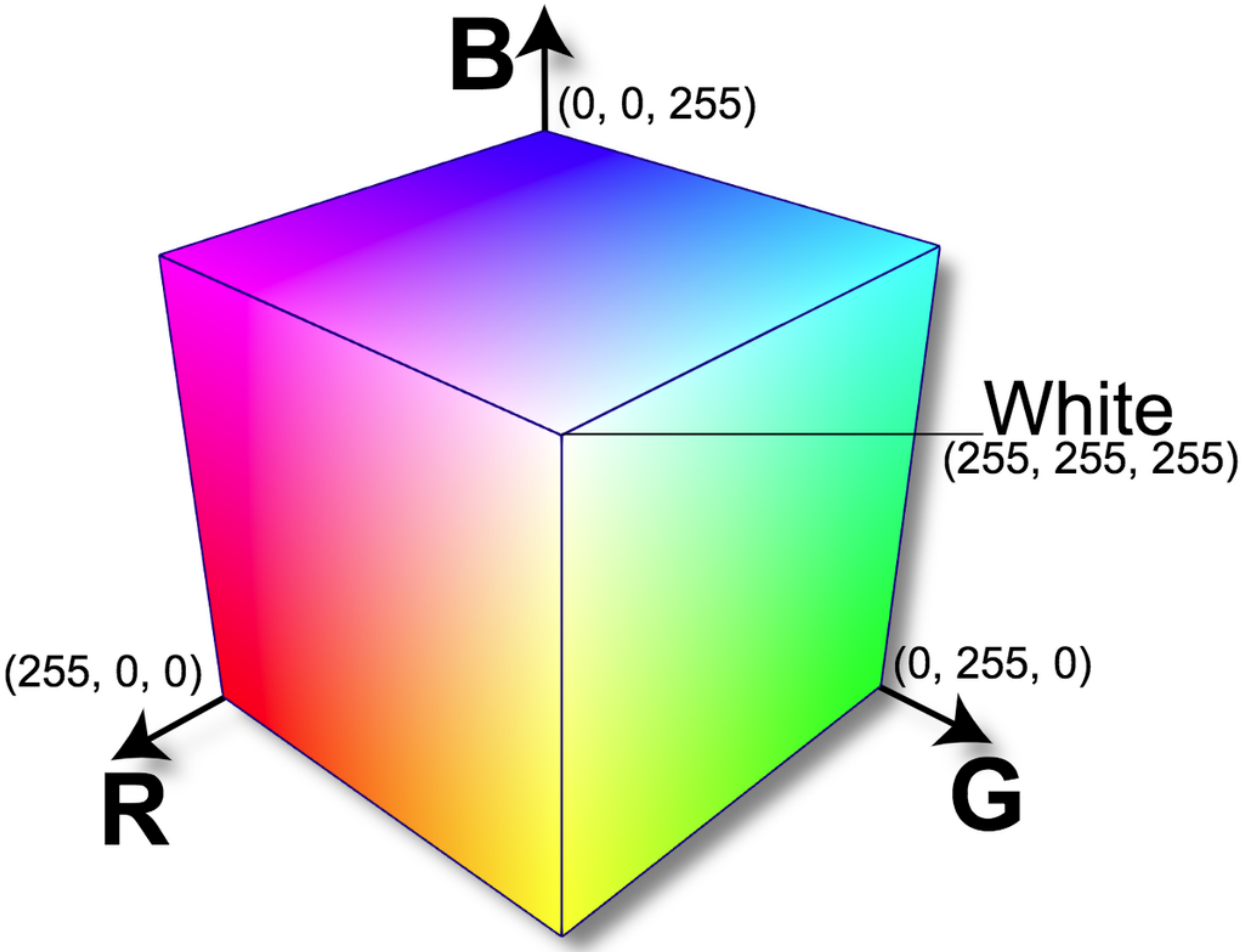
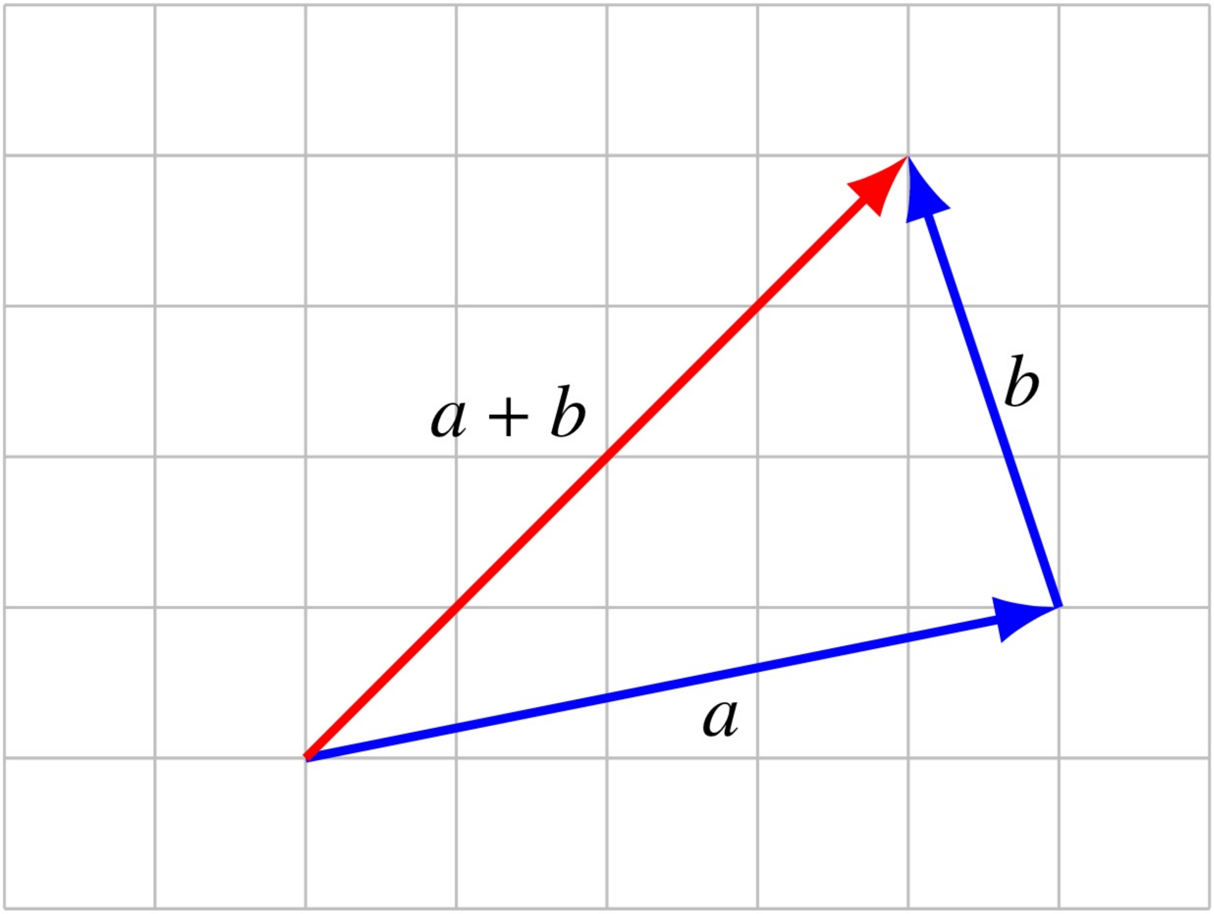
01 Vectors





