Geometric Transformations

CS 4620 Lecture 9

A little quick math background

- **Notation for sets, functions, mappings**
- **Linear and affine transformations**
- **Matrices**
  - Matrix-vector multiplication
  - Matrix-matrix multiplication
- **Implicit vs. explicit geometry**

Implicit representations

- **Equation to tell whether we are on the curve**

$$\{\mathbf{v} \mid f(\mathbf{v}) = 0\}$$

- **Example: line (orthogonal to u, distance *k* from 0)**

$$\{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{u} + k = 0\}$$ (**u** is a unit vector)

- **Example: circle (center p, radius *r*)**

$$\{\mathbf{v} \mid (\mathbf{v} - \mathbf{p}) \cdot (\mathbf{v} - \mathbf{p}) - r^2 = 0\}$$

- **Always define boundary of region**
  - (if ***f*** is continuous)

Explicit representations

- **Also called parametric**
- **Equation to map domain into plane**

$$\{f(t) \,|\, t \in D\}$$

- **Example: line (containing p, parallel to u)**

$$\{\mathbf{p} + t\mathbf{u} \,|\, t \in \mathbb{R}\}$$

- **Example: circle (center b, radius *r*)**

$$\{\mathbf{p} + r[\cos t \;\; \sin t\,]^T \,|\, t \in [0, 2\pi)\}$$

- **Like tracing out the path of a particle over time**
- **Variable *t* is the "parameter"**

Transforming geometry

- **Move a subset of the plane using a mapping from the plane to itself**

$$S \to \{T(\mathbf{v}) \,|\, \mathbf{v} \in S\}$$

- **Parametric representation:**

$$\{f(t) \,|\, t \in D\} \to \{T(f(t)) \,|\, t \in D\}$$

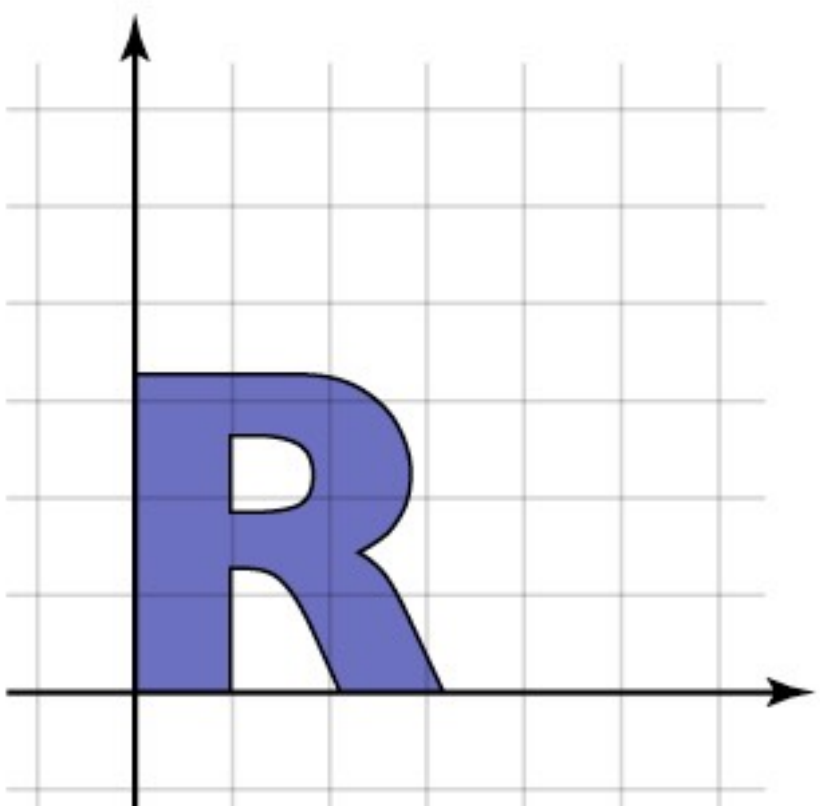
- **Implicit representation:**

$$\{\mathbf{v} \,|\, f(\mathbf{v}) = 0\} \to \{T(\mathbf{v}) \,|\, f(\mathbf{v}) = 0\}$$

$$= \{\mathbf{v} \,|\, f(T^{-1}(\mathbf{v})) = 0\}$$

Translation

- **Simplest transformation:** $T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$
- **Inverse:** $T^{-1}(\mathbf{v}) = \mathbf{v} - \mathbf{u}$
- **Example of transforming circle**



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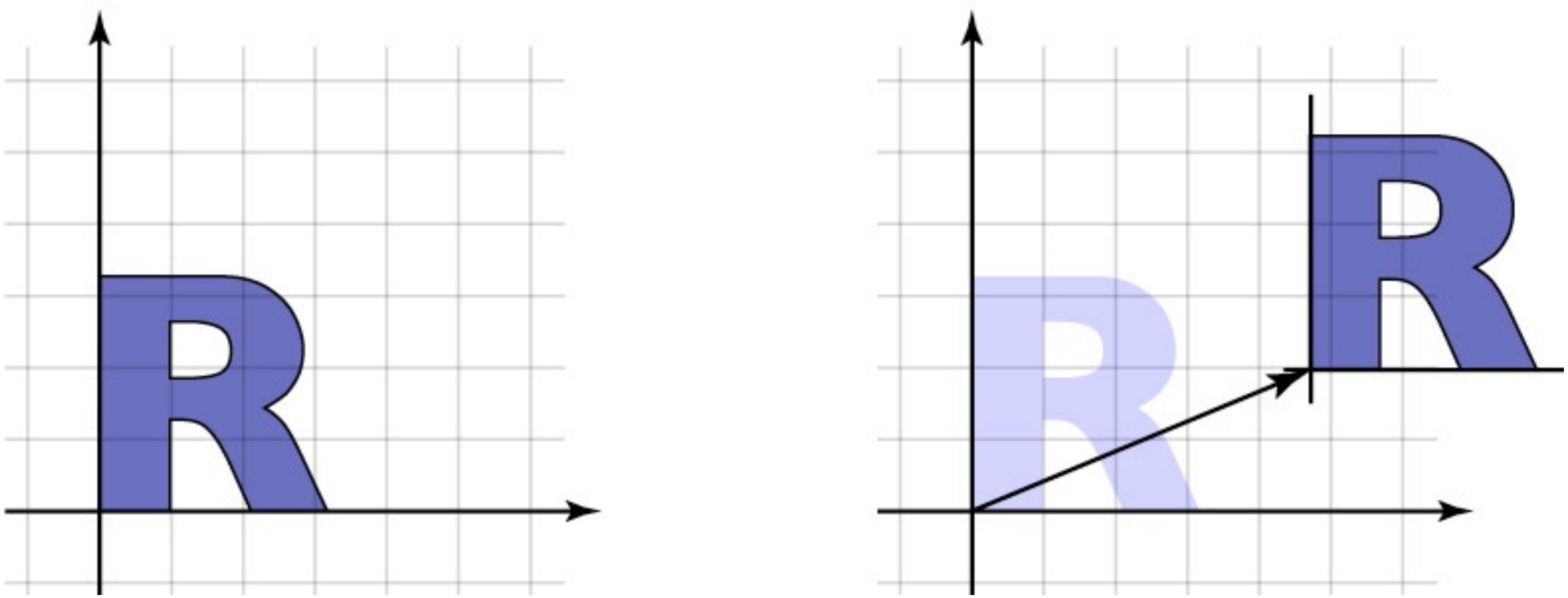


Linear transformations

- **One way to define a transformation is by matrix multiplication:**

$$T(\mathbf{v}) = M\mathbf{v}$$

- **Such transformations are** *linear*, **which is to say:**

$$T(a\mathbf{u} + \mathbf{v}) = aT(\mathbf{u}) + T(\mathbf{v})$$

(and in fact all linear transformations can be written this way)

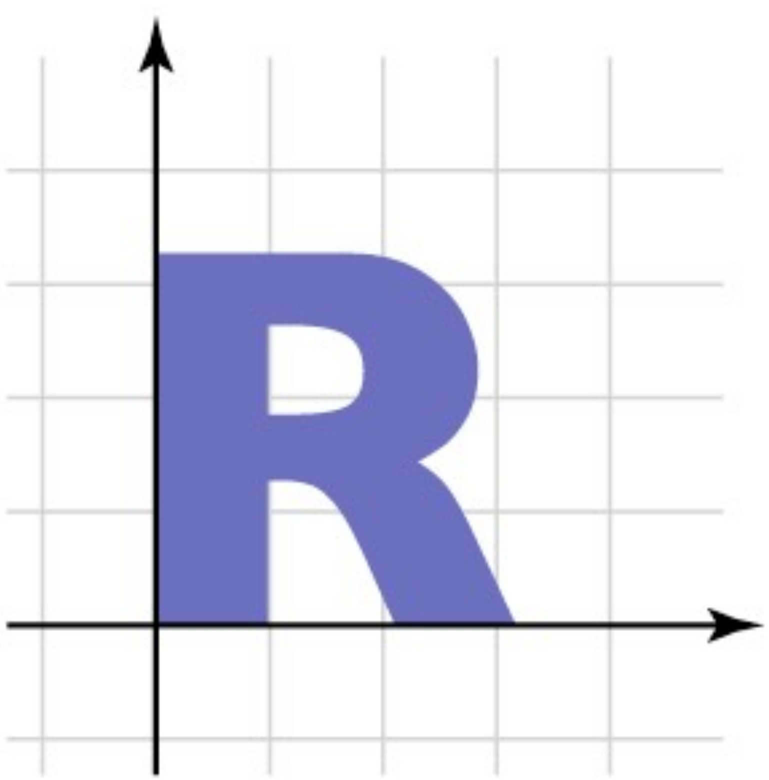
Geometry of 2D linear trans.

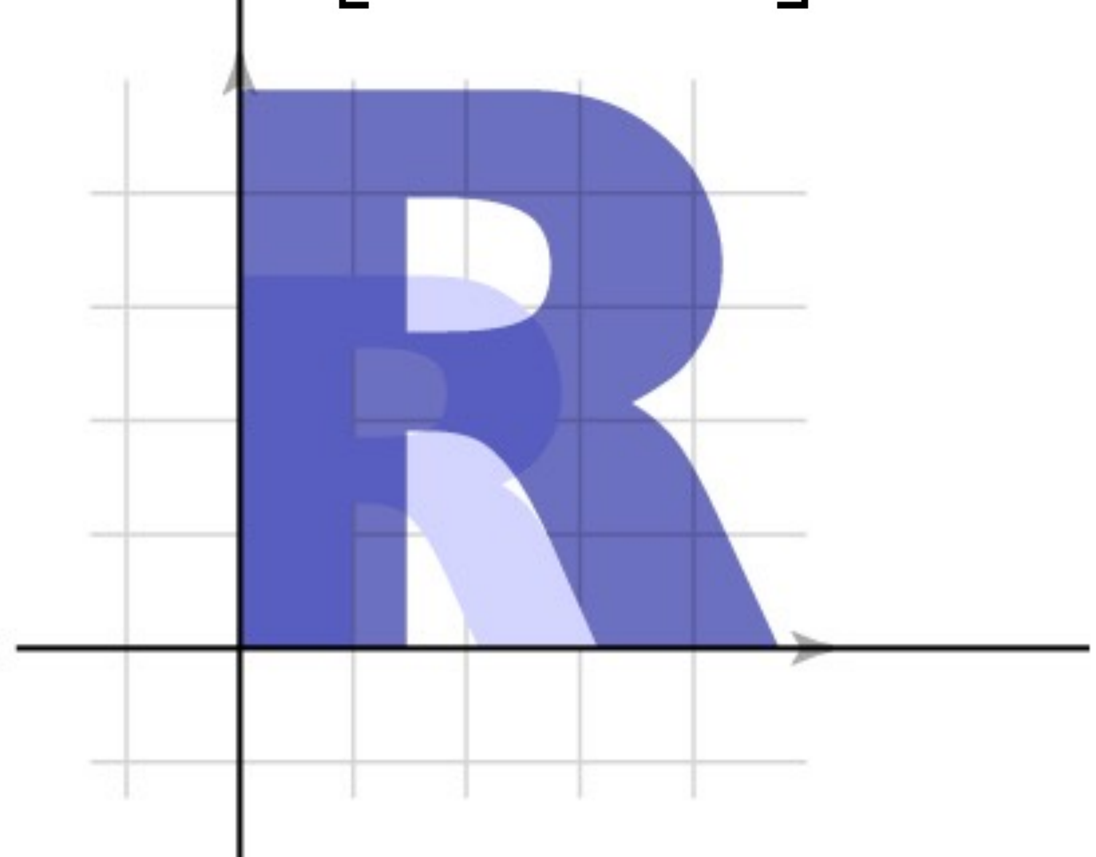
- **2x2 matrices have simple geometric interpretations**
  - uniform scale
  - non-uniform scale
  - rotation
  - shear
  - reflection
- **Reading off the matrix**

Linear transformation gallery

- **Uniform scale** $$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} sx \\ sy \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$



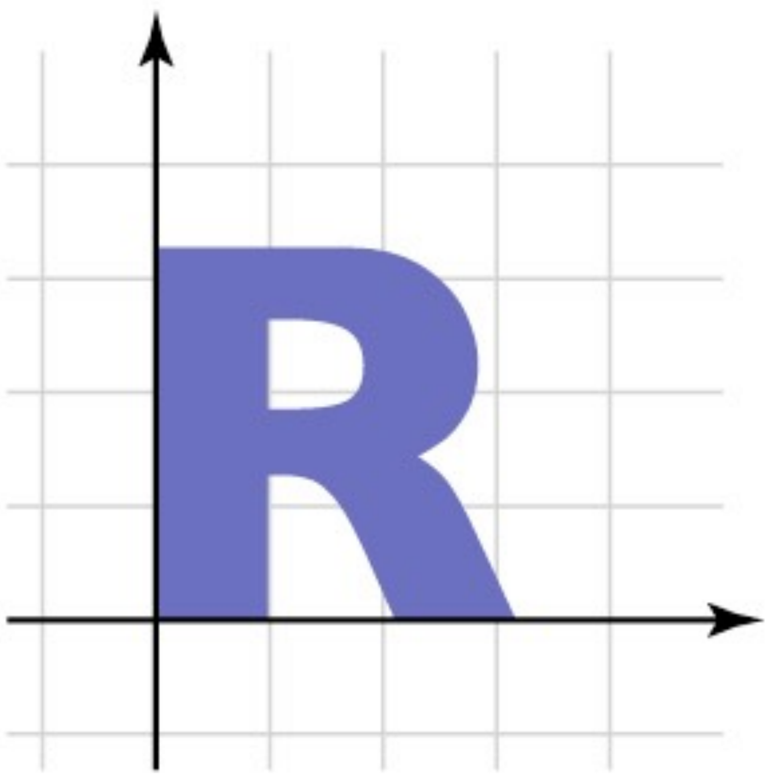


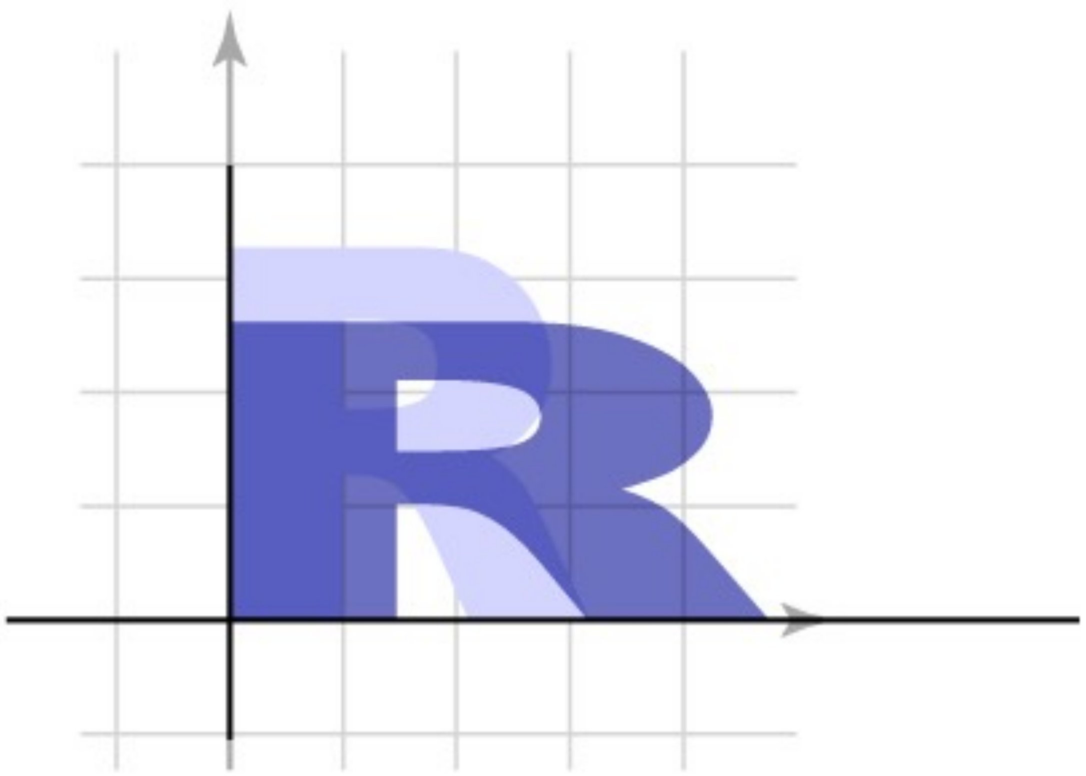
Linear transformation gallery

- **Nonuniform scale**

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.8 \end{bmatrix}$$

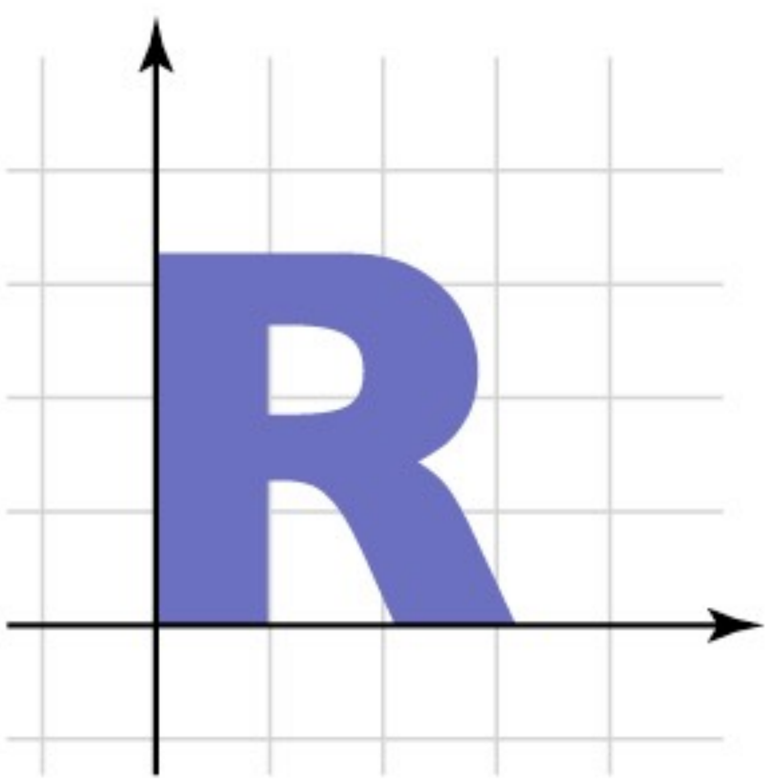


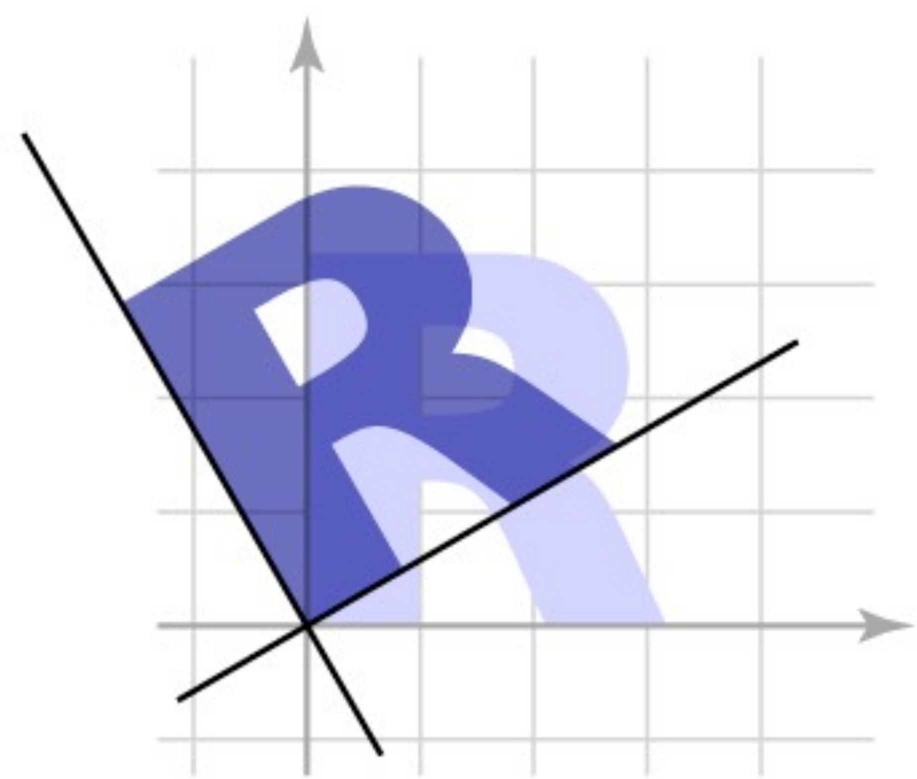


Linear transformation gallery

- **Rotation** $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$

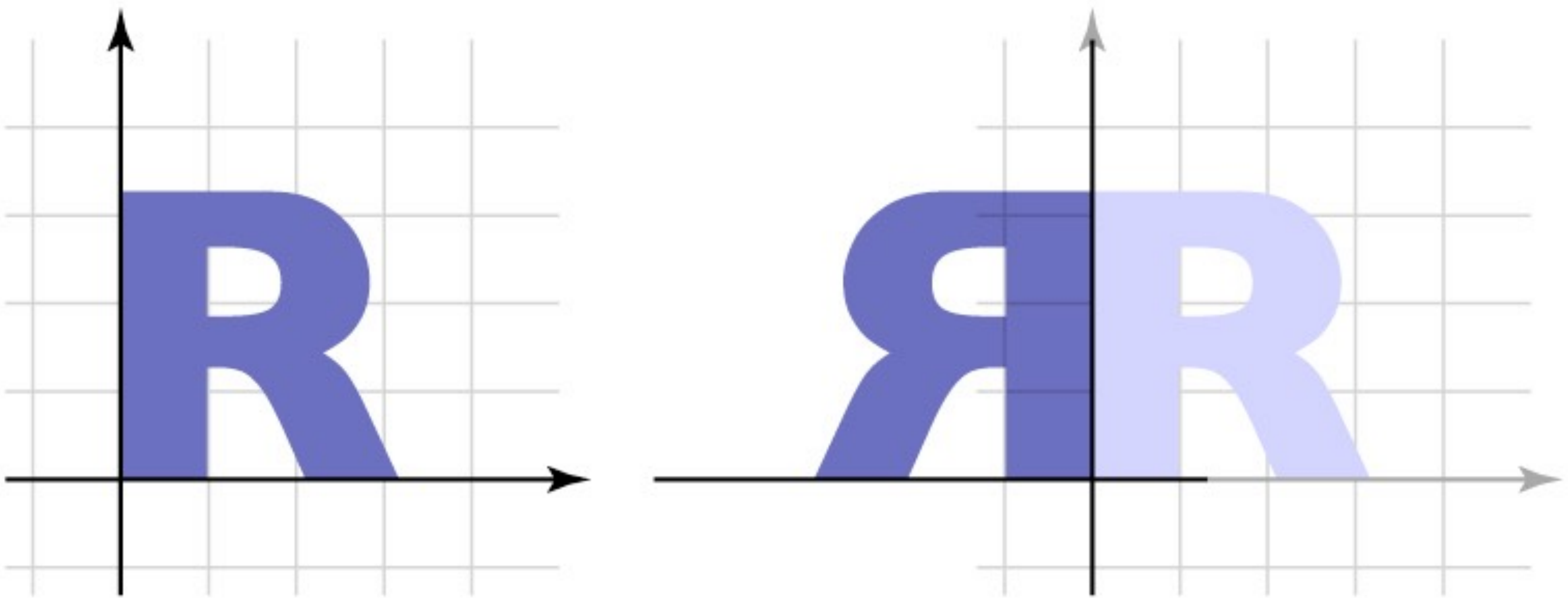




Linear transformation gallery

- **Reflection**
  - can consider it a special case of nonuniform scale

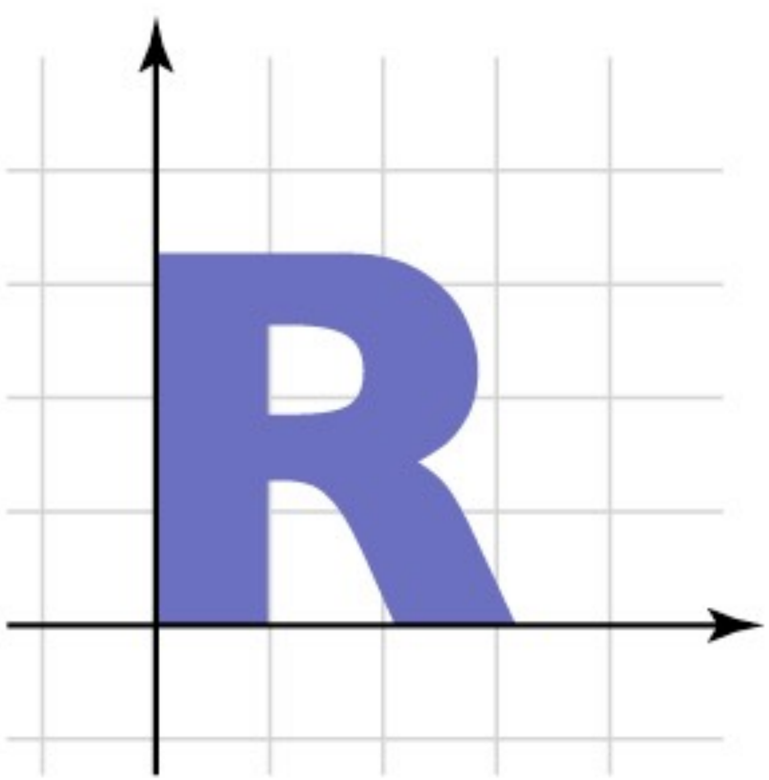
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

