Notes on Waves

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Contents

1 Simple Harmonic Motion

Simple Harmonic Motion
Periodic motion where the restoring force is directly proportional to the magnitude of the object's displacement from the equilibrium. [1]

$$F = -kx, k > 0 \tag{1}$$

[1] The restoring force is always directed towards the equilibrium, indicated by the negative sign and k>0

1.1 Equations of Motion for SHM

By Newton's Second Law and Hooke's Law

$$\begin{aligned} F = ma &= -kx \\ m\ddot{x} &= -kx \\ \ddot{x} + \frac{k}{m}x &= 0 \end{aligned}$$

Angular Frequency: Spring-Mass
For a spring-mass system, we set:

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} && (2) \\ &= 2\pi f && (3) \\ &= \frac{2\pi}{T} && (4) \end{aligned}$$

Angular Frequency: Pendulum For rotational motion, we can use the rotational form of Newton's Second Law.

$$\tau = I\alpha = mgx_{cm}$$

For a simple pendulum (massless rod of length l and point mass m), The torque by gravity is $-mg\sin\theta \cdot l$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

For small angles, approximate $\sin\theta \approx \theta$ and so we have the same 2nd Order ODE form as the spring-mass system:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Thus,

$$\begin{aligned} k &= \frac{mg}{l} \\ \omega &= \sqrt{\frac{g}{l}} && (5) \end{aligned}$$

If the pendulum has a moment of inertia, [2]

$$\begin{aligned} \ddot{\theta} + \frac{mgx_{cm}}{I}\theta &= 0 \\ \omega &= \sqrt{\frac{mgx_{cm}}{I}} && (6) \end{aligned}$$

[2] Usually we will have a rod with uniform density pivoting at the end (i.e. same as simple pendulum except now the rod has a mass), then $I = \frac{1}{3}ml^2$ and $x_{cm} = l/2$, so $\omega = \sqrt{\frac{mgl/2}{1/3ml^2}} = \sqrt{\frac{3g}{2l}}$

Angular Frequency: Electrical Circuits

For an LC circuit: By Kirchhoff's Voltage Law we can use the voltage of the inductor, $V_L = L\frac{dI}{dt}$, and that of the capacitor, $V_C = \frac{q}{C}$, to obtain:

$$\frac{q}{C} + L\frac{dI}{dt} = 0$$
$$\frac{q}{C} + L\frac{dq^2}{dt^2} = 0$$

Now we have the same form of 2nd order ODE as earlier:

$$\frac{dq^2}{dt^2} + \frac{1}{LC}q = 0 \therefore \omega = \sqrt{\frac{1}{LC}} \tag{7}$$

Expressing these oscillators as a differential equation (substitute x with θ or whatever),

$$\ddot{x} + \omega^2 x = 0$$

We obtain the solution,

$$x = A\cos(\omega t + \phi)$$

And because we can use the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, we can solve for ϕ given an x coordinate using

$$x = A_1\cos(\omega t) + A_2\sin(\omega t)$$

Taking the time derivatives:
Equations of Motion

$$x = A\cos(\omega t + \phi) \tag{8}$$
$$v = \dot{x} = -\omega A\sin(\omega t + \phi) \tag{9}$$
$$a = \ddot{x} = -\omega^2 A\cos(\omega t + \phi) \tag{10}$$
$$= -\omega^2 x \tag{11}$$

1.2 Energy in SHM

Energy Formulas: Mechanical Oscillators This needs to be explained better lol.
Kinetic Energy:

$$T = \frac{1}{2}mv^2 \tag{12}$$

Potential Energy: Potential energy is equal to

$$U = -\int F dx = -\int(-kx)dx \qquad = \frac{1}{2}kx^2 \tag{13}$$

For a pendulum, our restoring force points downward (y-axis) so

$$U = mgy = mgl(1 - cos\theta) \tag{14}$$

Total Energy: [3]

$$E = T + U \tag{15}$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{16}$$
$$= \frac{1}{2}kA^2 \tag{17}$$

[3] Note that total energy must be conserved in a simple harmonic oscillator since no forms of energy loss are present, simply a linear restoring force exists. At the oscillation amplitude, the velocity is 0 and so T=0 and U=E. At equilibrium, displacement is 0 and so U=0 and all the potential energy has been converted to kinetic (T=E)

Energy Formulas: Electrical Oscillators

Energy in capacitor:

$$E_C = \frac{1}{2}CV_C^2 = \frac{q^2}{2C} \tag{18}$$

Energy in inductor:

$$E_L = \frac{1}{2}LI^2 \tag{19}$$

Total Energy:

$$\begin{aligned} E &= E_C + E_L & (20) \\ &= \frac{1}{2}CV_C^2 + \frac{1}{2}LI^2 & (21) \end{aligned}$$

2 The Damped Harmonic Oscillator

2.1 Equations of Motion

Damping harmonic motion occurs when a damping force is applied to the oscillator:

$$F_D = -bv \tag{22}$$

Where,
b = drag coefficient

> Using Newton's Second Law we can equate forces and express the motion as a 2nd Order ODE:
>
> $$ma = -kx - bv$$
> $$\ddot{x} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

