

# A First Course in General Relativity - Selected Solutions

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## 1 Chapter 1: Special Relativity

## 2 Chapter 2: Vector Analysis in Special Relativity

**Exercise 2.3:** Prove Eq. (2.5)

**Solution:** By convention, Latin indices are not summed over 0 so if we are to interchange them with Greek indices as a dummy index, we must perform the following:

$$\Lambda_{\beta}^{\bar{\alpha}} \Delta x^{\beta} = \Lambda_0^{\bar{\alpha}} \Delta x^0 + \Lambda_i^{\bar{\alpha}} \Delta x^i \quad (1)$$

since  $\Lambda_{\beta}^{\bar{\alpha}} \Delta x^{\beta}$  implies a sum over all of the positive real numbers where  $\Lambda_i^{\bar{\alpha}} \Delta x^i$  is a sum over positive real numbers not including zero.

**Exercise 2.7a:** Prove Eq. (2.10) for all  $\alpha, \beta$

**Solution:** To verify that  $(\vec{e}_{\alpha})^{\beta} = \delta_{\alpha}^{\beta}$ , consider an arbitrary basis vector,  $\vec{e}_{\alpha}$ , meaning that the elements in its list are all zero except for the single entry at the  $\alpha$ th component. This can be written as:

$$\vec{e}_{\alpha} = (... , 0, 0, 1, 0, ...) \quad (2)$$

Where the index of each value in the list can be traced with respect to  $\alpha$

$$(\alpha - n, ..., \alpha - 2, \alpha - 1, \alpha, \alpha + 1, ..., \alpha + n) \quad (3)$$

Then  $(\vec{e}_{\alpha})^{\beta}$  indicates the  $\beta$ th component of the basis vector  $\vec{e}_{\alpha}$ . By the definition of a basis vector, we know that all entries in  $\vec{e}_{\alpha}$  are zero except the one at the  $\alpha$ th component. So if we choose  $\beta$  to be any non- $\alpha$  index, the result must be 0:

$$(\vec{e}_{\alpha})^{\alpha-1} = 0 \quad (4)$$

It's for this reason that we can define the  $\beta$ th component of the  $\vec{e}_{\alpha}$  basis vector to be equal to the Kronecker delta, meaning that  $(\vec{e}_{\alpha})^{\beta} = 1$  only when  $\alpha = \beta$ .

**Exercise 2.29:** Prove, using component expressions, Eqs. (2.24) and (2.26), that

$$\frac{d}{d\tau}(\vec{U} \cdot \vec{U}) = 2\vec{U} \cdot \frac{d\vec{U}}{d\tau} \quad (5)$$

**Solution:** By (2.26):

$$\begin{aligned} \vec{U} \cdot \vec{U} &= -U^0 U^0 + U^1 U^1 + U^2 U^2 + U^3 U^3 \\ &= -(U^0)^2 + (U^1)^2 + (U^2)^2 + (U^3)^2 \end{aligned} \quad (6)$$

and by (2.24):

$$\vec{U} \cdot \vec{U} = \vec{U}^2 \implies \frac{d}{d\tau}(\vec{U} \cdot \vec{U}) = \frac{d}{d\tau}(\vec{U}^2) = 2\vec{U} \cdot \frac{d\vec{U}}{d\tau} \quad (7)$$

### 3 Chapter 3: Tensor Analysis in Special Relativity

**Exercise (3.1a):** Given an arbitrary set of numbers  $\{M_{\alpha\beta}; \alpha = 0, \dots, 3; \beta = 0, \dots, 3\}$  and two arbitrary vector components  $\{A^\mu, \mu = 0, \dots, 3\}$  and  $\{B^\nu, \nu = 0, \dots, 3\}$ , show that the two expressions

$$M_{\alpha\beta}A^\alpha B^\beta \quad (8)$$

and

$$M_{\alpha\alpha}A^\alpha B^\alpha \quad (9)$$

are not equivalent.

**Solution:**

$$M_{\alpha\alpha}A^\alpha B^\alpha = M_{00}A^0B^0 + M_{11}A^1B^1 + M_{11}A^1B^1 + M_{11}A^1B^1 \quad (10)$$

where

$$M_{\alpha\beta}A^\alpha B^\beta = B^\beta(M_{0\beta}A^0 + M_{1\beta}A^1 + M_{2\beta}A^2 + M_{3\beta}A^3) \quad (11)$$

So  $M_{\alpha\alpha}A^\alpha B^\alpha$  only contains the diagonal terms of  $M_{\alpha\beta}A^\alpha B^\beta$ .

**Exercise (3.1b):** Show that  $A^\alpha B^\beta \eta_{\alpha\beta} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$

**Solution:**

Because

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (12)$$

Any component of  $A^\alpha B^\beta \eta_{\alpha\beta}$  where  $\alpha \neq \beta$  means multiplying  $A^\alpha B^\beta$  by an off-diagonal component of  $\eta_{\alpha\beta}$ , which are all 0. Treating  $A^\alpha$  and  $B^\beta$  as row/column matrices and carrying out their multiplication with  $\eta_{\alpha\beta}$  will result in  $-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$ .

**Exercise (3.3a):** Prove, by writing out all of the terms, the validity of the following:

$$\tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha) \quad (13)$$

**Solution:**

Since one-forms act on vector arguments, the scalar values associated with  $A^\alpha$  may be pulled out of the expression like so:

$$\begin{aligned} \tilde{p}(A^\alpha \vec{e}_\alpha) &= \tilde{p}(A^0 \vec{e}_0 + A^1 \vec{e}_1 + A^2 \vec{e}_2 + A^3 \vec{e}_3) = \tilde{p}(A^0 \vec{e}_0) + \tilde{p}(A^1 \vec{e}_1) + \tilde{p}(A^2 \vec{e}_2) + \tilde{p}(A^3 \vec{e}_3) \\ &= A^0 \tilde{p}(\vec{e}_0) + A^1 \tilde{p}(\vec{e}_1) + A^2 \tilde{p}(\vec{e}_2) + A^3 \tilde{p}(\vec{e}_3) = A^\alpha \tilde{p}(\vec{e}_\alpha) \end{aligned} \quad (14)$$

**Exercise 3.5:** Justify each step leading from Eqs. (3.10a) to (3.10d).

**Solution:** To establish the frame-independence of  $A^{\bar{\alpha}} p_{\bar{\alpha}}$ :

$$\begin{aligned} A^{\bar{\alpha}} p_{\bar{\alpha}} &= A^{\bar{\alpha}} \tilde{p}(\vec{e}_{\bar{\alpha}}), \\ \vec{e}_{\bar{\alpha}} &= \Lambda_{\bar{\alpha}}^\mu \vec{e}_\mu, \\ A^{\bar{\alpha}} &= \Lambda_{\bar{\alpha}}^\beta A^\beta \\ \implies A^{\bar{\alpha}} \tilde{p}(\vec{e}_{\bar{\alpha}}) &= \Lambda_{\bar{\alpha}}^\beta A^\beta \tilde{p}(\Lambda_{\bar{\alpha}}^\mu \vec{e}_\mu) = \Lambda_{\bar{\alpha}}^\beta \Lambda_{\bar{\alpha}}^\mu A^\beta \tilde{p}(\vec{e}_\mu) = \Lambda_{\bar{\alpha}}^\beta \Lambda_{\bar{\alpha}}^\mu A^\beta p_\mu \end{aligned} \quad (15)$$

and by Eq. (2.18):

$$\Lambda_{\bar{\alpha}}^\beta \Lambda_{\bar{\alpha}}^\mu A^\beta p_\mu = \delta_{\bar{\alpha}}^\mu A^\beta p_\mu = A^\beta p_\beta \implies \boxed{A^{\bar{\alpha}} p_{\bar{\alpha}} = A^\beta p_\beta} \quad (16)$$

which should not be a surprising result given that the one-form of a vector produces a scalar and scalars are invariant quantities under Lorentz transformations.

