A First Course in General Relativity - Selected Solutions

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1 Chapter 1: Special Relativity

2 Chapter 2: Vector Analysis in Special Relativity

Exercise 2.3: Prove Eq. (2.5)

Solution: By convention, Latin indices are not summed over 0 so if we are to interchange them with Greek indices as a dummy index, we must perform the following:

$$\Lambda^{\bar{\alpha}}_{\beta}\Delta x^{\beta} = \Lambda^{\bar{\alpha}}_{0}\Delta x^{0} + \Lambda^{\bar{\alpha}}_{i}\Delta x^{i} \tag{1}$$

since $\Lambda^{\bar{\alpha}}_{\beta}\Delta x^{\beta}$ implies a sum over all of the positive real numbers where $\Lambda^{\bar{\alpha}}_{i}\Delta x^{i}$ is a sum over positive real numbers not including zero.

Exercise 2.7a: Prove Eq. (2.10) for all α, β

Solution: To verify that $(\vec{e}_{\alpha})^{\beta} = \delta^{\beta}_{\alpha}$, consider an arbitrary basis vector, $\vec{e}_{\alpha}$, meaning that the elements in its list are all zero except for the single entry at the αth component. This can be written as:

$$\vec{e}_{\alpha} = (..., 0, 0, 1, 0, ...) \tag{2}$$

Where the index of each value in the list can be traced with respect to α

$$(\alpha - n, ..., \alpha - 2, \alpha - 1, \alpha, \alpha + 1, ..., \alpha + n) \tag{3}$$

Then $(\vec{e}_{\alpha})^{\beta}$ indicates the βth component of the basis vector $\vec{e}_{\alpha}$. By the definition of a basis vector, we know that all entries in $\vec{e}_{\alpha}$ are zero except the one at the αth component. So if we choose β to be any non-α index, the result must be 0:

$$(\vec{e}_{\alpha})^{\alpha-1} = 0 \tag{4}$$

It's for this reason that we can define the βth component of the $\vec{e}_{\alpha}$ basis vector to be equal to the Kronecker delta, meaning that $(\vec{e}_{\alpha})^{\beta} = 1$ only when $\alpha = \beta$.

Exercise 2.29: Prove, using component expressions, Eqs. (2.24) and (2.26), that

$$\frac{d}{d\tau}(\vec{U}\cdot\vec{U}) = 2\vec{U}\cdot\frac{d\vec{U}}{d\tau} \tag{5}$$

Solution: By (2.26):

$$\begin{aligned} \vec{U}\cdot\vec{U} &= -U^0U^0 + U^1U^1 + U^2U^2 + U^3U^3 \\ &= -(U^0)^2 + (U^1)^2 + (U^2)^2 + (U^3)^2 \end{aligned} \tag{6}$$

and by (2.24):

$$\vec{U}\cdot\vec{U} = \vec{U}^2 \implies \frac{d}{d\tau}(\vec{U}\cdot\vec{U}) = \frac{d}{d\tau}(\vec{U}^2) = 2\vec{U}\cdot\frac{d\vec{U}}{d\tau} \tag{7}$$

3 Chapter 3: Tensor Analysis in Special Relativity

Exercise (3.1a): Given an arbitrary set of numbers $\{M_{\alpha\beta}; \alpha = 0, ..., 3; \beta = 0, ..., 3\}$ and two arbitrary vector components $\{A^{\mu}, \mu = 0, ..., 3\}$ and $\{B^{\nu}, \nu = 0, ..., 3\}$, show that the two expressions

$$M_{\alpha\beta} A^{\alpha} B^{\beta} \tag{8}$$

and

$$M_{\alpha\alpha} A^{\alpha} B^{\alpha} \tag{9}$$

are not equivalent.

Solution:

$$M_{\alpha\alpha} A^{\alpha} B^{\alpha} = M_{00} A^0 B^0 + M_{11} A^1 B^1 + M_1 A^1 B^1 + M_{11} A^1 B^1 \tag{10}$$

where

$$M_{\alpha\beta} A^{\alpha} B^{\beta} = B^{\beta} (M_{0\beta} A^0 + M_{1\beta} A^1 + M_{2\beta} A^2 + M_{3\beta} A^3) \tag{11}$$

So $M_{\alpha\alpha} A^{\alpha} B^{\alpha}$ only contains the diagonal terms of $M_{\alpha\beta} A^{\alpha} B^{\beta}$.
Exercise (3.1b): Show that $A^{\alpha} B^{\beta} \eta_{\alpha\beta} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$

Solution:
Because

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

Any component of $A^{\alpha} B^{\beta} \eta_{\alpha\beta}$ where $\alpha \neq \beta$ means multiplying $A^{\alpha} B^{\beta}$ by an off-diagonal component of $\eta_{\alpha\beta}$, which are all 0. Treating A^{α} and B^{β} as row/column matrices and carrying out their multiplication with $\eta_{\alpha\beta}$ will result in $-A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$.
Exercise (3.3a): Prove, by writing out all of the terms, the validity of the following:

$$\tilde{p}(A^{\alpha} \vec{e}_{\alpha}) = A^{\alpha} \tilde{p}(\vec{e}_{\alpha}) \tag{13}$$

Solution:
Since one-forms act on vector arguments, the scalar values associated with A^{α} may be pulled out of the expression like so:

$$\begin{aligned} \tilde{p}(A^{\alpha} \vec{e}_{\alpha}) &= \tilde{p}(A^0 \vec{e}^0 + A^1 \vec{e}^1 + A^2 \vec{e}^2 + A^3 \vec{e}^3) = \tilde{p}(A^0 \vec{e}_0) + \tilde{p}(A^1 \vec{e}_1) + \tilde{p}(A^2 \vec{e}_2) + \tilde{p}(A^3 \vec{e}_3) \\ &= A^0 \tilde{p}(\vec{e}_0) + A^1 \tilde{p}(\vec{e}_1) + = A^2 \tilde{p}(\vec{e}_2) + A^3 \tilde{p}(\vec{e}_3) = A^{\alpha} \tilde{p}(\vec{e}_{\alpha}) \end{aligned} \tag{14}$$

Exercise (3.5): Justify each step leading from Eqs. (3.10a) to (3.10d).

Solution: To establish the frame-independence of $A^{\bar{\alpha}} p_{\bar{\alpha}}$:

$$\begin{aligned} A^{\bar{\alpha}} p_{\bar{\alpha}} &= A^{\bar{\alpha}} \tilde{p}(\vec{e}_{\bar{\alpha}}), \\ \vec{e}_{\bar{\alpha}} &= \Lambda^{\mu}_{\bar{\alpha}} \vec{e}_{\mu}, \\ A^{\bar{\alpha}} &= \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \\ \implies A^{\bar{\alpha}} \tilde{p}(\vec{e}_{\bar{\alpha}}) = \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \tilde{p}(\Lambda^{\mu}_{\bar{\alpha}} \vec{e}_{\mu}) &= \Lambda^{\bar{\alpha}}_{\beta} \Lambda^{\mu}_{\bar{\alpha}} A^{\beta} \tilde{p}(\vec{e}_{\mu}) = \Lambda^{\bar{\alpha}}_{\beta} \Lambda^{\mu}_{\bar{\alpha}} A^{\beta} p_{\mu} \end{aligned} \tag{15}$$

and by Eq. (2.18):

$$\Lambda^{\bar{\alpha}}_{\beta} \Lambda^{\mu}_{\bar{\alpha}} A^{\beta} p_{\mu} = \delta^{\mu}_{\beta} A^{\beta} p_{\mu} = A^{\beta} p_{\beta} \implies A^{\bar{\alpha}} p_{\bar{\alpha}} = A^{\beta} p_{\beta} \tag{16}$$

which should not be a surprising result given that the one-form of a vector produces a scalar and scalars are invariant quantities under Lorentz transformations.
Exercise (3.10a): Given a frame $\mathcal{O}$ whose coordinates are $\{x^{\alpha}\}$, show that:

$$\frac{\partial x^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} \tag{17}$$

Solution: Taking the partial with respect to x^{β} means holding all terms in x constant except for the β index. When

$\alpha \neq \beta$, you are taking a partial of a fixed value (a constant), meaning your derivative will be equal to zero. When you differentiate the β index with respect to β, you will always get 1.
Exercise (3.10b): For any two frames, we have Eq. (3.18):

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} = \Lambda^{\beta}_{\bar{\alpha}}. \tag{18}$$

Show that (a) and the chain rule imply

$$\Lambda^{\beta}_{\bar{\alpha}} \Lambda^{\bar{\alpha}}_{\mu} = \delta^{\beta}_{\mu} \tag{19}$$

Solution:

$$\begin{aligned} \Lambda^{\beta}_{\bar{\alpha}} &= \frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}}, \\ \Lambda^{\bar{\alpha}}_{\mu} &= \frac{\partial x^{\bar{\alpha}}}{\partial x^{\mu}} \\ \implies \Lambda^{\beta}_{\bar{\alpha}} \Lambda^{\bar{\alpha}}_{\mu} &= \frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\alpha}}}{\partial x^{\mu}} = \frac{\partial x^{\beta}}{\partial x^{\mu}} = \delta^{\beta}_{\mu} \end{aligned} \tag{20}$$

Exercise (3.11a): Use the notation $\partial \phi / \partial x^{\alpha} = \phi_{,\alpha}$ to rewrite Eqs. (3.14), (3.15), and (3.18).