Beacuse this is a second order differential equation, there are three possible solutions based on the discriminant ($b^2 - 4ac$) of the characteristic equation of the ODE: $r^2 + \gamma r + \omega_0^2$

$$\Delta = \gamma^2 - 4(1)(\omega_0^2)$$
$$\Delta = \omega_0^2 - \frac{\gamma^2}{4} \tag{23}$$

2.2 Frequency and Damping

Natural angular frequency is the angular frequency without any damping:

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{24}$$

There are a bunch of damping stuffs and we need to convert between them depending on what is given in a question:

$$\beta = \frac{\gamma}{2} = \frac{b}{2m} \tag{25}$$
$$\gamma = \frac{b}{m} = \frac{1}{\tau} \tag{26}$$

Where,
τ = lifetime of oscillation [s]

The "actual" angular frequency of the oscillation is given by the discriminant, [4]

> **Frequency of Damped Oscillators**
>
> $$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{27}$$
>
> This means we can solve for:
>
> - ω given period T or some damping coefficient (+ maybe mass) + nat-

[4] Since the frequency should be a positive value, our typical periodic motion is always underdamped. When $\omega <= 0$ it's not really oscillating anymore so we know its critically or overdamped case

ural frequency

- Obtain some damping coefficient or a variable that the ω_0 is a function of given ω

Solutions to Each Type of Dampened Harmonic Oscillation
Underdamped: In the first case, the discriminant is less than 0.

$$\omega^2 > 0 \tag{28}$$
$$x = A_0 e^{-\gamma t/2}\cos(\omega t + \phi) \tag{29}$$
$$\tag{30}$$

Critically Damped: The critically damped case is when the decay is quickest, so we can use (27) and (25) to find a value for γ or b that will give use the fastest time to equilibrium.

$$\omega^2 = 0 \tag{31}$$
$$x = Ae^{-\gamma t/2} + Bte^{-\gamma t/2} \tag{32}$$
$$v = (B - \frac{\gamma A}{2} - \gamma\frac{Bt}{2})e^{-\gamma t/2} \tag{33}$$
$$v_0 = B - \gamma x_0/2 \implies B = v_0 + \gamma x_0/2 \tag{34}$$
$$\therefore x = ((v_0 + \frac{\gamma x_0}{2})t + x_0)e^{-\gamma t/2} \tag{35}$$

Overdamped: This is so lame why would you do a question on this.

$$\omega^2 < 0 \tag{36}$$

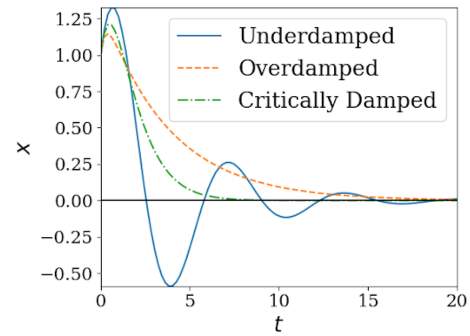

Figure 1: Comparison of different damped oscillations, source: http://www.physicsbootcamp.org/Damped-Harmonic-Oscillator.html

2.3 Energy

Damping removes energy from the system and this energy decay is exponential, decaying to $1/e$ of its original amplitude after $Q\pi$ cycles. The rate of energy change is power: [5]

[5] we can obtain this expression from P=Fv or taking the derivative of E = T + U

Power in DHO

$$P = Fv = -bv(v) = -bv^2 \tag{37}$$
$$= \frac{dE}{dt} = -\frac{kA_0^2}{\tau} = idk : (\tag{38}$$

Energy in DHO
The solutions shown earlier and the energy equation for underdamped oscillations are pretty much the same as SHM, it just adds an exponential decay term:

$$E = \frac{1}{2}kA_0^2 e^{-\gamma t} = E_0 e^{-\gamma t} \tag{39}$$

Note that amplitude decay squared is equal to energy decay.
Also, if we obtain the amplitudes at t_0 and 1 oscillation later at $t_0 + T$, the

cosines are equal:

$$\frac{A_{n+1}}{A_n} = e^{-\gamma T/2} \tag{40}$$

2.4 Quality Factor

The quality factor is how many times an underdamped oscillator can oscillate before dying, or the energy in oscillator / energy lost per radian

$$Q = \frac{\omega_0}{\gamma} = 2\pi\frac{\tau}{T} = \omega\tau \tag{41}$$

$$= 2\pi\frac{E}{\Delta E} = 2\pi\frac{A}{\Delta A} \tag{42}$$

We can add this to our list of damping stuffs in (23/24):

$$\omega^2 = \omega_0^2(1 - \frac{1}{4Q^2}) \tag{43}$$

We can also see that $\frac{Q}{\pi}$ is equal to the number of cycles it takes to reach 1/e of its original amplitude, and $\frac{Q}{2\pi}$ is equal to the number of cycles it takes to reach 1/e of its original energy

$$\begin{aligned}
\frac{A_i}{A_{i+n}} &= e^{-\frac{\gamma t}{2} - (-\frac{\gamma(t+nT)}{2})} \\
\frac{1}{e} &= e^{-\frac{\gamma nT}{2}} \\
-1 &= -\frac{\gamma nT}{2} \\
2 &= \frac{\omega}{Q}nT \\
Q &= 2\frac{2\pi}{T}nT \\
&= \pi n
\end{aligned}$$

Where, n = number of cycles to reach 1/e of original amplitude

2.5 Electrical Oscillators

Resistors act as damping in electrical circuits (dissipate power), so for an RLC circuit:

$$\gamma = \frac{R}{L} \tag{44}$$

$$\omega_0^2 = \frac{1}{LC} \tag{45}$$

$$\omega^2 = \frac{1}{L}(\frac{1}{C} - \frac{R^2}{4L}) \tag{46}$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{47}$$

3 Forced Oscillations

Forced or driven oscillators are those that have a (typically sinusoidal) force imposed on the system. The oscillator will *always* take on the frequency of the driving force, regardless of ω_0 and γ, at the cost of amplitude.

