Building Calculus From Scratch

With Terence Tao's *Analysis I*

October 14, 2025

1 Starting Axioms

1. 0 is a natural number.
2. if n is a natural number, then its successor $n{+}{+}$ is a natural number. We define the short-hand symbols
$$1 := 0{+}{+}$$
$$2 := (0{+}{+}){+}{+}$$
$$3 := ((0{+}{+}){+}{+}){+}{+}$$
$$\cdots$$
3. 0 is not the successor of any natural number. That is, $0 \neq n{+}{+}$ for any n.
4. Different natural numbers have different successors. That is, $[n \neq m] \iff [n{+}{+} \neq m{+}{+}]$ for any n, m. Equivalently, if we have that $n{+}{+} = m{+}{+}$, then we have that $n = m$.
5. The principle of induction holds. That is, let $P(n)$ be a property of the natural number n. Let $P(0)$ be true, and let $P(n) \implies P(n{+}{+})$. Then $P(n)$ is true for all natural numbers n.

2 Steps & Problems

1. Define the addition operation on two natural numbers, n and m.

We define incrementing m by zero as $0 + m := m$. Suppose inductively, that we know how to increment m by n. Then we can increment m by $n{+}{+}$ by defining

$$(n{+}{+}) + m := (n + m){+}{+}$$

This recursive definition allows us to now add numbers (perform repeated incrementation). For example, $2 + m = (1{+}{+}) + m = (1 + m){+}{+} = ((0{+}{+}) + m){+}{+} = ((0 + m){+}{+}){+}{+} = ((m){+}{+}){+}{+}$, which is exactly m incremented twice.

2. Prove that $n + 0 = n$ for any natural number n

We will induct on n. We have that $0 + 0 = 0$ by the fact that $0 + m := m$ for any natural number m, including 0. Suppose, inductively, that $k + 0 = k$ for some natural number k. Now consider the sum $(k{+}{+}) + 0 = (k + 0){+}{+} \overset{\text{I.H.}}{=} (k){+}{+}$. Hence for any natural number n, we have that $n + 0 = n$.

3. Prove that $n + (m{+}{+}) = (n + m){+}{+}$

We will induct on n. For $n = 0$, we have that $0 + (m{+}{+}) = m{+}{+} = (0 + m){+}{+}$. Assume inductively that $k + (m{+}{+}) = (k + m){+}{+}$ for some natural number k. Now for the next number, $k{+}{+}$, we have that $(k{+}{+}) + (m{+}{+}) = (k + (m{+}{+})){+}{+}$ (by the definition from problem 1) $\overset{\text{I.H.}}{=} ((k + m){+}{+}){+}{+} = ((k{+}{+}) + m){+}{+}$. Hence, $n + (m{+}{+}) = (n + m){+}{+}$ for all natural numbers n.

4. Prove that the natural numbers are closed under addition. That is, if a,b are natural numbers, then $a + b$ is a natural number.

We will fix b and induct on a. Base case: 0 is a natural number by Axiom 1 and let b be a natural number. Then $0 + b = b$ by our definition of incrementing by zero. But b is a natural number (by assumption). This closes the base case. Assume inductively that for some natural number a, if a and b are natural numbers then so is $a + b$. Consider the next natural number $a{+}{+}$. By Axiom 2, since a is a natural number (I.H), we must have that $a{+}{+}$ is also a natural number. Now consider the sum $(a{+}{+}) + b = (a + b){+}{+}$. We have that a and b are natural numbers, so $a + b$ is a natural number by the inductive hypothesis. By Axiom 2, we have that its successor $(a + b){+}{+}$ must also be a natural number. This closes the induction. Hence the natural numbers are closed under addition.

5. Prove that the addition operation is commutative. That is, for any two natural numbers, a and b, we have that $a + b = b + a$

We will fix b and induct on a. We have that $0+b = b$ from our first definition—incrementing a natural number by 0. Then $b = b + 0$, which we have from problem 2's proof. Hence we have our base case $0 + b = b + 0$. Suppose inductively that $a + b = b + a$ for some natural number a. Now consider the next number, $a{+}{+}$: we have $(a{+}{+}) + b = (a + b){+}{+} \overset{\text{I.H.}}{=} (b + a){+}{+} = b + (a{+}{+})$ by our previous proof. Hence for some natural number b, we have that $a + b = b + a$ for all natural numbers a.