Outline for ECE 3

Unit 1: Vectors, Book ILA Ch. 1-5

- ***01 Vectors***
- 02 Linear Functions
- 03 Norms and Distances
- 04 Clustering
- 05 Linear Independence

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

Unit 3: Least Squares, Book ILA Ch. 12-14 + Book IMC Ch. 8

Unit 4: Eigen-decomposition, Book IMC Ch. 10, 12, 19

Outline: 01 Vectors

- First definitions and notations
- Examples
- Addition, subtraction and scalar multiplication

- Inner product
- Complexity

Outline: 01 Vectors

- **First definitions and notations**
- Examples
- Addition, subtraction and scalar multiplication
- Inner product
- Complexity

Vector: Definition

Definition A *vector* is an ordered list of numbers, written as:

$$\begin{pmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{pmatrix} \text{ or } \begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix} \text{ or } (-1.1, 0.0, 3.6, -7.2).$$

- The numbers in the list are called: *components, elements, entries, or coefficients* of the vector.
- The number n of elements in the list is called: *size, dimension, or length* of the vector.
- If a vector has n elements, it is called a n-vector.

Exercise: What are the components of the vector above? What is its dimension?

Definition: In contrast to vectors, numbers are just called *scalars*. For example, 3.4 is a scalar.

Vector: Notations

Notation:

- We use symbols to denote vectors, e.g., $a, X, p, \beta, E^{\text{aut}}, \mathbf{g}, \vec{a}$.
- The ith element of n-vector a is denoted a_i.
- In a_i, the i is the index.

Remarks: What is really "a_i"?

- in Math: for an n-vector, indexes run from $i = 1$ to $i = n$,
- 😱 in Python: for an n-vector, indexes run from $i = 0$ to $i = n - 1$,

- 😱 Sometimes, a_i refers to the ith vector in a list of vectors.

Definition: Two vectors a and b of the same size n are equal if:

$$a_i = b_i \text{ for all } i \text{ in } \{1, \ldots, n\}.$$

Vector: Python

In Python, vectors can be represented as:

- a list of numbers, using `[]` ,
- a tuple of numbers, using `()` ,
- an array of numbers with the package NumPy.

In [4]:
```
a = [-1.1, 0.0, 333.6, -7.2]; print(a)
b = (-1.1, 0.0, 3.6, -7.2); print(b)
```

```
[-1.1, 0.0, 333.6, -7.2]
(-1.1, 0.0, 3.6, -7.2)
```

In [7]:
```
import numpy as np
c = np.array([-1.1, 0.0, 3.6, -7.2]); c
```

Out[7]:
```
array([-1.1,  0. ,  3.6, -7.2])
```

The size/length/dimension of a vector is computed with `len` .

In [8]:
```
len(a), len(b), len(c)
```

Out[8]:
```
(4, 4, 4)
```

Vector components: Python

In Python, we can "access" the components of a vector with the following syntax.

In [10]:
```
a[0], b[0], c[0]
```

Out[10]:
```
(-1.1, -1.1, -1.1)
```

😱- Be careful that the first element is indexed 0, and the last is indexed n - 1.

In [11]:
```
a[4]
```

```
---------------------------------------------------------------------------
IndexError                                Traceback (most recent call last)
/var/folders/dz/k1hb2xr94k558sjs416njdp40000gn/T/ipykernel_22722/3944406842.py i
n <module>
----> 1 a[4]
```

```
IndexError: list index out of range
```

Vector components: Python

In Python, the components of a vector can be "assigned" (except for the tuple 😱).

In [12]:
```
a[2] = 4.0; c[2] = 4.0
print(a); c
```

```
[-1.1, 0.0, 4.0, -7.2]
```

Out[12]:
```
array([-1.1,  0. ,  4. , -7.2])
```

In [7]:
```
b[2] = 4.0
b
```

```
---------------------------------------------------------------------------
TypeError                                 Traceback (most recent call last)
/var/folders/dz/k1hb2xr94k558sjs416njdp40000gn/T/ipykernel_21476/1212393371.py i
n <module>
----> 1 b[2] = 4.0
      2 b

TypeError: 'tuple' object does not support item assignment
```

😱 Special Warning with NumPy Arrays

If we:

- assign a vector a to another new vector y,
- change the components of a,

Then the components of y will be changed.

