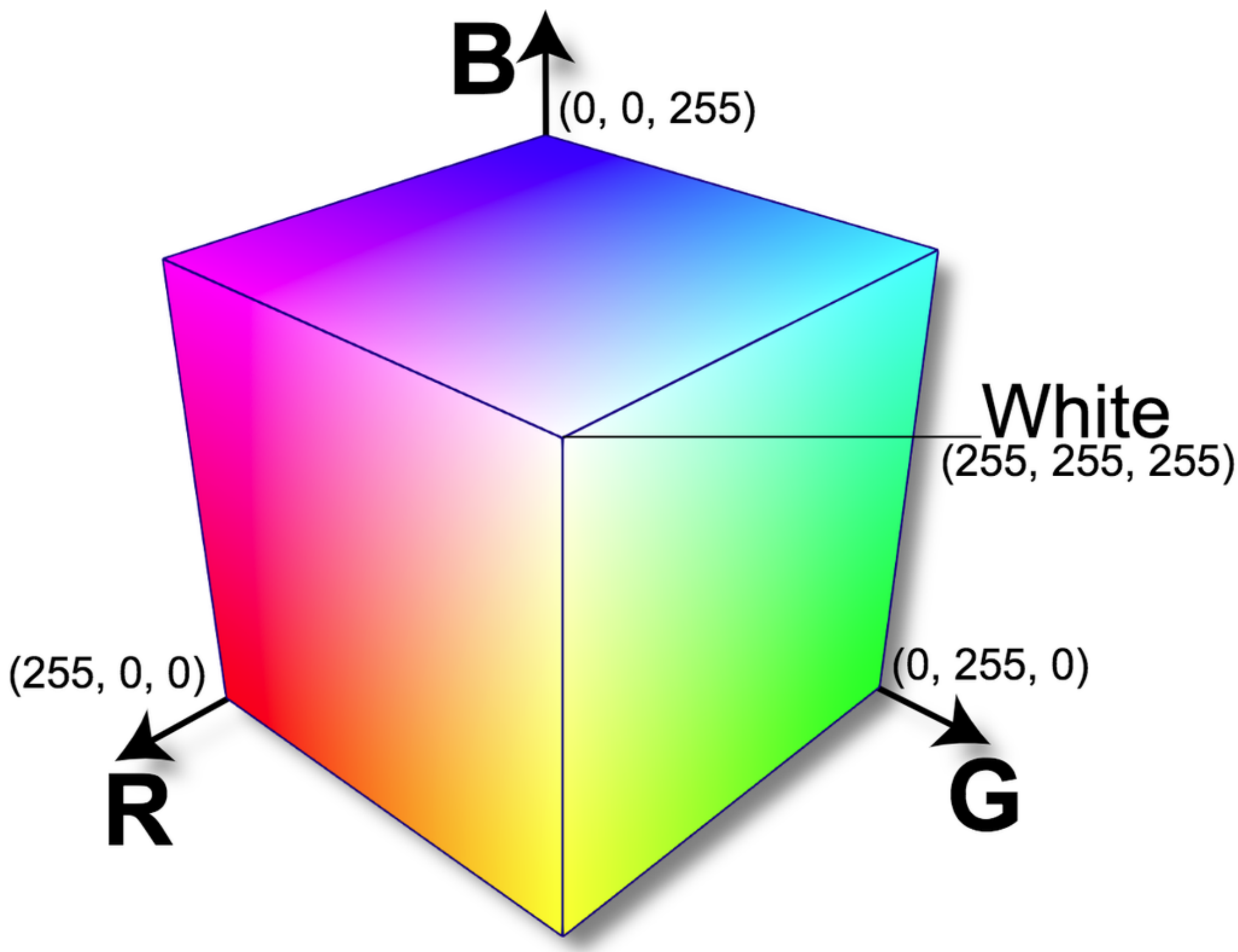
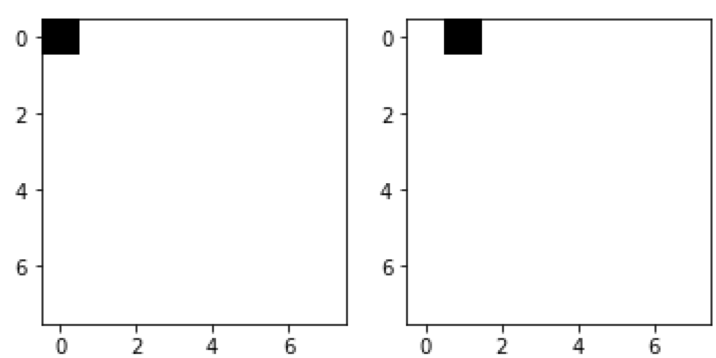
05 Linear Independence





First two elements of a basis of the space of images.

