Faculty of Mathematics
Part III Essays: 2017-18

Titles 1 – 54

Department of Pure Mathematics
& Mathematical Statistics

Titles 55 – 83

Department of Applied Mathematics
& Theoretical Physics

Titles 84 – 94

Additional Essays

Contents

Introductory Notes

General advice. Before attempting any particular essay, candidates are advised to meet the setter in person. Normally candidates may consult the setter up to three times before the essay is submitted. The first meeting may take the form of a group meeting at which the setter describes the essay topic and answers general questions.

Choice of topic. The titles of essays appearing in this list have already been announced in the Reporter. If you wish to write an essay on a topic not covered in the list you should approach your Part III Adviser or any other member of staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board, c/o the Undergraduate Office at the CMS (Room B1.28) **no later than 1 February**. The new essay title will require the approval of the Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. Additional Essays will be announced in the Reporter no later than 1 March and are open to all candidates. Even if you request an essay you do not have to do it. Essay titles cannot be approved informally: the only allowed essay titles are those which appear in the final version of this document (on the Faculty web site).

Originality. The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but often candidates will find novel approaches. All sources and references used should be carefully listed in a bibliography.

Length of essay. There is no prescribed length for the essay in the University Ordinances, but the general opinion seems to be that 5,000-8,000 words is a normal length. If you are in any doubt as to the length of your essay please consult your adviser or essay setter.

Presentation. Your essay should be legible and may be either hand written or produced on a word processor. Candidates are reminded that mathematical content is more important than style. Usually it is advisable for candidates to write an introduction outlining the contents of the essay. In some cases a conclusion might also be required. It is very important that you ensure that the pages of your essay are fastened together in an appropriate way, by stapling or binding them, for example.

Credit. The essay is the equivalent of one three-hour exam paper and marks are credited accordingly.

Final decision on whether to submit an essay. You are not asked to state which papers you have chosen for examination and which essay topic, if any, you have chosen until the beginning of the third term (Easter) when you will be sent the appropriate form to fill in and hand to your Director of Studies. Your Director of Studies should counter-sign the form and send it to the Chairman of Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive before the second Friday in Easter Full Term, which this year is **Friday 4 May 2018**. **Note that this deadline will be strictly adhered to.**

Date of submission. You should submit your essay, through your Director of Studies, to the Chairman of Examiners (c/o Undergraduate Office, CMS). Your essay should be sent with the completed essay submission form found on page 10 of this document. The first part of the form should be signed by your Director of Studies. The second part should be completed and signed by you.

Please do not bind or staple the essay submission form to your essay, but instead attach it loosely, e.g. with a paperclip.

Then either you or your Director of Studies should take your essay and the signed essay submission form to the Undergraduate Office (B1.28) at the Centre for Mathematical Sciences so as to arrive **not later than** the second Friday in Easter Full Term, which this year is **Friday 4 May 2018**. **Note that this deadline will be strictly adhered to.**

Title page. The title page of your essay should bear **ONLY** the essay title. Please **DO NOT** include your name or any other personal details on the title page or anywhere else on your essay.

Signed declaration. The essay submission form requires you to sign the following declaration. It is important that you read and understand this before starting your essay.

> I declare that this essay is work done as part of the Part III Examination. I have read and understood the *Statement on Plagiarism for Part III and Graduate Courses* issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Important note. The *Statement on Plagiarism for Part III and Graduate Courses* issued by the Faculty of Mathematics is reproduced starting on page 11 of this document. If you are in any doubt as to whether you will be able to sign the above declaration you should consult the member of staff involved in the essay. If they are unsure about your situation they should consult the Chairman of the Examiners as soon as possible. The examiners have the power to examine candidates **viva voce** (i.e. to give an oral examination) on their essays, although this procedure is not often used. However, you should be aware that the University takes a very serious view of any use of unfair means (plagiarism, cheating) in University examinations. The powers of the University Court of Discipline in such cases extend to depriving a student of membership of the University. Fortunately, incidents of this kind are very rare.

Return of essays. It is not possible to return essays. You are therefore advised to make your own copy before handing in your essay.

Further advice. It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.

Research. If you are interested in going on to do research you should, if possible, be available for consultation in the next few days after the results are published. If this is not convenient, or if you have any specific queries about PhD admissions, please contact the following addresses:

Applied Mathematics & Theoretical Physics
research@damtp.cam.ac.uk
DAMTP PhD Admissions,
Mathematics Graduate Office,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WA,
United Kingdom.

Pure Mathematics & Mathematical Statistics
research@dpmms.cam.ac.uk
DPMMS PhD Admissions,
Mathematics Graduate Office,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WB,
United Kingdom.

MATHEMATICAL TRIPOS, PART III 2018
Essay submission form

To the Chairman of Examiners for Part III Mathematics.

Dear Sir/Madam,

I enclose the Part III essay of ....

Signed .... (Director of Studies)

I declare that this essay is work done as part of the Part III Examination. I have read and understood the *Statement on Plagiarism for Part III and Graduate Courses* issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed: .... Date: ....

Title of Essay: ....Essay Number: ....

Name: .... College: ....

Home address: (for return of comments) ....

....

Appendix: Faculty of Mathematics Guidelines on Plagiarism

For the latest version of these guidelines please see

http://www.maths.cam.ac.uk/facultyboard/plagiarism/.

University Resources

The University publishes information on *Good academic practice and plagiarism*, including

- a *University-wide statement on plagiarism*;
- *Information for students*, covering
  - *Your responsibilities*
  - *Why does plagiarism matter?*
  - *Using commercial organisations and essay banks*
  - *How the University detects and disciplines plagiarism*;
- information about *Referencing and study skills*;
- information on *Resources and sources of support*;
- the *University's statement on proofreading*;
- *FAQs*.

There are references to the University statement

- in the Part IB and Part II Computational Project Manuals,
- in the Part III Essay booklet, and
- in the M.Phil. Computational Biology Course Guide.

Please read the University statement carefully; it is your responsibility to read and abide by this statement.

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However neither the University Statement nor the Faculty Guidelines supersede the University's Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

What is plagiarism?

Plagiarism can be defined as **the unacknowledged use of the work of others as if this were your own original work**. In the context of any University examination, this amounts to **passing off the work of others as your own to gain unfair advantage**.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.

Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to *Turnitin Plagiarism Detection Software*). See also the Board of Examinations' statement on How the University detects and disciplines plagiarism.

The scope of plagiarism

Plagiarism may be due to

- **copying** (this is using another person's language and/or ideas as if they are your own);
- **collusion** (this is collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed).

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.), are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a 'ghost writing service'). Furthermore, you should not deliberately reproduce someone else's work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essays, projects and course-work will be based on what you have read and learned from other sources, and it is important that in your essay or project or course-work you show exactly where, and how, your work is indebted to these other sources. The golden rule is that **the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else**.

A good guideline to avoid plagiarism is not to repeat or reproduce other people's words, diagrams or computer programs. If you need to describe other people's ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay or project or course-work, it is not sufficient merely to repeat or paraphrase someone else's view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

Quoting. A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:

- short quotations should be in inverted commas, and a reference given to the source;
- longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

Paraphrasing. Paraphrasing means putting someone else's work into your own words. Paraphrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. "Smith (2001) goes on to argue that ..." or "Smith (2001) provides further proof that ..."). As with quotation, the full details of the source should be given in the bibliography or reference list.

General indebtedness. When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

Use of web sources. You should use web sources as if you were using a book or journal article. The above rules for quoting (including 'cutting and pasting'), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

Collaboration. Unless it is expressly allowed, collaboration is collusion and counts as plagiarism. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

Links to University Information

- Information on *Good academic practice and plagiarism*, including
  - *Information for students*.
  - information on *Policy, procedure and guidance for staff and examiners*.

Table 1: **A Timetable of Relevant Events and Deadlines**

Thursday 1 February	Deadline for Candidates to request additional essays.
Friday 4 May	Deadline for Candidates to return form stating choice of papers and essays.
Friday 4 May	Deadline for Candidates to submit essays.
Thursday 31 May	Part III Examinations begin.

Comments. If you feel that these notes could be made more helpful please write to *The Chairman of Examiners, c/o the Undergraduate Office, CMS*.

Further information. Professor T.W. Körner (DPMMS) wrote an essay on Part III essays which may be useful (though it is slanted towards the pure side). It is available via his home page

https://www.dpmms.cam.ac.uk/~twk/Essay.pdf

1. Deformations of Algebras ....

Dr C. J. B. Brookes

Algebraic deformation theory is primarily concerned with the interplay between homological algebra and the perturbations of algebraic structures. For example one might want to deform a commutative algebra to give a non-commutative one via 'quantisation'.

In a series of papers [2], [3], [4] and [5] Gerstenhaber developed deformation theory for associative algebras. The homology theory required in this context is Hochschild cohomology and one finds that 2-cocycles arise from deformations. A more modern approach is to consider additional algebraic structures, including that of a Lie algebra defined on the Hochschild cocomplex of the associative algebra. Deformations correspond to Maurer-Cartan elements. This approach has arisen in the work of Konstsevich on the deformation quantisation of Poisson manifolds.

The introductory article by Fox [1] is a good place to start. It first describes the classical theory of deformations of associative algebras and then moves on to more general algebraic structures. The papers of Gerstenhaberare also very readable.

Relevant Courses

Essential: Commutative Algebra

References

[1] T. Fox, An introduction to algebraic deformation theory, J Pure and Applied Algebra, 84 (1993), 17-41.

[2] M.Gerstenhaber, On the deformation theory of rings and algebras, Ann. Math. 78 (1963) 267-288.

[3] M.Gerstenhaber, On the deformation theory of rings and algebras II, Ann. Math. 84 (1966) 1-19.

[4] M.Gerstenhaber, On the deformation theory of rings and algebras III, Ann. Math. 88 (1968) 1-34.

[5] M.Gerstenhaber, On the deformation theory of rings and algebras IV, Ann. Math. 99 (1974) 257-256.

2. Weyl Algebras ....

Dr C. J. B. Brookes

The Weyl algebras form a family of what may be thought of as non-commutative polynomial algebras. For a field k the first member of the family is the k-algebra generated by x and y subject to the relation that $xy - yx = 1$. In general in characteristic zero the nth Weyl algebra may be represented as the ring of differential operators $k[t_1, \ldots, t_n, \partial/\partial t_1, \ldots, \partial/\partial t_n]$ of $k[t_1, \ldots, t_n]$ of the polynomial algebra in n variables and therefore provide the prototype algebra in the study of algebraic D-modules. They also arise as images of enveloping algebras of nilpotent Lie algebras and play an important role in Lie representation theory.

Read the introduction of Coutinho's book [1] to get an idea of what is involved. Other texts are section 4.6 of Dixmier [2] and section 1.3 of [3]. Both these books have large bibliographies and lots of references to Weyl algebras. Other possible sources are [4], [5] and [6], especially for the homological properties avoided by Coutinho. There are other more recent books on D-modules but they go far beyond the Weyl algebras.

There are various possible approaches to the topic. For example, one might concentrate on the geometric aspects underlying the module theory, or think about the homological properties, taking note of the role of the non-commutativity.

Relevant Courses

Essential: Commutative Algebra

References

[1] S.C. Coutinho, A primer on algebraic D-modules, London Mathematical Society student texts 33, C.U.P. 1995.

[2] J. Dixmier, Enveloping algebras, North Holland 1977

[3] J.C. McConnell, J.C. Robson, Non-commutative Noetherian rings, Wiley 1987.

[4] G.R. Krause, T.H. Lenagan, Growth of algebras and Gelfand-Kirillov dimension, Research notes in mathematics 116, Pitman 1985

[5] J.E. Bjork, Rings of differential operators, North Holland 1979

[6] A. Borel et al, Algebraic D-modules, Perspectives in Mathematics Volume 2, Academic Press 1987

3. Counting Cubic Number Fields .... Dr T. A. Fisher

Let $N_d(X)$ be the number of degree d number fields whose discriminant has absolute value at most X. It is conjectured that $N_d(X) \sim c_d X$ for some constant c_d as $X \to \infty$. This was proved in the case $d = 3$ by Davenport and Heilbronn [4], and more recently extended to the cases $d = 4$ and $d = 5$ by Bhargava.

This essay should concentrate on explaining the proof for $d = 3$, for which the references [3] and [5] may also be helpful. If time and space permit, the essay might also discuss extensions to larger d, second order terms, or computational aspects (for the latter see [1] and [2]).

Relevant Courses

Essential: Algebraic Number Theory

References

[1] K. Belabas, A fast algorithm to compute cubic fields, *Math. Comp.* **66** (1997), no. 219, 1213-1237.

[2] K. Belabas, On quadratic fields with large 3-rank, *Math. Comp.* **73** (2004), no. 248, 2061-2074.

[3] M. Bhargava, A. Shankar and J. Tsimerman, On the Davenport-Heilbronn theorems and second order terms, *Invent. Math.* **193** (2013), no. 2, 439-499.

[4] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields. II, *Proc. Roy. Soc. London Ser. A* **322** (1971), no. 1551, 405-420.

[5] M.M. Wood, Asymptotics for number fields and class groups, *Directions in number theory*, 291-339, Assoc. Women Math. Ser., 3, Springer, 2016.

4. Wellquasiorders and Betterquasiorders .... Dr T. E. Forster

A well-quasi-order is a reflexive transitive relation with no infinite descending chains and no infinite antichains. Although this may not sound natural there are many natural examples, at least one of which is famous: the theorem of Seymour and Robertson that finite graphs under the graph minor relationj form a WQO. There is Laver's theorem that the isomorphism types of scattered total orders (orders in which the rationals cannot be embedded) form a WQO. Finite trees with nodes labelled with elements of a WQO are also WQO-ed. The class of WQO's lacks certain nice closure properties and this leads to a concept of *Better*-quasi-ordering. The class of BQOs is algebraicly nicer.

These combinatorial ideas have wide ramifications in graph theory, logic and computer science (lack of infinite descending chains is always liable to be connected with termination of processes) and the area has a good compact literature and some meaty theorems. Recommended for those of you who liked the Logic course and the Combinatorics course.

A Big plus for this topic is that there is no textbook! There is a wealth of literature, some of which I have photocopies of. Interested students should discuss this with me.

Relevant Courses

Essential: None

Useful: Combinatorics, Logic and Set Theory

5. Countable Ordinals, Proof theory and Fast-growing Functions .... Dr T. E. Forster

A good point of departure for this essay would be Exercise 10 on Sheet 4 of Prof Johnstone's Part II set Theory and Logic course in 2012/3, `https://www.dpmms.cam.ac.uk/study/II/Logic/` in which the student is invited to show how, for every countable ordinal, a subset of $\Re$ can be found that is wellordered to that length in the inherited order. The obvious way to do this involves induction on countable ordinals and leads swiftly to the discovery of fundamental sequences. These can be put to work immediately in the definition of hierarchies of fast-growing functions. This leads in turn to the Schmidt conditions, which the student should explain carefully. There is a wealth of material on how proofs of totality for the faster-growing functions in this hierarchy have significant—indeed *calibratable*—consistency strength. One thinks of Goodstein's function and Con(PA), or of Paris-Harrington. There is plenty here from which the student can choose what to cover.

The Doner-Tarski hierarchy of functions (addition, multiplication, exponentiation ...) invites a transfinite generalisation and supports a generalisation of Cantor Normal form for ordinals. Nevertheless, the endeavour to notate ordinals beyond ϵ_0 does not use those ideas, but rather the enumeration of fixed points: such is the *Veblen hierarchy*. From this one is led to the impredicative Bachmann notation, with Ω and the ϑ function.

Currently it is my intention to rerun (in Lent term?) the reading group on ordinals that ran in Easter term 2014. Any student attempting this essay would be well advised to sign up for it.

Relevant Courses

Essential:

Part II Set Theory and Logic

References

Schwichtenberg-Wainer Proofs and Computations Cambridge University Press http://www.cambridge.org/us/academic/subjects/mathematics/logic-categories-and-sets/proofs-and-computations

http://www.math.ucsb.edu/~doner/articles/Doner-Tarski.pdf

http://www.dpmms.cam.ac.uk/~tf/cam_only/fundamentalsequence.pdf

http://www.dpmms.cam.ac.uk/~tf/cam_only/TMStalk.pdf

6. Homotopy Type Theory .... Professor P. T. Johnstone

Those who work with a type-theoretical (as oppsed to set-theoretical) foundation for mathematics, such as category-theorists and theoretical computer scientists, have been aware for some time that, if types are viewed intensionally, they have the categorical structure of higher-dimensional groupoids. Such groupoids also arise in algebraic topology, as algebraic models of homotopy types. More recently, V. Voevodsky's introduction of the 'univalence axiom' in type theory has opened up new and much more formal connections between type theory and homotopy theory, which hold out the hope that it may soon be possible to use foundational methods from type theory to prove new results in homotopy theory. The recent publication of the first book-form account of this material [1] brings it within the scope of a Part III essay. More recent work on models of the univalence axiom may be found in the recent papers of Voevodsky [2].

Relevant Courses

Essential: Category Theory

Useful: Algebraic Topology

References

[1] *Homotopy Type Theory: Univalent Foundations for Mathematics*, Institute for Advanced Study, 2013. [2] V. Voevodsky, Papers in *Theory and Applications of Categories* 30 (2015), 1181–1215; 31 (2016), 1044–1094; 32 (2017), 53–112 and 113–121.

7. Categories of Relations ....
Professor P. T. Johnstone

Categories whose morphisms behave like relations rather than functions can be studied in various ways. The objective of the study is to identify those morphisms (commonly called maps) which correspond to actual functions, and to relate properties of the whole category to properties of its subcategory of maps. One highly successful approach, originally developed by Peter Freyd, is developed in detail in [1], and more succinctly in [2]; other approaches include that of Carboni and Walters [3], which makes more explicit use of 2-categorical ideas. It is suggested that an essay might take as its goal the characterization of those allegories (or cartesian bicategories) whose categories of maps are toposes; alternatively, one might give a detailed comparison of these two (and possibly other) approaches.

Relevant Courses

Essential: Category Theory

Useful:

References

[1] P.J. Freyd and A. Scedrov, *Categories, Allegories* (North–Holland, 1990).

[2] P.T. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium* (Oxford U.P., 2002), Chapter A3

[3] A. Carboni and R.F.C. Walters, Cartesian bicategories I, J. Pure Appl. Algebra 49 (1987), 11-32.

8. Almost-toric Fibrations and Exotic Lagrangians ....
Dr A. M. Keating

Lagrangian submanifolds are distinguished half-dimensional submanifolds of symplectic manifolds – for instance, in the case of $\mathbb{C}P^2$, the torus $\{(x : y : z) : |x| = |y| = |z|\}$. The goal of the essay is to give an account of recent work of Vianna, who produced an infinite collection of new examples of Lagrangian tori in $\mathbb{C}P^2$ using almost-toric geometry. This essay should readily build on parts of the Symplectic Geometry course, notably basic notions in toric geometry. After briefly recalling relevant definitions from the course, the essay should start by introducing almost-toric symplectic manifolds and almost-toric fibrations, giving some examples. It should then explain how to perform operations on almost-toric fibrations, such as node trades, node slides, cut transfers, and symplectic blow-ups. Once this is set-up, the essay should give some account of how to obtain Vianna's tori, and how to tell them apart; emphasis can be placed on different parts of this depending on background and inclination. In particular, you may choose to focus on a small number of examples, and / or to treat various amounts of Floer-theoretic background as a 'black-box'.

Relevant Courses

Essential: Differential geometry, symplectic geometry, basic notions from algebraic topology (homology and cohomology)

Useful: Algebraic geometry

References

[1] M. Symington, *Four dimensions from two in symplectic topology* Topology and geometry of manifolds (Athens, GA, 2001), 153–208, Proc. Sympos. Pure Math., 71, Amer. Math. Soc., Providence, RI, 2003.

[2] N. C. Leung and M. Symington, *Almost toric symplectic four-manifolds*, J. Symplectic Geom. 8 (2010), no. 2, 143–187.

[3] R. Vianna,*Infinitely many exotic monotone Lagrangian tori in* $\mathbb{C}P^2$, J. Topol. 9 (2016), no. 2, 535–551

For the symplectic geometry background (to be covered in the Lent term course):

[4] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd edition, Oxford University Press, 2017

[5] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer, 2nd edition (2008), also available at https://people.math.ethz.ch/$\sim$acannas/

9. Complements of Hyperplane arrangements .... Dr A. M. Keating

A hyperplane arrangement is a finite collection of affine hyperplanes in $\mathbb{C}^n$. These have been the object of considerable research, notably regarding the topological properties of their complements in $\mathbb{C}^n$. The goal of this essay is to study some of these properties. It should begin by discussing the fundamental group of the complement of a hyperplane arrangement, with starting point the Zariski–Van Kempen theorem. Several directions are then possible, for instance: Hattori's result on the topology of the complement of a generic arrangement; Deligne's proof that a simplicial arrangement gives a $K(\pi, 1)$ Eilenberg-MacLane space; the description of the cohomology ring of the complement in terms of generators and relations.

Relevant Courses

Essential: Algebraic topology

Useful: Algebraic geometry, differential geometry

References

[1] P. Orlik, H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, Berlin, 1992

[2] A. Dimca, *Hyperplane arrangements. An introduction*, Springer, Cham, 2017.

[3] R. Randell,*The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. 69 (1982), no. 1, 103–108

[4] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^n$, Invent. Math. 88 (1987), no. 3, 603–618.

[5] W. Arvola,*The fundamental group of the complement of an arrangement of complex hyperplanes*, Topology 31 (1992), no. 4, 757–765

[6] A. Hattori, *Topology of* $\mathbb{C}^n$ *minus a finite number of affine hyperplanes in general position*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), no. 2, 205–219.

[7] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972), 273–302.

10. Dirac Operators ....

Dr A. G. Kovalev

The Dirac operator, for smooth functions from $\mathbf{R}^n$ to $\mathbf{C}^N$, may be defined as a first order differential operator whose square is the Laplacian. (Thus the simplest example of Dirac operator is $i(d/dx)$ acting on the complex-valued functions on $\mathbf{R}$.) Unlike the Laplacian, which is well-defined on every oriented Riemannian manifold, the construction of Dirac operator requires the existence of a certain vector bundle, called spinor bundle, over the base manifold. The essay could begin by explaining the significance of spinor bundles (cf. [1]), and why a Dirac operator can always be constructed when the dimension of the base manifold is 3 or 4. The null-space of a Dirac operator arises in many geometric and topological applications, including symmetries (more precisely, the holonomy) of a Riemannian metric of the base manifold, deformations of volume-minimizing submanifolds, invariants of smooth 4-dimensional manifolds. The essay could have a closer look into one of these latter topics/examples and involves some choices. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` and discuss the possibilities. The first two or three of sections in [2] would be a good introductory reading (and a source of useful exercises!).

Relevant Courses

Essential: Differential Geometry, Algebraic Topology

Useful: Complex manifolds

References

[1] J. Roe, *Elliptic operators, topology and asymptotic methods*, Pitman Res. Notes in Math., Longman, 1988 (or second edition 1998).

[2] Hitchin, *The Dirac operator.* Invitations to geometry and topology, Oxford Univ. Press, Oxford, 2002, pages 208–232.

[3] N. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55.

[4] H.B. Lawson, Jr. and M.-L. Michelson, *Spin geometry*, Princeton University Press, 1989.