**Exercise (3.10a):** Given a frame  $\mathcal{O}$  whose coordinates are  $\{x^\alpha\}$ , show that:

$$\frac{\partial x^\alpha}{\partial x^\beta} = \delta_\beta^\alpha \quad (17)$$

**Solution:** Taking the partial with respect to  $x^\beta$  means holding all terms in  $x$  constant except for the  $\beta$  index. When

$\alpha \neq \beta$ , you are taking a partial of a fixed value (a constant), meaning your derivative will be equal to zero. When you differentiate the  $\beta$  index with respect to  $\beta$ , you will always get 1.

**Exercise (3.10b):** For any two frames, we have Eq. (3.18):

$$\frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda_{\bar{\alpha}}^\beta. \quad (18)$$

Show that (a) and the chain rule imply

$$\Lambda_{\bar{\alpha}}^\beta \Lambda_\mu^{\bar{\alpha}} = \delta_\mu^\beta \quad (19)$$

**Solution:**

$$\begin{aligned} \Lambda_{\bar{\alpha}}^\beta &= \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}}, \\ \Lambda_\mu^{\bar{\alpha}} &= \frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} \\ \implies \Lambda_{\bar{\alpha}}^\beta \Lambda_\mu^{\bar{\alpha}} &= \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} = \frac{\partial x^\beta}{\partial x^\mu} = \delta_\mu^\beta \end{aligned} \quad (20)$$

**Exercise (3.11a):** Use the notation  $\partial\phi/\partial x^\alpha = \phi_{,\alpha}$  to rewrite Eqs. (3.14), (3.15), and (3.18).

**Solution:**

Eq. (3.14):

$$\frac{\partial\phi}{\partial t}U^t + \frac{\partial\phi}{\partial t}U^x + \frac{\partial\phi}{\partial t}U^y + \frac{\partial\phi}{\partial t}U^z \implies \phi_{,t}U^t + \phi_{,x}U^x + \phi_{,y}U^y + \phi_{,z}U^z \quad (21)$$

Eq. (3.15):

$$\tilde{d}\phi \xrightarrow{\mathcal{O}} \left( \frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right) \implies \tilde{d}\phi \xrightarrow{\mathcal{O}} (\phi_{,t}, \phi_{,x}, \phi_{,y}, \phi_{,z}) \quad (22)$$

Eq. (3.18):

$$\frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda_{\bar{\alpha}}^\beta \implies x_{,\bar{\alpha}}^\beta = \Lambda_{\bar{\alpha}}^\beta \quad (23)$$

**Exercise (3.13):** Prove, by geometric or algebraic arguments, that  $\tilde{d}f$  is normal to surfaces of constant  $f$ .

**Solution:** If we consider any point  $P = (t_0, x_0, y_0, z_0)$  along a parameterized level curve  $f(\tau) = \phi(t(\tau), x(\tau), y(\tau), z(\tau)) = c$ , then we can take the gradient to be as follows:

$$\tilde{d}f = \partial\phi_{,t} \left| \frac{dt}{d\tau} \right|_P + \partial\phi_{,x} \left| \frac{dx}{d\tau} \right|_P + \partial\phi_{,y} \left| \frac{dy}{d\tau} \right|_P + \partial\phi_{,z} \left| \frac{dz}{d\tau} \right|_P = 0 \quad (24)$$

Since this is also the definition of the dot product between two vectors:

$$\left\langle \partial\phi_{,t} \Big|_P, \partial\phi_{,x} \Big|_P, \partial\phi_{,y} \Big|_P, \partial\phi_{,z} \Big|_P \right\rangle \cdot \left\langle \frac{dt}{d\tau} \Big|_{\tau_0}, \frac{dx}{d\tau} \Big|_{\tau_0}, \frac{dy}{d\tau} \Big|_{\tau_0}, \frac{dz}{d\tau} \Big|_{\tau_0} \right\rangle = 0 \quad (25)$$

whose product is equal to zero, we can say that  $\tilde{d}f$  and  $f$  are normal to each other at every point along a level curve.

**Exercise (3.14):** Let  $\tilde{p} \rightarrow_{\mathcal{O}} (1, 1, 0, 0)$  and  $\tilde{q} \rightarrow_{\mathcal{O}} (-1, 0, 1, 0)$  be two one-forms. Prove, by trying two vectors  $\vec{A}$  and  $\vec{B}$  as arguments, that  $\tilde{p} \otimes q \neq \tilde{q} \otimes \tilde{p}$ . Then find the components of  $\tilde{p} \otimes \tilde{q}$ .

**Solution:** Since a one-form supplied with a vector argument:  $\tilde{p}(\vec{A}) = p^\alpha A^\alpha$ , we can perform the following operations to show  $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$ :

$$\begin{aligned} \tilde{p} \otimes \tilde{q} &= \tilde{p}(\vec{A})\tilde{q}(\vec{B}) = (A^0 + A^1)(-B^0 + B^2) \\ \tilde{q} \otimes \tilde{p} &= \tilde{q}(\vec{A})\tilde{p}(\vec{B}) = (-A^0 + A^2)(B^0 + B^1) \end{aligned} \quad (26)$$

**Exercise (3.16a):** Prove that  $\mathbf{h}_{(s)}$  defined by

$$\mathbf{h}_{(s)}(\vec{A}, \vec{B}) = \frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) + \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) \quad (27)$$

is a symmetric tensor.

**Solution:** From Eq. (3.27), we know a tensor  $\mathbf{f}$  is symmetric if:

$$\mathbf{f}(\vec{A}, \vec{B}) = \mathbf{f}(\vec{B}, \vec{A}) \quad \forall \vec{A}, \vec{B} \quad (28)$$

So if  $\mathbf{h}$  is to be symmetric, then:

$$\frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) - \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) = 0 \quad (29)$$

Let  $\vec{A} = (A_0, A_1, A_2, A_3)$  and  $\vec{B} = (B_0, B_1, B_2, B_3)$  then:

$$\frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) = \frac{1}{2}(A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3) \quad (30)$$

and

$$\frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) = \frac{1}{2}(B_0A_0 + B_1A_1 + B_2A_2 + B_3A_3) \quad (31)$$

and since multiplication is commutative, Eq. 21 must be true, meaning that  $\mathbf{h}_{(s)}(\vec{A}, \vec{B})$  is a symmetric tensor.

## 4 Chapter 4: Perfect Fluids in Special Relativity

**Exercise (4.7):** Derive Eq. (4.21).

**Solution:** Using the fact that  $T^{\alpha\beta} = \rho U^\alpha U^\beta$  derived from Eq. (4.20), we can begin deriving the expressions in Eq. (4.21):

$$\begin{aligned} T^{00} &= \rho U^0 U^0 = \rho \frac{1}{\sqrt{1-v^2}} \frac{1}{\sqrt{1-v^2}} = \frac{\rho}{1-v^2} \\ T^{0i} &= \rho U^0 U^i = \rho \frac{1}{\sqrt{1-v^2}} \frac{v^i}{\sqrt{1-v^2}} = \frac{\rho v^i}{1-v^2} \\ T^{i0} &= \rho U^i U^0 = \frac{\rho v^i}{1-v^2} \\ T^{ij} &= \rho U^i U^j = \frac{\rho v^i v^j}{1-v^2} \end{aligned} \quad (32)$$

**Exercise (4.10):** Take the limit of Eq. (4.35) for  $|\vec{V}| \ll 1$  to get  $\partial n / \partial t + \partial(nv^i) / \partial x^i = 0$

**Solution:** Beginning with Eq. (4.35):

$$\frac{\partial}{\partial x^\alpha} (n U^\alpha) = 0 \quad (33)$$

With the knowledge that  $U^\alpha$  contains both spatial components and a temporal component, the previous expression must be separated into two parts:

$$\frac{\partial}{\partial t} (n U^t) + \frac{\partial}{\partial x^i} (n U^i) = 0 \quad (34)$$

Using the expressions give on page 93 for  $U^t$  and  $U^i$ , this becomes:

$$\frac{\partial}{\partial t} \left( \frac{n}{\sqrt{1-v^2}} \right) + \frac{\partial}{\partial x^i} \left( \frac{n v^i}{\sqrt{1-v^2}} \right) = 0 \quad (35)$$

Then taking the limit where the speed is much less than 1 makes  $1 - v^2 \approx 1$  so this expression becomes:

$$\frac{\partial n}{\partial t} + \frac{\partial(nv^i)}{\partial x^i} = 0 \quad (36)$$

**Exercise (4.17):** We have defined  $a^\mu = U^\mu_{,\beta} U^\beta$ . Go to the relativistic limit (small velocity) and show that  $a^i = \dot{v}^i + (\vec{v} \cdot \nabla) v^i = Dv^i / Dt$  where the operator  $D/Dt$  is the usual “total” or “advective” time derivative of fluid dynamics.

**Solution:** Writing out an initial expression for  $a^i$  while again keeping in mind that  $U^i$  contains a temporal component and spatial components:

$$a^i = \frac{\partial U^i}{\partial x^\beta} U^\beta = \frac{\partial U^i}{\partial t} U^t + \frac{\partial U^i}{\partial x^j} U^j = \frac{\partial}{\partial t} \left( \frac{v^i}{\sqrt{1-v^2}} \right) \frac{1}{\sqrt{1-v^2}} + \frac{\partial}{\partial x^j} \left( \frac{v^i}{\sqrt{1-v^2}} \right) \frac{v^j}{\sqrt{1-v^2}} = 0 \quad (37)$$

Taking the non-relativistic limit:

$$a^i = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x^j} v^j = 0 \quad (38)$$

And since  $\partial v^i / \partial x^j$  is just the dot product between the  $i$ th velocity component and the spatial derivatives under the Einstein summation convention, this expression becomes:

$$a^i = \dot{v}^i + (\vec{v} \cdot \nabla) v^i \quad (39)$$

Which is an expression defined to be the material or “advective” derivative used in fluid mechanics.

## 5 Chapter 5: Preface to Curvature

**Exercise (5.3a):** Show that the coordinate transformation  $(x, y) \rightarrow (\xi, \eta)$  with  $\xi = x$  and  $\eta = 1$  violates Eq. (5.6).

**Solution:** For a transformation to be reasonable, it must assign all coordinates in the source  $(x, y)$  to distinct coordinates in the target  $(\xi, \eta)$ . This property will be satisfied if the Jacobian is non-zero, which is the definition given by Eq. (5.6). So to show this transformation is not reasonable, it must be shown to violate Eq. (5.6):

$$\det \begin{pmatrix} \partial \xi / \partial x & \partial \xi / \partial y \\ \partial \eta / \partial x & \partial \eta / \partial y \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad (40)$$

So this transformation of coordinates is not reasonable.

**Exercise (5.7):** Calculate all elements of the transformation matrices  $\Lambda_{\beta}^{\alpha'}$  and  $\Lambda_{\mu}^{\nu'}$  for the transformation from Cartesian  $(x, y)$  - the unprimed indices - to polar  $(r, \theta)$  - the primed indices.

**Solution:** Since, by Eq. (5.8):

$$\Lambda_{\beta}^{\alpha'} = \begin{pmatrix} \partial \xi / \partial x & \partial \xi / \partial y \\ \partial \eta / \partial x & \partial \eta / \partial y \end{pmatrix} = \begin{pmatrix} \partial r / \partial x & \partial r / \partial y \\ \partial \theta / \partial x & \partial \theta / \partial y \end{pmatrix} \quad (41)$$

We can directly compute the transformation from Cartesian into Polar components by computing the terms of this matrix (knowing that  $\xi(x, y) = r = \sqrt{x^2 + y^2}$  and  $\eta(x, y) = \theta = \arctan(y/x)$  in polar coordinates). This results in:

$$\Lambda_{\beta}^{\alpha'} = \begin{pmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{pmatrix} \quad (42)$$

Since  $\Lambda_{\mu}^{\nu'}$  is defined as:

$$\Lambda_{\mu}^{\nu'} = \begin{pmatrix} \partial x / \partial \xi & \partial y / \partial \xi \\ \partial x / \partial \eta & \partial y / \partial \eta \end{pmatrix} = \begin{pmatrix} \partial x / \partial r & \partial y / \partial r \\ \partial x / \partial \theta & \partial y / \partial \theta \end{pmatrix} \quad (43)$$

in Eq. (5.13), the matrix is the following:

$$\Lambda_{\mu}^{\nu'} = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} \quad (44)$$

**Exercise (5.8a):** (Use the result of Exer.7.) Let  $f = x^2 + y^2 + 2xy$  and in Cartesian Coordinates  $\vec{V} \rightarrow (x^2 + 3y, y^2 + 3x)$ ,  $\vec{W} \rightarrow (1, 1)$ . Compute  $f$  as a function of  $r$  and  $\theta$ , and find the components of  $\vec{V}$  and  $\vec{W}$  on the polar basis, expressing them as functions of  $r$  and  $\theta$ .

**Solution:** Expressing  $f$  as a polar function is as simple as making the substitutions  $x = r \cos \theta$  and  $y = r \sin \theta$ , arriving at  $f = r^2 + 2r^2 \cos \theta \sin \theta$ . To express  $\vec{V}$  and  $\vec{W}$  as polar functions, the same process can be applied. This results in  $\vec{V} = (r^2 \cos^2 \theta + 3r \sin \theta, r^2 \sin^2 \theta + 3r \cos \theta)$  and  $\vec{W} = (1, 1)$ . To express  $\vec{V}$  and  $\vec{W}$  in a polar *basis*, though, you must use the transformations found in the previous problem:

$$V^{\alpha'} = \Lambda_{\beta}^{\alpha'} V^{\beta} \implies \vec{V} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + 3r \sin \theta \\ r^2 \sin^2 \theta + 3r \cos \theta \end{pmatrix} = \begin{pmatrix} r^2(\cos^3 \theta + \sin^3 \theta) + 6r \sin \theta \cos \theta \\ r(\cos \theta \sin^2 \theta - \cos^2 \theta \sin \theta) + 3(\cos^2 \theta - \sin^2 \theta) \end{pmatrix} \quad (45)$$

$$\vec{W} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta + \sin \theta \\ (\cos \theta - \sin \theta)/r \end{pmatrix} \quad (46)$$

**Exercise (5.8b):** Find the components of  $\tilde{d}f$  in Cartesian Coordinates and obtain them in polars (i) by direct calculation in polars, and (ii) by transforming components from Cartesian.

**Solution:** (i) To compute by direct calculation in polar:  $\tilde{d}f = (\partial f/\partial r, \partial f/\partial \theta)$  we can use the definition of  $f$  in polar that was derived in part (a):

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r} (r^2 + 2r^2 \cos \theta \sin \theta) = 2r + 4r \cos \theta \sin \theta \quad (47)$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta} (r^2 + 2r^2 \cos \theta \sin \theta) = 2r^2 \cos(2\theta) \quad (48)$$

(ii) To compute  $\tilde{d}f$  by transforming components from Cartesian,

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \xi} \frac{\partial \phi}{\partial y} \implies \frac{\partial f}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial f}{\partial y} \quad (49)$$

$$\frac{\partial f}{\partial r} = \cos \theta (2x + 2y) + \sin \theta (2x + 2y) = (\cos \theta + \sin \theta)(2r \cos \theta + 2r \sin \theta) = 2r + 4r \sin \theta \cos \theta \quad (50)$$

Similarly:

$$\frac{\partial f}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial f}{\partial y} \quad (51)$$

$$\frac{\partial f}{\partial \theta} = (-r \sin \theta)(2x + 2y) + (r \cos \theta)(2x + 2y) = 2r^2 \cos \theta \quad (52)$$

It should be noted that the expressions from (ii) match those derived from (i).

**Exercise (5.8c):** (i) Use the metric tensor in polar coordinates to find the polar components of the one-forms  $\tilde{V}$  and  $\tilde{W}$  associated with  $\vec{V}$  and  $\vec{W}$ . (ii) Obtain the polar components of  $\tilde{V}$  and  $\tilde{W}$  by transformation of their Cartesian components.

**Solution:** (i) By Eq. (5.31), the metric tensor in polar coordinates is:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad (53)$$

The metric in polar coordinates can be used to find the polar components of the one-forms by:

$$\tilde{W}_\alpha = g_{\alpha\beta} W^\beta = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} \cos \theta + \sin \theta \\ (\cos \theta - \sin \theta)/r \end{pmatrix} = \begin{pmatrix} \cos \theta + \sin \theta \\ r(\cos \theta - \sin \theta) \end{pmatrix} \quad (54)$$

The same can be done for  $\vec{V}$  as computed in polar form from part (a) of this problem.

(ii) Using the transformation matrix  $\Lambda_{\beta'}^\alpha$  to obtain  $\tilde{V}$  and  $\tilde{W}$ :

$$\Lambda_{\beta'}^\alpha \tilde{W}^\alpha = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta + \sin \theta \\ r(\cos \theta - \sin \theta) \end{pmatrix} \quad (55)$$

And the same process can be employed to solve for  $\tilde{V}$ . Note that  $\tilde{V}$  and  $\tilde{W}$  is just  $\vec{V}$  and  $\vec{W}$  in Cartesian coordinates since the metric tensor in Cartesian coordinates is the identity matrix.

**Exercise (5.11a):** For the vector field  $\vec{V}$  whose Cartesian components are  $(x^2 + 3y, y^2 + 3x)$ , compute  $V_{,\beta}^\alpha$  in Cartesian.

**Solution:** Since  $V_{,\beta}^\alpha \equiv \partial V^\alpha / \partial x^\beta$ :

$$V_{,\beta}^\alpha = \begin{pmatrix} \partial V^1 / \partial x & \partial V^1 / \partial y \\ \partial V^2 / \partial x & \partial V^2 / \partial y \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \quad (56)$$

**Exercise (5.11b):** Compute the transformation  $\Lambda_\alpha^{\mu'} \Lambda_\nu^\beta V_{,\beta}^\alpha$  to polars.

This computation is a straightforward usage of the transformation matrices  $\Lambda_\alpha^{\mu'}$  and  $\Lambda_\nu^\beta$ , from Cartesian to polar coordinates derived in Exercise 5.7 and the polar form of  $V_{,\beta}^\alpha$  found in the previous part to this problem. The order of

multiplication for these matrices should be noted, however, since computing  $\Lambda_{\alpha}^{\mu'} \Lambda_{\nu'}^{\beta} V_{,\beta}^{\alpha}$  would leave  $V_{,\beta}^{\alpha}$  unchanged. Computing  $\Lambda_{\alpha}^{\mu'} V_{,\beta}^{\alpha} \Lambda_{\nu'}^{\beta}$  results in:

$$\begin{aligned} \Lambda_{\alpha}^{\mu'} V_{,\beta}^{\alpha} \Lambda_{\nu'}^{\beta} &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{pmatrix} \begin{pmatrix} 2r \cos \theta & 3 \\ 3 & 2r \sin \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} 2(r \cos^3 \theta + 3 \cos \theta \sin \theta + r \sin^3 \theta) - r(\cos \theta - \sin \theta) & (-3 \sin \theta + \cos \theta)(-3 + 2r \sin \theta) \\ \frac{(\cos \theta - \sin \theta)(3 \cos \theta + 3 \sin \theta - r \sin(2\theta))}{r} & (-3 + r \cos \theta + r \sin \theta) \sin(2\theta) \end{pmatrix} \end{aligned} \quad (57)$$

**Exercise (5.11c):** Compute the components  $V_{;\nu'}^{\mu'}$  directly in polars using the Christoffel symbols.

**Solution:** Since  $\alpha, \beta \in \{x, y\}$  in Cartesian coordinates, there will be four components to compute:  $V_{;r}^r, V_{;r}^{\theta}, V_{;\theta}^r, V_{;\theta}^{\theta}$ . Beginning with  $V_{;r}^r$ :

$$\begin{aligned} V_{;r}^r &= \frac{\partial V^r}{\partial r} + V^{\mu} \Gamma_{\mu r}^r \quad \& \quad \Gamma_{rr}^{\mu} = \forall \mu \implies V_{;r}^r = \frac{\partial V^r}{\partial r} + V^{\theta} \Gamma_{\theta r}^r \\ \Gamma_{\theta r}^r &: \frac{\partial \vec{e}_{\theta}}{\partial r} = -\sin \theta \vec{e}_x + \cos \theta \vec{e}_y = \frac{1}{r} \vec{e}_{\theta} \implies \Gamma_{\theta r}^r = 0 \end{aligned} \quad (58)$$

$$\implies V_{;r}^r = \frac{\partial V^r}{\partial r} = \frac{\partial}{\partial r} (r^2(\cos^3 \theta + \sin^3 \theta) + 6r \sin \theta \cos \theta) = 2r(\cos^3 \theta + \sin^3 \theta) + 6 \sin \theta \cos \theta$$

$$\begin{aligned} V_{;r}^{\theta} &= \frac{\partial V^{\theta}}{\partial r} + V^{\mu} \Gamma_{\mu r}^{\theta} = \frac{\partial V^{\theta}}{\partial r} + \frac{1}{r} V^{\theta} = \\ \frac{\partial}{\partial r} (r(\cos \theta \sin^2 \theta - \cos^2 \theta \sin \theta) + 3(\cos^2 \theta - \sin^2 \theta)) &+ \frac{1}{r} (r(\cos \theta \sin^2 \theta - \cos^2 \theta \sin \theta) + 3(\cos^2 \theta - \sin^2 \theta)) \\ \implies V_{;r}^{\theta} &= \frac{(\cos \theta - \sin \theta)(3 \cos \theta + 3 \sin \theta) - r \sin(2\theta)}{r} \end{aligned} \quad (59)$$

$$\begin{aligned} V_{;\theta}^r &= \frac{\partial V^r}{\partial \theta} + V^{\mu} \Gamma_{\mu \theta}^r = \frac{\partial V^r}{\partial \theta} - r V^{\theta} \\ \implies V_{;\theta}^r &= -r(\cos \theta - \sin \theta)(-3 \cos \theta + 3 \sin \theta) + r \sin(2\theta) \end{aligned} \quad (60)$$

$$V_{;\theta}^{\theta} = \frac{\partial V^{\theta}}{\partial \theta} + \frac{1}{r} V^r = \sin(2\theta)(-3 + r \cos \theta + r \sin \theta) \quad (61)$$

**Exercise (5.11d):** Compute the divergence  $V_{,\alpha}^{\alpha}$  using results from part (a).

**Solution:**

$$V_{,\alpha}^{\alpha} = \frac{\partial V^{\alpha}}{\partial x^{\alpha}} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} = 2(x + y) = 2r(\cos \theta + \sin \theta) \quad (62)$$

**Exercise (5.11e):** Compute the divergence  $V_{;\mu'}^{\mu'}$  using results from either part (b) or (c).

**Solution:**

$$\begin{aligned} V_{;\mu'}^{\mu'} &= V_{;r}^r + V_{;\theta}^{\theta} = \frac{\partial V^r}{\partial r} + \Gamma_{rr}^r V^r + \Gamma_{\theta r}^r V^{\theta} + \frac{\partial V^r}{\partial r} + \Gamma_{\theta \theta}^{\theta} V^{\theta} + \Gamma_{r \theta}^{\theta} V^r \\ &= \frac{\partial V^r}{\partial r} + \frac{\partial V^{\theta}}{\partial \theta} + \frac{1}{r} V^r = 2r(\cos \theta + \sin \theta) \end{aligned} \quad (63)$$

**Exercise (5.11f):** Compute the divergence  $V_{;\mu}^{\mu'}$  using Eq. (5.55) directly.

$$V_{;\mu}^{\mu'} = \frac{1}{r} \frac{\partial}{\partial r} (r V^r) + \frac{\partial}{\partial \theta} V^{\theta} = 2r(\cos \theta + \sin \theta) \quad (64)$$

(This and the majority of the results given for Exercise 5.11 were computed in Mathematica)

**Exercise (5.12a):** For the one-form field  $\tilde{p}$  whose Cartesian coordinates are  $(x^2 + 3y, y^2 + 3x)$ , compute  $p_{\alpha, \beta}$  in Cartesian.

**Solution:**

$$p_{\alpha, \beta} = \begin{pmatrix} p_{rr} & p_{r\theta} \\ p_{\theta r} & p_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r \cos \theta & 3 \\ 3 & 2r \sin \theta \end{pmatrix} \quad (65)$$

**Exercise (5.12b):** Compute the transformation  $\Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\beta} p_{\alpha, \beta}$  to polars.

**Solution:**

$$\Lambda_{\mu'}^{\alpha} \Lambda_{\nu'}^{\beta} p_{\alpha,\beta} = (\Lambda_{\mu'}^{\alpha})^T p_{\alpha,\beta} \Lambda_{\nu'}^{\beta} = (2r(\cos^3 \theta - 3 \cos \theta \sin \theta + r^2 \sin^3 \theta)) \quad (66)$$

**Exercise (5.12c):** Compute the components  $p_{\mu';\nu'}$  directly in polars using the Christoffel symbols, Eq. (4.44), in Eq. (5.62).

**Solution:**

$$p_{r;r} = p_{r,r} - p_{\mu} \Gamma_{\alpha\beta}^{\mu} = \frac{\partial p_r}{\partial r} - p_r \Gamma_{rr}^r - p_{\theta} \Gamma_{rr}^{\theta} \implies p_{r;r} = \frac{\partial p_r}{\partial r} \quad (67)$$

Where  $p_r$  is the  $r$ -component of the one-form in a polar basis.

$$p_r = r^2(\cos^3 \theta + \sin^3 \theta) + 6r \sin \theta \cos \theta \implies p_{r;r} = \frac{\partial p_r}{\partial r} = 2r(\cos^3 \theta + \sin^3 \theta) + 6 \sin \theta \cos \theta \quad (68)$$

$$p_{r;\theta} = \frac{\partial p_r}{\partial \theta} - p_r \Gamma_{r\theta}^r - p_{\theta} \Gamma_{r\theta}^{\theta} = \frac{\partial p_r}{\partial \theta} - \frac{1}{r} p_{\theta} \quad (69)$$

**Exercise (5.14):** For the tensor whose polar coordinates are ( $A^{rr} = r^2$ ,  $A^{r\theta} = r \sin \theta$ ,  $A^{\theta r} = r \cos \theta$ ,  $A^{\theta\theta} = \tan \theta$ ), compute in Eq. (5.65) in polars for all possible indices:

$$\begin{aligned} \nabla_r A^{rr} &= \frac{\partial A^{rr}}{\partial r} + A^{\alpha r} \Lambda_{\alpha r}^r + A^{r\alpha} \Lambda_{\alpha r}^r = \frac{\partial A^{rr}}{\partial r} + A^{rr} \Lambda_{rr}^r + A^{\theta r} \Lambda_{rr}^r + A^{rr} \Lambda_{rr}^r + A^{r\theta} \Lambda_{rr}^r \\ &\implies \nabla_r A^{rr} = \frac{\partial A^{rr}}{\partial r} = \frac{\partial}{\partial r} (r^2) = 2r \end{aligned} \quad (70)$$

$$\begin{aligned} \nabla_{\theta} A^{rr} &= \frac{\partial A^{rr}}{\partial \theta} + A^{rr} \Lambda_{r\theta}^r + A^{\theta r} \Lambda_{r\theta}^r + A^{rr} \Lambda_{r\theta}^r + A^{r\theta} \Lambda_{r\theta}^r \\ \implies \nabla_{\theta} A^{rr} &= -r(A^{\theta r} + A^{r\theta}) + \frac{\partial A^{rr}}{\partial \theta} = -r(r \cos \theta + r \sin \theta) + \frac{\partial}{\partial \theta} (r^2) = -r^2(\cos \theta + \sin \theta) \end{aligned} \quad (71)$$

$$\begin{aligned} \nabla_{\theta} A^{r\theta} &= \frac{\partial A^{r\theta}}{\partial \theta} + A^{r\theta} \Gamma_{r\theta}^r + A^{\theta\theta} \Gamma_{r\theta}^r + A^{rr} \Gamma_{r\theta}^{\theta} + A^{r\theta} \Gamma_{r\theta}^{\theta} = \frac{\partial A^{r\theta}}{\partial \theta} - r(A^{\theta\theta}) + \frac{1}{r}(A^{rr}) \\ &\implies \nabla_{\theta} A^{r\theta} = r(\cos \theta - \tan \theta - 1) \end{aligned} \quad (72)$$

And the five remaining computations for all possible indices ( $\nabla_r A^{r\theta}$ ,  $\nabla_r A^{\theta r}$ ,  $\nabla_{\theta} A^{\theta r}$ ,  $\nabla_r A^{\theta\theta}$ ,  $\nabla_{\theta} A^{\theta\theta}$ ) can be computed in exactly the same manner.

**Exercise (5.16):** Fill in all the missing steps leading from Eq. (5.74) to Eq. (5.75).

**Solution:** Starting with Eq. (5.72):

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma_{\alpha\mu}^{\nu} g_{\nu\beta} - \Gamma_{\beta\mu}^{\nu} g_{\alpha\nu} \quad (73)$$

And using the fact that  $g_{\alpha'\mu';\beta'} = 0$ :

$$g_{\alpha'\beta',\mu'} = \Gamma_{\alpha'\mu'}^{\nu'} g_{\nu'\beta'} + \Gamma_{\beta'\mu'}^{\nu'} g_{\alpha'\nu'} \implies g_{\alpha\beta,\mu} = \Gamma_{\alpha\mu}^{\nu} g_{\nu\beta} + \Gamma_{\beta\mu}^{\nu} g_{\alpha\nu} \quad (74)$$

And since  $\alpha, \beta, \mu$  are dummy indices whose order can be rearranged in the previous expression, the following form can be arrived at by switching the  $\beta$  and  $\mu$  indices:

$$g_{\alpha\mu,\beta} = \Gamma_{\alpha\beta}^{\nu} g_{\nu\mu} + \Gamma_{\mu\beta}^{\nu} g_{\alpha\nu} \quad (75)$$

And the following expression can be arrived at by switching  $\alpha$  with  $\beta$  in Eq. (75) and multiplying the whole expression by a negative sign:

$$g_{\beta\mu,\alpha} = \Gamma_{\beta\alpha}^{\nu} g_{\nu\mu} + \Gamma_{\mu\alpha}^{\nu} g_{\beta\nu} \implies -g_{\beta\mu,\alpha} = -\Gamma_{\beta\alpha}^{\nu} g_{\nu\mu} - \Gamma_{\mu\alpha}^{\nu} g_{\beta\nu} \quad (76)$$

We can now consider the addition of the three terms,  $g_{\alpha\beta,\mu}$ ,  $g_{\alpha\mu,\beta}$ ,  $-g_{\beta\mu,\alpha}$ :

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \Gamma_{\alpha\mu}^{\nu} g_{\nu\beta} + \Gamma_{\beta\mu}^{\nu} g_{\alpha\nu} + \Gamma_{\alpha\beta}^{\nu} g_{\nu\mu} + \Gamma_{\mu\beta}^{\nu} g_{\alpha\nu} - \Gamma_{\beta\alpha}^{\nu} g_{\nu\mu} - \Gamma_{\mu\alpha}^{\nu} g_{\beta\nu} \quad (77)$$

And, using the fact that the indices of the metric can be interchanged ( $g_{\beta\nu} = g_{\nu\beta}$ ), we arrive at:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = (\Gamma_{\alpha\mu}^{\nu} - \Gamma_{\mu\alpha}^{\nu}) g_{\nu\beta} + (\Gamma_{\alpha\beta}^{\nu} - \Gamma_{\beta\alpha}^{\nu}) g_{\nu\mu} + (\Gamma_{\beta\mu}^{\nu} + \Gamma_{\mu\beta}^{\nu}) g_{\alpha\nu} \quad (78)$$

Since the lower indices of the Christoffel symbols may be interchanged, this leaves us with:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2\Gamma_{\beta\mu}^{\nu} g_{\alpha\nu} \quad (79)$$



Using the fact that inverting the metric just turns its covariant indices into contravariant indices ( $1/g_{\alpha\beta} = g^{\alpha\beta}$ ):

$$\Gamma_{\beta\mu}^{\nu} = \frac{1}{2}g^{\alpha\nu}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \quad (80)$$

It's important to remind the reader of the notation being used here to understand the meaning of this result. Recall that  $\phi_{,\alpha} \equiv \frac{\partial\phi}{\partial x^{\alpha}}$  so the previous expression becomes:

$$\Gamma_{\beta\mu}^{\nu} = \frac{1}{2}g^{\alpha\nu} \left( \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} + \frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}} \right) \quad (81)$$

Meaning the Christoffel symbols can be written in terms of derivatives of the metric.

**Exercise (5.22):** Show that if  $U^{\alpha}\nabla_{\alpha}V^{\beta} = W^{\beta}$ , then  $U^{\alpha}\nabla_{\alpha}V_{\beta} = W_{\beta}$

**Solution:** Recall the notation that  $\nabla_{\alpha}V^{\beta} \equiv V_{;\alpha}^{\beta}$  from Eq. (5.51). This turns the expression into:

$$U^{\alpha}V_{;\alpha}^{\beta} = W^{\beta} \quad (82)$$

We can then multiply both sides of the expression by the metric  $g_{\mu\beta}$ :

$$U^{\alpha}g_{\mu\beta}V_{;\alpha}^{\beta} = g_{\mu\beta}W^{\beta} \quad (83)$$

From Eq. (5.68),  $V_{\alpha;\beta} = g_{\alpha\mu}V_{;\beta}^{\mu}$  so we can transform the left hand side of this expression to be:

$$U^{\alpha}V_{\mu;\alpha} = g_{\mu\beta}W^{\beta} \quad (84)$$

And we can finally use  $V_{\alpha} = g_{\alpha\mu}V^{\mu}$  from Eq. (5.69) to simply the right side of the expression into:

$$U^{\alpha}V_{\mu;\alpha} = W_{\mu} \quad (85)$$

And since  $\mu$  is just a dummy index, it can be changed for  $\beta$ , resulted in the desired expression:

$$U^{\alpha}\nabla_{\alpha}V_{\beta} = W_{\beta} \quad (86)$$

## 6 Chapter 6: Curved Manifolds

**Exercise (6.6):** Prove that the first term in Eq. (6.37) vanishes.

**Solution:** Starting with Eq. (6.37):

$$\Gamma_{\mu\alpha}^{\alpha} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta}) + \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu} \quad (87)$$

We want to show that  $g^{\alpha\beta}g_{\beta\mu,\alpha} = g^{\alpha\beta}g_{\mu\alpha,\beta}$  to show that the first term vanishes. To do this, first use the fact that the metric is symmetric so the its indices can be interchanged as in following step:

$$g^{\alpha\beta}g_{\beta\mu,\alpha} \implies g^{\beta\alpha}g_{\beta\mu,\alpha} \quad (88)$$

Then use the fact that the indices considered are dummy indices so the swap  $\alpha \rightarrow \beta$  &  $\beta \rightarrow \alpha$  can be made:

$$g^{\beta\alpha}g_{\beta\mu,\alpha} \implies g^{\alpha\beta}g_{\alpha\mu,\beta} \quad (89)$$

And again use the fact that the metric is symmetric so the indices  $\alpha$  and  $\mu$  can be interchanged:

$$g^{\alpha\beta}g_{\alpha\mu,\beta} \implies g^{\alpha\beta}g_{\mu\alpha,\beta} \quad (90)$$

Reaching the desired result.

**Exercise (6.8):** Fill in the missing algebra leading to Eqs. (6.40) and (6.42).

**Solution:** Starting from Eq. (6.38):

$$\Gamma_{\mu\alpha}^{\alpha} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu} \quad (91)$$

We can swap the  $\alpha$  and  $\beta$  indices on  $g_{\alpha\beta,\mu}$  so that we can use Eq. (6.39):

$$g_{,\mu} = gg^{\alpha\beta}g_{\beta\alpha,\mu} \implies g_{,\mu} \frac{1}{g} = g^{\alpha\beta}g_{\beta\alpha,\mu} \quad (92)$$

as a substitution:

$$\Gamma_{\mu\alpha}^\alpha = \frac{1}{2} \frac{1}{g} g_{,\mu} \quad (93)$$

And in a step that I really don't understand, this becomes:

$$\Gamma_{\mu\alpha}^\alpha = \frac{1}{\sqrt{-g}} (\sqrt{-g})_{,\mu} \quad (94)$$

resulting in Eq. (6.40).

With this, Eq. (5.49) can be used where  $\beta \rightarrow \alpha$  since this is just a dummy index and using the new definition of  $\Gamma_{\mu\alpha}^\alpha$  in Eq. (6.40):

$$V_{;\alpha}^\alpha = V_{,\alpha}^\alpha + V^\mu \Gamma_{\mu\alpha}^\alpha = V_{,\alpha}^\alpha + V^\mu \left( \frac{1}{\sqrt{-g}} (\sqrt{-g})_{,\mu} \right) \quad (95)$$

$V_{;\alpha}^\alpha$  can then be multiplied by  $\frac{\sqrt{-g}}{\sqrt{-g}}$  so that it can be combined with the other term in the expression:

$$V_{;\alpha}^\alpha = \frac{1}{\sqrt{-g}} (\sqrt{-g} V_{,\alpha}^\alpha + V^\alpha \sqrt{-g}_{,\mu}) \quad (96)$$

And since the term in the parentheses is just the definition of the chain rule:

$$V_{;\alpha}^\alpha = \frac{1}{\sqrt{-g}} (\sqrt{-g} V^\alpha)_\alpha \quad (97)$$

Which results in Eq. (6.42).

**Exercise (6.13a):** Show that if  $\vec{A}$  and  $\vec{B}$  are parallel-transported along a curve, then  $\mathbf{g}(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}$  is constant on the curve.

**Solution:** If we parallel transport  $\vec{A}$  and  $\vec{B}$  along a curve,  $\vec{U}$ , then the following condition is satisfied:

$$\frac{d\vec{V}}{d\lambda} = \frac{\partial V^\alpha}{\partial x^\beta} \frac{dx^\beta}{d\lambda} = 0 \quad (98)$$

for both  $\vec{A}$  and  $\vec{B}$  where  $\frac{dx^\beta}{d\lambda}$  is the curve  $\vec{U}$ . If we parallel transport  $\vec{A} \cdot \vec{B}$  along the curve, then:

$$\begin{aligned} \frac{d}{d\lambda} (g_{\alpha\beta} A^\alpha B^\beta) &= \frac{\partial}{\partial x^\beta} (g_{\alpha\beta} A^\alpha B^\beta) \frac{dx^\beta}{d\lambda} \\ &= \left( A^\alpha B^\beta \frac{\partial g_{\alpha\beta}}{\partial x^\beta} + g_{\alpha\beta} B^\alpha \frac{\partial A^\alpha}{\partial x^\beta} + g_{\alpha\beta} A^\alpha \frac{\partial B^\beta}{\partial x^\beta} \right) \frac{dx^\beta}{d\lambda} \end{aligned} \quad (99)$$

Note that parallel transport requires the second and third term of this expression to equal zero and that the local flatness theorem ( $g_{\alpha\beta,\gamma} = 0$ ) requires the first term to equal zero, leaving us to conclude that  $\vec{A} \cdot \vec{B}$  is constant along the curve  $\vec{U}$ .

**Exercise (6.13b):** Conclude from this that if a *geodesic* is spacelike (or timelike or null) somewhere, it is spacelike (or timelike or null) everywhere.

**Solution:** The conditions for determining whether a geodesic  $\vec{U}$  is spacelike, timelike, or null are given below:

$$\vec{U} \cdot \vec{U} = \begin{cases} < 0 & \text{timelike} \\ = 0 & \text{null} \\ > 0 & \text{spacelike} \end{cases} \quad (100)$$

Since we have shown that  $\vec{U} \cdot \vec{U}$  is constant, we know that if the geodesic is defined to be spacelike, timelike, or null anywhere, it must satisfy this condition everywhere.

**Exercise (6.14):** The proper distance along a curve whose tangent is  $\vec{V}$  is given by Eq. (6.8). Show that if the curve is a geodesic, then the proper length is an affine parameter. (Use the result of Exer. 13.)

**Solution:** From Eq. (6.8), the proper distance is defined to be:

$$\ell = \int_{\lambda_0}^{\lambda_1} |\vec{V} \cdot \vec{V}|^{1/2} d\lambda \quad (101)$$

And since we know that  $\vec{V} \cdot \vec{V}$  is a constant from Exer. (6.13), this integral will just result in  $\vec{V} \cdot \vec{V}$  being multiplied by the length of the line:

$$\ell = |\vec{V} \cdot \vec{V}|^{1/2} \int_{\lambda_0}^{\lambda_1} d\lambda = |\vec{V} \cdot \vec{V}|^{1/2} \lambda \quad (102)$$

And since an affine parameter is defined to be  $\phi = a\lambda + b$  on page 167 of the text,  $\ell$  must be an affine parameter with  $a = |\vec{V} \cdot \vec{V}|^{1/2}$  and  $b = 0$  (since, again,  $\vec{V} \cdot \vec{V}$  was found to be a constant from the previous exercise).

**Exercise 6.19:** Prove that  $R_{\beta\mu\nu}^\alpha = 0$  for polar coordinates in the Euclidean plane. Use Eq. (5.44) or equivalent results.

**Solution:** Using the definition of the Riemann curvature tensor given by Eq. (6.63):

$$R_{\beta\mu\nu}^\alpha = \frac{\partial}{\partial x^\mu} \Gamma_{\beta\nu}^\alpha - \frac{\partial}{\partial x^\nu} \Gamma_{\beta\mu}^\alpha + \Gamma_{\sigma\mu}^\alpha \Gamma_{\beta\nu}^\sigma - \Gamma_{\sigma\nu}^\alpha \Gamma_{\beta\mu}^\sigma \quad (103)$$

Where the  $\alpha, \beta, \mu, \nu, \sigma$  indices will be summed over  $r, \theta$  in polar coordinates. It's best to consider these sums in parts since they will become so large. Starting with the first term in the expression for  $R_{\beta\mu\nu}^\alpha$ :

$$\frac{\partial}{\partial x^\mu} \Gamma_{\beta\nu}^\alpha = \frac{\partial}{\partial x^\mu} (\Gamma_{rr}^\alpha + \Gamma_{r\theta}^\alpha + \Gamma_{\theta r}^\alpha + \Gamma_{\theta\theta}^\alpha) = \frac{\partial}{\partial x^\mu} (\Gamma_\theta^r + \Gamma_{r\theta}^\theta + \Gamma_{\theta r}^\theta) = \frac{-2}{r^2} - r^2 \quad (104)$$

And since the  $-\frac{\partial}{\partial x^\nu} \Gamma_{\beta\mu}^\alpha$  component of the Riemann curvature tensor changes nothing but the sign of the result shown above,  $\frac{\partial}{\partial x^\mu} \Gamma_{\beta\nu}^\alpha - \frac{\partial}{\partial x^\nu} \Gamma_{\beta\mu}^\alpha = 0$ . Computing the  $\Gamma_{\sigma\mu}^\alpha \Gamma_{\beta\nu}^\sigma$  component of  $R_{\beta\mu\nu}^\alpha$ :

$$\Gamma_{\sigma\mu}^\alpha \Gamma_{\beta\nu}^\sigma = \Gamma_{\theta\theta}^r \Gamma_{\beta\nu}^\theta = \Gamma_{\theta\theta}^r \Gamma_{r\theta}^\theta + \Gamma_{\theta\theta}^r \Gamma_{\theta r}^\theta + \Gamma_{r\theta}^\theta \Gamma_{\theta\theta}^r = -3 \quad (105)$$

And, again, since  $-\Gamma_{\sigma\nu}^\alpha \Gamma_{\beta\mu}^\sigma$  changes nothing but the sign of the previous result,  $\Gamma_{\sigma\mu}^\alpha \Gamma_{\beta\nu}^\sigma - \Gamma_{\sigma\nu}^\alpha \Gamma_{\beta\mu}^\sigma = 0$ , resulting in  $R_{\beta\mu\nu}^\alpha = 0$ , which should be expected since we are computing the Riemann curvature tensor in the Euclidean plane, which is defined to have no curvature.

**Exercise (6.20):** Fill in the algebra necessary to establish Eq. (6.73).

**Solution:** Starting from:

$$\nabla_\alpha \nabla_\beta V^\mu \quad (106)$$

And using the definition of a covariant derivative built in the previous chapter, specifically Eq. (5.48):

$$\nabla_\alpha \nabla_\beta V^\mu = \nabla_\alpha \left( \frac{\partial V^\mu}{\partial x^\beta} + V^\nu \Gamma_{\nu\beta}^\mu \right) \quad (107)$$

The covariant derivative with respect to  $\alpha$  can then be distributed across this expression like so:

$$\nabla_\alpha \left( \frac{\partial V^\mu}{\partial x^\beta} + V^\nu \Gamma_{\nu\beta}^\mu \right) = \frac{\partial V^\mu}{\partial x^\alpha \partial x^\beta} + V^\nu \frac{\partial \Gamma_{\nu\beta}^\mu}{\partial x^\alpha} + \Gamma_{\nu\beta}^\mu \frac{\partial V^\nu}{\partial x^\alpha} \quad (108)$$

And since we are considering these covariant derivatives in a locally inertial frame at some point, the  $\Gamma_{\nu\beta}^\mu$  term goes to zero but its partial derivative does not, leaving us with:

$$\nabla_\alpha \nabla_\beta V^\mu = \frac{\partial V^\mu}{\partial x^\alpha \partial x^\beta} + V^\nu \frac{\partial \Gamma_{\nu\beta}^\mu}{\partial x^\alpha} \quad (109)$$

## 7 Chapter 7: Physics in a Curved Spacetime

**Exercise (7.2):** To first order in  $\phi$ , compute  $g^{\alpha\beta}$  for Eq. (7.8).

**Solution:** Eq. (7.8) gives the line element for the ordinary Newtonian potential to the first order to be:

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2) \quad (110)$$

so the metric can be inferred to be:

$$g^{\alpha\beta} = \begin{pmatrix} -(1 + 2\phi) & 0 & 0 & 0 \\ 0 & (1 - 2\phi) & 0 & 0 \\ 0 & 0 & (1 - 2\phi) & 0 \\ 0 & 0 & 0 & (1 - 2\phi) \end{pmatrix} \quad (111)$$

**Exercise (7.3):** Calculate all the Christoffel symbols for the metric given by Eq. (7.8), to first order in  $\phi$ . Assume  $\phi$  is a general function of  $t, x, y$ , and  $z$ .

**Solution:** Using Eq. (5.75), which allows us to compute the Christoffel symbols in terms of the metric and its derivatives, the components of the Christoffel symbols can be computed. Take  $\Gamma_{xx}^x$ , for example. This would be computed by:

$$\begin{aligned}\Gamma_{xx}^x &= \frac{1}{2}g^{tx}(g_{tx,x} + g_{tx,x} - g_{xx,t}) + \frac{1}{2}g^{xx}(g_{xx,x} + g_{xx,x} - g_{xx,x}) \\ &\quad + \frac{1}{2}g^{yx}(g_{yx,x} + g_{yx,x} - g_{xx,y}) + \frac{1}{2}g^{zx}(g_{zx,x} + g_{zx,x} - g_{xx,z})\end{aligned}\tag{112}$$

Which reduces nicely since the non-zero components of this metric are only found along the diagonal, meaning that only  $g^{tt}, g^{xx}, g^{yy}, g^{zz}$  will have non-zero components. This results in:

$$\begin{aligned}\Gamma_{xx}^x &= \frac{1}{2}g^{xx}(g_{xx,x} + g_{xx,x} - g_{xx,x}) = \frac{1}{2}g^{xx}\frac{\partial g_{xx}}{\partial x} \\ &= \frac{1}{2}(1 - 2\phi)\frac{\partial}{\partial x}(1 - 2\phi) = -(1 - 2\phi)\frac{\partial \phi}{\partial x}\end{aligned}\tag{113}$$

And the other 63 components of  $\Gamma_{\beta\mu}^\gamma$  can be computed in the same way, using the simplifications used above.