Unit 1: Vectors, Book ILA Ch. 1-5

Unit 2: Matrices, Book ILA Ch. 6-11 + Book IMC Ch. 2

Unit 3: Least Squares, Book ILA Ch. 12-14 + Book IMC Ch. 8

Unit 4: Eigen-decomposition, Book IMC Ch. 10, 12, 19

Outline: 05 Linear Independence

Linear Dependence

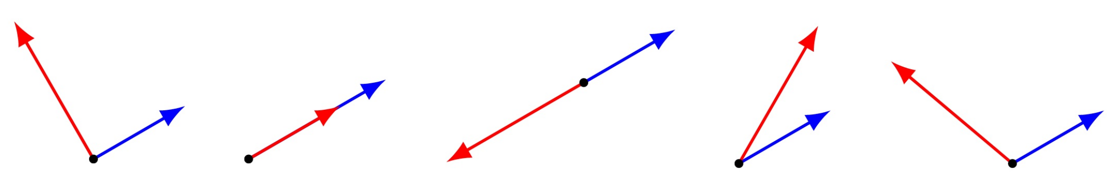
Definition: A set of n-vectors $a_1, \ldots, a_k$ (with $k \geq 1$) is linearly dependent if:

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0$$

holds for some $\beta_1, \ldots, \beta_k$ that are not all zero. This statement is equivalent to: at least one a_i is a linear combination of the others.

- a_1 is linearly dependent only if $a_1 = 0$.
- a_1, a_2 is linearly dependent only if one a_i is multiple of the other.
- for $k > 2$, there is no simple way to state condition.

Exercise: Which sets of vectors are linearly dependent?



Example: The following vectors:

$$a_1 = \begin{pmatrix} 0.2 \\ -7 \\ 8.6 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -0.1 \\ 2 \\ -1 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 \\ -1 \\ 2.2 \end{pmatrix}$$

are linearly dependent because:

- $a_1 + 2a_2 - 3a_3 = 0$,
- we can express any of them as linear combination of the other two, e.g., $a_2 = (-1/2)a_1 + (3/2)a_3$

Linear Independence

Definition: The set of n-vectors $a_1, \dots, a_k$ (with $k \geq 1$) is linearly independent if it is not linearly dependent, i.e. if

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0$$

holds only when $\beta_1 = \cdots = \beta_k = 0$. This statement is equivalent to: no a_i is a linear combination of the others.

Example: The one-hot n-vectors $e_1, \dots, e_n$ are linearly independent.

Property of linear independence

Property: Take linearly independent vectors $a_1, \dots, a_k$. Suppose x is linear combination of the $a_1, \dots, a_k$:

$$x = \beta_1 a_1 + \cdots + \beta_k a_k.$$

Then, the coefficients $\beta_1, \dots, \beta_k$ are unique.

Remark: This means that, in principle, we can compute the coefficients from x.

Linear (in)dependence & dimension

Propositions:

- A linearly independent set of n-vectors $a_1, \dots, a_k$ can have at most n elements, i.e. $k \leq n$.
- Any set of $n + 1$ or more n-vectors is linearly dependent.

Exercise: Are the vectors $(1, 2)$, $(2, 3)$ and $(3, 0)$ linearly dependent or independent?

Outline: 05 Linear Independence

Basis

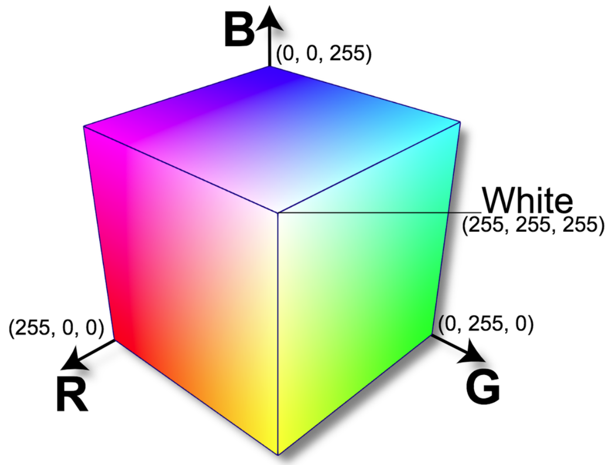
Definition: A set of n linearly independent n-vectors $a_1, \dots, a_n$ is called a basis of the space of n-vectors.

