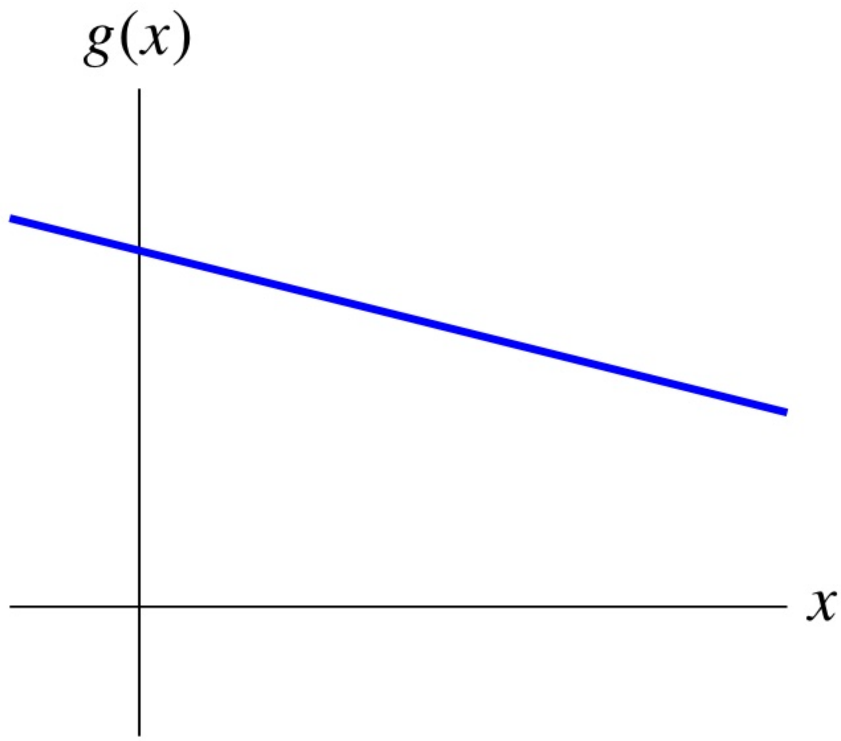
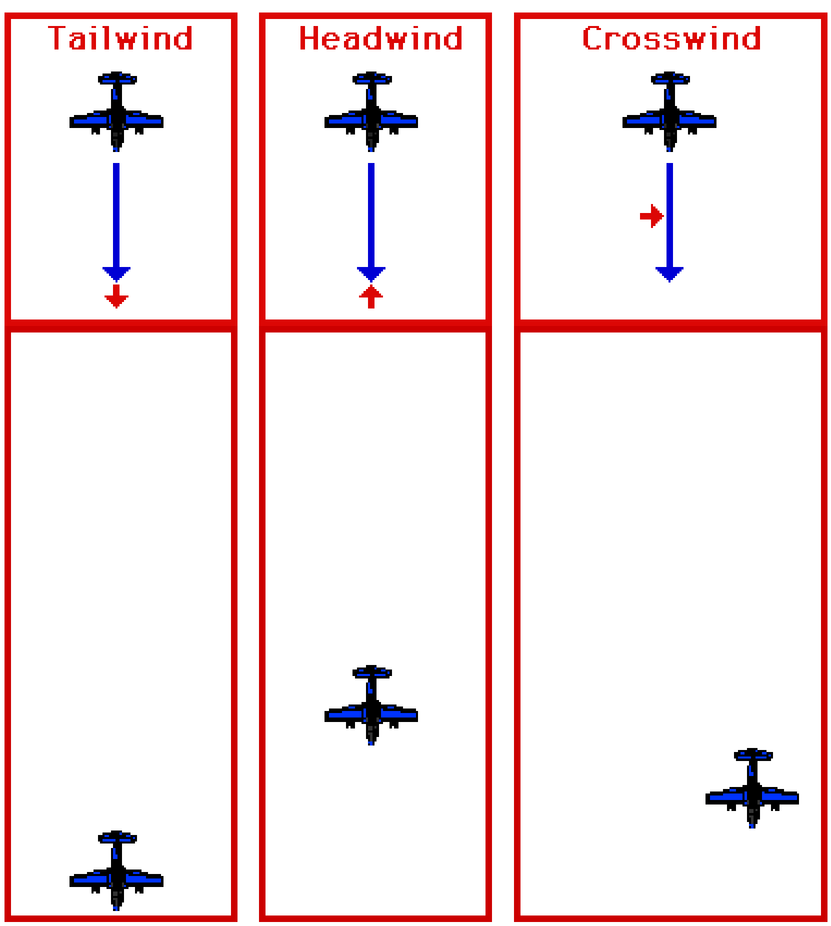
# 07 Linear Equations



Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

- 06 Matrices
- ***07 Linear Equations***
- 08 Linear Dynamical Systems
- 09 Matrix Multiplication
- 10 Matrix Inverse

Unit 3: Least Squares, Book ILA Ch. 12-14 + Book IMC Ch. 8

Unit 4: Eigen-decomposition, Book IMC Ch. 10, 12, 19

# Outline: 07 Linear Equations

- **Linear and affine functions**
- Linear equations

# Recall: Superposition and linear functions

Notation (function): The notation $f : \mathbb{R}^n \to \mathbb{R}$ means $f$ is a function mapping $n$-vectors to numbers. The space $\mathbb{R}^n$ denotes the space of all possible $n$-vectors.