Solution:
Eq. (3.14):

$$\frac{\partial \phi}{\partial t} U^t + \frac{\partial \phi}{\partial t} U^x + \frac{\partial \phi}{\partial t} U^y + \frac{\partial \phi}{\partial t} U^z \implies \phi_{,t} U^t + \phi_{,x} U^x + \phi_{,y} U^y + \phi_{,z} U^z \tag{21}$$

Eq. (3.15):

$$\tilde{d}\phi \underset{\mathcal{O}}{\rightarrow} \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \implies \tilde{d}\phi \underset{\mathcal{O}}{\rightarrow} (\phi_{,t}, \phi_{,x}, \phi_{,y}, \phi_{,z}) \tag{22}$$

Eq. (3.18):

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} = \Lambda^{\beta}_{\bar{\alpha}} \implies x^{\beta}_{,\bar{\alpha}} = \Lambda^{\beta}_{\bar{\alpha}} \tag{23}$$

Exercise (3.13): Prove, by geometric or algebraic arguments, that $\tilde{d}f$ is normal to surfaces of constant f.

Solution: If we consider any point $P = (t_0, x_0, y_0, z_0)$ along a parameterized level curve $f(\tau) = \phi(t(\tau), x(\tau), y(\tau), z(\tau) = c$, then we can take the gradient to be as follows:

$$\tilde{d}f = \partial \phi_{,t} \Big|_P \frac{dt}{d\tau}\Big|_{\tau_0} + \partial \phi_{,x} \Big|_P \frac{dx}{d\tau}\Big|_{\tau_0} + \partial \phi_{,y} \Big|_P \frac{dy}{d\tau}\Big|_{\tau_0} + \partial \phi_{,z} \Big|_P \frac{dz}{d\tau}\Big|_{\tau_0} = 0 \tag{24}$$

Since this is also the definition of the dot product between two vectors:

$$\left\langle \partial \phi_{,t} \Big|_P, \partial \phi_{,x} \Big|_P, \partial \phi_{,y} \Big|_P, \partial \phi_{,z} \Big|_P \right\rangle \cdot \left\langle \frac{dt}{d\tau}\Big|_{\tau_0}, \frac{dx}{d\tau}\Big|_{\tau_0}, \frac{dy}{d\tau}\Big|_{\tau_0}, \frac{dz}{d\tau}\Big|_{\tau_0} \right\rangle = 0 \tag{25}$$

whose product is equal to zero, we can say that $\tilde{d}f$ and f are normal to each other at every point along a level curve.
Exercise (3.14): Let $\tilde{p} \rightarrow_{\mathcal{O}} (1, 1, 0, 0)$ and $\tilde{q} \rightarrow_{\mathcal{O}} (-1, 0, 1, 0)$ be two one-forms. Prove, by trying two vectors $\vec{A}$ and $\vec{B}$ as arguments, that $\tilde{p} \otimes q \neq \tilde{\tilde{q}} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.
Solution: Since a one-form supplied with a vector argument: $\tilde{p}(\vec{A}) = p^{\alpha} A^{\alpha}$, we can perform the following operations to show $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$:

$$\begin{aligned} \tilde{p} \otimes \tilde{q} &= \tilde{p}(\vec{A}) \tilde{q}(\vec{B}) = (A^0 + A^1)(-B^0 + B^2) \\ \tilde{q} \otimes \tilde{p} &= \tilde{q}(\vec{A}) \tilde{p}(\vec{B}) = (-A^0 + A^2)(B^0 + B^1) \end{aligned} \tag{26}$$

Exercise (3.16a): Prove that $\mathbf{h}_{(s)}$ defined by

$$\mathbf{h}_{(s)}(\vec{A}, \vec{B}) = \frac{1}{2} \mathbf{h}(\vec{A}, \vec{B}) + \frac{1}{2} \mathbf{h}(\vec{B}, \vec{A}) \tag{27}$$

is a symmetric tensor.

Solution: From Eq. (3.27), we know a tensor $\mathbf{f}$ is symmetric if:

$$\mathbf{f}(\vec{A},\vec{B}) = \mathbf{f}(\vec{B},\vec{A}) \forall \vec{A},\vec{B} \tag{28}$$

So if $\mathbf{h}$ is to be symmetric, then:

$$\frac{1}{2}\mathbf{h}(\vec{A},\vec{B}) - \frac{1}{2}\mathbf{h}(\vec{B},\vec{A}) = 0 \tag{29}$$

Let $\vec{A} = (A_0, A_1, A_2, A_3)$ and $\vec{B} = (B_0, B_1, B_2, B_3)$ then:

$$\frac{1}{2}\mathbf{h}(\vec{A},\vec{B}) = \frac{1}{2}(A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3) \tag{30}$$

and

$$\frac{1}{2}\mathbf{h}(\vec{B},\vec{A}) = \frac{1}{2}(B_0A_0 + B_1A_0 + B_2A_2 + B_3A_3) \tag{31}$$

and since multiplication is commutative, Eq. 21 must be true, meaning that $\mathbf{h}_{(s)}(\vec{A},\vec{B})$ is a symmetric tensor.

4 Chapter 4: Perfect Fluids in Special Relativity

Exercise (4.7): Derive Eq. (4.21).

Solution: Using the fact that $T^{\alpha\beta} = \rho U^\alpha U^\beta$ derived from Eq. (4.20), we can begin deriving the expressions in Eq. (4.21):

$$\begin{aligned} T^{00} = \rho U^0 U^0 = \rho \frac{1}{\sqrt{1-v^2}} \frac{1}{\sqrt{1-v^2}} &= \frac{\rho}{1-v^2} \\ T^{0i} = \rho U^0 U^i = \rho \frac{1}{\sqrt{1-v^2}} \frac{v^i}{\sqrt{1-v^2}} &= \frac{\rho v^i}{1-v^2} \\ T^{i0} = \rho U^i U^0 &= \frac{\rho v^i}{1-v^2} \\ T^{ij} = \rho U^i U^j &= \frac{\rho v^i v^j}{1-v^2} \end{aligned} \tag{32}$$

Exercise (4.10): Take the limit of Eq. (4.35) for $|\vec{V}| << 1$ to get $\partial n/\partial t + \partial(nv^i)/\partial x^i = 0$

Solution: Beginning with Eq. (4.35):

$$\frac{\partial}{\partial x^\alpha}(nU^\alpha) = 0 \tag{33}$$

With the knowledge that U^α contains both spatial components and a temporal component, the previous expression must be separated into two parts:

$$\frac{\partial}{\partial t}(nU^t) + \frac{\partial}{\partial x^i}(nU^i) = 0 \tag{34}$$

Using the expressions give on page 93 for U^t and U^i, this becomes:

$$\frac{\partial}{\partial t}\left(\frac{n}{\sqrt{1-v^2}}\right) + \frac{\partial}{\partial x^i}\left(\frac{nv^i}{\sqrt{1-v^2}}\right) = 0 \tag{35}$$

Then taking the limit where the speed is much less than 1 makes $1 - v^2 \approx 1$ so this expression becomes:

$$\frac{\partial n}{\partial t} + \frac{\partial(nv^i)}{\partial x^i} = 0 \tag{36}$$

Exercise (4.17): We have defined $a^\mu = U^\mu_{,\beta} U^\beta$. Go to the relativistic limit (small velocity) and show that $a^i = \dot{v}^i + (\vec{v} \cdot \nabla)v^j = Dv^i/Dt$ where the operator D/Dt is the usual "total" or "advective" time derivative of fluid dynamics.