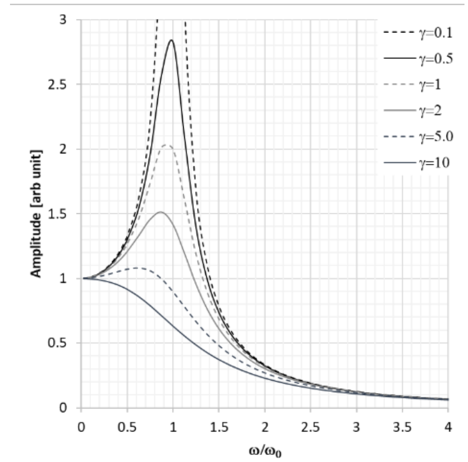

Figure 2: Amplitude vs driving frequency / natural frequency. Notice the amplification at resonance and how $A \to 0$ as $\omega \to \infty$

Resonance
When driven frequency is close to the natural frequency of the oscillation

- At resonance, the amplitude is the maximum (for undamped) because the driving force acts as an amplifier to the original:
  $A(\omega)_{max} \approxeq A(\omega_0)$, depending on damping
  This is because we have maximum power absorption at this point
- As the driving frequency increases, the amplitude decreases and the mass moves out of phase with the driving force
- As the driving frequency approaches infinity, the oscillator does not move due to the mass' inertia

3.1 Equations of Motion

3.1.1 Undamped

[6]

Using Newton's Second Law,

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos(\omega t) \tag{48}$$

$$\frac{d^2x}{dt^2} = \frac{-k}{m}(x - a\cos(\omega t)) \tag{49}$$

Where,
$F_0 = ka =$ Driving amplitude (without oscillation)

[6] We assume the driving force is sinusoidal and has no phase angle

Solutions and Time Derivatives of Undamped Driven Oscillations
The solution and its time derivatives are,

$$x = A(\omega)\cos(\omega t - \delta) \tag{50}$$

$$v = \dot{x} = -A(\omega) \cdot \omega \sin(\omega t - \delta) \tag{51}$$

$$a = \ddot{x} = -A(\omega) \cdot \omega^2 \cos(\omega t - \delta) \tag{52}$$

Where,
$\delta =$ phase angle between driving force and resultant displacement

[7]

By substituting x and a into the equation of motion, the simplifying, we obtain:

$$A(\omega)[-\omega^2 + \omega_0^2]\cos\delta = \omega_0^2 a$$
$$A(\omega)[-\omega^2 + \omega_0^2]\sin\delta = 0$$
$$\therefore \tan\delta = 0$$

[7] Note that the -δ implies the displacement always lags behind the driving force
$\delta \to 0$ as $\omega \to 0$, $\delta \to \pi/2$ as $\omega \to \infty$

From this, we can obtain expressions for the phase angle between the driver and oscillator, and the amplitude:

Phase Angle and Amplitude of Undamped Driven Oscillators

$$\delta = 0, \pi \tag{53}$$

$$A(\omega) = \frac{a}{1 - \frac{\omega^2}{\omega_0^2}}, \quad \omega < \omega_0 \tag{54}$$

$$A(\omega) = \frac{-a}{1 - \frac{\omega^2}{\omega_0^2}}, \quad \omega > \omega_0 \tag{55}$$

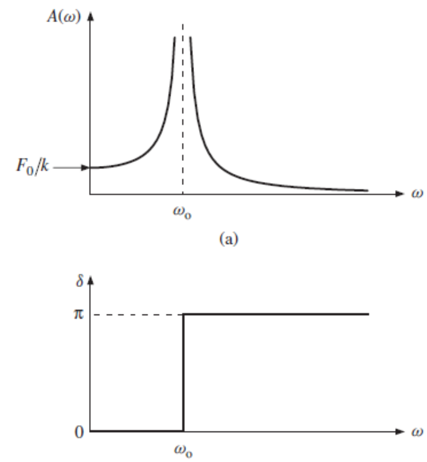

Figure 3: Undamped Amplitude and Phase Behaviour: Note that as $\omega \to 0$, $A \to a = \frac{F_0}{k}$

3.1.2 Damped

Using Newton's Second Law and skipping some steps,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t) \tag{56}$$

We obtain the same solutions as the undamped case, however our phase angle and amplitude functions have changed:

