Building Calculus From Scratch

With Terence Tao's *Analysis I*

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1 Starting Axioms

1. 0 is a natural number.
2. if n is a natural number, then its successor $n{+}{+}$ is a natural number. We define the short-hand symbols
$$1 := 0{+}{+}$$
$$2 := (0{+}{+}){+}{+}$$
$$3 := ((0{+}{+}){+}{+}){+}{+}$$
$$\ldots$$
3. 0 is not the successor of any natural number. That is, $0 \neq n{+}{+}$ for any n.
4. Different natural numbers have different successors. That is, $[n \neq m] \iff [n{+}{+} \neq m{+}{+}]$ for any n, m. Equivalently, if we have that $n{+}{+} = m{+}{+}$, then we have that $n = m$.
5. The principle of induction holds. That is, let $P(n)$ be a property of the natural number n. Let $P(0)$ be true, and let $P(n) \implies P(n{+}{+})$. Then $P(n)$ is true for all natural numbers n.

2 Steps & Problems

1. Define the addition operation on two natural numbers, n and m.

We define incrementing m by zero as $0 + m := m$. Suppose inductively, that we know how to increment m by n. Then we can increment m by $n{+}{+}$ by defining

$$(n{+}{+}) + m := (n + m){+}{+}$$

This recursive definition allows us to now add numbers (perform repeated incrementation). For example, $2 + m = (1{+}{+}) + m = (1 + m){+}{+} = ((0{+}{+}) + m){+}{+} = ((0 + m){+}{+}){+}{+} = ((m){+}{+}){+}{+}$, which is exactly m incremented twice.

2. Prove that $n + 0 = n$ for any natural number n

We will induct on n. We have that $0+0 = 0$ by the fact that $0+m := m$ for any natural number m, including 0. Suppose, inductively, that $k+0 = k$ for some natural number k. Now consider the sum $(k{++})+0 = (k+0){++} \overset{\text{I.H.}}{=} (k){++}$. Hence for any natural number n, we have that $n+0 = n$.

3. Prove that $n + (m{++}) = (n + m){++}$

We will induct on n. For $n = 0$, we have that $0+(m{++}) = m{++} = (0+m){++}$. Assume inductively that $k + (m{++}) = (k + m){++}$ for some natural number k. Now for the next number, $k{++}$, we have that $(k{++}) + (m{++}) = (k + (m{++})){++}$ (by the definition from problem 1) $\overset{\text{I.H.}}{=} ((k + m){++}){++} = ((k{++}) + m){++}$. Hence, $n + (m{++}) = (n + m){++}$ for all natural numbers n.

4. Prove that the natural numbers are closed under addition. That is, if a,b are natural numbers, then $a + b$ is a natural number.

We will fix b and induct on a. Base case: 0 is a natural number by Axiom 1 and let b be a natural number. Then $0 + b = b$ by our definition of incrementing by zero. But b is a natural number (by assumption). This closes the base case. Assume inductively that for some natural number a, if a and b are natural numbers then so is $a+b$. Consider the next natural number $a{++}$. By Axiom 2, since a is a natural number (I.H), we must have that $a{++}$ is also a natural number. Let b be a natural number by assumption. Now consider the sum $(a{++}) + b = (a + b){++}$. We have that a and b are natural numbers, so $a + b$ is a natural number by the inductive hypothesis. By Axiom 2, we have that its successor $(a + b){++}$ must also be a natural number. This closes the induction. Hence the natural numbers are closed under addition.

5. Prove that the addition operation is commutative. That is, for any two natural numbers, a and b, we have that $a + b = b + a$

We will fix b and induct on a. We have that $0+b = b$ from our first definition—incrementing a natural number by 0. Then $b = b+0$, which we have from problem 2's proof. Hence we have our base case $0 + b = b + 0$. Suppose inductively that $a + b = b + a$ for some natural number a. Now consider the next number, $a{++}$: we have $(a{++}) + b = (a + b){++} \overset{\text{I.H.}}{=} (b+a){++} = b+(a{++})$ by our previous proof. Hence for some natural number b, we have that $a + b = b + a$ for all natural numbers a.

6. Prove that the additive cancellation law holds. That is, for any three natural numbers a, b, and c such that $a + b = a + c$, we have $b = c$.