The assignment of *arrays* does not copy the original array to a new one, it creates a *reference* to the same values.

In [16]:
```
y = a; print(a, y)
y[2] = 666; print(a, y)
```

```
[-1.1, 0.0, 4.0, 2023] [-1.1, 0.0, 4.0, 2023]
[-1.1, 0.0, 666, 2023] [-1.1, 0.0, 666, 2023]
```

To avoid this problem, use the `.copy` syntax:

In [17]:
```
y = a.copy(); print(a, y)
a[3] = 202222; print(a, y)
```

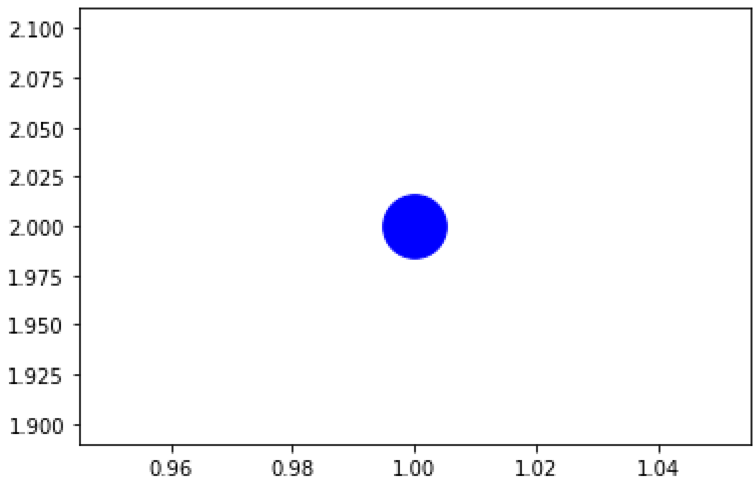
```
[-1.1, 0.0, 666, 2023] [-1.1, 0.0, 666, 2023]
[-1.1, 0.0, 666, 202222] [-1.1, 0.0, 666, 2023]
```

Visualization of vectors

In Python, we can use the package `matplotlib` to plot vectors.

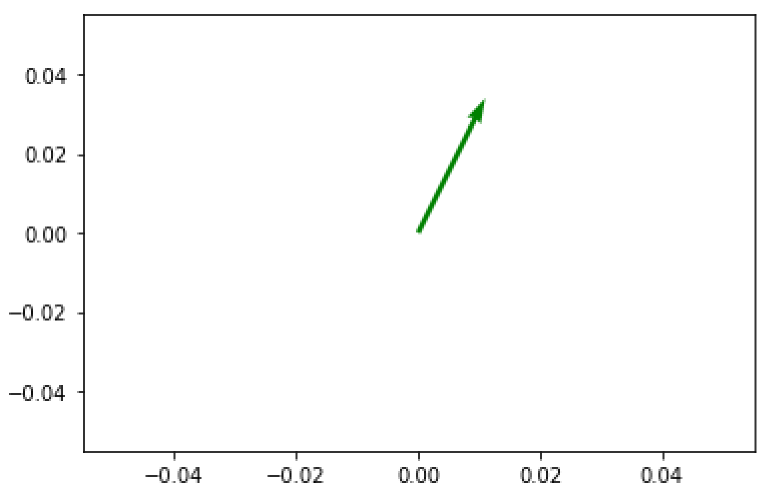
The function `scatter` only plots a point that represents the end tip of the vector.

In [20]:
```
import matplotlib.pyplot as plt

vector_2d = [1, 2]
plt.scatter(vector_2d[0], vector_2d[1], color='blue', s=1000);
```



The fonction `quiver` can actually plot a vector.

In [25]:
```
origin = [0, 0]

plt.quiver(origin[0], origin[1], vector_2d[0], vector_2d[1], color='green', scal
```



Block or Stacked Vectors

Definition: Suppose a, b, c are vectors with sizes m, n, p. We can create a new vector d as:

$$d = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

The vector d is called a *block vector or a stacked vector with entries* a, b, c, or simply the concatenation of a, b, c. d has size $m + n + p$ with the following components:

$$d = (a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p).$$

Block or Stacked Vectors

In Python, we can compute a block vector using `concatenate` .

In [36]:

```
a = np.array([1, -1]); b = np.array([2, -2, -2.2]); c = np.array([3, 3.3])
d = np.concatenate([a, b, c]); d
```

Out[36]:

```
array([ 1. , -1. ,  2. , -2. , -2.2,  3. ,  3.3])
```

Zero, Ones and One-Hot Vectors

Definition: The n-vector with all entries 0 is denoted 0_n or just 0 and is called a zero vector. The n-vector with all entries 1 is denoted 1_n or just 1 and is called a ones-vector.

Definition: A one-hot vector is a vector which has one entry 1 and all others 0. If i is the index of the non-zero entry, we denote it e_i.

Exercise: What are all the one-hot vectors of length 3?