Phase Angle and Amplitude of Damped Driven Oscillators

$$\tan(\delta) = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \tag{57}$$

$$A(\omega) = \frac{a\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \tag{58}$$

$$A(\omega)_{max} \text{ occurs when },\tag{59}$$

$$\omega = \frac{\omega_0}{\sqrt{1 - \frac{\gamma^2}{2\omega_0^2}}} = \frac{\omega_0}{\sqrt{1 - \frac{1}{2Q^2}}} \tag{60}$$

$$A_{max} = \frac{aQ}{\sqrt{1 - \frac{1}{4Q^2}}} \tag{61}$$

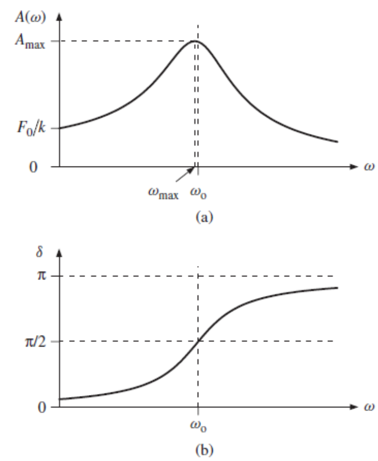

Figure 4: Damped Amplitude and Phase Behaviour: Substituting some values in to see end and resonance behaviour:
As $\omega \to 0$, $A \to a = \frac{F_0}{k}$ and $\delta \to 0$
As $\omega \to \omega_0$, $A \to \frac{a\omega_0}{\gamma}$ and $\delta \to \frac{\pi}{2}$
As $\omega \to \infty$, $A \to 0$ and $\delta \to \pi$

3.2 Energy

Maximum power occurs at the resonance frequency:

$$P = b[v_0(\omega)]^2 \sin^2(\omega t - \delta) \tag{62}$$

Where, $v_0(\omega) = A(\omega) \cdot \omega$
As expected by averaging the $\sin^2$ term, average power across a period is given by,

$$\bar{P}(\omega) = \frac{b[v_0(\omega)]^2}{2} \tag{63}$$

Power loss is greatest when $\omega = \omega_0$. Substituting this into average power and simplifying,

$$\bar{P}_{max} = \frac{F_0^2}{2b} \tag{64}$$

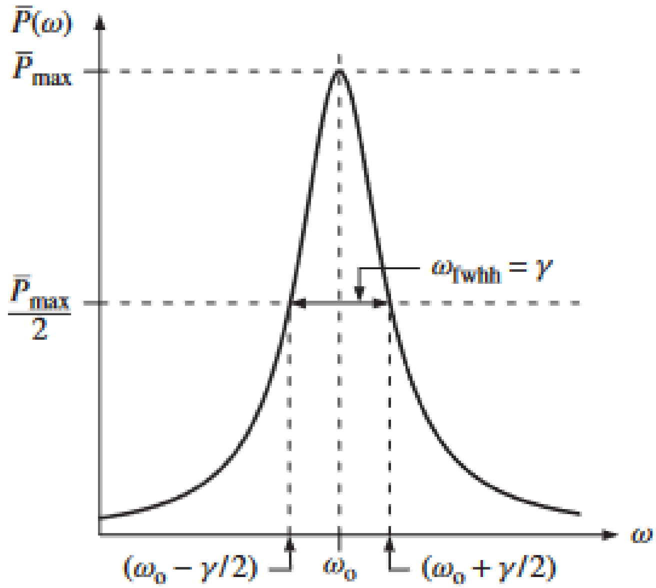

Figure 5: Power curve for driven damped oscillator with ω/ω_0 on x-axis

The power curve has one important parameter called the full width at half height (fwhh).

$$\omega_{fwhh} \approxeq \gamma \tag{65}$$

At half height ($\bar{P}_{max}/2$), $\omega \approx \omega_0$.
Thus we can call $\Delta\omega = \omega - \omega_0$, and

$$(\Delta\omega_{fwhh})^2 \approx \frac{\gamma^2}{4}$$

From the graph:

$$\omega_{fwhh} = 2\Delta\omega_{fwhh} \approx \gamma = \frac{\omega_0}{Q} \tag{66}$$

$$Q \approx \frac{\omega_0}{\omega_{fwhh}} \tag{67}$$

4 Coupled Oscillations

Coupling causes bodies to move with the same frequency. Normal modes are patterns of motion in which all parts of the system move in the same oscillatory motion with the same frequency and fixed phase relation:

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \vec{v} = -\omega^2 \vec{v} \tag{68}$$

More generally,

$$\begin{bmatrix} k_{11} & \dots & k_{1N} \\ \dots & \ddots & \dots \\ k_{N1} & \dots & k_{NN} \end{bmatrix} \cdot \vec{v} = -\omega^2 \vec{v} \tag{69}$$

Where,

ω = frequency of normal modes

4.1 Coupled Pendulums by a Spring: Equations of Motion

Case 1: displace both masses in the same direction by equal magnitude
Since both pendulums are in phase with the same frequency and amplitude, the spring remains unstretched and we just have two pendulums doing their thing. They will swing with identical phase and no relative change in position. The spring remains unstretched and exerts no force on either mass. We call this the **first mode of oscillation**.

