

Building Calculus From Scratch

With Terence Tao's *Analysis I*

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1 Starting Axioms

1. 0 is a natural number.
2. if n is a natural number, then its successor $n++$ is a natural number. We define the short-hand symbols

$$1 := 0++$$

$$2 := (0++)++$$

$$3 := ((0++)++)++$$

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3. 0 is not the successor of any natural number. That is, $0 \neq n++$ for any n .
4. Different natural numbers have different successors. That is, $[n \neq m] \iff [n++ \neq m++]$ for any n, m . Equivalently, if we have that $n++ = m++$, then we have that $n = m$.
5. The principle of induction holds. That is, let $P(n)$ be a property of the natural number n . Let $P(0)$ be true, and let $P(n) \implies P(n++)$. Then $P(n)$ is true for all natural numbers n .

2 Steps & Problems

1. Define the addition operation on two natural numbers, n and m .

We define incrementing m by zero as $0 + m := m$. Suppose inductively, that we know how to increment m by n . Then we can increment m by $n++$ by defining

$$(n++) + m := (n + m)++$$

This recursive definition allows us to now add numbers (perform repeated incrementation). For example, $2 + m = (1++) + m = (1 + m)++ = ((0++) + m)++ = ((0 + m)++)++ = ((m)++)++$, which is exactly m incremented twice.

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2. Prove that $n + 0 = n$ for any natural number n

We will induct on n . We have that $0 + 0 = 0$ by the fact that $0 + m := m$ for any natural number m , including 0. Suppose, inductively, that $k + 0 = k$ for some natural number k . Now consider the sum $(k++) + 0 = (k + 0)++ \stackrel{\text{I.H.}}{=} (k)++$. Hence for any natural number n , we have that $n + 0 = n$.

3. Prove that $n + (m++) = (n + m)++$

We will induct on n . For $n = 0$, we have that $0 + (m++) = m++ = (0 + m)++$. Assume inductively that $k + (m++) = (k + m)++$ for some natural number k . Now for the next number, $k++$, we have that $(k++) + (m++) = (k + (m++))++$ (by the definition from problem 1) $\stackrel{\text{I.H.}}{=} ((k + m)++)++ = ((k + m)++)++$. Hence, $n + (m++) = (n + m)++$ for all natural numbers n .

4. Prove that the natural numbers are closed under addition. That is, if a, b are natural numbers, then $a + b$ is a natural number.

We will fix b and induct on a . Base case: 0 is a natural number by Axiom 1 and let b be a natural number. Then $0 + b = b$ by our definition of incrementing by zero. But b is a natural number (by assumption). This closes the base case. Assume inductively that for some natural number a , if a and b are natural numbers then so is $a + b$. Consider the next natural number $a++$. By Axiom 2, since a is a natural number (I.H.), we must have that $a++$ is also a natural number. Let b be a natural number by assumption. Now consider the sum $(a++) + b = (a + b)++$. We have that a and b are natural numbers, so $a + b$ is a natural number by the inductive hypothesis. By Axiom 2, we have that its successor $(a + b)++$ must also be a natural number. This closes the induction. Hence the natural numbers are closed under addition.

5. Prove that the addition operation is commutative. That is, for any two natural numbers, a and b , we have that $a + b = b + a$

We will fix b and induct on a . We have that $0 + b = b$ from our first definition—incrementing a natural number by 0. Then $b = b + 0$, which we have from problem 2's proof. Hence we have our base case $0 + b = b + 0$. Suppose inductively that $a + b = b + a$ for some natural number a . Now consider the next number, $a++$: we have $(a++) + b = (a + b)++ \stackrel{\text{I.H.}}{=} (b + a)++ = b + (a++)$ by our previous proof. Hence for some natural number b , we have that $a + b = b + a$ for all natural numbers a .

6. Prove that the additive cancellation law holds. That is, for any three natural numbers a , b , and c such that $a + b = a + c$, we have $b = c$.

We will fix b, c and induct on a . For the base case $a = 0$, we will show that $[0 + b = 0 + c] \implies [c = b]$. Let us assume that $0 + b = 0 + c$. We have that $0 + b = b$ and that $0 + c = c$ from a previous proof. Then by transitive equality, we have $b = c$. This closes the base case. Assume inductively that for some natural number a , we have that $[a + b = a + c] \implies [b = c]$. Now consider the next natural number $a++$. Assume that

$(a++) + b = (a++) + c$. Re-writing both sides, we have $(a + b)++ = (a + c)++$. $a + b$ is a natural number and $a + c$ is a natural number (by our additive closure proof). Recall by Axiom 4 that if the successors of two natural numbers are equal, then the natural numbers themselves must be equal. Hence we have that $a + b = a + c$. By the inductive hypothesis ($[a + b = a + c] \implies [b = c]$), we must now have that $b = c$. This closes the inductive step. Thus, for any a, b, c such that $a + b = a + c$, we also have that $b = c$.

