Physics 4500 /6502

Outline (§7.3.1 – §7.3.5):

I. Electrodynamics before Maxwell

II. How Maxwell fixed Ampere's law

III. Maxwell's equations in matter

I. Electrodynamics before Maxwell

So far in this course, electric and magnetic fields are described by four equations, which specify the divergence and curl of $\vec{E}$ and $\vec{B}$:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss's law)}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \text{(Ampere's law)}$$

Maxwell noticed problem (fatal inconsistency) with Ampere's law:

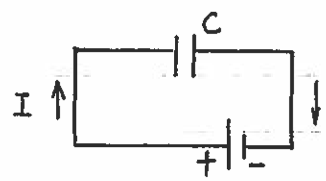
1. Problem seen from vector algebra

As we know, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$ (divergence of curl is zero)

However, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$

Where $\vec{\nabla}\cdot\vec{J} = 0$ for steady current but not when $\vec{J}$ is changing.

2. Problem found with Ampere's law with the example of a capacitor



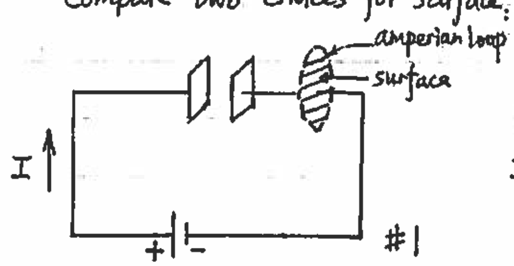
Look at current flowing while capacitor charges

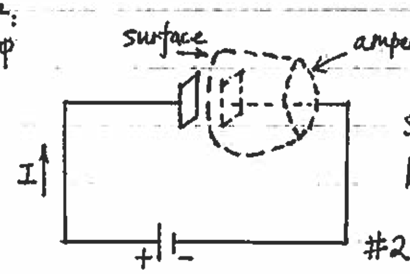
— Draw amperian loop around wire

— Apply $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$

Where $I_{enclosed}$ is current through ANY surface bounded by loop.

Compare two choices for surface:





Surface through space between plates

Same amperian loop $\Rightarrow \oint \vec{B}\cdot d\vec{l}$ are the same for both cases.

However, #1 : $I_{enclosed}$ is current through surface in wire

#2 : No wire through surface, but $\vec{E}$ field between plates changes as the capacitor charges.

II. How Maxwell fixed Ampere's law

Maxwell defined Displacement Current $J_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, which is proposed to be added to Ampere's law to give :

$$\vec{\nabla}\times\vec{B} = \underbrace{\mu_0 \vec{J}}_{\text{in wire}} + \mu_0\varepsilon_0 \underbrace{\frac{\partial \vec{E}}{\partial t}}_{\text{between capacitor plates}}$$

- Does this fix $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})$?

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{(continuity equation, 5.29)}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\therefore\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J}) + \mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = -\mu_0\frac{\partial \rho}{\partial t} + \mu_0\frac{\partial \rho}{\partial t} = 0$$

as required

- Does this fix Ampere's law for charging a capacity ?

Electric field between plates $E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{A}$ (Gauss's law)

So: $\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt} = \frac{1}{\varepsilon_0 A} I$ (for plates close together)

Then: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed} + \mu_0\varepsilon_0\int \frac{d\vec{E}}{dt}\cdot d\vec{a}$

Surface 1 :	$\mu_0 I$ for #1	0 for surface #1
Surface 2 :	0 for #2	$\mu_0\varepsilon_0 \times \frac{1}{\varepsilon_0 A} I \times A = \mu_0 I$ for #2

Significance of $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$:

A changing electric field induces a magnetic field.

Note: • Parallel to induction of $\vec{E}$ by changing $\vec{B}$.

• Important for electromagnetic waves.

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss's law)}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \text{(Ampere's law + Maxwell term)}$$

Together with the force law, $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ are principle content of classical electrodynamics.

III Maxwell's equations in matter

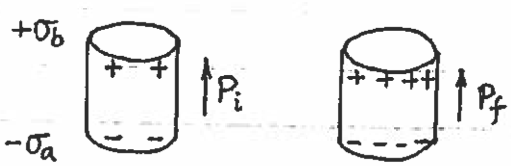
1. For electric and magnetic fields in materials, it will be convenient to rewrite Maxwell's equations to explicitly account for polarization and magnetization so that only directly controllable sources of the "free" charges and current in the equations.

2. For static cases:

   Non-uniform electric polarization $\Rightarrow$ bound charge density $\Rightarrow$ $\rho_b = -\vec{\nabla}\cdot\vec{P}$

   Non-uniform magnetization $\Rightarrow$ bound current density $\Rightarrow$ $\vec{J}_b = \vec{\nabla}\times\vec{M}$

3. There is just one new feature to consider in the nonstatic case, i.e., additional feature if polarization is changing:



For any change in P, for example, positive $\frac{d\vec{P}}{dt}$, it involves a flow of (bound) charge ($\vec{J}_p$) — positive charge and negative charge flow in opposite direction.

Result: additional contribution to current density

$$\vec{J}_b = \frac{\partial\vec{P}}{\partial t}$$

which satisfies continuity equation for bound charge:

$$\vec{\nabla}\cdot\vec{J}_b = \vec{\nabla}\cdot\left(\frac{\partial\vec{P}}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P}) = -\frac{\partial\rho_b}{\partial t}$$

So: we can write:

$$\rho = \rho_{free} - \vec{\nabla}\cdot\vec{P}$$

and $$\vec{J} = \vec{J}_{free} + \underbrace{\vec{\nabla}\times\vec{M}}_{\text{bound current}} + \underbrace{\frac{\partial\vec{P}}{\partial t}}_{\text{polarization current}}$$

Substitute $\rho = \rho_f - \vec{\nabla}\cdot\vec{P}$ into Gauss's law:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\rho_f - \vec{\nabla}\cdot\vec{P})$$

$$\therefore \vec{\nabla}\cdot\underbrace{(\varepsilon_0\vec{E}+\vec{P})}_{\equiv\vec{D}} = \rho_f \qquad \text{Note: } \vec{D}\equiv\varepsilon_0\vec{E}+\vec{P}$$

So: $\vec{\nabla}\cdot\vec{D} = \rho_f$ (already saw)

Substitute $\vec{J} = \vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}$ into Ampere's law:

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\therefore \vec{\nabla}\times\underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)}_{=\vec{H}} = \vec{J}_f + \frac{\partial}{\partial t}\underbrace{(\varepsilon_0\vec{E}+\vec{P})}_{=\vec{D}}$$

So: $$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

Therefore, in terms of free charges and free currents, Maxwell's equations read

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

P6502:

During this week, please complete the following requirements:

1. Self-reading of Sections 6.1 – 6.6 in Chapter 6 (p. 237-257) of Jackson's book on *Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws*.

2. Please carefully read the topics in Sections 6.3-6.5, which are not covered in an undergraduate course.