11. The Hodge Decomposition Theorem ....

Dr A. G. Kovalev

The concept of Laplace operator $\Delta = -(\partial/\partial x_1)^2 - \ldots - (\partial/\partial x_n)^2$ for functions on the Euclidean space $\mathbf{R}^n$ may be extended to oriented Riemannian manifolds. The construction uses a certain duality, called the Hodge star, and the resulting 2nd order differential operator is well-defined on the differential forms. The celebrated Hodge decomposition theorem asserts a natural isomorphism between the kernel of this Laplacian (i.e. the space of harmonic differential forms) and the de Rham cohomology of a compact oriented manifold without boundary [1]. The theory admits a nice extension to compact manifolds with boundary. The interplays between bounded harmonic forms and absolute and relative de Rham cohomology of non-compact Riemannian manifolds 'with boundary at infinity' are more challenging. But these ideas lead to some far-reaching relations between the topology and curvature in dimension 4 (say), including a generalization of the Gauss-Bonnet theorem for surfaces. Interested candidates are welcome to contact me (`A.G.Kovalev@dpmms`) for further details; section 3.5 of `http://www.dpmms.cam.ac.uk/∼agk22/riem1.pdf` could be a good preliminary reading.

Relevant Courses

Essential: Differential Geometry

Useful: Algebraic Topology, Complex manifolds, Elliptic Partial Differential Equations.

References

[1] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer 1983. Chapter 6.

[2] S. Cappell, D. DeTurck, H. Gluck, E. Y. Miller, *Cohomology of harmonic forms on Riemannian manifolds with boundary*, arxiv:math.DG/0508372 or Forum Math. **18** (2006), 923–931.

[3] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 97–118.

12. Lagrangians of Hypergraphs ....

Professor I. B. Leader

The Lagrangian of a hypergraph is a function that in some sense seems to measure how 'tightly packed' a subset of the hypergraph one can find. Lagrangians have beautiful properties and are of great interest, both in their own right and because they have several applications, most notably to the celebrated 'jumping hypergraphs' conjecture.

The main topic would be the way in which Lagrangians influence other properties, ranging from the fact, due to Motzkin, that Lagrangians provide a simple proof of Turan's theorem, right up to the relationship between Lagrangians and 'asymptotic density', with the disproof by Frankl and Rodl of the Erdos conjecture that the set of possible asymptotic densities is discrete. There would also be an examination of the Frankl-Furedi conjecture on maximising the Lagrangian, taking in the recent proof of this by Tyomkyn.

Relevant Courses

Essential: Combinatorics

Useful: None

References

[1] T. Motzkin and E. Strauss, Maxima for graphs and a new proof of a theorem of Turan, Canadian Journal of Mathematics, vol 17 (1965), 533-540.

[2] P. Frankl and V. Rodl, Hypergraphs do not jump, Combinatorica, vol 4 (1984), 149-159.

[3] M. Tyomkyn, Lagrangians of hypergraphs: the Frankl-Furedi conjecture holds almost everywhere (Arxiv 1703.04273)

13. Canonical Ramsey Theory ....

Professor I. B. Leader

Canonical Ramsey theorems extend classical Ramsey theorems, which typically involve colourings involving a specified finite number of colours, to arbitrary colourings. The flavour is often different to that of the classical Ramsey theorems, although there is usually some relationship between a canonical theorem and a higher-order classical theorem.

There would be two themes to the essay. One is the question of bounds for finite canonical Ramsey theorems, dealing with work of Duffus, Lefmann and Rodl and especially some bounds of Shelah that more or less completely answer the growth-speed questions. The other is the question of what the actual canonical version of a classical theorem should be, focusing on the canonical Gallai theorem due to Deuber, Graham, Promel and Voigt and the canonical Hindman theorem due to Taylor.

Relevant Courses

Essential: Ramsey Theory

Useful: None

References

[1] D. Duffus, H. Lefmann and V. Rodl, Shift graphs and lower bounds on Ramsey numbers $r_k(l;r)$, Discrete Math, vol 137 (1995), 177-187.

[2] H. Lefmann and V. Rodl, On Erdos-Rado numbers, Combinatorica, vol 15 (1995), 85-104.

[3] S. Shelah, Finite canonization, Comment. Math. Univ. Carolinae, vol 37 (1996), 445-456.

[4] W. Deuber, R. Graham, H. Promel and B. Voigt, A canonical partition theorem for equivalence relations on Z^t, Journal of Combinatorial Theory Series A, vol 34 (1983), 331-339.

[5] A. Taylor, A canonical partition relation for finite subsets of omega, Journal of Combinatorial Theory Series A, vol 21 (1976), 137-146.

14. Infinite Ehrenfeucht-Fraïssé Games .... Professor B. Löwe

Ehrenfeucht-Fraïssé games are finite two-player games characterising elementary equivalence. For two structures $\mathcal{A}$ and $\mathcal{B}$ and every n is a natural number, we have the n-round Ehrenfeucht-Fraïssé game $\mathrm{EF}_n(\mathcal{A},\mathcal{B})$. Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if and only if player II has a winning strategy in all games $\mathrm{EF}_n(\mathcal{A},\mathcal{B})$. The goal of this essay is to look at infinite variants of these games.

For transfinite ordinals α, the game $\mathrm{EF}_\alpha(\mathcal{A},\mathcal{B})$ is a transfinite game of length α. We aim to study the model-theoretic concepts that these games correspond to. In contrast to finite games, infinite games are not always determined (i.e., if could be that neither of the players has a winning strategy) and it is particularly interesting to explore the connection between determinacy of the games and their model-theoretic properties.

Moving back to the world of determined games, the *dynamic Ehrenfeucht-Fraïssé games* $\mathrm{EFD}_\alpha(\mathcal{A},\mathcal{B})$ are finite games that mimic playing for α many moves. In this essay, the standard games will be compared with the dynamic games in order to find out when they are equivalent.

The essay is based on recent textbook literature [1,3,2] and can explore current research on game characterisations of infinitary logics [4].

Relevant Courses

Essential: Part II *Logic and Set Theory* (or equivalent).

Useful: Part III *Topics in Set Theory* (or equivalent).

References

[1] David Marker. Model theory. An Introduction. Graduate Texts in Mathematics, Vol. 217 (Springer-Verlag, 2002).

[2] David Marker. Lectures on Infinitary Model Theory Lecture Notes in Logic, Vol. 46 (Cambridge University Press, 2016).

[3] Jouko Väänänen. Models and Games. Cambridge Studies in Advanced Mathematics (Cambridge University Press, 2011).

[4] Jouko Väänänen & Tong Wang. An Ehrenfeucht-Fraïssé game for $\mathcal{L}_{\omega_1\omega}$. *Mathematical Logic Quarterly* 59:4-5 (2013), 357–370.

15. The Minimality Principle for the Modal Logic of Forcing .... Professor B. Löwe

Hamkins [2] proposed to study the *modal logic of forcing* as the modal theory of the generic multiverse, the collection of transitive models of set theory connected by the relation defined by $M \leq N$ iff N is a set-forcing extension of M. One of the first motivating examples for the fruitfulness of this approach was Hamkins's *maximality principle* stating "everything that can be forced is true". This essay will consider its natural dual, the *minimality principle*, informally stated as "everything that can be unforced is false".

The first task is to transform this informal statement into a mathematically precise form, using the language of *control statements* due to Hamkins et al. [3]. The modality "can be unforced" has two possible interpretations, a weak reading and a strong reading. The strong reading of "φ can be unforced" is "there is an inner model in which φ is false". With the strong reading, the minimality principle turns out to be equivalent to **V**=**L**. The weak reading of of "φ can be unforced" is "there is a ground model in which φ is false". The analysis of the weak minimality principle is closely related to the area of *set-theoretic geology* [1] and can be linked to recent work from Piribauer's Master's thesis [4].

The techniques necessary for this work are developed in the paper [3] and a number of other papers that will be provided.

Relevant Courses

Essential: Part II *Logic and Set Theory* (or equivalent), Part III *Topics in Set Theory* (or equivalent).

References

[1] Gunter Fuchs, Joel David Hamkins, & Jonas Reitz. Set-theoretic geology. Annals of Pure and Applied Logic 166:4 (2015), 464–501.

[2] Joel D. Hamkins. A simple maximality principle. Journal of Symbolic Logic, 68:2 (2003), 527–550.

[3] Joel D. Hamkins, George Leibman and Benedikt Löwe, Structural connections between a forcing class and its modal logic, Israel Journal of Mathematics 207 (2015), 607–651.

[4] Jakob Piribauer. The Modal Logic of Generic Multiverses. Master's Thesis. (Universiteit van Amsterdam, 2017).

16. Transcendence Results in the Style of Solovay and Judah-Shelah .... Professor B. Löwe

Sets exhibiting lack of regularity (e.g., non-Lebesgue measurable sets or sets without the Baire property) are constructed using the Axiom of Choice.

In Gödel's constructive universe $\mathbf{L}$, there is a definable well-ordering of the real numbers which gives a definable way to use the Axiom of Choice and consequently get definable sets exhibiting lack of regularity. E.g., in $\mathbf{L}$, we have a $\mathbf{\Delta}^1_2$ set that is not Lebesgue measurable. Solovay, Judah, and Shelah showed that these constructions characterise the constructible universe in the following sense:

Theorem (Judah-Shelah; [2]). The statement "every $\mathbf{\Delta}^1_2$ set is Lebesgue measurable" is true if and only if for every real number x, there is a real generic over $\mathbf{L}[x]$ for Solovay's random forcing.

Such a theorem is called a *transcendence result* since it gives an equivalence between a regularity statement and a statement expressing that the universe transcends the constructible universe. In his doctoral dissertation, Ikegami [3] gave an abstract treatment of transcendence results of this type.

This essay aims to give an overview of Ikegami's results, taking into account recent work by Fischer et al. [1]. Ikegami's results are formulated for arboreal forcings whose conditions correspond to closed sets in Polish spaces; it would be interesting to generalise this to forcings living on *category bases*, a generalisation of topological spaces [4].

Relevant Courses

Essential: Part II *Logic and Set Theory* (or equivalent), Part III *Topics in Set Theory* (or equivalent).

Useful: Part Ib *Metric and Topological Spaces* (or equivalent), Part II *Probability and Measure* (or equivalent).

References

[1] Vera Fischer, Sy D. Friedman, & Yurii Khomskii. Cichon's diagram, regularity properties and $\mathbf{\Delta}^1_3$ sets of reals. Archive for Mathematical Logic 53:5-6 (2014) 695–729.

[2] Jaime I. Ihoda & Saharon Shelah. $\mathbf{\Delta}^1_2$-sets of reals. Annals of Pure and Applied Logic 42:3 (1989), 207–223.

[3] Daisuke Ikegami. Games in set theory and logic. Ph.D. Thesis, ILLC Publications DS-2010-04. (Universiteit van Amsterdam, 2010).

[4] John Morgan. Baire category from an abstract viewpoint. Fundamenta Mathematicae 94:1 (1977), 59–64.

17. Blocks with a Cyclic Defect Group ....
Dr S. Martin

This topic is perhaps the highlight of the classical theory of modular representation theory as initiated and developed by Richard Brauer, and then refined further by Sandy Green. In this theory a central role is played by the p-blocks of the modular group algebra and by certain p-subgroups of G called defect groups. As such, the more complicated the defect group, the more representation theoretically complicated is the p-block (for example, blocks of defect zero - where the defect group is trivial - are matrix algebras wherein everything is semisimple). Given a block of a finite group with a cyclic defect group, there is combinatorial gadget called a Brauer tree which describes the structure of the indecomposable projective modules completely - in the sense that we know their 'module diagram' [4,5]. One could adopt Green's original approach [2] and [1, Ch 5] or use some methods from the representation theory of algebras as in [3, Chapter 6] to give a construction of these trees. At least one non-trivial example should be included for illustration of the theory. Generalisations of these methods [4,5] could be mentioned if you have time.

Relevant Courses

Modular representation theory

References

[1] J.L. Alperin, Local representation theory, CUP (1986).

[2] J.A. Green, Walking around the Brauer tree, *J. Australian Math. Soc.* **17** (1974), 197–213.

[3] D.J. Benson, Representations and cohomology vol I. (Basic representation theory of finite groups and associative algebras), 2nd edn., Cambridge Studies in Advanced Mathematics **30**. CUP (1998).

[4] D.J. Benson and J.F Carlson, Diagrammatic methods for modular representations and cohomology, Commun. in Algebra **15**, 53–121 (1987).

[5] D.J. Benson and J.H. Conway, Diagrams for modular lattices, J. Pure Appl. Algebra 37 (1985), 111-116.

18. Modular Representations of Finite Groups of Lie Type in the Defining Characteristic ....
Dr S. Martin

To quote the preface of [2]: *whatever its therapeutic value may be, group theory offers plenty of diversions and challenges. In particular, the study of finite groups of Lie type and their representations (ordinary or modular) leads to deep questions, many of which are still unsolved.* As well as group theory there are deep links to number theory and topology. The theory of Deligne-Lusztig characters links the ordinary (characteristic 0) theory and the modular (characteristic p) theory in more or less subtle ways according as we are in the describing characteristic or in non-describing characteristic $\ell \neq p$. Concentrating on the former case there are amazing similarities between the representation theory of the ambient simple algebraic group and its so-called Frobenius kernels. Certain 'key' representations of the finite group over the field of p^r elements resemble those of the rth Frobenius kernel (when $r = 1$ corresponding to the Lie algebra of the algebraic group).

An essay treatment would define, classify and construct the finite groups of Lie type (probably going via the algebraic group, as done in [2] below) and then go on to describe their simple modules and the structure of the Weyl modules; the Tensor Product Theorem of Steinberg reduces the problem of determining characters to studying restricted weights lying in the fundamental region.

The sources were all written before Williamson had shown that the Lusztig conjecture was spectacularly false, but nevertheless this remains an intriguing area and one of continued research today.

Relevant Courses

Essential: Algebraic geometry; Lie algebras; modular representation theory.

Useful: Commutative algebra

References

[1] Jens Carsten Jantzen, 'Representations of Algebraic Groups' (2nd edn), AMS 2003; Part II

[2] Jim Humphreys, 'Modular representations of finite groups of Lie type', CUP 2006; Chapters 1–3

[3] Jens Carsten Jantzen, 'Representations of Chevalley groups in their own characteristic' in The Arcata Conference on Representations of Finite Groups, Vol 1, AMS 1987.

[4] George Lusztig 'Some problems in the representation theory of finite Chevalley groups' in The Santa Cruz Conference on Finite Groups, AMS 1980.

19. Non-commutative Character Theory of the Symmetric Group .... Dr S. Martin

Classical accounts [3] of the characteristic 0 representation theory of symmetric groups form a common currency amongst representation theorists. The theory based around the (commutative) $\mathbb{C}$-algebra of class functions (or, with an obvious change of reference frame, of symmetric functions [4]).

Following an idea of Solomon in his seminal paper [4] of 1976 in which he gives a non-commutative refinement of Mackey's Theorem, there is a certain non-commutative superstructure which maps onto the algebra of class functions. His idea is to transfer problems to be solved into the non-commutative setting, and to solve them in that setting. While the language is still combinatorial [2], in many ways this non-commutative world is more accessible than the classical theory, in fact more transparent and rather elementary, hence is perfect for novices to cut their teeth using the new tools offered in this approach. The theory has been developed by Jöllenbeck, Reutenauer, Lascoux, Leclerc, Thibon and others in Paris and beyond.

A splendid little book [1] gives an excellent and straightforward outline of the theory, with Parts I and II forming the main body of knowledge for the essay. Background knowledge also appears inSolomon's original paper [5].

Relevant courses

Essential Representation theory (Parts II and III), commutative algebra

Useful: ordinary character theory and basic group theory; Lie algebras

References

[1] D. Blessenohl and M. Schocker, Noncommutative character theory of the symmetric group, Imperial College Press (2005).

[2] W. Fulton, Young tableaux, CUP (1997).

[3] G.D. James, The representation theory of the symmetric groups, SLN 682, Springer (1978).

[4] I. G. Macdonald, Symmetric functions and Hall polynomials (2nd edn), Clarendon Press, Oxford (1995).

[5] Louis Solomon, A Mackey formula in the group ring of a Coxeter group, *J. Algebra* 41, 255–268 (1976).

20. The Local Geodesic X-ray Transform .... Professor G. P. Paternain

In this essay you will study an injectivity result for the local geodesic X-ray transform. The geodesic X-ray transform, where one integrates a function along geodesics of a Riemannian metric, appears naturally in several geometric inverse problems. A classical example is the standard X-ray or *Radon* transform [1,2], where one integrates a function along straight lines in the plane – it is the basis of medical imaging techniques such as CT (computer tomography) and PET (positron emission tomography).

Quite recently, G. Uhlmann and A. Vasy [4] proved a remarkable result: under a convexity assumption on the boundary they showed that the geodesic X-ray transform can be inverted locally in a stable manner assuming that the manifold has dimension ≥ 3.

The objective of this essay is to explain the main ideas that go into this paper. One striking feature is the use of the scattering calculus developed by Melrose [3] for the purpose of studying scattering problems for the Laplacian. The essay will explain the main features of this calculus and how is being used to prove the local injectivity result.

Relevant Courses

Differential Geometry, Analysis of Partial Differential Equations, Elliptic Partial Differential Equations

References

[1] S. Helgason, *Integral Geometry and Radon Transforms.* Springer, New York (2010).

[2] P. Kuchment, *The Radon transform and medical imaging.* CBMS-NSF Regional Conference Series in Applied Mathematics, **85**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2014.

[3] R.B. Melrose, *Spectral and Scattering Theory for the Laplacian on Asymptotically Euclidian Spaces.* Marcel Dekker, New York (1994).

[4] G. Uhlmann, A. Vasy, *The inverse problem for the local geodesic ray transform,* Invent. Math. **205** (2016) 83–120.

21. Topological Hochschild Homology .... Dr O. Randal-Williams

Hochschild homology is an invariant $HH_*(A)$ of a ring A, roughly analogous to the homology of a space, which arises in many parts of mathematics, such as algebraic geometry, representation theory, number theory, topology, and more. It is most naturally a relative invariant, where $HH_*(A)$ is the Hochschild homology of A relative to the initial ring $\mathbb{Z}$.

Stable homotopy theory concerns spectra, which are suitable stabilisations of spaces, and one may consider rings in this context: now $\mathbb{Z}$ is no longer the initial ring, and the sphere spectrum $\mathbb{S}$ takes this role instead. While $\mathbb{S} \to \mathbb{Z}$ becomes an isomorphism after rationalising, the torsion information in $\mathbb{S}$ is incredibly rich (it is the stable homotopy groups of spheres, whose computation is the motivating problem in stable homotopy theory). The topological Hochschild homology $THH_*(A)$ is obtained as the Hochschild homology of A relative to the initial ring spectrum $\mathbb{S}$; here A can be a ring or more generally a ring spectrum.

The goal of this essay is to understand the definition of THH, and to calculate it in some examples. You will first need a definition of a modern category of spectra which is symmetric monoidal, such as that of [1]. The details of this should be understood but not belaboured. The definition of THH is then quite easy, see p. 172 of [1]. The next goal is to compute $THH_*(\mathbb{F}_p)$, first done by Bökstedt [2], looking at Section 5.4 of [3] and its references for help. This will require you to learn about spectral sequences, the (dual) Steenrod algebra, and Dyer–Lashof operations. You should then explore further calculations, for example following [3].

Relevant Courses

Essential: Part III Algebraic Topology

Useful: Part III Category theory, or a willingness to read about it.

References

[1] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, *Rings, modules, and algebras in stable homotopy theory.* Mathematical Surveys and Monographs, 47., 1997.

[2] M. Bökstedt, *The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$.* Available from me.

[3] V. Angeltveit, J. Rognes, *Hopf algebra structure on topological Hochschild homology.* Algebr. Geom. Topol. 5 (2005), 1223-1290.

22. Geometric Topology .... Dr O. Randal-Williams

This essay will develop some of the fundamental tools and techniques of geometric topology, that is, the topology of manifolds. Many phenomena in this subject depend sensitively on the fundamental group, and this essay will exhibit some of these phenomena. Firstly, you should explain the notion of Whitehead torsion of an h-cobordism, and the s-cobordism theorem. You should then explain the theorem of Siebenmann concerning when a non-compact manifold is the interior of a compact manifold with boundary, which must include a discussion of Wall's finiteness obstruction.

You could then go any of in several directions, which you should discuss with me. One suggestion would be to explain when a map to S^1 can be made into a fibre bundle, following [4], [5], or

[6] (in increasing order of sophistication). Another would be to explain the second Whitehead group and its relation to pseudo-isotopies, see e.g. [7].

Relevant Courses

Essential: Part III Algebraic Topology

References

Section 1 of [2] gives an exhaustive proof of the s-cobordism theorem, but [3] is shorter. Sections 8, (the algebraic part of) 11, and 12 of [1] are directly relevant.

[1] S. Ferry, *Geometric topology notes*, `www.math.rutgers.edu/~sferry/ps/geotop.pdf`

[2] W. Lück, *A basic introduction to surgery theory*, `131.220.77.52/lueck/data/ictp.pdf`

[3] M. Kervaire, *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. 40 1965 31–42

[4] W. Browder, J. Levine, *Fibering manifolds over a circle.* Comment. Math. Helv. 40 1966 153-160.

[5] F. T. Farrell, *The obstruction to fibering a manifold over a circle.* Indiana Univ. Math. J. 21 1971/1972 315-346.

[6] L. C. Siebenmann, *A total Whitehead torsion obstruction to fibering over the circle.* Comment. Math. Helv. 45 1970 1-48.

[7] A. E. Hatcher, *Concordance spaces, higher simple-homotopy theory, and applications.* Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, pp. 3-21,

23. Soergel Bimodules and HOMFLY-PT Homology ....
Dr J. A. Rasmussen

The HOMFLY-PT polynomial is an invariant of oriented knots and links in S^3. The goal of the essay is to learn something about this polynomial and its categorification, as described in Khovanov's classic paper [3]. The process of categorification replaces polynomials with graded homology groups whose graded Euler characteristic recovers the original polynomial. Khovanov homology, which categrifies the Jones polynomial, is an important example. As preparation for reading [3], you should first learn about and the Khovanov homology of tangles, as described in [1].

A good essay will elaborate on the themes touched on in [3], including the relationship between the HOMFLY-PT polynomial and the Hecke algebra (as described in [2]), discuss Soergel bimodules and their relation to the Kazhdan-Lusztig basis, the connection between the HOMFLY-PT homology and Khovanov homology for tangles.

Relevant Courses

Essential:

Useful: Algebraic Topology, Lie Algebras and their Representations, Represenation Theory

References

[1] Dror Bar-Natan, Khovanov's homology for tangles and cobordisms, *Geom. Topol.* 9 (2005) 1443–1499, arXiv:math/0410495

[2] Vaughn Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* (2) 126 (1987), no. 2, 335-388

[3] Mikhail Khovanov, Triply-graded link homology and Hochschild homology of Soergel bimodules, *Internat. J. Math.*, 18 (2007) 869–885,
arXiv:math/0510265

24. L-Spaces and Left Orderings ....
Dr J. A. Rasmussen

This essay explores two invariants of 3-manifolds and a strange connection between them. The first invariant is easy to state: a 3-manifold Y is left-orderable if its fundamental group admits a total ordering which is invariant under left-multiplication: $x < y$ implies $ax < ay$. The second property involves an invariant of 3-manifolds called Heegaard Floer homology. If the Heegaard Floer homology of Y is as small as possible, Y is called an L-space. A surprising conjecture of Boyer, Gordon, and Watson says that a closed, prime, orientable 3-manifold Y is left-orderable if and only if Y is not an L-space. The goal of the essay is learn something about the ideas underlying the conjecture and the evidence for it. You should learn enough about left-orderings in order to explain Boyer, Rolfsen, and Wiest's theorem that a manifold with a nontrivial homomorphism to a left-orderable group is left-orderable, and enough about Heegaard Floer homology to explain Levine and Lewallen's work on strong L-spaces.

