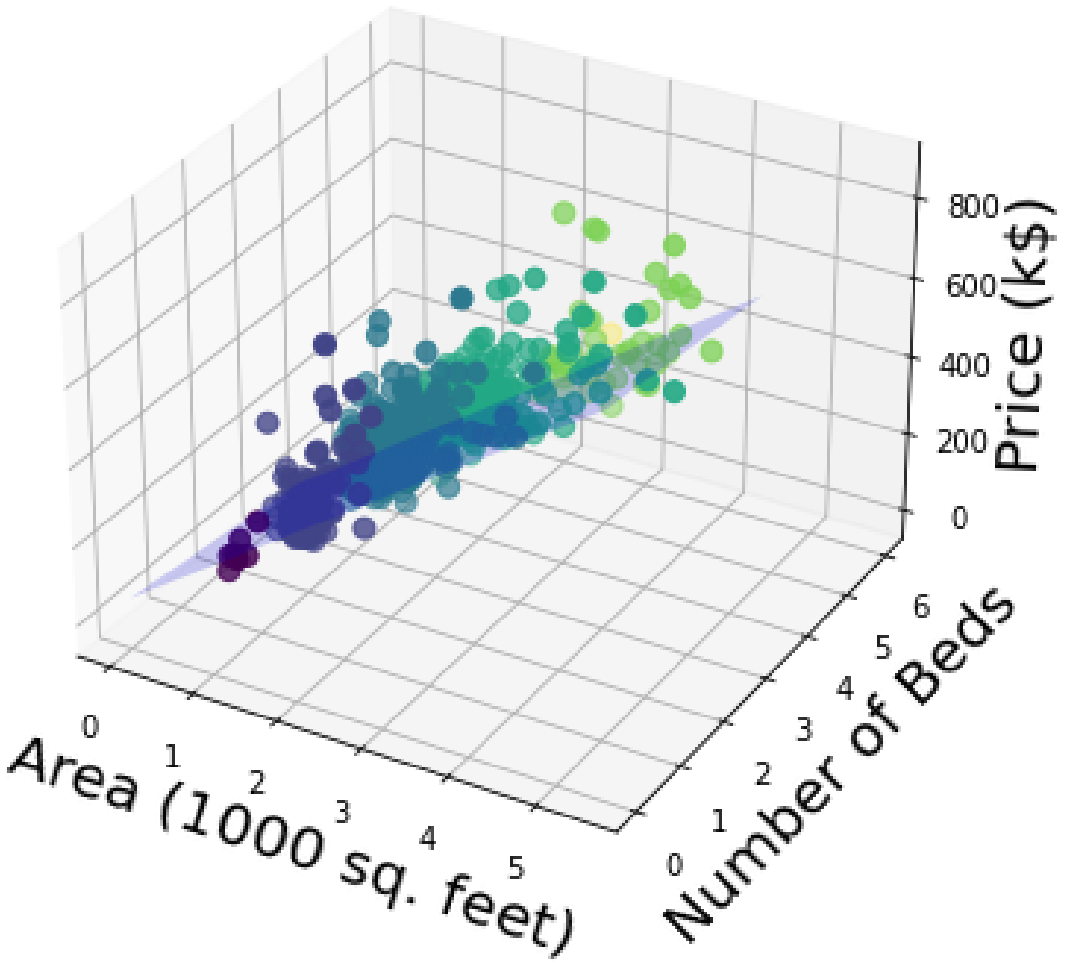
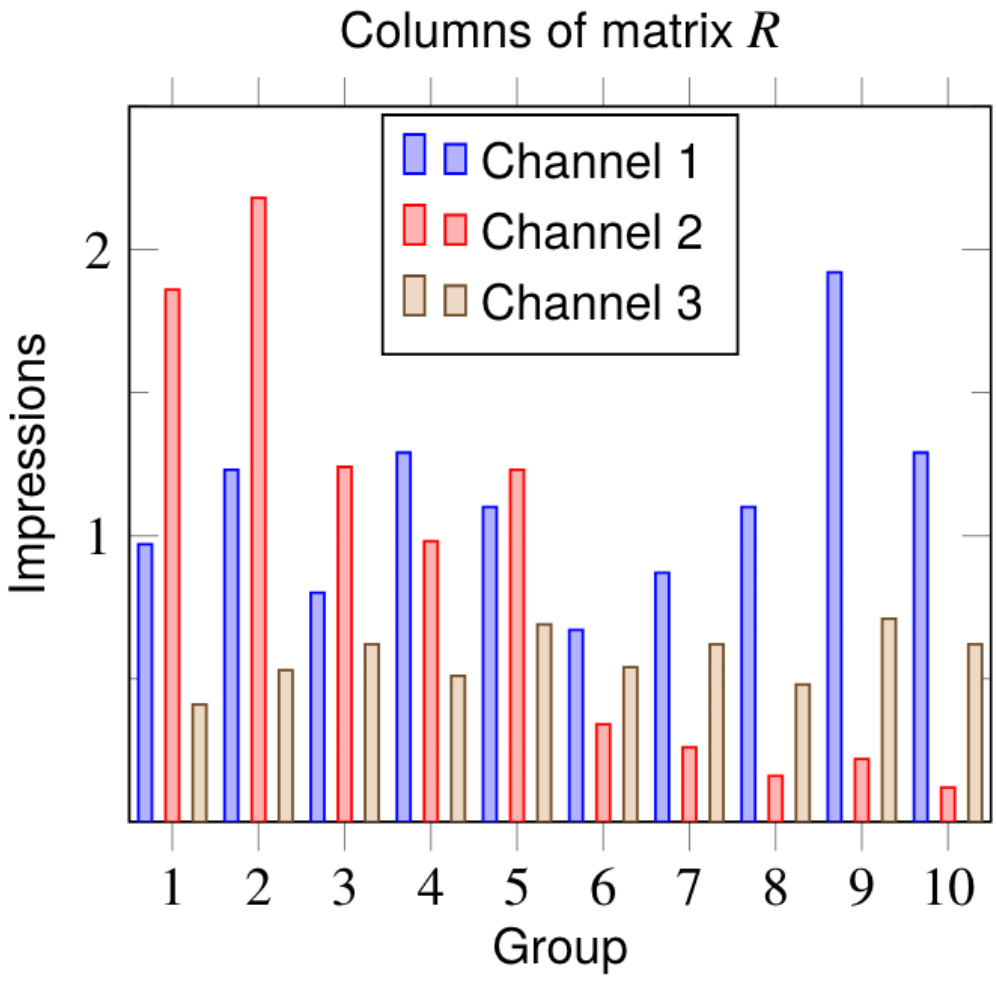
11 Least Squares





Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

Unit 3: Least Squares, Book ILA Ch. 12-14

- 11 Least Squares
- 12 Least Squares Data Fitting
- 13 Least Squares Classification

Outline: 11 Least Squares

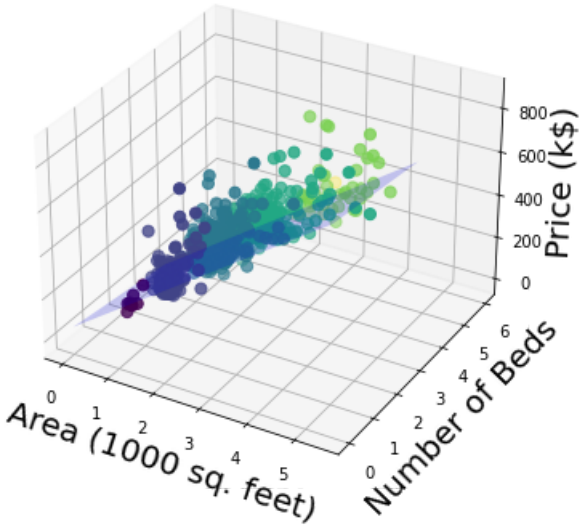
- Least Square Problem
- Solution of Least Square Problem
- Examples

Regression to Model House Prices

- Outcome is y, selling price of house in $1000.
- Regressors are x = (house area in 1000 sq.ft, # bedrooms).

- A regression model helps us predict the price from x.

Question: Find w, b given data? $\rightarrow$ Solve least square problem!