Definition (superposition): We say that the function $f : \mathbb{R}^n \to \mathbb{R}$ satistifies the superposition property if, for all scalars $\alpha, \beta$ and all $n$-vectors $x, y$:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Definition (linear function): A function that satisfies superposition is called linear.

Proposition: A function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if and only if, for all $n$-vector $x$, $f(x) = a^T x$ for some $n$-vector $a$.

# Functions $f : \mathbb{R}^n \to \mathbb{R}^m$

Notation: The notation $f : \mathbb{R}^n \to \mathbb{R}^m$ means $f$ is a function mapping $n$-vectors to $m$-vectors. We write:

- $f(x) = (f_1(x), \ldots, f_m(x))$ to emphasize components of $f(x)$,
- $f(x) = f(x_1, \ldots, x_n)$ to emphasize components of $x$,
- $f(x) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ to emphasize both.

Example: Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined as: $f(x_1, x_2, x_3) = (x_2^3 - x_1, -x_3^4)$, which we can also write:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2^3 - x_1 \\ -x_3^4 \end{bmatrix}$$

## Superposition and linear functions (General)

Definition: We say that the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satistifies the superposition property if, for all scalars $\alpha, \beta$ and all $n$-vectors $x, y$:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Definition: A function that satisfies superposition is called linear.

Proposition: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if and only if, for all $n$-vector $x$, $f(x) = Ax$ for some $m \times n$ matrix $A$.

Example: The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ defined by:

$$f(x_1, \ldots, x_n) = (x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})$$

is a linear function. It can also be written as:

$$f(x) = Dx$$

where $D$ is the difference matrix from the previous lecture.

Exercise: Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5x_2 - x_1 \\ 2x_3 \end{bmatrix}.$$

What is the matrix $A$ such that: $f(x) = Ax$?

Exercise: Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, that is linear. What is the matrix $A$ such that: $f(x) = Ax$?

## Recall: Affine functions

Definition (affine function): A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that is linear plus a constant is called affine. Its general form is:

$$f(x) = a^T x + b \quad \text{with } a \text{ an } n\text{-vector and } b \text{ a scalar}$$

Proposition: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is affine if and only if, for all scalars $\alpha, \beta$ with $\alpha + \beta = 1$ and all $n$-vectors $x, y$, we have:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

# Affine functions (General)

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ that is linear plus a constant is called affine. Its general form is:

$$f(x) = Ax + b \quad \text{with } A \text{ an } m \times n \text{ matrix and } b \text{ a } m\text{-vector}$$

Proposition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine if and only if, for all scalars $\alpha, \beta$ with $\alpha + \beta = 1$ and all $n$-vectors $x, y$, we have:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Exercise: Consider an affine function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5x_2 - x_1 + 3 \\ 2x_3 + 2 \end{bmatrix}.$$

What is the matrix $A$ and the vector $b$ such that: $f(x) = Ax + b$?

Exercise: Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$, that is affine. What is the matrix $A$ and the vector $b$ such that: $f(x) = Ax + b$?

# Outline: 07 Linear Equations

- Linear and affine functions
- Linear equations

Definition: A set (or system) of $m$ linear equations in $n$ variables $x_1, \ldots, x_n$ is defined as:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m \end{aligned}$$

In this system:

- the matrix $A$ and the vector $b$ are given, and the goal is to find $x_1, \ldots, x_n$.
- n-vector x is called the variable or unknown,
- $A_{ij}$ are the coefficients, $A$ is the coefficient matrix
- b is called the right-hand side

Remark: A system of linear equations can be expressed very compactly as $Ax = b$.

Definition: A system of linear equations is called:

- under-determined if $m < n$ ($A$ wide)
- square if $m = n$ ($A$ square)
- over-determined if $m > n$ ($A$ tall)

Definition: If we find a vector $x$ such that $Ax = b$, then $x$ is called a solution of the system of equations.

Definition: Depending on $A$ and $b$, there can be:

- many solutions
- one solution
- no solution

Exercise: An airplane travels 1200 miles in 4 hours with a tail wind. On the way back, the same trip takes 5 hours, now with a head wind (against the wind). What is the speed of the plane in still air, and what was the wind speed?

Remark: We will see later (after learning a bit more on operations on matrices) how to automatically solve such a system.

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Resources: Ch. 8 of ILA