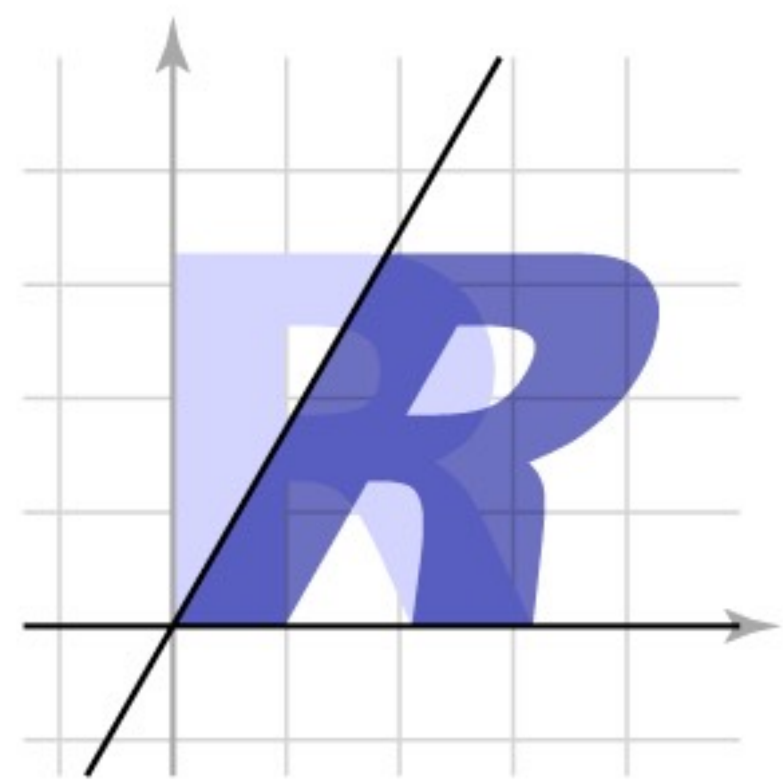


Linear transformation gallery

- **Shear** $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$





Composing transformations

- **Want to move an object, then move it some more**
  - $\mathbf{p} \to T(\mathbf{p}) \to S(T(\mathbf{p})) = (S \circ T)(\mathbf{p})$
- **We need to represent S o T ("S compose T")**
  - and would like to use the same representation as for S and T
- **Translation easy**
  - $T(\mathbf{p}) = \mathbf{p} + \mathbf{u}_T; S(\mathbf{p}) = \mathbf{p} + \mathbf{u}_S$

    $(S \circ T)(\mathbf{p}) = \mathbf{p} + (\mathbf{u}_T + \mathbf{u}_S)$
- **Translation by uT then by uS is translation by uT + uS**
  - commutative!

Composing transformations

- **Linear transformations also straightforward**
  - $$T(\mathbf{p}) = M_T\mathbf{p}; S(\mathbf{p}) = M_S\mathbf{p}$$
    $$(S \circ T)(\mathbf{p}) = M_S M_T \mathbf{p}$$
- **Transforming first by M_T then by M_S is the same as transforming by $M_S M_T$**
  - only sometimes commutative
    - e.g. rotations & uniform scales
    - e.g. non-uniform scales w/o rotation
  - Note $M_S M_T$, or $S \circ T$, is T first, then S

Combining linear with translation

- **Need to use both in single framework**
- **Can represent arbitrary seq. as** $T(\mathbf{p}) = M\mathbf{p} + \mathbf{u}$
  - $T(\mathbf{p}) = M_T\mathbf{p} + \mathbf{u}_T$
  - $S(\mathbf{p}) = M_S\mathbf{p} + \mathbf{u}_S$
  - $(S \circ T)(\mathbf{p}) = M_S(M_T\mathbf{p} + \mathbf{u}_T) + \mathbf{u}_S$
    $= (M_S M_T)\mathbf{p} + (M_S\mathbf{u}_T + \mathbf{u}_S)$
  - e.g. $S(T(0)) = S(\mathbf{u}_T)$
- **Transforming by M_T and $\mathbf{u}_T$, then by M_S and $\mathbf{u}_S$, is the same as transforming by $M_S M_T$ and $\mathbf{u}_S + M_S\mathbf{u}_T$**
  - This will work but is a little awkward

Homogeneous coordinates

- **A trick for representing the foregoing more elegantly**
- **Extra component *w* for vectors, extra row/column for matrices**
  - for affine, can always keep $w = 1$
- **Represent linear transformations with dummy extra row and column**

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

Homogeneous coordinates

- **Represent translation using the extra column**

$$\begin{bmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+t \\ y+s \\ 1 \end{bmatrix}$$

Homogeneous coordinates

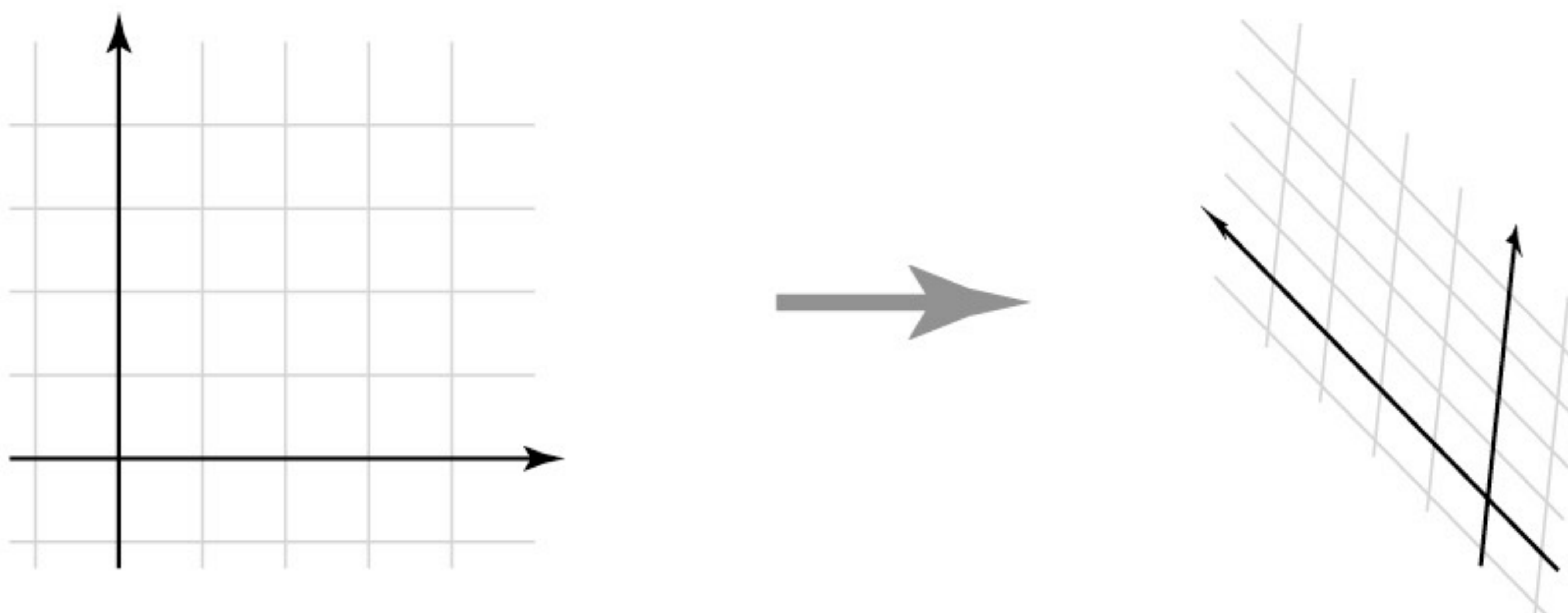
- **Composition just works, by 3x3 matrix multiplication**

$$\begin{bmatrix} M_S & \mathbf{u}_S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_T & \mathbf{u}_T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} (M_S M_T)\mathbf{p} + (M_S \mathbf{u}_T + \mathbf{u}_S) \\ 1 \end{bmatrix}$$

- **This is exactly the same as carrying around *M* and u**
  - but cleaner
  - and generalizes in useful ways as we'll see later

Affine transformations

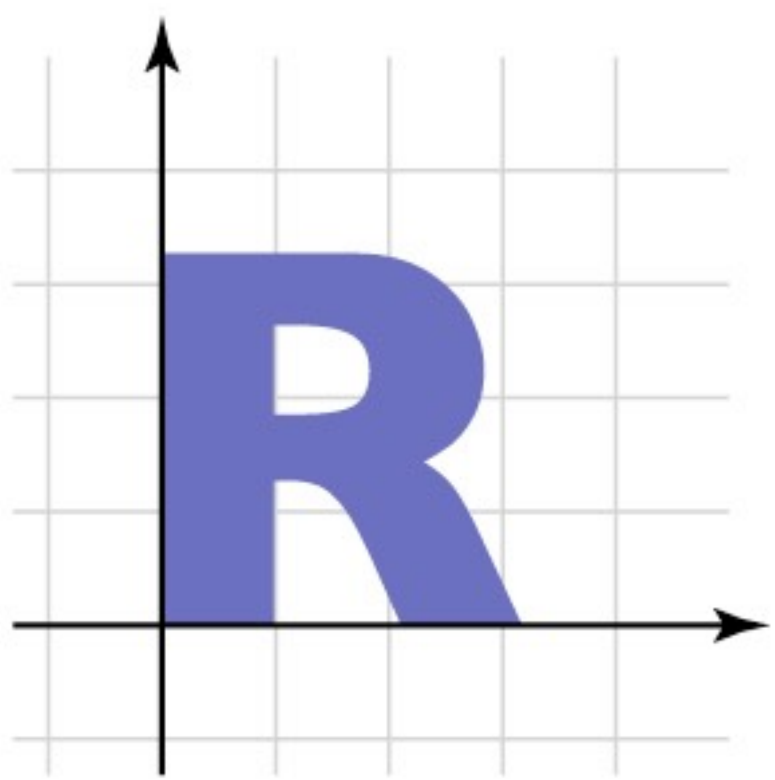
- **The set of transformations we have been looking at is known as the “affine” transformations**
  - straight lines preserved; parallel lines preserved
  - ratios of lengths along lines preserved (midpoints preserved)

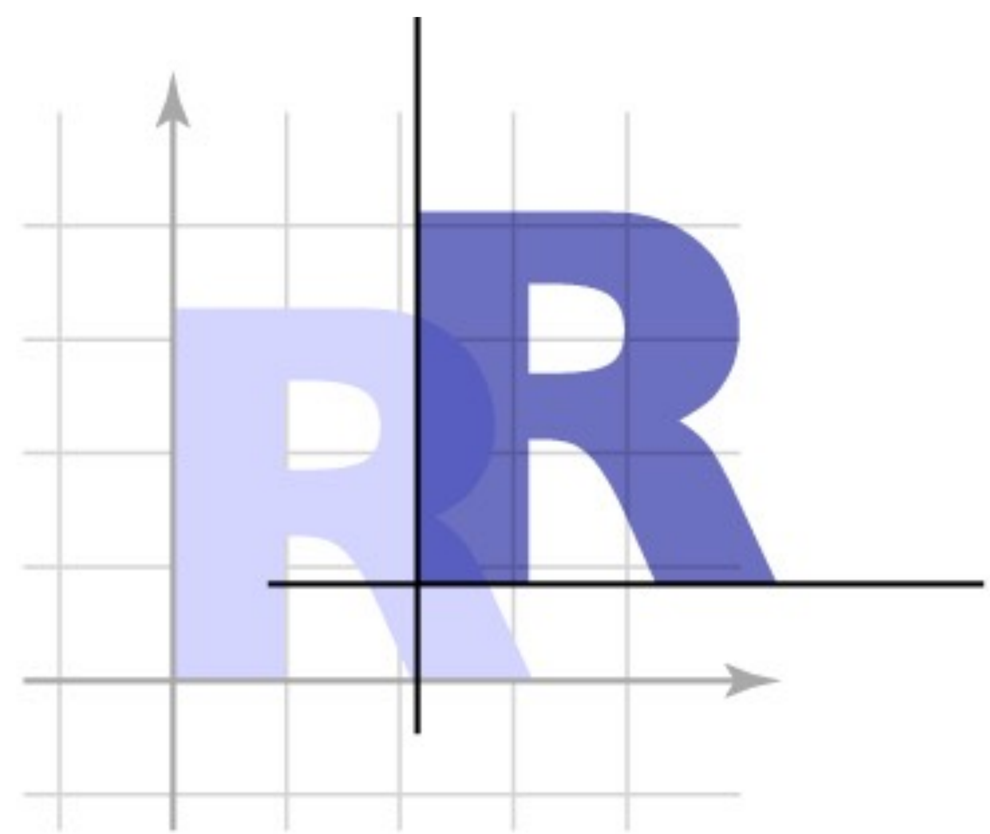


Affine transformation gallery

- **Translation**

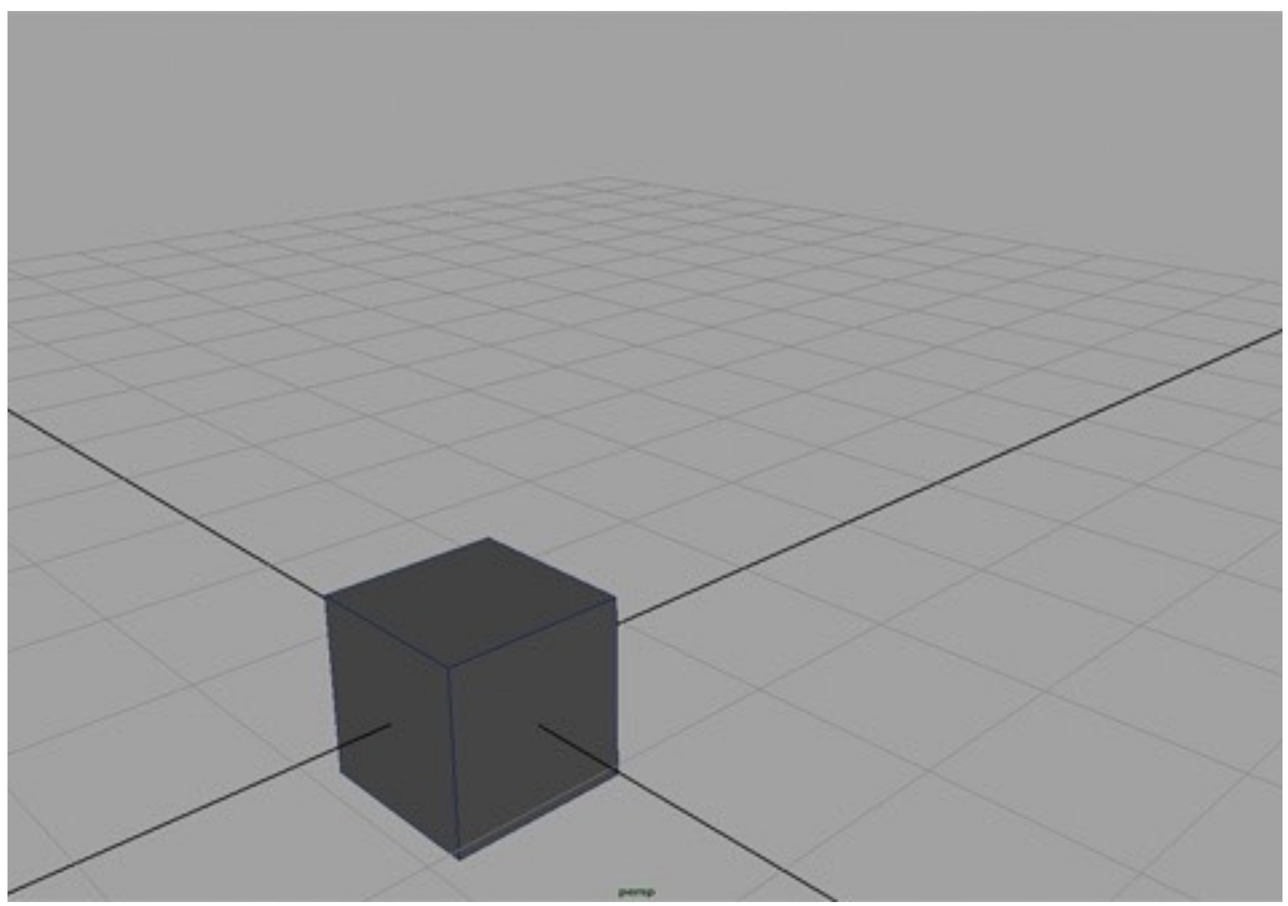
$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2.15 \\ 0 & 1 & 0.85 \\ 0 & 0 & 1 \end{bmatrix}$$





Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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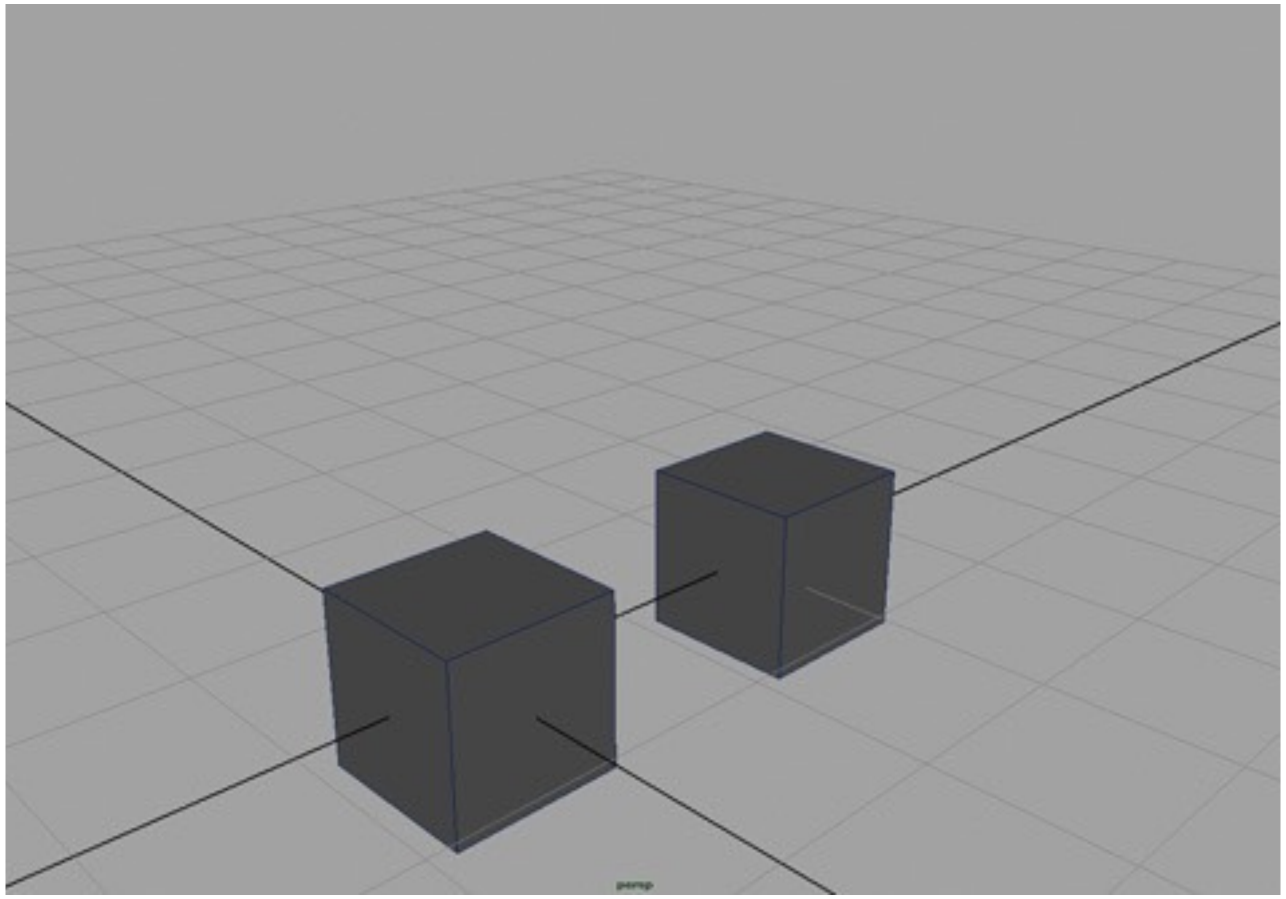

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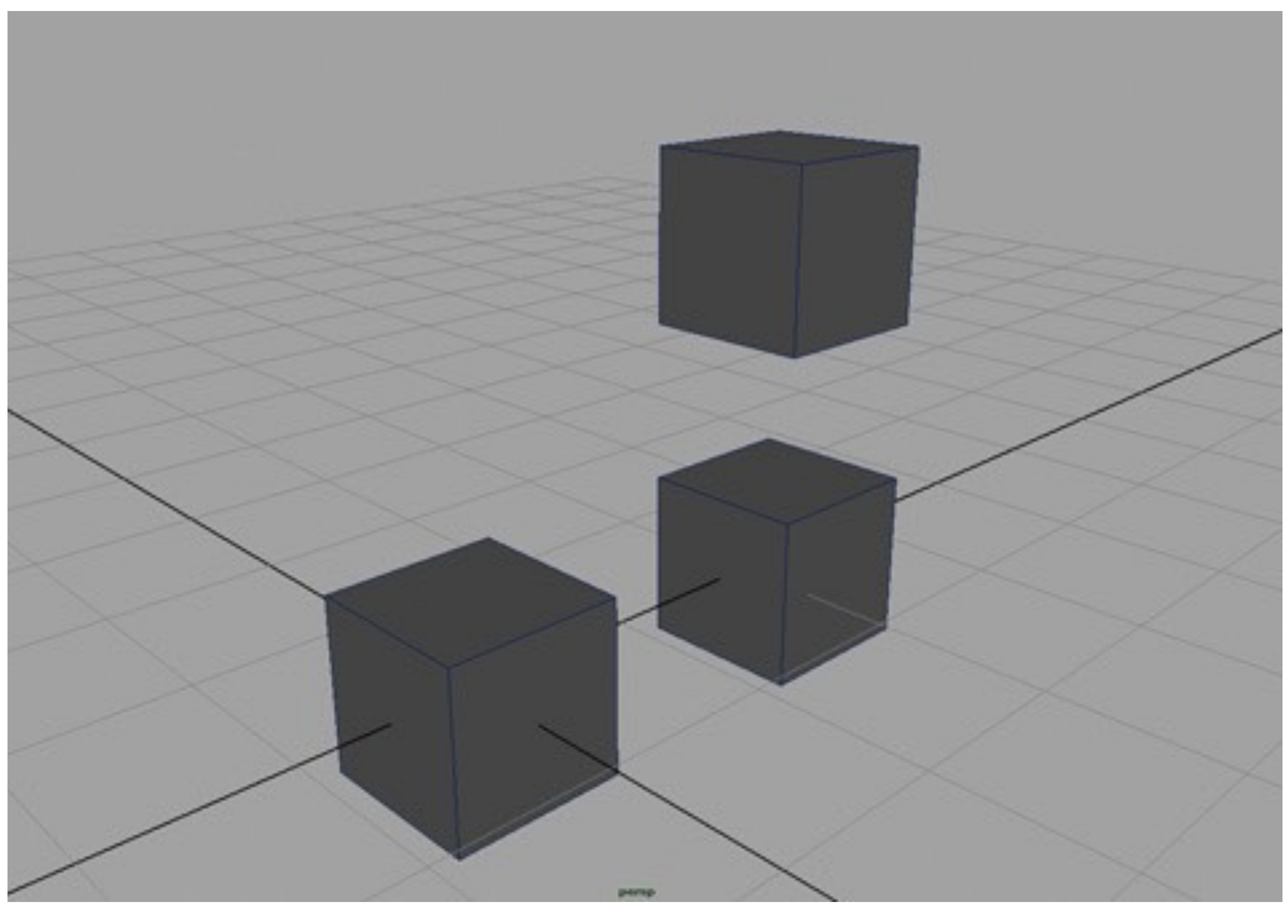