Least Squares Problem

Definition: Let be given a $m \times n$ matrix A and m-vector b. The least squares problem is the problem of choosing an n-vector x to minimize:

$$\|Ax - b\|^2.$$

- $\|Ax - b\|^2$ is called the objective function,
- If $\hat{x}$ is a solution of the linear equation $Ax = b$, then $\hat{x}$ is a solution of the least square problem. The converse is not true.
- $\hat{x}$ is a solution of least squares problem if $\|A\hat{x} - b\|^2 \leq \|Ax - b\|^2$ for any other n-vector x.

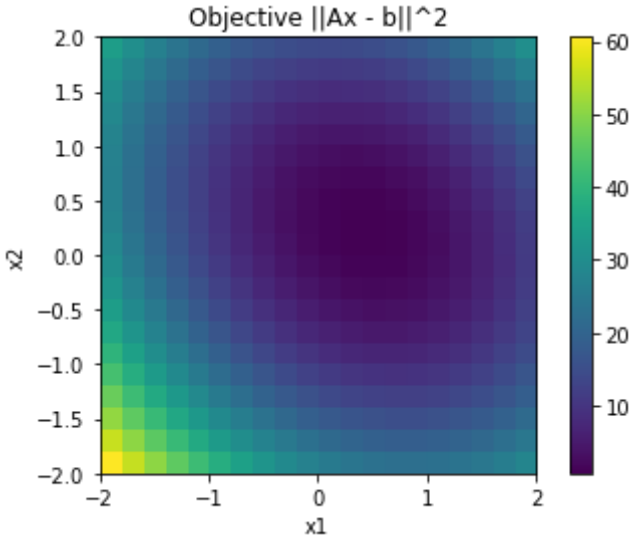
Exercise: Consider the matrix A and vector b as:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Write the objective function associated to the least square problem defined by A and b in terms of entries of x.

In [9]:

```
import numpy as np; import matplotlib.pyplot as plt
objective = lambda x : (2 * x[0] - 1) ** 2 + (- x[0] + x[1]) ** 2 + (2 * x[1] +
n_points, xmin, xmax, ymin, ymax = 20, -2, 2, -2, 2
x = np.arange(xmin, xmax, (xmax-xmin)/n_points); y = np.arange(ymin, ymax, (ymax
for i in range(n_points):
    for j in range(n_points):
        Z[i, j] = objective(xx[i, j])
plt.imshow(Z, extent=[xmin, xmax, ymin, ymax]); plt.colorbar(); plt.xlabel("x1")
```



Outline: 11 Least Squares

Least Square Solution

Proposition:

- Consider a least square problem $||Ax - b||^2$ for matrix A and vector b.
- Assume that A has linearly independent columns.

Then, there is a unique solution $\hat{x}$ to the least square problem, defined as:

$$\hat{x} = (A^T A)^{-1} A^T b = A^\dagger b.$$

- $A^\dagger = (A^T A)^{-1} A^T$ is called the pseudo-inverse of A.

Exercise (hard): Using the fact that:

$$||a + b||^2 = ||a||^2 + ||b||^2 + 2a^T b,$$

prove that $\hat{x}$ defined in the previous slide is indeed a solution.

- Hint: Show that for any other n-vector x, we have:

$$\|A\hat{x} - b\|^2 \leq \|Ax - b\|^2.$$

- Hint 2: You will need to show that $A^T(A\hat{x} - b) = 0$.

