Building Calculus From Scratch

With Terence Tao's *Analysis I*

October 16, 2025

1 Starting Axioms

1. 0 is a natural number.

2. if n is a natural number, then its successor $n{+}{+}$ is a natural number. We define the short-hand symbols

$$1 := 0{+}{+}$$
$$2 := (0{+}{+}){+}{+}$$
$$3 := ((0{+}{+}){+}{+}){+}{+}$$
$$\dots$$

3. 0 is not the successor of any natural number. That is, $0 \neq n{+}{+}$ for any n.

4. Different natural numbers have different successors. That is, $[n \neq m] \iff [n{+}{+} \neq m{+}{+}]$ for any n, m. Equivalently, if we have that $n{+}{+} = m{+}{+}$, then we have that $n = m$.

5. The principle of induction holds. That is, let $P(n)$ be a property of the natural number n. Let $P(0)$ be true, and let $P(n) \implies P(n{+}{+})$. Then $P(n)$ is true for all natural numbers n.

2 Steps & Problems

1. Define the addition operation on two natural numbers, n and m.

We define incrementing m by zero as $0 + m := m$. Suppose inductively, that we know how to increment m by n. Then we can increment m by $n{+}{+}$ by defining

$$(n{+}{+}) + m := (n + m){+}{+}$$

This recursive definition allows us to now add numbers (perform repeated incrementation). For example, $2 + m = (1{+}{+}) + m = (1 + m){+}{+} = ((0{+}{+}) + m){+}{+} = ((0 + m){+}{+}){+}{+} = ((m){+}{+}){+}{+}$, which is exactly m incremented twice.

2. Prove that $n + 0 = n$ for any natural number n

We will induct on n. We have that $0 + 0 = 0$ by the fact that $0 + m := m$ for any natural number m, including 0. Suppose, inductively, that $k + 0 = k$ for some natural number k. Now consider the sum $(k{+}{+}) + 0 = (k + 0){+}{+} \stackrel{\text{I.H.}}{=} (k){+}{+}$. Hence for any natural number n, we have that $n + 0 = n$.

3. Prove that $n + (m{+}{+}) = (n + m){+}{+}$

We will induct on n. For $n = 0$, we have that $0 + (m{+}{+}) = m{+}{+} = (0 + m){+}{+}$. Assume inductively that $k + (m{+}{+}) = (k + m){+}{+}$ for some natural number k. Now for the next number, $k{+}{+}$, we have that $(k{+}{+}) + (m{+}{+}) = (k + (m{+}{+})){+}{+}$ (by the definition from problem 1) $\stackrel{\text{I.H.}}{=} ((k + m){+}{+}){+}{+} = ((k{+}{+}) + m){+}{+}$. Hence, $n + (m{+}{+}) = (n + m){+}{+}$ for all natural numbers n.

4. Prove that the natural numbers are closed under addition. That is, if a,b are natural numbers, then $a + b$ is a natural number.

We will fix b and induct on a. Base case: 0 is a natural number by Axiom 1 and let b be a natural number. Then $0 + b = b$ by our definition of incrementing by zero. But b is a natural number (by assumption). This closes the base case. Assume inductively that for some natural number a, if a and b are natural numbers then so is $a + b$. Consider the next natural number $a{+}{+}$. By Axiom 2, since a is a natural number (I.H), we must have that $a{+}{+}$ is also a natural number. Now consider the sum $(a{+}{+}) + b = (a + b){+}{+}$. We have that a and b are natural numbers, so $a + b$ is a natural number by the inductive hypothesis. By Axiom 2, we have that its successor $(a + b){+}{+}$ must also be a natural number. This closes the induction. Hence the natural numbers are closed under addition.

5. Prove that the addition operation is commutative. That is, for any two natural numbers, a and b, we have that $a + b = b + a$

We will fix b and induct on a. We have that $0+b = b$ from our first definition—incrementing a natural number by 0. Then $b = b + 0$, which we have from problem 2's proof. Hence we have our base case $0 + b = b + 0$. Suppose inductively that $a + b = b + a$ for some natural number a. Now consider the next number, $a{+}{+}$: we have $(a{+}{+}) + b = (a + b){+}{+} \stackrel{\text{I.H.}}{=} (b + a){+}{+} = b + (a{+}{+})$ by our previous proof. Hence for some natural number b, we have that $a + b = b + a$ for all natural numbers a.

