Smooth Shape Interpolation with Multiple Keyframes

6.UAR SuperUROP Proposal
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February 8, 2017

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4.1 The Complex Derivatives

We observe the complex function $f : \Omega \to \mathbb{R}^2$. This function can also be represented as $f(x, y) = u(x, y) + iv(x, y)$ where u, v are real functions. Using a different notation, f can be written as $f = (u, v)$.

We deal very often with the 2x2 Jacobian of our function $f : \Omega \to \mathbb{R}^2$. In particular, we desire to decompose the Jacobian into two matrices: the similarity and anti-similarity matrices. Note that this decomposition is also called the "additive decomposition."

They are defined as:

$$S_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c & d \\ d & -c \end{bmatrix}$$

Suppose that the Jacobian is the 2x2 matrix

$$J_f = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

Then we have the decomposition $J_f = S_2 + A_2$ where

$$\begin{cases} a = \frac{p+s}{2} \\ b = \frac{r-q}{2} \\ c = \frac{p-s}{2} \\ d = \frac{r+q}{2} \end{cases}$$

What are these matrices intuitively though? The similarity matrices S_2 apply a similarity transformation by rotating and scaling $\mathbb{R}^2$. We can see this by recalling the definition of a 2x2 rotation matrix by angle θ:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The similarity matrix is thus the matrix MR_θ where $M = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. So we can see, that a multiplication of a coordinate vector by similarity matrix S_2 corresponds to a rotation by θ and a scaling by M.

On the other hand, the anti-similarity matrix is simply an application of the same rotation and scaling after a reflection about the x-axis is applied. As we can see:

$$A_2 = M' R_{\theta'} F_y$$

$$= \sqrt{c^2+d^2} \begin{bmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where $M' = \sqrt{c^2+d^2}$ and $\theta' = \tan^{-1}\left(\frac{d}{c}\right)$.

Now we note that multiplying a vector $z = [xy]^T$ by the similarity matrix S_2 is equivalent to multiplying the complex number $(x+iy)$ by $a+ib$. Let us quickly confirm.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xa - by \\ xb + ay \end{bmatrix}$$

$$(a+ib)(x+iy) = (xa-by) + i(xb+ay)$$

Analogously, multiplication by the anti-similarity matrix A_2 is equivalent to multiplying $\bar{z} = x - iy$ by $c + id$. This can be confirmed in the same fashion.

Now we come into the definitions of $f_z, f_{\bar{z}}$. These are also called the Wirtinger derivatives $\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}}$ respectively. The intuition is that we are effectively computing a change of variable from (x,y) to $(z,\bar{z})$. Let us use the chain rule to determine $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$.

Since $z = x + iy$, we have that:

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z}\dfrac{\partial z}{\partial x} + \dfrac{\partial}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}} \\ \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z}\dfrac{\partial z}{\partial y} + \dfrac{\partial}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial y} = i\dfrac{\partial}{\partial z} - i\dfrac{\partial}{\partial \bar{z}} \end{cases}$$

This gives us a linear system, which we can solve in order to get that

$$\begin{cases} \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) \\ \dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) \end{cases}$$

These are our definitions of the Wirtinger derivatives.

Now as it turns out, $f_z = a+ib, f_{\bar{z}} = c+id$ where a, b, c, d are the elements of the matrices in the additive decomposition of the Jacobian of f. In order to show this for f_z, recall that

$$f_z = \frac{1}{2}\left(f_x - if_y\right)$$

$$= \frac{1}{2}\left((u_x + iv_x) - i(u_y + iv_y)\right)$$

$$= \frac{1}{2}\left((u_x + v_y) + i(-u_y + v_x)\right)$$

$$= a + ib$$

And $f_{\bar{z}} = c + id$ follows by a similar argument.

4.2 Holomorphic & Anti-Holomorphic Mappings

Now we cover holomorphic and anti-holomorphic mappings, which are essential to the study of complex analysis. By definition, holomorphic functions are complex functions f such that $f_{\bar{z}} = 0$. Recall that we can write $f = (u(x, y), v(x, y)) = u(x, y) + iv(x, y)$ where u, v are real functions.

By expanding $f_{\bar{z}}$, we obtain this formula:

$$\frac{1}{2}(f_x + if_y) = 0$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

By matching real parts and imaginary parts, we get

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

This system of equations is also called the Cauchy-Riemann equations. Any function that satisfies the Cauchy-Riemann equations also satisfies $f_{\bar{z}} = 0$ and is therefore holomorphic.

Thus, for holomorphic mappings, we know that our jacobian J_f, which can be decomposed into similarity and anti-similarity matrices S_2, A_2 that are equivalent to $f_z, f_{\bar{z}}$, is a similarity matrix everywhere. The Jacobian would be equivalent to multiplying $\mathbb{C}$ by f_z. Note that this value f_z can also be written as f'.

