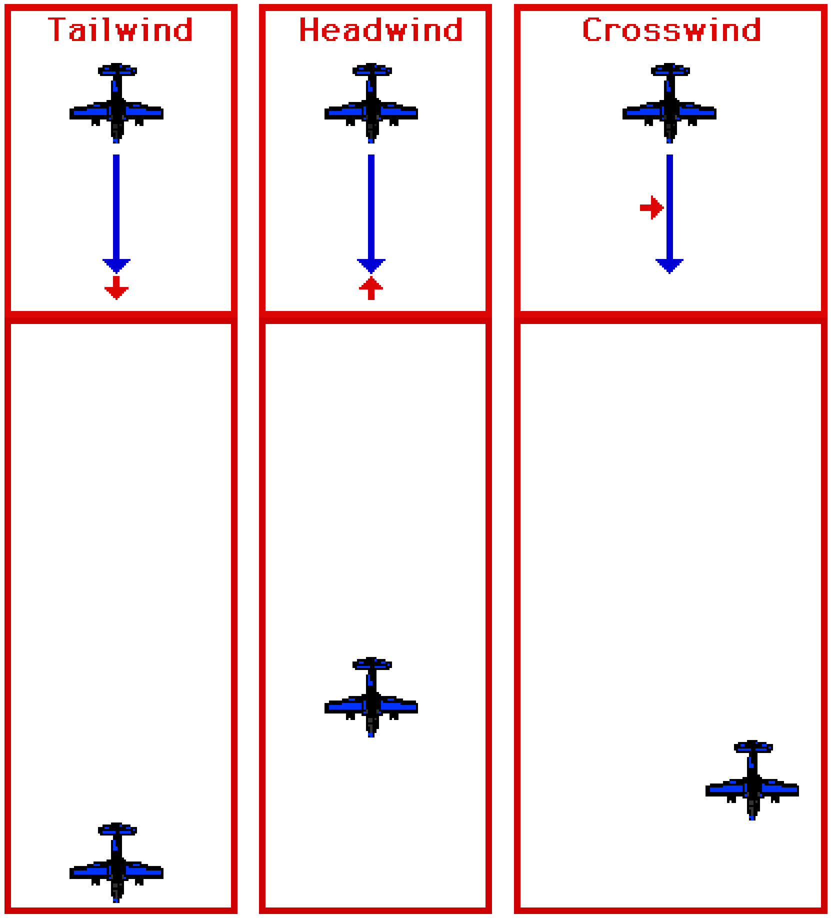
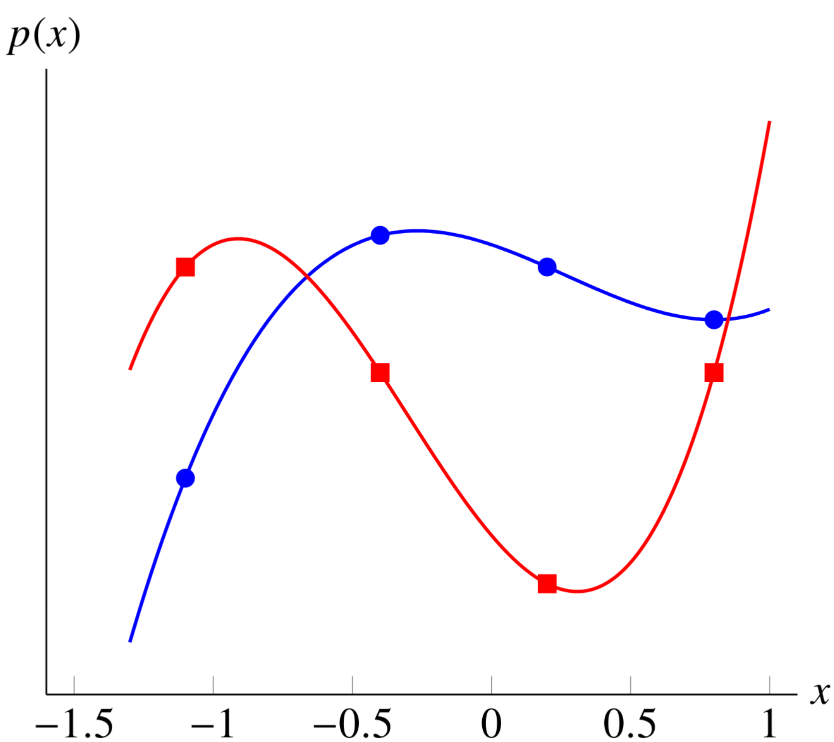
10 Matrix Inverse





Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

- 06 Matrices
- 07 Linear Equations
- 08 Linear Dynamical Systems
- 09 Matrix Multiplication
- ***10 Matrix Inverse***

Unit 3: Least Squares, Book ILA Ch. 12-14

Outline: 10 Matrix Inverse

- **Left and right inverses**
- Inverse
- Solving linear equations
- Examples

Left inverse

Definition: Consider a scalar a. A scalar x that satisfies $xa = 1$ is called the inverse of a.

- We have $x = \frac{1}{a}$, which exists and is unique if and only if $a \neq 0$.

Definition: Consider a matrix A. A matrix X that satistifies:

$$XA = I$$

is called a left-inverse of A. If a left inverse exists, A is left-invertible. The left-inverse might not be unique.

Exercise: Show that the matrix:

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

has two different left-inverses:

$$X_1 = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \quad X_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}.$$

Properties of left inverses

Properties:

- If A has a left inverse, then the columns of A are linearly independent.

- If A has a left inverse, then A is tall or square.

Exercise: Prove the above statement.

Solving linear equations with left inverses

Proposition: Consider the linear equation $Ax = b$. Consider C a left-inverse of A. Then, a solution to the linear equation is:

$$x = Cb.$$

Exercise: Prove the above statement.

Example: Consider the matrix $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$ from the previous slide, and $b = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

Give two solutions to the linear equation:

$$Ax = b.$$

Right inverses

Definition: Consider a matrix A. A matrix X that satistifies:

$$AX = I$$

is called a right-inverse of A. If a right inverse exists, A is right-invertible. The right-inverse might not be unique.

Properties of right inverses

Properties:

- A is right invertible if and only if A^T is left invertible.
- A is right invertible if and only if its rows are linearly independent.
- If A is right invertible, then A is wide or square.

Exercise: Prove the above statements.

Solving linear equations with right inverses

Proposition: Consider the linear equation $Ax = b$. Consider B a right-inverse of A. Then, a solution to the linear equation is:

$$x = Bb.$$

Exercise: Prove the above statement.

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Fill out this second anonymous survey ;)

https://tinyurl.com/2vaxhke9

Inverse

Definition: If A has a left and a right inverse, they are unique and equal. We say that A is invertible. We denote A^{-1} the (unique) inverse of A.

Properties:

- If A is invertible then A is square.
- The inverse of the inverse is: $(A^{-1})^{-1} = A$.

Which Matrices are Invertible?

Properties: Examples of matrices that are always invertible:

- Any lower triangular matrix L with nonzero diagonal entries is invertible.
- Any upper triangular R with nonzero diagonal entries is invertible.

Exercise: Give examples of invertible matrices.

Computing Inverses: 2×2 matrices

Properties: Consider A is a 2×2 matrix:

- A is invertible if and only if $A_{11}A_{22} \neq A_{12}A_{21}$.
- In this case: $A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

Exercise: Compute the inverse of $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$.

Computing Inverses

Properties:

- $I^{-1} = I$

- If Q is square matrix with $Q^TQ = I$:
  - Then $Q^{-1} = Q^T$.
- If $D = diag(a_1, \ldots, a_n)$ is a diagonal matrix with nonzero elements:
  - Then $D^{-1} = diag(\frac{1}{a_1}, \ldots, \frac{1}{a_n})$.

Computing Inverses

Properties: Consider invertible square matrices A, B with known inverses A^{-1}, B^{-1}.