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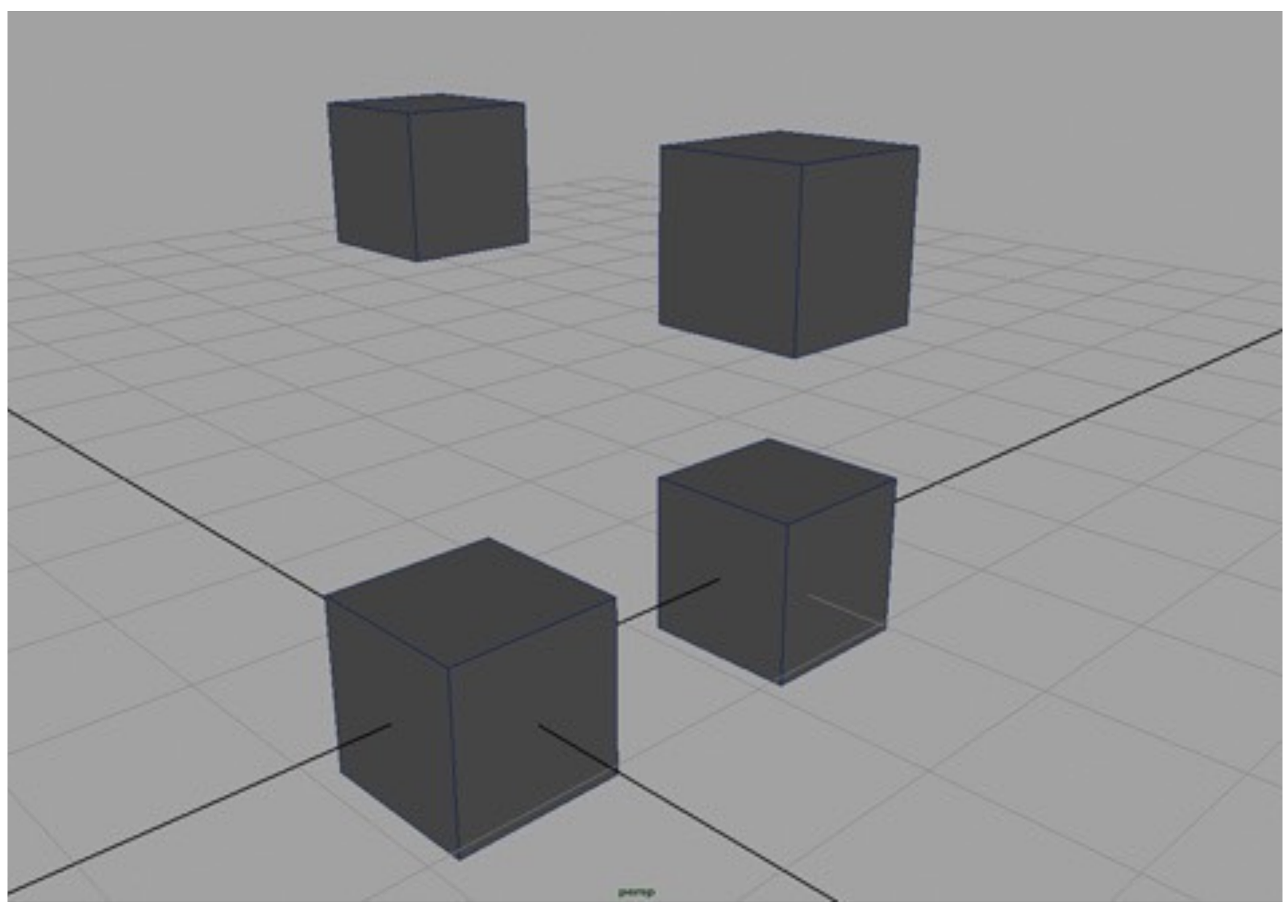
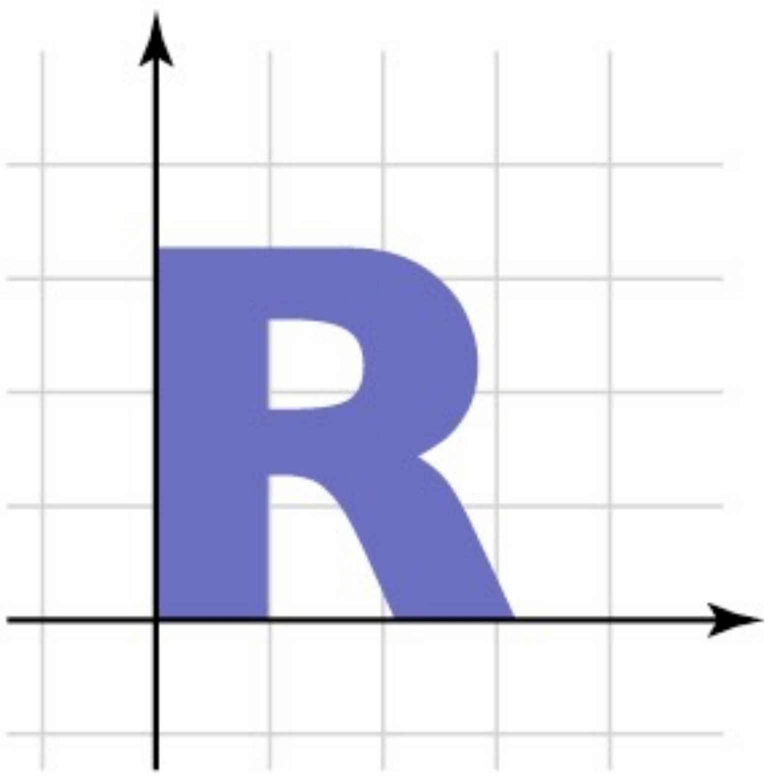
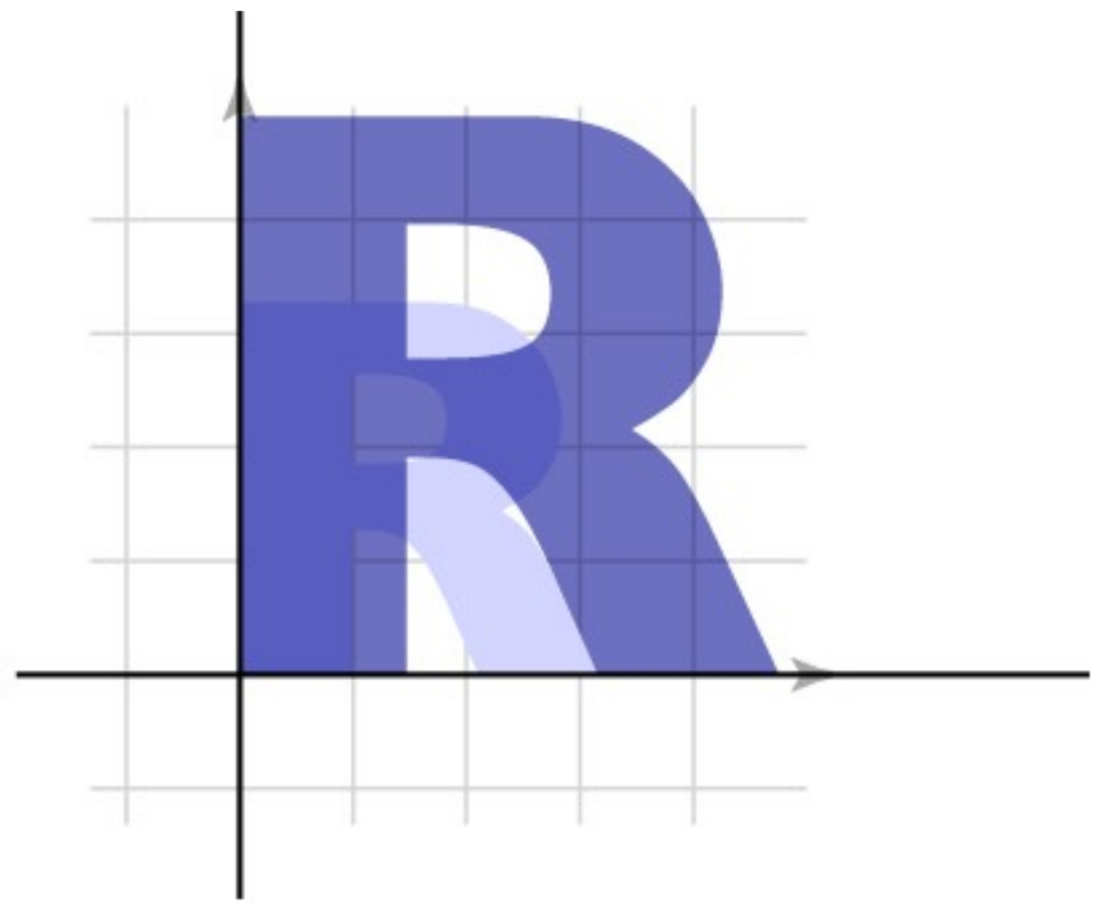


Affine transformation gallery

- **Uniform scale**

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

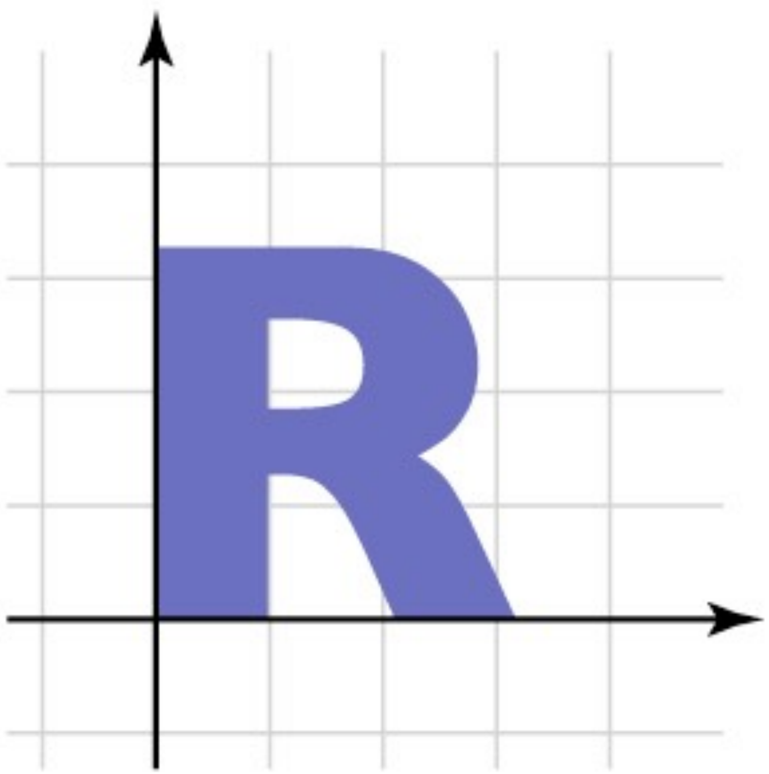


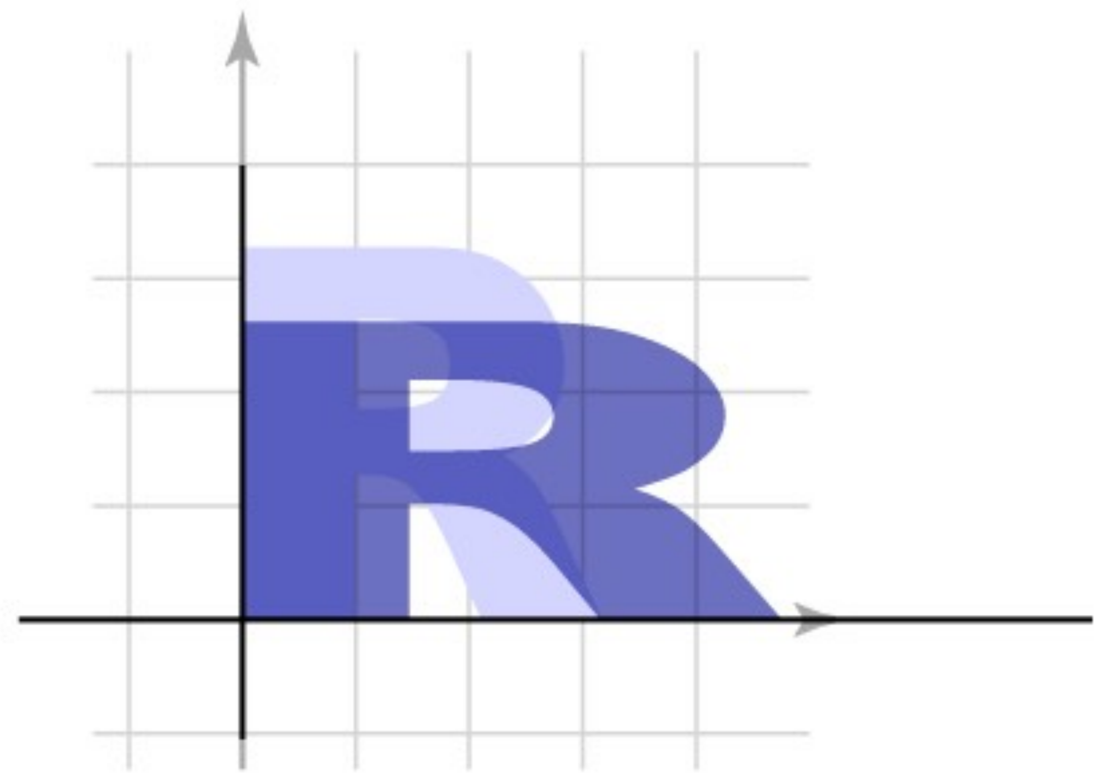


Affine transformation gallery

- **Nonuniform scale**

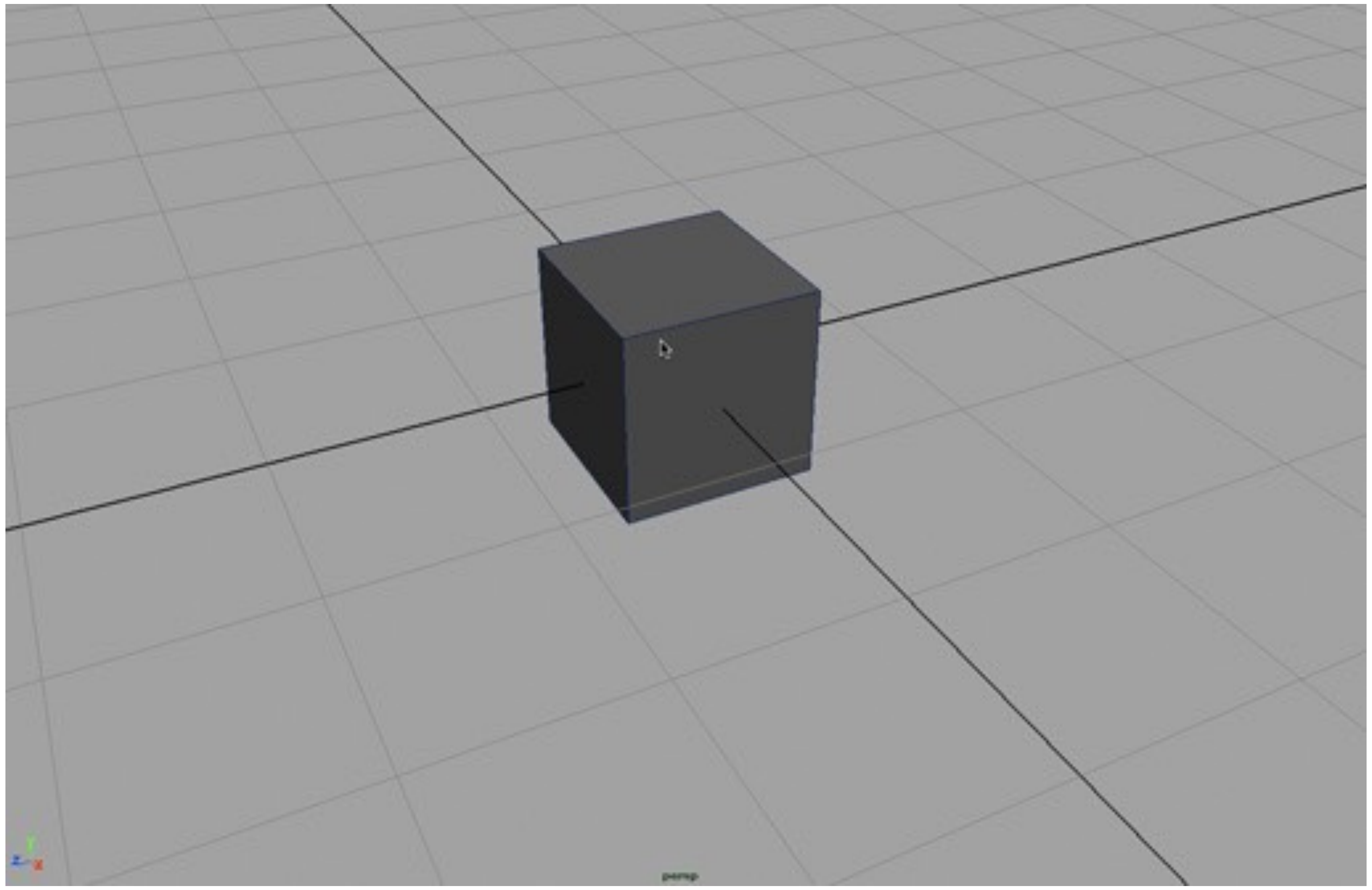
$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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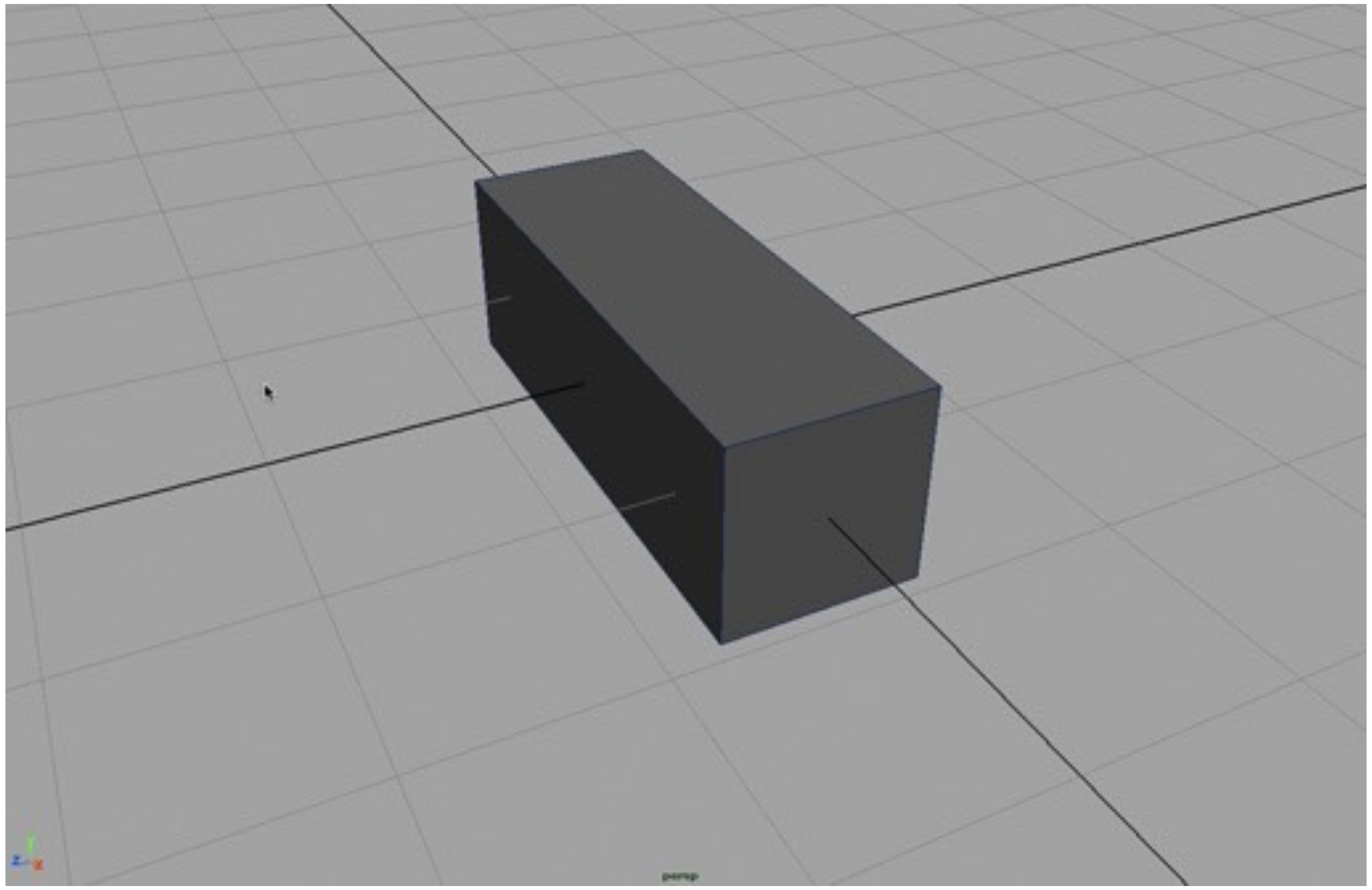