Relevant Courses

Essential: Algebraic Topology

Useful: 3-Manifolds

References

[1] S. Boyer, C. Gordon, and L. Watson, On L-spaces and left-orderable fundamental groups, *Math. Ann.* 356 (2013), 1213-1245, arXiv:1107.5016.

[2] S. Boyer, D. Rolfsen, and B. Wiest, Orderable 3-manifold groups, *Ann. Inst. Fourier* 55 (2005), 243-288, arXiv:math/0211110.

[3] A. Levine and S. Lewallen, Strong L-spaces and left orderability, *Math. Res. Lett.* 19 (2012), 1237–1244, arXiv:1110.0563.

[4] P. Ozsváth and Z. Szabó, An introduction to Heegaard Floer homology, in *Floer Homology, Gauge Theory, and Low-Dimensional Topology*, edited by D. Ellwood et al. Clay Mathematics Institute, 2006.

25. Tight Contact Structures on Lens Spaces ....
Dr S. D. Rasmussen

A contact structure is an everywhere-nonintegrable hyperplane distribution on an odd-dimensional manifold, and is one of the fundamental structures of geometric and topological interest on 3-

manifolds. Contact structures come in two classes: overtwisted and tight. Overtwisted contact structures provide no information about a 3-manifold beyond its homotopy classification of 2-plane fields. Tight contact structures, which satisfy an additional rigidity condition, are more idiosyncratic and informative; for example, they provide necessary conditions for the existence of taut foliations, and they govern the symplectic 4-manifold fillings a 3-manifold can admit.

Tight contact structures are much more difficult to classify, requiring intricate tools to identify and distinguish. One of the few classes of manifolds for which tight contact structures are classified are the lens spaces $L(p,q)$—certain smooth quotients of S^3 by cyclic group actions. There are various approaches to describing these tight contact structures, for example from the standpoint of Legendrian surgery [1], open book decompositions [5], or convex surface theory [2]. The essay should discuss Giroux's convex surfaces strategy, summarised in [3], for classifying the tight contact structures on lens spaces, and relate the answer to one or more other means of describing tight contact structures on lens spaces. See [4] for introductory material.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry

Useful: 3-Manifolds

References

[1] F. Ding, H. Geiges, A. Stipsicz, Surgery diagrams for contact 3-manifolds, *Turkish J. Math.* 28 (2004), 41-74. `arXiv:math/0307237`

[2] W. Kazez, A cut-and-paste approach to contact topology, Lecture Notes, 2002. `arXiv:math/0210091`

[3] P. Massot, Geodesible contact structures on 3–manifolds, *Geom. Topol.* 12 (2008) 1729-1776. `arXiv:0711.0377`

[4] P. Massot, Topological methods in 3-dimensional contact geometry, Lecture Notes, 2013. `arXiv:1303.1063`

[5] S. Schönenberger, Planar Open Books and Symplectic Fillings, PhD Thesis, Univerity of Pennsylvania, 2005.
`https://www.math.upenn.edu/grad/dissertations/SchoenenbergerThesis.pdf`

26. p-adic L-functions . . . . Professor A. J. Scholl

In the 19th century Kummer discovered a family of congruences involving Bernoulli numbers. Kubota and Leopoldt showed that these were equivalent to the existence of a p-adically continuous function interpolating the values of the Riemann ζ-function at negative integers. Since then numerous generalisations have been found, for characters of number fields, modular forms and other automorphic forms. This essay should explore one of these developments from Kubota and Leopoldt's original work. A suggested (but not the only) direction is to discuss one construction of p-adic L-functions of totally real fields, for which two completely different approaches were found: Deligne–Ribet (using arithmetic geometry of abelian varieties) and Barsky and Cassou–Noguès (using Shintani's representation of the Dedekind ζ-function).

Relevant Courses

Algebraic Number Theory

References

[1] J. Neukirch: “Algebraic Number Theory”, chapter VII (for classical L-functions and an account of Shintani’s unit theorem)

[2] L. Wasshington: “An introduction to cyclotomic fields” (for Kubota–Leopoldt theory)

[3] N. M. Katz: “Another look at p-adic L-functions for totally real fields”. Math. Annalen **255**, 33–43 (1981) and references cited therein.

27. Higher Regulators of Number Fields ....
Professor A. J. Scholl

The analytic class number formula gives an interpretation for the residue at $s = 1$ of the Dedekind zeta function of a number field in terms of the regulator, constructed from logarithms of absolute values of units.

In 1977, A. Borel [1] found a remarkable generalisation of this result, giving an interpretation of the values of the Dedekind zeta at arbitrary integers in terms of “higher regulators”, in which the group of units is replaced by a higher K-group of the rings of integers. The proof relies on the relation between K-theory and homology of general linear groups of rings of integers, which is then studied using Lie algebra cohomology.

The goal of this essay is to understand the statement of Borel’s theorem and some of the ingredients of the proof. The courses on Algebraic Number Theory and Lie Algebras (and perhaps Algebraic Topology) will help in tackling the essay.

Related courses

Algebraic Number Theory, Lie Algebras, Algebraic Topology

References

[1] A Borel: “Cohomology de SL(n) et valeurs de fonctions zeta aux points entiers”. Ann. Sc. Norm. Super. Pisa 4 (1977), 613-636 (also to be found in vol.III of Borel’s Collected Works)

28. Simple Singularities ....
Dr N. Sheridan

We consider holomorphic functions f defined on a neighbourhood of $0 \in \mathbb{C}^n$. We define an equivalence relation on such functions: $f_1 \sim f_2$ if there exist neighbourhoods U_1, U_2 of $0 \in \mathbb{C}^n$ with a biholomorphism $g : U_1 \to U_2$ such that $f_1 = f_2 \circ g$. A ‘singularity’ is an equivalence class of $\sim$. References on singularity theory include [1] and the more rapid [2].

We will focus on the ‘simple singularities’, which are those with no moduli (they can’t be deformed in a positive-dimensional family of inequivalent singularities). These can be classified (see [3] and [1, Part II]), and it turns out they can be indexed by simply-laced Dynkin diagrams.

Given $f : \mathbb{C}^3 \to \mathbb{C}$ with a simple singularity at the origin, we can get the corresponding Dynkin diagram in three ways:

(a) There is an isomorphism

$$f^{-1}(0) \cong \mathbb{C}^2/\Gamma$$

where $\Gamma \subset SU(2)$ is a binary polyhedral group, and these are indexed by simply-laced Dynkin diagrams (see [2, Chap. 1, §2.4] and references therein, [4, 6, 7]);

(b) $V = f^{-1}(0)$ admits a 'minimal resolution' $\pi : \tilde{V} \to V$, and $\pi^{-1}(0)$ is a union of $\mathbb{P}^1$s whose intersection graph determines a Dynkin diagram (see [2, Chap. 1, §2.5] and references therein, [4, 7]);

(c) The Milnor fibre $S := f^{-1}(\epsilon)$ gives us a lattice by equipping $H_2(S; \mathbb{Z})$ with the intersection pairing, and the homology classes of vanishing cycles of f form a root system in this lattice, which corresponds to a Dynkin diagram (see [2, Chap. 2, §2] and references therein, and [5]).

The essay should cover the classification of simple singularities and at least part of (a) above. A good essay will furthermore cover (b) or (c).

Relevant Courses

Essential: If you want to do (b): Part II algebraic geometry. If you want to do (c): Part II algebraic topology, differential geometry.

Useful: Part II algebraic geometry, algebraic topology, differential geometry, Part III commutative algebra, algebraic geometry.

References

[1] Vladimir I Arnold, S. M. Gusein-Zade, and A. Varchenko. Singularities of differentiable maps, Volume I, Monographs in Mathematics, 82, Birkhäuser, Boston, 1985.

[2] Vladimir I Arnold, Victor Goryunov, O Lyashko, and Victor Vasil'ev. *Singularity Theory. I. Translated from the 1988 Russian original by A. Iacob. Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [Dynamical systems. VI, Encyclopaedia Math. Sci., 6, Springer, Berlin, 1993].* Springer-Verlag, Berlin, 1998.

[3] Vladimir I Arnold. Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities. *Funkcional. Anal. i Priložen.*, 6(4):3–25, 1972.

[4] K. Lamotke. *Regular Solids and Isolated Singularities (Advanced Lectures in Mathematics Series).* Friedr. Vieweg und Sohn, 1986.

[5] John Milnor. *Singular points of Complex Hypersurfaces, Annals of Mathematics Studies 61.* Princeton University Press, 1968.

[6] Jerry Shurman. *Geometry of the quintic.* John Wiley and Sons, 1997.

[7] Joris van Hoboken. *Platonic solids, binary polyhedral groups, Kleinian singularities and Lie algebras of type A, D, E.* Masters thesis, University of Amsterdam, 2009.

29. Picard-Lefschetz theory .... Professor I Smith

Picard-Lefschetz theory involves studying an algebraic variety, or symplectic manifold, via a pencil of hypersurface sections. This essay will start by discussing the algebraic topology of Lefschetz pencils, Dehn twist monodromy, and the existence of Lefschetz pencils on algebraic varieties; the symplectic nature of Dehn twists should be clearly explained. There are then several possible directions: constraints on the topology of Lefschetz pencils on four-manifolds; the geometry and symplectic topology of varieties with small dual; or (more analytical and challenging) Donaldson's existence theorem for Lefschetz pencils on symplectic manifolds. If interested, talk to me, and a more precise focus can emerge depending on the background and interests of the writer.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry, Symplectic Geometry.

Useful: Complex Manifolds, Algebraic Geometry, Positivity in Algebraic geometry.

References

[1] *The topology of complex projective varieties after S. Lefschetz.* K. Lamotke, Topology 1981.

[2] *Cohomology and intersection cohomology of algebraic varieties.* E. Looijenga, Notes in Park City Volume "Complex algebraic geometry", AMS 1997.

[3] *Singular points of complex hypersurfaces.* J. Milnor, Princeton Univ. Press 1968.

[4] *Mapping class group factorizations and symplectic 4-manifolds: some open problems.* D. Auroux, available at https://math.berkeley.edu/ auroux/papers/index.html

[5] *Lefschetz pencils on symplectic manifolds*, S. Donaldson, J. Diff. Geom. 1999.

30. Containers for Hypergraphs .... Professor A. G. Thomason

The notion of containers is a simple but surprising one. Let G be an r-uniform hypergraph. An independent set in G is one which contains no edge. A set of containers for G is a collection $\mathcal{C}$ of subsets of the vertices, such that for every independent set I there is a container $C \in \mathcal{C}$ with $I \subset C$. The collection $\mathcal{C}$ is useful if the containers are not much bigger than the independent sets but the overall collection $\mathcal{C}$ is much smaller than the collection of all independent sets. There seems no reason to think a useful collection of containers might exist but it does: this is the basic discovery of Balogh, Morris and Samotij [1] and of Saxton and Thomason [3]. An essay might explore some ways to construct containers, as well as their many applications.

Relevant Courses

Essential: none

Useful: Extremal Graph Theory, Combinatorics

References

[1] J. Balogh, R. Morris and W. Samotij, Independent sets in hypergraphs, *J. Amer. Math. Soc.* **28** (2015), 669–709.

[2] A.A. Sapozhenko, Systems of containers and enumeration problems, in SAGA 2005, *Lecture Notes in Computer Science*, Springer (2005), 1–13.

[3] D. Saxton and A. Thomason, Hypergraph containers, *Inventiones Mathematicae* **201** (2015), 925–992.

31. Flag Algebras in Graph Theory . . . . Professor A. G. Thomason

A substantial number of questions in extremal graph theory have been attacked successfully in recent years by the method of flag algebras. The method's most spectacular early success was the determination of the minimum number of triangles in a graph subject to a given number of edges. The effectiveness of the method comes to a great extent from the fact that it permits a computer to find, in effect, an optimal proof by solving a practically feasible semi-definite program. In some cases the method can be pushed even further, to the point of establishing a calculus for finding a minimum. There are other limitations to the method too which are now understood better. An essay would describe the basic flag algebra method, together with some examples, and then explore further aspects.

Relevant Courses

Essential: none

Useful: Extremal Graph Theory, Combinatorics

References

[1] A. Razborov, On the minimal density of triangles in graphs, *Combinatorics, Probability and Computing* **17** (2008), 603–618.

[2] R. Baber and J. Talbot, Hypergraphs do jump, *Combinatorics, Probability and Computing* **20** (2011), 161–171.

[3] H. Hatami, J. Hladky, D. Král', S. Norine and A. Razborov, On the number of pentagons in triangle-free graphs, *J. Combinatorial Theory (Ser A)* **120** (2013), 723–732.

32. Modular Forms of Weight 1 . . . . Dr J. A. Thorne

Modular forms are holomorphic functions on the complex upper half plane which transform in a specified way under suitable subgroups of $\mathrm{SL}_2(\mathbb{Z})$, acting through Möbius transformations. The Langlands conjectures predict that modular forms are closely related to 2-dimensional representations of the absolute Galois group of $\mathbb{Q}$. A particular special case of this is the Shimura–Taniyama–Weil conjecture, which predicts a bijection between certain modular forms of weight 2 and isogeny classes of elliptic curves over $\mathbb{Q}$. The Shimura–Taniyama–Weil conjecture has now been proved, starting with the work of Wiles, who established the *modularity* of semistable elliptic curves over $\mathbb{Q}$ on his way to proving Fermat's Last Theorem.

Another important special case is that of modular forms of weight 1. These should correspond to odd Artin representations, i.e. representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ such that $\det \rho(c) = -1$ for any choice $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of complex conjugation. This conjectural correspondence has also now been proved to exist. The first steps were taken by Deligne and Serre, who showed how to attach an odd Artin representation to a modular form of weight 1.

The main goal of this essay will be to give a discussion of modular forms of weight 1 and to understand the construction of Deligne and Serre, which combines ideas from analysis, algebra and number theory in quite a surprising way. Interested students could go on to give some numerical examples, using algorithms for computing with weight 1 forms which have recently become widely available.

Relevant Courses

Essential: Modular Forms and L-Functions

Useful: Algebraic Number Theory

References

[1] Fred Diamond and Jerry Shurman. *A first course in modular forms.* Graduate Texts in Mathematics, 228. Springer-Verlag, New York, 2005.

[2] Jean-Pierre Serre, *Modular forms of weight one and Galois representations.* Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 193–268.

[3] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids 1.* Ann. Sci. École Norm. Sup. (4) 7 (1974), pp. 507–530.

[4] Kevin Buzzard and Alan Lauder. *A computation of modular forms of weight one and small level.* `arXiv` preprint, available at `https://arxiv.org/abs/1605.05346`.

33. Danzer Sets and Dense Forests ....

Dr P. P. Varjú

A Danzer set is a subset of $\mathbf{R}^d$ that intersects any convex set of volume 1. Danzer asked the question whether there is a Danzer set $A \subset \mathbf{R}^d$ that is sparse in the sense that

$$\#(A \cap B_R) \leq CR^d$$

for all $R \in \mathbf{R}_{>0}$, where B_R is the ball of radius R centered at the origin and $C > 0$ is a constant. This problem is still open, but there have been exciting new developments in a strongly related problem about dense forests. A dense forest with visibility function $f(\varepsilon)$ is a set $A \subset \mathbf{R}^d$ such that for any line segment of length $f(\varepsilon)$ in $\mathbf{R}^d$, there is a point in A at distance at most ε to the line segment. A Danzer set is a dense forest with visibility $f(\varepsilon) = C\varepsilon^{-(d-1)}$ and the converse is also true if $d = 2$.

Several recent papers construct sparse dense forests based on a variety of techniques including probability, Fourier analysis and dynamical systems. The essay will discuss some of these results.

Relevant Courses

The essay is not directly related to any particular courses and the essay writer has some freedom in choosing the material to suit her or his background. Some basic knowledge in one field among probability, Fourier analysis, ergodic theory or topological dynamics would be useful.

References

[1] R. P. Bambah and A. C. Woods, *On a problem of Danzer*, Pacific J. Math., 37 (1971), No. 2, 295–301.

[2] C. J. Bishop, *A set containing rectifiable arcs QC-locally but not QC-globally*, Pure Appl. Math. Q., 7 (2011), No. 1, 121–138.

[3] Y. Solomon and B. Weiss, *Dense forests and Danzer sets*, Ann. Scient. Éc. Norm. Sup. 4^e série, 49 (2016), 1053–1074.

[4] D. Simmons and Y. Solomon, *A Danzer set for axis parallel boxes*, Proc. Amer. Math. Soc., 144 (2016), No. 6, 2725–2729.

[5] F. Adiceam, *How far you can see in a forest?*, Int. Math. Res. Not. IMRN, 2016, No. 16, 4867–4881.

[6] O. Solan, Y. Solomon and B. Weiss, *On problems of Danzer and Gowers and dynamics on the space of closed subsets of* $\mathbb{R}^d$, Int. Math. Res. Not. IMRN, http://doi.org/10.1093/imrn/rnw204, 15 pp.

[7] N. Alon, *Uniformly discrete forests with poor visibility*, http://www.tau.ac.il/~nogaa/PDFS/forest2.pdf, 9 pp.

34. Combinatorial Morse Theory .... Dr H. Wilton

Classical Morse theory (as developed in [1]) is a way of understanding the topology of a manifold M via a *Morse function* $M \to \mathbb{R}$. Combinatorial Morse theory is an analogue of this theory for cell complexes, developed in parallel by Forman and Bestvina–Brady.

The latter were able to use it to provide counterexamples to various longstanding problems in topology: they constructed an infinitely presented group of type FP_2, and showed that at most one of the Whitehead and Eilenberg–Ganea conjectures hold [2].

The goal of this essay is to describe Bestvina–Brady's construction, and to give a proof of their main theorem, along with appropriate background material [3],[4].

Relevant Courses

Essential: Part II Algebraic topology

Useful: Topics in Geometric Group Theory

References

[1] J. Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[2] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Inventiones Mathematicae*, 129:445–470, 1997.

[3] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Gundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[4] Kai-Uwe Bux and Carlos Gonzalez. The Bestvina–Brady construction revisited: geometric computation of σ-invariants for right-angled Artin groups. *Journal of the London Mathematical Society. Second Series*, 60:793–801, 1999.

35. Hyperbolic Groups ....

Dr H. Wilton

Hyperbolic groups (as defined by Mikhail Gromov [1]) have become a powerful tool in modern group theory and low-dimensional topology. A metric space is called *hyperbolic* if it satisfies a coarse geometric inequality motivated by the negatively curved geometry of the hyperbolic plane. A group is called *word-hyperbolic*, or just *hyperbolic*, if it acts nicely on a hyperbolic space. The class of hyperbolic groups is so large that it is generic (in a suitable sense) among all finitely presented groups, and yet small enough that we can prove theorems about them: for instance, the word problem is solvable in hyperbolic groups.

The goal of this essay is to bring together a diverse group of sources [2],[3],[4] to develop the basic theory of these groups.

Relevant Courses

Essential: Part Ib Geometry

Useful: Part II Algebraic topology, Topics in Geometric Group Theory,

References

[1] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[2] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften.*
Springer-Verlag, Berlin, 1999.

[3] Cornelia Drutu and Michael Kapovich. *Geometric group theory*, preprint, 2017 https://www.math.ucdavis.edu/~kapovich/EPR/ggt.pdf

[4] E. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*, volume 83 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

36. Complexes of Curves .... Dr H. Wilton

The *mapping class group* $\mathrm{Mod}(\Sigma)$ of a compact surface Σ plays a fundamental role in diverse areas of mathematics. The structure of $\mathrm{Mod}(\Sigma)$ is typically extremely complicated, but one powerful technique for analysing it is provided by the *complex of curves* $\mathcal{C}(\Sigma)$.

The goal of this essay is to prove some geometric and topological properties of $\mathcal{C}(\Sigma)$, and to show how they can be applied to study $\mathrm{Mod}(\Sigma)$.

Relevant Courses

Essential: Part II Algebraic topology,

Useful: Topics in Geometric Group Theory,

References

[1] Benson Farb and Dan Margalit, *A primer on mapping class groups.* Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012. xiv+472 pp.

[2] Sebastian Hensel, Piotr Przytycki and Richard Webb, '1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs', *Journal of the European Mathematical Society,* to appear.

[3] W. B. R. Lickorish, 'A finite set of generators for the homeotopy group of a 2-manifold', *Proc. Cambridge Philos. Soc.* 60 1964 769778.

37. Central Limit Theorem for Additive Functionals of Reversible Markov Processes .... Dr S Andres

Let $\{X_n\}_{n\geq 0}$ be a stationary, reversible, ergodic Markov chain with initial distribution given by its invariant measure. In a seminal paper [1] Kipnis and Varadhan showed for a large class of functionals V on the state space that the process $\{V(X_n)\}_{n\geq 0}$ satisfies a central limit theorem in the sense that the sequence

$$Y_n(t) := n^{1/2} \sum_{j=1}^{\lfloor nt \rfloor} V(X_j)$$

converges weakly to a Brownian motion $\sigma B(t)$ with some variance σ^2 as $n \to \infty$. A main step of the proof – nowadays knows a the Kipnis-Varadhan technique – is the reduction to a weak convergence of a certain martingale towards Brownian motion. One key object introduced in this paper is the so-called process of the environment as seen from the particle. We also refer to the more recent monograph [2] on this topic.

A successful essay will review these results including some detailed proofs and discuss applications to simple exclusion processes and to homogenisation of diffusions in random environment (see Chapers 5 and 9 in [2]).

Relevant Courses

Essential: Markov Chains, Advanced Probability

Useful: Functional Analysis

References

[1] C. Kipnis, S.R.S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** (1986), no. 1, 1–19.

[2] T. Komorowski, C. Landim, S. Olla. Fluctuations in Markov processes. Time symmetry and martingale approximation. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 345. Springer, Heidelberg, 2012.

38. The Statistical Analysis of Distribution Function Data .... Professor J. A. D. Aston

Many observed data are constrained by their intrinsic features or geometry. This is especially true when the objects under analysis are distribution functions, where there is an inherent need for the data to obey certain properties to allow it to form a distribution. In addition, it is unusual that the distribution of interest is discrete, in which case the observation is only ever likely to be a finite dimensional approximation.

The idea of this essay will be to review some of the recent advances in the statistical analysis of distribution function data. There are possibilities to undertake a theoretical treatment of regression of distribution function data, or other common statistical techniques in the context of distribution functions. Alternatively, or in addition, a comprehensive statistical treatment of data which is inherently composed of distributions could be performed, using modern techniques from the literature.

Relevant Courses

Useful: Differential Geometry, Modern Statistical Methods, Topics in Statistical Theory

References

[1] IL Dryden and KV Mardia (1998). *Statistical Shape Analysis.* Wiley

[2] A Petersen and HG Müller. Functional data analysis for density functions by transformation to a Hilbert space (2016). *Annals of Statistics*, **44**:183–218.

[3] PZ Hadjipantelis, JAD Aston, HG Müller, and JP Evans. Unifying Amplitude and Phase Analysis: A Compositional Data Approach to Functional Multivariate Mixed-Effects Modeling of Mandarin Chinese. (2015) *Journal of the American Statistical Association* **110**:545–559.