Zero, Ones and One-Hot Vectors

In Python:

In [30]:

```
zeros_vec = np.zeros(3); display(zeros_vec)
ones_vec = np.ones(4); display(ones_vec)
```

```
array([0., 0., 0.])
array([1., 1., 1., 1.])
```

In [40]:

```
i = 1; n = 3
ei = np.zeros(n); ei[i] = 1
ei
```

Out[40]:

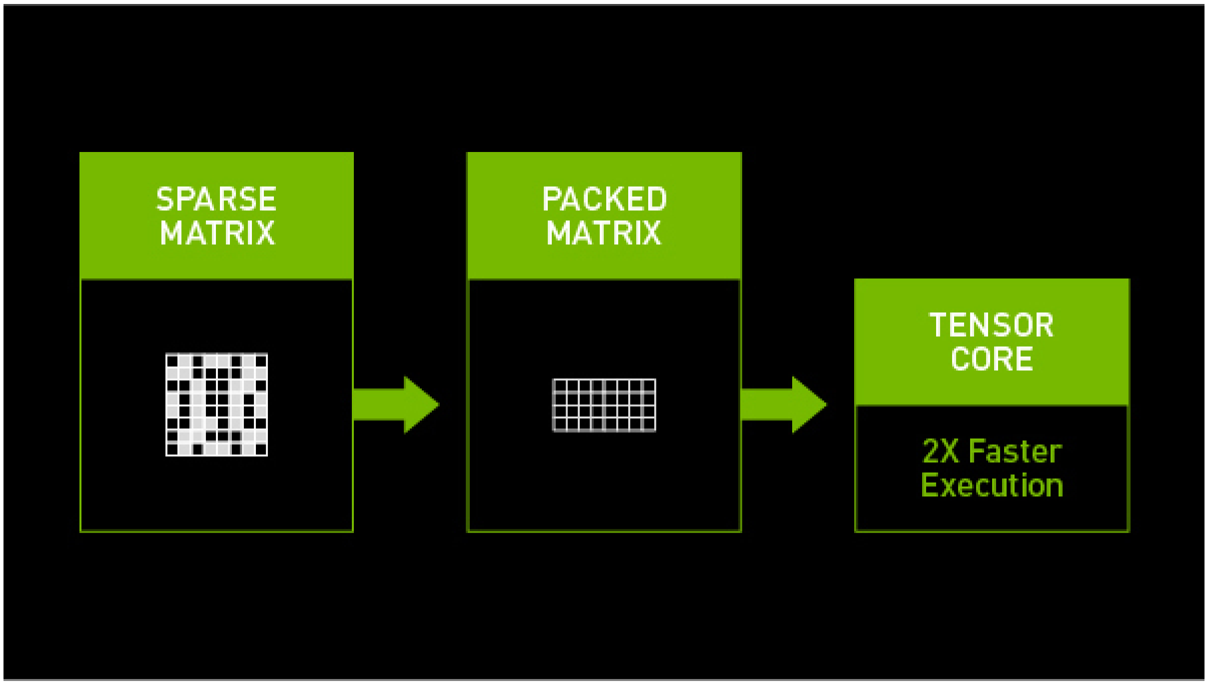
```
array([0., 1., 0.])
```

Sparsity

Definition: A vector is sparse if "many" of its entries are 0.

- Can be stored and manipulated efficiently on a computer.

Exercise: Give examples of sparse and non-sparse vectors.



The A100 packs sparse matrices to accelerate AI inference tasks

Outline: 01 Vectors

- First definitions and notations
- **Examples**
- Addition, subtraction and scalar multiplication
- Inner product
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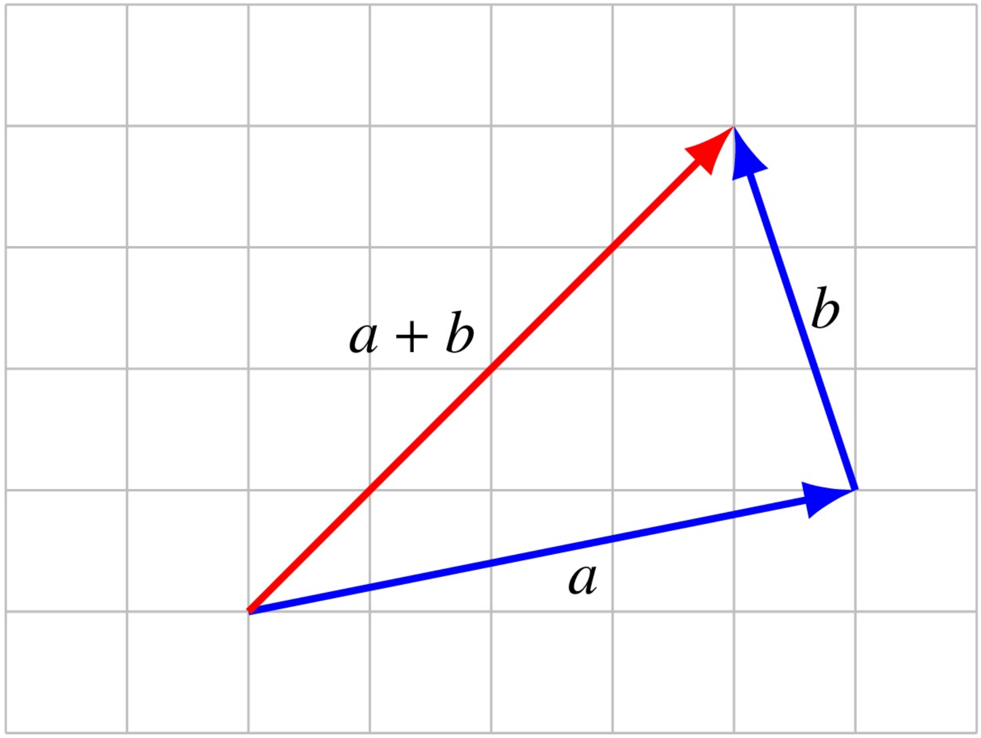
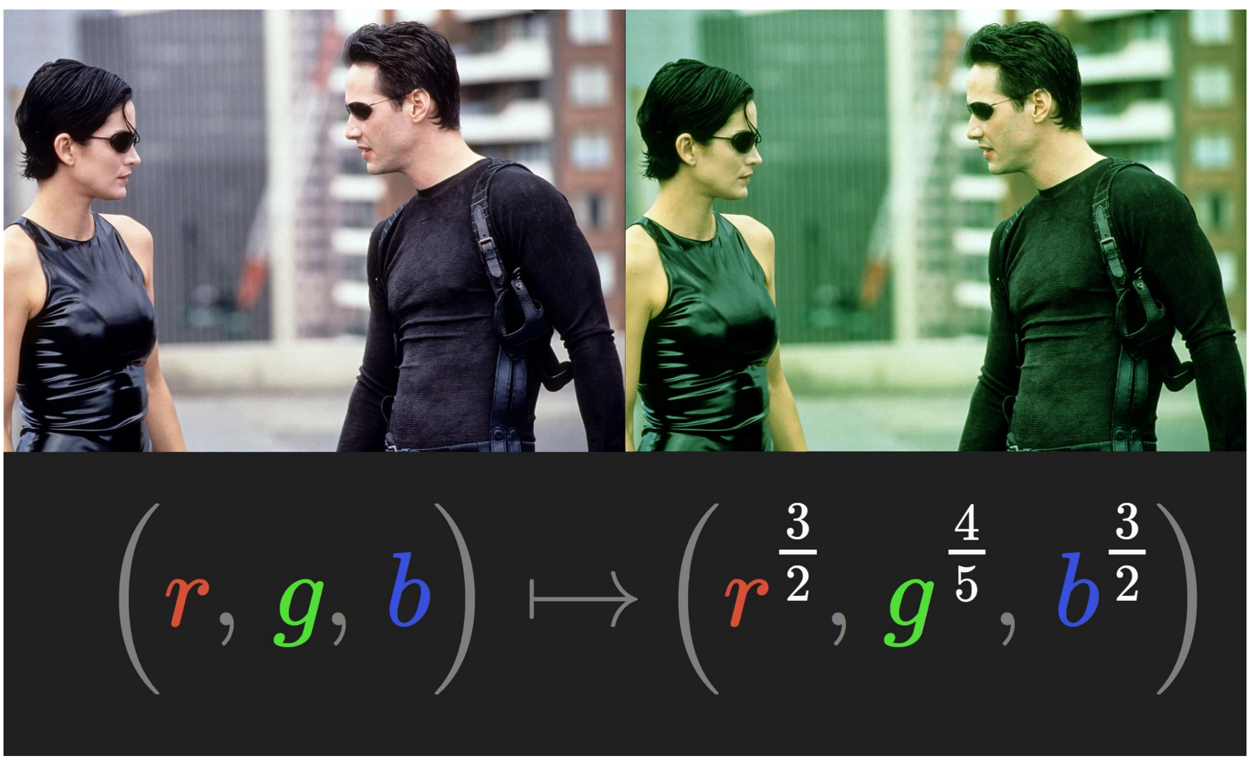
Example: Location or displacement in 2D or 3D