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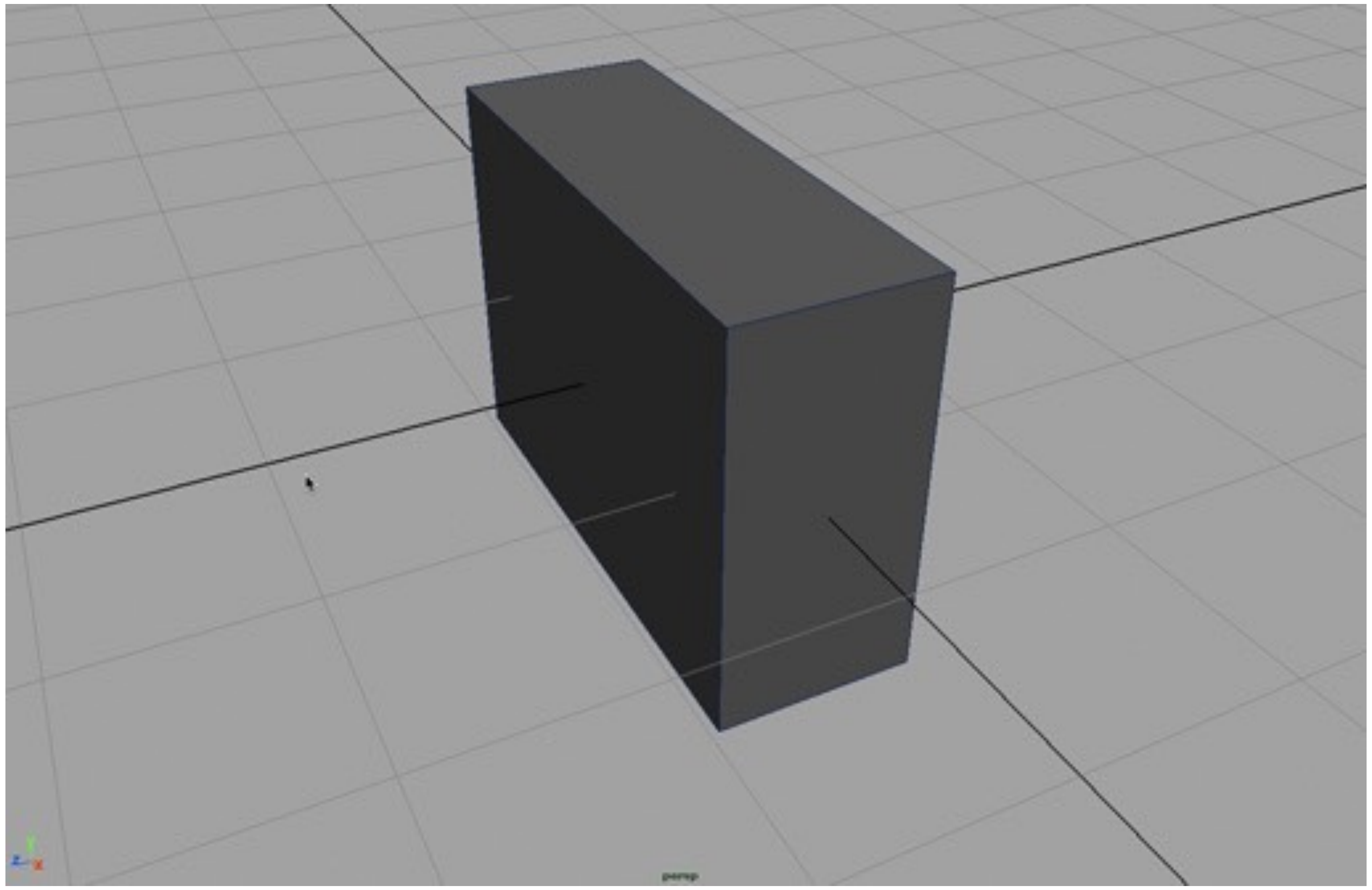

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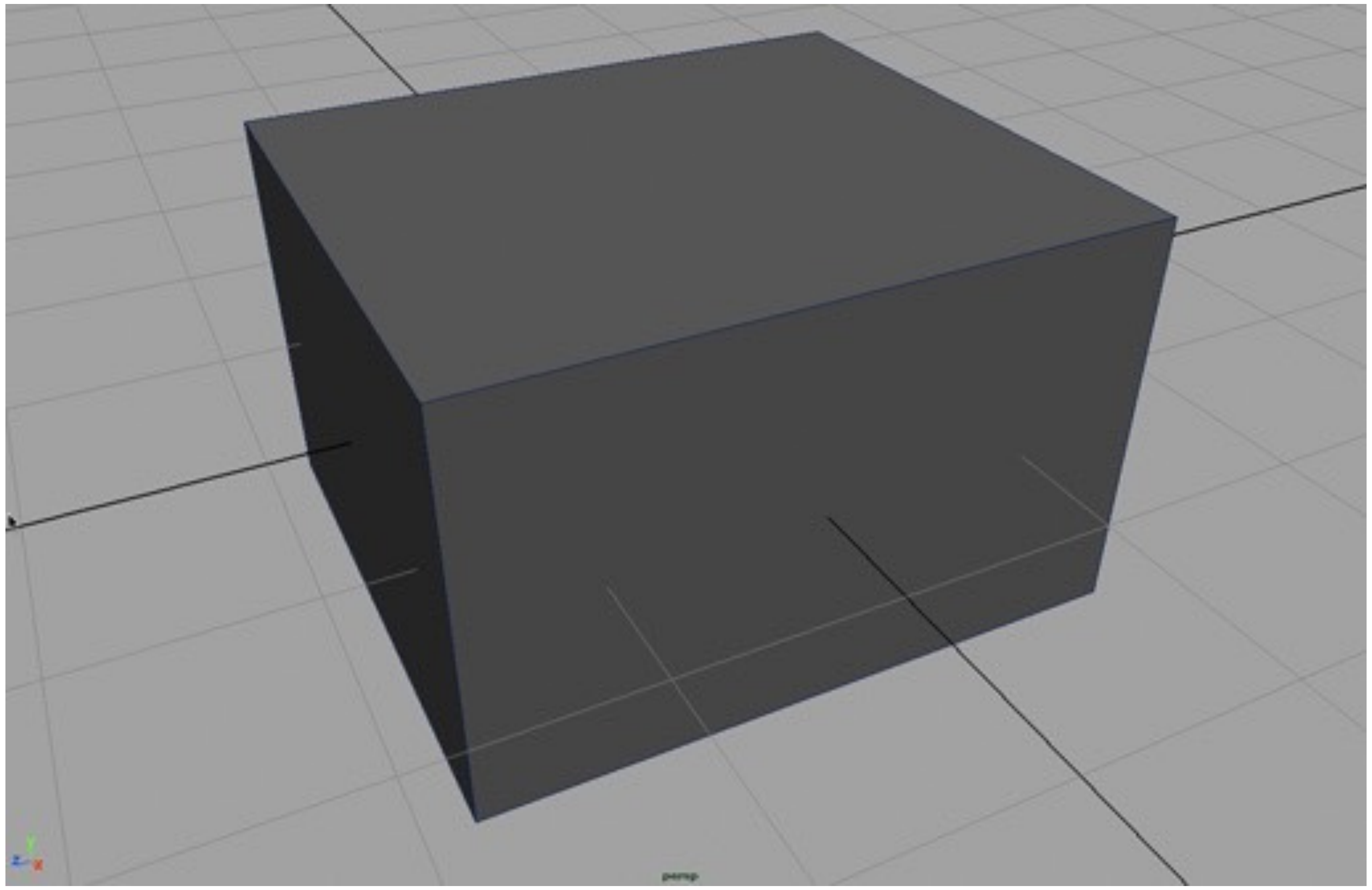
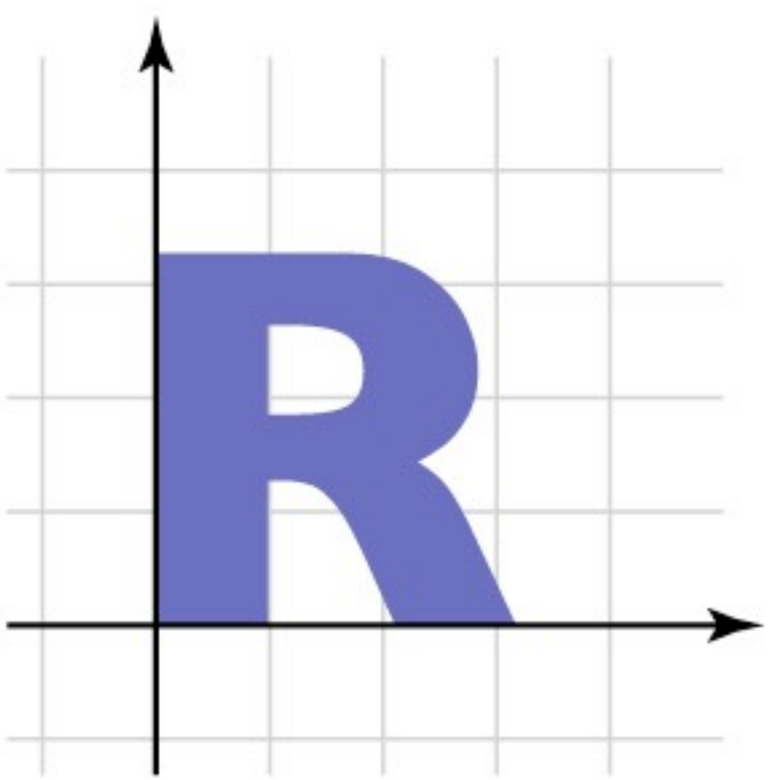
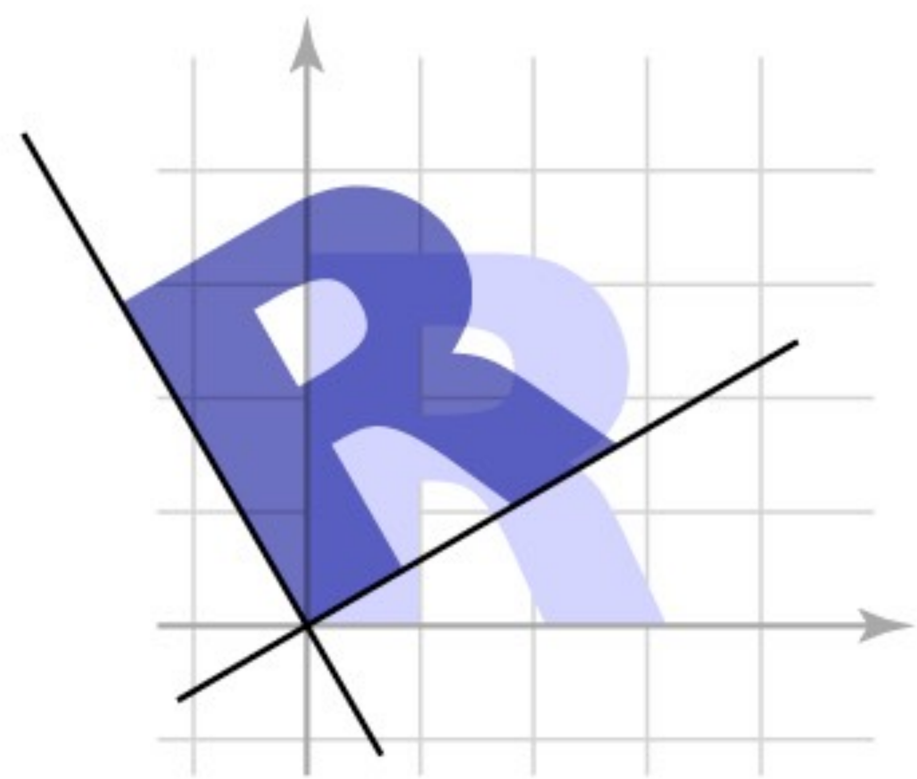


Affine transformation gallery

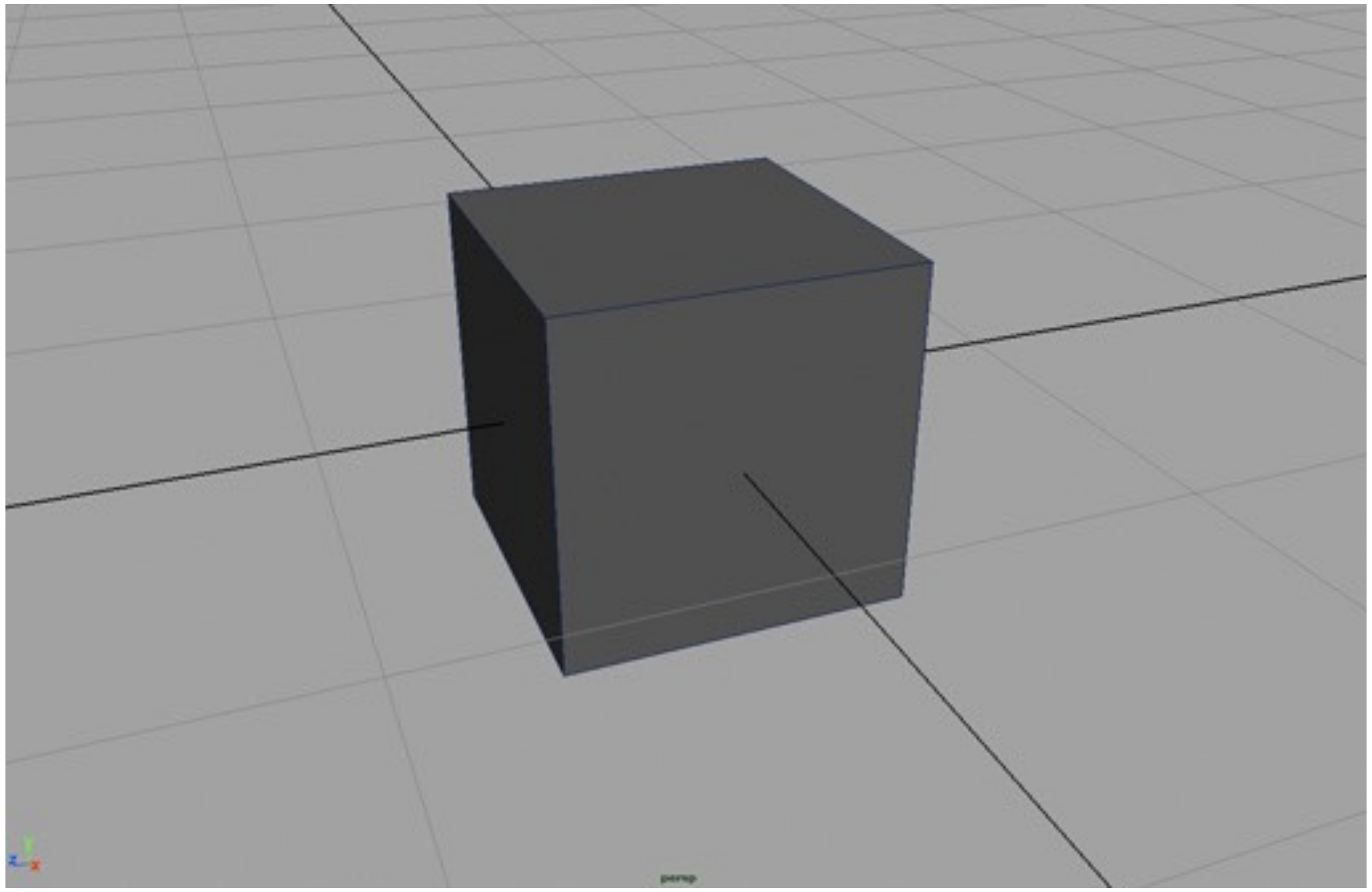
- **Rotation** $$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





Rotation about **z** axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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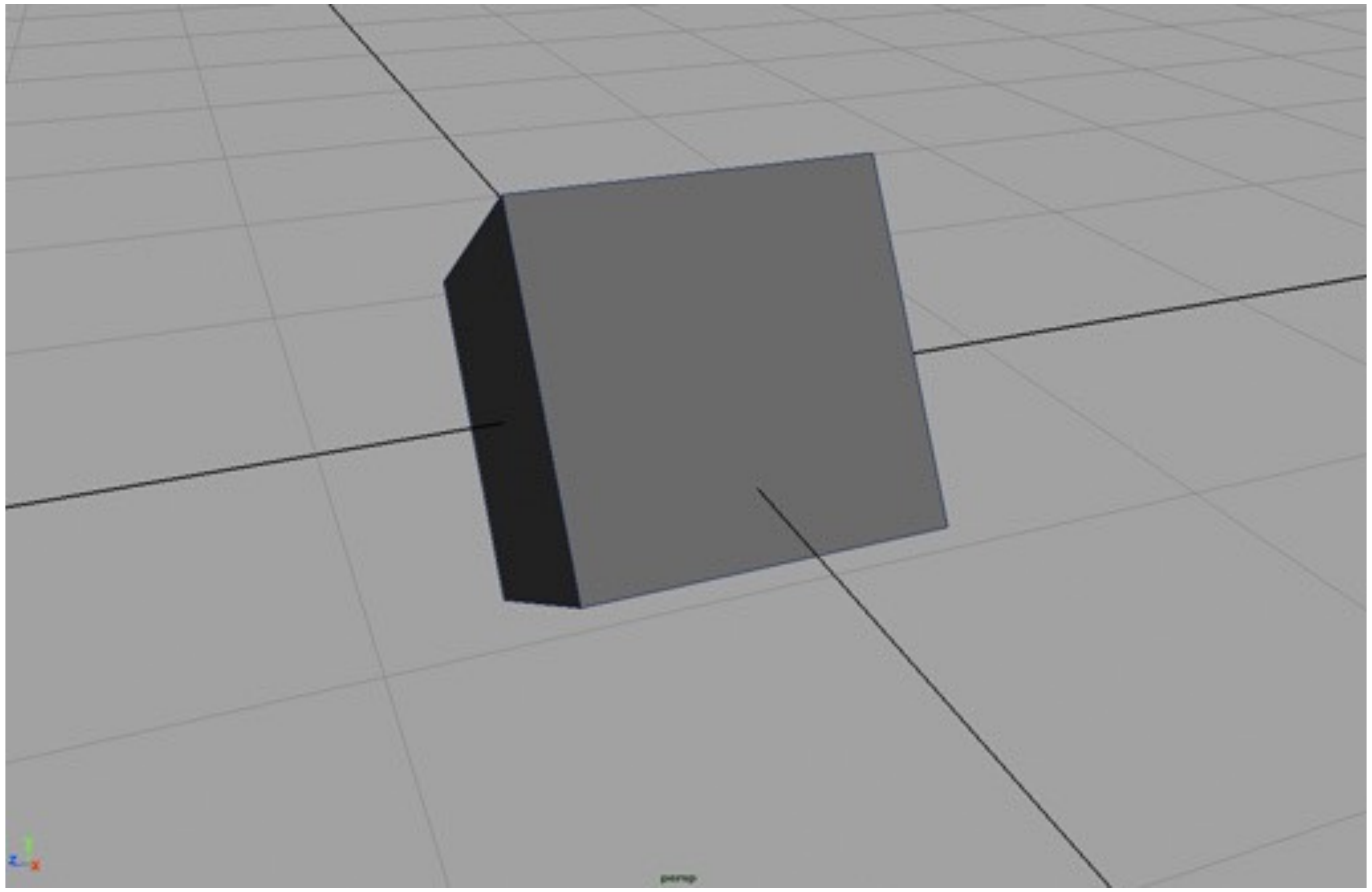
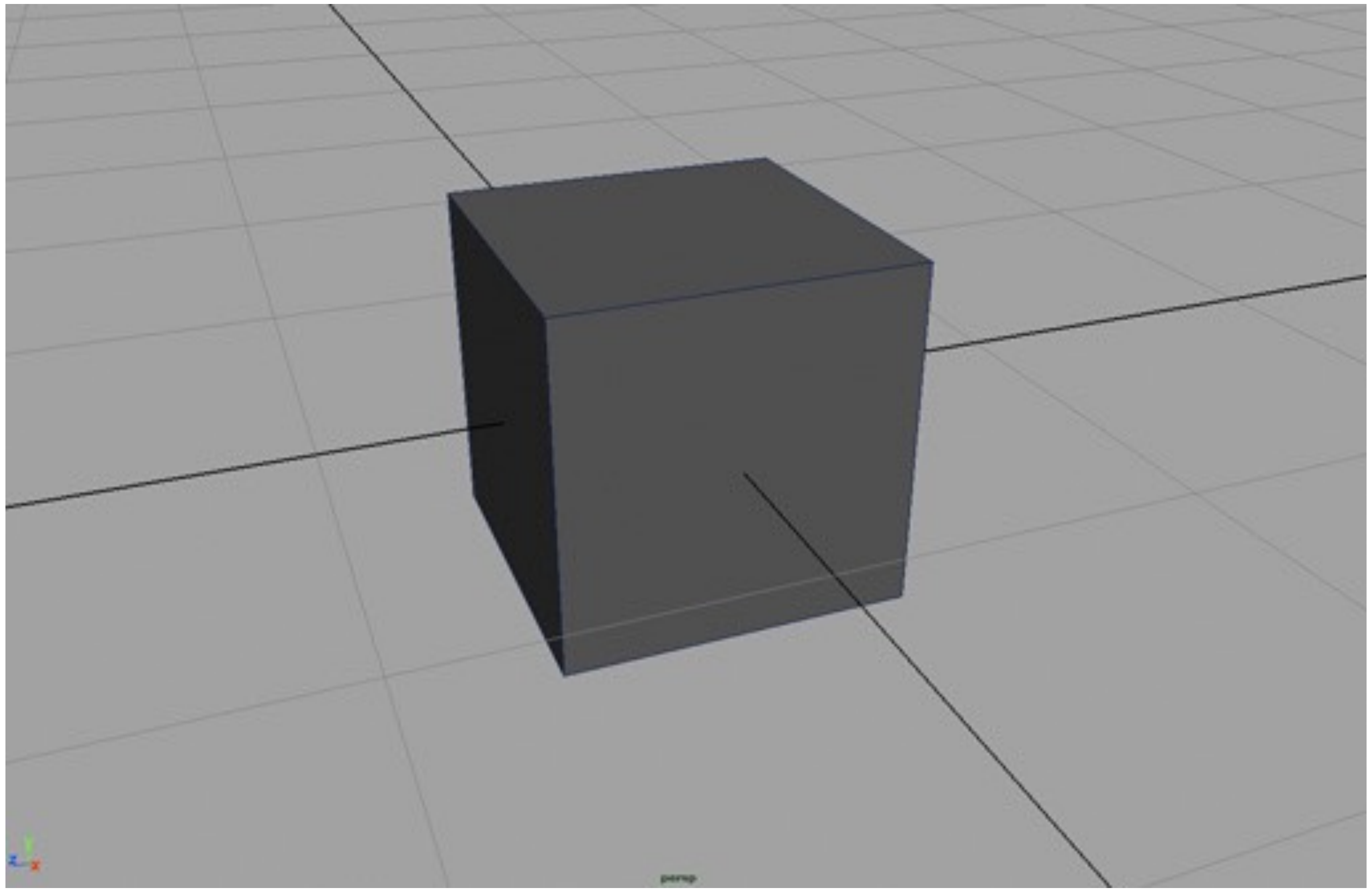


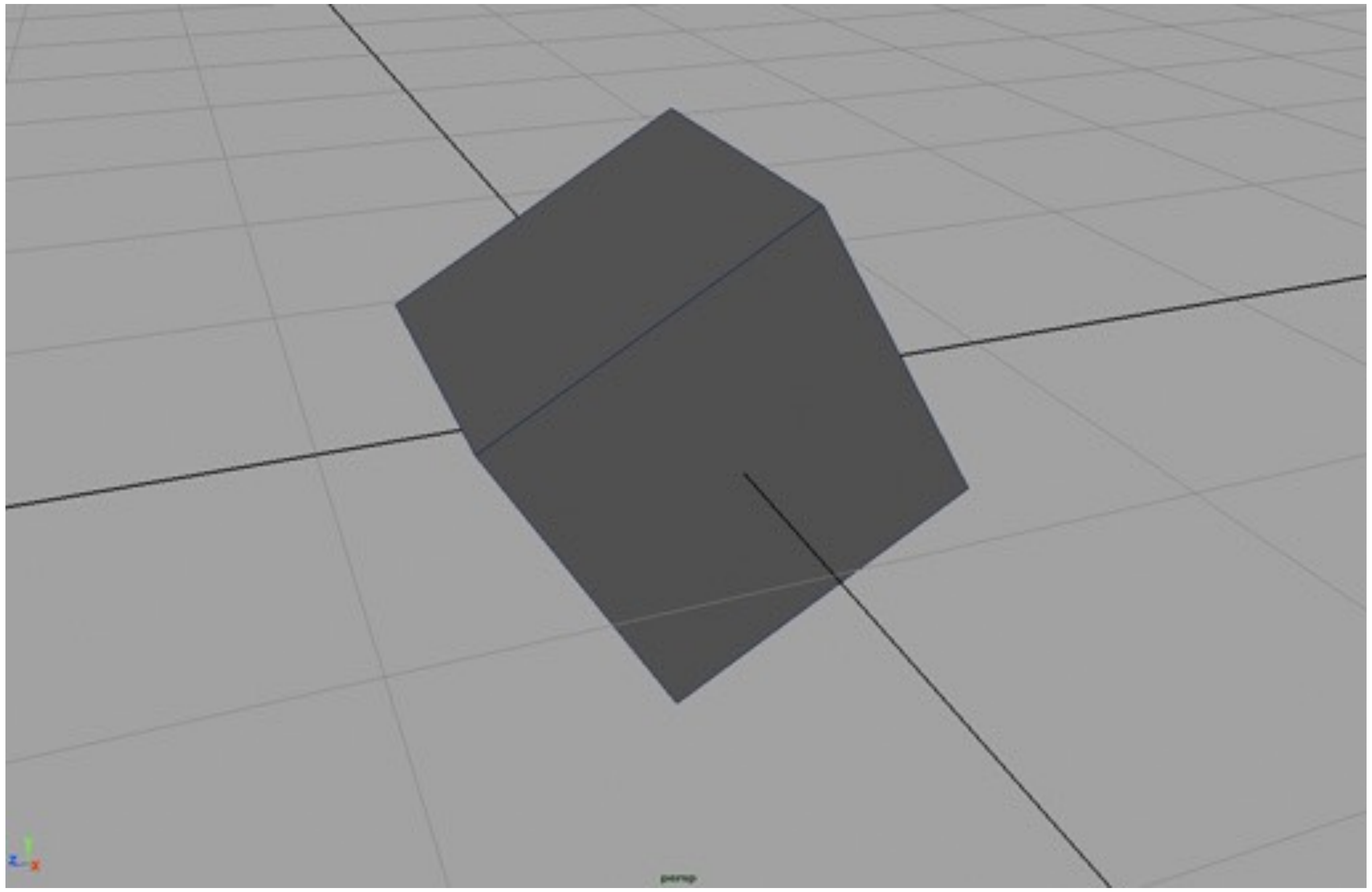
Rotation about ***x*** axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