Solution: Writing out an initial expression for a^i while again keeping in mind that U^i contains a temporal component and spatial components:

$$a^i = \frac{\partial U^i}{\partial x^\beta} U^\beta = \frac{\partial U^i}{\partial t} U^t + \frac{\partial U^i}{\partial x^j} U^j = \frac{\partial}{\partial t}\left(\frac{v^i}{\sqrt{1-v^2}}\right) \frac{1}{\sqrt{1-v^2}} + \frac{\partial}{\partial x^j}\left(\frac{v^i}{\sqrt{1-v^2}}\right) \frac{v^j}{\sqrt{1-v^2}} = 0 \tag{37}$$

Taking the non-relativistic limit:

$$a^i = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x^j} v^j = 0 \tag{38}$$

And since $\partial v^i / \partial x^j$ is just the dot product between the ith velocity component and the spatial derivatives under the Einstein summation convention, this expression becomes:

$$a^i = \dot{v}^i + (\vec{v} \cdot \nabla) v^j \tag{39}$$

Which is an expression defined to be the material or "advective" derivative used in fluid mechanics.

5 Chapter 5: Preface to Curvature

Exercise (5.3a): Show that the coordinate transformation $(x, y) \to (\xi, \eta)$ with $\xi = x$ and $\eta = 1$ violates Eq. (5.6).

Solution: For a transformation to be reasonable, it must assign all coordinates in the source (x, y) to distinct coordinates in the target (ξ, η). This property will be satisfied if the Jacobian is non-zero, which is the definition given by Eq. (5.6). So to show this transformation is not reasonable, it must be shown to violate Eq. (5.6):

$$\det \begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \tag{40}$$

So this transformation of coordinates is not reasonable.

Exercise (5.7): Calculate all elements of the transformation matrices $\Lambda^{\alpha'}_{\beta}$ and $\Lambda^{\nu'}_{\mu}$ for the transformation from Cartesian (x, y) - the unprimed indices - to polar (r, θ) - the primed indices.

Solution: Since, by Eq. (5.8):

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} \tag{41}$$

We can directly compute the transformation from Cartesian into Polar components by computing the terms of this matrix (knowing that $\xi(x, y) = r = \sqrt{x^2 + y^2}$ and $\eta(x, y) = \theta = \arctan(y/x)$ in polar coordinates). This results in:

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \tag{42}$$

Since $\Lambda^{\nu'}_{\mu}$ is defined as:

$$\Lambda^{\mu}_{\nu'} = \begin{pmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{pmatrix} = \begin{pmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{pmatrix} \tag{43}$$

in Eq. (5.13), the matrix is the following:

$$\Lambda^{\mu}_{\nu'} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \tag{44}$$

Exercise (5.8a): (Use the result of Exer.7.) Let $f = x^2 + y^2 + 2xy$ and in Cartesian Coordinates $\vec{V} \to (x^2 + 3y, y^2 + 3x)$, $\vec{W} \to (1, 1)$. Compute f as a function of r and θ, and find the components of $\vec{V}$ and $\vec{W}$ on the polar basis, expressing them as functions of r and θ.

Solution: Expressing f as a polar function is as simple as making the substitutions $x = r\cos\theta$ and $y = \sin\theta$, arriving at $f = r^2 + 2r^2\cos\theta\sin\theta$. To express $\vec{V}$ and $\vec{W}$ as polar functions, the same process can be applied. This results in $\vec{V} = (r^2\cos^2\theta + 3r\sin\theta, r^2\sin^2\theta + 3r\cos\theta)$ and $\vec{W} = (1, 1)$. To express $\vec{V}$ and $\vec{W}$ in a polar *basis*, though, you must use the transformations found in the previous problem:

$$V^{\alpha'} = \Lambda^{\alpha'}_{\beta} V^{\beta} \implies \vec{V} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix} = \begin{pmatrix} r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta \\ r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta - \sin^2\theta) \end{pmatrix} \tag{45}$$

$$\vec{W} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ (\cos\theta - \sin\theta)/r \end{pmatrix} \tag{46}$$

Exercise (5.8b): Find the components of $\tilde{d}f$ in Cartesian Coordinates and obtain them in polars (i) by direct calculation in polars, and (ii) by transforming components from Cartesian.

Solution: (i) To compute by direct calculation in polar: $\tilde{d}f = (\partial f/\partial r, \partial f/\partial \theta)$ we can use the definition of f in polar that was derived in part (a):

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left(r^2 + 2r^2\cos\theta\sin\theta\right) = 2r + 4r\cos\theta\sin\theta \tag{47}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta}\left(r^2 + 2r^2\cos\theta\sin\theta\right) = 2r^2\cos(2\theta) \tag{48}$$

(ii) To compute $\tilde{d}f$ by transforming components from Cartesian,

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial \phi}{\partial y} \implies \frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} \tag{49}$$

$$\frac{\partial f}{\partial r} = \cos\theta(2x+2y) + \sin\theta(2x+2y) = (\cos\theta + \sin\theta)(2r\cos\theta + 2r\sin\theta) = 2r + 4r\sin\theta\cos\theta \tag{50}$$

Similarly:

$$\frac{\partial f}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y} \tag{51}$$

$$\frac{\partial f}{\partial \theta} = (-r\sin\theta)(2x+2y) + (r\cos\theta)(2x+2y) = 2r^2\cos\theta \tag{52}$$

It should be noted that the expressions from (ii) match those derived from (i).

Exercise (5.8c): (i) Use the metric tensor in polar coordinates to find the polar components of the one-forms $\tilde{V}$ and $\tilde{W}$ associated with $\vec{V}$ and $\vec{W}$. (ii) Obtain the polar components of $\tilde{V}$ and $\tilde{W}$ by transformation of their Cartesian components.

Solution: (i) By Eq. (5.31), the metric tensor in polar coordinates is:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{53}$$

The metric in polar coordinates can be used to find the polar components of the one-forms by:

$$\tilde{W}_\alpha = g_{\alpha\beta}W^\beta = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}\begin{pmatrix} \cos\theta + \sin\theta \\ (\cos\theta - \sin\theta)/r \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ r(\cos\theta - \sin\theta) \end{pmatrix} \tag{54}$$

The same can be done for $\vec{V}$ as computed in polar form from part (a) of this problem.

(ii) Using the transformation matrix $\Lambda^\alpha_{\beta'}$ to obtain $\tilde{V}$ and $\tilde{W}$:

$$\Lambda^\alpha_{\beta'}\tilde{W}^\alpha = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ r(\cos\theta - \sin\theta) \end{pmatrix} \tag{55}$$

And the same process can be employed to solve for $\tilde{V}$. Note that $\tilde{V}$ and $\tilde{W}$ is just $\vec{V}$ and $\vec{W}$ in Cartesian coordinates since the metric tensor in Cartesian coordinates is the identity matrix.

Exercise (5.11a): For the vector field $\vec{V}$ whose Cartesian components are $(x^2 + 3y, y^2 + 3x)$, compute $V^\alpha_{,\beta}$ in Cartesian.

Solution: Since $V^\alpha_{,\beta} \equiv \partial V^\alpha/\partial x^\beta$:

$$V^\alpha_{,\beta} = \begin{pmatrix} \partial V^1/\partial x & \partial V^1/\partial y \\ \partial V^2/\partial x & \partial V^2/\partial y \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \tag{56}$$

Exercise (5.11b): Compute the transformation $\Lambda^{\mu'}_\alpha \Lambda^\beta_{\nu'} V^\alpha_{,\beta}$ to polars.

This computation is a straightforward usage of the transformation matrices $\Lambda^{\mu'}_\alpha$ and $\Lambda^\beta_{\nu'}$ from Cartesian to polar coordinates derived in Exercise 5.7 and the polar form of $V^\alpha_{,\beta}$ found in the previous part to this problem. The order of multiplication for these matrices should be noted, however, since computing $\Lambda^{\mu'}_{\alpha}\Lambda^{\beta}_{\nu'}V^{\alpha}_{,\beta}$ would leave $V^{\alpha}_{,\beta}$ unchanged. Computing $\Lambda^{\mu'}_{\alpha}V^{\alpha}_{,\beta}\Lambda^{\beta}_{\nu'}$ results in:

$$\Lambda^{\mu'}_{\alpha}V^{\alpha}_{,\beta}\Lambda^{\beta}_{\nu'} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} 2(r\cos^3\theta + 3\cos\theta\sin\theta + r\sin^3\theta) - r(\cos\theta - \sin\theta) & (-3\sin\theta + \cos\theta)(-3 + 2r\sin\theta) \\ \frac{(\cos\theta - \sin\theta)(3\cos\theta + 3\sin\theta - r\sin(2\theta))}{r} & (-3 + r\cos\theta + r\sin\theta)\sin(2\theta) \end{pmatrix} \tag{57}$$

Exercise (5.11c): Compute the components $V^{\mu'}_{;\nu'}$ directly in polars using the Christoffel symbols.

