Building Calculus From Scratch

With Terence Tao's *Analysis I*

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1 Starting Axioms

1. 0 is a natural number.

2. if n is a natural number, then its successor $n{+}{+}$ is a natural number. We define the short-hand symbols

$$1 := 0{+}{+}$$

$$2 := (0{+}{+}){+}{+}$$

$$3 := ((0{+}{+}){+}{+}){+}{+}$$

$$\dots$$

3. 0 is not the successor of any natural number. That is, $0 \neq n{+}{+}$ for any n.

4. Different natural numbers have different successors. That is, $[n \neq m] \iff [n{+}{+} \neq m{+}{+}]$ for any n, m. Equivalently, if we have that $n{+}{+} = m{+}{+}$, then we have that $n = m$.

5. The principle of induction holds. That is, let $P(n)$ be a property of the natural number n. Let $P(0)$ be true, and let $P(n) \implies P(n{+}{+})$. Then $P(n)$ is true for all natural numbers n.

2 Steps & Problems

1. Define the addition operation on two natural numbers, n and m.

We define incrementing m by zero as $0 + m := m$. Suppose inductively, that we know how to increment m by n. Then we can increment m by $n{+}{+}$ by defining

$$(n{+}{+}) + m := (n + m){+}{+}$$

This recursive definition allows us to now add numbers (perform repeated incrementation). For example, $2 + m = (1{+}{+}) + m = (1 + m){+}{+} = ((0{+}{+}) + m){+}{+} = ((0 + m){+}{+}){+}{+} = ((m){+}{+}){+}{+}$, which is exactly m incremented twice.

2. Prove that $n+0=n$ for any natural number n

We will induct on n. We have that $0+0=0$ by the fact that $0+m := m$ for any natural number m, including 0. Suppose, inductively, that $k+0=k$ for some natural number k. Now consider the sum $(k{+}{+})+0 = (k+0){+}{+} \overset{\text{I.H.}}{=} (k){+}{+}$. Hence for any natural number n, we have that $n+0=n$.

3. Prove that $n+(m{+}{+}) = (n+m){+}{+}$

We will induct on n. For $n=0$, we have that $0+(m{+}{+}) = m{+}{+} = (0+m){+}{+}$. Assume inductively that $k+(m{+}{+}) = (k+m){+}{+}$ for some natural number k. Now for the next number, $k{+}{+}$, we have that $(k{+}{+})+(m{+}{+}) = (k+(m{+}{+})){+}{+}$ (by the definition from problem 1) $\overset{\text{I.H.}}{=} ((k+m){+}{+}){+}{+} = ((k{+}{+})+m){+}{+}$. Hence, $n+(m{+}{+}) = (n+m){+}{+}$ for all natural numbers n.

4. Prove that the natural numbers are closed under addition. That is, if a,b are natural numbers, then $a+b$ is a natural number.

We will fix b and induct on a. Base case: 0 is a natural number by Axiom 1 and let b be a natural number. Then $0+b=b$ by our definition of incrementing by zero. But b is a natural number (by assumption). This closes the base case. Assume inductively that for some natural number a, if a and b are natural numbers then so is $a+b$. Consider the next natural number $a{+}{+}$. By Axiom 2, since a is a natural number (I.H), we must have that $a{+}{+}$ is also a natural number. Now consider the sum $(a{+}{+})+b = (a+b){+}{+}$. We have that a and b are natural numbers, so $a+b$ is a natural number by the inductive hypothesis. By Axiom 2, we have that its successor $(a+b){+}{+}$ must also be a natural number. This closes the induction. Hence the natural numbers are closed under addition.

5. Prove that the addition operation is commutative. That is, for any two natural numbers, a and b, we have that $a+b=b+a$

We will fix b and induct on a. We have that $0+b=b$ from our first definition—incrementing a natural number by 0. Then $b=b+0$, which we have from problem 2's proof. Hence we have our base case $0+b=b+0$. Suppose inductively that $a+b=b+a$ for some natural number a. Now consider the next number, $a{+}{+}$: we have $(a{+}{+})+b = (a+b){+}{+} \overset{\text{I.H.}}{=} (b+a){+}{+} = b+(a{+}{+})$ by our previous proof. Hence for some natural number b, we have that $a+b=b+a$ for all natural numbers a.