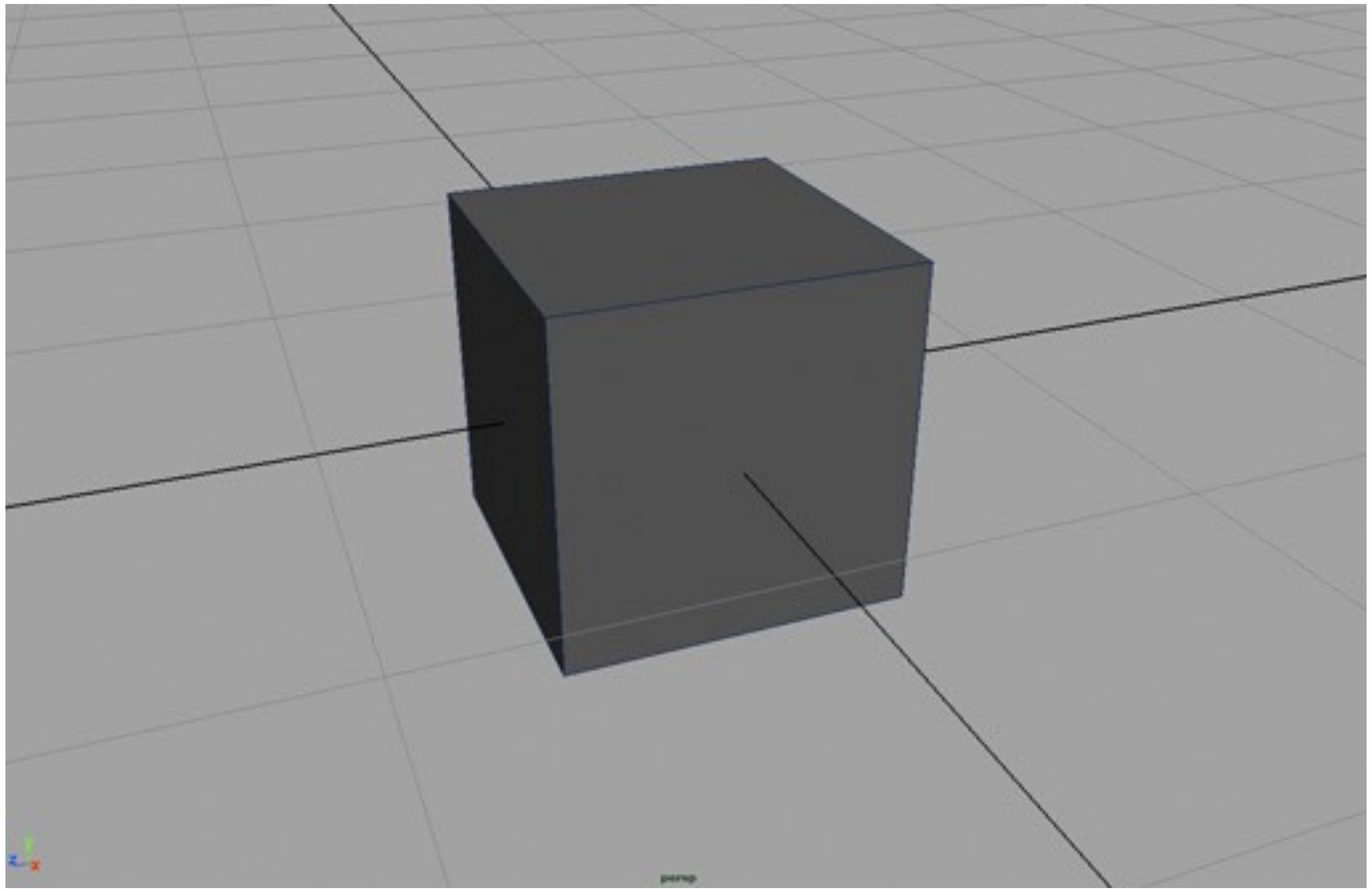
Rotation about ***x*** axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



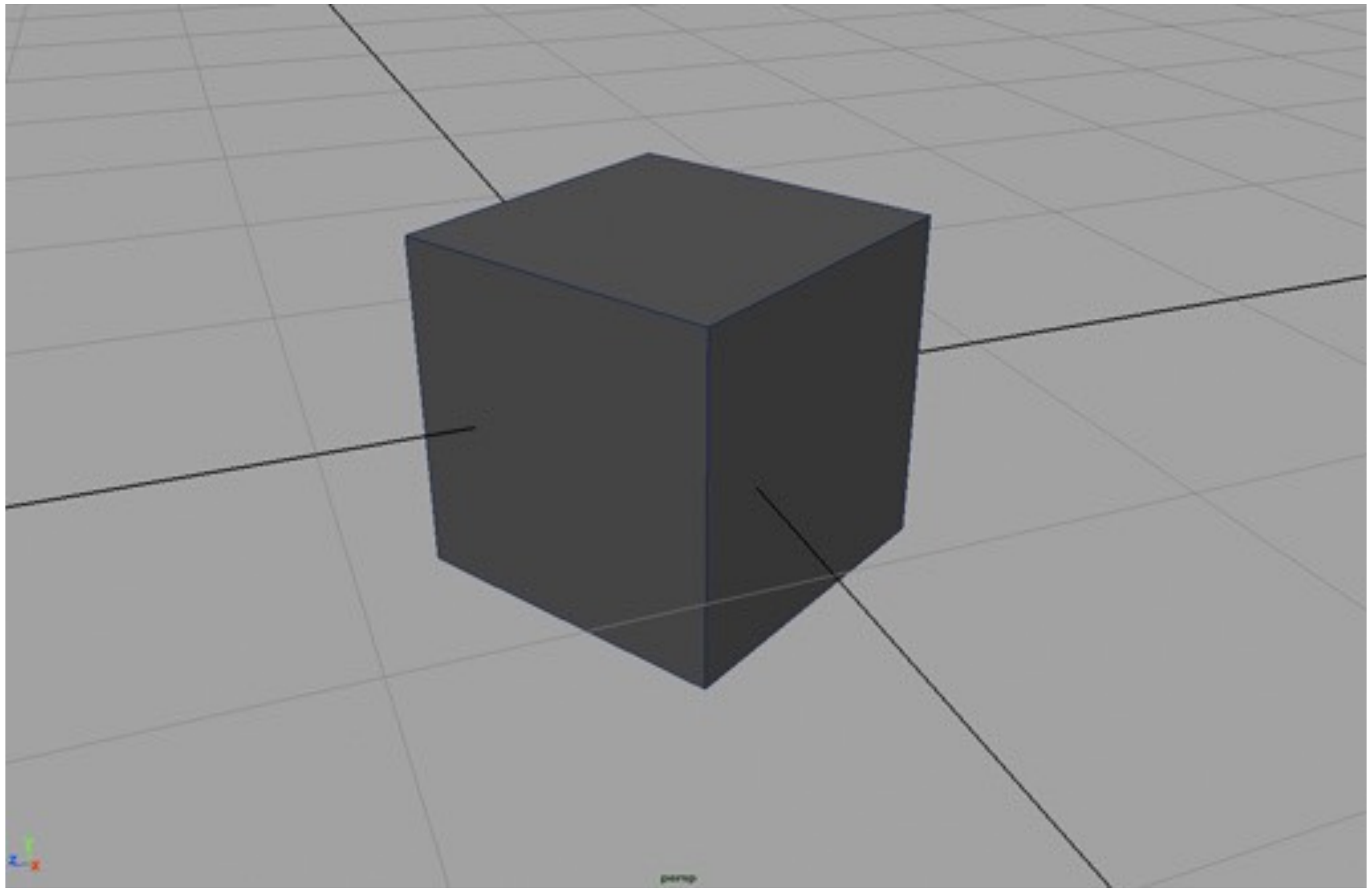
Rotation about **y** axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Rotation about **y** axis

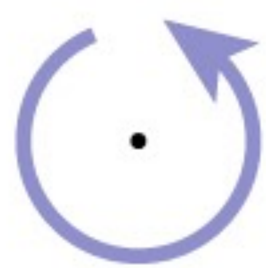
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



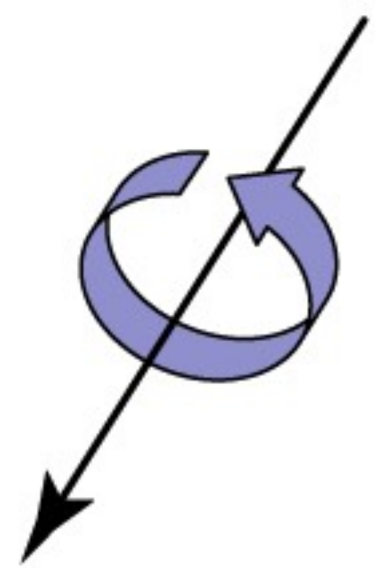
General Rotation Matrices

- **A rotation in 2D is around a point**
- **A rotation in 3D is around an axis**
  - so 3D rotation is w.r.t a line, not just a point
  - there are many more 3D rotations than 2D
    - a 3D space around a given point, not just 1D

convention: positive rotation is CCW



2D

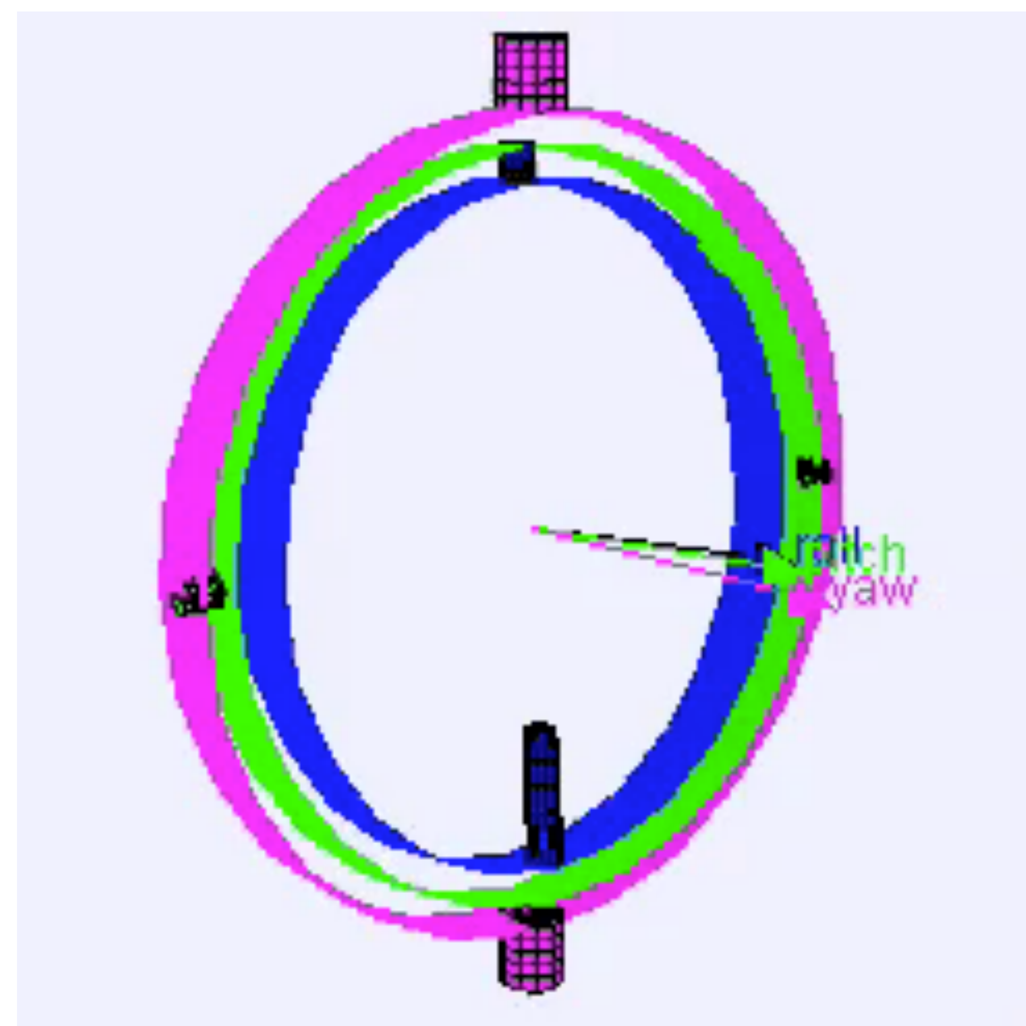


3D

convention: positive rotation is CCW when axis vector is pointing at you

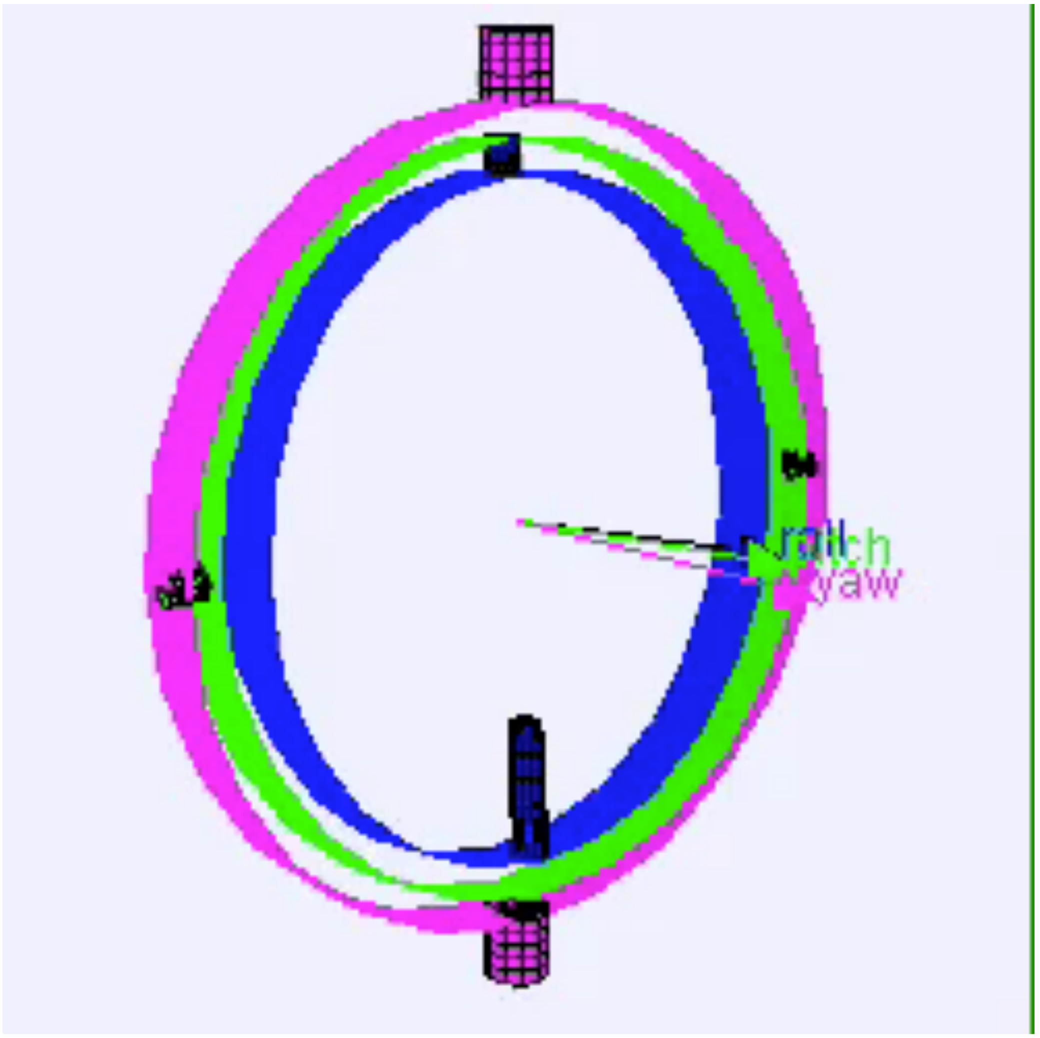
Euler angles

- **An object can be oriented arbitrarily**
- **Euler angles: simply compose three coord. axis rotations**
  - e.g. x, then y, then z: $R(\theta_x, \theta_y, \theta_z) = R_z(\theta_z) R_y(\theta_y) R_x(\theta_x)$
  - "heading, attitude, bank" (common for airplanes)
  - "roll, pitch, yaw" (common for vehicles)
  - "pan, tilt, roll" (common for cameras)

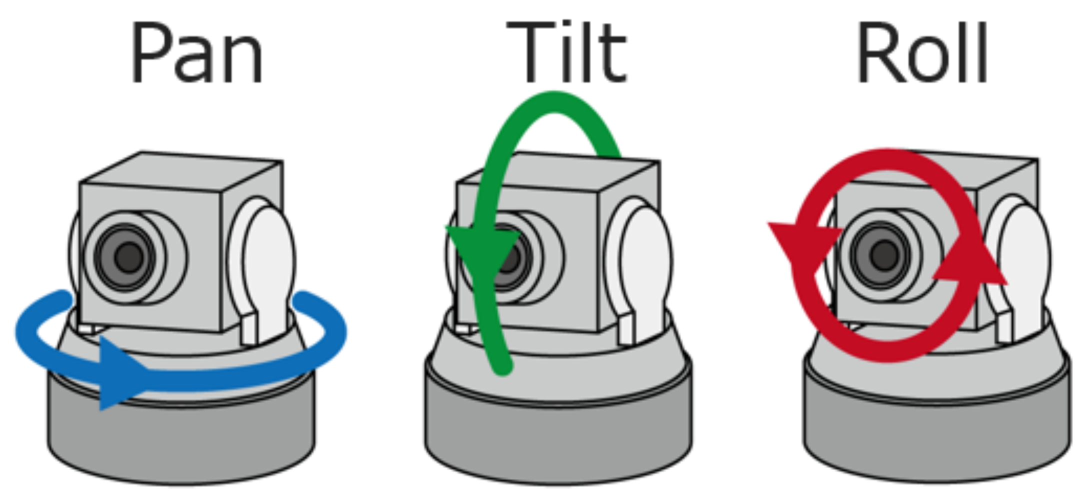


Euler angles

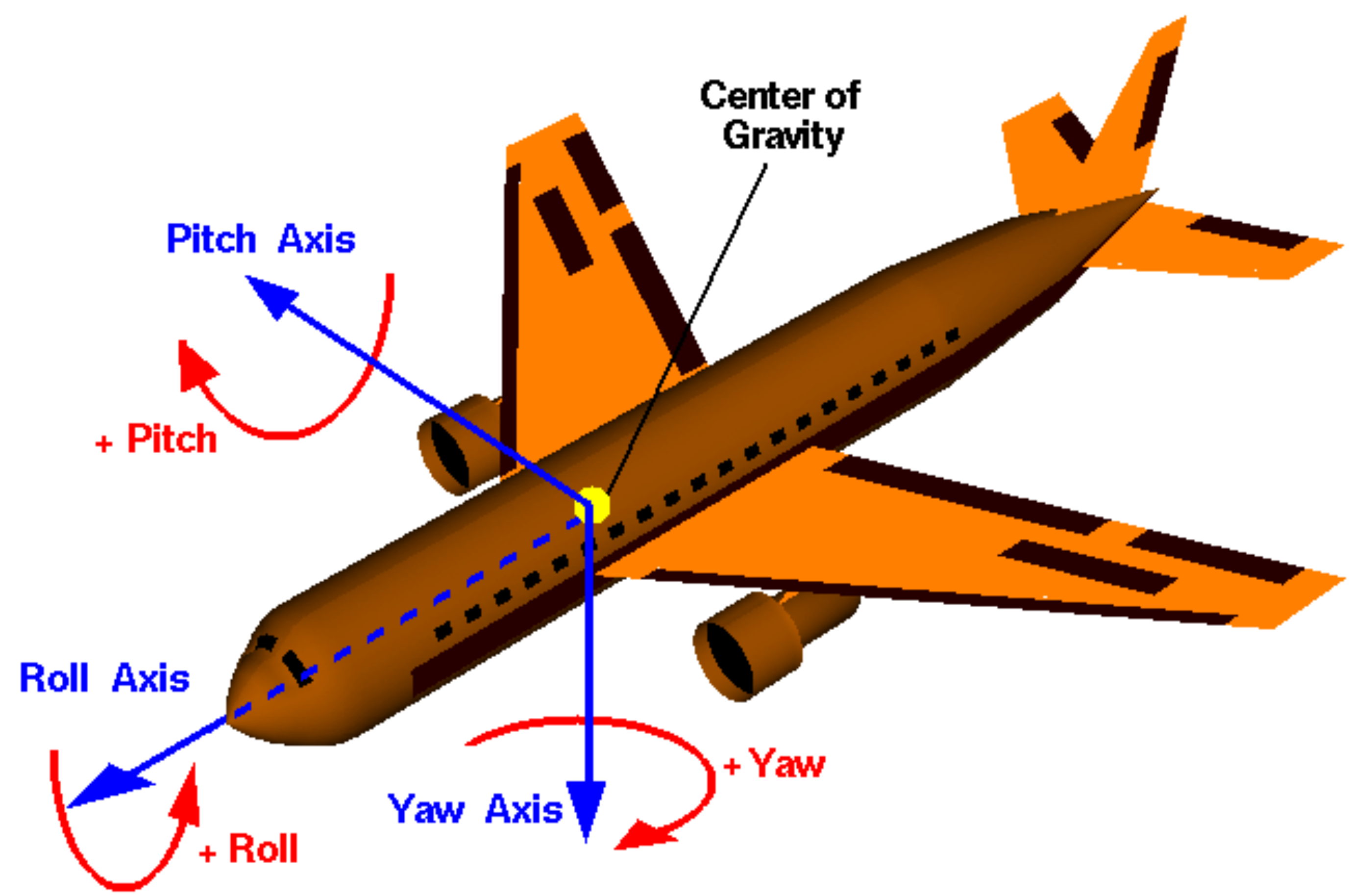
- **An object can be oriented arbitrarily**
- **Euler angles: simply compose three coord. axis rotations**
  - e.g. x, then y, then z: $R(\theta_x, \theta_y, \theta_z) = R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$
  - "heading, attitude, bank" (common for airplanes)
  - "roll, pitch, yaw" (common for vehicles)
  - "pan, tilt, roll" (common for cameras)



