Building Calculus From Scratch

With Terence Tao's *Analysis I*

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1 Starting Axioms

1. 0 is a natural number.

2. if n is a natural number, then its successor $n{+}{+}$ is a natural number. We define the short-hand symbols
$$1 := 0{+}{+}$$
$$2 := (0{+}{+}){+}{+}$$
$$3 := ((0{+}{+}){+}{+}){+}{+}$$
$$\ldots$$

3. 0 is not the successor of any natural number. That is, $0 \neq n{+}{+}$ for any n.

4. Different natural numbers have different successors. That is, $[n \neq m] \iff [n{+}{+} \neq m{+}{+}]$ for any n, m. Equivalently, if we have that $n{+}{+} = m{+}{+}$, then we have that $n = m$.

5. The principle of induction holds. That is, let $P(n)$ be a property of the natural number n. Let $P(0)$ be true, and let $P(n) \implies P(n{+}{+})$. Then $P(n)$ is true for all natural numbers n.

2 Steps & Problems

1. Define the addition operation on two natural numbers, n and m.

We define incrementing m by zero as $0 + m := m$. Suppose inductively, that we know how to increment m by n. Then we can increment m by $n{+}{+}$ by defining

$$(n{+}{+}) + m := (n + m){+}{+}$$

This recursive definition allows us to now add numbers (perform repeated incrementation). For example, $2 + m = (1{+}{+}) + m = (1 + m){+}{+} = ((0{+}{+}) + m){+}{+} = ((0 + m){+}{+}){+}{+} = ((m){+}{+}){+}{+}$, which is exactly m incremented twice.

2. Prove that $n + 0 = n$ for any natural number n

We will induct on n. We have that $0 + 0 = 0$ from $0 + m := m$ for any natural m, including 0. Suppose, inductively, that $k + 0 = k$ for some natural number k. Then

$$\begin{aligned}(k{+}{+}) + 0 &= (k + 0){+}{+} \quad \text{(definition of addition)} \\ &= (k){+}{+} \quad \text{(I.H)}\end{aligned}$$

Hence for any natural number n, we have that $n + 0 = n$.

3. Prove that $n + (m{+}{+}) = (n + m){+}{+}$

We will induct on n. For $n = 0$,

$$\begin{aligned}0 + (m{+}{+}) &= m{+}{+} \\ &= (0 + m){+}{+}\end{aligned}$$

Assume inductively that $k + (m{+}{+}) = (k + m){+}{+}$ for some natural number k. Then

$$\begin{aligned}(k{+}{+}) + (m{+}{+}) &= (k + (m{+}{+})){+}{+} \quad \text{(definition of addition)} \\ &= ((k + m){+}{+}){+}{+} \quad \text{(I.H)} \\ &= ((k{+}{+}) + m){+}{+}\end{aligned}$$

Hence, $n + (m{+}{+}) = (n + m){+}{+}$ for all natural numbers n.

4. Prove that the natural numbers are closed under addition. That is, if a,b are natural numbers, then $a + b$ is a natural number.

Fix a natural number b. For $a = 0$, we have $a + b = b$ (natural). Assume inductively that $a + b$ is natural for some natural a. Then $(a{+}{+}) + b = (a + b){+}{+}$ where $a + b$ is natural by the inductive hypothesis. Hence $(a+b){+}{+}$ is natural by Axiom 2 (if a number is natural then its successor must also be natural). Hence the natural numbers are closed under addition.

5. Prove that the addition operation is commutative. That is, for any two natural numbers, a and b, we have that $a + b = b + a$

Fix a natural number b. For $a = 0$,

$$\begin{aligned}a + b &= b \\ &= b + 0 \quad \text{(from proof that } n + 0 = n \text{ for any } n) \\ &= b + a\end{aligned}$$

Assume inductively that $a+b=b+a$ for some natural a. Then

$$\begin{aligned}(a{+}{+})+b &= (a+b){+}{+} \text{ (definition of addition)}\\ &= (b+a){+}{+} \text{ (I.H)}\\ &= b+(a{+}{+}) \text{ (alternate-order proof of addition)}\end{aligned}$$

Hence for any a, b, we have that $a+b=b+a$.

6. Prove that the additive cancellation law holds. That is, for any three natural numbers a, b, and c such that $a+b=a+c$, we have $b=c$.

Fix natural numbes b and c. For $a=0$,

$$\begin{aligned}a+b=b \text{ (0-incrementing)}\\ a+c=c \text{ (0-incrementing)}\\ a+b=a+c \implies b=c \text{ (substitution)}\end{aligned}$$

Assume inductively that for some natural a, we have $[a+b=a+c] \implies [b=c]$. Then

$$\begin{aligned}(a{+}{+})+b=(a{+}{+})+c &\implies (a+b){+}{+}=(a+c){+}{+} \text{ (definition of addition)}\\ &\implies a+b \text{ and } a+c \text{ are naturals (additive closure)}\\ &\implies a+b=a+c \text{ (Axiom 4)}\\ &\implies b=c \text{ (I.H)}\end{aligned}$$

Hence, for any a, b, c, we have $[a+b=a+c] \implies [b=c]$

7. A natural number a is said to be positive iff $a \neq 0$. Prove that the sum of a natural number and a positive number must be positive.