39. Registration and Warping in Statistics ....
Professor J. A. D. Aston

In one or higher dimensions, when comparing data, it is sometimes the case that the unit against which the data is measured is not comparable across observations. For example, it might be possible to make one time-based observation look identical to another simply by transforming the notion of "clock time" to another (possibly nonlinear) measure of time. This is an example of registration or warping. In higher dimensions, examples of registration include trying to match landmarks of one person's brain to another.

There are a number of possibilities with this essay. A formal theoretical understanding of registration can be framed in terms of statistics on quotient groups. Developing some theoretical results for some simple statistical methods in such settings could be one possible avenue for the essay. Another might be to find data sets where registration is an issue and show how the registration problem can be mathematically formulated and accounted for in the final analysis.

Relevant Courses

Useful: Differential Geometry, Modern Statistical Methods, Topics in Statistical Theory

References

[1] IL Dryden and KV Mardia (1998). *Statistical Shape Analysis.* Wiley

[2] A Srivastava and EP Klassen (2016). *Functional and Shape Data Analysis*, Springer

[3] E Lila and JAD Aston. Functional and Geometric Statistical Analysis of Textured Surfaces with an application to Medical Imaging (2017) *arXiv*:1707.00453.

40. Generative Adversarial Networks ....
Dr S. A. Bacallado

The deep learning community has shown great interest in generative models in recent years, because, among other reasons, they can learn representations of data which are useful in supervised tasks without relying on costly labelled examples. Generative models make it possible to simulate "the dreams" of a deep neural network, producing synthetic data, such as images of faces, which are sometimes indistinguishable from natural signals. Applications include transfer learning, image segmentation and inpainting, style transfer, video generation, and bayesian optimisation, among others.

This essay will focus on one of the dominant approaches to generative modelling, generative adversarial networks (GANs) ([1] introduced the idea, and [2] reviews recent developments). Your essay must provide an overview of GANs for an audience of mathematicians, emphasising a specific strand of research. For example, you might focus on a specific application domain, and perhaps implement a model and apply it to a dataset of your choice. You may also review and discuss recent theoretical developments, such as those in [3, 4].

Relevant Courses

Useful: Modern Statistical Methods, Bayesian Modelling and Computation.

References

[1] Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, Yoshua Bengio. Generative Adversarial Nets.
NIPS 2014, arXiv:1406.2661.

[2] Ian Goodfellow. NIPS 2016 Tutorial: Generative Adversarial Networks. arXiv:1701.00160.

[3] Sanjeev Arora, Rong Ge, Yingyu Liang, Tengyu Ma, and Yi Zhang. Generalization and Equilibrium in Generative Adversarial Nets (GANs). ICML 2017, arXiv:1703.00573.

[4] Sanjeev Arora and Yi Zhang. Do GANs actually learn the distribution? An empirical study. arXiv:1706.08224.

[5] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. Deep Learning. MIT Press, 2016. Available at http://www.deeplearningbook.org.

41. Delocalisation of Eigenfunctions ....
Dr R. Bauerschmidt

Let $H = -\Delta$ be the Laplacian matrix of a finite regular graph X, or let $H = -\Delta$ be the Laplace operator on a compact Riemann surface X. What can be said about the eigenfunctions of H when X is generic, say random under a suitable measure on graphs or surfaces? It is believed that eigenfunctions with large eigenvalue are delocalised for large classes of surfaces, and similarly for the eigenfunctions of large graphs. Delocalised means that the eigenfunctions extend over most of the space in a suitable sense. One such sense is described in [1].

A closely related question is that of the Anderson transition [2]. For $\lambda > 0$, let $H = -\Delta + \lambda V$ be a Schrödinger operator on a regular bounded degree graph with a random diagonal potential V that is independent for different vertices. For λ large enough, it is understood that the eigenfunctions of H are localised, i.e., concentrated in a small region of space. For large subsets of $\mathbb{Z}^d$, $d > 2$, it is is a major open problem to prove that eigenfunctions can delocalise for small λ. The essentially only existing results for Anderson delocalisation concern the regular tree.

A successful essay will present a selected result for eigenfunction delocalisation in detail, including the required background. Possibilities include results for eigenfunctions on the sphere [3], for the Anderson model on the tree [4],[5], or results for the Anderson model on large finite regular graphs [6].

Relevant Courses

Essential: Advanced Probability

Useful: Differential Geometry

References

[1] P. Sarnak. Recent progress on the quantum unique ergodicity conjecture. *Bull. Amer. Math. Soc. (N.S.)*, 48(2):211–228, 2011.

[2] M. Aizenman and S. Warzel. *Random operators*, volume 168 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015. Disorder effects on quantum spectra and dynamics.

[3] S. Brooks, E. Le Masson, and E. Lindenstrauss. Quantum ergodicity and averaging operators on the sphere. *International Mathematics Research Notices. IMRN*, (19):6034–6064, 2016.

[4] A. Klein. Extended states in the Anderson model on the Bethe lattice. *Adv. Math.*, 133(1):163–184, 1998.

[5] M. Aizenman, R. Sims, and S. Warzel. Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs. *Probab. Theory Related Fields*, 136(3):363–394, 2006.

[6] N. Anantharaman and M. Sabri. Quantum ergodicity on graphs : from spectral to spatial delocalization.

42. Matrix Completion .... Dr Q. Berthet

In various applications, from movie or product ratings, to an analysis of marks on a multiple question examination, we seek to estimate a *low-rank* matrix $M^\star = UV^\top$, where $M^\star \in \mathbf{R}^{m\times n}$ and $U \in \mathbf{R}^{m\times r}, V \in \mathbf{R}^{n\times r}$ for $r \leq m, n$. The observations consists in noisy evaluations of some entries, i.e. for $(i,j) \in \Omega$

$$Y_{ij} = M^\star_{ij} + Z_{ij} = u_i^\top v_j + Z_{ij}$$

Solving the constrained least-squares of this problem

$$\min_{M\,:\,\mathbf{rank}(M)\leq r} \|\mathcal{P}_\Omega(M-Y)\|_2^2$$

is NP-hard in general. In a pioneering line of work, [1,2] showed that convex relaxations can be used to alleviate the computational hardness of this problem, and obtained theoretical guarantees on the statistical properties of this estimator. Even though this made the problem tractable, it was still a computationally intensive task, and more recent advances [3,4] have focused on simpler approaches, formulating the least-squares as

$$\min_{U'\in\mathbf{R}^{m\times r}, V'\in\mathbf{R}^{n\times r}} \|\mathcal{P}_\Omega(U'V'^\top - Y)\|_2^2 \,.$$

Even though this function is not jointly convex in U', V', it has been shown that gradient descent leads to good local minima, under various conditions.

The objective of this project is to study the literature, and to work on an application or a generalisation.

Relevant Courses

Useful: Topics in Convex Optimisation, Topics in Statistical Theory, Modern Statistical Methods.

References

[1] Exact Matrix Completion via Convex Optimization, Emmanuel Candès, Benjamin Recht, *Foundations of Computational Mathematics*, (2009)

[2] Matrix Completion With Noise, Emmanuel Candès, Yaniv Plan, *Proc. of the IEEE*, (2010)

[3] Guaranteed matrix completion via nonconvex factorization, Ruoyu Sun and Zhi-Quan Luo, *FOCS 2015*

[4] Matrix Completion has No Spurious Local Minimum, Rong Ge, Jason Lee, Tengyu Ma, *NIPS 2016*

43. Bandit Problems .... Dr Q. Berthet

In a casino with multiple slot machines (one-armed bandits), we would like to play the one with the highest mean reward. To increase our confidence that we have identified the best one, we have to play more often suboptimal arms, and there is a natural trade-off between *exploration* and *exploitation*. This framework finds its roots in a statistical formulation of experimental drug treatments, and is nowadays applied in various contexts, from artificial intelligence in games to advertising.

The objective of this project is to study the literature, and to work on an application or a generalisation.

Relevant Courses

Useful: Topics in Convex Optimisation, Topics in Statistical Theory, Modern Statistical Methods.

References

[1] Regret Analysis of Stochastic and Nonstochastic Multi-armed Bandit Problems, S. Bubeck and N. Cesa-Bianchit, *Foundations and Trends in Machine Learning*, (2012)

[2] Finite-time Analysis of the Multiarmed Bandit Problem, Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer, *Machine Learning*, (2002)

[3] Stochastic Linear Optimization under Bandit Feedback, Varsha Dani, Thomas P. Hayes, and Sham M. Kakade?, *COLT 2008*

44. Lorentz/Ehrenfest wind–tree model .... Professor G. R. Grimmett

A light particle moving around a Euclidean space is subject to reflections by heavy particles. If the heavy particles are distributed at random, what can be said about the trajectory of the light particle? In particular, is it almost surely bounded? If not, is it asymptotically Gaussian?

This essay will contain a summary of the existing rigorous theory and its limitations, together with a deeper study of some specific aspect such as: Harris's theorem for two-dimensional Poissonian systems, Gaussianity for periodic systems, and Lorentzian models in the presence of extra noise.

References

[1] Grimmett, G. R. (1999), *Percolation*, Springer-Verlag, Section 13.3.

[2] Spohn, H. (1991), *Large Scale Dynamics of Interacting Particles*, Springer-Verlag, Berlin.

[3] Grimmett, G. R. (1999), Stochastic pin-ball, number 20 of `http://www.statslab.cam.ac.uk/~grg/preprints.html`.

[4] Harris, M. (2001), Nontrivial phase transition in a continuum mirror model, J. Theoret. Probab. 14, 299–317.

45. Percolation on Random Planar Maps .... Dr J. Miller

A *planar map* is a finite graph together with an embedding in the plane, defined up to continuous deformation. The *faces* of a planar map are the connected components of the complement of its edges. A planar map is called a *triangulation* if each face has exactly three adjacent edges (including the unbounded face). Since there are only a finite number of triangulations with n faces, one can talk about picking one uniformly at random and this is what is known as a random planar map. The study of large uniformly random triangulations have been the focus of a considerable amount of literature in recent years. It is also natural to consider random quadrangulations, pentagulations, or maps with mixed face sizes.

One of the most famous models in statistical mechanics is *percolation*, introduced in the mathematics literature by Broadbent and Hammersley (1957). It is defined by starting with a graph (G, V) and then removing edges at random independently with some probability p. One is then interested in the connectivity properties of the resulting random graph. Percolation is very interesting when the underlying graph is a random planar map.

A successful essay will review the computation of percolation thresholds on random planar maps and discuss more recent developments on scaling limits for percolation.

Relevant Courses

Advanced Probability, Percolation.

References

[1] O. Angel and O. Schramm (2003). Uniform infinite planar triangulations. *Comm. Math. Phys.* **241,** no. 2-3, 191–213.

[2] O. Angel (2003). Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.* **13**, no. 5, 935–974.

[3] O. Angel and N. Curien. Percolations on random planar maps I: Half-plane models. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** (2015), no. 4, 405–431.

[4] M. Laurent and P. Nolin. Percolation on uniform infinite planar maps. *Electron. J. Probab.* 19, no. 79.

[5] N. Curien and I. Kortchemski. Percolation on random triangulations and stable looptrees. *Probab. Theory and Related Fields* **163**, no. 1–2, 303–337.

[6] E. Gwynne and J. Miller. Convergence of percolation on uniform quadrangulations with boundary to SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity. arXiv:1701.05175.

46. Conformal Loop Ensembles .... Dr J. Miller

A *conformal loop ensemble* (CLE) is a random collection of non-crossing loops in a simply connected planar domain D and it is the loop version of the Schramm-Loewner evolution (SLE). Just like SLE describes the scaling limit of a *single* interface in a number of statistical mechanics models (e.g. percolation, the Ising model, uniform spanning trees), CLE describes the scaling limit of *all* of the interfaces. CLE's also turn out to be closely connected to a number of other constructions in probability, such as planar Brownian motion and Brownian loops, as well as the Gaussian free field.

A successful essay will review the construction of CLE and explain its connection to Brownian loops.

Relevant Courses

Advanced Probability, Stochastic Calculus, Schramm-Loewner Evolutions.

References

[1] S. Sheffield (2009). Exploration trees and conformal loop ensembles. *Duke Math. J.* **147**, no. 1, 79–129.

[2] O. Schramm and D.B. Wilson. SLE coordinate changes. *New York J. Math.* 11, 659–669.

[3] S. Sheffield and W. Werner (2012). Conformal loop ensembles: the Markovian characterization and the loop-soup constructon. *Ann. of Math.* **176**, no. 3, 1827–1917.

[4] J. Miller, N. Sun, and D.B. Wilson (2014). The Hausdorff dimension of the CLE gasket. *Ann. Probab.* **42**, no. 4, 1644-1665.

47. The Bayesian Approach to Statistical Inverse Problems .... Professor R. Nickl

In many scientific application areas of modern mathematics, key problems are of 'inverse' type. One may want to recover an image by a 'non-invasive' technique such as a computer tomography scan, try to detect a 'cooling effect' modelled by some potential term in the heat equation, to find the drift of the dynamics of a particle evolving according to some diffusive pattern, or to find the parameters of a discretely observed pure jump process. Real world measurements in such inverse problems typically contain statistical noise, and it is of fundamental importance to solve these inverse problems in a way that is robust to the presence of noise.

A natural approach to such statistical inverse problems is the Bayesian one, which provides a powerful new algorithmic tool in this area. Its computational feasibility has seen much progress, and the references [1,7] contain an overview of many important developments in the area. An intriguing open question is whether these Bayesian algorithms provide objective (prior-free) solutions of these inverse problems in the large sample (or small noise) limit. Using tools from the recently emerged area of Bayesian Nonparametrics, some progress has recently been made, and references can be found in [3,4,5,6]. Particularly for non-linear inverse problems, many open problems remain.

The purpose of this essay is to summarise recent developments in this area in a coherent way. There is a possibility to delve a bit deeper by considering a computational implementation project, or by looking into some theoretical aspects of the proofs of recent results.

Relevant Courses

Some background in Analysis and Statistics will be helpful, basically at the level of the Part II course on Principles of Statistics. Interest in partial differential equations can provide synergies but is by no means a requirement. Some background on (Bayesian) nonparametric statistics is in reference [2].

References

[1] M. Dashti and A. Stuart. The Bayesian approach to inverse problems. In: Handbook of Uncertainty Quantification, Editors R. Ghanem, D. Higdon and H. Owhadi, Springer, 2016. Online at arXiv:1302.6989

[2] E. Giné, R. Nickl, *Mathematical foundations of infinite-dimensional statistical models.* Cambridge University Press, Cambridge (2016).

[3] F. Monard, R. Nickl, G. Paternain, Efficient nonparametric Bayesian inference for X-ray transforms. (with F. Monard, G.P. Paternain). arXiv:1708.06332

[4] R. Nickl, Bernstein-von Mises theorems for statistical inverse problems I: Schrödinger equation. arXiv:1707.01764

[5] R. Nickl, J. Söhl, Bernstein - von Mises theorems for statistical inverse problems II: compound Poisson processes. arXiv:1709.07752

[6] R. Nickl, J. Söhl, Nonparametric Bayesian posterior contraction rates for discretely observed scalar diffusions. Annals of Statistics 45 (2017) 1664-1693.

[7] Stuart, A. M., Inverse problems: a Bayesian perspective. *Acta Numerica* **19** (2010) 451-559.

48. Algorithms and Sufficient Conditions for Faithfulness of Probability Distributions and Graphs ....

Dr K. Sadeghi

In causal inference, one tries to infer *causation* rather than *correlation* in observational or interventional studies; see [2,4]. One of the main assumptions used for causal inference is *faithfulness* [4], which implies that the set of conditional independence statements induced by a probability distribution is exactly the same as the set of separations induced by the graph (that is used for causal inference) as discussed in graphical models [1]. Recently, some necessary and sufficient conditions for faithfulness of probability distributions and graphs have been found [3]. These conditions are a generalization of so-called *graphoid* axioms of Pearl's [2].

This essay should investigate how one can test the known conditions for faithfulness. This can be achieved by finding sufficient conditions that ensure the conditions for faithfulness are satisfied, or by proposing algorithms to test these conditions.

Relevant Courses

Essential: Modern Statistical Methods

References

[1] Lauritzen, S. L., *Graphical models*, Clarendon Press, Oxford, United Kingdom, 1996

[2] Pearl, J., *Probabilistic Reasoning in Intelligent Systems : networks of plausible inference*, Morgan Kaufmann Publishers, San Mateo, CA, USA, 1988

[3] Sadeghi K., *Faithfulness of Probability Distributions and Graphs*, https://arxiv.org/abs/1701.08366, 2017

[4] Spirtes, P. and Glymour, C. and Scheines, R., *Causation, Prediction, and Search, 2nd edition*, MIT press, 2000

49. Manifold Learning .... Dr T. B. Berrett and Professor R. J. Samworth

In many modern applications, it is very common for the dimension of data sets to exceed the sample size. In such circumstances, we can only hope to estimate unknown population quantities if there is an underlying low-dimensional structure to the population. A very natural such structure in applications such as face, speech and handwriting recognition ([1], [2], [3]), human age estimation ([4]) and others is to assume that the data lie on, or close to, a low-dimensional manifold within the high-dimensional ambient space. Such ideas underpin image, video and audio compression algorithms such as MR3, JPEG and MPEG. Manifold learning can be regarded as a generalisation to non-affine spaces of principal components analysis.

Two key challenges in manifold learning are the estimation of the intrinsic dimension and the intrinsic entropy, the latter of which measures the difficulty of data compression and is defined as an integral over the manifold ([5]). Some recent work has also examined the problem of estimating the entire manifold under different noise models ([6]). In addition to surveying this broad literature, an applications-oriented candidate might explore the use of manifold learning techniques in one or more real data problems. A more theoretically-inclined candidate could consider the performance of different estimators of the intrinsic manifold quantities; such estimators may be based on extensions of k-nearest neighbour entropy estimation approaches for Euclidean data ([7]).

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods, Statistical Learning in Practice

References

[1] He, X., Yan, S., Hu, Y., Niyogi, P. and Zhang, H.-J. (2005) Face recognition using Laplacianfaces. *IEEEE Transactions on Pattern Recognition*, **27**, 328–340.

[2] Bregler, C. and Omohundro, S. M. (1995) Nonlinear manifold learning for visual speech recognition. In *Proc. Fifth International Conference on Computer Vision*, 494–499.

[3] Hinton, G. E., Dayan, P. and Revow, M. (1997) Modeling the manifolds of images of handwritten digits. *IEEE Transactions on Neural Networks*, **8**, 65–74.

[4] Guo, G., Fu, Y., Dyer, C. R. and Huang, T. S. (2008) Image-based human age estimation by manifold learning and locally adjusted robust regression. *IEEE Transactions on Image Processing*, **17**, 1178–1188.

[5] Costa, J. A. and Hero, A. O. (2004) Geodesic entropic graphs for dimension and entropy estimation in manifold learning. *IEEE Transactions on Signal Processing*, **52**, 2210–2221.

[6] Genovese, C. R., Perone-Pacifico, M., Verdinelli, I. and Wasserman, L. (2012) Manifold estimation and singular deconvolution under Hausdorff loss. *Ann. Statist.*, **40**, 941–963.

[7] Berrett, T. B., Samworth, R. J. and Yuan, M. (2017) Efficient multivariate entropy estimation via k-nearest neighbour distances. `https://arxiv.org/abs/1606.00304v3`.

50. Data Perturbation for High-Dimensional Statistical Inference .... Dr T. Wang and Professor R. J. Samworth

Traditional statistical models, often based on rather stylised parametric assumptions, are typically seriously misspecified in the context of Big Data, which may be extremely heterogeneous. In such settings, instead of running an algorithm once on the whole data set, an attractive and increasingly popular approach is to apply it to many different perturbations of the original data, and to aggregate the results appropriately. Potential perturbations include subsamples, bootstrap samples, random projections, dropout training and adding artificial noise, among others. This broad class of techniques have been used in a wide range of statistical problems, including variable selection ([1], [2], [3]), high-dimensional classification and regression ([4], [5], [6]), confidence interval construction ([7]), Sparse Principal Components Analysis ([8]) and measurement error ([9]). In some cases, it can be shown that these methods are highly robust to different data generating mechanisms.

One possible approach for this essay would be to survey this range of techniques, and the theoretical guarantees they enjoy. Alternatively, an ambitious candidate might like to explore the use of a data perturbation technique on a new problem of their choice.

Relevant Courses

Essential: Modern Statistical Methods

Useful: Topics in Statistical Theory, Statistical Learning in Practice

References

[1] Meinshausen, N. and Bühlmann, P. (2010) Stability selection. *J. Roy. Statist. Soc., Ser. B (with discussion)*, **72**, 417–473.

[2] Shah, R. D. and Samworth, R. J. (2017) Variable selection with error control: Another look at Stability Selection. *J. Roy. Statist. Soc., Ser. B*, **75**, 55–80.

[3] Barber, R. F. and Candès, E. J. (2015) Controlling the false discovery rate via knockoffs. *Ann. Statist.*, **43**, 2055–2085.

[4] Cannings, T. I. and Samworth, R. J. (2017) Random-projection ensemble classification. *J. Roy. Statist. Soc., Ser. B (with discussion)*, **79**, 959–1035.

[5] Bishop, C. M. (2008) Training with noise is equivalent to Tikhonov regularization. *Neural Computation*, **7**, 108–116.

[6] Wager, S., Wang, S. and Liang, P. S. (2013) Dropout training as adaptive regularization. *Advances in Neural Information Processing Systems*, 351–359.

[7] Kleiner, A., Talwalkar, A., Surkar, P. and Jordan, M. I. (2014) A scalable bootstrap for massive data. *J. Roy. Statist. Soc., Ser. B*, **76**, 795–816.

[8] Gataric, M., Wang, T. and Samworth R. J. (2017) Sparse principal components analysis via random projections. *In preparation.*

[9] Cook, J. R. and Stefanski, L. A. (1994) Simulation-extrapolation estimation in parametric measurement error models. *J. Amer. Statist. Assoc.*, **89**, 1314–1328.

51. Causal Inference with High-Dimensional Data .... Dr R. D. Shah

Causal inferences are often the ultimate goal in statistical analyses. Whilst in medicine and epidemiology, randomised controlled trials are the gold standard for delivering causal conclusions, in many settings they are either infeasible or prohibitively expensive to implement. Performing causal inference on observational data is thus of real importance, and has been the subject of a great deal of research in statistics and related disciplines.

One way of setting up the problem is as follows: we have n units characterised by a pair of *potential* outcomes $(Y_i(0), Y_i(1))$ corresponding to 'control' and 'treatment'. Known to us is the treatment indicator $W_i \in \{0,1\}$ and we observe $Y_i(W_i)$. Also associated with each unit is a vector of covariates $x_i \in \mathbb{R}^p$. One goal may then be to estimate the conditional average treatment effect

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Y_i(1) - Y_i(0)|x_i);$$

the challenge is that we only observe one of $Y_i(0)$ and $Y_i(1)$. There are a variety of approaches to this problem, as well as variations on the problem setup, with some interesting connections to semi-parametric statistical theory and methods for missing data ([1], [2], [3], [4]). Whilst much progress has been made, the big data revolution now poses greater challenges to the field as applications demand causal inferences from high-dimensional data with p large ([5], [6]).

There are a number of approaches to this essay. One approach would be to first review some of the methods designed for the low-dimensional setting, and then move on to describe some of the more recent advances suited to high-dimensional settings, making connections and drawing comparisons (empirically or theoretically) between different methods and ideas. The focus could be either on the theoretical framework and efficiency results or more on the methods and applications. There is also some scope for modifying or extending existing methods, though this is not a requirement.