6. Prove that the additive cancellation law holds. That is, for any three natural numbers a, b, and c such that $a + b = a + c$, we have $b = c$.

We will fix b,c and induct on a. For the base case $a = 0$, we will show that $[0 + b = 0 + c] \implies [c = b]$. Let us assume that $0 + b = 0 + c$. We have that $0 + b = b$ and that $0 + c = c$ from a previous proof. Then by transitive equality, we have $b = c$. This closes the base case. Assume inductively that for some natural number a, we have that $[a + b = a + c] \implies [b = c]$. Now consider the next natural number $a{+}{+}$. Assume that $(a{+}{+}) + b = (a{+}{+}) + c$. Re-writing both sides, we have $(a + b){+}{+} = (a + c){+}{+}$. $a + b$ is a natural number and $a + c$ is a natural number (by our additive closure proof). Recall by Axiom 4 that if the successors of two natural numbers are equal, then the natural numbers themselves must be equal. Hence we have that $a + b = a + c$. By the inductive hypothesis ($[a + b = a + c] \implies [b = c]$), we must now have that $b = c$. This closes the inductive step. Thus, for any a, b, c such that $a + b = a + c$, we also have that $b = c$.

7. A natural number a is said to be positive iff $a \neq 0$. Prove that the sum of a natural number and a positive number must be positive.

Fix a positive number b. We will show that $a+b$ is positive for all a. Base case: $0+b = b$ by our definition of incrementing by zero, and b is positive (by assumption). This closes the base case. Assume inductively that $a+b$ is positive for some natural number a. Then consider the statement for the next natural number $a{+}{+}$: we have $(a{+}{+})+b = (a+b){+}{+}$. By the inductive hypothesis, we have that $a+b$ is positive. But this means that $(a+b){+}{+}$, the succesor of $a + b$, must also be positive—it cannot be 0 because of Axiom 3: zero is not the successor of any natural number. This closes the induction.

8. Prove that, for natural numbers a and b, if $a + b = 0$ then $a = 0$ and $b = 0$.

We will prove the equivalent contraposition statement: if $a \neq 0$ or $b \neq 0$, then $a + b \neq 0$. Assume that either $a \neq 0$ or $b \neq 0$. In the case that $a \neq 0$, $a + b$ must be positive (according to the result of the previous proof) and hence non-zero. In the case that $b \neq 0$, $a + b$ must similarly be positive and hence non-zero.

9. (**EXERCISE 1**) Prove that the addition operation is associative. That is, for any three natural numbers, a, b, c, we have that $(a + b) + c = a + (b + c)$

We will fix $b, c \in \mathbb{N}$ and induct on a. Base case:

$$\begin{aligned}(0 + b) + c &= b + c \\ &= 0 + (b + c)\end{aligned}$$

Assume inductively that $(a + b) + c = a + (b + c)$ for some natural number a. Then

$$\begin{aligned}((a{+}{+}) + b) + c &= ((a + b){+}{+}) + c \\ &= ((a + b) + c){+}{+} \\ &= (a + (b + c)){+}{+} \quad \text{(I.H)} \\ &= (a{+}{+}) + (b + c)\end{aligned}$$

This closes the induction.

10. (**EXERCISE 2**) Let a be a positive number. Prove that there exists exactly one natural number b such that $b{+}{+} = a$

We effectively need to prove that each positive a has a unique predecessor b. Hence, it suffices to show that each a has both: 1) *at least* 1 predecessor and 2) *at most* 1 predecessor. If a is positive, then $a \neq 0$, which guarantees the existence of a b such that $b{+}{+} = a$ (a predecessor of a). Axiom 4 (different natural numbers have different successors) guarantees that no two distinct natural numbers have the same successor. Hence, a cannot have more than 1 predecessor $\implies$ b must be the only predecessor of a.