The 2-vector $a = (a_1, a_2)$ can represent a location or a displacement in 2-D.

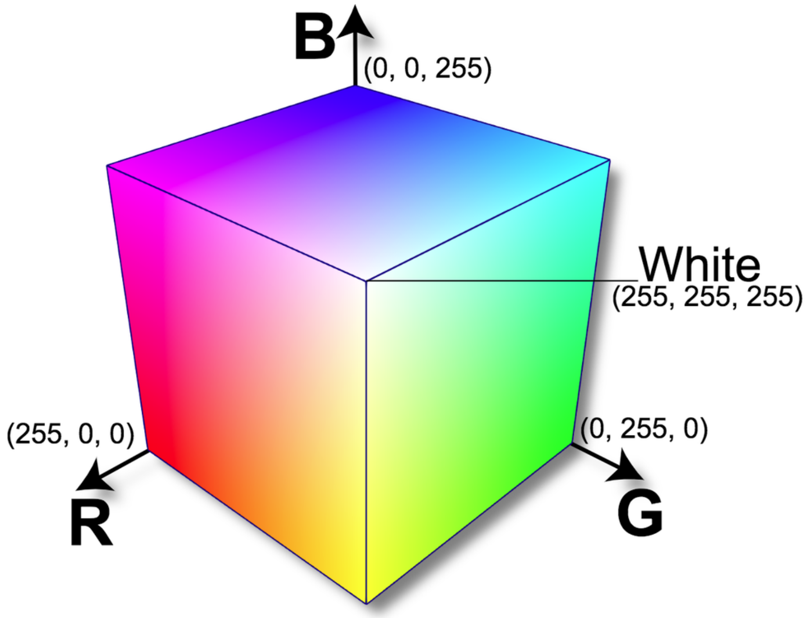


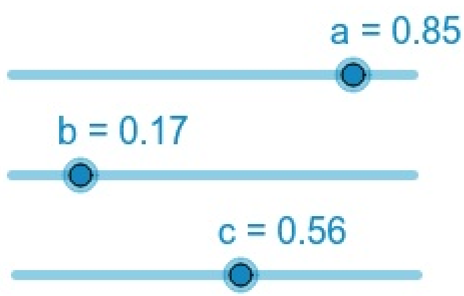
Example: Color in RGB

The 3-vector a can represent a color in RGB: $a = (R, G, B)$



The matrix coloring is an operation on vectors





red: $\vec{u} = (255, 0, 0)$
green: $\vec{v} = (0, 255, 0)$
blue: $\vec{w} = (0, 0, 255)$

Example: Word count vectors

The n-vector a can represent the count of n words in a given document.

