Tuesday, October 15, 2024 1:53 PM

Complex eigenvalues and eigenvectors

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad 0 = \det(A - \lambda I)$$

$$= \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$\lambda = \pm i \qquad 0 = \lambda^2 + 1$$

$$\lambda = i : A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} -d \\ c \end{bmatrix}$$

$$\lambda = -i : A - (-i)I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = i, \ \vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \lambda = -i, \ \vec{x} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda = \pm i, \ \vec{x} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} P^{-1}$$

$$0 = (a - \lambda)^2 + b^2$$

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad -b^2 = (a - \lambda)^2$$

$$C=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$-b^2=(a-\lambda)^2$$
$$-b^2=(\lambda-a)^2$$
$$\pm bi=\lambda-a$$

$$\lambda=a\pm ib$$
$$\mathrm{Re}\,\lambda=a$$
$$\mathrm{Im}\,\lambda=\pm b$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}=\boxed{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}\boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

contraction/dilation

counterclockwise rotation around origin of angle θ

$$a=r\cos\theta$$
$$b=r\sin\theta$$
$$r=\sqrt{a^2+b^2}$$
$$\theta=\tan^{-1} b/a$$

$$A=\begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$0=(-1-\lambda)(2-\lambda)+3$$
$$=\lambda^2+\lambda-2\lambda-2+3$$
$$0=\lambda^2-\lambda+1$$

$$\lambda=\frac{1\pm\sqrt{1-4\cdot 1}}{2}$$

$$=\frac{1\pm\sqrt{3}\,i}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i=a\pm ib$$

$$A=PCP^{-1}=\begin{bmatrix} \mathrm{Re}\,\vec{\varphi} & \mathrm{Im}\,\vec{\varphi} \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1}$$

$\vec{\varphi}$ is the eigenvector corresponding to $\lambda=a-ib$

$$\lambda=\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$\lambda = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$A - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)I = \begin{bmatrix} -1-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) & -3 \\ 1 & 2-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) \end{bmatrix}$

$= \begin{bmatrix} -\frac{3}{2}+\frac{\sqrt{3}}{2}i & -3 \\ 1 & \frac{3}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix} \begin{bmatrix} 3 \\ -\frac{3}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 6 \\ -3+\sqrt{3}i \end{bmatrix}$ $\quad \begin{bmatrix} \frac{3}{2}+\frac{\sqrt{3}}{2}i \\ -1 \end{bmatrix}$

$\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)\begin{bmatrix} \frac{3}{2}+\frac{\sqrt{3}}{2}i \\ -1 \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right) \\ -\frac{3}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix}$

$= \begin{bmatrix} \frac{9}{4}+\frac{3\sqrt{3}}{4}i-\frac{3\sqrt{3}}{4}i-\frac{3}{4}i^2 \\ -\frac{3}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix} = \begin{bmatrix} \frac{9}{4}+\frac{3}{4} \\ -\frac{3}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{3}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix}$

$a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$ $\quad \vec{\varphi} = \begin{bmatrix} 3 \\ -\frac{3}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix}$

any real number

>0

$A = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix} = PCP^{-1} = \begin{bmatrix} 3 & 0 \\ -3/2 & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{bmatrix}P^{-1}$

$r = \sqrt{a^2+b^2} = 1$

$\theta = \tan^{-1}(\sqrt{3}) = \pi/3$

$\begin{bmatrix} 3 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} \end{bmatrix} \frac{2}{\ldots}\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 \\ -\frac{3}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \frac{2}{3\sqrt{3}} \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{3}{2} & 3 \end{bmatrix}$$

$$= \frac{2}{3\sqrt{3}} \begin{bmatrix} 3 & 0 \\ -\frac{3}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{3\sqrt{3}}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

$$= \frac{2}{3\sqrt{3}} \begin{bmatrix} -\frac{3\sqrt{3}}{2} & -\frac{9\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} & 3\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \qquad A = PCP^{-1}$$

$$0 = (1-\lambda)(5-\lambda) + 8$$

$$0 = \lambda^2 - \lambda - 5\lambda + 5 + 8$$

$$0 = \lambda^2 - 6\lambda + 13$$

$$0 = \lambda^2 - 6\lambda + 9 + 4$$

$$-4 = (\lambda - 3)^2 \qquad \lambda = 3 \pm 2i$$

$\lambda = 3 - 2i$:

$$A - (3-2i)I = \begin{bmatrix} 1-(3-2i) & 2 \\ -4 & 5-(3-2i) \end{bmatrix}$$

$$= \begin{bmatrix} -2+2i & 2 \\ -4 & 2+2i \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$A = PCP^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} P^{-1} \qquad \lambda = a \pm ib$$

$$\vec{\varphi} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \begin{bmatrix} 1+1i \\ 2+0i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \operatorname{Re}\varphi & \operatorname{Im}\varphi \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} P^{-1}$$

$$\vec{\varphi} \rightarrow A\vec{\varphi} = (a-ib)\vec{\varphi} \qquad \lambda = a \pm ib$$

