Worth: 15%

1. [10 marks]
   Recall that an undirected graph $G = (V, E)$ is **3-colorable** iff there is a map $f : V \to \{red, green, blue\}$ such that no edge is assigned the same color to both its end points. Define

   $$\text{3-Col} = \{\langle G\rangle \mid G \text{ is a 3-colorable graph}\}.$$

   Give an explicit reduction showing that 3-Col $\leq_P$ 3-Sat. Justify your answer.

2. [15 marks]
   Given an undirected graph $G = (V, E)$, we say that a subset $S \subseteq V$ is **triangle-free** if for every size three subset $\{u, v, w\} \subseteq S$, at least one of the edges uv, vw, uw is not in E. Define

   $$\text{TriangleFree} = \{\langle G, k\rangle \mid G \text{ has a triangle-free subset of size at least } k\}.$$

   Show that TriangleFree is $\mathcal{NP}$-complete by reducing from IndependentSet.

3. [25 marks]
   A Boolean formula φ is a **tautology** if φ evaluates to true on every possible truth assignment. Define

   $$\text{Tautology} = \{\langle\varphi\rangle \mid \varphi \text{ is a tautology}\}$$

   Prove the following:

   (a) [15 marks] Tautology is $co\mathcal{NP}$-complete.
   (b) [10 marks] If Tautology $\in \mathcal{NP}$, then $\mathcal{NP} = co\mathcal{NP}$.

4. [30 marks]
   Let Factor be the following search problem: given $\langle x\rangle$, where x is an integer ≥ 2, output $\langle p_1, p_2, \ldots, p_m\rangle$, where each p_i is a prime number and x is the product $p_1 p_2 \cdots p_m$.
   Let

   $$\text{Div} = \{\langle x, y\rangle \mid x, y \in \mathbb{N} \text{ and } x \text{ has a divisor } d \text{ with } 1 < d \leq y\}.$$

   Prove the following:

   (a) [15 marks] Factor $\xrightarrow{p}$ Div.
   (b) [15 marks] Div $\in \mathcal{NP} \cap co\mathcal{NP}$.

   Note: All integers are represented in binary notation.