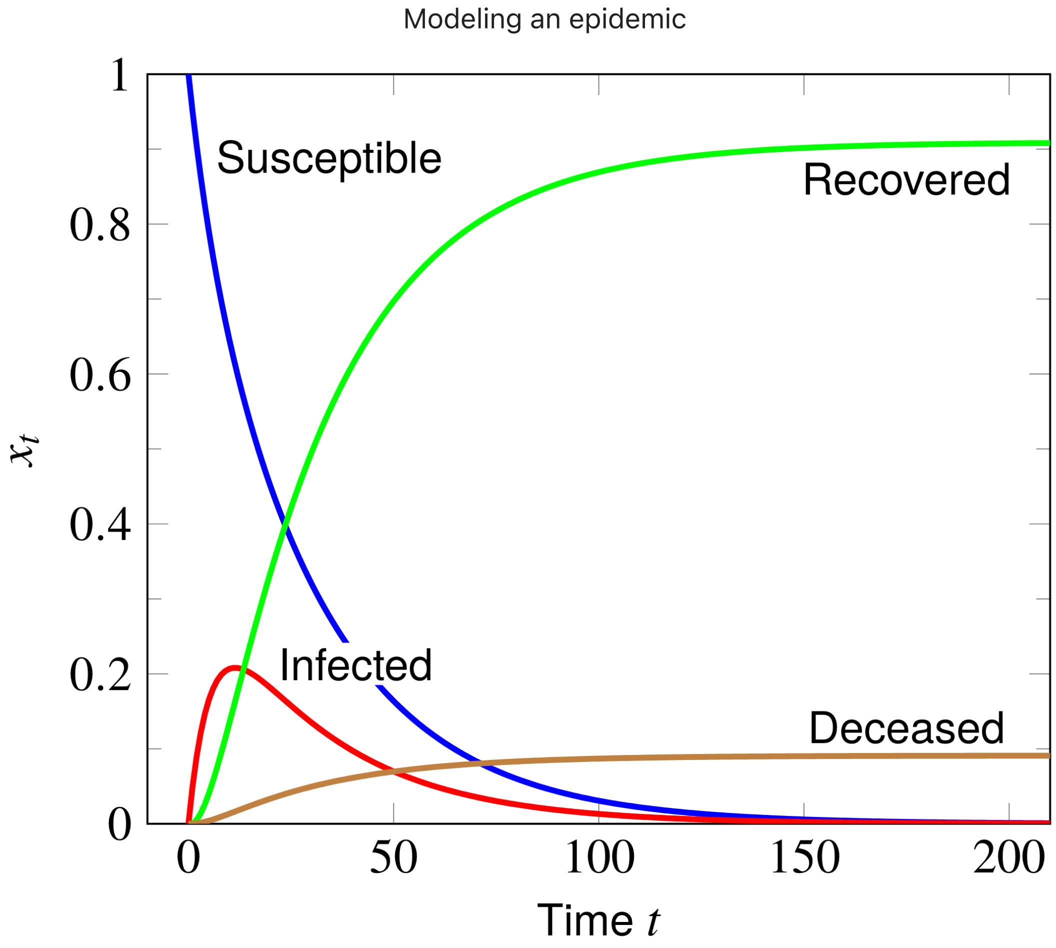
# 08 Linear Dynamical Systems



Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

- ***06 Matrices***
- ***07 Linear Equations***
- 08 Linear Dynamical Systems
- 09 Matrix Multiplication
- 10 Matrix Inverse

Unit 3: Least Squares, Book ILA Ch. 12-14 + Book IMC Ch. 8

Unit 4: Eigen-decomposition, Book IMC Ch. 10, 12, 19

# Exercises

Exercise: Let $0_n$ be the $n \times n$ zeroes matrix and $x$ an $n$-vector. Compute $0_n x$.

Exercise: Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, that is affine. What is the matrix $A$ and the vector $b$ such that: $f(x) = Ax + b$?

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# Outline: 08 Linear Dynamical Systems

- **Linear dynamical systems**
- Epidemic dynamics

# State sequence

Definition: Consider a sequence of $n$-vectors $x_1, x_2, \ldots, x_t, \ldots$ where $t$ denotes the time or period. Then:

- $x_t$ is called a state at time $t$, or a time-point
- the sequence $x_1, x_2, \ldots, x_t, \ldots$ is called a state trajectory or a time-series.

Assuming $t$ is the current time:

- $x_t$ is the current state,
- $x_{t-1}$ is the previous state,
- $x_{t+1}$ is the next state.

# Covid-19 in Santa Barbara

In [2]:

```
import pandas as pd

filename = 'https://raw.githubusercontent.com/CSSEGISandData/COVID-19/master/css
df_covid = pd.read_csv(filename)
df_sb = df_covid[df_covid["Admin2"] == "Santa Barbara"]
df_sb.head()
```

Out[2]:

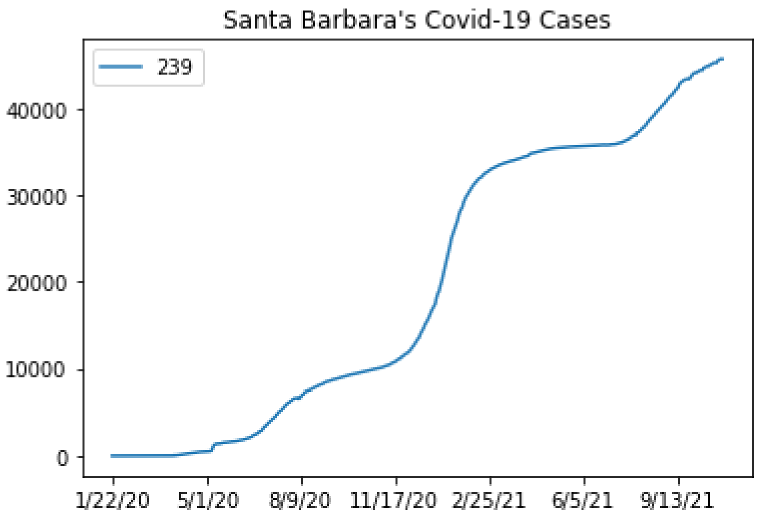
| | UID | iso2 | iso3 | code3 | FIPS | Admin2 | Province_State | Country_Region | Lat |
|---|---|---|---|---|---|---|---|---|---|

|  | UID | iso2 | iso3 | code3 | FIPS | Admin2 | Province_State | Country_Region | Lat |
|---|---|---|---|---|---|---|---|---|---|
| **239** | 84006083 | US | USA | 840 | 6083.0 | Santa Barbara | California | US | 34.653295 |

1 rows × 660 columns

In [71]:

```
df_timeseries = df_sb.drop(
    columns=[
        "Combined_Key", "UID", "iso2", "iso3", "code3",
        "FIPS", "Admin2", "Province_State", "Country_Region",
        "Lat", "Long_"])
df_timeseries.transpose().plot(
    title="Santa Barbara's Covid-19 Cases");
```



# Other Examples

- epidemiology: susceptible, infected, recovered and deaths at $t$
- mechanics: position and velocity of plane through time $t$
- finance: return of $n$ assets at time $t$

Exercise: In these examples, what is the state $x_t$?

# Linear Dynamical System

Definition: A linear dynamical system is a linear equation describing the evolution of a state over time, and written as:

$$x_{t+1} = A_t x_t \quad t = 1, 2, \ldots$$

where:

- $A_t$ are $n \times n$ dynamics matrices,
- $(A_t)_{ij}(x_t)_j$ is contribution to $(x_{t+1})_i$ from $(x_t)_j$.

The system is called time-invariant if $A_t = A$ does not depend on time.

In Python: We can simulate evolution of $x_t$ using for loops to compute:

$$x_{t+1} = A_t x_t$$

recursively (see later slide on the epidemics model).

# Variations: Linear Dynamical System with Input

Definition: The following system is called a linear dynamical system with input:

$$x_{t+1} = A_t x_t + B_t u_t + c_t \quad t = 1, 2, \ldots$$

- $u_t$ is an input $m$-vector
- $B_t$ is $n \times m$ input matrix
- $c_t$ is offset.

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# SIR Model

Definition: The Susceptible-Infected-Recovered (SIR) model of an epidemic is a linear dynamical system defined on a $4$-vector state $x_t$, that gives the proportion of the population in $4$ infection states:

- Susceptible: can acquire the disease the next day
- Infected: have the disease
- Recovered: had the disease, recovered, now immune
- Deceased: had the disease, and unfortunately died

Example of state: $x_t = (0.75, 0.10, 0.10, 0.05)$

Exercise Write the linear dynamical system associated to this epidemic.

- among susceptible population
  - 5% acquires the disease
  - 95% remain susceptible
- among infected population
  - 1% dies
  - 10% recovers with immunit
  - 4% recover without immunity (i.e., become susceptible)
  - 85% remain infected
- 100% of immune and dead people remain in their state

# Simulation from $x_0 = (1, 0, 0, 0)$

In [6]:

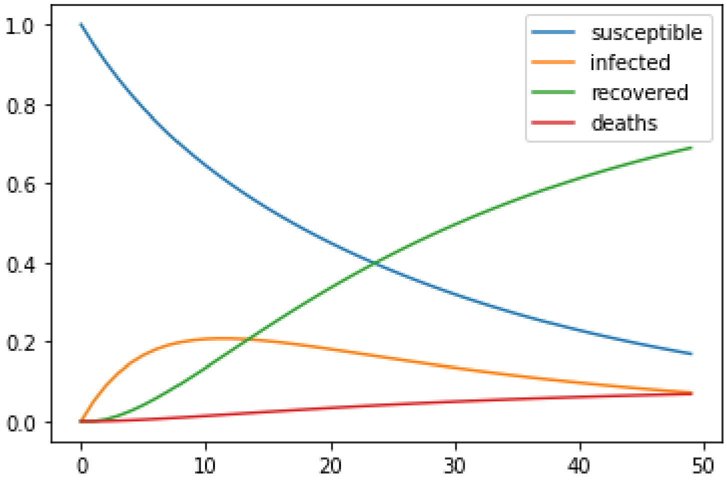
```
import matplotlib.pyplot as plt
import numpy as np
x0 = np.array([1, 0, 0, 0])
A = np.array([[0.95, 0.04, 0, 0],[0.05, 0.85, 0, 0],[0, 0.1, 1, 0],[0, 0.01, 0,
n_times = 50
times = np.arange(0, n_times, 1); #print(times)
x_current = x0 ; states = []
for time in times:
    x_next = A @ x_current
    states.append(x_current)
    x_current = x_next

plt.plot(times, [x[0] for x in states], label="susceptible")
plt.plot(times, [x[1] for x in states], label="infected")
plt.plot(times, [x[2] for x in states], label="recovered")
plt.plot(times, [x[3] for x in states], label="deaths")

plt.legend()
```

Out[6]:

```
<matplotlib.legend.Legend at 0x126728d60>
```



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Resources: Book ILA Ch. 09