These holomorphic functions are infinitely differentiable. This is not a trivial theorem to prove, and is commonly derived as a corollary of Cauchy's Integral Formula, which states that a holomorphic function defined on a disk is completely determined by its values on the boundary of the disk. In addition, the derivatives and anti-derivatives of holomorphic functions are holomorphic as well. The sums, products, compositions are holomorphic. And the quotients are holomorphic wherever the denominator does not vanish or evaluate to 0.

4.3 Harmonic Planar Mappings

Harmonic planar mappings are defined in this section. But first, we begin with harmonic real functions. A real function $u(x, y) : \Omega \to \mathbb{R}$ is harmonic if it satisfies the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Harmonic planar mappings are mappings $f : \Omega \to \mathbb{R}^2$ where $f = (u, v)$ and the u, v are both harmonic real functions. A mapping is simply another name for a function, although in analysis, functions typically are more restrictive and only include mappings from $\mathbb{R}$ to $\mathbb{C}$.

Now, we prove that holomorphic and anti-holomorphic functions are harmonic through the Cauchy Riemann equations. For any holomorphic function $f = (u, v)$, we have that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y}$$

$$= 0$$

The argument is similar for v. We can then use the same logic to prove that anti-holomorphic functions are also harmonic.

However, the converse is not necessarily true: Harmonic planar mappings are not necessarily holomorphic nor anti-holomorphic. It is simple to think up an example.

Now we prove that any harmonic planar mapping f can be written as the sum of a holomorphic function ϕ and anti-holomorphic function $\bar{\psi}$:

$$f(z) = \phi(z) + \bar{\psi}(z) \tag{4.1}$$

In order to accomplish this, we bring in harmonic conjugates. Suppose that a real function $u(x, y)$ is harmonic. Then the harmonic conjugate $\hat{v}(x, y)$ is a harmonic real function such that $u + iv$ is holomorphic and thus satisfies the Cauchy-Riemann equations. Look to a complex analysis textbook to see the formula for $\hat{v}$ given u. It is not as relevant here. We only need the fact that such a $\hat{v}$ exists.

Since u is harmonic, there exists a harmonic conjugate $\hat{v}$ such that $u + i\hat{v}$ that satisfies the Cauchy-Riemann equations. Since v is harmonic, there exists a harmonic conjugate $\hat{u}$ such that $\hat{u} + iv$ that satisfies the Cauchy-Riemann equations.

If we let $\phi = \frac{u+\hat{u}}{2} + i\frac{v+\hat{v}}{2}$ and $\bar{\psi} = \frac{u-\hat{u}}{2} + i\frac{v-\hat{v}}{2}$, then we can confirm that ϕ is holomorphic and $\bar{\psi}$ is anti-holomorphic. In addition $f = \phi + \psi$.

It also goes that $f_z = \phi'$ and $f_{\bar{z}} = \bar{\psi}'$. We can find this by taking the derivative of Equation 4.1 with respect to $z, \bar{z}$ and recalling that $f_{\bar{z}} = 0$ for holomorphic functions and $f_z = 0$ for anti-holomorphic functions.

Also, intuitively, the parts of the additive decomposition of J_f can be integrated separately to obtain $\phi, \bar{\psi}$. Note that because these functions are integrable because they are holomorphic.

4.4 Local Geometric Quantities

Now we show that some important quantities are representable in terms of the Wirtinger derivatives. First we wish to prove that

$$\det(J_f) = |f_z|^2 - |f_{\bar{z}}|^2$$

Let us approach from the right side:

$$|f_z|^2 - |f_{\bar{z}}|^2 = \frac{1}{4}\left((a^2+b^2) - (c^2+d^2)\right)$$

$$= \frac{1}{4}\left((p+s)^2 + (r-q)^2 - (p-s)^2 - (r+q)^2\right)$$

$$= \frac{1}{4}(4ps - 4rq)$$

$$= ps - rq$$

$$= \det(J_f)$$

and we are done.

Now, a mapping f is locally injective and sense-preserving at point z if $\det(J_f) > 0$. What does it mean to be locally injective? What does it mean to be sense-preserving?

Therefore, f is locally injective and sense-preserving whenever

$$\det(J_f) = ps - rq > 0$$

$$ps > rq$$

$$(p+s)^2 + (r-q)^2 > (p-s)^2 + (r+q)^2$$

$$a^2 + b^2 > c^2 + d^2$$

$$|f_z|^2 > |f_{\bar{z}}|^2$$

$$|f_z| > |f_{\bar{z}}|$$

Since we desire our mapping f to be locally injective and sense-preserving as part of our desire to produce natural looking animations, we look only at mappings f that satisfy this inequality everywhere.

Since $|f_{\bar{z}}| \geq 0$, we have that $|f_z| > 0$.

In the special case of a holomorphic function g, recall that $g_{\bar{z}} = 0$ at every point, thus $g' \neq 0$ everywhere and g is a conformal mapping. In other words, it preserves the angle between any two intersection curves.