In a short document:

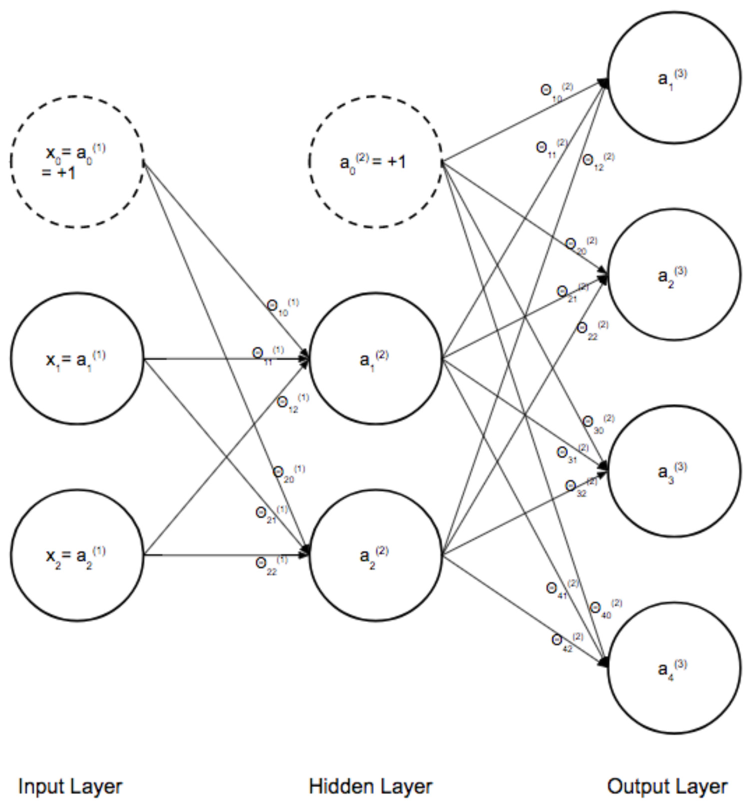
Word count vectors are used in computer based document analysis. Each entry of the word count vector is the number of times the associated dictionary word appears in the document.

Exercise: Give the word count vector associated to the text in italic above using the following dictionnary:

$$D = \{\text{word}, \text{in}, \text{number}, \text{horse}, \text{the}, \text{document}\}.$$

Other examples in ECE and beyond

- Education: Grades of n different exams
- Audio: a_i is the acoustic pressure at sample time i (sample times are spaced 1/44100 seconds apart)
- Deep Learning: Layers of a neural network



Each layer is represented as a vector in the computer.

Outline: 01 Vectors

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- Inner product
- Complexity

Vector Addition and Subtraction

Definition: Two (or more) n-vectors a and b can be added, with sum denoted a + b.

- The sum is computed by adding corresponding entries.
- Similary, a and b can be subtracted, with subtraction denoted a - b, and subtracting entries.

Exercise: Compute:

$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Vector Addition and Subtraction

In Python:

In [88]:

```
a = np.array([0, 7, 3]) + np.array([1, 2, 0]); display(a)
a = np.array([1, 9]) - np.array([1, 1]); display(a)
```

```
array([1, 9, 3])
array([0, 8])
```

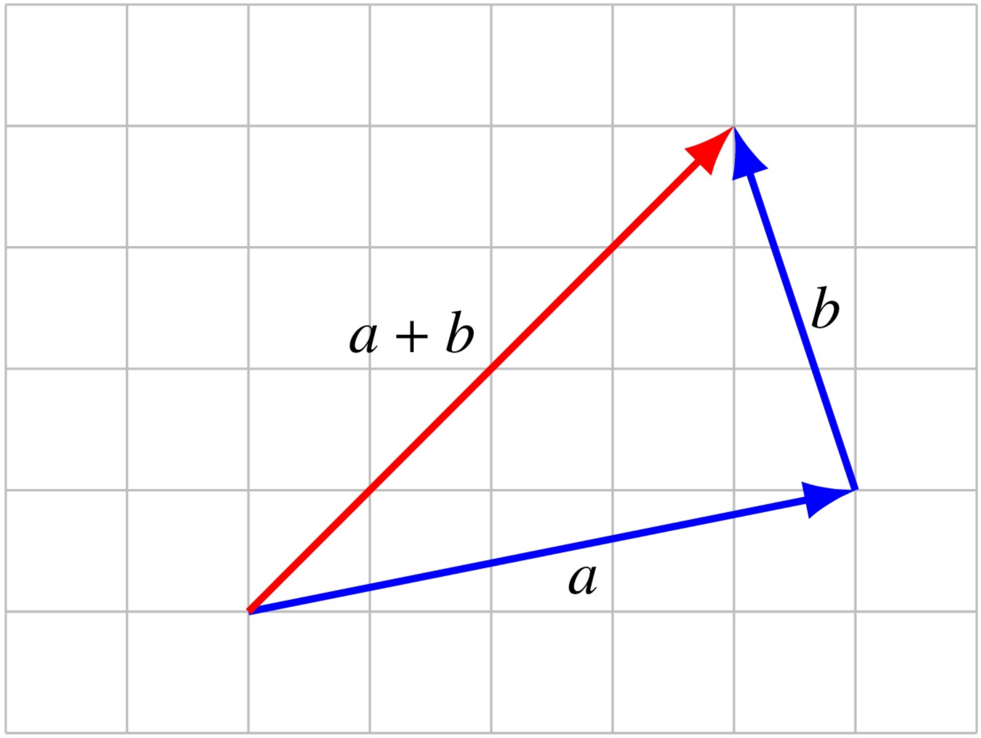
Properties of Vector Addition

Properties: For any n-vectors a, b we have:

- commutative: $a + b = b + a$
- associative: $(a + b) + c = a + (b + c) = a + b + c$
- $a + 0 = a$
- $a - a = 0$

