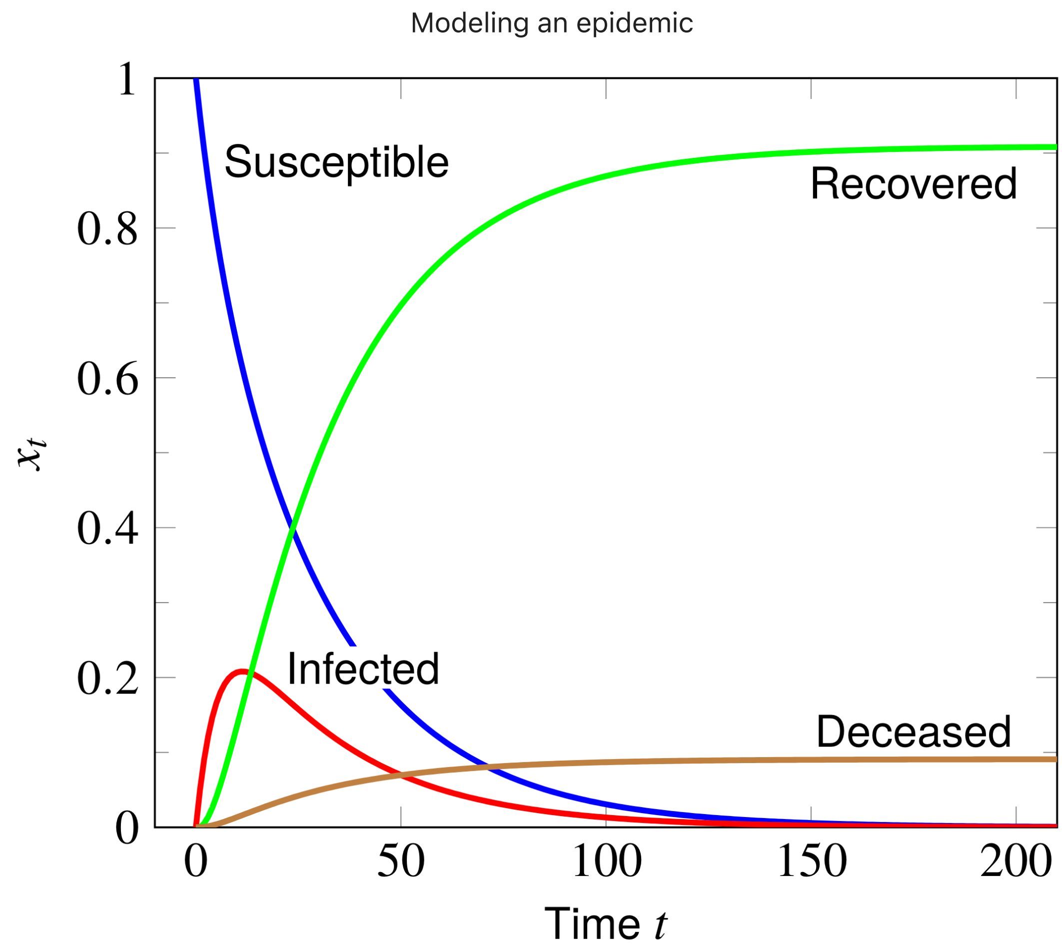
07 Linear Dynamical Systems



Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

- ***06 Matrices***
- ***07 Linear Equations***
- 08 Linear Dynamical Systems
- 09 Matrix Multiplication
- 10 Matrix Inverse

Unit 3: Least Squares, Book ILA Ch. 12-14 + Book IMC Ch. 8

Unit 4: Eigen-decomposition, Book IMC Ch. 10, 12, 19

Exercises

Exercise: Let 0_n be the $n \times n$ zeroes matrix and x an n-vector. Compute $0_n x$.

Exercise: Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$, that is affine. What is the matrix A and the vector b such that: $f(x) = Ax + b$?

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Outline: 08 Linear Dynamical Systems

- **Linear dynamical systems**
- Epidemic dynamics

State sequence

Definition: Consider a sequence of n-vectors $x_1, x_2, \ldots, x_t, \ldots$ where t denotes the time or period. Then:

- x_t is called a state at time t, or a time-point
- the sequence $x_1, x_2, \ldots, x_t, \ldots$ is called a state trajectory or a time-series.

Assuming t is the current time:

- x_t is the current state,
- x_{t-1} is the previous state,
- x_{t+1} is the next state.