Relevant Courses

Essential: Modern Statistical Methods

Useful: Topics in Statistical Theory

References

[1] Rosenbaum, P. and Rubin, D. (1983) The central role of the propensity score in observational studies for causal effects. *Biometrika*, **70**, 41–55.

[2] Imbens, G. (2004) Nonparametric estimation of average treatment effects under exogeneity: A review. *The review of Economics and Statistics* **86.1**, 4–29.

[3] Rubin, D. (2005) Causal inference using potential outcomes: design, modeling, decisions. *JASA*, **100**, 322–331.

[4] Van Der Laan, M. and Rubin, D. (2006) Targeted maximum likelihood learning. *The International Journal of Biostatistics*, 2(1).

[5] Athey, S., Imbens, G. and Wager, S. (2016) Approximate residual balancing: De-biased inference of average treatment effects in high dimensions. *arXiv:1604.07125.*

[6] Chernozhukov, V. et al. (2016) Double machine learning for treatment and causal parameters. *arXiv:1608.00060.*

52. New Advances in Multiple Testing ....

Dr R. D. Shah

Problems involving the testing of thousands of hypotheses simultaneously now occur routinely across a variety of scientific disciplines, so much so that the landmark paper introducing the false discovery rate [1] is perhaps the most cited statistics paper of the last 25 years.

Since this seminal work, there have been a number of developments, and multiple testing is currently a highly active area within in methodological statistics. In particular, there is an interesting line of work that looks at providing error control using procedures that work with particular test statistics rather than simply thresholding p-values at an appropriate point ([2], [3], [4]). Other approaches have considered additional natural assumptions on the hypotheses in question [5], or restrictions on the sets of possible rejected hypotheses [6].

This essay could either review some of the innovations in multiple testing that have been introduced in recent years and compare them theoretically and / or empirically. Another option would be to combine or slightly modify existing procedure or ideas with the aim of obtaining improved performance in certain settings.

Relevant Courses

Essential: Modern Statistical Methods

References

[1] Benjamini, Y. and Hochberg, Y. (1995) Controlling the false discovery rate: a practical and powerful approach to multiple testing. *Journal of the Royal Statistical Society, Series B*, **57**, 289–300.

[2] Barber, R. F. and Candes, E. J. (2015) Controlling the false discovery rate via knockoffs. *The Annals of Statistics* **43**, 2055–2085.

[3] Candes, E. et al. (2016) Panning for gold: Model-free knockoffs for high-dimensional controlled variable selection. *arXiv:1610.02351.*

[4] Hemerik, J. and Goeman, J. J. (2017) False discovery proportion estimation by permutations: confidence for significance analysis of microarrays. *Journal of the Royal Statistical Society, Series B*, `doi:10.1111/rssb.12238`

[5] Cai, T. T., Sun, W., and Wang, W. (2016) CARS: Covariate Assisted Ranking and Screening for Large-Scale Two-Sample Inference.
Available at `http://www-stat.wharton.upenn.edu/ tcai/paper/CARS.pdf`.

[6] G'Sell, M. G., et al. (2016) Sequential Selection Procedures and False Discovery Rate Control. *Journal of the Royal statistical society, Series B* **78**, 423–444.

53. Internal Diffusion Limited Aggregation ....

Dr V. Silvestri

Internal Diffusion Limited Aggregation (IDLA) is a model for the growth of a random subset of $\mathbb{Z}^d$, called cluster, by subsequent aggregation of particles. Fix a connected subset $A(0) \subset \mathbb{Z}^d$ to be the initial cluster. At each discrete time $t \geq 1$ a simple random walk is released from a point in the current cluster, and its exit location X_t is added to it, that is $A(t) = A(t-1) \cup \{X_t\}$. This defines a Markov chain in the space of connected subsets of $\mathbb{Z}^d$.

In the classical setting, where $A(0) = \{0\}$ and all random walks are released from the origin, the long term behaviour of IDLA clusters is by now well understood. In 1992 Lawler, Bramson and Griffeath [5] showed that large IDLA clusters have a limiting shape, given by a Euclidean ball. Understanding the fluctuations around this shape remained an open question for many years, until two recent breakthrough works by Asselah and Gaudillière [1] and independently by Jerison, Levine and Sheffield [3], who showed that these are logarithmic in the size of the cluster.

Different rules for the starting positions of the random walks give different but related dynamics, whose behaviour can be very far from the one of classical IDLA. This is believed to be the case if, for example, the next random walk is released close to the exit location of the previous one. Understanding how robust the Euclidean ball theorem is with respect to different starting rules is object of active research.

A first result in this direction was recently obtained by Benjamini, Duminil-Copin, Kozma and Lucas in [2], where they studied IDLA with uniform starting points. This model should be thought of as interpolating between classical IDLA and the case of starting locations close to the boundary. Combining standard IDLA techniques with ideas from First Passage Percolation, they proved that large clusters are again close to Euclidean balls. That is, releasing particles from a uniformly chosen point in the current cluster rather than from the origin does not change the limiting shape.

A successful essay will consist of an overview of the above results, together with detailed proofs of selected theorems.

Relevant Courses

Useful: Advanced Probability

References

[1] Amine Asselah, and Alexandre Gaudillière. Sublogarithmic fluctuations for Internal DLA. *The Annals of Probability*, 41(3A):1160–1179, 2013.

[2] Itai Benjamini, Hugo Duminil-Copin, Gady Kozma, and Cyrille Lucas. Internal diffusion-limited aggregation with uniform starting points. *arXiv preprint arXiv:1707.03241*, 2017.

[3] David Jerison, Lionel Levine, and Scott Sheffield. Logarithmic fluctuations for Internal DLA. *J. Amer. Math. Soc.*, 25(1):271–301, 2012.

[4] Gregory F. Lawler. Subdiffusive fluctuations for internal diffusion limited aggregation. *Ann. Probab.*, 23(1):71–86, 1995.

[5] Gregory F. Lawler, Maury Bramson, and David Griffeath. Internal diffusion limited aggregation. *Ann. Probab.*, 20(4):2117–2140, 1992.

54. The Ant in the Labyrinth ....

Dr P. Sousi

Perform supercritical bond percolation on $\mathbb{Z}^d$ and let $\mathcal{C}_0$ be the cluster containing 0. We now condition on $\{|\mathcal{C}_0| = \infty\}$ and consider a random walk X on $\mathcal{C}_0$ with bias depending on a parameter λ in one direction of $\mathbb{Z}^d$.

The speed of the walk is defined to be the limit as $n \to \infty$ of X_n/n. This is of course a function of the bias λ. The study of this function was initiated in the physics literature by Barma and Dhar [1].

The first rigorous results due to Berger, Gantert and Peres [3] for $d = 2$ and to Sznitman [5] for $d \geq 3$ proved the surprising phenomenon that for high values of the bias, the speed is equal to 0, while for small values of the bias, the speed is positive. This was later strengthened by Fribergh and Hammond [4] who established a phase transition, i.e. there exists a critical value λ_c so that for all biases $\lambda > \lambda_c$, the speed is equal to 0, while below λ_c, the speed is positive. Intuitively, this phenomenon occurs because when the bias is high, the walk gets stuck in traps created by the percolation cluster and spends a lot of its time there. The proof of Fribergh and Hammond [4] reveals a lot of information about the geometry of these traps.

There are many open questions relating to the speed as a function of the percolation probability and the bias. For instance, for a fixed bias, is the speed increasing in the percolation parameter?

A successful essay will give an account of the current state of the art and include proofs (or overviews of proofs) of the important results.

Relevant Courses

Useful: Advanced Probability

References

[1] Mustansir Barma and Deepak Dhar. (1982). Directed diffusion in a percolation network. *J. Phys. C: Solid State Phys.* 16 1451-1458

[2] Gérard Ben Arous and Alexander Fribergh. (2016). Biased random walks on random graphs. *Probability and statistical physics in St. Petersburg*, volume 91 of *Proc. Sympos. Pure Math.*, pages 99–153. Amer. Math. Soc., Providence, RI.

[3] Noam Berger, Nina Gantert and Yuval Peres.(2003). The speed of biased random walk on percolation clusters. *Probab. Theory Relat. Fields.* 126 (2), 221-242.

[4] Alexander Fribergh and Alan Hammond. (2014). Phase transition for the speed of the biased random walk on the supercritical percolation cluster. *Comm. Pure Appl. Math.*, 67(2):173–245.

[5] Alain-Sol Sznitman. (2003). On the anisotropic random walk on the percolation cluster. *Commun. Math. Phys.* 240 (1-2), 123-148.

55. Precision Higgs Mass Predictions in Minimal Supersymmetry .... Professor B. C. Allanach

The Minimal Supersymmetric Standard Model (MSSM) is still regarded by many to be an attractive TeV-scale extension to the Standard Model (however, in this essay, it is not necessary to review supersymmetry or the MSSM at all).

The prediction of a Higgs boson whose properties match those of the experimentally discovered particle is an obvious priority. The calculation of its mass in particular has rather large radiative corrections, and is calculated to a relatively high order in perturbation theory, in various different schemes and approximations. It is a subject of active research as to which scheme or approximation links the Higgs boson mass prediction most precisely to the rest of the model.

The purpose of this essay is to find out and present the issues in the precision Higgs mass prediction in the MSSM, while providing an overall context.

The first half of the essay will set the scene in terms of experimental data for the Higgs boson discovery, and some analytical predictions for the lightest CP even Higgs boson mass prediction in terms of the other parameters of the model. The second half should address the important issues coming from approximations and different schemes, along with current attempts to address them, and their short-comings.

Relevant Courses

Essential: Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory

Useful:

References

[1] D. de Florian *et al.* [LHC Higgs Cross Section Working Group], arXiv:1610.07922 [hep-ph].

[2] P. Kant, R. V. Harlander, L. Mihaila and M. Steinhauser, JHEP **1008** (2010) 104 [arXiv:1005.5709 [hep-ph]].

[3] R. V. Harlander, J. Klappert and A. Voigt, arXiv:1708.05720 [hep-ph].

[4] S. P. Martin, Adv. Ser. Direct. High Energy Phys. **21** (2010) 1 [Adv. Ser. Direct. High Energy Phys. **18** (1998) 1] [hep-ph/9709356].

(and references therein)

56. Edge-Turbulence Interaction and the Generation of Sound .... Dr L. J. Ayton

An unavoidable source of noise occurs when hydrodynamic pressure fluctuations move over a surface and encounter sudden changes to that surface, resulting in the fluctuations refracting and scattering into acoustic waves. A simple example occurs when a turbulent boundary layer above an aerodynamic wing encounters the sharp trailing edge.

This essay will discuss early analytical models of turbulence-edge interaction noise [1-3] then present ideas for how to update these models to include more realistic effects, for example the inclusion of surface roughness which could occur due to the inevitable damage of surfaces over time [4].

Relevant Courses

Useful: Perturbation Methods

References

[1] M. S. Howe (1978) A review of the theory of trailing edge noise. Journal of Sound and Vibration.

[2] B. Noble (1958) Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations.

[3] D. G. Crighton & F. G. Leppington (1970) On the scattering of aerodynamic noise.

[4] Y. Liu, A. P. Dowling & H-C. Shin (2006) Effects of surface roughness on airframe noise. 27th AIAA Aeroacoustics Conference.

57. From Quantum Fluctuations to Galaxies: Non-linear Structure Formation and Perturbative Treatments . . . . Dr T. Baldauf

In the past two decades our picture of the Universe has been dramatically refined by observational campaigns that led to the measurement of the temperature fluctuations in the Cosmic Microwave Background (CMB) and the discovery of the accelerated expansion using supernovae. These new insights led to new ideas about the origin and evolution of the Universe and raised a plethora of new questions. For instance, what are the processes that seeded the rich structures that we observe today and what is causing the Universe to accelerate?

Large Scale Structure (LSS), the distribution of matter and galaxies in the late time Universe has been shown to have the potential to answer some of these questions. For instance, LSS is able to put an upper bound on the total neutrino mass, it can rule out some modifications of gravity and constrain the properties of dark energy models. Finally, deviations from the simplest models for the accelerated expansion and production of quantum fluctuations during inflation can be probed on the largest scales observable in LSS. Extracting these signals from upcoming galaxy and weak lensing surveys is a non-trivial task: galaxies are an imperfect tracer of the non-linear matter distribution and for most of the above mentioned processes, only the imprint on the linear matter distribution is understood to date.

In this essay you will understand how the small density perturbations generated from zero point fluctuations grow to form the filamentary distribution of matter in the late time Universe, how galaxies form in collapsed dark matter haloes and how we can relate the observable distribution of galaxies to the dynamics of the Universe and learn about fundamental physics.

In the first part of the essay you will describe the phenomenology of the system and explore non-linear structure formation in general, whereas in the second part you can focus on one of the following advanced subjects

- redshift space distortions
- non-linear tracer clustering in the threshold and peak models
- effective field theory treatments of tracers of LSS
- large-scale relativistic effects in LSS

Relevant Courses

- *Essential:* Cosmology
- *Useful:* Advanced Cosmology, QFT

References

[1] F. Bernardeau, S. Colombi, E. Gaztanaga and R. Scoccimarro, "Large scale structure of the universe and cosmological perturbation theory", Phys. Rept. **367** (2002) 1 [astro-ph/0112551].

[2] A. J. S. Hamilton, "Linear redshift distortions: A Review", [astro-ph/9708102]

[3] V. Assassi, D. Baumann, D. Green and M. Zaldarriaga, "Renormalized Halo Bias", JCAP **1408** (2014) 056 [arXiv:1402.5916 [astro-ph.CO]].

[4] J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, "The Statistics of Peaks of Gaussian Random Fields", Astrophys. J. 304 (1986) 1561

[5] J. Yoo, A. L. Fitzpatrick, and M. Zaldarriaga, "A New Perspective on Galaxy Clustering as a Cosmological Probe: General Relativistic Effects", Phys. Rev. D80 (2009) 083514, [arXiv:0907.0707]

58. Kent's Relativistic Solution to the Quantum Measurement Problem .... Dr J. N. Butterfield

Kent has proposed, in three recent papers [1], a philosophically realist solution to the quantum measurement problem, that is distinctive in being: (i) relativistic, unlike the pilot-wave theory and dynamical reduction models (in most versions), yet (ii) 'one-world' (unlike the Everett interpretation). The key idea is to choose as a preferred quantity (or 'beable' in John Bell's terminology) the electromagnetic stress-energy distribution on a late-time hypersurface: which one thinks of as registering the arrival on the hypersurface of photons that have scattered off macroscopic bodies at earlier times, so that the stress-energy distribution records the positions of those bodies (including in particular, the positions of pointers on measurement apparatuses). Thus Kent proposes that the actual values of this late-time beable make definite: (i) the values of stress-energy at appropriate earlier spacetime points (by orthodox quantum correlations between the two regions); and thus also (ii) the values of quantities like the position of the centre-of-mass of macroscopic bodies. The aim of the essay is to assess, and if possible develop, this proposal.

There are three natural topics to be addressed:
(A): *Development*: Kent describes various alternative versions of the proposal. He also gives, especially in his second and third papers, toy-models which illustrate how the proposal works. Some of these models use the formalism of photon wave mechanics, as developed by Bialynicki-Birula, Sipe and others. So it is natural to investigate these alternatives, and to develop these models.
(B): *Comparison*: It is natural to compare Kent's proposal with other programmes for solving the measurement problem. The most obvious comparison is with the pilot-wave theory (e.g. [2]), since it also selects a preferred quantity (in most versions, the positions of point-particles) and retains orthodox quantum theory's unitary evolution.
(C): *Non-locality*: It is natural to ask how Kent's framework describes quantum nonlocality: in particular, to ask what are its verdicts for the various locality conditions that are distinguished in the foundational literature. Two well-known conditions are Outcome Independence and Parameter Independence (in Shimony's terminology [3]). Butterfield argues that Kent's proposal violates Outcome Independence (a verdict that agrees with orthodox quantum theory; cf. [3]). But he also argues that the verdict about Parameter Independence remains open: and that settling the matter, for example by giving a toy-model of a Bell experiment, would give an interesting application of two important recent theorems: one by Colbeck, Renner and Landsman [4], and one by Leegwater [5].

Relevant Courses

Essential: None

Useful: Philosophical aspects of quantum field theory.

References

[1]: A. Kent: (1): Solution to the Lorentzian quantum reality problem, *Physical Review A* **90**, 012107; arxiv: 1311.0249; (2014). (2): Lorentzian quantum reality: postulates and toy models, *Philosophical Transactions of the Royal Society A* **373**, 20140241; arxiv: 1411.2957; (2015). (3): Kent, A.: Quantum reality via late time photodetection; arxiv: 1608.04805 (2016).

[2]: P. Holland: *The Quantum Theory of Motion*, C.U.P. 1993; D. Bohm and B. Hiley, *The Undivided Universe*, Routledge 1992.

[3] A. Shimony: (1) Controllable and uncontrollable nonlocality. In: Kamefuchi , S. et al. (eds) *Foundations of Quantum Mechanics in the Light of New Technology*, Tokyo: Physical Society of Japan (1984). (2): Bell's theorem, in *The Stanford Encyclopedia of Philosophy.* Available at: https://plato.stanford.edu/entries/bell-theorem/ (2009).
J. Butterfield: Peaceful Coexistence: examining Kent's relativistic solution to the quantum measurement problem; http://arxiv.org/abs/1710.07844; http://philsci-archive.pitt.edu/14040; forthcoming in *Proceedings of the Nagoya 2015 Conference on Foundations of Quantum Theory*, ed. M.Ozawa et al., Springer.

[4] R. Colbeck and R. Renner: (1) No extension of quantum theory can have improved predictive power, *Nature Communications* **2**, 411. http://dx.doi.org/10.1038/ncomms1416 (2011). (2) The completeness of quantum theory for predicting measurement outcomes, arxiv:1208.4123 (2012).
N. Landsman: (1) The Colbeck-Renner theorem, *Journal of Mathematical Physics* **56**, 122103; (2015). (2): *Foundations of Quantum Theory*, (Chapter 6.6) Springer 2017: freely downloadable anywhere, as a whole, or Chapter by Chapter, from https://link.springer.com/book/10.1007/978-3-319-51777-3

[5]: G. Leegwater: An impossibility theorem for parameter independent hidden variable theories, *Studies in the History and Philosophy of Modern Physics*, **54** 18-34; http://philsci-archive.pitt.edu/12067/; (2016).

59. Spontaneous Symmetry Breaking and Quantum Measurement .... Dr J. N. Butterfield

Spontaneous symmetry breaking (SSB)—roughly speaking, a system's ground state (or equilibrium state at a low enough temperature) not being invariant under a symmetry of the laws—is a very large subject, with many aspects. The foundational and philosophical literature tends to discuss SSB using algebraic formulations of quantum theory, which are well adapted to treating rigorously quantum systems with infinitely many degrees of freedom (such as in quantum field theory and quantum statistical mechanics). In this framework, SSB is a matter of unitarily inequivalent representations of the relevant algebra: cf. e.g. [1].

Recently, Landsman (and coauthors) has used such formulations to analyse spontaneous symmetry breaking in quantum measurement processes. This work represents the quantum measurement problem as an example of the impossibility of SSB in a finite quantum system, i.e. one with finitely many degrees of freedom: such as the systems of interest in a measurement process. Landsman describes how the appropriate limit (as $\hbar \to 0$, or the number N of degrees of freedom goes to infinity) of the ground state (or equilibrium state at a low enough temperature) of such a system is mixed—and does not display the SSB we actually see. He proposes a solution based on the idea that perturbations prevent the bad limiting behaviour, and yield SSB of the appropriate kind in a finite system. The first paper takes a toy model of measurement using a double-well potential [2]; the second considers spin-chains (Ising and Curie-Weisz) [3]. These analyses also show how a classical system, i.e. a commutative algebra of observables, can be a rigorous limit of a sequence of quantum systems (non-commutative algebras): a large theme that is dubbed *asymptotic Bohrification* in Landsman's review [4], and book [5]. The overall view is well summarized in Chapter 11 of [5].

There are various natural questions about this proposal that can be pursued. There are 'external' questions, e.g. about its conception of what the measurement problem really is, and about comparison with other models of measurement (Landsman favours that of Spehner and Haake, e.g. [6]). There are also 'internal' questions. These tend to focus on how best to couple the perturbation that yields the appropriate 'collapsed' state in the apparatus to the measured quantity on the measured system—and how to make the statistics of the perturbation sensitive to the amplitudes, in the system's state, for the various eigenvalues of that quantity. Such questions have been pursued by van Heugten and Wolters [7]. So the aim of the essay is to assess this framework for understanding SSB, and-or for understanding quantum measurement.

Relevant Courses

Essential: None

Useful: Quantum field theory, Statistical field theory, Philosophical aspects of quantum field theory.

References

[1]: F. Strocchi. *Symmetry Breaking*, Lecture Notes on Physics 643 (Springer Berlin Heidelberg 2008); L. Ruetsche *Interpreting Quantum Theories* O.U.P. (2011), Chapters 12-14; D. Baker and H. Halvorson, How is spontaneous symmetry breaking possible? Understanding Wigner's theorem in light of unitary inequivalence, *Studies in the History and Philosophy of Modern Physics* **44**, 464-469 (2013).

[2] N. Landsman and R. Reuvers, A flea on Schrödinger's Cat, *Foundations of Physics* **43** 373407 (2013)

[3] N. Landsman, Spontaneous symmetry breaking in quantum systems: Emergence or reduction? *Studies in History and Philosophy of Modern Physics* **44**, 379394 (2013).

[4] N. Landsman, Bohrification: From classical concepts to commutative algebras. To appear in Faye, J. and Folse, J. (eds.) *Niels Bohr and Philosophy of Physics: Twenty-First Century Perspectives*, London: Bloomsbury; (2017) arXiv:1601.02794.

[5] N. Landsman, *Foundations of Quantum Theory*, Springer 2017: freely downloadable anywhere, as a whole, or Chapter by Chapter, from https://link.springer.com/book/10.1007/978-3-319-51777-3

[6] D. Spehner and F. Haake, Quantum measurements without macroscopic superpositions. *Physical Review A* **77**, 052114 (2008).

[7] J. van Heugten and S. Wolters, Obituary for a flea, *Proceedings of the Nagoya Winter Workshop 2015: Reality and Measurement in Algebraic Quantum Theory*, to appear. Ozawa, M., et al. (ed.) arXiv:1610.06093.

60. Instability and Perturbation Growth in Stratified Shear Flows .... Professor C. P. Caulfield

Stratified shear flows, where both the fluid density and the velocity vary with height, are extremely common both in the environment and in industrial contexts. It is of great practical importance to understand how such flows undergo the transition to turbulence, as turbulence typically hugely increases mixing, transport and dissipation within such flows. It is commonly believed that 'normal' mode flow instabilities play a central role in this transition process, and the conventional argument is that the 'most unstable' normal mode will dominate the nonlinear evolution of the flow, and hence lead the flow to transition. However, the underlying linearized operator is non-normal, and so it is possible for substantial transient growth of perturbations to occur. Although this has been widely studied in unstratified flows, [1] the transient behaviour of stratified flows has been much less-studied. Also, stratified shear flows are prone to multiple, qualitatively different primary and secondary instabilities (particularly when the density distribution develops layers [2]) and it appears that the transition to turbulence is typically associated with **secondary** instabilities which only develop once the primary instability has saturated [3]. There are also several interesting mathematical issues about the 'optimal' measures of perturbation growth to use, as the potential energy as well as the kinetic energy of the perturbation varies in a stratified flow [4], and this essay could approach the general issue of perturbation growth in stratified shear flows from a variety of mathematical and computational directions.

