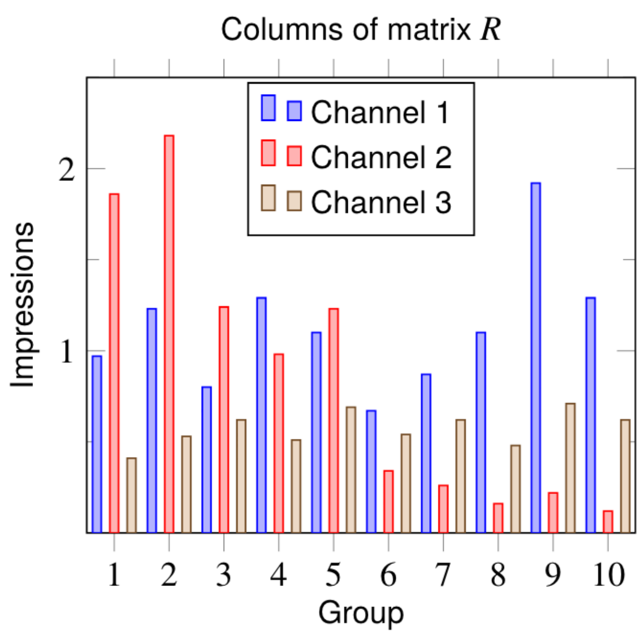
# 11 Least Squares


Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

Unit 3: Least Squares, Book ILA Ch. 12-14

- ***11 Least Squares***
- 12 Least Squares Data Fitting
- 13 Least Squares Classification

# Outline: 11 Least Squares

- Least Square Problem
- Solution of Least Square Problem

- Examples

# Survey Results

Videos summarizing some of the concepts of this class:

- https://www.3blue1brown.com/topics/linear-algebra

Running the code from the lectures by clicking on Binder:

- https://github.com/bioshape-lab/ece3

Final preparation:

- Exercises from HW and class
- Review session with exercises
- Mock exam

# Least Squares Problem

Definition: Let be given a $m \times n$ matrix $A$ and $m$-vector $b$. The least squares problem is the problem of choosing an $n$-vector $x$ to minimize:

$$||Ax - b||^2.$$

- $||Ax - b||^2$ is called the objective function,
- If $\hat{x}$ is a solution of the linear equation $Ax = b$, then $\hat{x}$ is a solution of the least square problem. The converse is not true.
- $\hat{x}$ is a solution of least squares problem if $||A\hat{x} - b||^2 \leq ||Ax - b||^2$ for any other $n$-vector $x$.

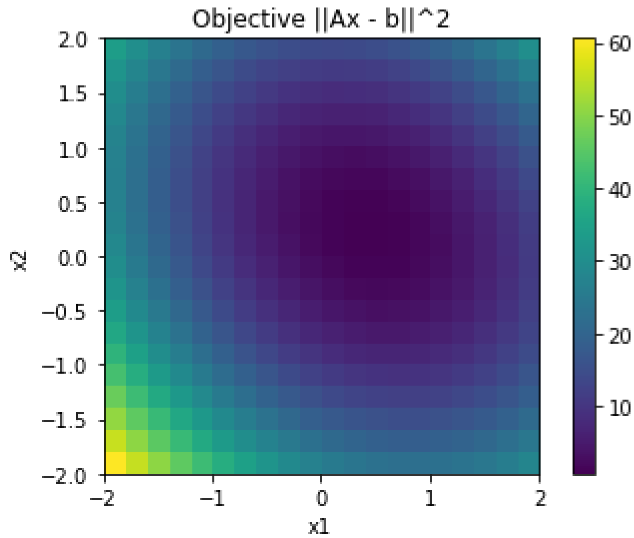
Exercise: Consider the matrix $A$ and vector $b$ as:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Write the objective function associated to the least square problem defined by $A$ and $b$ in terms of entries of $x$.