7. A natural number a is said to be positive iff $a \neq 0$. Prove that the sum of a natural number and a positive number must be positive.

8. Prove that, for natural numbers a and b , if $a + b = 0$ then $a = 0$ and $b = 0$.

We will prove the equivalent contraposition statement: if $a \neq 0$ or $b \neq 0$, then $a + b \neq 0$. Assume that either $a \neq 0$ or $b \neq 0$. In the case that $a \neq 0$, $a + b$ must be positive and hence non-zero. In the case that $b \neq 0$, $a + b$ must similarly be positive and hence non-zero.

9. (**EXERCISE 1**) Prove that the addition operation is associative. That is, for any three natural numbers, a, b, c , we have that $(a + b) + c = a + (b + c)$

We will fix $b, c \in \mathbb{N}$ and induct on a . Base case: $(0 + b) + c = b + c = 0 + (b + c)$. Assume inductively that $(a + b) + c = a + (b + c)$ for some natural number a . Then $((a++) + b) + c = ((a + b)++) + c = ((a + b) + c)++ \stackrel{\text{I.H.}}{=} (a + (b + c))++ = (a++) + (b + c)$. Hence, we have $(a + b) + c = a + (b + c)$ for all natural numbers a .

10. (**EXERCISE 2**) Let a be a positive number. Prove that there exists exactly one natural number b such that $b++ = a$

We will induct on a . Base case: for $a = 1$, we have $b = 0 \implies b++ = a$. This is the *only* such b for $a = 1$ because a different $b \neq 0$ would have a different successor $b++ \neq 1$ according to Axiom 4: different natural numbers have different successors. Suppose inductively that for some natural number a there exists a b such that $b++ = a$. Now consider the next natural number $a++$: the number $a++$, by definition, is the successor of a . So we have found one $b = a$ such that $b++ = a++$. This is the only such b because, similarly, by Axiom 4, any other $b \neq a$ has a different successor $b++ \neq a++$. This closes the induction.

11. Define the ordering of the natural numbers.

Let n and m be natural numbers. We say that n is *greater than or equal to* m , and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a . We say that n is *strictly greater than* m , and write $n > m$ or $m < n$, iff we have $n = m + a$ for some *positive* number a .

12. (**EXERCISE 3**) Prove properties of order.

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- (a) (Reflexivity) Prove that $a \geq a$ for any natural number a .

We can find a natural number a such that $0 = 0 + a$. Choose $a = 0$ and we have $0 = 0 + 0$. Hence $0 \geq 0$. Assume inductively that $n \geq n$ for some natural number n . Consider the next natural number $n++$. We can find an a such that $n++ = (n++) + a$. Chose $a = 0$ again, and then we have $(n++) + 0 = (n+0)++ = (n)++$. Hence $n++ \geq n++$. This closes the induction. Note that this is degenerate induction, because the inductive hypothesis was not needed.

- (b) (Transitivity) Prove that if $a \geq b$ and $b \geq c$, then $a \geq c$.

Assume that $a \geq b$ and $b \geq c$ for some natural numbers a, b, c . Then we have some natural number k_1 such that $a = b + k_1$, and we have some natural number k_2 such that $b = c + k_2$. Substituting the second equation into the first, we have that $a = (c + k_2) + k_1$, which, by associativity, is $a = c + (k_2 + k_1)$. By additive closure, $k_2 + k_1$ is some natural number, call it m . But since we have that $a = c + m$ for some natural number m , then we must have that $a \geq c$.

- (c) (Anti-symmetry) Prove that if $a \geq b$ and $b \geq a$, then $a = b$

Assume that $a \geq b$ and $b \geq a$ for some a, b . Then we have some k_1 such that $a = b + k_1$ and we have some k_2 such that $b = a + k_2$. Substituting the second equation into the first, we have $a = (a + k_2) + k_1$ which is $a = a + (k_2 + k_1)$ by associativity. Re-writing this as $a + 0 = a + (k_2 + k_1)$ allows us to apply our proven cancellation law to get that $0 = (k_2 + k_1)$.