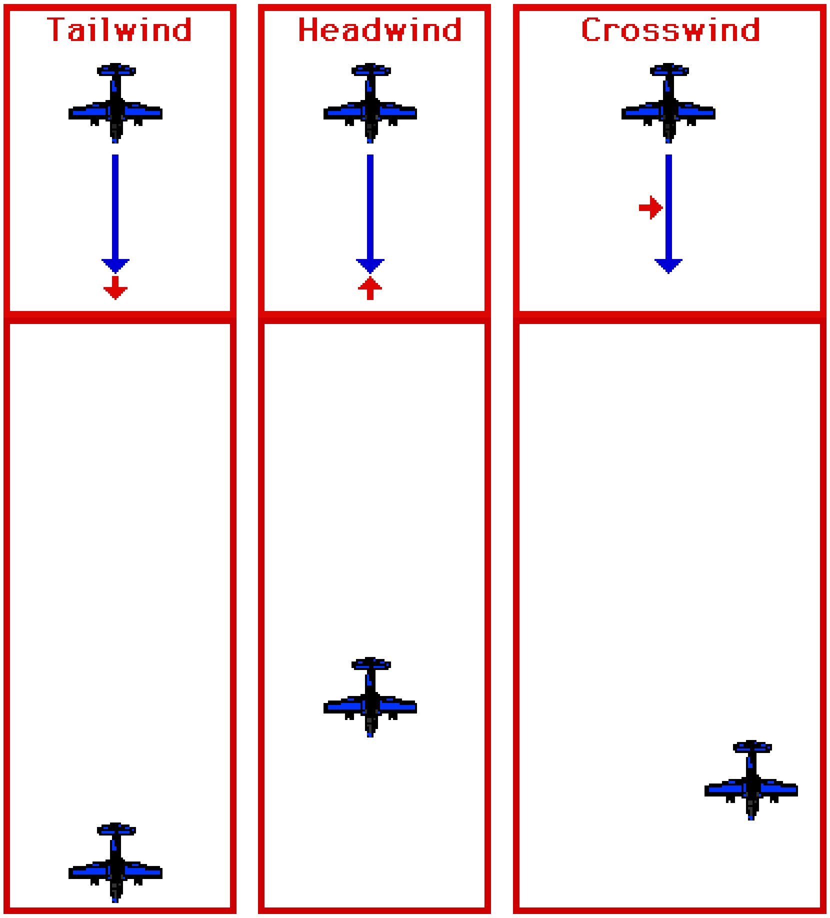
10 Matrix Inverse



Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

- 06 Matrices
- 07 Linear Equations
- 08 Linear Dynamical Systems
- ***09 Matrix Multiplication***
- 10 Matrix Inverse

Unit 3: Least Squares, Book ILA Ch. 12-14

Outline: 10 Matrix Inverse

Left inverse

Definition: Consider a scalar a. A scalar x that satisfies $xa = 1$ is called the inverse of a.

- We have $x = \frac{1}{a}$, which exists and is unique if and only if $a \neq 0$.

Definition: Consider a matrix A. A matrix X that satistifies:

$$XA = I$$

is called a left-inverse of A. If a left inverse exists, A is left-invertible. The left-inverse might not be unique.

Exercise: Show that the matrix:

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

has two different left-inverses:

$$X_1 = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \quad X_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}.$$

Properties of left inverses

Properties:

- If A has a left inverse, then the columns of A are linearly independent.
- If A has a left inverse, then A is tall or square.

Exercise: Prove the above statement.

Solving linear equations with left inverses

Proposition: Consider the linear equation $Ax = b$. Consider C a left-inverse of A. Then, a solution to the linear equation is:

$$x = Cb.$$

Exercise: Prove the above statement.

Example: Consider the matrix $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$ from the previous slide, and $b = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

Give two solutions to the linear equation:

$$Ax = b.$$

Right inverses

Definition: Consider a matrix A. A matrix X that satistifies:

$$AX = I$$

is called a right-inverse of A. If a right inverse exists, A is right-invertible. The right-inverse might not be unique.

Properties of right inverses

Properties:

- A is right invertible if and only if A^T is left invertible.
- A is right invertible if and only if its rows are linearly independent.
- If A is right invertible, then A is wide or square.

Solving linear equations with right inverses

Proposition: Consider the linear equation $Ax = b$. Consider B a right-inverse of A. Then, a solution to the linear equation is:

$$x = Bb.$$

Exercise: Prove the above statement.

Outline: 10 Matrix Inverse

Inverse

Definition: If A has a left and a right inverse, they are unique and equal. We say that A is invertible. We denote A^{-1} the (unique) inverse of A.

Properties:

- If A is invertible then A is square.
- The inverse of the inverse is: $(A^{-1})^{-1} = A$.

In Python, we use `np.linalg.inv` to compute the inverse.

In [7]:
```
import numpy as np

A = np.array([
    [-3, -4],
    [4, 6]
])

...
```

Out[7]:
```
Ellipsis
```

In Python, a system of linear equations can be solved in two ways:

- using `np.linalg.inv`
- using `np.linalg.solve`

In [6]:
```
A = np.array([[-3, -4], [4, 6]]); b = np.array([1, 2])
```

```
[-7.  5.]
[-7.  5.]
```

Well-known inverses

Properties:

- $I^{-1} = I$
- If Q is square matrix with $Q^TQ = I$, then $Q^{-1} = Q^T$. E.g. rotation matrices.
- Consider A is a 2×2 matrix:
  - A is invertible if and only if $A_{11}A_{22} \neq A_{12}A_{21}$.
  - In this case: $A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

Properties: Consider invertible square matrices A, B.

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- Negative powers! $A^{-k} = (A^k)^{-1}$

Triangular Matrices

Properties: Any lower triangular matrix L with nonzero diagonal entries is invertible. Any upper triangular R with nonzero diagonal entries is invertible.

Properties: Consider A, a square and invertible matrix. Consider the QR factorization $A = QR$. Then, the inverse of A can be written: $A^{-1} = R^{-1}Q^T$.

Outline: 10 Matrix Inverse

- Left and right inverses

- Inverse
- **Solving linear equations**
- Examples
- Pseudo-inverse

In [ ]:
```

```