Fix a positive number b. Let $a=0 \implies a+b=b \neq 0$. Then $(a{+}{+})+b=(a+b){+}{+} \neq 0$ since $a+b$ is a natural number (by our proof of additive closure), and the successor of any natural number is positive by Axiom 3 (zero is not the successor of any natural number). Hence $a+b \neq 0$ for all a.

8. Prove that, for natural numbers a and b, if $a+b=0$ then $a=0$ and $b=0$.

We will prove the contrapositive:

$$\begin{aligned}&\text{Let } a \neq 0 \text{ or } b \neq 0 \implies\\ &\qquad \text{if } a \neq 0 \implies a+b \neq 0 \text{ (by previous proof)}\\ &\qquad \text{if } b \neq 0 \implies a+b \neq 0 \text{ (by previous proof)}\end{aligned}$$

Hence $a \neq 0$ or $b \neq 0 \implies a+b \neq 0$. Equivalently,
$a+b=0 \implies a=0$ and $b=0$.

9. (**EXERCISE 1**) Prove that the addition operation is associative. That is, for any three natural numbers, a, b, c, we have that $(a+b)+c = a+(b+c)$

We will fix $b, c \in \mathbb{N}$ and induct on a. Base case:

$$\begin{aligned}(0+b)+c &= b+c \\ &= 0+(b+c)\end{aligned}$$

Assume inductively that $(a+b)+c = a+(b+c)$ for some natural number a. Then

$$\begin{aligned}((a{+}{+})+b)+c &= ((a+b){+}{+})+c \\ &= ((a+b)+c){+}{+} \\ &= (a+(b+c)){+}{+} \quad \text{(I.H)} \\ &= (a{+}{+})+(b+c)\end{aligned}$$

This closes the induction.

10. (**EXERCISE 2**) Let a be a positive number. Prove that there exists exactly one natural number b such that $b{+}{+} = a$

For the first positive number $a = 1$, choose $b = 0$ and we have $b{+}{+} = a$. Then for the next natural $a{+}{+}$, choose $b = a$ and we have $b{+}{+} = a{+}{+}$ (no induction hypothesis needed). Hence, every positive a has at least 1 predecessor b. Axiom 4 (different natural numbers have different successors) guarantees that no two distinct natural numbers have the same successor. Hence, for every a, the corresponding b that satisfies $b{+}{+} = a$ must be the *only* such b.

11. Define the ordering of the natural numbers.

Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

12. (**EXERCISE 3**) Prove properties of order.

(a) (Reflexivity) Prove that $a \geq a$ for any natural number a.

For any natural a, we can find an m such that $a = a + m$ $(a \geq a)$ $\implies$ choose $m = 0$

(b) (Transitivity) Prove that if $a \geq b$ and $b \geq c$, then $a \geq c$.

$$\begin{aligned} a \geq b, b \geq c &\implies a = b + k_0, b = c + k_1 \ \text{ for some } k_0, k_1 \in \mathbb{N} \\ &\implies a = (c + k_1) + k_0 \\ &\implies a = c + (k_1 + k_0) \ \ \text{(associativity)} \\ &\implies a = c + m \ \text{ for some natural } m = k_0 + k_1 \ \ \text{(additive closure)} \\ &\implies a \geq c \end{aligned}$$

(c) (Anti-symmetry) Prove that if $a \geq b$ and $b \geq a$, then $a = b$

$$\begin{aligned} a \geq b, b \geq a &\implies a = b + k_0, b = a + k_1 \ \text{ for some } k_0, k_1 \in \mathbb{N} \\ &\implies a = (a + k_1) + k_0 \\ &\implies a = a + (k_1 + k_0) \ \ \text{(associativity)} \\ &\implies a + 0 = a + (k_1 + k_0) \\ &\implies 0 = k_1 + k_0 \ \ \text{(cancellation law)} \\ &\implies k_0 = 0 \ \text{ and } \ k_1 = 0 \ \ \text{(zero-sum proof)} \\ &\implies a = b \end{aligned}$$

(d) (Order preservation under addition) Prove that $a \geq b$ if and only if $a + c \geq b + c$.

$$\begin{aligned} a \geq b &\iff \exists k (a = b + k) \\ &\iff a + c = (b + k) + c \\ &\iff a + c = b + (k + c) \\ &\iff a + c = b + (c + k) \\ &\iff a + c = (b + c) + k \\ &\iff a + c \geq b + c \end{aligned}$$

(e) Prove that $a < b$ if and only if $a{++} \leq b$.

