CSC265 Fall 2020 Homework Assignment 10

Solutions

The list of people with whom I discussed this homework assignment:

1. Suppose that, during the BFS of an undirected graph $G = (V, E)$, node a is first visited before node b, which is first visited before node c. Prove that if $\{a, c\} \in E$, but $\{a, b\} \notin E$, then there exists a neighbour d of b which is visited before node a.

   Solution: We will use the following pseodocode from the text (page 595).

```
BFS(G, s)
 1  for each vertex u ∈ V \ {s}
 2      u.color = WHITE
 3      u.d = ∞
 4      u.π = NIL
 5  s.color = GRAY
 6  s.d = 0
 7  s.π = NIL
 8  Q = ∅
 9  ENQUEUE(Q, s)
10  while Q is nonempty
11      u = DEQUEUE(Q)
12      for each v such that {u, v} ∈ E
13          if v.color == WHITE
14              v.color = GRAY
15              v.d = u.d + 1
16              v.π = u
17              ENQUEUE(Q, v)
18      u.color = BLACK
```

   Suppose $\{a, c\} \in E$, $\{a, b\} \notin E$. We know from lemma in class that if a is visited before b, which is visited before c, then $a.d \le b.d \le c.d$. However, since $\{a, c\} \in E$ and c is visited after a, we know when a is dequeued, $c.color = \text{WHITE}$, so $c.d = a.d + 1$ by line 15. This means we must have $a.d < b.d = c.d$ or $a.d = b.d < c.d$.

   Case 1: $a.d < b.d = c.d$. We know that c is visited when a is dequeued but b cannot be visited when a is dequeued because $\{a, b\} \notin E$. This means b is visited before a is dequeued. However, notice that b is visited also when its parent $b' = b.\pi$ is dequeued. This means b' is enqueued before a, and thus b' is visited before a.

   Case 2: $a.d = b.d < c.d$. We know from line 15 and 16 that $(b.\pi).d + 1 = b.d$, so $(b.\pi).d < b.d = a.d$. By lemma in class, we know this implies that $b.\pi$ is visited before a.

   In either case, we know that $b.\pi$ (which is a neighbour of b) is visited before a, so we are done.

2. Suppose that, during the DFS of an undirected graph $G = (V, E)$, node a is first visited before node b, which is first visited before node c. Complete the following sentence and prove it is correct (I can think of two versions, so I will show both are true): If $\{a, c\} \in E$, but $\{a, b\} \notin E$, then there exists a neighbour d of b such that when a is first visited, there is a path of unvisited nodes from a to d (this path is of length ≥ 1, since $\{a, b\} \notin E$) except for a (call this version 1). Another version (call this version 2) says that there exists a neighbour d of b such that d is visited before node c.

Solution: We will use the following pseudocode from the text (page 604).

```
DFS(G)
1  for each vertex u ∈ V
2      u.color = WHITE
3      u.π = NIL
4  time = 0
5  for each vertex u ∈ V
6      if u.color == WHITE
7          DFS-Visit(G, u)
```

```
DFS-Visit(G, u)
 1  time = time + 1
 2  u.d = time
 3  u.color = GRAY
 4  for each v such that {u, v} ∈ E
 5      if v.color == WHITE
 6          v.π = u
 7          DFS-Visit(G, v)
 8  u.color = BLACK
 9  time = time + 1
10  u.f = time
```

Suppose $\{a, c\} \in E$, $\{a, b\} \notin E$. From the nested/disjoint interval theorem in class (and page 606, theorem 22.7 in CLRS), we know that for any two nodes u, v, $(v.d, v.f) \subset (u.d, u.f)$ if and only if v is a proper descendant of u in the DFS tree. Since $\{a, c\} \in E$ and $a.d < c.d$, we know that c must be visited before DFS-Visit(G, a) is completed by line 5 of DFS-Visit. So we know $(c.d, c.f) \subset (a.d, a.f)$. We are given $a.d < b.d < c.d$ in the question. Combining with the fact that $b.f < c.f$, we have $(b.d, b.f) \subset (a.d, a.f)$. Thus, b is a proper descendant of a. Let $a_0 = a, a_1, \ldots, a_k = b$ be the simple path from a to b in the DFS tree. Since $\{a, b\} \notin E$, we know that $k \geq 2$. Let d be a_{k-1}. It remains to show that $a_1, \ldots, a_{k-1}$ are all unvisited. This is clearly true since $a_{i+1}.d = a_i.d + 1$ for $i = 0, \ldots, k-1$ by deifinition of DFS tree. So $a_j.d \geq a_1.d > a.d$ for all $j \in \{1, \ldots, k-1\}$, meaning that a_1 to $a_{k-1} = d$ is a path containing unvisited nodes. This proves version 1.

Moreover, this implies that a_{k-1} is visited before c because $(a_{k-1}).d < b.d < a.d$. This proves version 2.