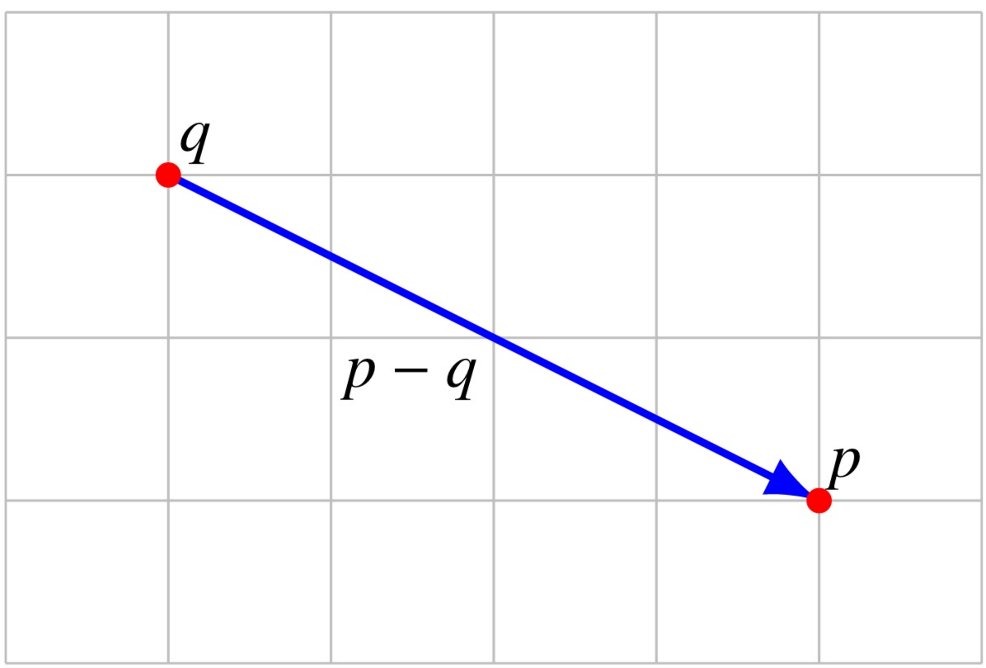
Adding displacement vectors

If vectors a and b are displacements, a + b is the sum displacement



Displacement from one point to another

Displacement from point q to point p is p − q



Scalar-vector multiplication

Definition: A scalar β and an n-vector a can be multiplied: $\beta a = (\beta a_1, \ldots, \beta a_n)$.

Exercise: Compute:

$$(-2)\begin{bmatrix}1\\9\\6\end{bmatrix}.$$

In Python:

In [67]:
```
x = np.array([0, 2, -1])
2.2 * x
```

Out[67]:
```
array([ 0. ,  4.4, -2.2])
```

Properties of scalar-vector multiplication

Properties: For scalars β, γ and n-vectors a, b:

- associative: $(\beta\gamma)a = \beta(\gamma a)$
- left distributive: $(\beta + \gamma)a = \beta a + \gamma a$
- right distributive: $\beta(a + b) = \beta a + \beta b$

In Python:

In [68]:
```
a = np.array([1, 2]); b = np.array([3, 4]); beta = 0.5

lhs = beta * (a + b)
rhs = beta * a + beta * b
lhs, rhs
```

Out[68]:
```
(array([2., 3.]), array([2., 3.]))
```

Linear combinations

Definition: For n-vectors $a_1, \dots, a_m$ and scalars $\beta_1, \dots, \beta_m$:

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

is a linear combination of the vectors.

- $\beta_1, \dots, \beta_m$ are the coefficients.

Exercise: Write a n-vector $b = (b_1, \dots, b_n)$ as a linear combination of the one-hot n-vectors $e_1, \dots, e_n$.

Linear combinations

In Python:

In [69]:
```
a, b = np.array([1, 2]), np.array([3, 4])
```

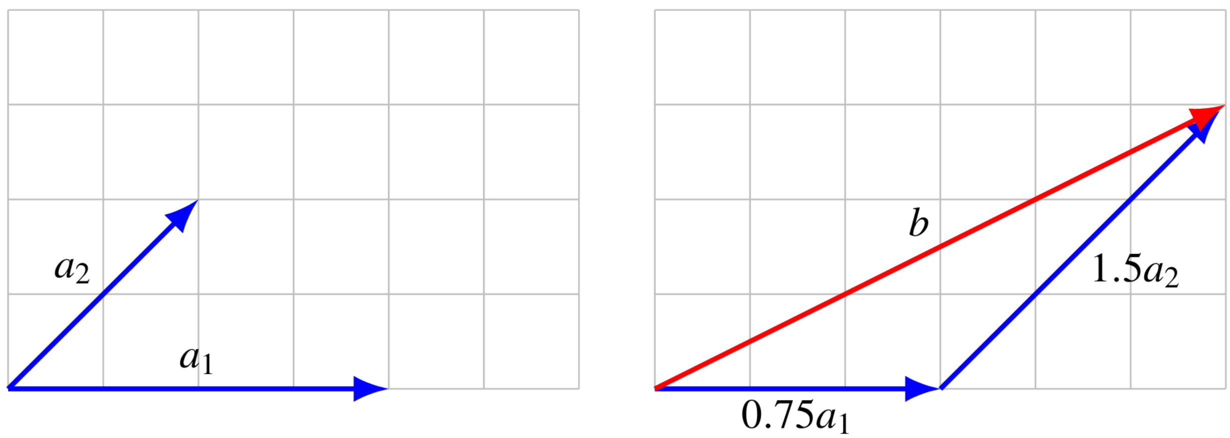
```
alpha, beta = -.5, 1.5
alpha * a + beta * b
```

Out[69]:

```
array([4., 5.])
```

Displacements and Linear Combination

Two vectors a_1 and a_2, and linear combination $b = 0.75a_1 + 1.5a_2$



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- **Inner product**
- Complexity

Inner Product

Definition: The inner product (or dot product) of n-vectors a and b is

$$a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

- Other notations: $\langle a, b\rangle, \langle a|b\rangle, (a, b), a \cdot b.$

Exercise: Compute the inner product of $a = (1, 0, 2)$ and $b = (-1, 1, 2)$.