6. Prove that the additive cancellation law holds. That is, for any three natural numbers a, b, and c such that $a+b=a+c$, we have $b=c$.

Fix natural numbes b and c. For $a=0$,

$$a+b=b \quad \text{(0-incrementing)}$$
$$a+c=c \quad \text{(0-incrementing)}$$
$$a+b=a+c \implies b=c \quad \text{(transitive equality)}$$

Assume inductively that for some natural a, $[a+b=a+c] \implies [b=c]$

then

we will show that $[0+b=0+c] \implies [c=b]$. Let us assume that $0+b=0+c$. We have that $0+b=b$ and that $0+c=c$ from a previous proof. Then by transitive equality, we have $b=c$. This closes the base case. Assume inductively that for some natural a, $[a+b=a+c] \implies [b=c]$. Now consider the next natural number $a{+}{+}$. Assume that $(a{+}{+})+b=(a{+}{+})+c$. Re-writing both sides, we have $(a+b){+}{+}=(a+c){+}{+}$. $a+b$ is a natural number and $a+c$ is a natural number (by our additive closure proof). Recall by Axiom 4 that if the successors of two natural numbers are equal, then the natural numbers themselves must be equal. Hence we have that $a+b=a+c$. By the inductive hypothesis ($[a+b=a+c] \implies [b=c]$), we must now have that $b=c$. This closes the inductive step. Thus, for any a, b, c such that $a+b=a+c$, we also have that $b=c$.

7. A natural number a is said to be positive iff $a \neq 0$. Prove that the sum of a natural number and a positive number must be positive.

Fix a positive number b. Let $a=0 \implies a+b=b \neq 0$. Then $(a{+}{+})+b=(a+b){+}{+} \neq 0$ since $a+b$ is a natural number (by our proof of additive closure), and the successor of any natural number is positive by Axiom 3 (zero is not the successor of any natural number). Hence $a+b \neq 0$ for all a.

8. Prove that, for natural numbers a and b, if $a+b=0$ then $a=0$ and $b=0$.

We will prove the contrapositive:

$$\begin{aligned} \text{Let } a \neq 0 \text{ or } b \neq 0 \implies & \\ & \text{if } a \neq 0 \implies a+b \neq 0 \ \ \text{(by previous proof)} \\ & \text{if } b \neq 0 \implies a+b \neq 0 \ \ \text{(by previous proof)} \end{aligned}$$

Hence $a \neq 0$ or $b \neq 0 \implies a+b \neq 0$. Equivalently,
$a+b=0 \implies a=0$ and $b=0$.

9. (**EXERCISE 1**) Prove that the addition operation is associative. That is, for any three natural numbers, a, b, c, we have that $(a+b)+c=a+(b+c)$

We will fix $b, c \in \mathbb{N}$ and induct on a. Base case:

$$\begin{aligned} (0+b)+c &= b+c \\ &= 0+(b+c) \end{aligned}$$

Assume inductively that $(a+b)+c = a+(b+c)$ for some natural number a. Then

$$\begin{aligned}((a{+}{+})+b)+c &= ((a+b){+}{+})+c \\ &= ((a+b)+c){+}{+} \\ &= (a+(b+c)){+}{+} \quad \text{(I.H)} \\ &= (a{+}{+})+(b+c)\end{aligned}$$

This closes the induction.

10. (**EXERCISE 2**) Let a be a positive number. Prove that there exists exactly one natural number b such that $b{+}{+} = a$

For the first positive number $a = 1$, choose $b = 0$ and we have $b{+}{+} = a$. Then for the next natural $a{+}{+}$, choose $b = a$ and we have $b{+}{+} = a{+}{+}$ (no induction hypothesis needed). Hence, every positive a has at least 1 predecessor b. Axiom 4 (different natural numbers have different successors) guarantees that no two distinct natural numbers have the same successor. Hence, for every a, the corresponding b that satisfies $b{+}{+} = a$ must be the *only* such b.