Euler angles in applications



Ishikawa Watanabe Laboratory

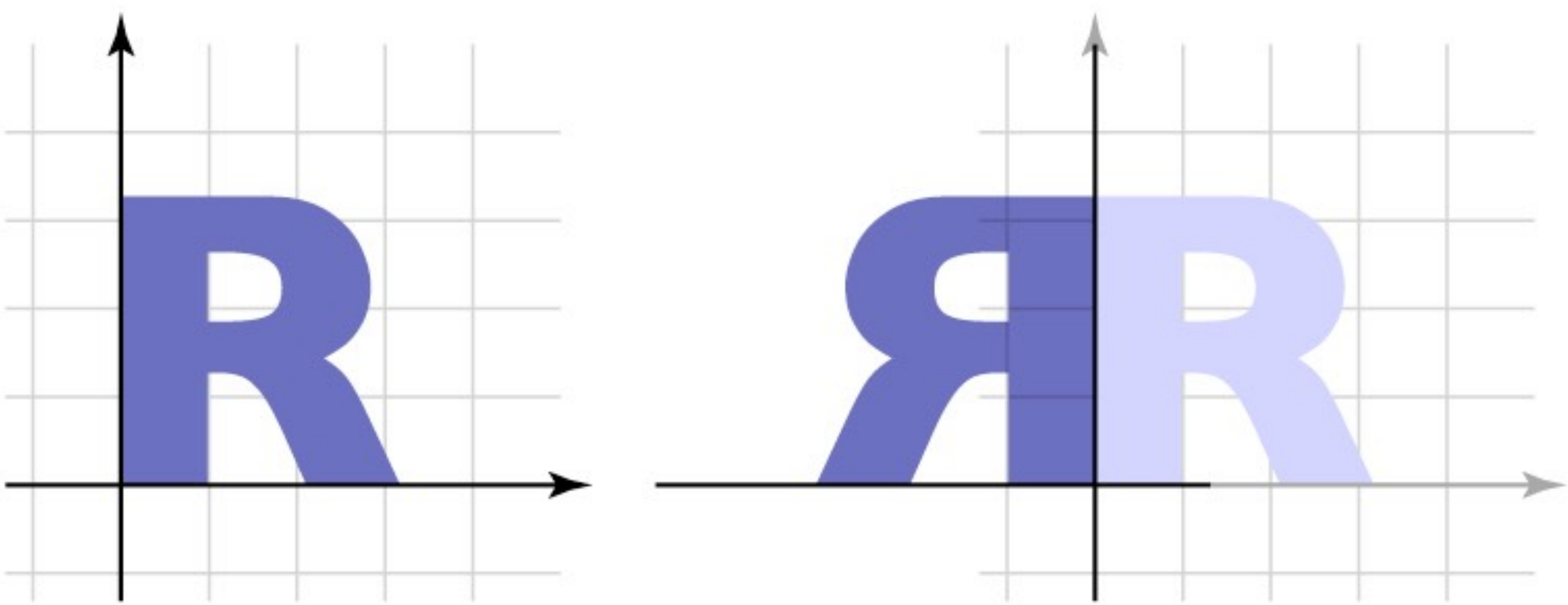


Wikipedia

Affine transformation gallery

- **Reflection**
  – can consider it a special case of nonuniform scale

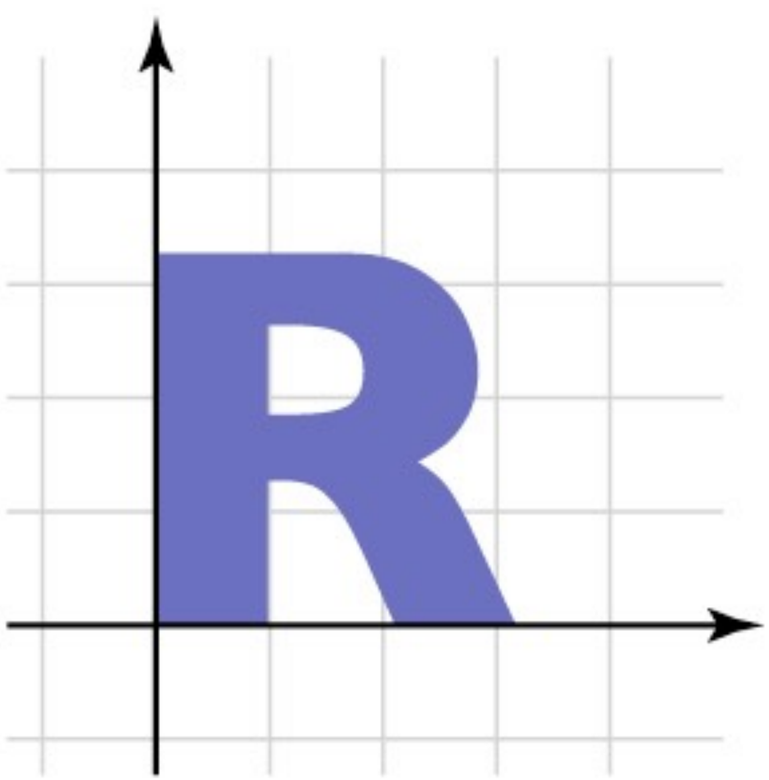
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

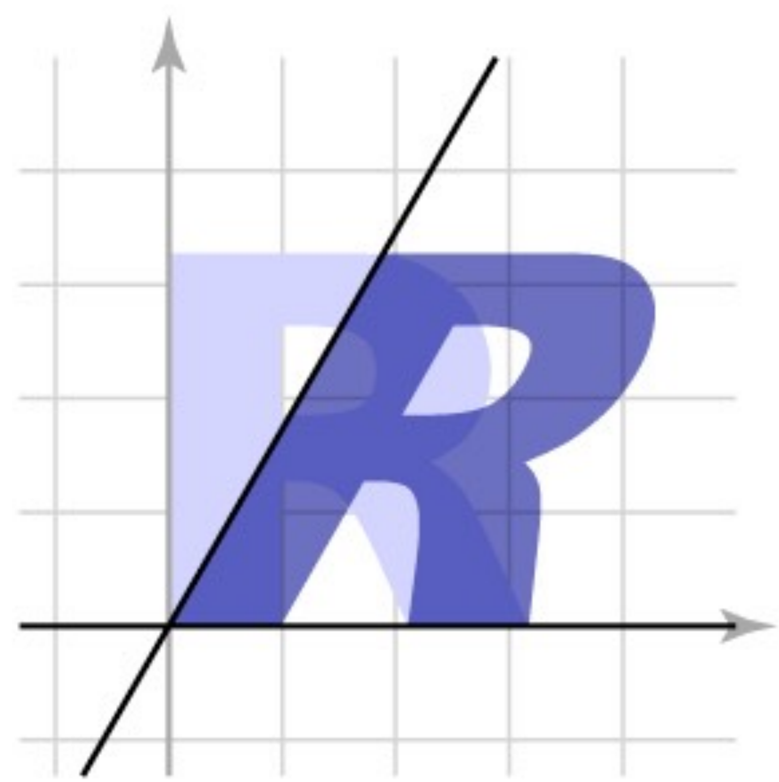


Affine transformation gallery

- **Shear**

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





Properties of Matrices

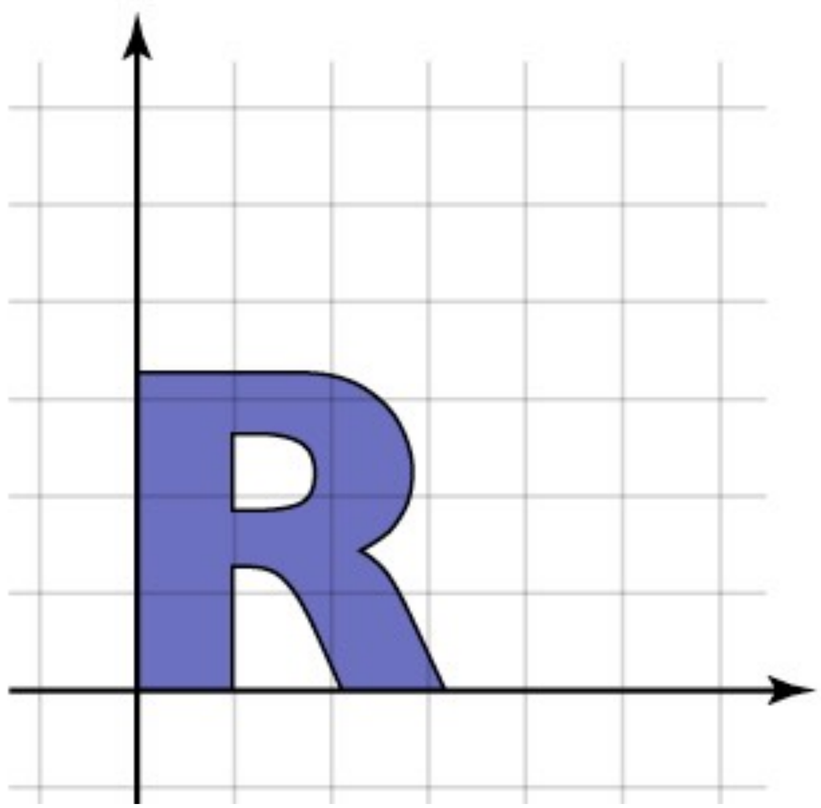
- **Translations: linear part is the identity**
- **Scales: linear part is diagonal**
- **Rotations: linear part is orthogonal**
  - Columns of R are mutually orthonormal: $RR^T=R^TR=I$
  - Also, determinant of R is 1.0 [$\det(R) = 1$]

General affine transformations

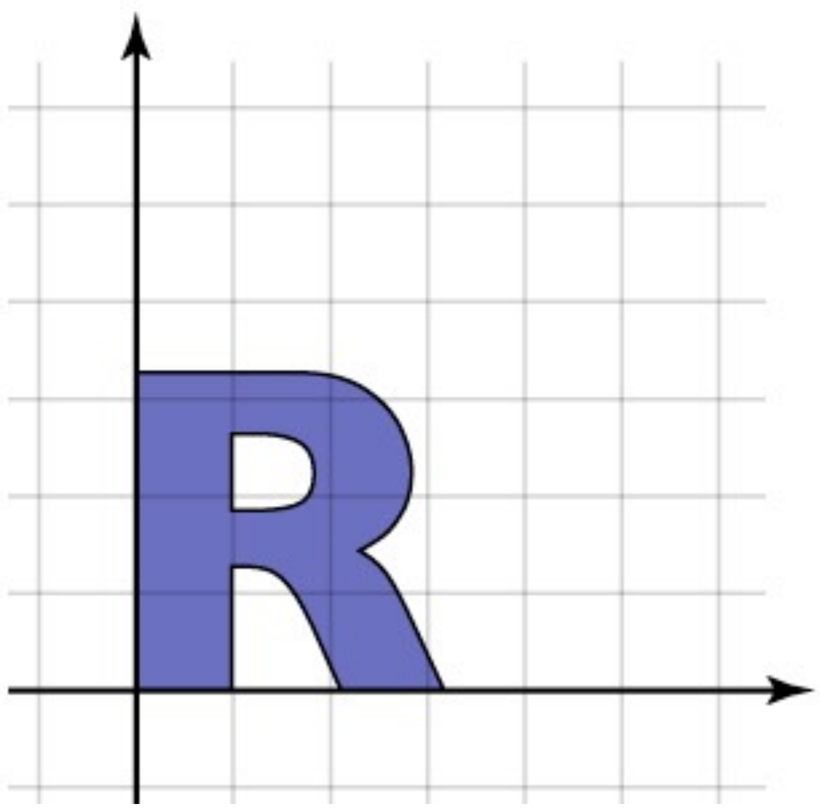
- **The previous slides showed “canonical” examples of the types of affine transformations**
- **Generally, transformations contain elements of multiple types**
  - often define them as products of canonical transforms
  - sometimes work with their properties more directly

Composite affine transformations

- **In general not commutative: order matters!**



rotate, then translate



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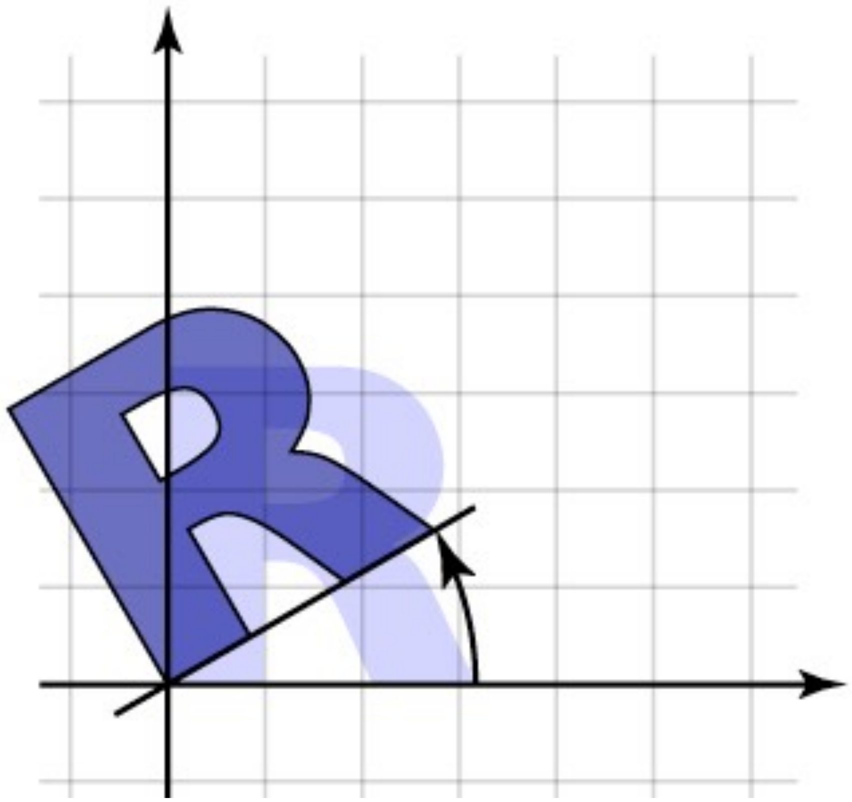
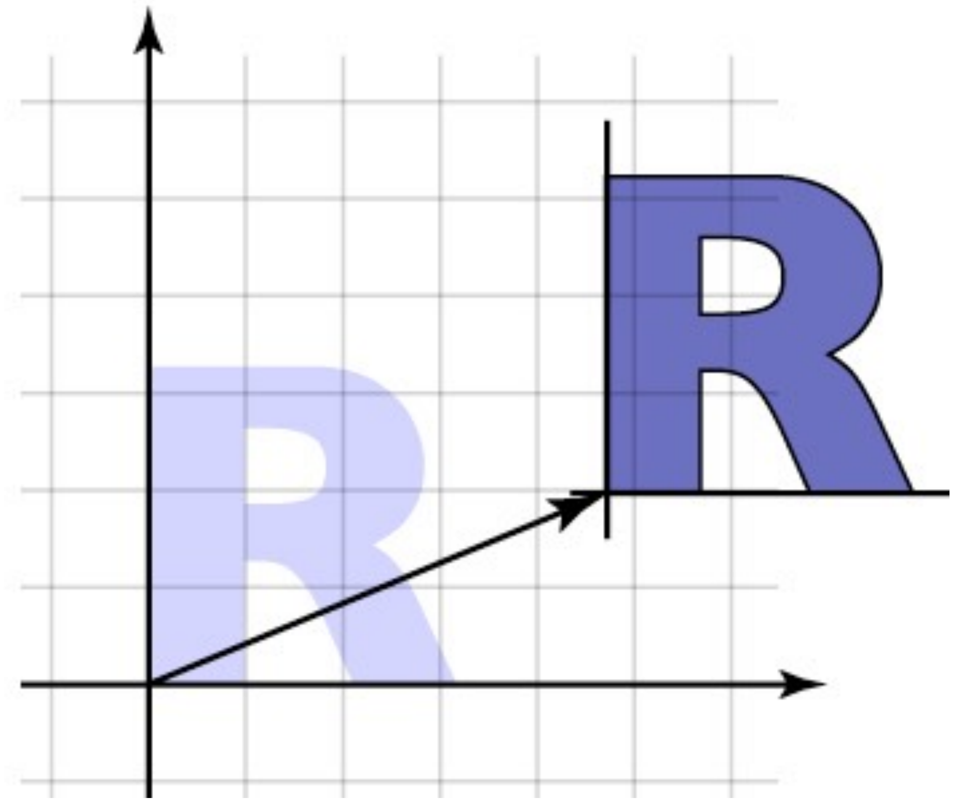


rotate, then translate



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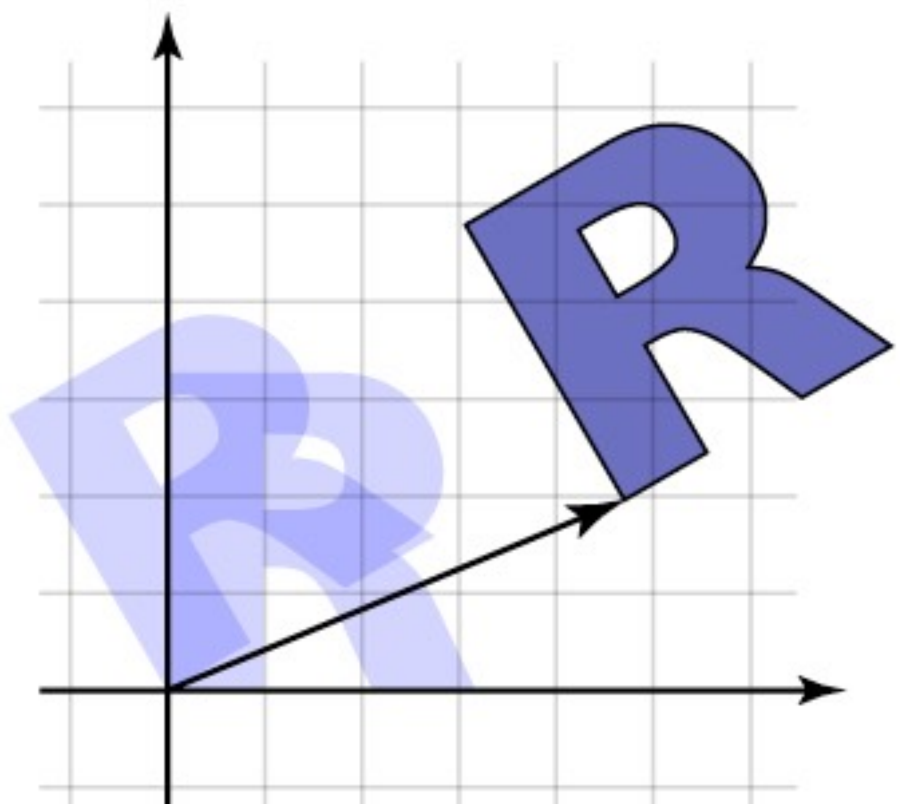

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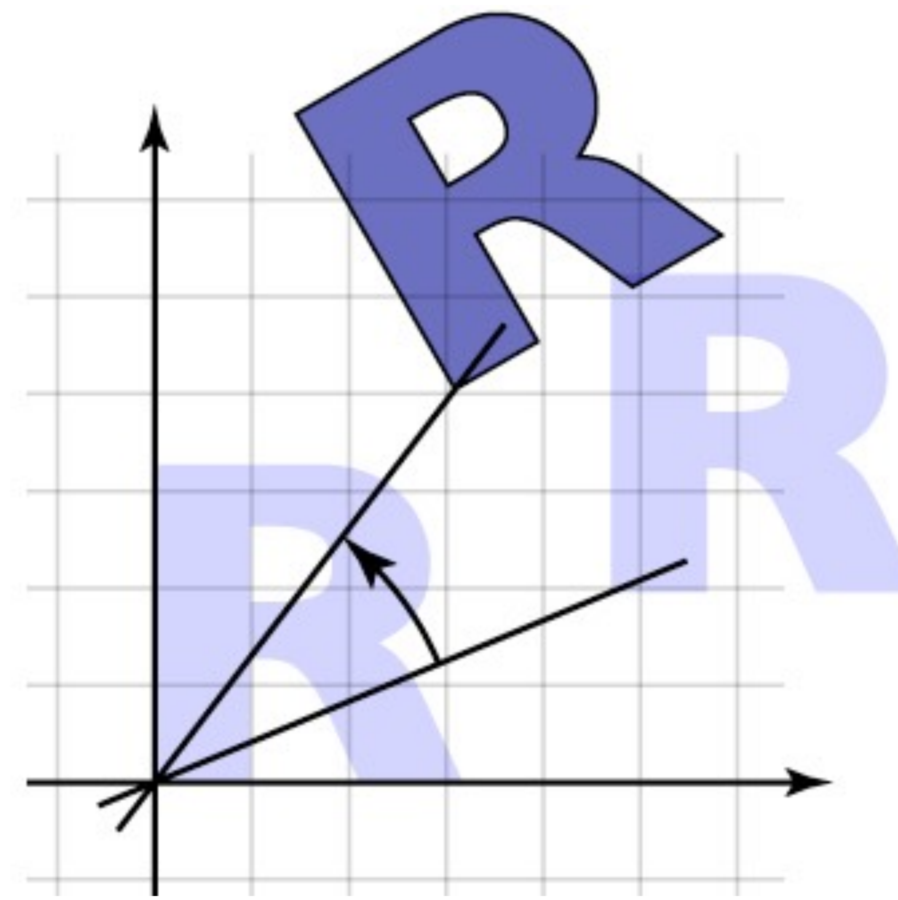

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Composite affine transformations

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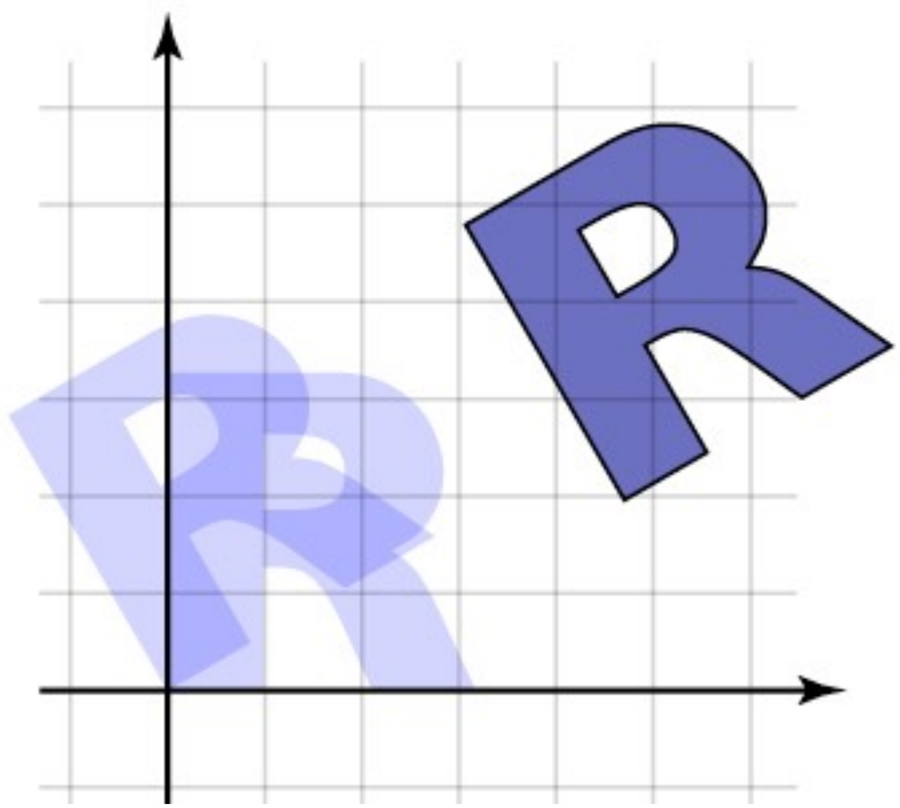
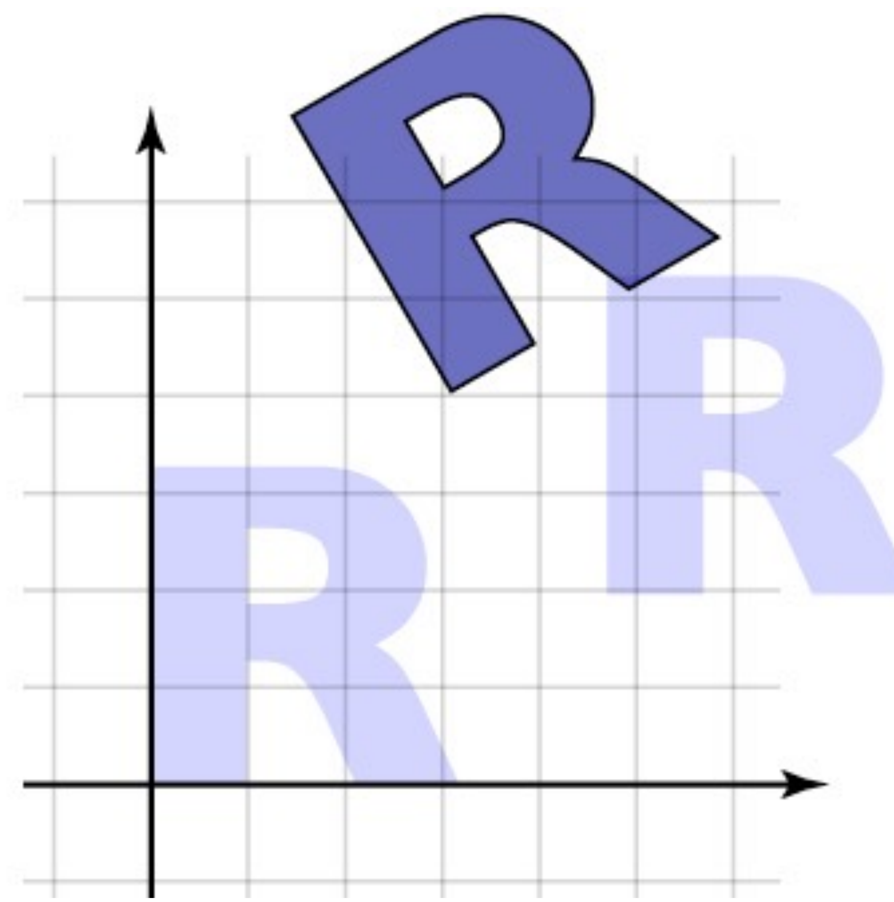


