06 Matrices

Class Survey

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Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

- ***06 Matrices***
- 07 Linear Equations
- 08 Linear Dynamical Systems
- 09 Matrix Multiplication
- 10 Matrix Inverse

Unit 3: Least Squares, Book ILA Ch. 12-14 + Book IMC Ch. 8

Unit 4: Eigen-decomposition, Book IMC Ch. 10, 12, 19

Outline: 06 Matrices

- **Matrices**
- Matrix-vector multiplication
- Examples

Matrices

Definition: A matrix is a rectangular array of numbers, e.g.:

$$A = \begin{bmatrix} 0 & 1 & -2.3 \\ 1.3 & 4 & -0.1 \end{bmatrix}$$

- Its size, or shape, is: (row dimension) x (column dimension).
  - Example: Matrix above has size 2 x 3.
- Its elements are called: entries, coefficients.
- $A_{i,j}$ refers to element at ith row and jth column in matrix A.
  - i is the row index and j is the column index.

In Python, we use `numpy` and `np.array` to build matrices. The shape of the matrix can be accessed via the function `shape`.

In [7]:
```
"Implement matrix A and returns its shape"
```

Out[7]:
```
'Implement matrix A and returns its shape'
```

In Python, we can access the elements of the matrix.

In [8]:
```
"Access and print elements of vectors."
"Access and print elements of a matrix"
```

Out[8]:
```
' Access and print elements of a matrix'
```

Sizes/Shapes of Matrices

Definitions: A m x n matrix A is:

- tall if m > n,
- wide if m < n,
- square if m = n.

Matrices, Vectors and Scalars

Definitions:

- A 1 x 1 matrix is a number or scalar.
- A n x 1 matrix is an n-vector.
- A 1 x n matrix is a n-row-vector.

Starting now, we will distinguish vectors and row vectors.

Columns and rows of a matrix

Notations: Take A a $m \times n$ matrix with entries A_{ij} for $i = 1, \ldots, m$ and $j = 1..., n$.

- Its jth column is the m-vector:

$$\begin{bmatrix} A_{1j} \\ \dots \\ A_{mj} \end{bmatrix}$$

- Its ith row is the n-row-vector: $[A_{i1}, \dots, A_{in}]$.

Slices of a matrix

Definition The slice of matrix $A_{p:q,r:s}$ is the matrix:

$$\begin{bmatrix} A_{pr} & A_{p,r+1} & \dots & A_{ps} \\ \dots & \dots & \dots & \dots \\ A_{qr} & A_{q,r+1} & \dots & A_{qs} \end{bmatrix}$$

In Python, we can extract rows, columns and slices:

In [9]:

```
"""Extract rows, columns and slices from matrix A."""
```

Out[9]:

```
'Extract rows, columns and slices from matrix A.'
```

Block matrices

Definition: A matrix A composed from other matrices is called a block matrix:

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

where B, C, D, E are called submatrices or blocks of A.

Example: Build a block-matrix.

Column and row representation of matrix

Notations: Take A a $m \times n$ matrix with entries A_{ij} for $i = 1, \dots, m$ and $j = 1..., n$.

- A is the block matrix of its columns $a_1, \dots, a_n$:
  - $A = [a_1 \dots a_n]$
- A is the block matrix of its rows $b_1, \dots, b_m$:
  - $A = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

Examples in ECE and beyond

- Images: A_{ij} is intensity value at i, j.
- Weather: A_{ij} is rainfall data at location i on day j.
- Finances: A_{ij} is the return of asset i in period j

Exercise: In each of these, what do the rows and columns mean?

Special Matrices

Definition: The $m \times n$ zero matrix (resp. ones-matrix) is the matrix with all entries equal to 0 (resp. to $!$).

Definition: The identity matrix I is the square matrix with $I_{ii} = 1$ and $I_{ij} = 0$ if $i \neq j$, for example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In Python:

In [11]:

```
"""Code zeroes matrices, ones matrices and the identity matrix"""
```

Out[11]:

```
'Code zeroes matrices, ones matrices and the identity matrix'
```

Diagonal Matrices

Definition: A diagonal matrix A is a square matrix with $A_{ij} = 0$ for $i \neq j$.

- diag$(a_1, \ldots, a_n)$ denotes the diagonal matrix with $A_{ii} = a_i$, for example:

$$\text{diag}(0.2, -3, 1.2) = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}$$

In Python:

In [12]:

```
np.diag([0.2, -3, 1.2])
```

Out[12]:

```
array([[ 0.2,  0. ,  0. ],
       [ 0. , -3. ,  0. ],
       [ 0. ,  0. ,  1.2]])
```

Triangular Matrices

Definition: A lower triangular matrix A is a matrix such that $A_{ij} = 0$ for $i < j$. An upper triangular matrix A is a matrix such that $A_{ij} = 0$ for $i > j$.

Example: $\begin{bmatrix} 0.2 & 1.2 & 10 \\ 0 & -3 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}$ (upper-triangular)

Transpose

Definition: The transpose of an $m \times n$ matrix A is written A^T and is defined by:

$$(A^T)_{ij} = A_{ji}, \quad i = 1, \ldots, n \quad j = 1, \ldots, m$$

Example: $\begin{bmatrix} 0.2 & 1.2 & 10 \\ 0 & -3 & 0 \end{bmatrix}^T = \begin{bmatrix} 0.2 & 0 \\ 1.2 & -3 \\ 10 & 0 \end{bmatrix}$

In Python:

In [10]:
```
"""Transpose a vector or a matrix."""
```

Out[10]:
```
'Transpose a vector or a matrix.'
```

Addition, Substraction and Scalar Multiplication

Just like vectors:

- we can add or subtract matrices of the same size
- we can multiply a matrix by a scalar.

Property: The transpose verifies:

- $(A^T)^T = A$.
- $(A+B)^T = A^T + B^T$.

Matrix norm

Definition: For a $m \times n$ matrix A, we define the matrix norm as:

$$||A|| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}.$$

Remark: This definition agrees with the definition of norm of vectors when $n = 1$ or $m = 1$.

Distance between two matrices

Definition: The distance between two matrices A and B is defined as:

$$dist(A, B) = ||A - B||.$$

Outline: 06 Matrices