The aim of the paper is to control the amount of conformal (or angular) and isometric (or metric / distance) distortion induced by a mapping. Where we introduce a rigorous definition of these distortion values using singular values of the Jacobian $J_f : 0 \leq \sigma_b \leq \sigma_a$.

If you look at this ARTICLE, then you can find out how to retrieve a 2x2 matrix's singular value decomposition (SVD). This gives you the formula for the two singular values σ_a, σ_b:

$$\sigma_a = |f_z| + |f_{\bar{z}}|, \qquad \sigma_b = ||f_z| - |f_{\bar{z}}|| \tag{4.2}$$

In our case though, we consider mappings that are locally injective and sense-preserving that have $|f_z| > |f_{\bar{z}}|$ so that we can drop the external absolute value sign in σ_b to get $\sigma_b = |f_z| - |f_{\bar{z}}| > 0$.

Now we define the first complex dilation as

$$\mu = \frac{f_{\bar{z}}}{f_z}$$

We also define the little dilation as $|\mu|$ or the modulus of the first complex dilation:

$$k = \frac{|f_{\bar{z}}|}{|f_z|}$$

This is what we will use to quantify the amount of conformal distortion.

Observe that our requirement $|f_z| > |f_{\bar{z}}|$ implies that $k = \dfrac{|f_{\bar{z}}|}{|f_z|} < 1$ so that $0 \leq k < 1$ throughout the domain and $k = 0$ when $f_{\bar{z}} = 0$ or f is holomorphic and therefore conformal.

5 Interpolation Problem

5.1 Basic Approach

We would like to interpolate the Jacobian J_f to preserve local geometric quantities. The process for this consists of interpolating J_f's decomposition into similarity and anti-similarity parts: $f_z = \phi', f_{\overline{z}} = \psi'$. Then by integrating these values, we can sum to obtain an interpolation for f.

Since $f_z, f_{\overline{z}}$ are holomorphic and anti-holomorphic, they are integrable and result in, again, holomorphic and anti-holomorphic results ϕ, ψ. Recall that $f = \phi + \overline{\psi}$, so that summing these values gives us f.

Now we can prove a couple lemmas:

Lemma 1 *If f_z and $f_{\overline{z}}$ have Property 4 (smoothness), then f will have Property 4 as well.*

Lemma 2 *If $|f_z| > |f_{\overline{z}}| \geq 0 \forall (t, z) \in [0, 1] \times \Omega$, f will have Property 3 (local injectivity).*

6 Parallel Methods

In this section, we introduce two parallel methods for computing the interpolation. Both of them interpolate f_z logarithmically. The only difference is the way that they compute $f_{\overline{z}}$.

When we interpolate f_z logarithmically, this means that we interpolate $\arg f_z$ linearly.

6.1 Logarithmic Interpolation of f_z

In order to interpolate f_z logarithmically, we follow this formula where the input are f_z^0, f_z^1. It is expressed as:

$$\begin{aligned} f_z^t &= (f_z^0)^{1-t}(f_z^1)^t \\ &= e^{(1-t)\log f_z^0 + t \log f_z^1} \\ &= |f_z^0|^{1-t}|f_z^1|^t e^{i\left((1-t)\arg(f_z^0)+t\arg(f_z^1)\right)} \end{aligned}$$

Note that in the second equation, we need to precisely define what our log function is.

6.2 Bounding Conformal Distortion

The second complex dilation ν is defined as:

$$\nu = \frac{\overline{g_{\overline{z}}}}{g_z} \tag{6.1}$$

where g is a planar mapping.

This means that we can calculate $f_{\overline{z}}$ as

$$f_{\overline{z}} = \nu g_z$$

Now if we linearly interpolate ν with respect to time, we obtain:

$$\nu^t = (1-t)\nu^0 + t\nu^1$$

$$f_{\overline{z}}^t = \overline{\nu^t f_z^t}$$

Note that the derived formula for $f_{\overline{z}}^t$ is anti-holomorphic as expected.

This is the first method, also known as the ν method.

6.3 Interpolating Stretch Direction

Short summary: We want to interpolate the stretch direction. This property is our 8th property and is not fulfilled by the ν method.

6.4 Introducing η

For this reason, we introduce the second method, called the η method. For a planar mapping g, we define

$$\eta = g_{\bar{z}}\overline{g_z} = \mu|g_z|^2$$

We linearly interpolate η in order to obtain the equations:

$$\eta^t = (1-t)\eta^0 + t\eta^1$$

$$f^t_{\bar{z}} = \frac{\eta^t}{\overline{f^t_z}}$$

This method allows us to interpolate $f^t_{\bar{z}}$ with the 8th property.

6.5 Scaling η

7 Metric Pullback

7.1 The Metric Tensor and Linear Interpolation

7.2 Interpolation on the Boundary

7.3 Variant Validation

8 Implementation