Solution: Since $\alpha, \beta \in \{x, y\}$ in Cartesian coordinates, there will be four components to compute: $V^r_{;r}, V^\theta_{;r}, V^r_{;\theta}, V^\theta_{;\theta}$. Beginning with $V^r_{;r}$:

$$V^r_{;r} = \frac{\partial V^r}{\partial r} + V^\mu\Gamma^r_{\mu r} \text{ \& } \Gamma^\mu_{rr} = \forall\mu \implies V^r_{;r} = \frac{\partial V^r}{\partial r} + V^\theta\Gamma^r_{\theta r}$$
$$\Gamma^r_{\theta r} : \frac{\partial\vec{e_\theta}}{\partial r} = -\sin\theta\vec{e_x} + \cos\theta\vec{e_y} = \frac{1}{r}\vec{e_\theta} \implies \Gamma^r_{\theta r} = 0 \tag{58}$$
$$\implies V^r_{;r} = \frac{\partial V^r}{\partial r} = \frac{\partial}{\partial r}\left(r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta\right) = 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta$$

$$V^\theta_{;r} = \frac{\partial V^\theta}{\partial r} + V^\mu\Gamma^\theta_{\mu r} = \frac{\partial V^\theta}{\partial r} + \frac{1}{r}V^\theta =$$
$$\frac{\partial}{\partial r}\left(r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta - \sin^2\theta)\right) + \frac{1}{r}\left(r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta) + 3(\cos^2\theta - \sin^2\theta)\right) \tag{59}$$
$$\implies V^\theta_{;r} = \frac{(\cos\theta - \sin\theta)(3\cos\theta + 3\sin\theta) - r\sin(2\theta)}{r}$$

$$V^r_{;\theta} = \frac{\partial V^r}{\partial\theta} + V^\mu\Gamma^r_{\mu\theta} = \frac{\partial V^r}{\partial\theta} - rV^\theta$$
$$\implies V^r_{;\theta} = -r(\cos\theta - \sin\theta)(-3\cos\theta + 3\sin\theta) + r\sin(2\theta) \tag{60}$$

$$V^\theta_{;\theta} = \frac{\partial V^\theta}{\partial\theta} + \frac{1}{r}V^r = \sin(2\theta)(-3 + r\cos\theta + r\sin\theta)) \tag{61}$$

Exercise (5.11d): Compute the divergence $V^\alpha_{,\alpha}$ using results from part (a).

Solution:

$$V^\alpha_{,\alpha} = \frac{\partial V^\alpha}{\partial x^\alpha} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} = 2(x + y) = 2r(\cos\theta + \sin\theta) \tag{62}$$

Exercise (5.11e): Compute the divergence $V^{\mu'}_{;\mu'}$ using results from either part (b) or (c).

Solution:

$$V^{\mu'}_{;\mu'} = V^r_{;r} + V^\theta_{;\theta} = \frac{\partial V^r}{\partial r} + \Gamma^r_{rr}V^r + \Gamma^r_{\theta r}V^\theta + \frac{\partial V^r}{\partial r} + \Gamma^\theta_{\theta\theta}V^\theta + \Gamma^\theta_{r\theta}V^r$$
$$= \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial\theta} + \frac{1}{r}V^r = 2r(\cos\theta + \sin\theta) \tag{63}$$

Exercise (5.11f): Compute the divergence $V^{\mu'}_{;\mu'}$ using Eq. (5.55) directly.

$$V^{\mu'}_{;\mu'} = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial\theta}V^\theta = 2r(\cos\theta + \sin\theta) \tag{64}$$

(This and the majority of the results given for Exercise 5.11 were computed in Mathematica)

Exercise (5.12a): For the one-form field $\tilde{p}$ whose Cartesian coordinates are $(x^2 + 3y, y^2 + 3x)$, compute $p_{\alpha,\beta}$ in Cartesian.

Solution:

$$p_{\alpha,\beta} = \begin{pmatrix} p_{rr} & p_{r\theta} \\ p_{\theta r} & p_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix} \tag{65}$$

Exercise (5.12b): Compute the transformation $\Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'}p_{\alpha,\beta}$ to polars.

Solution:

$$\Lambda^{\alpha}_{\ \mu'}\Lambda^{\beta}_{\ \nu'}p_{\alpha,\beta} = (\Lambda^{\alpha}_{\ \mu'})^T p_{\alpha,\beta}\Lambda^{\beta}_{\ \nu'} = \begin{pmatrix} 2r(\cos^3\theta - 3\cos\theta\sin\theta + r^2\sin^3\theta) & \end{pmatrix} \tag{66}$$

Exercise (5.12c): Compute the components $p_{\mu';\nu'}$ directly in polars using the Christoffel symbols, Eq. (4.44), in Eq. (5.62).

Solution:

$$p_{r;r} = p_{r,r} - p_\mu \Gamma^{\mu}_{\alpha\beta} = \frac{\partial p_r}{\partial r} - p_r\Gamma^{r}_{rr} - p_\theta\Gamma^{\theta}_{rr} \implies p_{r;r} = \frac{\partial p_r}{\partial r} \tag{67}$$

Where p_r is the r-component of the one-form in a polar basis.

$$p_r = r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta \implies p_{r;r} = \frac{\partial p_r}{\partial r} = 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta \tag{68}$$

$$p_{r;\theta} = \frac{\partial p_r}{\partial \theta} - p_r\Gamma^{r}_{r\theta} - p_\theta\Gamma^{\theta}_{r\theta} = \frac{\partial p_r}{\partial \theta} - \frac{1}{r}p_\theta \tag{69}$$

Exercise (5.14): For the tensor whose polar coordinates are $(A^{rr} = r^2, A^{r\theta} = r\sin\theta, A^{\theta r} = r\cos\theta, A^{\theta\theta} = \tan\theta)$, compute in Eq. (5.65) in polars for all possible indices:

$$\begin{aligned} \nabla_r A^{rr} &= \frac{\partial A^{rr}}{\partial r} + A^{\alpha r}\Lambda^{r}_{\alpha r} + A^{r\alpha}\Lambda^{r}_{\alpha r} = \frac{\partial A^{rr}}{\partial r} + A^{rr}\Lambda^{r}_{rr} + A^{\theta r}\Lambda^{r}_{\theta r} + A^{rr}\Lambda^{r}_{rr} + A^{r\theta}\Lambda^{r}_{\theta r} \\ &\implies \nabla_r A^{rr} = \frac{\partial A^{rr}}{\partial r} = \frac{\partial}{\partial r}\left(r^2\right) = 2r \end{aligned} \tag{70}$$

$$\begin{aligned} \nabla_\theta A^{rr} &= \frac{\partial A^{rr}}{\partial \theta} + A^{rr}\Lambda^{r}_{r\theta} + A^{\theta r}\Lambda^{r}_{\theta\theta} + A^{rr}\Lambda^{r}_{r\theta} + A^{r\theta}\Lambda^{r}_{\theta\theta} \\ \implies \nabla_\theta A^{rr} &= -r\left(A^{\theta r} + A^{r\theta}\right) + \frac{\partial A^{rr}}{\partial \theta} = -r(r\cos\theta + r\sin\theta) + \frac{\partial}{\partial\theta}\left(r^2\right) = -r^2(\cos\theta + \sin\theta) \end{aligned} \tag{71}$$

$$\begin{aligned} \nabla_\theta A^{r\theta} &= \frac{\partial A^{r\theta}}{\partial \theta} + A^{r\theta}\Gamma^{r}_{r\theta} + A^{\theta\theta}\Gamma^{r}_{\theta\theta} + A^{rr}\Gamma^{\theta}_{r\theta} + A^{r\theta}\Gamma^{\theta}_{\theta\theta} = \frac{\partial A^{r\theta}}{\partial\theta} - r(A^{\theta\theta}) + \frac{1}{r}(A^{rr}) \\ &\implies \nabla_\theta A^{r\theta} = r(\cos\theta - \tan\theta - 1) \end{aligned} \tag{72}$$

And the five remaining computations for all possible indices $(\nabla_r A^{r\theta}, \nabla_r A^{\theta r}, \nabla_\theta A^{\theta r}, \nabla_r A^{\theta\theta}, \nabla_\theta A^{\theta\theta})$ can be computed in exactly the same manner.

Exercise (5.16): Fill in all the missing steps leading from Eq. (5.74) to Eq. (5.75).