Case 2: displace both masses in the opposite direction by equal magnitude
Now we have displaced pendulum A by x_A and B by $x_B = -x_A$, meaning the spring is stretched by $\Delta x = 2x_A$. So Newton's Second Law has the restoring force equal to the pendulum's usual gravitational force and an extra term created by the spring force.

$$m_A \ddot{x}_A = -\frac{mg}{l} x_A - k(2x_A) = -(\frac{mg}{l} + 2k) x_A \tag{70}$$

Bringing this to the same 2nd Order ODE form:

$$\ddot{x}_A = -(\frac{g}{l} + \frac{2k}{m}) x_A = -\omega_2^2 x_A \tag{71}$$

$$\therefore \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}} \tag{72}$$

We expect that the motions of these two pendulums are mirrored, affected by the spring's tension and compression. We call this the **second mode of oscillation**

[8]

These two modes of oscillation are the **normal modes**. For any system of two coupled pendulums, we can describe their motion as a superposition of the two normal modes of oscillations.

[8] Note that $\omega_2 > \omega_1$

Case 3: general motion of coupled oscillator with variable displacement
For displacement $x_a \neq \pm x_b$, we have a spring extension $\Delta x = (x_a - x_b)$. Thus our restoring forces become,

$$\ddot{x}_a + \frac{g}{l}x_a + \frac{k}{m}(x_a - x_b) = 0 \tag{73}$$

$$\ddot{x}_b + \frac{g}{l}x_b - \frac{k}{m}(x_a - x_b) = 0 \tag{74}$$

Symmetry Method
We can sum or substract these expressions to get stuff in terms of $(x_A - x_B)$ and $(x_A + x_B)$

$$\frac{d^2(x_A + x_B)}{dt^2} + \frac{g}{L}(x_A + x_B) = 0$$

$$\frac{d^2(x_A - x_B)}{dt^2} + (\frac{g}{L} + \frac{2k}{m})(x_A - x_B) = 0$$

We obtain two equations are just the two normal modes from Case 1 and Case 2: these equations are just SHM in terms of $x_A + x_B$ and $x_A - x_B$. We can treat these terms as the independent variable and we get the familiar SHM solutions:

$$\omega_1 = \sqrt{\frac{g}{L}}, \quad \omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}}$$

$$x_1 + x_2 = C_1 \cos(\omega_1 t)$$

$$\omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}}$$

$$x_1 - x_2 = C_2 \cos(\omega_2 t)$$

If we now isolate for each individual variable, we get the general solution:

$$x_1 = \frac{1}{2}[C_1 \cos(\omega_1 t) + C_2 \cos(\omega_2 t)]$$

$$x_2 = \frac{1}{2}[C_2 \cos(\omega_2 t) - C_2 \cos(\omega_2 t)]$$

Using an initial condition, we can fully solve the initial condition.
Let the initial displacements be $x_1(0) = a$, $x_2(0) = b$. Since cos(0) is 1

$$a = \frac{1}{2}(C_1 + C_2), \quad b = \frac{1}{2}(C_1 - C_2)$$

Then $C_1 = \frac{1}{2}(a + b)$ and $C_2 = \frac{1}{2}(a - b)$.

[9]

4.2 Coupled Springs

If we have two springs connected to each other and rigid walls by three springs, by Newton's Second Law:

[9] Note that if a = b (i.e. Case 1), $C_2 = 0$ and it's just the two pendulums of the motion $x = a\cos(\omega_1 t)$ which makes sense because $\omega_1 = \sqrt{\frac{k}{m}}$ is the usual SHM angular frequency. Similarly, if a = -b (i.e. Case 2), $C_1 = 0$ and we get the same equations of motion as in Case 2.

$$m\ddot{x}_a = -kx_a + k(x_b - x_a) = kx_b - 2kx_a \tag{75}$$
$$m\ddot{x}_b = -k(x_b - x_a) - kx_b = kx_a - 2kx_b \tag{76}$$

Normal frequencies of the coupled system
$\omega_1^2 = \frac{k}{m}, A = B$
$\omega_2^2 = \frac{3k}{m]}, A = -B$

Solutions

$$x_a = C_1 \cos(\omega_1 t) + C_2 \cos(\omega_2 t) \tag{77}$$
$$x_b = C_1 \cos(\omega_1 t) - C_2 \cos(\omega_2 t) \tag{78}$$
$$\tag{79}$$

[10]

4.3 Coupled Oscillations in Matrix Representation

We can rewrite the equations in the form

$$\frac{2k}{m}A - \frac{k}{m}B = A\omega^2 \tag{80}$$
$$\frac{-k}{m}A + \frac{2k}{m}B = B\omega^2 \tag{81}$$
$$\begin{bmatrix} \frac{2k}{m} - \omega^2 & \frac{-k}{m} \\ \frac{-k}{m} & \frac{2k}{m} - \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{82}$$

We can solve this system for omega to obtain the same values as earlier.

[10] Derivation: For normal mode of oscillation, they have the same frequency and are of the form $x_a = A\cos(\omega t), x_b = B\cos(\omega t)$ Substituting x_a into the equation of motion for x_a and x_b into the equation of motion for x_b

$$-Am\omega^2\cos(\omega t) = kB\cos(\omega t) - 2kA\cos(\omega t)$$
$$\therefore \frac{A}{B} = \frac{k}{2k - m\omega^2}$$

$$-Bm\omega^2\cos(\omega t) = kA\cos(\omega t) - 2kB\cos(\omega t)$$
$$\therefore \frac{A}{B} = \frac{2k - m\omega^2}{k}$$

$$\therefore \frac{2k - m\omega^2}{k} = \frac{k}{2k - m\omega^2}$$
$$(2k - m\omega^2)^2 = k^2$$
$$\omega^2 = \frac{k}{m}, \frac{3k}{m}$$

5 Travelling Waves

Travelling Wave
Organized disturbance traveling with a well-defined wave speed

A *transverse wave* is one in which the displacement is perpendicular to the direction of travel
Ex: moving the end of a rope up and down
A *longitudinal wave* is one in which displacement is parallel to the direction of travel
Ex: pushing back and forth on air-filled pipe

$$y(x,t) = f(x \pm vt) = f(kx \pm t)$$

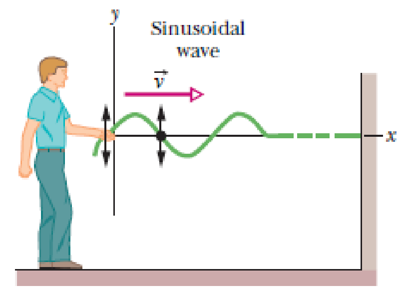

Figure 6: Transverse Wave Example. Source: HRW, Fundamentals of Physics

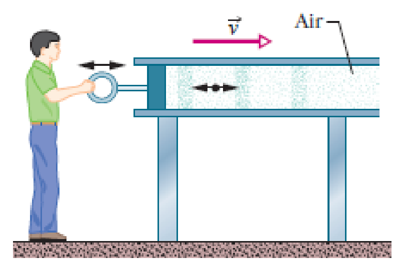

Figure 7: Longitudnal Wave Example. Source: ibid

If f is sinusoidal, then

$$y(x,t) = A\sin(kx \pm wt + \phi)$$

Where,
- y(x,t) = displacement as a function of time and x position
- A = displacement amplitude
- $kx \pm \omega t$ = phase of the wave
- k = angular wave number = $\frac{2\pi}{\lambda}$
- ω = angular frequency
- ϕ_0 = phase constant

Note that the phase argument changes linearly with time at a given x position, when the sin evaluates to 1, that means the y at that value of x and t is the amplitude
The velocity and acceleration are

$$v_y(x,t) = \frac{\partial y}{\partial t} = \pm A\omega\cos(kx \pm \omega t + \phi)$$

$$a_y(x,t) = \frac{\partial(v_y(x,t)}{\partial t} = \mp A\omega^2\sin(kx \pm t + \phi_0)$$

Wavelength and Angular Wave Number
The *wavelength* $\lambda[m]$ is the distance parallel to the direction of the wave's travel between repetitions of the wave shape.
Since the displacement y must be the same at both ends of this wavelength by definition:

$$y_m\sin(kx_1) = y_m\sin(k(x_1+\lambda)) = y_m\sin(kx_1 + k\lambda))$$

But we know that the sin function repeats every 2π radians, so $k\lambda = 2\pi$. We define this as the angular wave number:

$$k = \frac{2\pi}{\lambda}, \quad [\text{rad}^{-1}\text{m}]$$

Speed of Travelling Wave
Since $kx - \omega t = c$ is constant at the amplitude, taking the time derivative

gives $kv = \omega$
Thus, the wave speed is

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f$$

5.1 Waves on a String

The speed of a wave on a string is

$$v = \sqrt{\frac{\tau}{\mu}}, \quad [\mathrm{LT}^{-1}]$$

5.2 Energy

Consider an infinitesimally small subsection of a string. Each segment oscillates vertically and thus will have kinetic energy:

$$\delta K = \frac{1}{2}\delta m(\frac{\partial y}{\partial t})^2 = \frac{1}{2}\mu\delta x(\frac{\partial y}{\partial t})^2$$

For sinusoidal motion:

$$\delta K = \frac{1}{2}(\mu dx)(-\omega A)^2 \cos^2(kx - \omega t)$$

For small angles, we can obtain the potential energy, which is equal to the string's extension (stretch) times the tension T:

$$\delta s = \delta x(1 + \frac{1}{2}\theta^2) = \delta x(1 + \frac{1}{2}(\frac{\partial y}{\partial x})^2)$$

$$U = T(\delta s - \delta x) = \frac{1}{2}T\delta x(\frac{\partial y}{\partial t})^2$$

Then for the total energy, we have

$$E = \frac{1}{2}\int_a^b ((\frac{\partial y}{\partial t})^2 + T(\frac{\partial y}{\partial x})^2)dx$$

$$= \frac{1}{2}\mu\omega^2 A^2 \lambda$$

If we divide by the wavelength (v/λ), we end up with the average energy propagated per unit time, AKA power:

$$\bar{P} = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}\sqrt{\mu\tau}A^2\omega^2$$

$$P_{\text{max}} = \mu\omega^2 A^2 v$$

[11]

[11] Note the resemblance to electrical power: $P = \frac{1}{2}RI^2$, where R is the impedance or resistance of the medium

5.3 Wave Equation

One-Dimensional Wave Equation

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

The general solution of the above is $\phi = f(x - vt) + g(x + vt)$. We can verify if an equation represents a wave by differentiating twice with respect to each x or t. The general form of any wave motion is given by

$$y(x,t) = y_m \sin(kx - \omega t) \tag{83}$$

Where y_m is the amplitude of the wave. We can verify if an equation is indeed a wave by substituting it into this equation.