Will will induct on a. Base case: for $a = 1$, we have $b = 0 \implies b{+}{+} = a$. This is the *only* such b for $a = 0$ because a different $b \neq 0$ would have a different successor $b{+}{+} \neq 1$ acccording to Axiom 4: different natural numbers have different successors. Suppose inductively that for some natural number a there exists a b such that $b{+}{+} = a$. Now consider the next natural number $a{+}{+}$: the number $a{+}{+}$, by definition, is the successor of a. So we have found one $b = a$ such that $b{+}{+} = a{+}{+}$. This is the only such b because, similarly, by Axiom 4, any other $b \neq a$ has a different successor $b{+}{+} \neq a{+}{+}$. This closes the induction.

11. Define the ordering of the natural numbers.

Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

12. (**EXERCISE 3**) Prove properties of order.

(a) (Reflexivity) Prove that $a \geq a$ for any natural number a.

For any natural a, we can find an m such that $a = a + m$ $(a \geq a)$ $\implies$ choose $m = 0$

(b) (Transitivity) Prove that if $a \geq b$ and $b \geq c$, then $a \geq c$.

$$\begin{aligned} a \geq b, b \geq c &\implies a = b + k_0, b = c + k_1 \text{ for some } k_0, k_1 \in \mathbb{N} \\ &\implies a = (c + k_1) + k_0 \\ &\implies a = c + (k_1 + k_0) \text{ (associativity)} \\ &\implies a = c + m \text{ for some natural } m = k_0 + k_1 \text{ (additive closure)} \\ &\implies a \geq c \end{aligned}$$

(c) (Anti-symmetry) Prove that if $a \geq b$ and $b \geq a$, then $a = b$

$$\begin{aligned}
a \geq b, b \geq a &\implies a = b + k_0, b = a + k_1 \text{ for some } k_0, k_1 \in \mathbb{N} \\
&\implies a = (a + k_1) + k_0 \\
&\implies a = a + (k_1 + k_0) \text{ (associativity)} \\
&\implies a + 0 = a + (k_1 + k_0) \\
&\implies 0 = k_1 + k_0 \text{ (cancellation law)} \\
&\implies k_0 = 0 \text{ and } k_1 = 0 \text{ (zero-sum proof)} \\
&\implies a = b
\end{aligned}$$

(d) (Order preservation under addition) Prove that $a \geq b$ if and only if $a + c \geq b + c$.

$$\begin{aligned}
a \geq b &\iff \exists k(a = b + k) \\
&\iff a + c = (b + k) + c \\
&\iff a + c = b + (k + c) \\
&\iff a + c = b + (c + k) \\
&\iff a + c = (b + c) + k \\
&\iff a + c \geq b + c
\end{aligned}$$

(e) Prove that $a < b$ if and only if $a{+}{+} \leq b$.

$$\begin{aligned}
a < b &\iff \exists k(b = a + k \text{ and } a \neq b) \\
&\iff a + k \neq a \\
&\iff k \neq 0 \\
&\iff b = a + m{+}{+} \text{ with } k = m{+}{+} \\
&\iff b = a + (m + 1) \\
&\iff b = a + (1 + m) \\
&\iff b = (a + 1) + m \\
&\iff b = (a{+}{+}) + m \\
&\iff a{+}{+} \leq b
\end{aligned}$$

(f) Prove that $a < b$ if and only if $b = a + d$ for some positive number d.

$$\begin{aligned} a < b &\iff \exists k(b = a + k \text{ and } a \neq b) \\ &\iff a + k \neq a \\ &\iff k \neq 0 \\ &\iff \text{choose } d = k \\ &\iff b = a + d \text{ and } d \text{ is positive} \end{aligned}$$

13. EXERCISE 4 Justify the following:

(a) $0 \leq b$ for all b.

For any b, we can find an m such that $b = 0 + m$ $(0 \leq b)$ $\implies$ choose $m = b$

(b) If $a > b$, then $a{+}{+} > b$

$$\begin{aligned} a > b &\implies a{+}{+} > a > b \ \text{(definition of successor)} \\ &\implies a{+}{+} > b \ \text{(transitivity)} \end{aligned}$$

(c) If $a = b$, then $a{+}{+} > b$

$$\begin{aligned} a = b &\implies a{+}{+} > a = b \ \text{(definition of successor)} \\ &\implies a{+}{+} > b \end{aligned}$$

14. EXERCISE 5 Prove the principle of strong induction.

$$x \implies x$$