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- **Examples**

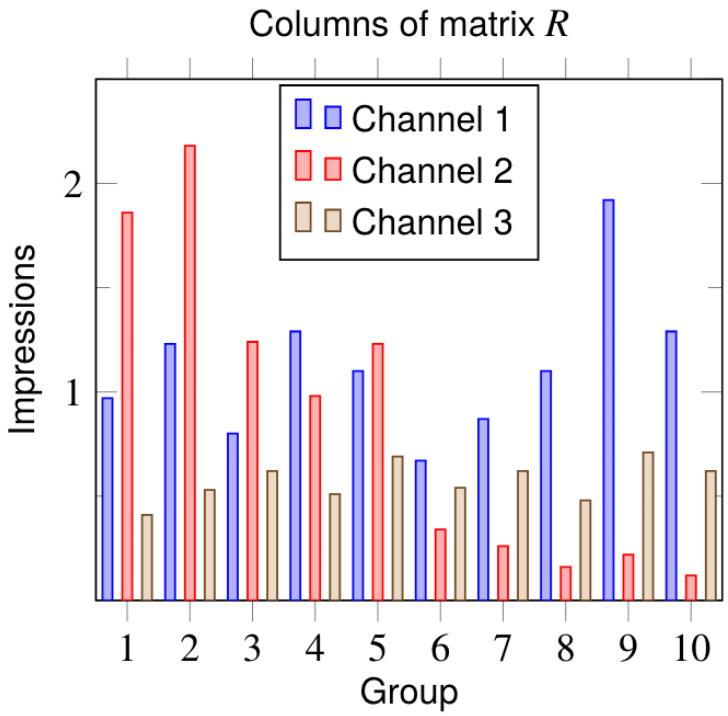
Political Advertising

Example: Political Advertising

- A company wants to advertize to potential voters.
  - m demographics groups, n advertising channels
  - v^{target} is m-vector of target views ("impressions") per group
  - s is n-vector of spending per channel
  - R is $m \times n$ matrix of demographic reach of channels:
    - R_{ij} is number of views per dollar spent (in 1000/$)
- How much should be spent to be as close as possible to v^{target}?

Example: What is the optimal spending $\hat{s}$?

- $m = 10$ groups and $n = 3$ channels,
- $v^{target} = 1000.\, 1_m$.



Example: Illumination

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What is the power of the lamps needed to reach a target illumination?

- n lamps illuminate an area divided in m regions
- A_{ij} is illumination in region i if lamp j is has power 1, other lamps are off
- x_j is power of lamp j
- $(Ax)_i$ is illumination level at region i
- b_i is target illumination level at region i

Find the associated least-square problem.

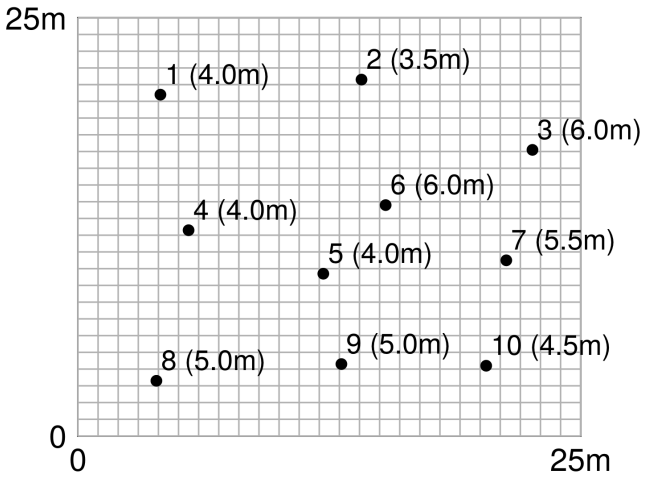
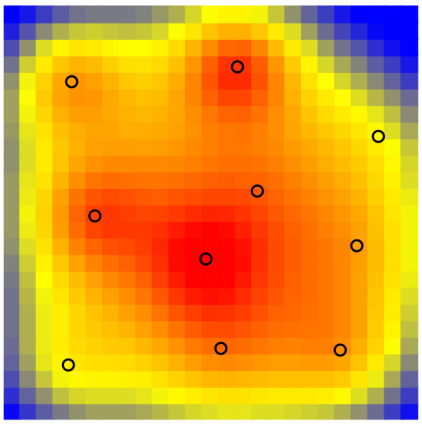


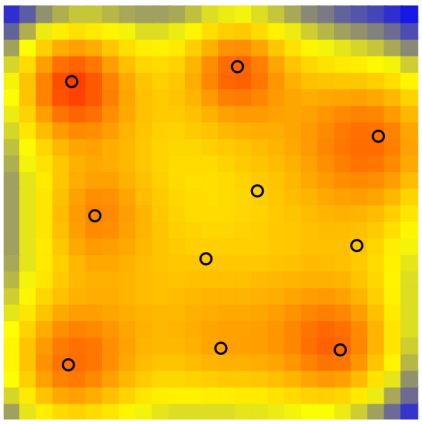
figure shows lamp positions for example with

$$m = 25^2, \quad n = 10$$

- equal lamp powers ($x = \mathbf{1}$)



- least squares solution $\hat{x}$, with $b = \mathbf{1}$



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Resources: Book ILA Ch. 11