6 Standing Waves

If two sinusoidal waves of the same wavelength are travelling in the opposite direction, we can apply the superposition principal to find a resulting wave that has points along the wave that do not move.

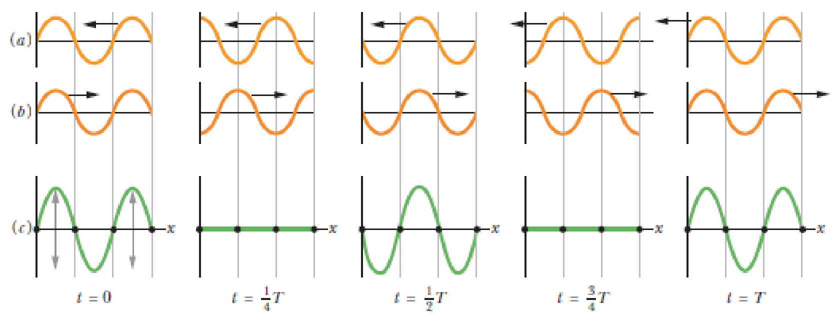

Figure 8: Standing Wave

The points at which the wave is "fixed" are called **nodes**.
Halfway between two adjacent nodes are **antinodes**, where the amplitude is at a maximum. These waves are called **standing waves** because the wave pattern itself doesn't move across the x axis over time. Thus, the motion of each point along the string can be described by two separate function:

1. At a point on the x-axis, what is the amplitude? (f(x))
2. At a moment in time, at what point are we in its oscillation? (sinusoidal oscillation)

Standing Wave
Fixed at both ends
The displacement of each particle in a standing wave takes on an equation of the form:

$$y(x,t) = A_n \sin(\frac{n\pi}{L}x) \cdot \cos(\omega_n t)$$

$$y(x,t) = A_n \sin(k_n x) \cos(\omega_n t)$$

Where,
$A_n \sin(\frac{n\pi}{L}x)$ = Amplitude Function

n = harmonic or mode of vibration (natural number)

n=1 is the fundamental mode or first harmonic, n=2 is the second, etc.

Ex: fixed string, closed pipe

Open at both ends

$$y(x,t) = A_n \cos(\frac{n\pi}{L}x) \cdot \cos(\omega_n t)$$

Ex: open pipe

For a standing wave pattern, the angular frequencies are discrete resonant frequencies:

$$\omega_n = \frac{n\pi v}{L}$$

$$f = \frac{nv}{2L}$$

[12] The fundamental frequency is the frequency of the first harmonic, which we can also solve for using the formulas for angular frequency and the velocity of a wave on a string:

$$f_1 = \frac{1}{2L}\sqrt{\frac{\tau}{\mu}}$$

The period is simply the inverse of the frequency:

$$T = \frac{2\pi}{\omega_n} = \frac{2L}{nv}$$

The wavelength is:

$$\lambda_n = \frac{2L}{n}$$

The fundamental wavelength is $2L$, then the 2nd harmonic is L, 3rd is $2L/3$, etc. [13] The angular wave number is:

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$$

[12] Note that the difference between any two consecutive harmonics is always equal to the fundamental frequency

[13] If we rearrange for L, we can see that a standing wave can only exist if there is an integer number of half-wavelengths between the fixed-ends of the string

Both ends fixed: antinode and node positions

For standing waves where both ends are fixed, the maximum and minimum amplitudes occurs at the x values for which $|\sin(kx)| = 0$ and $|\sin(kx)| = 1$ respectively.

Thus, nodes occur at $x = n\frac{\lambda}{2}, n = 0, 1, 2, ...$

Thus, antinodes occur at $kx = (n + \frac{1}{2})\frac{\lambda}{2}, n = 0, 1, 2, ..$

Both ends open: antinode and node positions

Nodes occur at $x = (2n+1)\frac{\lambda}{4}, n = 0, 1, 2, ...$

Antinodes occur at $x = n\frac{\lambda}{2}, n = 0, 1, 2...$

Standing Wave where one side is fixed and the other is open

For these standing waves, there is a different set of wavelengths that can satisfy the resonant conditions:

Note that these increment in *odd natural numbers*, that is only odd harmonics are present

$$\lambda = \frac{4L}{n}, n = 1, 3, 5, ...$$
$$f = \frac{nv}{4L}, n = 1, 3, 5, ...$$

[14]

6.1 Beats

When we hear two signals at different frequencies, their interference causes the amplitude or "intensity" to vary in a sinusoidal pattern. This wavering is what we know as a *beat* and is equal to the difference between the two frequencies.

$$f_{\text{beat}} = f_1 - f_2$$

[15]

6.2 Determination of Amplitudes

Fourier Amplitude

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx \tag{84}$$

$$f(x) = \Sigma_n A_n \sin(\frac{n\pi}{L}x)$$

[16]
[17]

[14] Note that the difference between two consecutive harmonics is 2 * fundamental frequency

[15] Beat frequency is the difference between the two frequencies we are exposed to. If the beat frequency decreases as we increase the tension of the string, then the frequency of a string must be smaller than the source frequency.