In [19]:

```
import numpy as np; import matplotlib.pyplot as plt
objective = lambda x : (2 * x[0] - 1) ** 2 + (- x[0] + x[1]) ** 2 + (2 * x[1] +
n_points, xmin, xmax, ymin, ymax = 20, -2, 2, -2, 2
x = np.arange(xmin, xmax, (xmax-xmin)/n_points; y = np.arange(ymin, ymax, (ymax-
for i in range(n_points):
    for j in range(n_points):
        Z[i, j] = objective(xx[i, j])
plt.imshow(Z, extent=[xmin, xmax, ymin, ymax]); plt.colorbar(); plt.xlabel("x1")
```




# Outline: 11 Least Squares

- Least Square Problem
- **Solution of Least Square Problem**
- Examples

# Least Square Solution

Proposition:

- Consider a least square problem $||Ax - b||^2$ for matrix $A$ and vector $b$.
- Assume that $A$ has linearly independent columns.

Then, there is a unique solution $\hat{x}$ to the least square problem, defined as:

$$\hat{x} = (A^T A)^{-1} A^T b = A^{\dagger} b.$$

- $A^{\dagger} = (A^T A)^{-1} A^T$ is called the pseudo-inverse of $A$.

Exercise (hard): Using the fact that:

$$||a + b||^2 = ||a||^2 + ||b||^2 + 2a^T b,$$

prove that $\hat{x}$ defined in the previous slide is indeed a solution.

- Hint: Show that for any other $n$-vector $x$, we have:

$$||Ax - b||^2 \geq ||A\hat{x} - b||^2.$$

- Hint 2: You will need to show that $A^T(A\hat{x} - b) = 0$.

In Python, we use:

- the function `np.linalg.lstsq` : returns the solution as the first element of the returned tuple
- the formula of the solution using transpose `.T` , inverse and matrix multiplication.

In [20]:
```
A = np.array([
    [2, 0],
    [-1, 1],
    [0, 2]
])
b = np.array([1, 0, -1])
```

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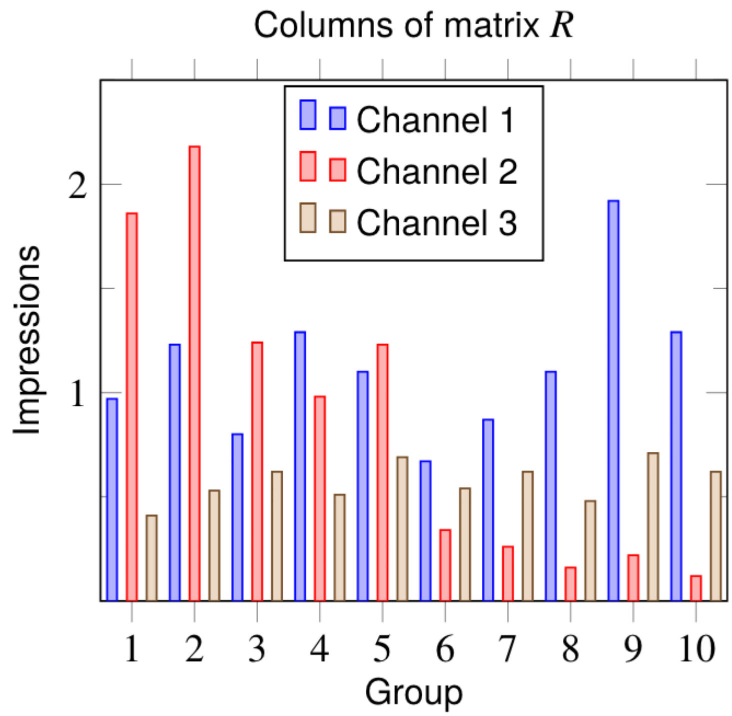
# Political Advertising


# Example: Political Advertising

- A company wants to advertize to potential voters.
  - $m$ demographics groups, $n$ advertising channels
  - $v^{target}$ is $m$-vector of target views ("impressions") per group
  - $s$ is $n$-vector of spending per channel
  - $R$ is $m \times n$ matrix of demographic reach of channels:
    - $R_{ij}$ is number of views per dollar spent (in 1000/$)
- How much should be spent to be as close as possible to $v^{target}$?

