A First Course in Loop Quantum Gravity - Selected Solutions

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1 Chapter 1: Lagrangians

Exercise (1.5): For a three-dimensional elastic medium, the potential energy is

$$V = \frac{\mathcal{T}}{2} \int d^3x \left(\nabla\psi\right)^2,$$

and the kinetic energy is

$$T = \frac{\rho}{2} \int d^3x \left(\frac{\partial\psi}{\partial t}\right)^2.$$

Use these results and show the functional derivative approach to show that ψ obeys the wave equation:

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2},$$

where v is the velocity of the wave.

Solution: Beginning by constructing the Lagrangian for this system, one finds that:

$$L = \int d^3x \frac{\mathcal{T}}{2} \left(\nabla\psi\right)^2 - \frac{\rho}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 \tag{1}$$

which means that the Lagrangian density is:

$$\mathcal{L} = \frac{\mathcal{T}}{2} \left(\nabla\psi\right)^2 - \frac{\rho}{2}\left(\frac{\partial\psi}{\partial t}\right)^2. \tag{2}$$

Applying Hamilton's principle of least action to this system:

$$\frac{\delta S}{\delta\psi} = 0 \implies \frac{\partial\mathcal{L}}{\partial\psi} - \partial_b \frac{\partial\mathcal{L}}{\partial\left(\partial_b\psi\right)} = 0 = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial t\right)} - \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x^i\right)} = 0 \tag{3}$$

Where $x^1 = x, x^2 = y$, and $x^3 = z$. Because $\mathcal{L}$ has no dependence on a ψ term, this functional derivative becomes:

$$-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial t\right)} = \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial x^i\right)}. \tag{4}$$

Taking these partial derivative of the Lagrangian density results in:

$$\begin{gathered} -\frac{\partial}{\partial t}\left(-\rho\frac{\partial\psi}{\partial t}\right) = \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left(\mathcal{T}\frac{\partial\psi}{\partial x^i}\right) \\ \implies \rho\frac{\partial^2\psi}{\partial t^2} = \mathcal{T}\nabla^2\psi \implies \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi \end{gathered} \tag{5}$$

when $v = \sqrt{\mathcal{T}/\rho}$.