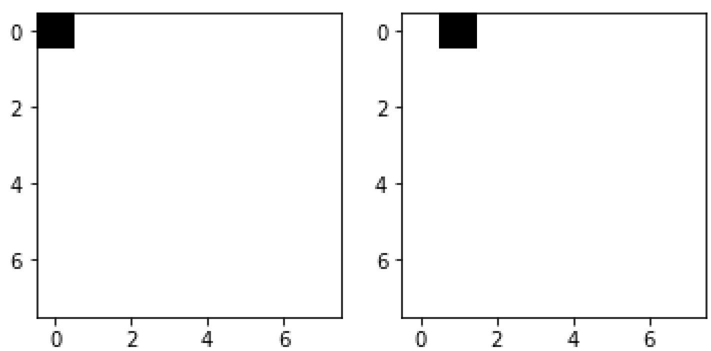
Example: The set of n one-hot n-vectors is a basis of the space of n-vectors. This is why we can also call them "basis vectors".

Example: What is a possible basis:

- For the space of colors represented as vectors (r, g, b)?
- For the space of documents with dictionary D?
- For the space of 8×8 images?

- Space of colors as 3-vectors: (r, g, b)
  - A basis: set of vectors representing red, green and blue.
- Space of documents as n-vectors on a dictionary of n words
  - A basis: set of vectors representing on dictionary word each.
- Space of 8x8 images as 64-vectors
  - A basis: set of images with only one black pixel.





Properties of a basis

Properties: If $a_1, \ldots, a_n$ is a basis of the space of n-vectors, then:

- Any n-vector b:
  - can be expressed as a linear combination of them: $b = \beta_1 a_1 + \cdots + \beta_n a_n$ for some $\beta_1, \ldots, \beta_n$.
  - and the coefficients $\beta_1, \ldots, \beta_n$ are unique.

Definition: The formula above is called expansion of b in the $a_1, \ldots, a_n$ basis.

Example: Compute expansion of $b = (b_1, \ldots, b_n)$ in basis $e_1, \ldots, e_n$

Outline: 05 Linear Independence

- Linear Independence
- Basis
- **Orthonormal Vectors**

Orthogonal, Normalized, Orthonormal

Definitions: The n-vectors $a_1, ..., a_k$ are:

- (mutually) orthogonal if $a_i \perp a_j$ for all i, j
- normalized if $||a_i|| = 1$ for $i = 1, ..., k$
- orthonormal if they are orthogonal and normalized.

Example: Give examples of orthogonal, normalized and orthogonal vectors in 2D. Draw them.

Properties of Orthonormal Vectors

Properties: If $a_1, ..., a_k$ are orthonormal, then:

- $a_i^T a_j = 0$ if $i \neq j$ and $a_i^T a_i = 1$,
- $k \leq n$,
- $a_1, ..., a_k$ are linearly independent.

Definition: A set of n orthonormal n-vectors is called an orthonormal basis.

Examples of Orthonormal Basis

Examples:

- One-hot n-vectors: $e_1, ..., e_n$
- The 3-vectors:

$$a_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

Exercise: Show that the 3-vectors above form an orthonormal basis using math computations.

Exercise: Show that these form an orthonormal basis using Python:

$$a_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

In [ ]:
```
a1 = ...
```

Orthonormal Expansion

Properties: If $a_1, \ldots, a_n$ is an orthonormal basis of the n-vectors, then:

- Any n-vector x can be written:

$$x = (a_1^T x)a_1 + \cdots + (a_n^T x)a_n.$$

Definition: The formula above is called orthonormal expansion of x in the orthonormal basis.

Outline: 05 Linear Independence

- Linear Independence
- Basis
- Orthonormal Vectors

Resources: Book ILA, Ch. 5