6. Prove that the additive cancellation law holds. That is, for any three natural numbers a, b, and c such that $a + b = a + c$, we have $b = c$.

We will fix b,c and induct on a. For the base case $a = 0$, we will show that $[0 + b = 0 + c] \implies [c = b]$. Let us assume that $0 + b = 0 + c$. We have that $0 + b = b$ and that $0 + c = c$ from a previous proof. Then by transitive equality, we have $b = c$. This closes the base case. Assume inductively that for some natural number a, we have that $[a + b = a + c] \implies [b = c]$. Now consider the next natural number $a{+}{+}$. Assume that $(a{+}{+}) + b = (a{+}{+}) + c$. Re-writing both sides, we have $(a + b){+}{+} = (a + c){+}{+}$. $a + b$ is a natural number and $a + c$ is a natural number (by our additive closure proof). Recall by Axiom 4 that if the successors of two natural numbers are equal, then the natural numbers themselves must be equal. Hence we have that $a + b = a + c$. By the inductive hypothesis ($[a + b = a + c] \implies [b = c]$), we must now have that $b = c$. This closes the inductive step. Thus, for any a, b, c such that $a + b = a + c$, we also have that $b = c$.

7. A natural number a is said to be positive iff $a \neq 0$. Prove that the sum of a natural number and a positive number must be positive.

Fix a positive number b. We will show that $a+b$ is positive for all a. Base case: $0+b = b$ by our definition of incrementing by zero, and b is positive (by assumption). This closes the base case. Assume inductively that $a+b$ is positive for some natural number a. Then consider the statement for the next natural number $a{+}{+}$: we have $(a{+}{+})+b = (a+b){+}{+}$. By the inductive hypothesis, we have that $a+b$ is positive. But this means that $(a+b){+}{+}$, the succesor of $a + b$, must also be positive—it cannot be 0 because of Axiom 3: zero is not the successor of any natural number. This closes the induction.

8. Prove that, for natural numbers a and b, if $a + b = 0$ then $a = 0$ and $b = 0$.

We will prove the equivalent contraposition statement: if $a \neq 0$ or $b \neq 0$, then $a + b \neq 0$. Assume that either $a \neq 0$ or $b \neq 0$. In the case that $a \neq 0$, $a + b$ must be positive (according to the result of the previous proof) and hence non-zero. In the case that $b \neq 0$, $a + b$ must similarly be positive and hence non-zero.

9. (**EXERCISE 1**) Prove that the addition operation is associative. That is, for any three natural numbers, a, b, c, we have that $(a + b) + c = a + (b + c)$

We will fix $b, c \in \mathbb{N}$ and induct on a. Base case: $(0 + b) + c = b + c = 0 + (b + c)$. Assume inductively that $(a + b) + c = a + (b + c)$ for some natural number a. Then $((a{+}{+})+b)+c = ((a+b){+}{+})+c = ((a+b)+c){+}{+} \overset{\text{I.H.}}{=} (a+(b+c)){+}{+} = (a{+}{+})+(b+c)$. Hence, we have $(a + b) + c = a + (b + c)$ for all natural numbers a.

10. (**EXERCISE 2**) Let a be a positive number. Prove that there exists exactly one natural number b such that $b{+}{+} = a$

Will will induct on a. Base case: for $a = 1$, we have $b = 0 \implies b{+}{+} = a$. This is the *only* such b for $a = 0$ because a different $b \neq 0$ would have a different successor $b{+}{+} \neq 1$ acccording to Axiom 4: different natural numbers have different successors. Suppose inductively that for some natural number a there exists a b such that $b{+}{+} = a$. Now consider the next natural number $a{+}{+}$: the number $a{+}{+}$, by definition, is the successor of a. So we have found one $b = a$ such that $b{+}{+} = a{+}{+}$. This is the only such b because, similarly, by Axiom 4, any other $b \neq a$ has a different successor $b{+}{+} \neq a{+}{+}$. This closes the induction.

11. Define the ordering of the natural numbers.

Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

12. (**EXERCISE 3**) Prove properties of order.

(a) (Reflexivity) Prove that $a \geq a$ for any natural number a.

We can find a natural number a such that $0 = 0 + a$. Choose $a = 0$ and we have $0 = 0 + 0$. Hence $0 \geq 0$. Assume inductively that $n \geq n$ for some natural number n. Consider the next natural number $n{+}{+}$. We can find an a such that $n{+}{+} = (n{+}{+}) + a$. Chose $a = 0$ again, and then we have $(n{+}{+}) + 0 = (n + 0){+}{+} = (n){+}{+}$. Hence $n{+}{+} \geq n{+}{+}$. This closes the induction. Note that this is degenerate induction, because the inductive hypothesis was not needed.

(b) (Transitivity) Prove that if $a \geq b$ and $b \geq c$, then $a \geq c$.

Assume that $a \geq b$ and $b \geq c$ for some natural numbers a, b, c. Then we have some natural number k_1 such that $a = b + k_1$, and we have some natural number k_2 such that $b = c + k_2$. Substituting the second equation into the first, we have that $a = (c + k_2) + k_1$, which, by associativity, is $a = c + (k_2 + k_1)$. By additive closure, $k_2 + k_1$ is some natural number, call it m. But since we have that $a = c + m$ for some natural number m, then we must have that $a \geq c$.

(c) (Anti-symmetry) Prove that if $a \geq b$ and $b \geq a$, then $a = b$

Assume that $a \geq b$ and $b \geq a$ for some a, b. Then we have some k_1 such that $a = b + k_1$ and we have some k_2 such that $b = a + k_2$. Substituting the second equation into the first, we have $a = (a + k_2) + k_1$ which is $a = a + (k_2 + k_1)$ by associativity. Re-writing this as $a + 0 = a + (k_2 + k_1)$ allows us to apply our proven cancellation law to get that $0 = (k_2 + k_1)$, which means that $k_1 = 0$ and $k_2 = 0$ (by our previous zero-sum proof). But this means that $a = b + 0$, or $a = b$.

(d) (Order preservation under addition) Prove that $a \geq b$ if and only if $a + c \geq b + c$.

$$\begin{aligned} a \geq b &\iff \exists k(a = b + k) \\ &\iff a + c = (b + k) + c \\ &\iff a + c = b + (k + c) \\ &\iff a + c = b + (c + k) \\ &\iff a + c = (b + c) + k \\ &\iff a + c \geq b + c \end{aligned}$$

(e) Prove that $a < b$ if and only if $a{+}{+} \leq b$.

$$\begin{aligned} a < b &\iff \exists k(b = a + k \ \text{ and } \ a \neq b) \\ &\iff k \neq 0 \\ &\iff \exists m(k = m{+}{+}) \\ &\iff b = a + m{+}{+} \\ &\iff b = a + (m + 1) \\ &\iff b = a + (1 + m) \\ &\iff b = (a + 1) + m \\ &\iff b = (a{+}{+}) + m \\ &\iff a{+}{+} \leq b \end{aligned}$$

(f) Prove that $a < b$ if and only if $b = a + d$ for some positive number d.

$$\begin{aligned} a < b &\iff \exists k(b = a + k \ \text{ and } \ a \neq b) \\ &\iff k \neq 0 \\ &\iff \text{choose } \ d = k \\ &\iff b = a + d \ \text{ and } d \text{ is positive} \end{aligned}$$

13. EXERCISE 4 Justify the following:

(a) $0 \leq a$ for all a.

$0 \leq 0$ since $0 = 0 + 0$. Assume inductively that $0 \leq a$ for some a. By the inductive hypothesis, we have that $1 \leq a{+}{+}$ by order-preservation under addition (proven previously). But we certainly have that $0 \leq 1 \leq a{+}{+}$, and by transitivity (proven previously), we have $0 \leq a{+}{+}$. This closes the induction.

(b) If $a > b$, then $a{+}{+} > b$

Let $a > b$. We have that $a{+}{+} > a$ by definition ($a{+}{+} = a + 1$). Putting these together, we have $a{+}{+} > a > b$. By transitivity (proved earlier), $a{+}{+} > b$.

(c) If $a = b$, then $a{+}{+} > b$

Let $a = b$. Then $a{+}{+} = b{+}{+} > b \implies a{+}{+} > b$.

14. EXERCISE 5 Prove the principle of strong induction.

$$
\begin{aligned}
x = 2 &\implies x + 1 = 3 \\
&\implies x + 2 = 4
\end{aligned}
$$