

Worth: 15%

1. [20 marks]

Let $EXP = \bigcup_k TIME(2^{n^k})$ and $NEXP = \bigcup_k NTIME(2^{n^k})$ be the classes of languages decidable by respectively deterministic and nondeterministic Turing machines with running time $O(2^{n^k})$ for some constant k .

Both $\mathcal{P} \stackrel{?}{=} \mathcal{NP}$ and $EXP \stackrel{?}{=} NEXP$ are open questions. However, it is known that if $\mathcal{P} = \mathcal{NP}$, then $EXP = NEXP$. Prove this fact!

Hint: For a language $A \in NTIME(2^{n^k})$, consider the “padded” language

$$A' = \{x\#^{2^{|x|^k}} \mid x \in A\},$$

where $x\#^{2^{|x|^k}}$ is the string formed by x followed by $2^{|x|^k}$ many #'s.

2. [20 marks]

The (m, n, k) -game is a game that generalizes the familiar game of Tic-Tac-Toe. There are two players — Player X and Player O . Player X plays first. Each player takes turns to place their marker “ X ” or “ O ” on an $m \times n$ grid G . The first player to get k markers consecutively in a row — horizontally, vertically, or diagonally — wins.

Let GT be the following language:

$$GT = \{\langle G, k \rangle \mid \text{Player } X \text{ has a winning strategy on the } (m, n, k)\text{-game } G\}.$$

Show that GT is in $PSPACE$.

3. [40 marks]

The purpose of this problem is to show that 2-SAT is NL -complete.

Given a 2-CNF formula φ , we associate a directed graph $G_\varphi = (V, E)$, where V is the set of all literals ℓ such that either ℓ or $\neg\ell$ occurs in φ , and for every clause $(\ell_1 \vee \ell_2)$ in φ we put the directed edges $(\neg\ell_1, \ell_2)$ and $(\neg\ell_2, \ell_1)$ in E .

(The idea is that if a truth assignment τ satisfies the clause $(\ell_1 \vee \ell_2)$, then if τ makes ℓ_1 FALSE, then ℓ_2 must be TRUE; and if τ makes ℓ_2 FALSE, then ℓ_1 must be TRUE.)

- (a) [10 marks] Suppose that ℓ_1 and ℓ_2 are two literals such that there is a directed path from ℓ_1 to ℓ_2 in G_φ . Then show that there is a directed path from $\neg\ell_2$ to $\neg\ell_1$ in G_φ . Also, show that every truth assignment to φ which satisfies φ and ℓ_1 also satisfies ℓ_2 .
- (b) [10 marks] Use part (a) to prove that φ is unsatisfiable iff G_φ has a directed cycle which includes both x and $\neg x$ for some variable x .
- (c) [20 marks] Use the previous observations to show that 2-SAT is NL -complete. You may use the fact that PATH is NL -complete.