[16] If we let

$$f(x) = A$$
$$A_n = \frac{2}{L}\int_0^L A\sin(\frac{n\pi}{L}x)dx$$
$$= \frac{2}{L}A\frac{L}{n\pi}[-\cos(\frac{n\pi}{L}x)]_0^L$$

[17] Some useful formulae here:

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$$

$$\int x\sin(ax)dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

7 Intensity

Power in a sound wave is omnidirectional and so we use the spatial distribution of power by defining the intensity as the power per unit area.

$$I = \frac{\bar{P}}{S}$$

$$I = \frac{1}{2}\sqrt{\rho B}A^2\omega^2$$

Attenuation (damping)

$$\frac{dI}{dx} = -\alpha I$$

$$I(x) = I(r_0)e^{-\alpha(x-r_0)}$$

Where, r_0 is a reference point
then combining attentuation and spreading, we have the following expressions for intensity

$$I(r) = I(r_0)[e^{-\alpha(r-r_0)}][\frac{r}{r_0}]^{N-1}$$

For example: 3d point source

$$I(r) = I(r_0)[e^{-\alpha(r-r_0)}]\frac{r_0^2}{r^2}$$

7.1 Mechanical Impedance

Impedance

$$Z = \frac{\tau_y(x,t)}{v_y(x,t)}$$

Mechanical impedance is a property of the medium that is similar to inertia: how much the medium resists motion when subjected to a force.

$$\tau_y = -\tau\frac{\partial y}{\partial x}$$

$$v_y = \frac{\partial y}{\partial t}$$

For sinusoid:

$$Z = \frac{\tau}{v} = \sqrt{\mu\tau}$$

Mechanical Impedance (Waves)

$$Z = \sqrt{\mu\tau} \implies \bar{P} = \frac{1}{2}ZA^2\omega^2$$

Acoustical Impedance (Fluids)

$$Z_a = \sqrt{\rho B} \implies I = \frac{1}{2}Z_a A^2\omega^2$$

Acoustical Impedance (Solids)

$$Z_a = \sqrt{\rho Y}$$

Electrical Impedance

$$Z_E = \frac{V}{I}$$

Where,
μ = linear density [kg m^{-1}]
τ = tension [N]
B = Bulk Modulus [Pa=Nm^{-2}]
Y = Young's Modulus [Pa]

Why is this important? At boundaries between two materials, there will be an incident wave that transmits into the second medium and one wave that reflects back (and interferes with incident wave). [18] At the boundary, the derivatives must exist and the following must hold:

$$f_1(x - c_1 t) + g_1(x + c_1 t) = f_2(x - c_2 t)$$

That is, incident + reflected = transmitted.
If the waves are sinusoidal and A_i, A_r, A_t are the incident, reflected, and transmitted amplitudes respectively, such that

$$A_i + A_r = A_t$$

Again, this seems counterintuitive because although the incident carried all of the initial energy, the transmitted amplitude is larger (sum of incident and reflected).
Also,

$$A_i Z_1 - A_r Z_1 = A_t Z_2$$

Where, Z_1 is impedance of first medium and Z_2 is that of second.
Then combining, [19] [20]

Reflection Coefficient

$$R = \frac{A_r}{A_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \in [-1, 1]$$

Reflection is 1 when everything gets reflected (mirrored). Reflection is -1 when everything gets reflected but with 90° phase shift (amplitude is negative now).

Transmission Coefficient

$$T = \frac{A_t}{A_i} = \frac{2Z_1}{Z_1 + Z_2} \in [0, 2]$$

Transmission is 2 when Z_2 is 0 (the impedance is so small in the second medium that we have more amplitude that we started with -> energy going against the original impedance is now going to the amplitude)
Transmission is 0 when everything is reflected because Z_2 is too big

Example
A wave is incident from the left on a boundary where the mass density triples. Find the reflection and transmission coefficients.

$$\mu_1 = \mu, \quad \mu_2 = 3\mu$$

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{\sqrt{\tau\mu} - \sqrt{\tau 3\mu}}{\sqrt{\tau\mu} + \sqrt{3\mu\tau}} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

$$T = \frac{2Z_1}{Z_1 + Z_2} = \frac{1 - \sqrt{3}}{2}$$

[18] the wave number and wavelength change during this boundary but the frequency does not

[19] If we match the impedances, just like in electrical circuits, there is no reflection (R=0) which prevents interference

[20] We can get a transmission coefficient larger than 1 because the amplitude can increase while energy is conserved if the original impedance is much greater

Average power of incident: $\frac{1}{2}Z_1(A_i\omega)^2$
Average power of reflected: $\frac{1}{2}Z_1R(A_i\omega)^2$
Ratio of reflected energy to incident energy: $R_e = R^2$

Average power of transmitted: $\frac{1}{2}Z_2A_t^2\omega^2 = \frac{1}{2}Z_2(A_iT)^2\omega^2$
Ratio of transmitted energy to incident energy: $T_e = \frac{Z_2}{Z_1}T^2$
Note that $T_e + R_e = 1$

Example
Designing acoustical foam to perfect absorb sound normally incident on it.
$\rho_{foam} = 29$ kgm^{-3}
$B_{air} = 1.42e5$ Pa
$\rho_{air} = 1.21$ kgm^{-3}
If perfectly absorbing, reflection coefficient must be 0 so the impedances must match.

$$Z_{air} = Z_{foam}$$

$$\sqrt{\rho_{air}B_{air}} = \sqrt{\rho_{foam}B_{foam}}$$

$$B_{foam} = 5.9kPa$$