Math 164 HW 7

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April 21, 2016

Problem 1

Let $X_1, ..., X_n$ be a random sample from an expontial distribution with mean θ where $0 < \theta$.

(a) Let $\hat{\theta}_n$ denote the maximum likelihood estimator of θ. Show that $\{\hat{\theta}_n\}$ is a consistent sequence of estimators for θ.
First let's find the MLE for an exponential distribution.

$$f(X_1, ..., X_n; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{X_i}{\theta}} = \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} X_i}{\theta}}$$

Take the derivative of log[f(x)]

$$\frac{d}{d\theta} log[\frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} X_i}{\theta}}] = \frac{d}{d\theta} - nlog(\theta) - \frac{\sum_{i=1}^{n} X_i}{\theta} = -n\frac{1}{\theta} + \frac{\sum_{i=1}^{n} X_i}{\theta^2}$$

Set equal to 0

$$-n\frac{1}{\hat{\theta}} + \frac{\sum_{i=1}^{n} X_i}{\hat{\theta}^2} = 0 \Rightarrow n\frac{1}{\hat{\theta}} = \frac{\sum_{i=1}^{n} X_i}{\hat{\theta}^2}$$

$$\Rightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Since $\hat{\theta}_n$ is the average of the X_i's it must be consistent by the weak law of large numbers

(b) Let $\hat{\tau}_n$ denote the maximum likelihood estimator of $\tau(\theta) = \frac{1}{\theta}$ show that $\{\hat{}_n\}$ is a consistent sequence of estimators of $\tau(\theta)$.
from the invariance property we know that

$$\hat{\tau} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i}$$

Let

$$g(u) = \frac{1}{u}$$

From theorem 6 we know since $\hat{\theta}_n$ is consistent $g(\hat{\theta}_n)$ is also consistent

(c) Use the method of moment generating function to find the distribution of

$$Y = \frac{2n\bar{X}_n}{\theta}$$

$$Y = \frac{2\sum_{i=1}^{n} X_i}{\theta}$$

Let $V_i = \frac{2X_i}{\theta}$ Thus Y $= \sum_{i=1}^{n} V_i$

$$M_{V_1+...+V_n}(t) = E(e^{t(V_1+...+V_n)}) = E(e^{tV_1})...E(e^{tV_n}) = [M_{V_1}]...[M_{V_n}] = [M_{V_1}]^n$$

$$= [M_{\frac{2X_1}{\theta}}]^n = [\frac{1}{1 - \frac{2t}{\theta}\theta}]^n = [\frac{1}{1-2t}]^n$$

This is a chi square with 2n degrees of freedom

(d) Show that $\{\hat{\tau}_n\}$ is a mean-squared-error consistent sequence of estimators of $\tau(\theta)$.

$$\hat{\tau}_n = \frac{1}{\frac{1}{n}\sum_{i=1}^n X_i}$$

We want to show that

$$\lim_{n\to\infty} bias(\hat{\tau}_n) = 0 \text{ and } \lim_{n\to\infty} var(\hat{\tau}_n) = 0$$

First $E(\hat{\tau}_n)$

$$E(\hat{\tau}_n) = E[\frac{1}{\bar{X}_n}] = \frac{2n}{\theta}E[\frac{\theta}{2n\bar{X}_n}] = \frac{2n}{\theta}E[(Y)^{-1}] = \frac{2n}{\theta}\int_0^\infty (Y)^{-1}\frac{1}{2^n\Gamma(n)}Y^{n-1}e^{-\frac{Y}{2}}$$

$$= \frac{2n}{\theta}\frac{1}{2^n\Gamma(n)}\int_0^\infty Y^{n-2}e^{-\frac{Y}{2}}dY = \frac{2n}{\theta}\frac{1}{2^n\Gamma(n)}[\frac{\Gamma(n-1)}{0.5^{n-1}}] = \frac{2n}{\theta}\frac{1}{2^n\Gamma(n)}\frac{(n-1)\Gamma(n)}{0.5^{n-1}}$$

$$= \frac{2n}{\theta}\frac{(n-1)}{2} = \frac{n}{(n-1)\theta}$$

Now $E[\hat{\tau}_n^2]$

$$E[\hat{\tau}_n^2] = \frac{(2n)^2}{\theta^2}E[(Y)^{-2}] = \frac{2n}{\theta}\int_0^\infty (Y)^{-2}\frac{1}{2^n\Gamma(n)}Y^{n-1}e^{\frac{Y}{2}}$$

$$= \frac{(2n)^2}{\theta^2}\frac{1}{2^n\Gamma(n)}\int_0^\infty Y^{n-3}e^{\frac{Y}{2}}dY = \frac{(2n)^2}{\theta^2}\frac{1}{2^n\Gamma(n)}[\frac{\Gamma(n-2)}{0.5^{n-2}}] = \frac{(2n)^2}{\theta^2}\frac{1}{2^n\Gamma(n)}\frac{(n-2)\Gamma(n-1)}{0.5^{n-2}} = \frac{(2n)^2}{\theta^2}\frac{1}{2^n\Gamma(n)}(n-1$$

$$= \frac{(2n)^2}{\theta^2}\frac{1}{2^2}(n-1)(n-2) = \frac{n^2(n-1)(n-2)}{\theta^2}$$

