\frac{\partial}{\partial t} \left(-\rho \frac{\partial \psi}{\partial t} \right) &= \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \left(\mathcal{T} \frac{\partial \psi}{\partial x^i} \right) \\ \implies \rho \frac{\partial^2 \psi}{\partial t^2} = \mathcal{T} \nabla^2 \psi &\implies \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi \end{aligned} \tag{5}$$

when $v = \sqrt{\mathcal{T}/\rho}$.

2 Chapter 2: Simple Harmonic Oscillators

Exercise (2.1): For the one-dimensional harmonic oscillator, show that with creation and annihilation operators defined as in Eq. (2.9) and Eq. (2.10), $[\hat{a},\hat{a}]=0$, $\left[\hat{a}^\dagger,\hat{a}^\dagger\right]=0$, $\left[\hat{a},\hat{a}^\dagger\right]=1$, and $\hat{H}=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)$.

Solution: With the definitions:

$$\begin{aligned}\hat{a}&=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)\\ \hat{a}^\dagger&=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right),\end{aligned}\tag{6}$$

$[\hat{a},\hat{a}]$ will be defined as:

$$[\hat{a},\hat{a}]=\frac{m\omega}{2\hbar}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)-\frac{m\omega}{2\hbar}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)=0\tag{7}$$

and same logic will apply to $[\hat{a}^\dagger,\hat{a}^\dagger]$:

$$[\hat{a}^\dagger,\hat{a}^\dagger]=\frac{m\omega}{2\hbar}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)-\frac{m\omega}{2\hbar}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)=0.\tag{8}$$

Now for $[\hat{a},\hat{a}^\dagger]$ (using the fact that $[\hat{x},\hat{p}]=i\hbar$):

$$\begin{aligned}\left[\hat{a},\hat{a}^\dagger\right]&=\frac{m\omega}{2\hbar}\left[\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)-\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)\right]\\ &\implies\frac{m\omega}{2\hbar}\left[\frac{-i}{m\omega}[\hat{x},\hat{p}]+\frac{i}{m\omega}[\hat{p},\hat{x}]\right]=\frac{m\omega}{2\hbar}\left[\frac{\hbar}{m\omega}+\frac{\hbar}{m\omega}\right]=1\end{aligned}\tag{9}$$

For the Hamiltonian, first invert the definitions of $\hat{a}$ and $\hat{a}^\dagger$ to find that:

$$\begin{aligned}\hat{x}&=\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)\\ \hat{p}&=-i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}-\hat{a}^\dagger\right)\end{aligned}\tag{10}$$

and plug these forms into the expression for the Hamiltonian:

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2=\frac{-\omega\hbar}{4}\left(\hat{a}-\hat{a}^\dagger\right)^2+\frac{\omega\hbar}{4}\left(\hat{a}+\hat{a}^\dagger\right)=\frac{\omega\hbar}{2}\left(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right)\tag{11}$$

and use the previously derived commutation relation $\left[\hat{a},\hat{a}^\dagger\right]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=1$:

$$\hat{H}=\frac{\omega\hbar}{2}\left(\hat{a}^\dagger\hat{a}+1+\hat{a}^\dagger\hat{a}\right)=\omega\hbar\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\tag{12}$$