We will fix b,c and induct on a. For the base case $a = 0$, we will show that $[0 + b = 0 + c] \implies [c = b]$. Let us assume that $0 + b = 0 + c$. We have that $0 + b = b$ and that $0 + c = c$ from a previous proof. Then by transitive equality, we have $b = c$. This closes the base case. Assume inductively that for some natural number a, we have that $[a + b = a + c] \implies [b = c]$. Now consider the next natural number $a{++}$. Assume that

$(a{+}{+})+b = (a{+}{+})+c$. Re-writing both sides, we have $(a+b){+}{+} = (a+c){+}{+}$. $a+b$ is a natural number and $a+c$ is a natural number (by our additive closure proof). Recall by Axiom 4 that if the successors of two natural numbers are equal, then the natural numbers themselves must be equal. Hence we have that $a+b = a+c$. By the inductive hypothesis ($[a+b = a+c] \implies [b = c]$), we must now have that $b = c$. This closes the inductive step. Thus, for any a, b, c such that $a+b = a+c$, we also have that $b = c$.

7. A natural number a is said to be positive iff $a \neq 0$. Prove that the sum of a natural number and a positive number must be positive.

Fix a positive number b. We will show that $a+b$ is positive for all a. Base case: $0+b = b$ by our definition of incrementing by zero, and b is positive (by assumption). This closes the base case. Assume inductively that $a+b$ is positive for some natural number a. Then consider the statement for the next natural number $a{+}{+}$: we have $(a{+}{+})+b = (a+b){+}{+}$. By the inductive hypothesis, we have that $a+b$ is positive. But this means that $(a+b){+}{+}$, the succesor of $a+b$, must also be positive—it cannot be 0 because of Axiom 3: zero is not the successor of any natural number. This closes the induction.

8. Prove that, for natural numbers a and b, if $a+b = 0$ then $a = 0$ and $b = 0$.

We will prove the equivalent contraposition statement: if $a \neq 0$ or $b \neq 0$, then $a+b \neq 0$. Assume that either $a \neq 0$ or $b \neq 0$. In the case that $a \neq 0$, $a+b$ must be positive (according to the result of the previous proof) and hence non-zero. In the case that $b \neq 0$, $a+b$ must similarly be positive and hence non-zero.

9. (**EXERCISE 1**) Prove that the addition operation is associative. That is, for any three natural numbers, a, b, c, we have that $(a+b)+c = a+(b+c)$

We will fix $b, c \in \mathbb{N}$ and induct on a. Base case: $(0+b)+c = b+c = 0+(b+c)$. Assume inductively that $(a+b)+c = a+(b+c)$ for some natural number a. Then $((a{+}{+})+b)+c = ((a+b){+}{+})+c = ((a+b)+c){+}{+} \overset{\text{I.H.}}{=} (a+(b+c)){+}{+} = (a{+}{+})+(b+c)$. Hence, we have $(a+b)+c = a+(b+c)$ for all natural numbers a.

10. (**EXERCISE 2**) Let a be a positive number. Prove that there exists exactly one natural number b such that $b{+}{+} = a$

Will will induct on a. Base case: for $a = 1$, we have $b = 0 \implies b{+}{+} = a$. This is the *only* such b for $a = 0$ because a different $b \neq 0$ would have a different successor $b{+}{+} \neq 1$ acccording to Axiom 4: different natural numbers have different successors. Suppose inductively that for some natural number a there exists a b such that $b{+}{+} = a$. Now consider the next natural number $a{+}{+}$: the number $a{+}{+}$, by definition, is the successor of a. So we have found one $b = a$ such that $b{+}{+} = a{+}{+}$. This is the only such b because, similarly, by Axiom 4, any other $b \neq a$ has a different successor $b{+}{+} \neq a{+}{+}$. This closes the induction.

11. Define the ordering of the natural numbers.

Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

12. (**EXERCISE 3**) Prove properties of order.

(a) (Reflexivity) Prove that $a \geq a$ for any natural number a.

We can find a natural number a such that $0 = 0 + a$. Choose $a = 0$ and we have $0 = 0 + 0$. Hence $0 \geq 0$. Assume inductively that $n \geq n$ for some natural number n. Consider the next natural number $n{+}{+}$. We can find an a such that $n{+}{+} = (n{+}{+}) + a$. Chose $a = 0$ again, and then we have $(n{+}{+}) + 0 = (n + 0){+}{+} = (n){+}{+}$. Hence $n{+}{+} \geq n{+}{+}$. This closes the induction. Note that this is degenerate induction, because the inductive hypothesis was not needed.