$$
\begin{aligned}
a < b &\iff \exists k(b = a + k \text{ and } a \neq b) \\
&\iff a + k \neq a \\
&\iff k \neq 0 \\
&\iff b = a + m{+}{+} \text{ with } k = m{+}{+} \\
&\iff b = a + (m + 1) \\
&\iff b = a + (1 + m) \\
&\iff b = (a + 1) + m \\
&\iff b = (a{+}{+}) + m \\
&\iff a{+}{+} \leq b
\end{aligned}
$$

(f) Prove that $a < b$ if and only if $b = a + d$ for some positive number d.

$$
\begin{aligned}
a < b &\iff \exists k(b = a + k \text{ and } a \neq b) \\
&\iff a + k \neq a \\
&\iff k \neq 0 \\
&\iff \text{choose } d = k \\
&\iff b = a + d \text{ and } d \text{ is positive}
\end{aligned}
$$

13. EXERCISE 4 Justify the following:

(a) $0 \leq b$ for all b.

For any b, we can find an m such that $b = 0 + m$ $(0 \leq b)$ $\implies$ choose $m = b$

(b) If $a > b$, then $a{+}{+} > b$

$$
\begin{aligned}
a > b &\implies a{+}{+} > a > b \text{ (definition of successor)} \\
&\implies a{+}{+} > b \text{ (transitivity)}
\end{aligned}
$$

(c) If $a = b$, then $a{+}{+} > b$

$$\begin{aligned} a = b &\implies a{+}{+} > a = b \text{ (definition of successor)} \\ &\implies a{+}{+} > b \end{aligned}$$

14. **EXERCISE 5** (Prove the principle of strong induction). Let m_0 be a natural number and let $P(m)$ be a property of an arbitrary natural number m. Suppose that for each $m \geq m_0$, if $P(m')$ is true for all $m_0 \leq m' < m$, then $P(m)$ is also true. Prove that $P(m)$ is then true for all $m \geq m_0$.

We will prove the principle of strong induction by re-notating (and effectively reducing) the multi-variable inductive hypothesis to a single-variable inductive hypothesis. Then we will prove the strong induction principle by simply invoking the ordinary induction principle (Axiom 5).

proof. Suppose that for each $m \geq m_0$, if $P(m')$ is true for all $m_0 \leq m' < m$, then $P(m)$ is also true. Put formally, we are assuming:

$$[\forall m \geq m_0 (\forall m' (m_0 \leq m' < m) \implies P(m'))] \implies P(m) \tag{1}$$

Let $Q(n)$ be the property that $P(m)$ is true for all $m_0 \leq m < n$. Note that $Q(n) \implies P(n)$ by (1). Hence, to show that $P(m)$ is true for all $m \geq m_0$, it suffices to show that $Q(n)$ holds for all $n \geq m_0$.

$Q(m_0)$ is true (vacuously). Suppose inductively that $Q(n)$ is true for some $n \geq m_0$: that is, $P(m)$ is true for all $m_0 \leq m < n$. Then $P(n)$ is true by (1), meaning that $P(m)$ is true for all all of $m_0 \leq m < n + 1$. But this is exactly the statement $Q(n + 1)$. This closes the induction.

15. (**EXERCISE 6**) (Prove the principle of backwards induction). Let n be a natural number and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is true. Prove that $P(m)$ is true for all $m \leq n$.

The principle of backward induction starting on n can be written as the statement

$$Q(n) := [P(n) \wedge \forall m (P(m{+}{+}) \implies P(m))] \implies P(m) \forall m \leq n \tag{2}$$

We effectively want to prove that $Q(n)$ is true for all n, and we can use ordinary forward induction to do this. We have that

$$Q(0) : [P(0) \wedge \forall m (P(m{+}{+}) \implies P(m))] \implies P(0)$$

is trivially true. Suppose inductively that for some $n \geq 0$, $Q(n)$ is true. Now we will prove that the following statement is true:

$$Q(n+1) : [P(n+1) \wedge \forall m(P(m{+}{+}) \implies P(m))] \implies P(m)\forall m \leq n+1$$

Since $Q(n)$ is true by the inductive hypothesis, we have that

$$[P(n) \wedge \forall m(P(m{+}{+}) \implies P(m))] \implies P(m)\forall m \leq n$$

Appending a condition to both sides of an implication maintains its truth. Hence we can append $P(n+1)$ to both sides and maintain the statement's truth:

$$[P(n+1) \wedge P(n) \wedge \forall m(P(m{+}{+}) \implies P(m))] \implies [P(n+1) \wedge P(m)\forall m \leq n]$$

$P(n+1) = P(n{+}{+})$ already implies $P(n)$ by the condition $\forall m(P(m{+}{+}) \implies P(m))$. Hence $P(n)$ is a redundant condition on the left-hand side. Removing $P(n)$ and re-writing the right-hand side, we get:

$$[P(n+1) \wedge \forall m(P(m{+}{+}) \implies P(m))] \implies P(m)\forall m \leq n+1]$$

Which is exactly the statement $Q(n+1)$. This closes the induction.