Relevant Courses

Essential:

Hydrodynamic stability.

Useful:

Fluid dynamics of the environment.

References

[1] P. J. Schmid 2007. Non-modal stability theory. *Ann. Rev. Fluid Mech.* **39** 129-162.

[2] C. P. Caulfield 1994. Multiple linear instability of layered stratified shear-flow. *J. Fluid Mech.* **258** 255-285.

[3] W. R. Peltier & C. P. Caulfield 2003. Mixing efficiency in stratified shear flows. *Ann. Rev. Fluid Mech.* **35** 135-167.

[4] A. Kaminski, J. R. Taylor & C. P. Caulfield 2014. Transient growth in strongly stratified shear layers. *J. Fluid Mech.* **758** R4, 12 pages.

61. Quantum Groups and their Physical Applications .... Professor N. Dorey

Quantum groups [1,2] first arose as symmetries of exactly solvable quantum systems such as the Heisenberg spin chain. Mathematically they correspond to a non-commutative deformation or "quantisation" of the algebra of functions on a Lie group. Many aspects of conventional Lie groups and their associated Lie algebras have analogs for quantum groups. Apart from their role in integrable systems, quantum groups have significant applications to rational conformal field theory [2,3] and to topological field theory. In the context of Chern-Simons gauge theory in three dimensions [4], they have applications to the computation of topological invariants of knots and links. The essay should cover basic aspects of the mathematical framework as well as describing one or more physical applications.

Relevant Courses

Essential: Quantum Field Theory; Symmetries, Fields and Particles; *Useful:* Advanced Quantum Field Theory; String Theory.

References

[1] V. Chari and A. Pressley, "A guide to quantum groups," (CUP 1994).

[2] J. Fuchs, "Affine Lie algebras and quantum groups: An Introduction, with applications in conformal field theory," (CUP 1995)

[3] L. Alvarez-Gaume, C. Gomez and G. Sierra, "Duality and Quantum Groups," Nucl. Phys. B **330** (1990) 347.

[4] E. Guadagnini, M. Martellini and M. Mintchev, "Braids and Quantum Group Symmetry in Chern-Simons Theory," Nucl. Phys. B **336** (1990) 581. doi:10.1016/0550-3213(90)90443-H

62. Gravitational Instantons Old and New ....

Dr M. Dunajski

Gravitational instantons are solutions to the Einstein equations in Riemannian signature which give complete metrics whose curvature is concentrated in a finite region of a space-time. The noncompact gravitational instantons asymptotically look like flat space. There are some explicit examples, like the Eguchi Hanson metric, or the Euclidean Schwarzchild solution.

The essay will review the subject, with focus on explicit examples, and the classification of gravitational instantons into ALE, ALF, ALG and ALH classes by different volume growths of geodesic balls

Relevant Courses

Useful: General Relativity

References

[1] Cherkis, S. A. and Kapustin,A. (1999) Singular monopoles and gravitational instantons. Comm. Math. Phys. **203**.

[2] Dunajski, M. (2009) *Solitons, Instantons & Twistors.* Oxford Graduate Texts in Mathematics **19**, Oxford University Press. (Chapter 9)

[3] Gibbons, G. W. and Hawking, S. W. (1979). Classification of Gravitational Instanton symmetries. Comm. Math. Phys. **66** 291310

[4] Minerbe, V. (2010). On the asymptotic geometry of gravitational instantons. Annales scientifiques de l'Ecole Normale Superieure. 883-924

[5] Ward R. S. & Wells R. (1990) Twistor Geometry and Field Theory, CUP (Chapter 6).

63. Lifts of Polytopes .... Dr H. Fawzi

A polytope is the convex hull of a finite number of points. A *lift* of a polytope is a representation of that polytope, call it P, as the projection of another higher-dimensional polytope Q which is "simpler" – typically we want Q to have much fewer facets than P. The idea of lifting plays a crucial role in mathematical optimisation since the existence of a "small" lift means we can efficiently solve optimisation problems over P. Good introductions to the idea of lifts are [1] and [2].

There has been a lot of progress recently in understanding when a polytope has (or not) a small lift. This essay will discuss some of the recent developments in this topic and its connection to some matrix factorisation problems [2]. A possible direction also is to investigate certain particular polytopes, like e.g., cyclic polytopes: these polytopes play an important role in polyhedral combinatorics but their extension complexity (size of smallest lift) is not yet known; see [3,4] for more on these polytopes.

Relevant Courses

Useful: Topics in Convex Optimisation

Knowledge of convex analysis and convex geometry would be useful.

References

[1] V. Kaibel, *Extended Formulations in Combinatorial Optimization*, https://arxiv.org/abs/1104.1023.

[2] J. Gouveia, P. A. Parrilo, R. R. Thomas, *Lifts of Convex Sets and Cone Factorizations.* Math. of O.R., https://arxiv.org/abs/1111.3164.

[3] Y. Bogomolov, S. Fiorini, A. Maksimenko, K. Pashkovich, *Small extended formulations for cyclic polytopes*, Discrete Comput. Geom., https://arxiv.org/abs/1401.8138.

[4] H. Fawzi, J. Saunderson, P. A. Parrilo, *Sparse sums of squares on finite abelian groups and improved semidefinite lifts*, Math. Prog., http://dx.doi.org/10.1007/s10107-015-0977-z.

64. Non-Gaussianity as a Probe of Fundamental Physics .... Dr J. Fergusson

The recent Planck satellite results [1] provide strong evidence for the inflationary Λ-CDM model which has become the standard cosmology. Despite this these are a multitude of possible inflationary models and our ability to distinguish between them using the power spectrum of the Cosmic Microwave Background is largely limited to two parameters, the tensor to scalar ratio, r, and the spectral index, n_s. One area where there is significant hope for improvement is to constrain (or perhaps even measure) non-Gaussian statistics of the primordial cosmological fluctuations generated during inflation. Non-Gaussian correlations encode an enormous amount of information about the physics of the early universe, and provide a powerful tool to discriminate between different microscopic models of inflation.

This essay will explore the prospects for primordial non-Gaussianity to probe models of inflation. For the introductory part of the essay, students will review the key concepts and theoretical underpinnings. Students should understand why self interactions do not give rise to an observable non-Gaussian signature in single field slow roll inflation models [2] and should appreciate the range of inflationary mechanisms which *can* give rise to interesting non-Gaussian phenomenology. (A useful place to start might be the review [3], or the models section of the 2015 Planck non-Gaussianity paper][4] and references therein. Useful reviews with emphasis on observational status and methodology include [5] and [6].

In the main part of the essay, students are free to choose a specific class of non-Gaussianian models and discuss them more detail describing how non-Gaussianity is generated in these models, what shape it takes and what the Planck constraints mean for these types of models.

Relevant Courses

Essential: Cosmology

Useful: Advanced cosmology, QFT, GR

References

[1] Ade, P. A. R. and others, Planck 2015 results. XX.
"Constraints on inflation", [arXiv:1502.02114].

[2] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," JHEP **0305**, 013 (2003) [astro-ph/0210603].

[3] X. Chen, "Primordial Non-Gaussianities from Inflation Models," Adv. Astron. **2010**, 638979 (2010) [arXiv:1002.1416 [astro-ph.CO]].

[4] Ade, P. A. R. and others, "Planck 2015 results. XVII. Constraints on primordial non-Gaussianity", [arXiv:1502.01592].

[5] M. Liguori, E. Sefusatti, J. R. Fergusson and E. P. S. Shellard, "Primordial non-Gaussianity and Bispectrum Measurements in the Cosmic Microwave Background and Large-Scale Structure," Adv. Astron. **2010**, 980523 (2010) [arXiv:1001.4707 [astro-ph.CO]].

[6] E. Komatsu, "Hunting for Primordial Non-Gaussianity in the Cosmic Microwave Background," Class. Quant. Grav. **27**, 124010 (2010) [arXiv:1003.6097 [astro-ph.CO]].

65. β-plane turbulence and jets ....

Professor P. H. Haynes

The β-plane is a mathematical construction that includes the important dynamical effect of spherical geometry on a fluid on the surface of a rotating planet, which is that the vertical component of rotation varies with latitude, but excludes less important purely geometric effects. The simplest form of β-plane dynamics is plane two-dimensional flow with the standard property that relative vorticity ζ is materially conserved in the absence of forcing/dissipation applied instead to the absolute vorticity $\zeta+\beta y$, where β is a constant and y is the north-south coordinate. Turbulent flow on the β-plane, where the 'turbulence' must be forced externally if the flow has no vertical structure, but may be forced by dynamical instabilities when there is vertical variation, has the striking property of self-organisation into alternating westward and eastward jets. There are many naturally occurring examples of this type of behaviour, in the Earth's atmosphere and ocean and in the atmospheres of giant planets.

An essay on this topic should review the dynamics of jet formation in two-dimensional flows where the turbulence is generated by external forcing. A good starting point for a student who chooses this essay would be [1] , a relatively recent research paper which cites several earlier papers such as [2] and [3] (which suggested a 'new paradigm' for jet formation). The later part of an essay might focus on a particular sub-topic, such as ocean jets, e.g. [4], or the application of quasilinear models, e.g. [5]. The equations for 2-D flow on a β-plane are relatively easy to code and an essay might include some research based on numerical simulations. It would be important to choose problems for which the computational resources required are not too large and any student who is potentially interested in this approach is advised to discuss their ideas with the essay setter.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Hydrodynamic Stability, Fluid Dynamics of Climate (Neither is essential.)

References

[1] Scott, R.K, Dritschel, D.G., 2012: The structure of zonal jets in geostrophic turbulence. J. Fluid Mech., 711, 576–598.

[2] Maltrud, M. E. and Vallis, G. K. 1991 Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence. J. Fluid Mech. 228, 321–342.

[3] Baldwin, M. P., Rhines, P. B., Huang, H.-P. and McIntyre, M. E. 2007 The jet-stream conundrum. Science 315, 467?468.

[4] Berloff, P., Karabasov, S., Farrar, J. T. and Kamenkovich, I. 2011 On latency of multiple zonal jets in the oceans. J. Fluid Mech. 686, 534?567.

[5] Srinivasan K., Young, W.R., 2012: Zonostrophic Instability. J. Atmos. Sci., 69, 1633?1656.

66. The Stratospheric Quasi-biennial Oscillation (QBO) .... Professor P. H. Haynes

The QBO is an oscillation of the longitudinal winds in the equatorial stratosphere (the layer of the atmosphere from 15km to 50km). The winds change from westward to eastward to westward with each cycle lasting about 28 months. There are continuous observations of the QBO winds since the early 1950s. (See [1] .) Finding an explanation for the QBO greatly exercised leading atmospheric dynamicists in the 1950s and 1960s. It is now accepted that the QBO is the result of a two-way interaction between waves and longitudinal-mean flow. Wave propagation is associated with momentum transport and therefore potentially gives a force on the mean flow, but also depends on the structure of the mean flow.

A significant first part of an essay on this topic should be devoted to the basic dynamics of the QBO, perhaps using the review [2] as a starting point, but if possible taking account of more recent research and identifying points that might be changed if that review was to be updated now. Important aspects of the dynamics include the types of waves that are important, what causes any asymmetries between eastward phase and westward phase and how the temperature structure of the QBO is related to the wind structure.

A second part might go beyond the basic dynamics to address one or two further topics, such as (i) the effect of the QBO on other parts of the atmosphere [4], (ii) the challenges of representing the QBO in global climate models [5] or (iii) the remarkable disruption of the QBO in early

2016 when a shallow layer of westward winds appeared within a deep layer of eastward winds [6,7].

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Fluid Dynamics of the Environment, Fluid Dynamics of Climate (Neither is essential.)

References

[1] http://www.geo.fu-berlin.de/met/ag/strat/produkte/qbo/qbo_wind_pdf.pdf

[2] Baldwin, M.P., Gray, L.J., Dunkerton, T.J., Hamilton, K., Haynes, P.H., Randel, W.J., Holton, J.R., Alexander, M.J., Hirota, I., Horinuchi, T., Jones, D.B.A., Kinnersley, J.S., Marquadt, C., Sato, K., Takahashi, M., 2001: The quasi-biennial oscillation. Revs. Geophys., 39, 179-229.

[3] Plumb, R. A., The interaction of two internal waves with the mean flow: Implications for the theory of the quasi-biennial oscillation, J. Atmos. Sci., 34, 1847?1858, 1977.

[4] Anstey JA, Shepherd TG. 2014. High-latitude influence of the quasi-biennial oscillation. Q. J. R. Meteorol. Soc. 140: 1?21. DOI:10.1002/qj.2132.

[5] Yao, W., and C. Jablonowski, 2015: Idealized quasi-biennial oscillations in an ensemble of dry GCM dynamical cores. J. Atmos. Sci., 72, 2201?2226, doi:10.1175/JAS-D-14-0236.1.

[6] Osprey, S. M., N. Butchart, J. R. Knight, A. A. Scaife, K. Hamilton, J. A. Anstey, V. Schenzinger, and C. Zhang, 2016: An unexpected disruption of the atmospheric quasi-biennial oscillation. Science, 353, 1424?1427, doi:10.1126/science.aah4156.

[7] Newman, P. A., L. Coy, S. Pawson, and L. R. Lait, 2016: The anomalous change in the QBO in 2015-2016. Geophys. Res. Lett., 43, 8791?8797, doi:10.1002/2016GL070373.

67. Phase Transitions in the Space of Trajectories .... Dr R. Jack

In statistical field theory, one commonly computes averages over all possible configurations of a system (for example, an Ising model). On changing interaction parameters (such as magnetic field or temperature), one may induce (classical) phase transitions, between different states of matter. In theories of soft matter, one may use similar techniques to describe how systems change in time: configurations in d dimensions are replaced by *trajectories* in $d+1$ dimensions, where the extra dimension is that of time. The resulting field theories support several different kinds of *dynamical phase transition.* Here, we concentrate on dynamical transitions that are directly analogous to classical phase transitions, but now in $(d+1)$ dimensions – these have been called “phase transitions in trajectory space” [1]. They are related to certain dynamical rare events that can occur in soft-matter systems, and have implications for glassy materials [2] and for non-equilibrium models [3].

This connection between phase transitions and rare events is related to the probabilistic theory of large deviations. The relevant phase transitions can be studied either within field theory or (more commonly) by analysing directly how the system evolves in time, and estimating probabilities of the relevant rare events. In some cases, the resulting phase transitions have a clear connection to classical phase transitions [2], but in other cases the situation is more subtle [3].

This essay will compare these dynamical phase transitions with their classical counterparts, and (if time permits) with other kinds of dynamical phase transition. A suitable approach might be to discuss the dynamical phase transitions and their relation to large deviation theory, and then to show how equilibrium phase transitions can also be understood within this framework. This should the reveal the similarities and differences between these dynamical phase transitions and their classical counterparts in $d + 1$ spatial dimensions.

Relevant Courses

Essential: Statistical Field Theory

Useful: Theoretical Physics of Soft Condensed Matter, Advanced Probability

References

[1] J. P. Garrahan *et al.*, *"First-order dynamical phase transition in models of glasses: an approach based on ensembles of histories"*, J. Phys. A **42**, 075007 (2009).

[2] L. O. Hedges, R. L. Jack, J. P. Garrahan and D. Chandler, *"Dynamic Order-Disorder in Atomistic Models of Structural Glass Formers"*, Science **323**, 1309 (2009).

[3] R. L. Jack, I. R. Thompson and P. Sollich, *"Hyperuniformity and Phase Separation in Biased Ensembles of Trajectories for Diffusive Systems"*, Phys. Rev. Lett. **114**, 060601 (2015)

68. Adding Weak Stochastic Noise to Deterministic Descriptions of Diffusive Systems . . . . Dr R. Jack

A single Brownian motion is a stochastic process. On the other hand, if we take many particles, each described by an independent Brownian motion, then their density ρ is well-known to evolve by a deterministic diffusion equation: $(\partial\rho/\partial t) = D\nabla^2\rho$. How can it be that the stochastic properties of the particle process do not appear in the diffusion equation?

In fact, the deterministic diffusion equation is not the whole story: as we pass from a stochastic system of particles to a deterministic partial differential equation, we arrive at a theory that can only describe the *typical behaviour* of the density ρ. The full behaviour of ρ should is encoded in a stochastic partial differential equation, with a noise strength that vanishes in the limit where the number of particles (N) is large. Equations of this type are often used in studies of soft matter.

However, making precise statements about the large-N limit is not a trivial matter: in the case of independent Brownian motions, a suitable stochastic PDE was proposed over 20 years ago [1], but the mathematical properties of this equation are not still understood in detail. However, precise statements about the large-N limit are available, both for independent Brownian motions and (sometimes) for systems with many *interacting* particles. This is the content of Macroscopic Fluctuation Theory [2].

This essay will compare the approaches of [1,2], which lead to similar conclusions, but use quite different techniques. It would also be interesting to explore the connections to field-theoretic descriptions of soft matter, or – perhaps – the mathematical properties of the stochastic PDE proposed in [1].

Relevant Courses

Essential: Theoretical Physics of Soft Condensed Matter.

Useful: Stochastic Calculus and Applications, Advanced Probability.

References

[1] D. S. Dean, *"Langevin equation for the density of a system of interacting Langevin processes"*, J. Phys. A **29**, L613 (1996)

[2] L. Bertini *et al.*, *"Macroscopic fluctuation theory"*, Rev. Mod. Phys. **87**, 593 (2015).

69. Quasilinear Approximation to the Navier-Stokes Equations .... Professor R. R. Kerswell

It is now well accepted that the Navier-Stokes equations contain the phenomenon of turbulence yet building an understanding of this state remains elusive. Recently it has become popular to adopt a quasilinear approximation to the Navier-Stokes equations in which certain aspects of its nonlinearity are ignored to gain insight. One motivation for this has been to capture the essential physics of transition to turbulence in the simplest possible framework [1]. Another is to allow a closed set of equations to be derived which govern the evolution of certain statistical properties of a fluid. Here the hope is that by solving these, certain aspects of the flow statistics can be generated directly rather than indirectly via many lengthy (and costly) simulations of the underlying Navier-Stokes equations [2,3]. And yet another motivation has been to focus attention on particular aspects of the Navier-Stokes equations with a view to interpreting experimental data [4]. Very recently, the same approximation is being used to illustrate how exact solutions to the Navier-Stokes equations can be found with ever smaller lengthscales as the Reynolds number increases. There are many possibilities a part III essay could take exploring one or more aspects of this approximation and possibly performing some novel calculations.

Relevant Courses

Essential:

Hydrodynamic Instability

References

[1] Farrell, B.F. and Ioannou, P.J. "Dynamics of streamwise rolls and streaks in turbulent wall-bounded shear flow" *J. Fluid Mech.* **708**, 149-196, 2012.

[2] Tobias, S.M. and Marston, J.B."Direct statistical simulation of out-of-equilibrium jets" *Phys. Rev. Lett.* **110**, 104502, 2013.

[3] Tobias, S.M., Dagon, K. and Marston, J.B. "Astrophysical fluid dynamics via direct statistical simulation" *Ap. J.* **727**, 127-138, 2011.

[4] McKeon, B.J. and Sharma, A.S. "A critical-layer framework for turbulent pipe flow" *J. Fluid Mech.* **658**, 336-382, 2010.

70. Optimal Metric of Douglas–Rachford Splitting Method for Subspaces .. Dr. J. Liang

Douglas–Rachford (DR) operator splitting method was originally proposed in [1] for solving a system of linear equations obtained from the discretisation of a partial differential equation. Later on, the method was generalised to the non-linear case, and nowadays, it is widely used in many disciplines, such as signal/image processing, statistics, and inverse problems.

Consider a very simple problem in $\mathbb{R}^2$ which is finding the intersection of two zero-crossing lines (of course for this case the point is 0, but things will become much more complicated in higher dimension and with affine spaces). If we solve this problem with DR method, then owing to the result of [2], the method will converge linearly with a rate which is the *cosine* of the angle between the two lines (this angle is called Friedrichs angle).

The target of this project is to accelerate the linear convergence rate. The basic idea is to find a linear (possibly shear) transform, after which the angle between the two transformed lines will be as big as possible, hence as faster as possible convergence rate. For this $2D$ example, the ideal linear transform would be the one which makes the two transformed lines perpendicular to each other.

The basic requirement of choosing this essay is a good knowledge of linear algebra, background with convex analysis will be very beneficial. Through this essay, the student will get familiar with the modern optimisation techniques which are widely used in imaging science, inverse problems, statistics and machine learning. Coding is needed for this essay, and the student is expected to have basic skills on using MATLAB.

Relevant Courses

Essential: Topics in Convex Optimisation (M16)

Useful: Numerical Solution of Differential Equations (L24); Inverse Problems (L24);

References

[1] Douglas, J. and Rachford, H.H., 1956. On the numerical solution of heat conduction problems in two and three space variables. Transactions of the American mathematical Society, 82(2), pp.421-439.

[2] Bauschke, H.H., Cruz, J.B., Nghia, T.T., Phan, H.M. and Wang, X., 2014. The rate of linear convergence of the Douglas–Rachford algorithm for subspaces is the cosine of the Friedrichs angle. Journal of Approximation Theory, 185, pp.63-79.

[3] Bauschke, H.H. and Combettes, P.L., 2011. Convex analysis and monotone operator theory in Hilbert spaces (Vol. 408). New York: Springer.

[4] Liang, J., Fadili, J. and Peyr, G., 2017. Local Convergence Properties of Douglas–Rachford and Alternating Direction Method of Multipliers. Journal of Optimization Theory and Applications, 172(3), pp.874-913.

71. The Inflation Consistency Condition .... Dr P. D. Meerburg

One of scientific targets of current and next generation CMB polarization experiments is the possible signature of primordial gravitational waves. A detection could inform us about the energy scales present in the early Universe and possibly provide conclusive evidence for the theory of inflation.

This essays aims to to explore the landscape of inflationary and non-inflationary models and their predictions for production of gravitational waves and to establish a distinction between large and small field models of inflation. Specifically, the students should explore the so-called inflation consistency condition, a relation that connects the power in gravitational waves to the scale dependence of that power. The students should understand under which conditions this relation holds and how it relates to the null energy condition. They are encouraged to explore deviations from this condition in more exotic models of inflation or scenarios in which primordial gravitational waves are not generated by inflationary dynamics (for example in a bounce or through classical production).

A conclusive part of the essay will be for the student to explore the observability of this consistency condition, both in the CMB as well as other possible tracers of primordial gravitational waves.

I recommend the following literature on this topic: a review of the science case [1], [2] , background on the inflation consistency condition [3] , a few examples where this condition is altered [4] , [5] and [6] and possible routes for detection [7] .

Essential: Cosmology

Useful: QFT, Advanced Cosmology, GR

References

[1] Abazajian:2016yjj K. N. Abazajian *et al.* [CMB-S4 Collaboration], arXiv:1610.02743 [astro-ph.CO].

[2] Guzzetti:2016mkm M. Guzzetti, C., N. Bartolo, Liguori, M. and S. Matarrese, Riv. Nuovo Cim. **39**, no. 9, 399 (2016) doi:10.1393/ncr/i2016-10127-1 [arXiv:1605.01615 [astro-ph.CO]].

[3] Dodelson:2014exa S. Dodelson, Phys. Rev. Lett. **112**, 191301 (2014) doi:10.1103/PhysRevLett.112.191301 [arXiv:1403.6310 [astro-ph.CO]].

[4] Baumann:2015xxa D. Baumann, H. Lee and G. L. Pimentel, JHEP **1601**, 101 (2016) doi:10.1007/JHEP01(2016)101 [arXiv:1507.07250 [hep-th]].

[5] Price:2014ufa L. C. Price, H. V. Peiris, J. Frazer and R. Easther, Phys. Rev. Lett. **114**, no. 3, 031301 (2015) doi:10.1103/PhysRevLett.114.031301 [arXiv:1409.2498 [astro-ph.CO]].

[6] Ben-Dayan:2016iks I. Ben-Dayan, JCAP **1609**, no. 09, 017 (2016) doi:10.1088/1475-7516/2016/09/017 [arXiv:1604.07899 [astro-ph.CO]].

[7] Meerburg:2015zua P. D. Meerburg, R. Hlo?ek, B. Hadzhiyska and J. Meyers, Phys. Rev. D **91**, no. 10, 103505 (2015) doi:10.1103/PhysRevD.91.103505 [arXiv:1502.00302 [astro-ph.CO]].

72. Time-reversal imaging in inhomogeneous media . . . . Dr. O. Rath Spivack

In time reversal experiments, a signal emitted from a localised source is recorded, time-reversed, and retransmitted into the medium. The retramsnitted signal than refocuses onto the original source. The refocusing properties are much enhanced in inhomogeneous media, somewhat contrary to the intuitive idea of ”smoothing out” all information through multiple random scattering.

Time reversal acoustics and the phenomena of focusing and ‘super-resolution’ in random media were first documented by Mathias Fink and his group [1] [2]. Time reversal has found many different applications, from early medical applications in lithotripsy, to destroy kidney stones, to medical imaging for cancer detection, underwater acoustics. remote sensing, and enhanced wireless communication. Main applications in imaging are based on three different methods: (a) iterated time reversal using iterations of the appropriate integral operator [3] [4], (b) solution of an appropriate wave equation for the back-propagated wavefield [5] [6] [7], and (c) minimisation of a functional related to the appropriate integral operator [8]. Method (a) is used for the identification of multiple point scatterers, whilst (b) and (c) are used for reconstruction of the initial values of the field inside the medium, and other properties of the medium, from boundary measurements.

After initial developments, when only experimental results and qualitative explanation of the phenomena associated with time reversal were available, more rigorous quantitative theories have been developed, based on the Wigner transform of the wave fields [9] [10].

This essay should provide first a presentation of time-reversal and the qualitative explanation of the phenomena associated with it, then a brief broad summary of quantitative theories and applications, before focusing on any one of

* quantitative theories of ‘super-resolution’ and focusing in random media,

* application to imaging of point scatterers,

* application to the solution of the acoustic inverse problem in photoacoustic imaging.

Relevant Courses

Essential: Basic knowledge of Linear Algebra, and basic knowledge of the wave equation and Green’s functions, from any undergraduate course.

Useful: The Part III courses ‘Direct and Inverse Scattering of Waves’ and ‘Inverse problems ’.

References

[1] M Fink *Time Reversal of Ultrasonic Fields - Part I: Basic Principles*, IEEE Trans. Ultrason. Ferroelectr. Freq. Control. 39, 555 (1992)

[2] M Fink and C Prada *Acoustic time-reversal mirrors* Inverse Problems 17, R1 (2001)

[3] A J. Devaney, E A. Marengo and F K. Gruber *Time-reversal-based imaging and inverse scattering of multiply scattering point targets* J. Acoust. Soc. Am. 118, 3129 (2005)

[4] Z J Waters and P E Barbone *Discriminating resonant targets from clutter using Lanczos iterated single-channel time reversal* J. Acoust. Soc. Am. 131, EL468 (2012)

[5] Y Xu and L V Wang em Time Reversal and Its Application to Tomography with Diffracting Sources Phys. Rev. Lett. 92, 033902 (2004)

[6] Y Hristova, P Kuchment and L Nguyen *Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media* Inverse Problems 24, 055006 (2008)

[7] B E Treeby and B T Cox *A k-space Green’s function solution for acoustic initial value problems in homogeneous media with power law absorption* J Acoust Soc Am. 129 3652 (2011)

[8] L Borcea and I Kocyigit *Imaging in Random Media with Convex Optimization* SIAM J. Imaging Sci. 10, 147 (2017)

[9] G Bal and L Ryzhik *Time Reversal and Refocusing in Random Media* IAM J. Appl. Math. 63, 1475 (2003)

[10] G Papanicolaou, K Solna and L Ryzhik *Statistical Stability in Time Reversal* SIAM J. Appl. Math. 64, 1133 (2004)

73. Extremal Black Holes ....

Professor H. S. Reall

For any time-independent black hole one can define a quantity called the surface gravity. Hawking showed that a black hole emits thermal radiation with temperature proportional to the surface gravity. An extremal black hole is one with vanishing surface gravity. Such black holes do not emit Hawking radiation so they can be quantum mechanically stable. This is why extremal black holes play an important role in attempts to construct a microscopic description of black holes in string theory.

This essay should review some aspects of extremal black hole physics. It might consider one or more of the following topics. 1. Black hole entropy calculations in string theory [1]; 2. Classification of near-horizon geometries of extremal black holes [2]; 3. The classical instability of extremal black holes [3]; 4. The Kerr/CFT correspondence [4].

The essay should be written at a level that would be understood by another Part III student who had attended the Black Holes course.

Relevant Courses

Essential: General Relativity, Black Holes.

Useful: Quantum Field Theory.

References

[1] A. Strominger and C. Vafa, *Phys. Lett.* **B379**, 99 (1996)

[2] H. Kunduri and J. Lucietti, *Living Reviews in Relativity* **16**, 8 (2013)

[3] S. Aretakis, *Adv. Theor. Math. Phys.* **19**, 507 (2015)

[4] G. Compere, *Living Reviews in Relativity* **15**, 11 (2012)

74. T-Duality and the Geometry of String Theory ....

Dr R. A. Reid-Edwards

The spacetime of string theory possesses remarkable properties that cannot be realised in a conventional field theory such as General Relativity. Probably the most famous example is T-duality. In its simplest setting, T-duality is the observation that a string theory describing a background with compact circle directions of radius R is physically equivalent to another string theory, in a very different background, with compact directions of radius R^{-1}. This equivalence between strikingly different descriptions of the same physics provides perhaps the clearest evidence that a deep understanding of string theory requires a radically different perspective on conventional notions of spacetime.

The first part of this essay would provide a derivation, at the worldsheet level, of the duality in a simple setting. The connection with B field transformations and the interpretation of the

$O(D,D;\mathbb{Z})$ symmetry group would then be studied. The essay could then go on to explore a topic in more depth, such as T-duality in the presence of more general background fields and the limitations of various proposals to extend the original construction. The more ambitious author may choose to include a discussion of the geometrisation of the $O(D,D;\mathbb{Z})$ symmetry and could provide an introduction to doubled geometry constructions, wherein the original and dual backgrounds appear on a manifestly equivalent footing. These constructions have been the source of speculations that string theory may be consistently formulated on 'non-geometric' backgrounds which have no analogue in conventional field theories such as General Relativity. An alternative route might choose to focus on the related phenomenon of Mirror Symmetry.

Relevant Courses

Essential:

Part III courses on Quantum Field Theory and and String Theory.

Useful:

Part III courses on Advanced Quantum Field Theory, Supersymmetry and General Relativity.

References

[1] String theory. Vol. 1: An introduction to the bosonic string, J. Polchinski, Cambridge University Press, 1998 - 402 pages

[2] A. Giveon, E. Rabinovici and G. Veneziano, "Duality in String Background Space," Nucl. Phys. B **322** (1989) 167.

[3] C. M. Hull and R. A. Reid-Edwards, "Non-geometric backgrounds, doubled geometry and generalised T-duality," JHEP **0909** (2009) 014.

75. Black Holes in Higher Dimensions ....

Dr J. E. Santos

Gravitational dynamics appears particularly simple in four spacetime dimensions. When the number of spacetime dimensions exceeds four, novel phenomena can occur. For instance, higher-dimensional black holes are not uniquely specified by their asymptotic charges, they can have many topologies, they can be unstable in large portions of their parameter space and play a key role in scenarios where the weak cosmic censorship conjecture appears to be violated.

The essay should do a thorough review of recent discoveries involving higher-dimensional black holes with asymptotically flat boundary conditions, and should be written in a language accessible to other Part III students taking similar courses.

Essential Courses

General Relativity and Black Holes

References

[1] Roberto Emparan and Harvey S. Reall, "Black Holes in Higher Dimensions", Living Rev. Rel. 11 (2008), 6.

[2] G. T. Horowitz, *et. al*, "Black Holes in Higher Dimensions", Cambridge University Press, 2012.

76. Deformation Quantization ....
Dr D. Skinner

Quantum mechanics involves studying self–adjoint operators on Hilbert space. Deformation quantization works instead with the usual Poisson manifold M of a classical system, but seeks a new multiplication law $\star$ for functions $f : M \to \mathbb{C}$ such that $f \star g = fg + O(\hbar)$ and $[f, g] = f \star g - g \star f = \mathrm{i}\hbar\{f, g\} + O(\hbar^2)$ and $\{f, g\}$ is the Poisson bracket. The simplest example of a star product, which works when $M \cong \mathbb{R}^{2n}$ with the standard symplectic structure, is the Moyal product due to Gronewald. The proof that deformation quantization can be carried out on an arbitrary Poisson manifold is due to Kontsevich. Kontsevich gave a formula for his star product that can be interpreted in terms of Feynman graphs, and a formal path integral derivation of his results was obtained by Cataneo & Felder. This essay will explore deformation quantization, understanding in what sense it is and is not the same as quantization,

Relevant Courses

Essential:

Any good course in quantum mechanics, covering path integrals.

Useful:

To explore the Cattaneo–Felder paper, some familiarity with QFT and worldsheet String Theory would be helpful.

References

[1] Kontsevich, M., *Deformation Quantization of Poisson Manifolds*, Lett. Math. Phys. **66** (2003), 157–216.

[2] Cattaneo, A. and Felder, G., *A Path Integral Approach to the Kontsevich Quantization Formula*, Commun. Math. Phys. **212** (2000), 591–611.

[3] Gutt, S., *Variations on Deformation Quantization*, in *Quantization, Deformations and Symmetries* Math. Phys. Stud. **21** (2000), 217–254. Kluwer.

[4] Weinstein, A., *Deformation Quantization*, Astérisque **227** (1994), 389–409.

77. Effective Potentials and Morphological Transitions for Binary Black Hole Spin Precession ....
Dr U. Sperhake

The dynamics of precessing binary systems of spinning black holes in the post-Newtonian regime exhibits a hierarchy of three distinct time scales: the orbital time scale which is much shorter than the precessional time scale which, in turn, is much shorter than the radiation reaction time scale on which the binary orbit shrinks due to emission of gravitational waves. These timescales can be exploited to derive an orbit *and* precession averaged formalism which leads to the identification of three distinct morphologies of spin-precessing black-hole binaries. This essay should give a clear derivation of the precession-averaged formalism as developed in Refs. [1,2]. For this purpose, however,it is not required to derive the post-Newtonian equations of motion – as for example Eqs. (25a,b) in [2] – which,instead, may be assumed as given in the references.

The essay should also discuss the morphologies and the transitions binaries may undergo between these morphologies over the course of the gravitational-wave driven inspiral. A brief discussion of the regime of validity of the formalism should also be included. One prediction by the precession averaged formalism is the existence of a precessional instability in binaries with aligned spins [3]. This instability should be described together with a brief discussion of astrophysical implications. Finally, the essay should present the equal-mass limit of the precession-averaged formalism as described in Ref. [4], highlighting how the binary dynamics differ in this limit from the general, unequal-mass case.

Relevant Courses

Essential:

Useful: General Relativity, Black Holes.

References

[1] M. Kesden, D. Gerosa, R. O'Shaughnessy, E. Berti & U. Sperhake, *Effective potentials and morphological transitions for binary black-hole spin precession*, Phys. Rev. Lett. **114** (2015) 081103; arXiv:1411.0674 .

[2] D. Gerosa, M. Kesden, U. Sperhake, E. Berti & R. O'Shaughnessy, *Multi-timescale analysis of phase transitions in precessing black-hole binaries*, Phys. Rev. D **92** (2015) 064016, arXiv:1506.03492 .

[3] D. Gerosa, M. Kesden, R. O'Shaughnessy, A. Klein, E. Berti, U. Sperhake & D. Trifirò, *Precessional instability in binary black holes with aligned spins*, Phys. Rev. Lett. **115** (2015) 141102; arXiv:1506.091106 .

[4] D. Gerosa, U. Sperhake & J. Vošmera, *On the equal-mass limit of precessing black-hole binaries*, Class. Quant. Grav. **34** (2017) 064004; arXiv:1612.05263 .

78. A Stronger Subadditivity Relation ....
Dr S. Strelchuk

Strong subadditivity of the von Neumann entropy is one of the most fundamental results with numerous applications in Quantum Information Processing [1]. It states that for a tripartite quantum state ρ_{CRB}:

$$S(CB)_\rho + S(RB)_\rho \geq S(CRB)_\rho + S(B)_\rho,$$

where $S(A)_\rho = -\mathrm{Tr}\rho \log \rho$ denotes the von Neumann entropy of the designated system. Alternatively, one may re-write it as $I(C : R|B)_\rho \geq 0$, where $I(C : R|B)_\rho$ is the quantum conditional mutual information evaluated on ρ_{CRB}. In recent years, there were several improvements to this inequality which presented a non-trivial function $g(\rho_{BCR})$, so that $I(B : C|R)_\rho \geq g(\rho_{BCR})$ [2]. One such function is based on the trace distance to the set of separable states [3]. Another refinement connects it with the ability to reconstruct the state from its bipartite reductions: the conditional mutual information is an upper bound on the regularized relative entropy distance between the quantum state and its reconstructed version [4-7].

This essay should discuss these refinements and their applications.

Relevant Courses

Part III Quantum Information Theory (M24) and Quantum Computing (M16) are recommended.

Useful:

The following textbooks may be used for guidance:

a) Nielsen, M. A., Chuang, I. L. (2000). Quantum Computation and Quantum Information. CUP.

b) Wilde, M. M. (2013). Quantum Information Theory. CUP.

References

[1] Lieb, E. H., Ruskai, M. B. (1973). Proof of the strong subadditivity of quantum-mechanical entropy. Journal of Mathematical Physics, 14(12), 1938–1941.

[2] Fawzi, O., Renner, R. (2015). Quantum Conditional Mutual Information and Approximate Markov Chains. Communications in Mathematical Physics, 1-37.

[3] Brandão F. G. S. L., Christandl M., Yard J. (2011). Faithful Squashed Entanglement. Communications in Mathematical Physics 306, 805.

[4] Brandão, F. G. S. L., Harrow, A. W., Oppenheim, J., Strelchuk, S. (2015). Quantum Conditional Mutual Information, Reconstructed States, and State Redistribution. Physical Review Letters, 115(5), 050501.

[5] Sutter, D., Tomamichel, M., Harrow, A. W. (2015). Monotonicity of Relative Entropy via Pinched Petz Recovery Map.

[6] Li, K., Winter, A. (2014). Squashed entanglement, k-extendibility, quantum Markov chains, and recovery maps.

79. The Muon Anomalous Magnetic Moment and Lattice QCD .... Dr C. E. Thomas

Quantum loop effects cause the magnetic moment of the muon to deviate from the value predicted by the Dirac equation and this deviation is parameterized by a quantity called the muon anomalous magnetic moment, a_μ. Comparing the high-precision experimental measurement of a_μ with theoretical calculations provides a stringent test of the Standard Model of particle physics and constraints on new physics. Currently there is a discrepancy of 3.6 standard deviations between experiment and theory – tantalizing but not yet conclusive.

In the Standard Model, a_μ can be expressed in terms of a QED component, an Electroweak component and a hadronic component, a_μ^{Had}. The first two components can be computed in perturbation theory. However, a_μ^{Had}, involving the strongly-interacting regime of Quantum Chromodynamics (QCD), can not and instead a number of methods have been used to relate it to other measurable quantities. Recently, there have been attempts to reduce the uncertainty in a_μ^{Had} by performing first-principles non-perturbative calculations in a numerical approach called lattice QCD.

The essay should give an introduction to the muon anomalous magnetic moment and describe the current theoretical and experimental situation. It should then discuss relevant lattice QCD calculations, focusing mainly on what can be computed and how this can be used as an input to a_μ^{Had} rather than the technical details of lattice QCD.

Relevant Courses

Essential: Quantum Field Theory.

Useful: Advanced Quantum Field Theory; Symmetries, Fields and Particles; Standard Model.

References

Background on the muon anomalous magnetic moment:

[1] Chapter V of J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model*, Cambridge University Press, 2nd edn. 2014.

[2] Review on "*Muon Anomalous Magnetic Moment*" in C. Patrignani et al. (Particle Data Group), *2016 Review of Particle Physics*, Chin. Phys. C, 40, 100001 (2016) [http://pdg.lbl.gov/].

Recent relevant lattice QCD calculations:

[3] Hartmut Wittig at the 34th International Symposium on Lattice Field Theory, 24-30 July 2016 [https://conference.ippp.dur.ac.uk/event/470/session/1/contribution/31].

[4] M. Della Morte *et al.*, JHEP **1710** (2017) 020 [arXiv:1705.01775 [hep-lat]].

[5] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung and C. Lehner, Phys. Rev. Lett. **118** (2017) no.2, 022005 [arXiv:1610.04603 [hep-lat]].

General background on lattice QCD:

[6] H. Rothe, *Lattice Gauge Theories: An Introduction*, World Scientific, 1992.

[7] I. Montvay & G. Münster, *Quantum fields on a lattice*, Cambridge University Press, 1994.

[8] T. Degrand & C. DeTar, *Lattice Methods for Quantum Chromodynamics*, World Scientific, 2006.

[9] C. Gattringer & C. Lang, *Quantum Chromodynamics on the Lattice: An Introductory Presentation*, Springer, 2010.

80. Particle-Vortex Duality and 3d Bosonization .... Professor D. Tong

Sometimes, two seemingly different quantum field theories describe the same low-energy physics. This is often referred to as a duality, but is better thought of as a manifestation of the universality seen in statistical mechanics. A prominent example, known as particle-vortex duality, occurs in d=2+1 dimensional Abelian gauge theories, where the quantum excitations of one theory arise as the vortices of another. Over the past year, it has become clear that there is something deeper underlying this duality, known as “3d bosonization”. This is the idea that one can turn bosons into fermions through the attachment of flux. The purpose of this essay is to review the ideas behind particle-vortex duality and bosonization and to show how they are related.

Relevant Courses

Essential: Quantum Field Theory, Statistical Field Theory, Advanced Quantum Field Theory

Useful: Standard Model

References

[1] Particle-vortex duality was first discovered by Michael Peskin in “*Mandelstam 't Hooft Duality in Abelian Lattice Models*,” Annals Phys. 113, 122 (1978). Impressive numerical evidence was later given in Dasgupta and Halperin, “*Phase Transition in a Lattice Model of Superconductivity*,” Phys. Rev. Lett. 47, 1556 (1981).

[2] A review of particle-vortex duality a can be found in chapter 5 of my lectures on the Quantum Hall Effect, which can be downloaded from http://www.damtp.cam.ac.uk/user/tong/qhe.html.

[3] The relationship between 3d bosonization and particle-vortex duality was described by Karch and Tong in “*Particle-Vortex Duality from 3d Bosonization*”, arXiv:1606.01893 and in Seiberg, Senthil, Wang and Witten, “*A Duality Web in 2+1 Dimensions and Condensed Matter Physics*”, arXiv:1606.01989.

81. Bogomol'nyi Bounds . . . .

Professor P. K. Townsend

In 1976 E.B. Bogomol'nyi showed how the Hamiltonian functional of many non-linear relativistic field theories of physical interest, including some non-Abelian gauge theories, could be written as the sum of a manifestly positive integral and a “topological charge” [1]. For fixed boundary conditions, the absolute value of this charge is a lower bound on the energy, and this “Bogomol'nyi bound” is typically saturated by solutions of a set of *first-order* PDEs. These “Bogomol'nyi equations” imply the second-order field equations but are much simpler and can often be solved exactly. Around the same time, M.K. Prasad and C.M. Sommerfield found exact magnetic monopole and “dyon” solutions of a Yang-Mills Higgs theory in a particular limit, and their stability was then proved by Sommerfield and others by using an argument similar to that of Bogomol'nyi.

In 1978 E. Witten and D. Olive showed that the field theories considered by Bogomol'nyi, Prasad and Sommerfield (BPS) were the “bosonic sectors” of supersymmetric field theories for which the topological charge appears as a central charge in the supersymmetry algebra, which then implies the Bogomol'nyi bound [3]. This new derivation is not limited to topological charges, and any energy bound in terms of central charges appearing in a supersymmetry algebra is now called a BPS bound.

There are many variants of the original Bogomol'nyi bound. The simplest is one in which the energy is bounded by the square root of the sums of squares of several conserved charges (topological or Noether). In another the energy is bounded by the sum of the absolute values of two charges [4]; this type of bound also arises in the context of intersecting domain walls [5], which lead to a tiling of the plane by "almost-BPS junctions" [6], and in a field theory analog of a string ending on a D-brane [7]. In all these cases, there is some maximal supersymmetric extension, in which context, solutions saturating the energy bound preserve some fraction of supersymmetry. The simplest solutions preserve 1/2, and the next simplest 1/4.

Many BPS solitons are solutions of field theories that have a simpler formulation in a higher-dimensional spacetime, in which context they become BPS branes [8]. The topological charge is then a non-central tensor charge [9]. In the context of the maximal supersymmetry algebra in 11 dimensions, these charges are carried by the strings/branes of string/M-theory [10]. Another variant of the Bogomol'nyi bound arises in the context of soliton solutions on the worldvolume of a relativistic p-brane (occcupying some p-plane in a higher-dimensional spacetime) [11, 12].

There are many other facets to Bogomol'nyi bounds. The papers cited here are appropriate to an essay that starts with Bogomol'nyi's paper and reviews selected applications, with some appreciation (but without technical details) of the relation to supersymmetry and of the relevance of the ideas to string/M-theory, but other essays on the same topic are possible, e.g. the connection to minimal surface theory and the mathematics of calibrations.

Relevant Courses

Essential:

Useful: String Theory, Supersymmetry, QFT, Black Holes

References

[1] E.B. Bogomol'nyi, "Stability of Classical Solutions", Soviet J. of Nucl. Phys. **24** (1976) 449.

[2] M.K. Prasad and M.C. Sommerfield, "Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon", Phys. Rev. Lett. **35** (1975) 760.

[3] E. Witten and D. Olive, "Supersymmetry algebras that include topological charges", Phys. Lett. **78B** (1978) 97.

[4] E.R.C. Abraham, "Non-linear sigma-models and their Q-lump solutions", Phys. Lett. **278B**, (1992) 291.

[5] G.W. Gibbons and P.K. Townsend, "A Bogomol'nyi equation for intersecting domain walls", Phys. Rev. Lett. **83** (1999) 1727.

[6] P.M. Saffin, "Tiling with almost-BPS junctions", Phys. Rev. Lett. **83** (1999) 4249.

[7] J.P. Gauntlett, R. Portugues, D. Tong and P.K. Townsend, "D-brane solitons in supersymmetric sigma-models", Phys. Rev. **D63** (2001) 085002.

[8] P.K. Townsend, "Supersymmetric extended solitons", Phys. Lett.**202B** (1988) 53.

[9] E.R.C. Abraham and P.K. Townsend, "Intersecting extended objects in supersymmetric field theories", Nucl. Phys. **B351** (1991) 313.

[10] P.K. Townsend, "M-theory from its superalgebra", Cargèse lectures 1997; hep-th/9712004.

[11] J.P. Gauntlett, J. Gomis and P.K. Townsend, "BPS bounds for worldvolume branes", JHEP 9801 (1983) 003.

[12] P.K. Townsend, "Brane theory solitons", Cargèse lectures 1999; hep-th/0004039

82. Cosmology with Clustering Ratios of Large Scale Structure .... Dr C. Uhlemann

Context. Recent observations of the CMB provide strong evidence for the standard model of cosmology with a cosmological constant and dark matter. However, to reach theoretically interesting benchmarks for alternative models of the early universe and to better pin down cosmological parameters, we need to extract yet more information than two-dimensional CMB maps at one redshift. This information is available within the large-scale structure of matter which probes three-dimensional information across time. In order to extract this information, we need to understand gravitational clustering and disentangle its effect on observables from fundamental physics. One particular property of nonlinear gravitational evolution is the generation of non-Gaussian correlations even for Gaussian initial conditions. The gravitationally induced non-Gaussianity can be captured with hierarchical clustering ratios that can be used to probe cosmological parameters.

Essay. The essay will explore the ability of hierarchical ratios as simple clustering statistics to probe cosmology. The introductory part should review the key theoretical concepts for gravitational collapse in terms of perturbation theory and the spherical collapse model. The students shall understand how hierarchical clustering ratios arise from there and how they can be related to observables in galaxy surveys. (A good starting point are parts of the pedagogical review [1] together with research articles [2] or [3].) In the main part of the essay, students can choose a specific aspect of cosmology and discuss in more detail how clustering ratios can help to constrain associated cosmological parameters (such as total matter density Ω_m, amplitude of density fluctuations σ_8, primordial non-Gaussianity $f_{\rm NL}$, equation of state of dark energy w).

Relevant Courses

Essential: Cosmology

Useful: Advanced Cosmology, General Relativity

References

[1] F. Bernardeau, S. Colombi, E. Gaztanaga, R. Scoccimarro,
"Large-Scale Structure of the Universe and Cosmological Perturbation Theory", Phys.Rept.367:1-248, 2002, arXiv:astro-ph/0112551

[2] F. Bernardeau,"The gravity-induced quasi-Gaussian correlation hierarchy", Astrophysical Journal, v. 390, no. 2, 1992, p. L61-L64.

[3] I. Szapudi, S. Colombi, "Cosmic Error and Statistics of Large-Scale Structure"", Astrophysical Journal v.470, p.131, 1995, arXiv:astro-ph/9510030

83. Spherically Symmetric Models in General Relativity .... Dr Claude Warnick

When faced with a complicated nonlinear PDE a natural thing to do is to first try to understand the most symmetric solutions. In the case of the vacuum Einstein equations of General Relativity, this approach falters as the Birkhoff theorem rules out dynamical spherically symmetric solutions. We can circumvent this problem by introducing a matter field. The simplest such field to consider is a scalar field satisfying the wave equation, but one can also consider more involved matter models such as a charged scalar field interacting with a Maxwell field.

Starting with Christodoulou in the '80s, spherically symmetric models have been used to study many features of General Relativity, including black hole formation and stability, and the related questions of weak and strong cosmic censorship, the stability of the anti-de Sitter spacetime and questions relating to extremal black holes. To get started in the literature, the introduction to [2] is a readable exposition of the basic set-up. [3] includes more general models, and has a fairly comprehensive reference list, [4] gives an example with non-zero cosmological constant.

This essay will give an overview of several results for Einstein's equations in spherical symmetry but should discuss at least one result in greater depth. Possible material to cover includes: reduction of the equations in double-null coordinates; local well-posedness and extension principles; trapped surface formation; weak and/or strong cosmic censorship; black hole stability; non-zero cosmological constant; numerical studies of spherically symmetric models.

Relevant Courses

Essential: General Relativity

Useful: Analysis of PDE, Black Holes

References

[1] "The problem of a self-gravitating scalar field", D. Christodoulou, Comm. Math. Phys. **105**, 337-361 (1986).

[2] "Spherically symmetric space-times with a trapped surface," M. Dafermos, Class. Quant. Grav. **22** (2005) 2221

[3] "The Global structure of spherically symmetric charged scalar field spacetimes," J. Kommemi, Commun. Math. Phys. **323** (2013) 35

[4] "Stability of Schwarzschild-AdS for the spherically symmetric Einstein-Klein-Gordon system," G. Holzegel, J. Smulevici Commun. Math. Phys. **317** (2013) 205

84. Viscous Fingering Instabilities . . . . Dr K. N. Kowal

The interface between two fluids can be made morphologically unstable, resulting in complex pattern formation frequently encountered in porous media and biological systems. Such phenomena are widespread in nature and industry, ranging from crude oil recovery, hydrology, and filtration, to the self-organisation of collective biological systems and medical applications. In most cases, these instabilities occur when a less viscous fluid displaces a more viscous one, for example water displacing syrup, either by injection or by gravity when the interface separates two fluids of different densities. In rectangular geometries, such a system becomes unstable to the formation of a single finger known as the Saffman-Taylor finger. In radial geometries, multiple fingers form by successive tip-splitting. The onset of such instabilities can be examined through linear stability analyses.

The essay should review the mechanism of the instability using linear stability theory. The candidate may choose to investigate various stabilising mechanisms, including the effects of surface tension or mixing at the interface, anisotropy, or the effects of non-Newtonian fluids (e.g. power-law fluids, viscoplastic fluids). A simplified Hele-Shaw geometry may be used. The exact direction of the essay depends on the interests of the candidate.

Relevant Courses

Essential: Undergraduate fluid mechanics

Useful: Slow Viscous Flow

References

[1] Saffman, P. G., Taylor, G. I. (1958) The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. London Ser. A*, 245, 312-329

[2] Homsy, G. M. (1987) Viscous Fingering in Porous Media. *Ann. Rev. Fluid Mech.* 19, 271-311.

[3] Lindner, A., Bonn, D., Meunier, J. (2000) Viscous fingering in a shear-thinning fluid. *Phys. Fluids*, 12, 256-261

[4] Kowal, K. N., Worster, M. G. (2015) Lubricated viscous gravity currents. J. Fluid Mech. 766, 626-655.

85. Basal Conditions and the Dynamics of Large-scale, Glacial Ice Sheets .. Dr K. N. Kowal

Ice sheets are large bodies of ice, such those of Greenland and Antarctica, that slowly deform, or spread, under their own weight. Glacial ice appears to behave as a solid on small length and time scales; however, over large scales and under substantial pressure due to their own weight, ice sheets begin to flow as a viscous fluid, much like the viscous fluids we regularly see and eat, like honey and syrup. As such, understanding the flow of thin films of viscous fluids helps us to understand large-scale ice-sheet dynamics. These dynamics are also strongly affected by what is going on beneath ice sheets. The presence of meltwater and glacial till greatly accelerates the flow and results in rapid ice discharge towards the oceans. Such effects are frequently modelled using appropriate basal boundary conditions, also known as sliding laws, or examined by coupling the flow to an underlying layer of material, such as a less viscous fluid. However, limited access to the underside of ice sheets makes it difficult to predict the exact sliding mechanisms.

The essay should review the fluid mechanics of ice sheets and the influence of different basal conditions on their large-scale dynamics. The candidate may choose to illustrate their discussion by considering simple models of steady, uniform viscous shear flow down an incline using lubrication theory, subject to various basal boundary conditions, for which exact solutions are obtainable. Newtonian rheology may be used. The exact direction of the essay depends on the interests of the candidate.

Relevant Courses

Essential: Undergraduate fluid mechanics

Useful: Fluid Dynamics of the Solid Earth, Slow Viscous Flow

References

[1] Weertman, J. (1957) On the sliding of glaciers. *J. Glaciol.* 3, 33-38.

[2] Boulton, G. S., Hindmarsh, R. C. A. (1987) Sediment deformation beneath glaciers: rheology and geological consequences. *J. Geophys. Res.* 92, 9059-9082.

[3] Schoof, C. , Hewitt, I. (2013) Ice-sheet dynamics. *Annu. Rev. Fluid Mech.* 45, 217-239.

[4] Kowal, K. N., Worster, M. G. (2015) Lubricated viscous gravity currents. *J. Fluid Mech.* 766, 626-655.

[5] Kowal, K. N., Pegler, S.S., Worster, M. G. (2016) Dynamics of laterally confined marine ice sheets. J. Fluid Mech. 790.

86. Flow Past an Obstacle: Small Reynolds Number Asymptotics .... Dr K. N. Kowal

The solution for flow past three dimensional bodies, such as a sphere, at zero Reynolds number is uniformly valid, both close to the body and away from it. However, any attempt at improving it using a regular asymptotic expansion in the Reynolds number fails. This is known as the Whitehead Paradox. The problem becomes more pronounced in two dimensions. One can quickly convince oneself that there exist no solutions to the two-dimensional analogue problem that do not diverge in the far field, for flow past a cylinder at zero Reynolds number. This is known as the Stokes Paradox. The paradoxes can be resolved by realizing that the limit of zero Reynolds number, is not, in fact, regular in an infinite domain, and by, instead, turning to Oseen flow.

The essay should review the paradoxes of Stokes and Whitehead, and discuss the motivation behind their resolution. The candidate may examine the resolution of either problem more fully via matched asymptotic expansions, and discuss the implications for obtaining corrections to the drag on a sphere or cylinder as a function of the Reynolds number.

Relevant Courses

Essential: Perturbation Methods, Undergraduate fluid mechanics

Useful: Slow Viscous Flow.

References

[1] Van Dyke (1964), Perturbation methods in fluid mechanics. *Academic Press.*

[2] Kaplun, Lagerstrom (1957) Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers, *J. Math. Mech.* 6. 585-593

[3] Proudman, Pearson (1957) Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid. Mech.* 2. 237-262

87. D-Modules, Hodge Theory, Representation Theory .... Professor I. Grojnowski

The aim of this essay is to understand the basics of the theory of D-modules, and some concrete applications to either representation theory or the topology of algebraic varieties.

Begin by learning the basic properties of holonomic D-modules on algebraic varieties—Bernstein's lemma on b-functions, and its consequences (the formalism of the six operations). This is the basic language of algebraic geometry and modern reprepresentation theory, and has applications throughout mathematics and physics.

Then, either

1) Prove the Beilinson-Bernstein theorem, describing the category of representations of a semisimple Lie algebra in terms of D-modules on the flag variety. This is an extraordinary result, which generalizes the Borel-Weil-Bott theorem (describing finite dimensional representations in terms of line bundles on the flag variety).

or

2) Study the mixed Hodge structure on D-modules, beginning by computing the Kashiwara-Malgrange filtration on vanishing cycles in some interesting cases. (If you do this, you'll have to learn about Hodge theory, too!).

References

Many textbook expositions of D-modules now exist. The two best are by the originators of the subject—Kashiwara and Bernstein (the latter are printed notes, available on the web somewhere).

The original Beilinson-Bernstein paper is 3 pages long, it is:

A. Beilinson, J Bernstein, Localisation de $\mathfrak{g}$-modules, C. R. Acad. Sci. Paris. 292 (1981), no. 1, 15–18.

but there are many expositions which are probably easier to read; for background on Hodge theory there are Deligne's extraordinary papers:

P. Deligne, Theorie des Hodges II, III.
Inst. Hautes Etudes Sci. Publ. Math. No. 40 (1971), 5–57;
Inst. Hautes Etudes Sci. Publ. Math. No. 44 (1974), 5–77.

P. Deligne, Travaux de Griffiths, Seminar Bourbaki 376, Lecture Notes in Math 180, Springer Verlag 1970, 213–237

88. Nonstandard Analysis . . . . Dr. T. E. Forster

Description of Essay

Developments in C20 Logic have made possible a rigorous treatment of infinitesimals, and thereby opened the door to a natural rational reconstruction of C17 calculus. However the interest of this material is not just historical but emphatically mathematical. Although the C20 discoveries that started this renascence lie in Logic, the Logic involved is not terribly recondite, and the material is accessible also to people in Analysis; it deserves to be more widely known.

The *locus classicus* for C20 Nonstandard Analysis is of course Abraham Robinson *op cit*, but nowadays there are are other approaches to infinitesimals that have become available. Rather than cover all of them the student may prefer to concentrate on just one.

There is a wealth of available literature, most of the essentials of which is listed below.

Relevant Courses

Essential: Part II Logic and Set Theory or equivalent.

Familiarity with Undergraduate Analysis is essential.

References

[1] John Bell "A Primer of Infinitesimal Analysis" CUP

[2] H. Jerome Keisler "Elementary Calculus: An Infinitesimal approach". http://www.math.wisc.edu/~keisler/calc.html

[3] André Pétry "Analyse Infinitesimale—Une Presentation Nonstandard" Céfal 2010

[4] Edward Nelson "Internal set theory" https://web.math.princeton.edu/~nelson/books/1.pdf

[5] Sergio Albeverio, Raphael Hegh-Krohn, Jens Erik Fenstad and Tom Lindstrm, Nonstandard methods in stochastic analysis and mathematical physics

Bull. Amer. Math. Soc. (N.S.) Volume 17, Number 2 (1987), 385-389.

[6] Cutland Neves Oliveira and Pinto "Developments in Nonstandard Mathematics " Longman 1995

[7]; J. Avigad, J. Helzner, Transfer principles for intuitionistic nonstandard arithmetic, Arch. Math. Logic. Archive for Mathematical Logic August 2002, Volume 41, Issue 6, pp 581-602

[8] An Introduction to Nonstandard Real Analysis, Volume 118 (Pure and Applied Mathematics) by Albert E. Hurd, Peter A. Loeb. Academic Press

[9] Non-standard Analysis. By Abraham Robinson. Princeton University Press, 1974.

89. Elastocapillary Coalescence . . . . Professor J. R. Lister

Elastocapillary effects occur when surface-tension forces are large enough to deform elastic structures. Examples include the clumping of wet hair [1] and the deformation of micro-pillar arrays on patterned surfaces [8], and there are many other applications to biology and novel micro-fabrication technologies. The essay should review recent progress in analysing such problems, beginning with a discussion of the natural length scales arising from various balances between surface tension, gravity and elasticity. The review should then address the dynamics of capillary attraction and coalescence, and various mechanisms for determining cluster size. Alternatively, the essay could contain less review element, and instead explore ideas for novel theoretical or numerical modelling of the *dynamics* of capillary attraction between two flexible cylinders, or of 2D extensions to the model of [7].

Relevant courses

Useful: Slow viscous flow

References

[1] Bico, J. et al. 2004 Elastocapillary coalescence in wet hair. *Nature* **432**, 690.

[2] Boudaoud, A., Bico, J. & Roman, B. 2007 Elastocapillary coalescence: Aggregation and fragmentation with a maximal size. *Phys. Rev. E.* **76**, 060102.

[3] Cambau, T., Bico, J. & Reyssat, E. 2011 Capillary rise between flexible walls. *Europhys. Lett.* **96**, 24001.

[4] Duprat, C., Aristoff, J.M. & Stone, H.A. 2011 Dynamics of elastocapillary rise. *J. Fluid Mech.* **679**, 641.

[5] Duprat, C. et al. 2012 Wetting of flexible fibre arrays. *Nature* **482**, 510.

[6] Gat, A.D. & Gharib, M. 2013 Elasto-capillary coalescence of multiple parallel sheets. *J. Fluid. Mech.* **723**, 692.

[7] Singh, K., Lister, J.R. & Vella, D. 2014 A fluid-mechanical model of elastocapillary coalescence. *J. Fluid. Mech.* **745**, 621.

[8] Pokroy, B., Kang, S.H., Mahadevan, L. & Aizenberg, J. 2009 Self-organisation of a mesoscale bristle into ordered hierarchical helical assemblies. *Science* **323**, 237.

[9] Py, C. et al. 2007 3D-aggregation of wet fibers. *EPL* **77**, 44005.

90. The Shape of a (Viscous) Chocolate Fountain .... Professor J. R. Lister

In a chocolate fountain, molten chocolate flows down over a vertical stack of dome-shaped tiers of increasing size (Try googling images for ‘chocolate fountain’). The flow on each tier can be described very simply using lubrication theory. The chocolate flows over the circular edge of each tier to form a thin axisymmetric curtain of falling fluid, which falls until it lands on the next tier below. Observation shows that the curtain contracts inwards as it falls, with radius looking almost linear as a function of height. The theory for an inviscid curtain of fluid is well-established from the study of ‘water bells’, but molten chocolate is viscous!

This essay would likely take the form of a mini-project to calculate the shape of a falling very viscous Newtonian curtain by using the equations of viscous extensional flow, as adapted to the axisymmetric geometry. The equations of axisymmetric shell theory in elasticity would be a useful comparison. Inertia should be neglected, but surface tension is (probably) relevant. The problem differs from tube-drawing in that radial and axial variations are comparable. Further references and guidance are available on request.

Relevant courses

Essential: Slow viscous flow

References

Townsend, A.K. & Wilson, H.J. 2015 The fluid dynamics of the chocolate fountain *Euro. J. Phys.* **37**.

Audoly, B. & Pomeau, Y. 2010 *Elasticity and geometry* OUP

Ribe, N.M. 2002 A general theory for the dynamics of thin viscous sheets. *J. Fluid Mech.* **457**, 255.

91. Volatility Surface Modelling ....

Dr M. R. Tehranchi

Let $C_0(T, K)$ be the time-0 price of a call option with maturity T and strike K. Since call options of many maturities and strikes are liquidly traded, they can be used as hedging instruments. Consequently, there is great practical interest in modelling call prices, either directly or via the Black–Scholes implied volatility surface.

The fundamental theorem of asset pricing says, under some assumptions, that

$$C_0(T, K) = \mathbb{E}[(S_T - K)^+] \tag{1}$$

where $(S_t)_{t\geq 0}$ is a martingale. Note that this function C_0 has the property that it is increasing in the first argument, and is decreasing and convex in the second.

Conversely, a theorem of Kellerer implies that if a given function C_0 in two arguments is increasing in the first, and decreasing and convex in the second then there exists a martingale $(S_t)_{t\geq 0}$ such that equation (1) holds.

This essay will review the recent literature on modelling call prices. One direction is to explore the notion of a peacock and its connection to Kellerer's theorem. Another possibility is to discuss practical issues around popular parametrisations of the volatility surface such as the widely-used SVI.

Relevant Courses

Essential: Advanced Financial Models

Useful: Advanced Probability, Stochastic Calculus & Applications

References

[1] J. Gatheral and A. Jacquier. Arbitrage-free SVI volatility surfaces. *Quantitative Finance* **14**(1): 59–71. (2014)

[2] F. Hirsh, Ch. Profeta, B. Roynette and M. Yor. *Peacocks and Associated Martingales, with Explicit Constructions*. Bocconi & Springer Series. (2011)

[3] M. Tehranchi. A Black–Scholes inequality: applications and generalisation.
`http://arxiv.org/abs/1701.03897` (2017)

92. Polynomial Preserving Processes ....

Dr M. .R. Tehranchi

A real-valued Markov process X is polynomial preserving if the function u defined by

$$u(t, x) = \mathbb{E}[f(X_t)|X_0 = x]$$

is a polynomial in x for all t whenever f is a polynomial. There is growing interest in modelling financial quantities with such processes since the computations involved in pricing certain derivative contracts are reasonably tractable.

This essay will survey the literature on polynomial preserving processes and related variants. Focus can be on the mathematical properties, such as characterisations of their generators, or can be on a particular application in finance, exploring their advantages and disadvantages compared to other modelling frameworks.

Relevant Courses

Essential: Advanced Financial Models

Useful: Advanced Probability, Stochastic Calculus & Applications

References

[1] S. Cheng and M. Tehranchi. Polynomial term structure models.
`http://arxiv.org/abs/1504.03238` (2016)

[2] Ch. Cuchiero, M. Keller-Ressel, and J. Teichmann. Polynomial processes and their applications to mathematical finance. *Finance and Stochastics* **16**: 711-740 (2012)

[3] D. Filipović and M. Larsson. Polynomial preserving diffusions and applications in finance. *Finance and Stochastic* **20**: 931972 (2016)

93. Change and Observables in the Constrained Hamiltonian Formalism . . . Dr J. B. Pitts

Consider the Hamiltonian formulation of General Relativity with a matter field that isn't just a scalar(s). (Feel free to simplify by discarding spatial dependence, arriving roughly at Bianchi I cosmologies.) How is 4-dimensional coordinate covariance (or its most interesting remnant, reparametrization invariance) implemented, along with any gauge freedom of matter? Under what circumstances is there change? How should observables be defined, and how reasonable are the lists of things that do and that do not satisfy this definition? Do observables change? If a simplified model was used, to what extent should these conclusions carry over to full General Relativity?

Relevant Courses

Essential: Hamiltonian General Relativity

Useful: General Relativity

References

Besides references from the essential course, see:

1. K. Sundermeyer, *Constrained Dynamics: With Applications to Yang–Mills Theory, General Relativity, Classical Spin, Dual String Model.* Springer, Berlin (1982).

2. J. M. Pons, D. C. Salisbury, and L. C. Shepley, *Journal of Mathematical Physics* **41** (2000), 5557; gr-qc/9912086.

3. J. M. Pons, *Studies in History and Philosophy of Modern Physics*
36 (2005), 491;physics/0409076v2.

4. K. Sundermeyer, *Symmetries in Fundamental Physics*, second edition. Springer, Heidelberg (2014), Appendix C.

5. J. B. Pitts, *Classical and Quantum Gravity* **34** (2017), 055008,
arXiv:1609.04812 [gr-qc].

6. T. Thiemann, *Modern Canonical Quantum General Relativity.* Cambridge University Press, Cambridge (2007).

94. Highly Oscillatory Quadrature ....

Dr A. Iserles

The main concern of this essay is the computation of integrals of the form

$$\int_\Omega f(x)e^{i\omega g(x)}dV,$$

where f and g are real, smooth functions, $\omega \gg 1$ and Ω is a bounded multivariate domain. Integrals of this form occur in numerous applications of mathematics, e.g. in electromagnetics, fluid dynamics, quantum physics and numerical analysis.

Since the integrand oscillates rapidly, standard methods of integration are useless or, at best, exceedingly expensive. In the last two several algorithms have been proposed, all based on an asymptotic expansion in inverse powers of ω. They all share the (perhaps counterintuitive) feature that *the error decays very rapidly when ω is large!* Although originally proposed for univariate quadrature, these methods are currently available in a multivariate setting. They all exploit the important feature of the above integral, namely that its asymptotic behaviour is in large measure determined by the integrand at *critical points,* a feature of the function f and of the geometry of the underlying domain.

The purpose of the essay is to review these developments. The monograph [1] is a comprehensive review of the subject. The essay will be expected to review briefly the asymptotic theory of highly oscillatory integrals, followed by a detailed exposition of *one* of the following approaches,

1. Extended Filon methods,
2. Numerical steepest descent, or
3. Complex-valued Gaussian quadrature,

based on [1] and references therein.

Relevant Courses

Essential: None

Useful: Numerical solution of differential equations, Perturbation and stability methods

References

[1] Alfredo Deaño, Daan Huybrechs & Arieh Iserles, *Computing Highly Oscillatory Integrals,* SIAM, Philadelphia, January 2018.