Covid-19 in Santa Barbara

In [69]:

```
import pandas as pd

filename = 'https://raw.githubusercontent.com/CSSEGISandData/COVID-19/master/css
df_covid = pd.read_csv(filename)
df_sb = df_covid[df_covid["Admin2"] == "Santa Barbara"]
df_sb.head()
```

Out[69]:

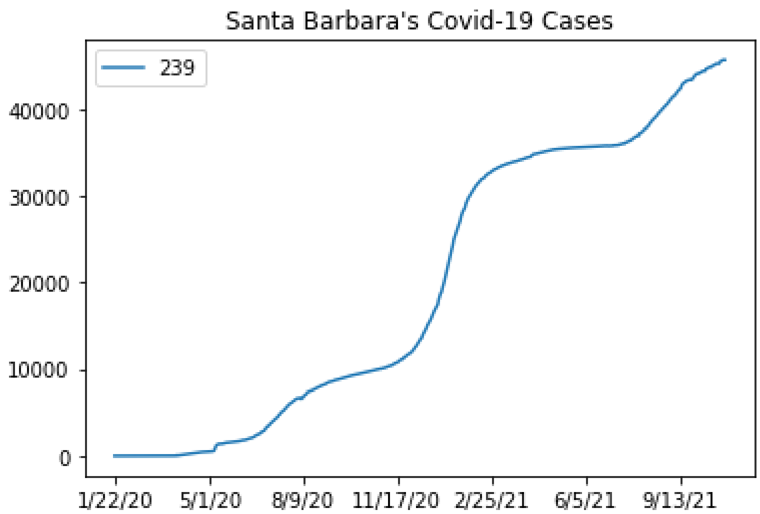
	UID	iso2	iso3	code3	FIPS	Admin2	Province_State	Country_Region	Lat

	UID	iso2	iso3	code3	FIPS	Admin2	Province_State	Country_Region	Lat
239	84006083	US	USA	840	6083.0	Santa Barbara	California	US	34.653295

1 rows × 660 columns

In [71]:

```
df_timeseries = df_sb.drop(
    columns=[
        "Combined_Key", "UID", "iso2", "iso3", "code3",
        "FIPS", "Admin2", "Province_State", "Country_Region",
        "Lat", "Long_"])
df_timeseries.transpose().plot(
    title="Santa Barbara's Covid-19 Cases");
```



Other Examples

- epidemiology: susceptible, infected, recovered and deaths at t
- mechanics: position and velocity of plane through time t
- finance: return of n assets at time t

Exercise: In these examples, what is the state x_t?

Linear Dynamical System

Definition: A linear dynamical system is a linear equation describing the evolution of a state over time, and written as:

$$x_{t+1} = A_t x_t \quad t = 1, 2, \ldots$$

where:

- A_t are $n \times n$ dynamics matrices,
- $(A_t)_{ij}(x_t)_j$ is contribution to $(x_{t+1})_i$ from $(x_t)_j$.

The system is called time-invariant if $A_t = A$ does not depend on time.

In Python: We can simulate evolution of x_t using for loops to compute:

$$x_{t+1} = A_t x_t$$

recursively (see later slide on the epidemics model).

Variations: Linear Dynamical System with Input

Definition: The following system is called a linear dynamical system with input:

$$x_{t+1} = A_t x_t + B_t u_t + c_t \quad t = 1, 2, \ldots$$

- u_t is an input m-vector
- B_t is $n \times m$ input matrix
- c_t is offset.

Variations: K- Markov Model

Definition: The following system is called a K-Markov model or an auto-regresssive model :

$$x_{t+1} = A_1 x_t + \ldots + A_K x_{t-K+1} \quad t = K, K+1, \ldots$$

- next state depends on current state and K – 1 previous states
- for K = 1:
  - time-invariant linear dynamical system $x_{t+1} = Ax_t$.

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SIR Model

Definition: The Susceptible-Infected-Recovered (SIR) model of an epidemic is a linear dynamical system defined on a 4-vector state x_t, that gives the proportion of the population in 4 infection states:

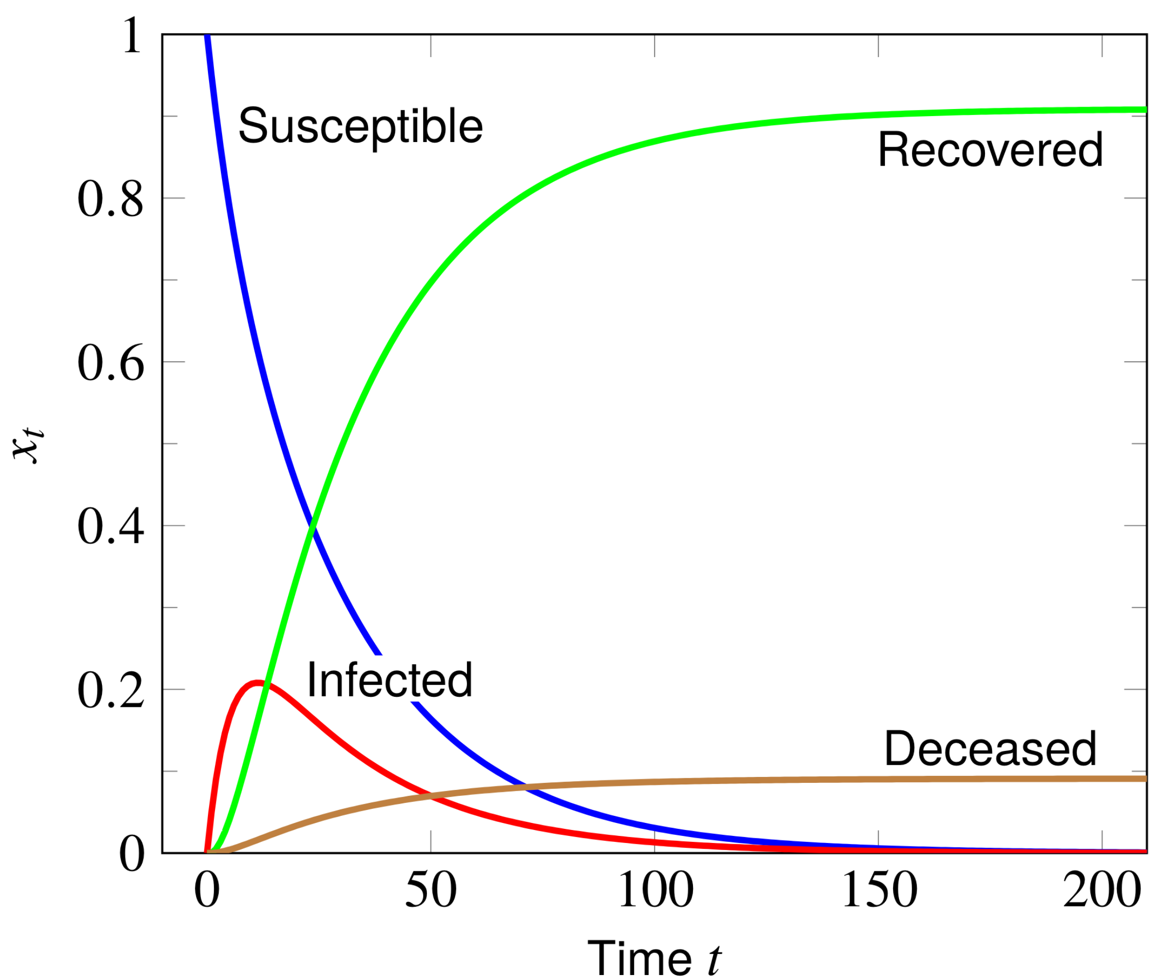
- Susceptible: can acquire the disease the next day
- Infected: have the disease
- Recovered: had the disease, recovered, now immune
- Deceased: had the disease, and unfortunately died

Example of state: $x_t = (0.75, 0.10, 0.10, 0.05)$

Exercise Write the linear dynamical system associated to this epidemic.

- among susceptible population
  - 5% acquires the disease
  - 95% remain susceptible
- among infected population
  - 1% dies
  - 10% recovers with immunit
  - 4% recover without immunity (i.e., become susceptible)
  - 85% remain infected
- 100% of immune and dead people remain in their state

Simulation from $x_0 = (1, 0, 0, 0)$



Exercise: Recreate this plot.

In [72]:

```
import matplotlib.pyplot as plt
import numpy as np

x0 = np.array([1, 0, 0, 0])
A = np.array([
    [0.95, 0.04, 0, 0],
    [0.05, 0.85, 0, 0],
```

```
    [0, 0.1, 1, 0],
    [0, 0.01, 0, 1]
])

...
```

Out[72]:
```
Ellipsis
```

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Resources: Book ILA Ch. 09