11. Define the ordering of the natural numbers.

Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

12. (**EXERCISE 3**) Prove properties of order.

(a) (Reflexivity) Prove that $a \geq a$ for any natural number a.

For any natural a, we can find an m such that $a = a + m$ $(a \geq a)$ $\implies$ choose $m = 0$

(b) (Transitivity) Prove that if $a \geq b$ and $b \geq c$, then $a \geq c$.

$$\begin{aligned}a \geq b, b \geq c &\implies a = b + k_0, b = c + k_1 \text{ for some } k_0, k_1 \in \mathbb{N} \\ &\implies a = (c + k_1) + k_0 \\ &\implies a = c + (k_1 + k_0) \text{ (associativity)} \\ &\implies a = c + m \text{ for some natural } m = k_0 + k_1 \text{ (additive closure)} \\ &\implies a \geq c\end{aligned}$$

(c) (Anti-symmetry) Prove that if $a \geq b$ and $b \geq a$, then $a = b$

$$\begin{aligned}
a \geq b, b \geq a &\implies a = b + k_0, b = a + k_1 \text{ for some } k_0, k_1 \in \mathbb{N} \\
&\implies a = (a + k_1) + k_0 \\
&\implies a = a + (k_1 + k_0) \text{ (associativity)} \\
&\implies a + 0 = a + (k_1 + k_0) \\
&\implies 0 = k_1 + k_0 \text{ (cancellation law)} \\
&\implies k_0 = 0 \text{ and } k_1 = 0 \text{ (zero-sum proof)} \\
&\implies a = b
\end{aligned}$$

(d) (Order preservation under addition) Prove that $a \geq b$ if and only if $a + c \geq b + c$.

$$\begin{aligned}
a \geq b &\iff \exists k(a = b + k) \\
&\iff a + c = (b + k) + c \\
&\iff a + c = b + (k + c) \\
&\iff a + c = b + (c + k) \\
&\iff a + c = (b + c) + k \\
&\iff a + c \geq b + c
\end{aligned}$$

(e) Prove that $a < b$ if and only if $a{++} \leq b$.

$$\begin{aligned}
a < b &\iff \exists k(b = a + k \text{ and } a \neq b) \\
&\iff a + k \neq a \\
&\iff k \neq 0 \\
&\iff b = a + m{++} \text{ with } k = m{++} \\
&\iff b = a + (m + 1) \\
&\iff b = a + (1 + m) \\
&\iff b = (a + 1) + m \\
&\iff b = (a{++}) + m \\
&\iff a{++} \leq b
\end{aligned}$$

(f) Prove that $a < b$ if and only if $b = a + d$ for some positive number d.

$$\begin{aligned} a < b &\iff \exists k(b = a + k \text{ and } a \neq b) \\ &\iff a + k \neq a \\ &\iff k \neq 0 \\ &\iff \text{choose } d = k \\ &\iff b = a + d \text{ and } d \text{ is positive} \end{aligned}$$

13. EXERCISE 4 Justify the following:

(a) $0 \leq b$ for all b.

For any b, we can find an m such that $b = 0 + m$ $(0 \leq b)$ $\implies$ choose $m = b$

(b) If $a > b$, then $a{++} > b$

$$\begin{aligned} a > b &\implies a{++} > a > b \text{ (definition of successor)} \\ &\implies a{++} > b \text{ (transitivity)} \end{aligned}$$

(c) If $a = b$, then $a{++} > b$

$$\begin{aligned} a = b &\implies a{++} > a = b \text{ (definition of successor)} \\ &\implies a{++} > b \end{aligned}$$

14. EXERCISE 5 Prove the principle of strong induction.

$$x \implies x$$