Example: What is the optimal spending $\hat{s}$?

- $m = 10$ groups and $n = 3$ channels,
- $v^{target} = 1000.\,1_m$.




In [9]:

```
import numpy as np

R = np.array([
    [0.97, 1.86, 0.41],
    [1.23, 2.18, 0.53],
    [0.8, 1.24, 0.62],
    [1.29, 0.98, 0.51],
    [1.1, 1.23, 0.69],
    [0.67, 0.34, 0.54],
    [0.87, 0.26, 0.62],
    [1.1, 0.16, 0.48],
    [1.92, 0.22, 0.71],
    [1.29, 0.12, 0.62]
])
```

# Example: Illumination


# Example: Illumination (Hard)

- $n$ lamps illuminate an area divided in $m$ regions
- $A_{ij}$ is illumination in region $i$ if lamp $j$ is has power 1, other lamps are off
  - inversely proportional to square distance to the lamp
  - average of $A$ is 1.
- $x_j$ is power of lamp $j$
- $(Ax)_i$ is illumination level at region $i$
- $b_i$ is target illumination level at region $i$

What is the power that each lamp should have in order to reach the target illumination? We just explain how to set up the problem.

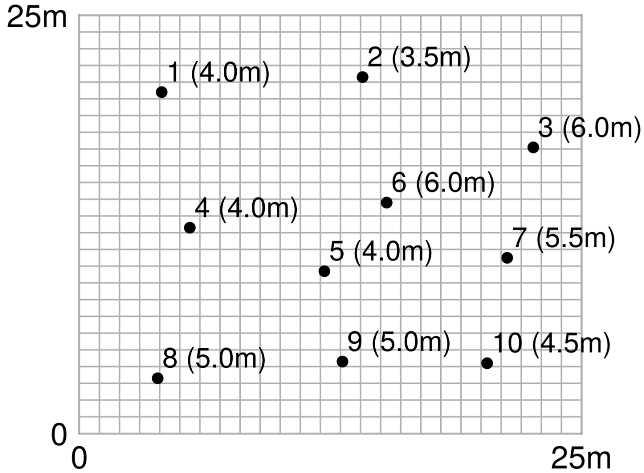
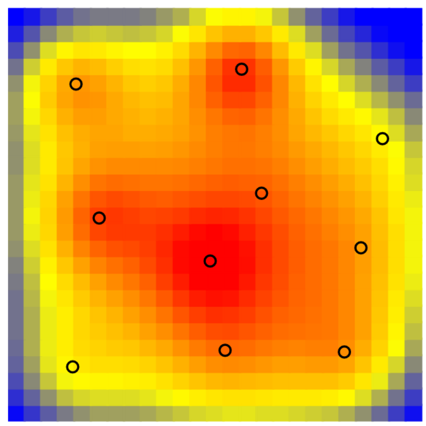



figure shows lamp positions for example with

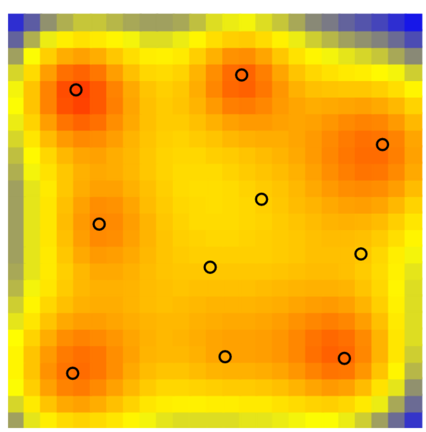
$$m = 25^2, \quad n = 10$$

The number in parenthesis indicates the height of the lamp (not taken into account for this exercise).

- equal lamp powers ($x = \mathbf{1}$)



- least squares solution $\hat{x}$, with $b = \mathbf{1}$



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Resources: Book ILA Ch. 11