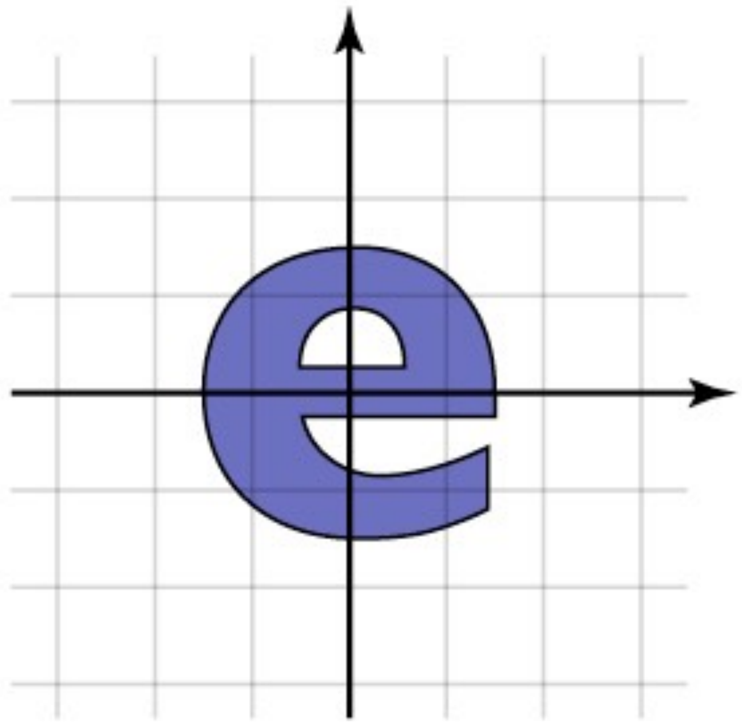
rotate, then translate



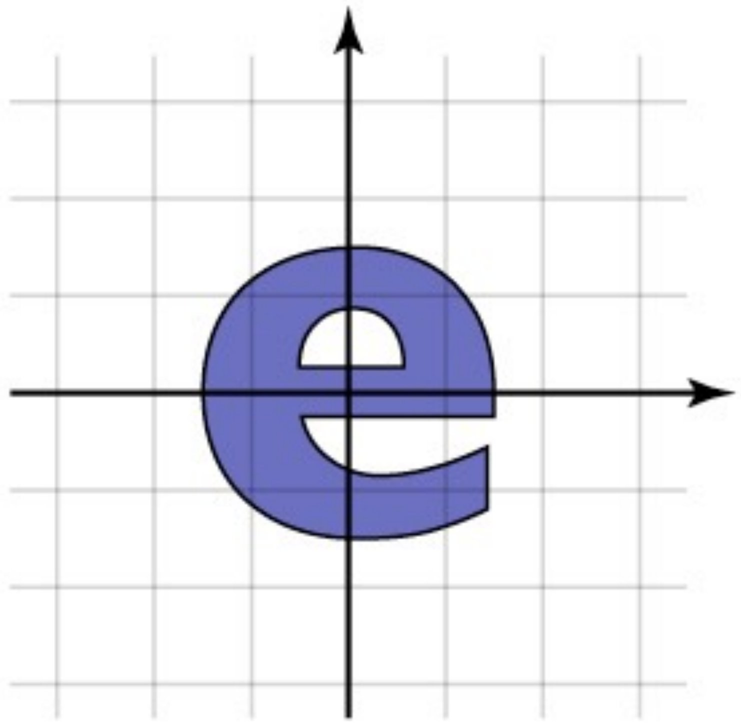
translate, then rotate

Composite affine transformations

- **Another example**



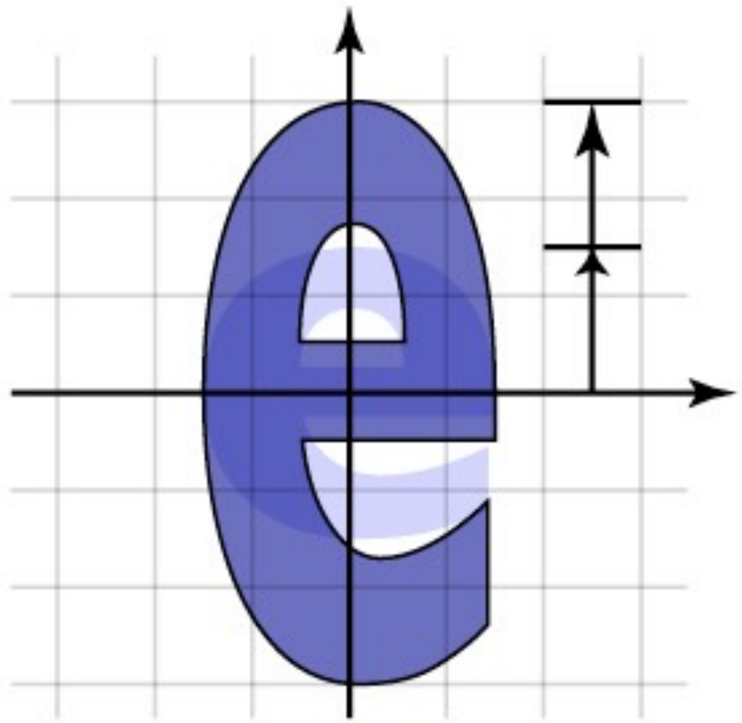
scale, then rotate



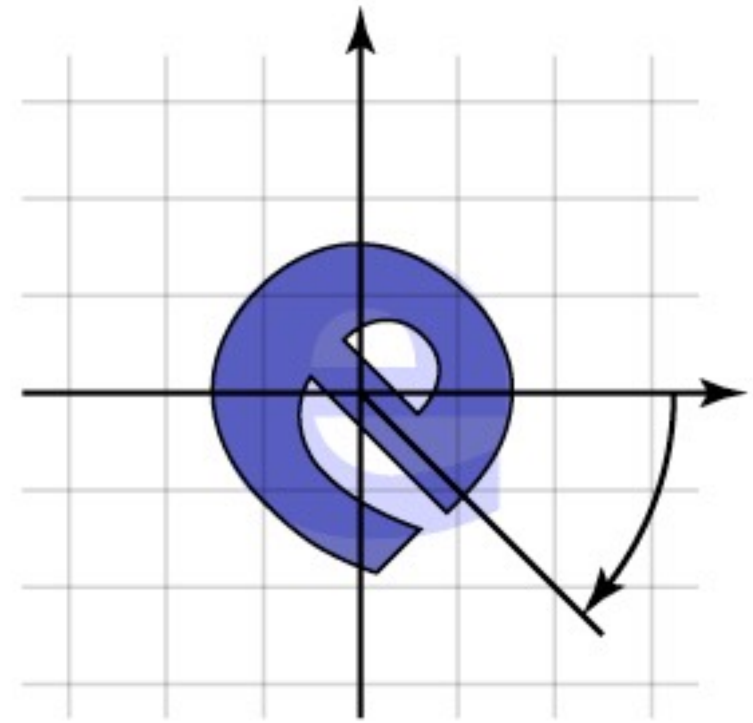
rotate, then scale

Composite affine transformations

- **Another example**



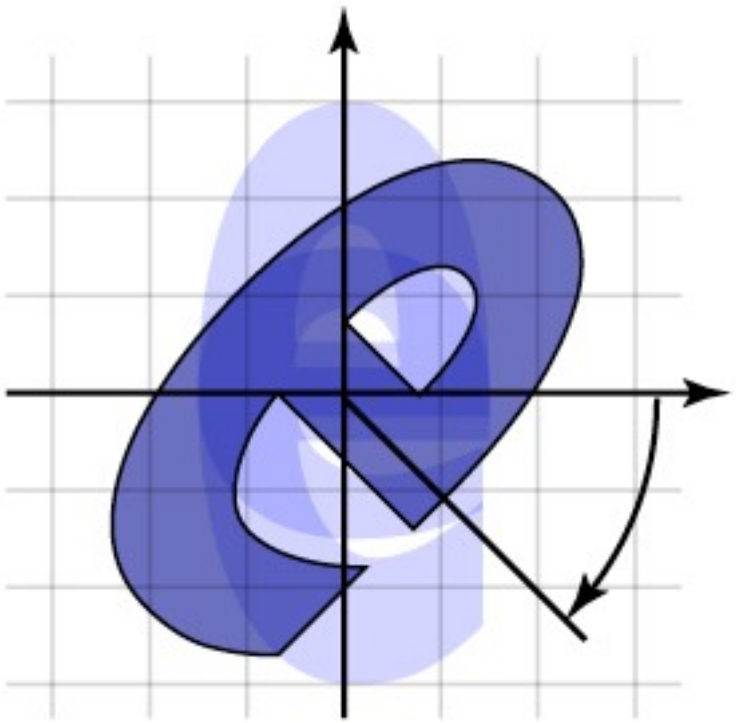
scale, then rotate



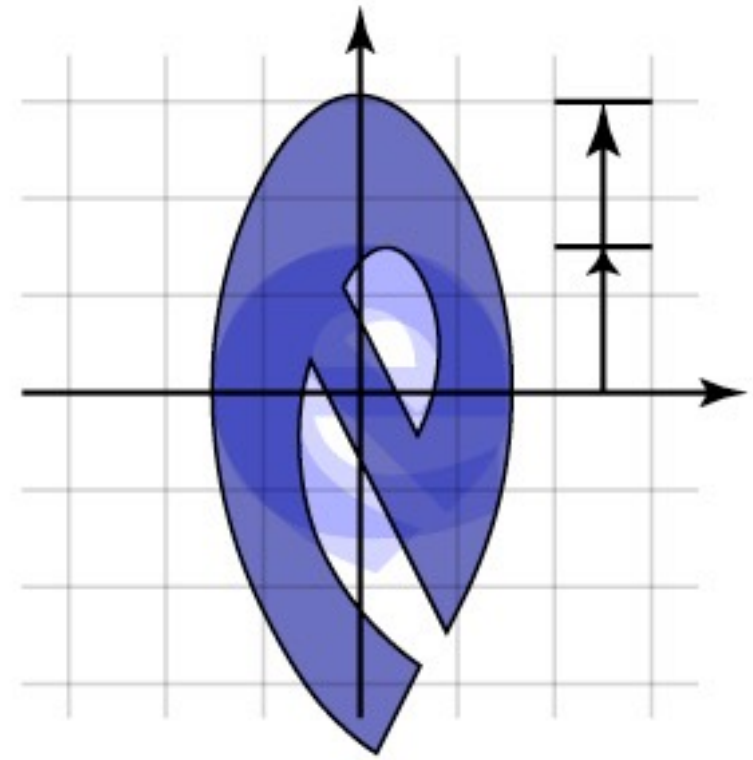
rotate, then scale

Composite affine transformations

- **Another example**



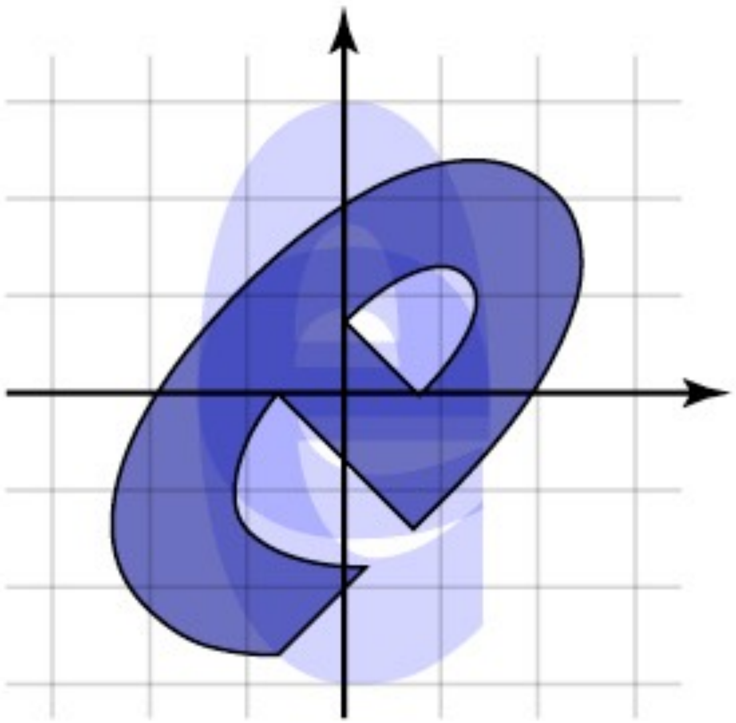
scale, then rotate



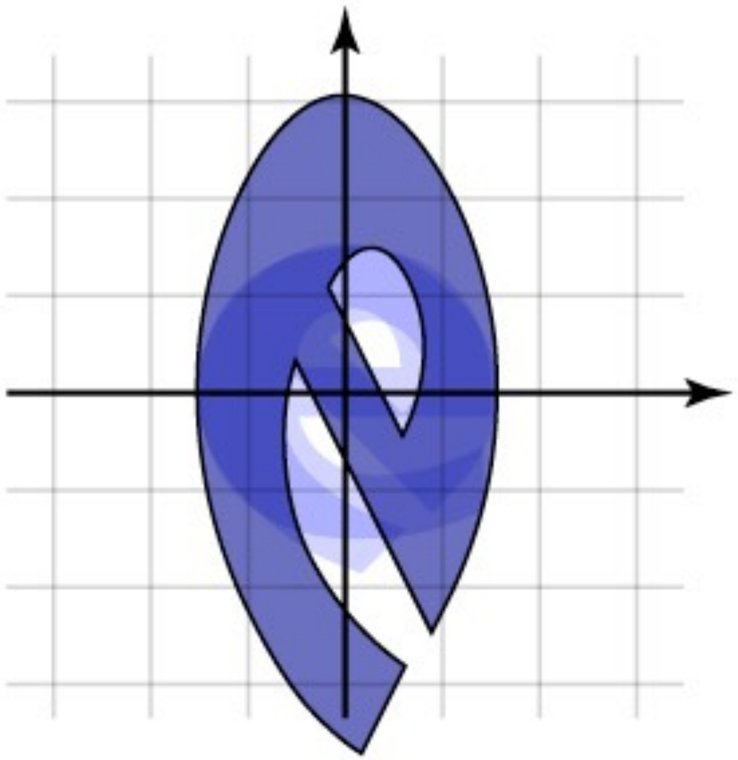
rotate, then scale

Composite affine transformations

- **Another example**



scale, then rotate



rotate, then scale

Rigid motions

- **A transform made up of only translation and rotation is a *rigid motion* or a *rigid body transformation***
- **The linear part is an orthonormal matrix**

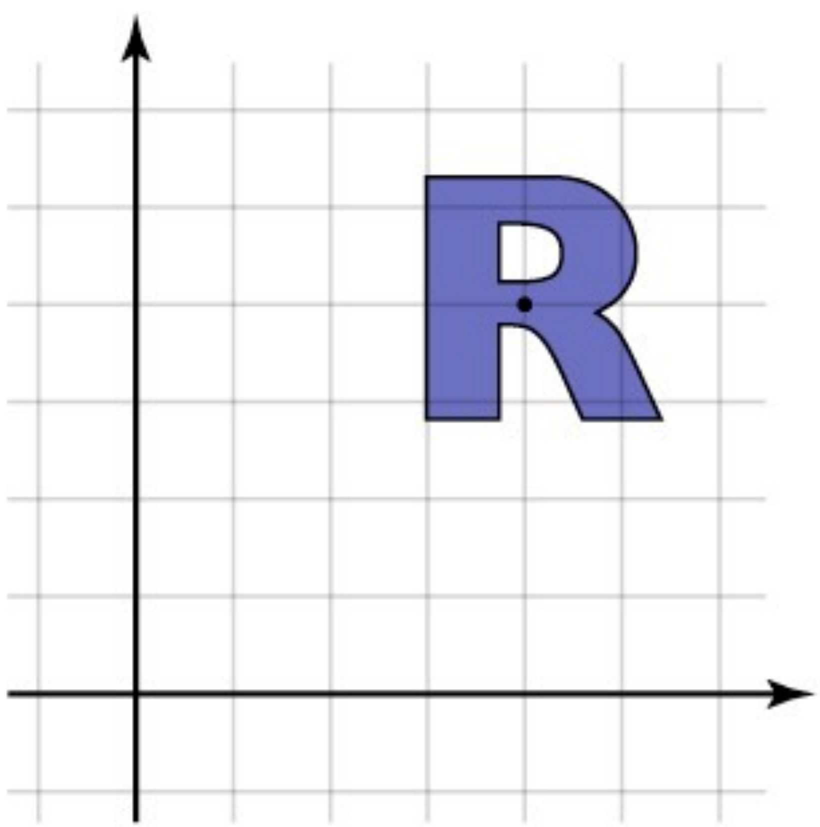
$$R = \begin{bmatrix} Q & \mathbf{u} \\ 0 & 1 \end{bmatrix}$$

- **Inverse of orthonormal matrix is transpose**
  - so inverse of rigid motion is easy:

$$R^{-1}R = \begin{bmatrix} Q^T & -Q^T\mathbf{u} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q & \mathbf{u} \\ 0 & 1 \end{bmatrix}$$

Composing to change axes

- **Want to rotate about a particular point**
  - could work out formulas directly…
- **Know how to rotate about the origin**
  - so translate that point to the origin



$$M = T^{-1}RT$$

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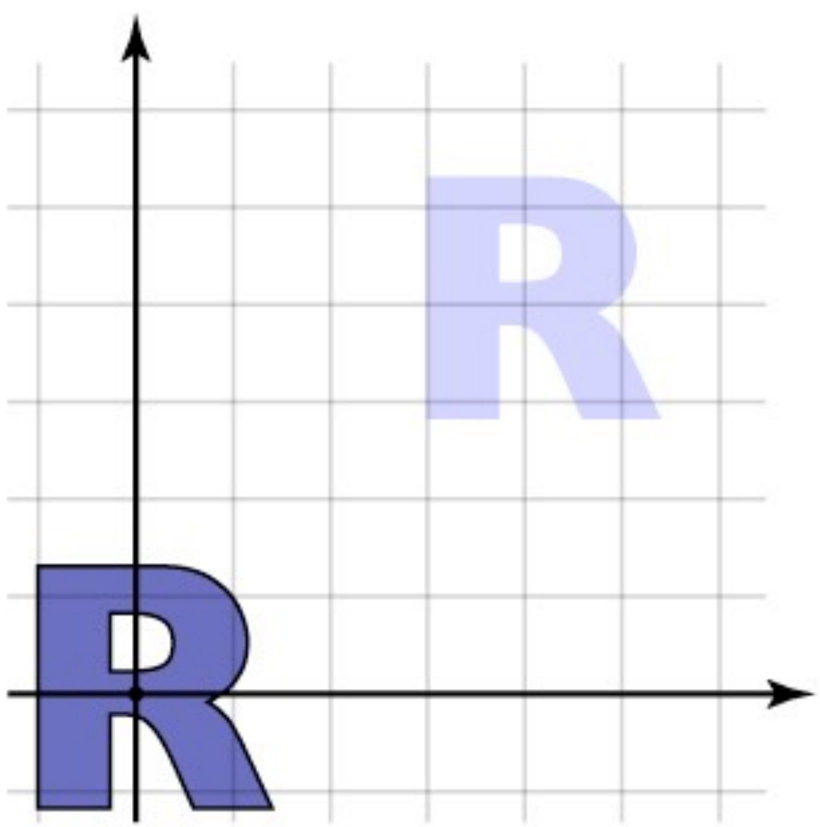

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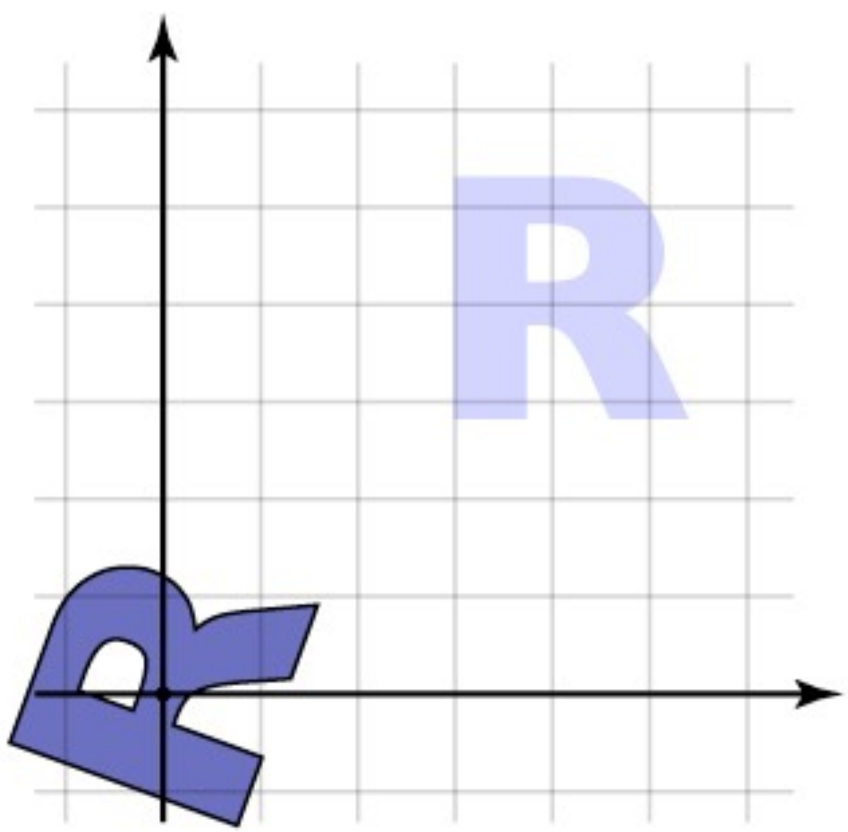