Solution: Starting with Eq. (5.72):

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} - \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu} \tag{73}$$

And using the fact that $g_{\alpha'\mu';\beta'} = 0$:

$$g_{\alpha'\beta',\mu'} = \Gamma^{\nu'}_{\alpha'\mu'}g_{\nu'\beta'} + \Gamma^{\nu'}_{\beta'\mu'}g_{\alpha'\nu'} \implies g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu} \tag{74}$$

And since α, β, μ are dummy indices whose order can be rearranged in the previous expression, the following form can be arrived at by switching the β and μ indices:

$$g_{\alpha\mu,\beta} = \Gamma^{\nu}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta}g_{\alpha\nu} \tag{75}$$

And the following expression can be arrived at by switching α with β in Eq. (75) and multiplying the whole expression by a negative sign:

$$g_{\beta\mu,\alpha} = \Gamma^{\nu}_{\beta\alpha}g_{\nu\mu} + \Gamma^{\nu}_{\mu\alpha}g_{\beta\nu} \implies -g_{\beta\mu,\alpha} = -\Gamma^{\nu}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}_{\mu\alpha}g_{\beta\nu} \tag{76}$$

We can now consider the addition of the three terms, $g_{\alpha\beta,\mu}, g_{\alpha\mu,\beta}, -g_{\beta\mu,\alpha}$:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu} + \Gamma^{\nu}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta}g_{\alpha\nu} - \Gamma^{\nu}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}_{\mu\alpha}g_{\beta\nu} \tag{77}$$

And, using the fact that the indices of the metric can be interchanged $(g_{\beta\nu} = g_{\nu\beta})$, we arrive at:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \left(\Gamma^{\nu}_{\alpha\mu} - \Gamma^{\nu}_{\mu\alpha}\right)g_{\nu\beta} + \left(\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\beta\alpha}\right)g_{\nu\mu} + \left(\Gamma^{\nu}_{\beta\mu} + \Gamma^{\nu}_{\mu\beta}\right)g_{\alpha\nu} \tag{78}$$

Since the lower indices of the Christoffel symbols may be interchanged, this leaves us with:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2\Gamma^{\nu}_{\beta\mu}g_{\alpha\nu} \tag{79}$$

Using the fact that inverting the metric just turns its covariant indices into contravariant indices ($1/g_{\alpha\beta} = g^{\alpha\beta}$):

$$\Gamma^{\nu}_{\beta\mu} = \frac{1}{2}g^{\alpha\nu}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right) \tag{80}$$

It's important to remind the reader of the notation being used here to understand the meaning of this result. Recall that $\phi_{,\alpha} \equiv \frac{\partial \phi}{\partial x^{\alpha}}$ so the previous expression becomes:

$$\Gamma^{\nu}_{\beta\mu} = \frac{1}{2}g^{\alpha\nu}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} + \frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\right) \tag{81}$$

Meaning the Christoffel symbols can be written in terms of derivatives of the metric.

Exercise (5.18): Verify Eq. (5.78).

Solution: Since we are working in polar coordinates, $\vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}}$ can only take on the forms $\vec{e}_r \cdot \frac{1}{r}\vec{e}_\theta$, $\vec{e}_r \cdot \vec{e}_r$, $\frac{1}{r}\vec{e}_\theta \cdot \frac{1}{r}\vec{e}_\theta$, or $\frac{1}{r}\vec{e}_\theta \cdot \vec{e}_r$. Since $\vec{e}_\theta$ will always be orthogonal to $\vec{e}_r$, meaning that $\vec{e}_r \cdot \vec{e}_\theta = \vec{e}_\theta \cdot \vec{e}_r = 0$. This also tells us that $\vec{e}_\theta \cdot \vec{e}_\theta = \vec{e}_r \cdot \vec{e}_r = 1$, satisfying the first part of Eq. (5.78):

$$\vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}} \equiv g_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}} \tag{82}$$

A similar argument can be made to prove the second half of Eq. (5.78) since the basis differentials $\tilde{d}r$ and $\tilde{d}\theta$ are orthogonal to each other, resulting in

$$\tilde{\omega}^{\hat{\alpha}} \cdot \tilde{\omega}^{\hat{\beta}} \equiv g^{\hat{\alpha}\hat{\beta}} = \delta^{\hat{\alpha}\hat{\beta}} \tag{83}$$

being a verified statement.

Exercise (5.22): Show that if $U^{\alpha}\nabla_{\alpha}V^{\beta} = W^{\beta}$, then $U^{\alpha}\nabla_{\alpha}V_{\beta} = W_{\beta}$

Solution: Recall the notation that $\nabla_{\alpha}V^{\beta} \equiv V^{\beta}_{;\alpha}$ from Eq. (5.51). This turns the expression into:

$$U^{\alpha}V^{\beta}_{;\alpha} = W^{\beta} \tag{84}$$

We can then multiply both sides of the expression by the metric $g_{\mu\beta}$:

$$U^{\alpha}g_{\mu\beta}V^{\beta}_{;\alpha} = g_{\mu\beta}W^{\beta} \tag{85}$$

From Eq. (5.68), $V_{\alpha;\beta} = g_{\alpha\mu}V^{\alpha}_{;\beta}$ so we can transform the left hand side of this expression to be:

$$U^{\alpha}V_{\mu;\alpha} = g_{\mu\beta}W^{\beta} \tag{86}$$

And we can finally use $V_{\alpha} = g_{\alpha\mu}V^{\mu}$ from Eq. (5.69) to simply the right side of the expression into:

$$U^{\alpha}V_{\mu;\alpha} = W_{\mu} \tag{87}$$

And since μ is just a dummy index, it can be changed for β, resulted in the desired expression:

$$U^{\alpha}\nabla_{\alpha}V_{\beta} = W_{\beta} \tag{88}$$

6 Chapter 6: Curved Manifolds

Exercise (6.6): Prove that the first term in Eq. (6.37) vanishes.

Solution: Starting with Eq. (6.37):

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta}) + \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu} \tag{89}$$

We want to show that $g^{\alpha\beta}g_{\beta\mu,\alpha} = g^{\alpha\beta}g_{\mu\alpha,\beta}$ to show that the first term vanishes. To do this, first use the fact that the metric is symmetric so the its indices can be interchanged as in following step:

$$g^{\alpha\beta}g_{\beta\mu,\alpha} \implies g^{\beta\alpha}g_{\beta\mu,\alpha} \tag{90}$$

Then use the fact that the indices considered are dummy indices so the swap $\alpha \to \beta$ & $\beta \to \alpha$ can be made:

$$g^{\beta\alpha}g_{\beta\mu,\alpha} \implies g^{\alpha\beta}g_{\alpha\mu,\beta} \tag{91}$$

And again use the fact that the metric is symmetric so the indices α and μ can be interchanged:

$$g^{\alpha\beta}g_{\alpha\mu,\beta} \implies g^{\alpha\beta}g_{\mu\alpha,\beta} \tag{92}$$

Reaching the desired result.

Exercise (6.8): Fill in the missing algebra leading to Eqs. (6.40) and (6.42).

Solution: Starting from Eq. (6.38):

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu} \tag{93}$$

We can swap the α and β indices on $g_{\alpha\beta,\mu}$ so that we can use Eq. (6.39):

$$g_{,\mu} = gg^{\alpha\beta}g_{\beta\alpha,\mu} \implies g_{,\mu}\frac{1}{g} = g^{\alpha\beta}g_{\beta\alpha,\mu} \tag{94}$$

as a substitution:

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}\frac{1}{g}g_{,\mu} \tag{95}$$

And in a step that I really don't understand, this becomes:

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g})_{\mu} \tag{96}$$

resulting in Eq. (6.40).

With this, Eq. (5.49) can be used where $\beta \to \alpha$ since this is just a dummy index and using the new definition of $\Gamma^{\alpha}_{\mu\alpha}$ in Eq. (6.40):

$$V^{\alpha}_{;\alpha} = V^{\alpha}_{,\alpha} + V^{\mu}\Gamma^{\alpha}_{\mu\alpha} = V^{\alpha}_{,\alpha} + V^{\mu}\left(\frac{1}{\sqrt{-g}}(\sqrt{-g})_{\mu}\right) \tag{97}$$

$V^{\alpha}_{,\alpha}$ can then be multiplied by $\frac{\sqrt{-g}}{\sqrt{-g}}$ so that it can be combined with the other term in the expression:

$$V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}V^{\alpha}_{,\alpha} + V^{\alpha}\sqrt{-g}_{\mu}\right) \tag{98}$$

And since the term in the parentheses is just the definition of the chain rule:

$$V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}V^{\alpha}\right)_{\alpha} \tag{99}$$

Which results in Eq. (6.42).

Exercise (6.13a): Show that if $\vec{A}$ and $\vec{B}$ are parallel-transported along a curve, then $\boldsymbol{g}(\vec{A},\vec{B}) = \vec{A}\cdot\vec{B}$ is constant on the curve.