In Python:

In [4]:

```
import numpy as np
```

```
x = np.array([-1, 2, 2])
y = np.array([1, 0, -3])
np.inner(x, y)
```

Out[4]:
```
-7
```

Properties of inner product

Properties: For n-vectors a, b, c and scalars γ:

- $a^T b = b^T a$
- $(\gamma a)^T b = \gamma(a^T b)$
- $(a+b)^T c = a^T c + b^T c$

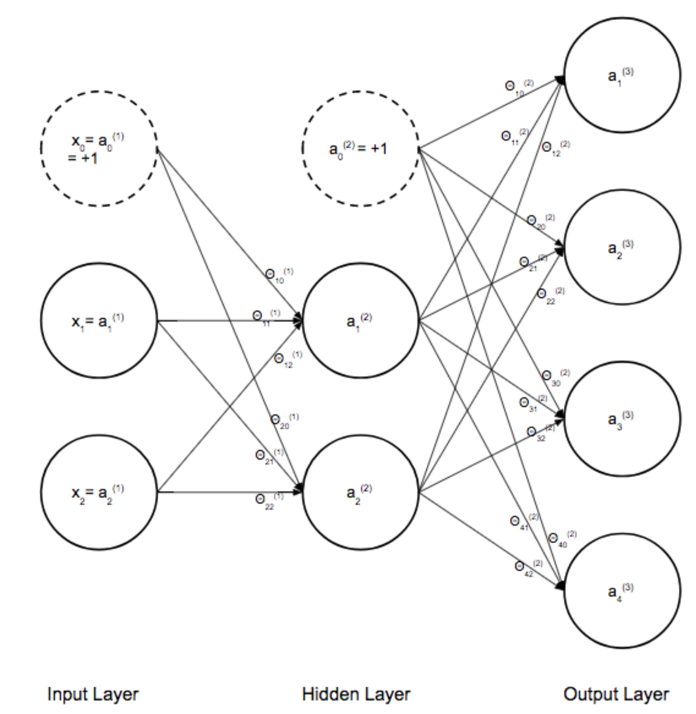
Exercise (at home):

- Show that for n-vectors a, b, c, d: $(a+b)^T(c+d) = a^T c + a^T d + b^T c + b^T d$
- Given $a = (a_1, \ldots, a_n)$: compute $e_i^T a$, $1^T a$, $a^T a$ using the entries of a.

Examples in ECE and beyond

- Education: p grades, q weights; $p^T q$ is the total grade

- Deep Learning:



Inner product in neural networks.

Outline: 01 Vectors

Complexity = Flop Counts

Computers store (real) numbers in floating-point format.

In [1]:
```
a = 1.4
type(a)
```

Out[1]:
```
float
```

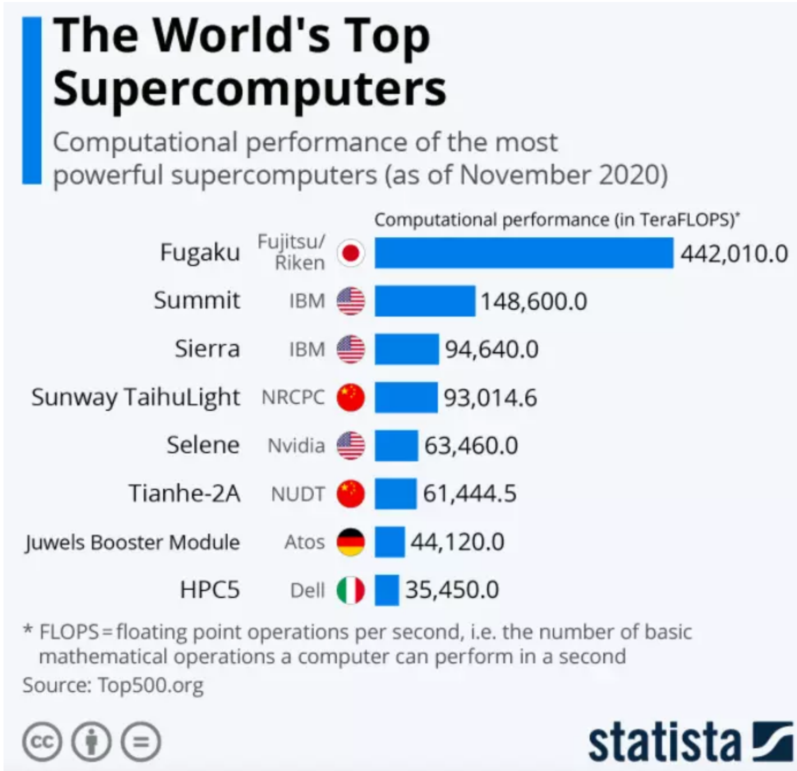
Definition: Basic arithmetic operations (addition, multiplication, . . .) are called floating point operations or flops.

Complexity = Flop Counts

Definition: The complexity of an algorithm or operation is the total number of flops needed, as function of the input dimension(s).

- Complexity allows to estimate the execution time: time to execute $\simeq$ (flops needed)/(computer speed)

- Current computers are around 1Gflop/sec (10^9 flop/sec)
- But the supercomputer can have many TeraFlop/sec (10^{12} flop/sec)



Source: https://www.weforum.org/agenda/2021/01/supercomputer-world-technology-computer-japan-fugaku/

Complexity of addition, inner product

For the n-vectors x and y:

- $x + y$ needs n additions, so: n flops
- $x^T y$ needs n multiplications and $n-1$ additions so: $2n-1$ flops --> approximated to $2n$

🤩 Much less flops are needed when x or y is sparse.

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Resources

- Ch. 1 of Book ILA

In [3]:

```
from IPython.display import Audio,Image, YouTubeVideo
id='fNk_zzaMoSs'
YouTubeVideo(id=id,width=600,height=300)
```

Out[3]: