What's wrong with standard polynomials?

Polynomials $p(x)$ are very useful approximating functions. They have good approximating properties, by Weierstrass' theorem. They are easy to evaluate, differentiate and integrate. However, they do have drawbacks.

- Polynomials do not have asymptotes—vertical or horizontal;
- Polynomials are finite on the finite real axis and tend to $\pm\infty$ as $x \to \pm\infty$.
- Polynomials have a tendency to oscillate

Rational polynomials

Rational functions $r(x)$ are **ratios of polynomials** and are more versatile. A rational function with equal degree numerator and denominator is bounded as $x \to \pm\infty$. A rational function has poles, i.e. it is infinite for finite values of x (the roots of the denominator polynomial). Rationals can also be free of oscillations.

The theory of best approximation in the L_2 and L_∞ norms can be extended to rational functions but we will explore a different type of approximation. We motivate the idea by thinking first about Taylor series expansions.

Suppose the scalar function f is smooth enough that it has the Taylor series expansion

$$f(x) = \sum_{i=0}^{\infty} f_i x^i, \qquad f_i = \frac{f^{(i)}(0)}{i!}, \tag{1}$$

(about $x = 0$). Then we can truncate the series to obtain the polynomial $P_k(x) = \sum_{i=0}^{k} f_i x^i$ of degree k, which satisfies[1]

$$f(x) - P_k(x) = \sum_{i=k+1}^{\infty} f_i x^i = O(x^{k+1}).$$

The approximation $P_k(x)$ has the property that it agrees with f and its first k derivatives at the origin:

$$f^{(i)}(0) = P_k^{(i)}(0), \qquad i = 0, 1, \dots, k.$$

Looking for rational approximations $r(x)$ to f with the same property as $P_k(x)$, leads us to **Padé approximation**.

Definition The rational function

$$r_{km}(x) = p_{km}(x)/q_{km}(x)$$

is a $[k/m]$ **Padé approximant** of f if p_{km} is a polynomial in x of degree at most k, q_{km} is a polynomial in x of degree at most m with $q_{km}(0) = 1$, and

$$f(x) - r_{km}(x) = O\big(x^{k+m+1}\big). \tag{2}$$

[1] Recall that $f(x) = O(x^k)$ means that there are (usually unknown) constants x_0 and C such that $|f(x)| \le Cx^k$ for all $0 < x \le x_0$.

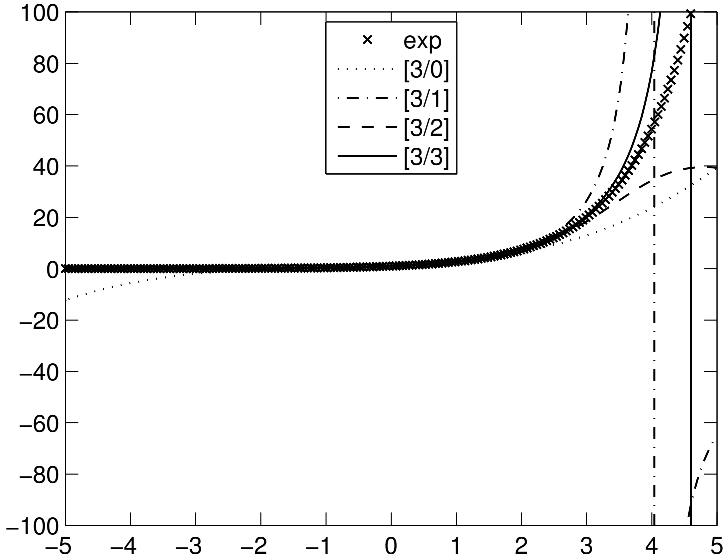

Figure 1: $f(x) = e^x$ and four Padé approximants.

Example Here are some examples of Padé approximants of $f(x) = e^x$:

$$r_{22}(x) = \frac{12 + 6x + x^2}{12 - 6x + x^2}, \qquad r_{13}(x) = \frac{24 + 6x}{24 - 18x + 6x^2 - x^3},$$

$$r_{31}(x) = \frac{24 + 18x + 6x^2 + x^3}{24 - 6x}, \qquad r_{33}(x) = \frac{120 + 60x + 12x^2 + x^3}{120 - 60x + 12x^2 - x^3},$$

$$r_{04}(x) = \frac{1}{1 - x + x^2/2 - x^3/6 + x^4/24}.$$

Note first, as a quick check, that $r_{km}(0) = 1 = e^0$ in each case. Notice also that r_{04} is just the reciprocal of the Taylor polynomial T_4 for e^{-x}.

Figure 1 plots the exponential function along with the Padé approximants r_{km} for $k = 3$ and $m = 0, 1, 2, 3$. We see that the approximations are good near the origin but start to diverge from e^x as $|x|$ gets larger. Indeed, the Padé approximant $r_{31}(x)$ to $f(x) = e^x$ not only diverges from f as x approaches 4, but it becomes infinite.

The Padé approximants to the exponential are known explicitly for all k and m:

$$p_{km}(x) = \sum_{j=0}^{k} \frac{(k+m-j)!\,k!}{(k+m)!\,(k-j)!} \frac{x^j}{j!}, \quad q_{km}(x) = \sum_{j=0}^{m} \frac{(k+m-j)!\,m!}{(k+m)!\,(m-j)!} \frac{(-x)^j}{j!}. \tag{3}$$

These formulae were discovered by Padé in his 1892 thesis.