(b) (Transitivity) Prove that if $a \geq b$ and $b \geq c$, then $a \geq c$.

Assume that $a \geq b$ and $b \geq c$ for some natural numbers a, b, c. Then we have some natural number k_1 such that $a = b + k_1$, and we have some natural number k_2 such that $b = c + k_2$. Substituting the second equation into the first, we have that $a = (c + k_2) + k_1$, which, by associativity, is $a = c + (k_2 + k_1)$. By additive closure, $k_2 + k_1$ is some natural number, call it m. But since we have that $a = c + m$ for some natural number m, then we must have that $a \geq c$.

(c) (Anti-symmetry) Prove that if $a \geq b$ and $b \geq a$, then $a = b$

Assume that $a \geq b$ and $b \geq a$ for some a, b. Then we have some k_1 such that $a = b + k_1$ and we have some k_2 such that $b = a + k_2$. Substituting the second equation into the first, we have $a = (a + k_2) + k_1$ which is $a = a + (k_2 + k_1)$ by associativity. Re-writing this as $a + 0 = a + (k_2 + k_1)$ allows us to apply our proven cancellation law to get that $0 = (k_2 + k_1)$, which means that $k_1 = 0$ and $k_2 = 0$ (by our previous zero-sum proof). But this means that $a = b + 0$, or $a = b$.

(d) (Order preservation under addition) Prove that $a \geq b$ if and only if $a + c \geq b + c$.

We will prove the forward direction first—that is, if $a \geq b$, then $a + c \geq b + c$. Assume that $a \geq b$ for some a, b. Then we have some k_1 such that $a = b + k_1$. We can add a natural number c to both sides to obtain $a + c = (b + k_1) + c = b + (k_1 + c) = b + (c + k_1) = (b + c) + k_1$ by applications of associativity and commutativity. In all, we have that $a + c = (b + c) + k_1$, which is the definition of $a + c \geq b + c$.

Let us prove the backwards direction now—that is, if $a + c \geq b + c$, then $a \geq b$. Assume that $a + c \geq b + c$ for some natural numbers a, b, c. Then we have that $a + c = (b + c) + k$ for some natural number k. With some associativity and commutativity manipulations, we can obtain $a + c = (b + k) + c$. Apply our cancellation law to obtain $a = b + k$, which is the definition of $a \geq b$.

(e) Prove that $a < b$ if and only if $a{+}{+} \leq b$.

Forward direction: assume that $a < b$. That is, $a \leq b$ and $a \neq b$. Then we have that $b = a + k$ for some k. Since $a \neq b$, we must have that $k \neq 0$. Hence k must be positive. But this means that k is the successor of some natural number, say m. That is, $k = m{+}{+}$. In total, we now have $b = a + (m{+}{+}) = a + (m + 1) = a + (1 + m) = (a + 1) + m = (a{+}{+}) + m$. Hence $b = (a{+}{+}) + m$, which is the definition of $a{+}{+} \leq b$.

Backward direction: assume that $a{+}{+} \leq b$. Then $b = (a{+}{+}) + m$ for some natural number m. Then $b = (a+1)+m = a+(1+m) = a+(m+1) = a+m{+}{+}$. Together, we have $b = a + (m{+}{+})$, which is the definition of $a \leq b$. Since $m{+}{+}$ is positive by definition, then $a \neq b$. If $a \leq b$ and $a \neq b$, then we must have that $a < b$.

(f) Prove that $a < b$ if and only if $b = a + d$ for some positive number d.

Forward: $a < b \implies b = a + k$ for some k and $a \neq b$. Temporarily assume that $k = 0$. Then $b = a + 0$ and $b = a$ (a contradiction). Hence $k \neq 0$ and k is positive. Let $d = k$, then we have $b = a + d$ where d is positive.

Backward: Let $b = a + d$ for some positive d. Then $a \leq b$. Since d is positive, then $d \neq 0$. Temporarily assume that $b = a$. Then $b = a + d \implies d = 0$ (contradiction). Hence, $b \neq a$, and we have in total that $a < b$.

13. EXERCISE 4 Justify the following:

(a) $0 \leq a$ for all a.