Discrete Dynamical Systems

$\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_k, \ldots$

$\vec{x}_{k+1} = A\vec{x}_k$, A is diagonalizable w/ e-vals $\lambda_1, \lambda_2, \ldots, \lambda_n$ w/ e-vects $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

$\vec{x}_0$ is known

x_0 is known

$$\vec{x}_1 = A\vec{x}_0$$

$$\vec{x}_2 = A\vec{x}_1 = A^2\vec{x}_0$$

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$\vec{x}_1 = A\vec{x}_0 = A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)$$

$$= c_1 A\vec{v}_1 + \cdots + c_n A\vec{v}_n$$

$$= c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n$$

$$\vec{x}_2 = A\vec{x}_1 = A(c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n)$$

$$= c_1\lambda_1^2\vec{v}_1 + \cdots + c_n\lambda_n^2\vec{v}_n$$

$$\boxed{\vec{x}_k = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \cdots + c_n\lambda_n^k\vec{v}_n}$$

$$\vec{x}_{k+1} = \begin{bmatrix} -0.2 & -0.2 \\ -1.8 & 0.3 \end{bmatrix}\vec{x}_k, \quad \vec{x}_0 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$0 = (-0.2-\lambda)(0.3-\lambda) - (-0.2)(-1.8)$$

$$= \lambda^2 + 0.2\lambda - 0.3\lambda - 0.06 - 0.36$$

$$0 = \lambda^2 - 0.1\lambda - 0.42$$

$$(\lambda - 0.7)(\lambda + 0.6) \qquad \lambda = -0.6,\ 0.7$$

$$\lambda = -0.6: \quad A - (-0.6)I = \begin{bmatrix} -0.2+0.6 & -0.2 \\ -1.8 & 0.3+0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.2 \\ -1.8 & 0.9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = 0.7: \quad A - 0.7I = \begin{bmatrix} -0.9 & -0.2 \\ -1.8 & -0.4 \end{bmatrix} \begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_k = c_1 (-0.6)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix} -5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$13 = 13 c_2$$

$$1 = c_2 \qquad c_1 = -3$$

$$\vec{x}_k = -3(-0.6)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1(0.7)^k \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$x_k \quad c(0.6) \begin{bmatrix} \\ 2 \end{bmatrix} + 1(0.7) \begin{bmatrix} \\ 9 \end{bmatrix}$$

$k \to \infty$

$$0 \qquad\qquad 0$$

the origin is an attractor

$$\vec{x}_{k+1} = \begin{bmatrix} 0.2 & -0.3 \\ 1.8 & 1.7 \end{bmatrix} \vec{x}_k, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$0 = (0.2 - \lambda)(1.7 - \lambda) + 0.3(1.8)$$

$$0 = \lambda^2 - 0.2\lambda - 1.7\lambda + 0.34 + 0.54$$

$$0 = \lambda^2 - 1.9\lambda + 0.88$$

$$0 = (\lambda - 1.1)(\lambda - 0.8) \qquad \lambda = 1.1, 0.8$$

$$\lambda = 1.1: A - 1.1I = \begin{bmatrix} -0.9 & -0.3 \\ 1.8 & 0.6 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\lambda = 0.8: A - 0.8I = \begin{bmatrix} -0.6 & -0.3 \\ 1.8 & 0.9 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{x}_k = c_1(1.1)^k \begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2(0.8)^k \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{x}_k = c_1(1.1)^k\begin{bmatrix}-1\\3\end{bmatrix} + c_2(0.8)^k\begin{bmatrix}-1\\2\end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix}1\\1\end{bmatrix} = c_1\begin{bmatrix}-1\\3\end{bmatrix} + c_2\begin{bmatrix}-1\\2\end{bmatrix}$$

$$c_2 = 3 \qquad c_1 = -4$$

$$\vec{x}_k = -4\underbrace{(1.1)^k}_{\infty}\begin{bmatrix}-1\\3\end{bmatrix} + 3\underbrace{(0.8)^k}_{0}\begin{bmatrix}-1\\2\end{bmatrix}$$

as $k \to \infty$

saddle point

$|\lambda_1|, |\lambda_2| < 1 \to$ attractor

$|\lambda_1| < 1,\ |\lambda_2| > 1 \to$ saddle point

$|\lambda_1|, |\lambda_2| > 1 \to$ repeller