Therefore

$$\lim_{n\to\infty} bias(\hat{\tau}_n) = \lim_{n\to\infty}[\frac{n}{(n-1)\theta} - \frac{1}{\theta}] = 0$$

$$var(\hat{\tau}_n) = E(\hat{\tau}_n^2)[E(\hat{\tau}_n)]^2 = \frac{n^2}{(n-2)(n-1)^2\theta^2}$$

$$lim_{n\to\infty}var(\hat{\tau}_n) = lim_{n\to\infty}\frac{n^2}{(n-2)(n-1)^2\theta^2} = 0$$

(e) Find the Cramer-Rao lower bound for variances of unbiased estimators of $\tau(\theta)$, and show that $\{t\hat{a}u_n\}$ is an asymptotically efficient sequence of estimators of $\tau(\theta)$.

Problem 2

Let $X_1, ..., X_n$ be a random sample from a distribution with probability mass function

$$f(x;\theta) = \begin{cases} \theta & \text{if x} = -1,; \\ (1-\theta)^2\theta^x & \text{if } x = 0, 1, ...; \end{cases}$$

(a) Show that the maximum likelihood estimator of θ is

$$\hat{\theta}_n = \frac{2\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i}{2n + \sum_{i=1}^n X_i}$$

$$f(X_1, ..., X_n;\theta) = \theta^{\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i)}(1-\theta)^{2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))}\theta^{\sum_{i=1}^n X_i(1-\mathbb{1}_{\{-1\}}(X_i))}$$

$$log[f(X_1, ..., X_n;\theta)] = \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i)log(\theta) + 2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))log(1-\theta) + \sum_{i=1}^n X_i(1-\mathbb{1}_{\{-1\}}(X_i))log(\theta)$$

$$= [\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)]log(\theta) + 2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))log(1-\theta)$$

Take the derivative of the log likelihood

$$\frac{d}{d\theta}\mathcal{L}(\theta) = \frac{\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)}{\theta} - \frac{2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))}{1-\theta}$$

Set equal to zero

$$\frac{\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)}{\hat{\theta}} - \frac{2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))}{1-\hat{\theta}} = 0$$

$$\Rightarrow \frac{\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)}{\hat{\theta}} = \frac{2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))}{1-\hat{\theta}}$$

$$\Rightarrow (1-\hat{\theta})[\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)] = \hat{\theta}2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))$$

$$\Rightarrow \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i) =$$

$$\hat{\theta}2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i)) + \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \hat{\theta}[\sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)]$$

$$\Rightarrow (\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i) =$$

$$\hat{\theta}[2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i)) + \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)]$$

$$\Rightarrow \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i) =$$

$$\hat{\theta}[2n - \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i - \sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i)]$$

Since $X_i = -1$ when $\mathbb{1}_{\{-1\}}(X_i)$ is 1 $\sum_{i=1}^n X_i\mathbb{1}_{\{-1\}}(X_i) = -\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i)$

$$\Rightarrow \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i + \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) = \hat{\theta}[2n - \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i + \sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i)]$$

$$\Rightarrow 2\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i = \hat{\theta}[2n + \sum_{i=1}^n X_i]$$

$$\Rightarrow \frac{2\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i}{2n + \sum_{i=1}^n X_i} = \hat{\theta}$$

check the second derivative

$$\frac{d^2}{d\theta^2}\mathcal{L}(\theta) = \frac{d}{d\theta}\frac{2\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i}{\theta} - \frac{2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))}{1-\theta}$$

$$= -\frac{2\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i}{\theta^2} - \frac{2\sum_{i=1}^n(1-\mathbb{1}_{\{-1\}}(X_i))}{(1-\theta)^2}$$

It might not be immediately obvious but this is always less than 0 there for our likelihood function is convex and our MLE must be a global max.

(b) Show that $\{\hat{\theta}_n\}$ is a consistent sequence of estimators of θ.

$$\hat{\theta}_n = \frac{2\sum_{i=1}^n \mathbb{1}_{\{-1\}}(X_i) + \sum_{i=1}^n X_i}{2n + \sum_{i=1}^n X_i}$$

$$= \frac{n}{2n + \sum_{i=1}^{n} X_i}[2\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{-1\}}(X_i) + \frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{2 + \frac{1}{n}\sum_{i=1}^{n} X_i}[2\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{-1\}}(X_i) + \frac{1}{n}\sum_{i=1}^{n} X_i]$$

What is $E[X]$

$$E[X] = -\theta + \sum_{x=0}^{\infty} x(1-\theta)^2\theta^x = -\theta + (1-\theta)^2\sum_{x=0}^{\infty} x\theta^x = -\theta + (1-\theta)^2\frac{\theta}{(1-\theta)^2} = -\theta + \theta = 0$$

$$g(u) = \frac{1}{2+u}$$

This means that $\frac{1}{2+\frac{1}{n}\sum_{i=1}^{n} X_i}$ is consistent for $\frac{1}{2}$

This means that $\hat{\theta}_n$ is consistent for

$$= \frac{1}{2}[2\theta + 0] = \theta$$