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- New notation: Negative powers! $A^{-k} = (A^k)^{-1}$

Computing Inverses from QR decomposition

Properties: Consider A, a square and invertible matrix. Consider the QR factorization $A = QR$
.

- Then, the inverse of A can be written: $A^{-1} = R^{-1}Q^T$.

Computing Inverses in Python

In Python, we use `np.linalg.inv` to compute the inverse.

In [1]:

```
import numpy as np

A = np.array([
    [1, 2],
    [0, 4]
])

np.linalg.inv(A)
```

Out[1]:

```
array([[ 1.  , -0.5 ],
       [ 0.  ,  0.25]])
```

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- Inverse
- **Solving linear equations**
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Recall: Linear equations

Definition: A set (or system) of m linear equations in n variables $x_1, \ldots, x_n$ is defined as:

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\
&\vdots \\
A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m
\end{aligned}
$$

and can be written compactly as: $Ax = b$.

Proposition: Consider the linear equation $Ax = b$. If A is invertible with inverse A^{-1}, then the equation has a unique solution: $x = A^{-1}b$.

In what follows, we see methods to solve $Ax = b$ in several special cases, i.e. methods to compute A^{-1}:

- when we can compute A^{-1}
- when A is upper-triangular invertible
- when we know the QR decomposition of A
- using Python.

Special case: we know A^{-1}

Method: Consider A an invertible matrix and the linear equation $Ax = b$. Assume that we know A^{-1}.

- Then the unique solution of $Ax = b$ is given by $A^{-1}b$.

Example: An airplane travels 1200 miles in 4 hours with a tail wind. On the way back, the same trip takes 5 hours, now with a head wind (against the wind). What is the speed of the plane in still air, and what was the wind speed?

Special case: $A = R$ upper triangular invertible

Method: Consider R an upper triangular matrix with nonzero entries and the linear equation: $Rx = b$, which can be re-written as:

$$
\begin{aligned}
R_{11}x_1 + \quad R_{12}x_2 + \quad \ldots + \quad R_{1,n}x_n \quad &= b_1 \\
&\vdots \\
R_{nn}x_n \quad &= b_n
\end{aligned}
$$

The solution of the linear equation can be found by back-substitution:

- Last equation gives: $x_n = b_n / R_{nn}$
- Second to last equation gives: $x_{n-1} = (b_{n-1} - R_{n-1,n}x_n)/R_{n-1,n-1}$
- Iterate.

Special case: $A = QR$ via QR Factorization

Method: Consider A an invertible matrix and the linear equation $Ax = b$. Assume that the QR factorization of A is given: $A = QR$.

The solution of the linear equation can be found by using these steps:

- Compute $Q^T b$
- Solve the linear equation $Rx = Q^T b$ by back substitution.

Example: We will see an example in a next slide.

General case in Python

In Python, a system of linear equations can be solved in several ways:

- using `np.linalg.qr`
- using `np.linalg.inv`
- using `np.linalg.solve`

In [11]:

```
A = np.array([[-3, -4], [4, 6]]); b = np.array([1, 2])

...
```

Out[11]:

```
Ellipsis
```

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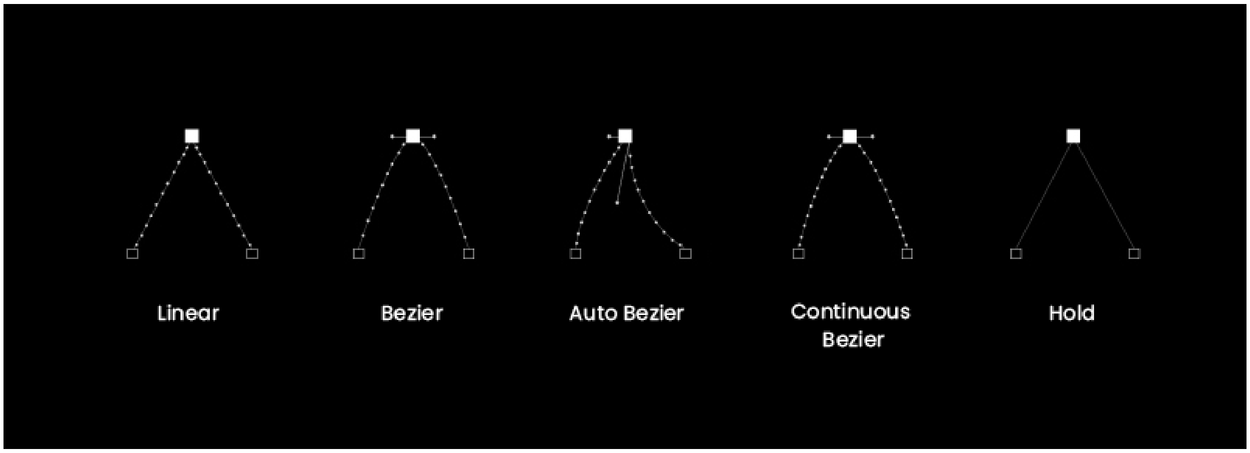
Example: Interpolation

In [14]:

```
from IPython.display import Video; Video("figs/11_aftereff.mp4")
```

Out[14]:

0:00



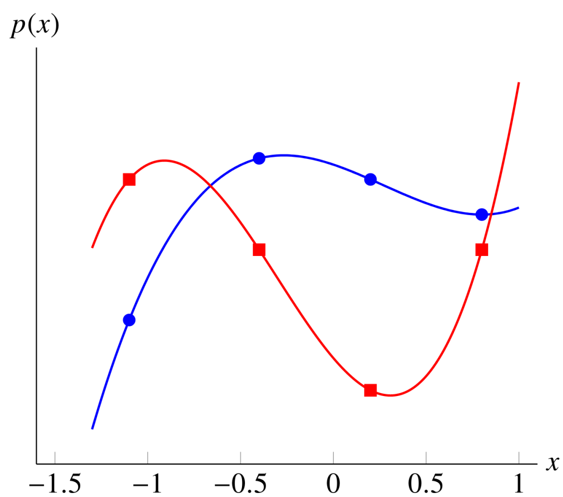
Example: Polynomial Interpolation

Example: Consider a cubic polynomial with unknown coefficients $c_0, \ldots, c_3$:

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3,$$

that satisfies: $p(-1.1) = 1, p(-0.4) = 2, p(0.1) = 4, p(0.8) = 1$.

Find the polynomial that interpolates these 4 points. You can only use `np.linalg.qr`.



In [13]:

```
import numpy as np
A = np.array([
    [1, -1.1, (-1.1) ** 2, (-1.1) ** 3],
    [1, -0.4, (-0.4) ** 2, (-0.4) ** 3],
    [1, 0.1, (0.1) ** 2, (0.1) ** 3],
    [1, 0.8, (0.8) ** 2, (0.8) ** 3]
])
b = np.array([1, 2, 4, 1])
print(A)

q, r = np.linalg.qr(A)
print("R = \n", r)

print(q.T @ b)
```

```
np.linalg.solve(A, b)
```

```
[[ 1.000e+00 -1.100e+00  1.210e+00 -1.331e+00]
 [ 1.000e+00 -4.000e-01  1.600e-01 -6.400e-02]
 [ 1.000e+00  1.000e-01  1.000e-02  1.000e-03]
 [ 1.000e+00  8.000e-01  6.400e-01  5.120e-01]]
R =
 [[-2.          0.3        -1.01        0.441     ]
 [ 0.         -1.3892444   0.41677332 -1.27198641]
 [ 0.          0.          0.84       -0.378     ]
 [ 0.          0.          0.         -0.28720648]]
[-4.         -0.35990788 -2.          1.36764993]
```

Out[13]:

```
array([ 3.72380952,  3.26190476, -4.52380952, -4.76190476])
```

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Resources: Book ILA Ch. 11