Solution: If we parallel transport $\vec{A}$ and $\vec{B}$ along a curve, $\vec{U}$, then the following condition is satisfied:

$$\frac{d\vec{V}}{d\lambda} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}\frac{dx^{\beta}}{d\lambda} = 0 \tag{100}$$

for both $\vec{A}$ and $\vec{B}$ where $\frac{dx^{\beta}}{d\lambda}$ is the curve $\vec{U}$. If we parallel transport $\vec{A}\cdot\vec{B}$ along the curve, then:

$$\begin{aligned}\frac{d}{d\lambda}\left(g_{\alpha\beta}A^{\alpha}B^{\beta}\right) &= \frac{\partial}{\partial x^{\beta}}\left(g_{\alpha\beta}A^{\alpha}B^{\alpha}\right)\frac{dx^{\beta}}{d\lambda}\\ &= \left(A^{\alpha}B^{\alpha}\frac{\partial g_{\alpha\beta}}{\partial x^{\beta}} + g_{\alpha\beta}B^{\alpha}\frac{\partial V^{\alpha}}{\partial x^{\beta}} + g_{\alpha\beta}A^{\alpha}\frac{\partial B^{\alpha}}{\partial x^{\beta}}\right)\frac{dx^{\beta}}{d\lambda}\end{aligned} \tag{101}$$

Note that parallel transport requires the second and third term of this expression to equal zero and that the local flatness theorem ($g_{\alpha\beta,\gamma} = 0$) requires the first term to equal zero, leaving us to conclude that $\vec{A}\cdot\vec{B}$ is constant along the curve $\vec{U}$.

Exercise (6.13b): Conclude from this that if a *geodesic* is spacelike (or timelike or null) somewhere, it is spacelike (or timelike or null) everywhere.

Solution: The conditions for determining whether a geodesic $\vec{U}|$ is spacelike, timelike, or null are given below:

$$\vec{U}\cdot\vec{U}=\begin{cases}<0 & \text{timelike}\\ =0 & \text{null}\\ >0 & \text{spacelike}\end{cases} \tag{102}$$

Since we have shown that $\vec{U}\cdot\vec{U}$ is constant, we know that if the geodesic is defined to be spacelike, timelike, or null anywhere, it must satisfy this condition everywhere.

Exercise (6.14): The proper distance along a curve whose tangent is $\vec{V}$ is given by Eq. (6.8). Show that if the curve is a geodesic, then the proper length is an affine parameter. (Use the result of Exer. 13.)

Solution: From Eq. (6.8), the proper distance is defined to be:

$$\ell=\int_{\lambda_0}^{\lambda_1}|\vec{V}\cdot\vec{V}|^{1/2}d\lambda \tag{103}$$

And since we know that $\vec{V}\cdot\vec{V}$ is a constant from Exer. (6.13), this integral will just result in $\vec{V}\cdot\vec{V}$ being multiplied by the length of the line:

$$\ell=|\vec{V}\cdot\vec{V}|^{1/2}\int_{\lambda_0}^{\lambda_1}d\lambda=|\vec{V}\cdot\vec{V}|^{1/2}\lambda \tag{104}$$

And since an affine parameter is defined to be $\phi=a\lambda+b$ on page 167 of the text, ℓ must be an affine parameter with $a=|\vec{V}\cdot\vec{V}|^{1/2}$ and $b=0$ (since, again, $\vec{V}\cdot\vec{V}$ was found to be a constant from the previous exercise).

Exercise 6.19: Prove that $R^{\alpha}_{\beta\mu\nu}=0$ for polar coordinates in the Euclidean plane. Use Eq. (5.44) or equivalent results.

Solution: Using the definition of the Riemann curvature tensor given by Eq. (6.63):

$$R^{\alpha}_{\beta\mu\nu}=\frac{\partial}{\partial x^{\mu}}\Gamma_{\beta\nu}-\frac{\partial}{\partial x^{\nu}}\Gamma^{\alpha}_{\beta\mu}+\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}-\Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} \tag{105}$$

Where the $\alpha,\beta,\mu,\nu,\sigma$ indices will be summed over r,θ in polar coordinates. It's best to consider these sums in parts since they will become so large. Starting with the first term in the expression for $R^{\alpha}_{\beta\mu\nu}$:

$$\frac{\partial}{\partial x^{\mu}}\Gamma_{\beta\nu}=\frac{\partial}{\partial x^{\mu}}\left(\Gamma^{\alpha}_{rr}+\Gamma^{\alpha}_{r\theta}+\Gamma^{\alpha}_{\theta r}+\Gamma^{\alpha}_{\theta\theta}\right)=\frac{\partial}{\partial x^{\mu}}\left(\Gamma^{r}_{\theta}+\Gamma^{\theta}_{r\theta}+\Gamma^{\theta}_{\theta r}\right)=\frac{-2}{r^2}-r^2 \tag{106}$$

And since the $-\frac{\partial}{\partial x^{\nu}}\Gamma^{\alpha}_{\beta\mu}$ component of the Riemann curvature tensor changes nothing but the sign of the result shown above, $\frac{\partial}{\partial x^{\mu}}\Gamma_{\beta\nu}-\frac{\partial}{\partial x^{\nu}}\Gamma^{\alpha}_{\beta\mu}=0$. Computing the $\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}$ component of $R^{\alpha}_{\beta\mu\nu}$:

$$\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}=\Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\beta\nu}=\Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{r\theta}+\Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r}+\Gamma^{\theta}_{r\theta}\Gamma^{r}_{\theta\theta}=-3 \tag{107}$$

And, again, since $-\Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$ changes nothing but the sign of the previous result, $\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}-\Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}=0$, resulting in $R^{\alpha}_{\beta\mu\nu}=0$, which should be expected since we are computing the Riemann curvature tensor in the Euclidean plane, which is defined to have no curvature.

Exercise (6.20): Fill in the algebra necessary to establish Eq. (6.73).

Solution: Starting from:

$$\nabla_{\alpha}\nabla_{\beta}V^{\mu} \tag{108}$$

And using the definition of a covariant derivative built in the previous chapter, specifically Eq. (5.48):

$$\nabla_{\alpha}\nabla_{\beta}V^{\mu}=\nabla_{\alpha}\left(\frac{\partial V^{\mu}}{\partial x^{\beta}}+V^{\nu}\Gamma^{\mu}_{\nu\beta}\right) \tag{109}$$

The covariant derivative with respect to α can then be distributed across this expression like so:

$$\nabla_{\alpha}\left(\frac{\partial V^{\mu}}{\partial x^{\beta}}+V^{\nu}\Gamma^{\mu}_{\nu\beta}\right)=\frac{\partial V^{\mu}}{\partial x^{\alpha}x^{\beta}}+V^{\nu}\frac{\partial\Gamma^{\mu}_{\nu\beta}}{\partial x^{\alpha}}+\Gamma^{\mu}_{\nu\beta}\frac{\partial V^{\nu}}{\partial x^{\alpha}} \tag{110}$$

And since we are considering these covariant derivatives in a locally inertial frame at some point, the $\Gamma^{\mu}_{\nu\beta}$ term goes to zero but its partial derivative does not, leaving us with:

$$\nabla_{\alpha}\nabla_{\beta}V^{\mu}=\frac{\partial V^{\mu}}{\partial x^{\alpha}x^{\beta}}+V^{\nu}\frac{\partial\Gamma^{\mu}_{\nu\beta}}{\partial x^{\alpha}} \tag{111}$$

Exercise (6.28a): Derive Eq. (6.19) by using the usual coordinate transformation from Cartesian to spherical polars.

Solution: Using the transformation rule:

$$\vec{e}_{\beta'} = \Lambda^{\beta'}_{\alpha}\vec{e}_{\beta} \tag{112}$$

and the coordinates:

$$x = r\sin\theta\cos\phi \tag{113}$$
$$y = r\sin\theta\sin\phi \tag{114}$$
$$z = r\cos\theta \tag{115}$$

this implies that $\vec{e}_r$ will be of the form:

$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z \tag{116}$$
$$\implies \vec{e}_r = \sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y + \cos\theta\vec{e}_z \tag{117}$$

and that $\vec{e}_\theta$ and $\vec{e}_\phi$ will be of the forms:

$$\vec{e}_\theta = \frac{\partial x}{\partial r\theta}\vec{e}_x + \frac{\partial y}{\partial\theta}\vec{e}_y + \frac{\partial z}{\partial\theta}\vec{e}_z \tag{118}$$
$$\implies \vec{e}_\theta = r\cos\theta\vec{e}_x + r\cos\theta\sin\phi\vec{e}_y - r\sin\theta\vec{e}_z \tag{119}$$

$$\vec{e}_\phi = -r\sin\theta\sin\phi\vec{e}_x + r\sin\theta\cos\phi\vec{e}_y \tag{120}$$

From these, the following terms can be computed:

$$\vec{e}_r\cdot\vec{e}_r = 1 \tag{121}$$
$$\vec{e}_\theta\cdot\vec{e}_\theta = r^2 \tag{122}$$
$$\vec{e}_\phi\cdot\vec{e}_\phi = r^2\sin^2\theta \tag{123}$$
$$\tag{124}$$

and that $\vec{e}_r\cdot\vec{e}_\theta = \vec{e}_\theta\cdot\vec{e}_r = \vec{e}_r\cdot\vec{e}_\phi = \vec{e}_\phi\cdot\vec{e}_r = \vec{e}_\theta\cdot\vec{e}_\phi = \vec{e}_\phi\cdot\vec{e}_\theta = 0$. These terms give the metric in spherical coordinates the form:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{125}$$

Exercise (6.28b): Deduce from Eq. (6.19) that the metric of the surface of a sphere of radius r has components $g_{rr} = r^2, g_{\phi\phi} = r^2\sin^2\theta, g_{\theta\phi} = 0$ in the usual spherical coordinates.

Solution: It's clear from the metric shown in the previous expression that $g_{rr} = r^2, g_{\phi\phi} = r^2\sin^2\theta$, and $g_{\theta\phi} = 0$.

Exercise (6.28c): Find the components $g^{\alpha\beta}$ for the sphere.

Solution: Since the metric for spherical coordinates is diagonal, its inverse will just invert the non-zero components like so:

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{126}$$

Exercise (6.33a): A three-sphere is the three-dimensional surface in four-dimensional Euclidean space (coordinates x, y, z, w) given by the equation $x^2 + y^2 + z^2 + w^2 = r^2$, where r is the radius of the sphere. Define new coordinates (r, θ, ϕ, χ) by the equations $w = r\cos\theta$, $z = r\sin\chi\cos\theta$, $x = r\sin\chi\sin\theta\cos\phi$, and $y = r\sin\chi\sin\theta\sin\phi$. Show that (θ, ϕ, χ) are coordinates for the sphere. These generalize the familiar polar coordinates.

Solution: To show that the (θ, ϕ, χ) coordinates define a four-dimensional sphere, we must compute $x^2 + y^2 + z^2 + w^2$. If this result results in the radius of the sphere, r, then we have defined coordinates that describe it. This computation is as follows:

$$x^2 + y^2 + z^2 + w^2 = r^2\sin^2\chi\sin^2\theta\cos^2\phi + r^2\sin^2\chi\sin^2\theta\sin^2\phi + r^2\sin^2\chi\cos^2\theta + r^2\cos^2\chi \tag{127}$$

Using the trigonometric identity $\sin^2\eta + \cos^2\eta = 1$ many times reduces this expression to the desired result that $x^2 + y^2 + z^2 + w^2 = r$, which means that our (θ, ϕ, χ) coordinates do in fact define a four-dimensional sphere.

Exercise (6.33b): Show that the metric of the three-sphere of radius r has components in these coordinates $g_{\chi\chi} = r^2, g_{\theta\theta} = r^2 \sin^2\chi, g_{\phi\phi} = r^2 \sin^2\chi \sin^2\theta$, all other components vanishing. (Use the same method as in Exer. 28.)

Solution: Using the same method as in Exer. 28, it's found that:

$$\vec{e}_r = \sin\chi\sin\theta\cos\phi\vec{e}_x + \sin\chi\sin\theta\sin\phi\vec{e}_y + \sin\chi\cos\theta\vec{e}_z + \cos\chi\vec{e}_w \tag{128}$$

$$\vec{e}_\theta = r\sin\chi\cos\theta\cos\phi\vec{e}_x + r\sin\chi\cos\theta\sin\phi\vec{e}_y - r\sin\chi\sin\theta\vec{e}_z \tag{129}$$

$$\vec{e}_\phi = -r\sin\chi\sin\theta\vec{e}_x + r\sin\chi\sin\theta\cos\phi\vec{e}_y \tag{130}$$

$$\vec{e}_\chi = r\cos\chi\sin\theta\cos\phi\vec{e}_x + r\cos\chi\sin\theta\sin\phi\vec{e}_y + r\cos\chi\cos\theta\vec{e}_z - r\sin\chi\vec{e}_w \tag{131}$$

Dotting all of these terms with each other results in the metric:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 & 0 \\ 0 & 0 & r^2\sin^2\chi\sin^2\theta & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \tag{132}$$

which is the desired result.

7 Chapter 7: Physics in a Curved Spacetime

Exercise (7.2): To first order in ϕ, compute $g^{\alpha\beta}$ for Eq. (7.8).

Solution: Eq. (7.8) gives the line element for the ordinary Newtonian potential to the first order to be:

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2) \tag{133}$$

so the metric can be inferred to be:

$$g_{\alpha\beta} = \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & (1-2\phi) & 0 & 0 \\ 0 & 0 & (1-2\phi) & 0 \\ 0 & 0 & 0 & (1-2\phi) \end{pmatrix} \tag{134}$$

Since this metric is diagonal, the inverse of it will just invert the components, meaning that:

$$g^{\alpha\beta} = \begin{pmatrix} \frac{-1}{1+2\phi} & 0 & 0 & 0 \\ 0 & \frac{1}{1-2\phi} & 0 & 0 \\ 0 & 0 & \frac{1}{1-2\phi} & 0 \\ 0 & 0 & 0 & \frac{1}{1-2\phi} \end{pmatrix} \tag{135}$$

Exercise (7.3): Calculate all the Christoffel symbols for the metric given by Eq. (7.8), to first order in ϕ. Assume ϕ is a general function of t, x, y, and z.

Solution: Using Eq. (5.75), which allows us to compute the Christoffel symbols in terms of the metric and its derivatives, the components of the Christoffel symbols can be computed. Take Γ^x_{xx}, for example. This would be computed by:

$$\begin{aligned} \Gamma^x_{xx} = &\frac{1}{2}g^{tx}\left(g_{tx,x} + g_{tx,x} - g_{xx,t}\right) + \frac{1}{2}g^{xx}\left(g_{xx,x} + g_{xx,x} - g_{xx,x}\right) \\ &+\frac{1}{2}g^{yx}\left(g_{yx,x} + g_{yx,x} - g_{xx,y}\right) + \frac{1}{2}g^{zx}\left(g_{zx,x} + g_{zx,x} - g_{xx,z}\right) \end{aligned} \tag{136}$$

Which reduces nicely since the non-zero components of this metric are only found along the diagonal, meaning that only $g^{tt}, g^{xx}, g^{yy}, g^{zz}$ will have non-zero components. This results in:

$$\begin{aligned} \Gamma^x_{xx} &= \frac{1}{2}g^{xx}\left(g_{xx,x} + g_{xx,x} - g_{xx,x}\right) = \frac{1}{2}g^{xx}\frac{\partial g_{xx}}{\partial x} \\ &= \frac{1}{2}(1-2\phi)\frac{\partial}{\partial x}(1-2\phi) = -(1-2\phi)\frac{\partial\phi}{\partial x} \end{aligned} \tag{137}$$

And the other 63 components of $\Gamma^\gamma_{\beta\mu}$ can be computed in the same way, using the simplifications used above.

8 Chapter 8: The Einstein Field Equations

Exercise (8.1): Show that Eq. (8.2) is a solution of Eq. (8.1) by the following method. Assume the point particle to be at the origin, $r = 0$, and to produce a spherically symmetric field. Then use Gauss' law on a sphere of radius r to conclude

$$\frac{d\phi}{dr} = \frac{Gm}{r^2}$$

Deduce Eq. (8.2) from this. (consider the behavior at infinity.)

Solution: First considering the acceleration experienced due to the gravitational field around this point particle, one gets:

$$g = \frac{Gm}{r^2} \tag{138}$$

When considering the application of Gauss' law to this object, it's clear that the direction of g will be opposite to that of dA, meaning $\vec{g} \cdot d\vec{A} = -gdA$. Using this in the integral form of Gauss' law when the integrated area is the surface area of a sphere concentric around the point at the origin:

$$\phi = \iint \vec{g} \cdot d\vec{A} = -\iint gdA = g - 4\pi r^2 = -4\pi Gm \tag{139}$$

Since the integral form of Gauss' law must be equal to the differential form:

$$\iiint \nabla \cdot gdV = -4\pi Gm \tag{140}$$

Since mass is just the density of some object integrated over a volume:

$$\iiint \nabla \cdot gdV = -4\pi G \iiint \rho dV \tag{141}$$

Which means by inspection that:

$$\nabla \cdot g = -4\pi G\rho \tag{142}$$

Given the form of the scalar gravitational potential $\phi = Gm/r$ and Eq. (138), it's clear that

$$g = -\frac{d\phi}{dr} \tag{143}$$

And since ϕ is only a function of r, $-d\phi/dr \equiv -\nabla\phi$ which means we can simplify Eq. (142) to the desired form:

$$\nabla^2\phi = 4\pi G\rho \tag{144}$$

Exercise (7.5a): Show that if $h_{\alpha\beta} = \xi_{\alpha,\beta} + \xi_{\beta,\alpha}$, then Eq. (8.25) vanishes.

Solution: Starting with Eq. (8.25):

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}\right) \tag{145}$$

All for components of $R_{\alpha\beta\mu\nu}$ must be computed in the following way to show that it vanishes:

$$h_{\alpha\nu,\beta\mu} = \xi_{\alpha,\beta\mu\nu} + \xi_{\nu,\alpha\beta\mu} \tag{146}$$

$$h_{\beta\mu,\alpha\nu} = \xi_{\beta,\alpha\mu\nu} + \xi_{\mu,\alpha\beta\nu} \tag{147}$$

$$-h_{\alpha\mu,\beta\nu} = -\xi_{\alpha,\beta\mu\nu} - \xi_{\mu,\alpha\beta\nu} \tag{148}$$

$$-h_{\beta\nu,\alpha\mu} = -\xi_{\beta,\alpha,\mu\nu} - \xi_{\nu,\alpha\beta\mu} \tag{149}$$

It's apparent that when you add Eq. (115) - Eq. (118) together, you will get 0, meaning $R_{\alpha\beta\mu\nu} = 0$.

Exercise (7.5b): Argue from this that Eq. (8.25) is gauge invariant.

Solution: A gauge transformation is defined as a small change in coordinates where $h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$. Since we have just shown that $h_{\alpha\beta} = \xi_{\alpha,\beta} + \xi_{\beta,\alpha}$ results in $R_{\alpha\beta\mu\nu} = 0$, if we use this expression for $h_{\alpha\beta}$ in the gauge transformation, we get $h_{\alpha\beta} \to \xi_{\alpha,\beta} + \xi_{\beta,\alpha} + \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$, which means that the gauge transformation becomes $h_{\alpha\beta} \to 0$. Using this expression for the transformed coordinates in Eq. (8.25) will obviously result in $R_{\alpha\beta\mu\nu} = 0$, which means that $R_{\alpha\beta\mu\nu}$ has been unchanged under this gauge transformation, meaning it is gauge invariant.

9 Chapter 9: Gravitational Radiation

Exercise (9.2): Show that the real and imaginary parts of Eq. (9.2) at a fixed spatial position $\{x^i\}$ oscillate sinusoidally in time with frequency $\omega = k^0$.

Solution: Starting with Eq. (9.2):

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\alpha x^\alpha} \tag{150}$$

With the use of Euler's formula, this becomes:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \left(\cos(k_\alpha x^\alpha) + i \sin(k_\alpha x^\alpha)\right) \tag{151}$$

Isolating the $\alpha = 0$ component in this expression leaves us with:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \left(\cos(k_0 x^0) + i \sin(k_0 x^0) + \cos(k_i x^i) + i \sin(k_i x^i)\right) \tag{152}$$

Which shows that $\bar{h}^{\alpha\beta}$ oscillates as a sinusoid in time with an angular frequency $\omega = k_0$:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \left(\cos(\omega t) + i \sin(\omega t) + \cos(k_i x^i) + i \sin(k_i x^i)\right) \tag{153}$$

10 Chapter 10: Spherical Solutions for Stars

Exercise (10.1): Starting with $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$, show that the coordinate transformation $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \cos^{-1}(z/r)$, $\phi = \tan^{-1}(y/x)$ leads to Eq. (10.1), $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

Solution: To derive this line element, we must first compute the metric for a flat spacetime in spherical coordinates. This metric will follow the form:

$$g_{\alpha\beta} = \begin{pmatrix} \vec{e}_t \cdot \vec{e}_t & \vec{e}_t \cdot \vec{e}_r & \vec{e}_t \cdot \vec{e}_\theta & \vec{e}_t \cdot \vec{e}_\phi \\ \vec{e}_r \cdot \vec{e}_t & \vec{e}_r \cdot \vec{e}_r & \vec{e}_r \cdot \vec{e}_\theta & \vec{e}_r \cdot \vec{e}_\phi \\ \vec{e}_\theta \cdot \vec{e}_t & \vec{e}_\theta \cdot \vec{e}_r & \vec{e}_\theta \cdot \vec{e}_\theta & \vec{e}_\theta \cdot \vec{e}_\phi \\ \vec{e}_\phi \cdot \vec{e}_t & \vec{e}_\phi \cdot \vec{e}_r & \vec{e}_\phi \cdot \vec{e}_\theta & \vec{e}_\phi \cdot \vec{e}_\phi \end{pmatrix} \tag{154}$$

Since this transformation from Cartesian to spherical coordinates does not depend on t, $\vec{e}_t$ will be of the form $\langle dt, 0, 0, 0\rangle$ and $\vec{e}_i$ will be of the form $\langle 0, ..., ..., ...\rangle$, which means that $\vec{e}_t \cdot \vec{e}_i = 0$. This turns our metric into:

$$g_{\alpha\beta} = \begin{pmatrix} \vec{e}_t \cdot \vec{e}_t & 0 & 0 & 0 \\ 0 & \vec{e}_r \cdot \vec{e}_r & \vec{e}_r \cdot \vec{e}_\theta & \vec{e}_r \cdot \vec{e}_\phi \\ 0 & \vec{e}_\theta \cdot \vec{e}_r & \vec{e}_\theta \cdot \vec{e}_\theta & \vec{e}_\theta \cdot \vec{e}_\phi \\ 0 & \vec{e}_\phi \cdot \vec{e}_r & \vec{e}_\phi \cdot \vec{e}_\theta & \vec{e}_\phi \cdot \vec{e}_\phi \end{pmatrix} \tag{155}$$

To compute $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$, the transformation $\vec{e}_{\alpha'} = \Lambda^\beta_{\alpha'} \vec{e}_\beta$ will be used with $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$:

$$\begin{gathered} \vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z \\ \implies \vec{e}_r = \sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y + \cos\theta\vec{e}_z \end{gathered} \tag{156}$$

$$\begin{gathered} \vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y + \frac{\partial z}{\partial \theta}\vec{e}_z \\ \implies \vec{e}_\theta = r\cos\theta\cos\phi\vec{e}_x + r\cos\theta\sin\phi\vec{e}_y - r\sin\theta\vec{e}_z \end{gathered} \tag{157}$$

$$\begin{gathered} \vec{e}_\phi = \frac{\partial x}{\partial \phi}\vec{e}_x + \frac{\partial y}{\partial \phi}\vec{e}_y + \frac{\partial z}{\partial \phi}\vec{e}_z \\ \implies \vec{e}_\phi = -r\sin\theta\sin\phi\vec{e}_x + r\sin\theta\cos\phi\vec{e}_y \end{gathered} \tag{158}$$

By direct computation, it can then be shown that $\vec{e}_\theta \cdot \vec{e}_r = \vec{e}_r \cdot \vec{e}_\theta = \vec{e}_r \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_r = \vec{e}_\theta \cdot \vec{e}_\phi = \vec{e}_\phi \cdot \vec{e}_\theta = 0$, which are notably all of the off-diagonal elements in $g_{\alpha\beta}$. It can also then be shown by direct computation that $\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2$, and $\vec{e}_\phi \cdot \vec{e}_\phi = r^2\sin^2\theta$. This results in the metric:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{159}$$

Reading the line element ds^2 off of this metric results in:

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{160}$$

11 Chapter 12: Cosmology

Exercise (12.7a): Find the coordinate transformation leading to Eq. (12.20).

Solution: To consider the Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right) \tag{161}$$

and when $k = -1$ requires the coordinate transformation:

$$d\chi^2 = \frac{dr^2}{1+r^2} \tag{162}$$

This implies that

$$d\chi = \frac{dr}{\sqrt{1+r^2}} \tag{163}$$

Integrating this to get $\chi(r)$ results in

$$\chi = \sinh^{-1}(r) \implies r = \sinh(\chi) \tag{164}$$

With this transformation, the Robertson-Walker metric when $k = -1$ becomes:

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \sinh^2(\chi) d\Omega\right) \tag{165}$$