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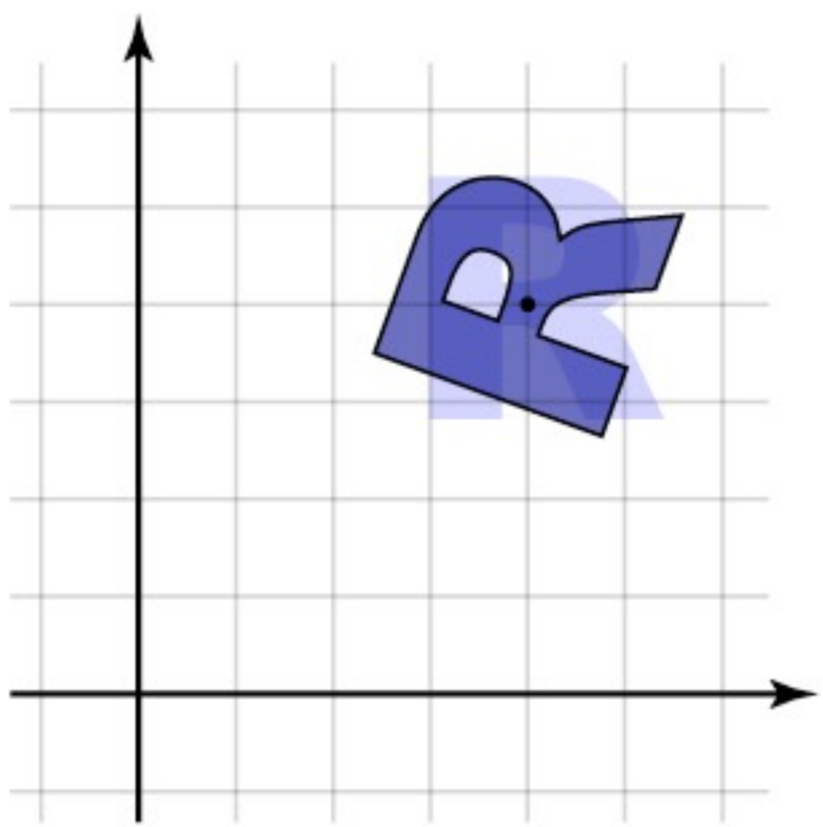
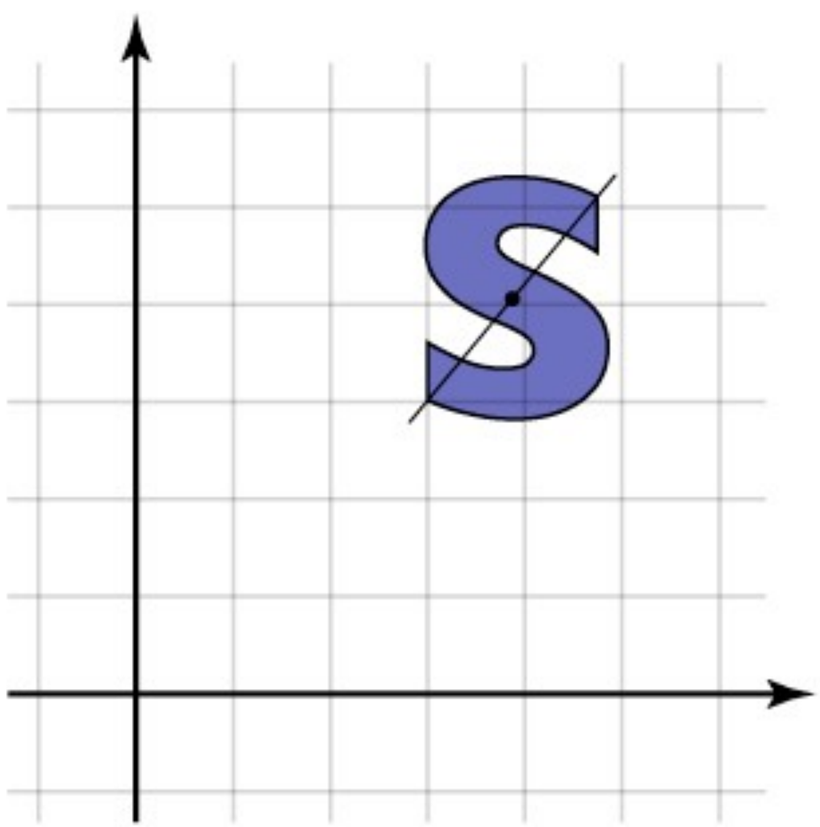


$$M = T^{-1} R T$$

Composing to change axes

- **Want to scale along a particular axis and point**
- **Know how to scale along the y axis at the origin**
  - so translate to the origin and rotate to align axes



$$M = T^{-1}R^{-1}SRT$$

Composing to change axes

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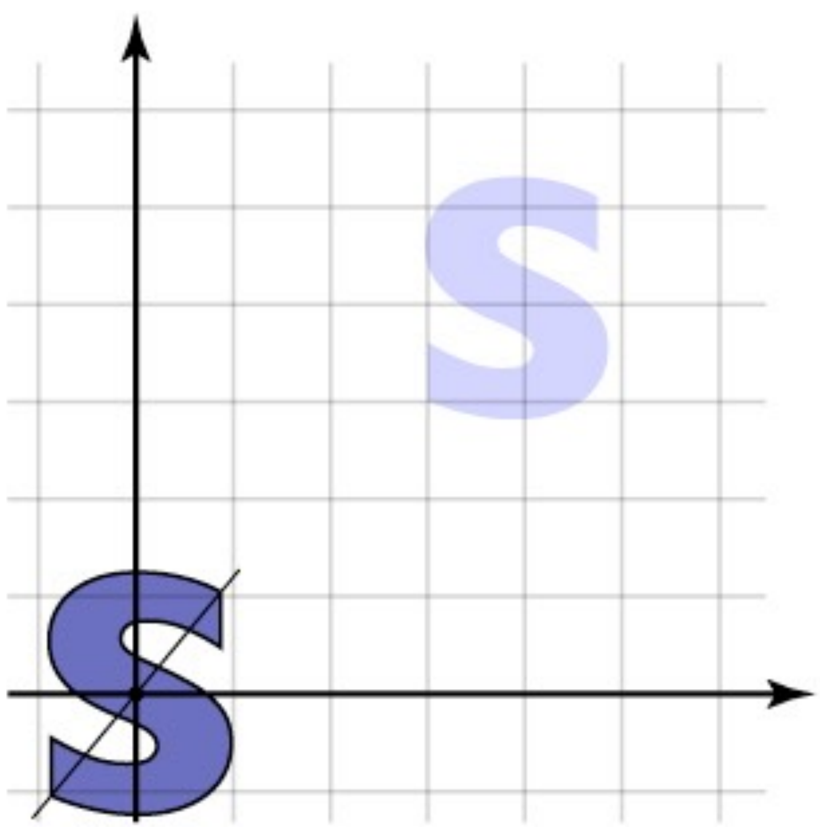

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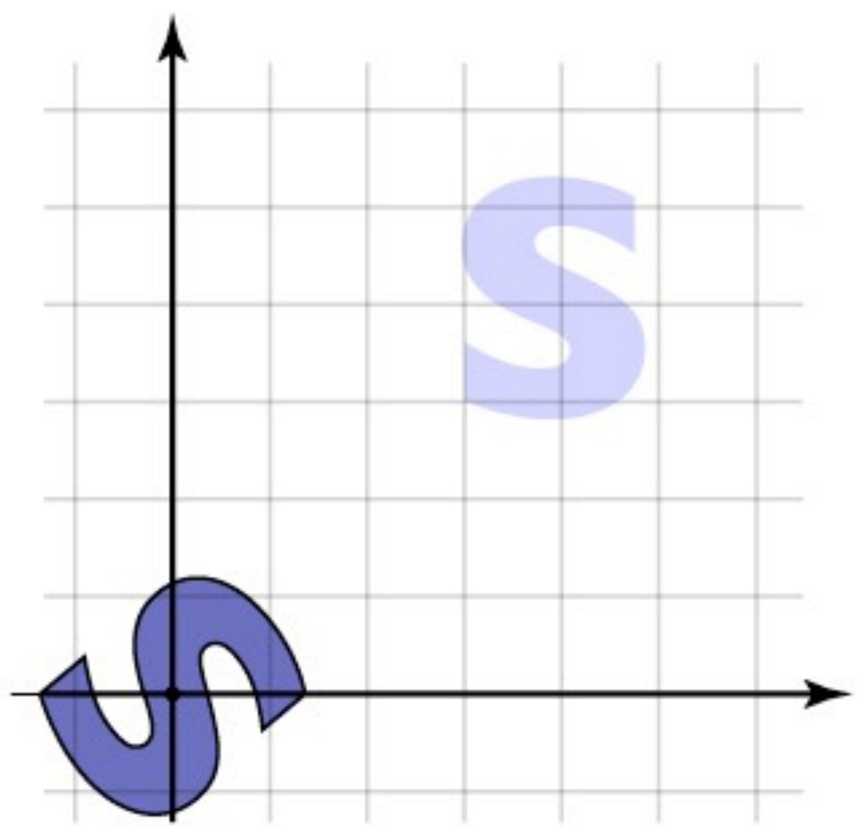

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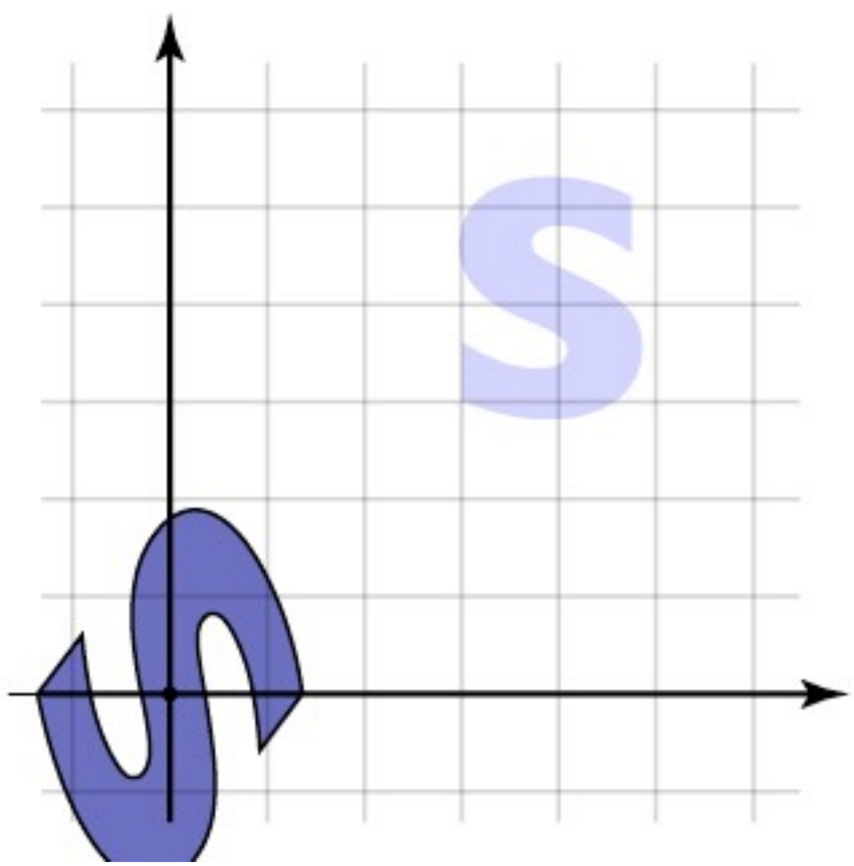

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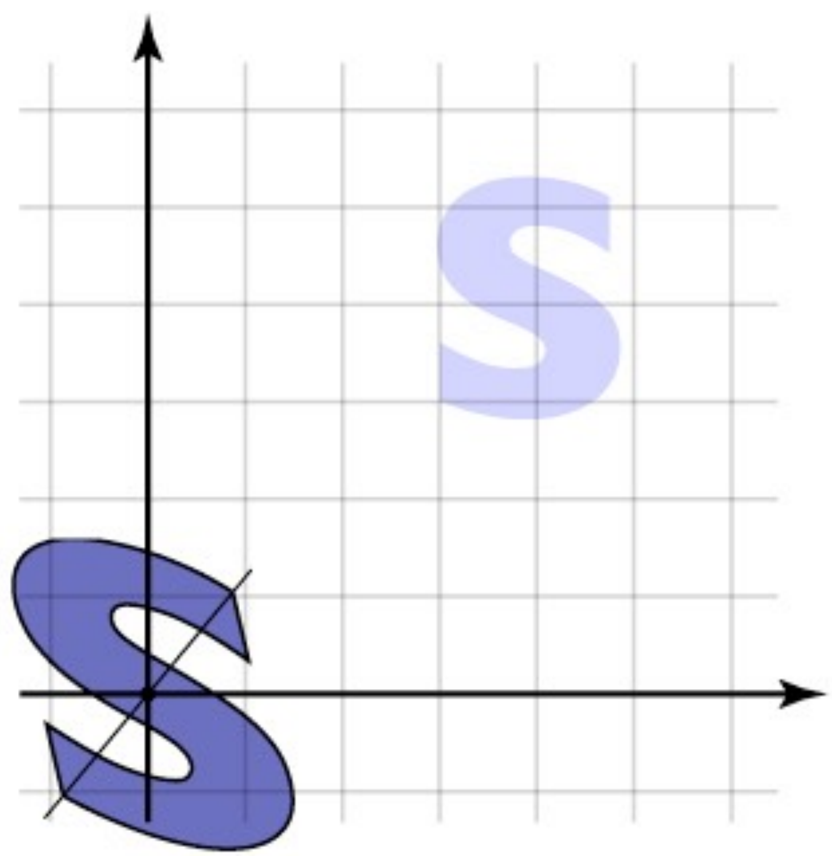

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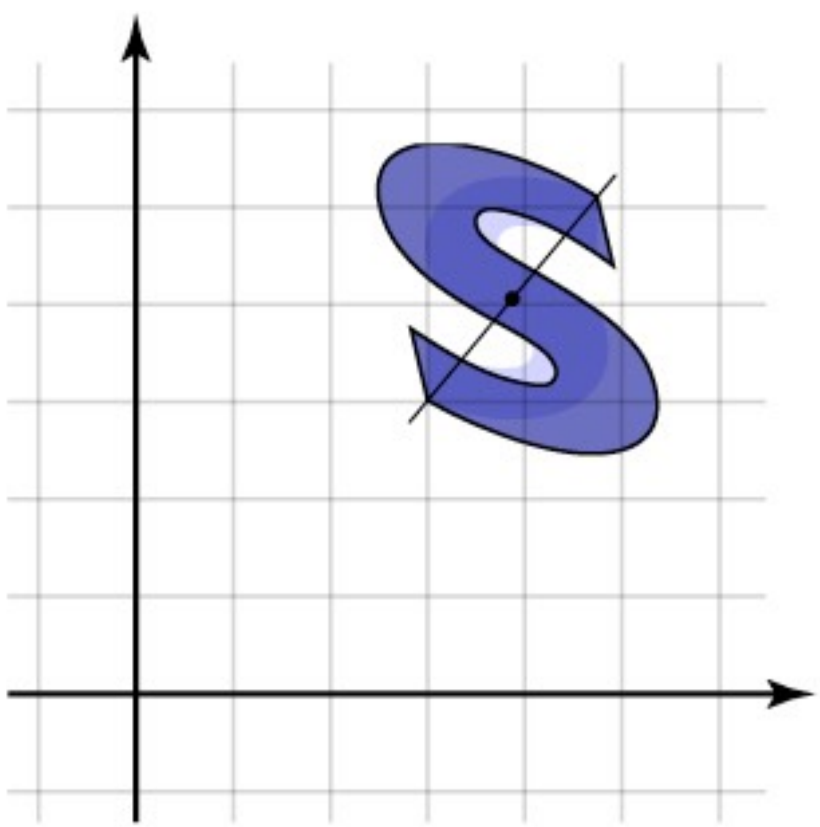


$$M = T^{-1}R^{-1}SRT$$

Transforming points and vectors

- **Recall distinction points vs. vectors**
  - vectors are just offsets (differences between points)
  - points have a location
    - represented by vector offset from a fixed origin
- **Points and vectors transform differently**
  - points respond to translation; vectors do not

$$\mathbf{v} = \mathbf{p} - \mathbf{q}$$

$$T(\mathbf{x}) = M\mathbf{x} + \mathbf{t}$$

$$\begin{aligned} T(\mathbf{p} - \mathbf{q}) &= M\mathbf{p} + \mathbf{t} - (M\mathbf{q} + \mathbf{t}) \\ &= M(\mathbf{p} - \mathbf{q}) + (\mathbf{t} - \mathbf{t}) = M\mathbf{v} \end{aligned}$$

Transforming points and vectors

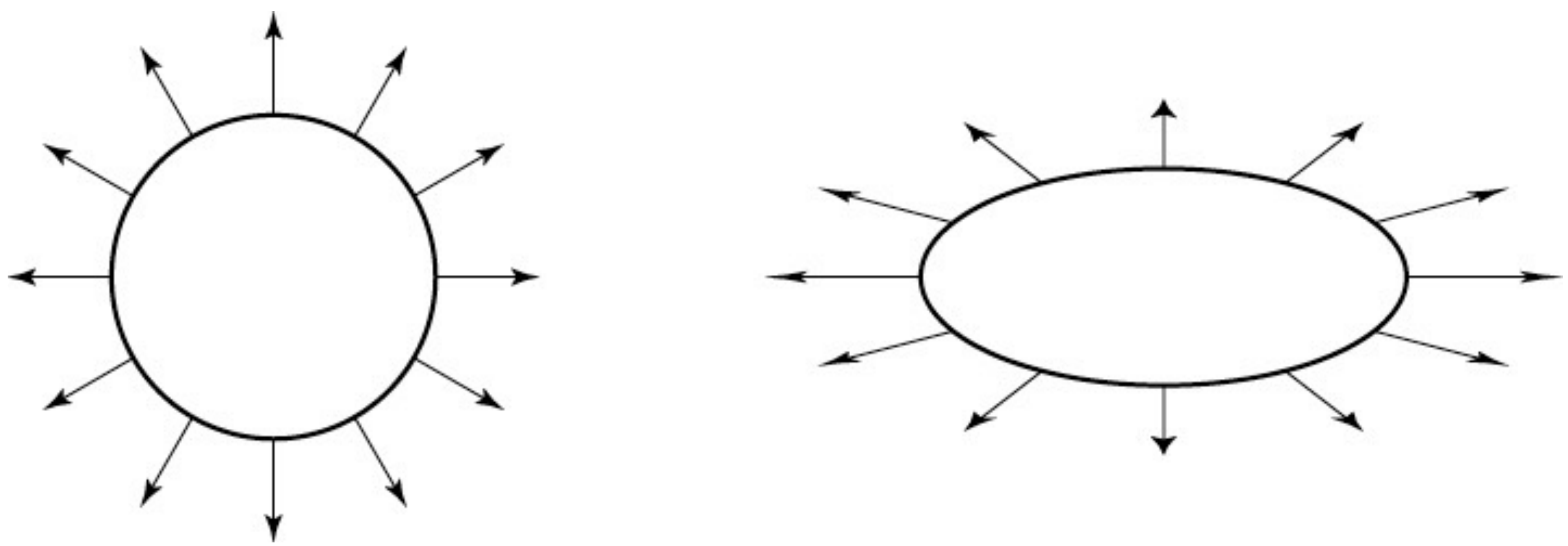
- **Homogeneous coords. let us exclude translation**
  - just put 0 rather than 1 in the last place

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} M\mathbf{p} + \mathbf{t} \\ 1 \end{bmatrix} \qquad \begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} M\mathbf{v} \\ 0 \end{bmatrix}$$

  - and note that subtracting two points cancels the extra coordinate, resulting in a vector!
- **Preview: projective transformations**
  - what's really going on with this last coordinate?
  - think of R^2 embedded in R^3: all affine xfs. preserve z=1 plane
  - could have other transforms; project back to z=1

Transforming normal vectors

- **Transforming surface normals**
  - differences of points (and therefore tangents) transform OK
  - normals do not; therefore use inverse transpose matrix



have: $\mathbf{t} \cdot \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$
want: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T X \mathbf{n} = 0$
so set $X = (M^T)^{-1}$
then: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T (M^T)^{-1} \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$

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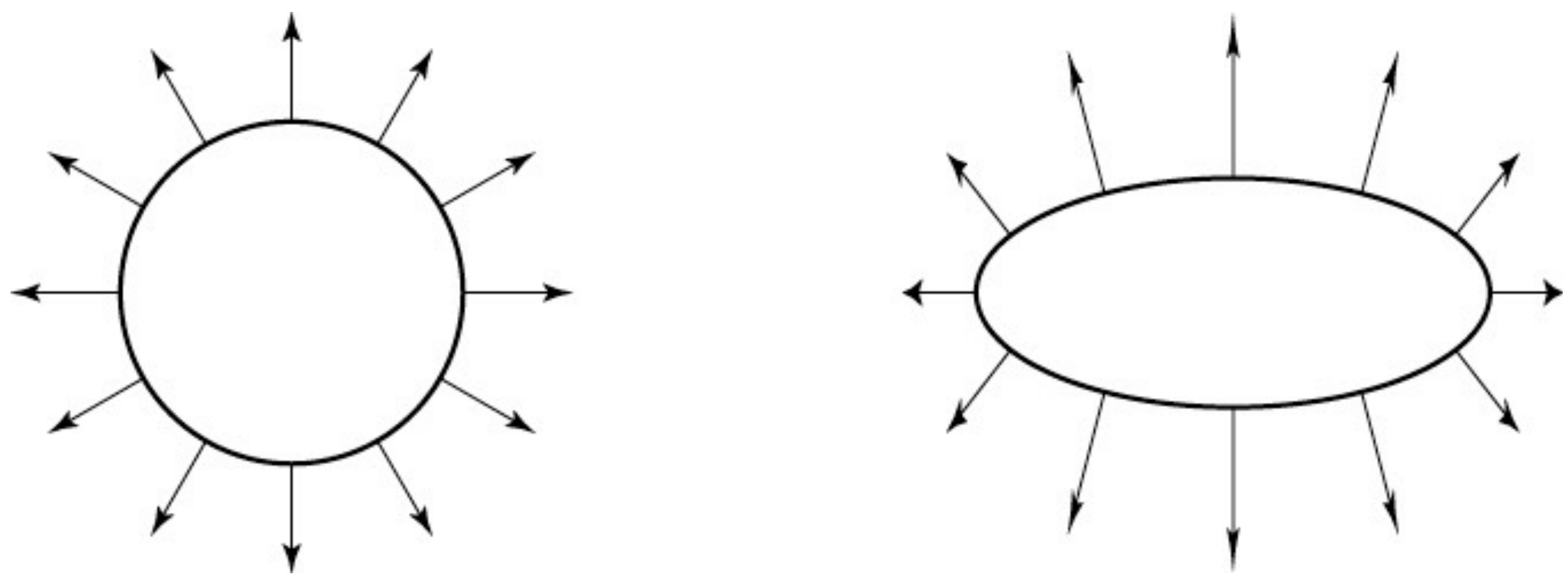

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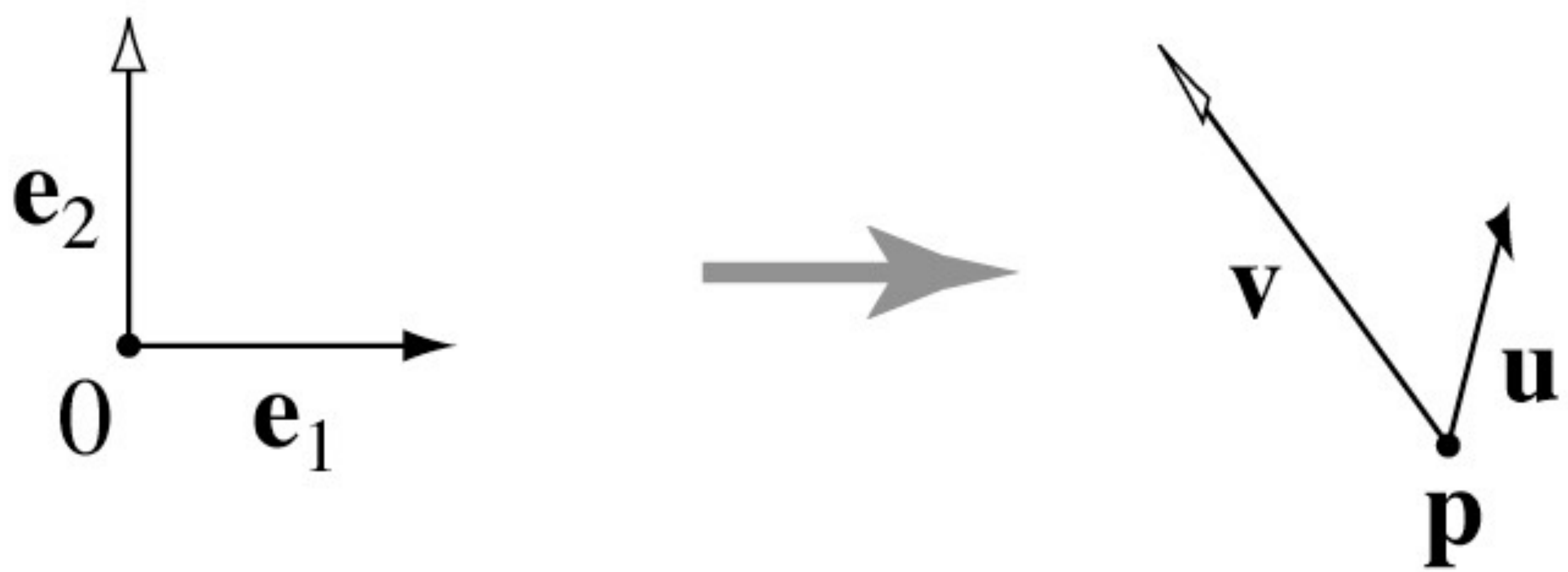
More math background

- **Coordinate systems**
  - Expressing vectors with respect to bases
  - Linear transformations as changes of basis

Affine change of coordinates

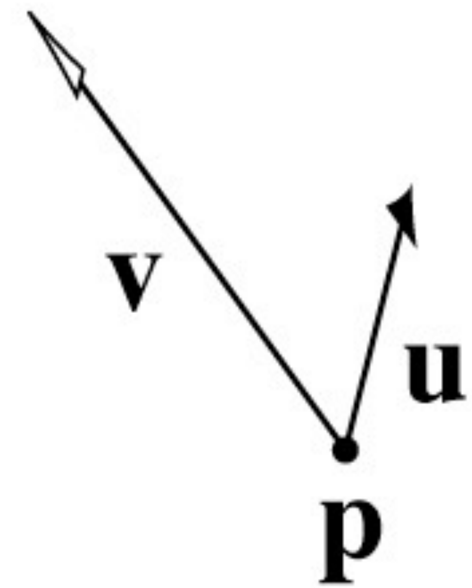
- **Six degrees of freedom**

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$



Affine change of coordinates

- **Coordinate frame: point plus basis**
- **Interpretation: transformation changes representation of point from one basis to another**
- **"Frame to canonical" matrix has frame in columns**
  - takes points represented in frame
  - represents them in canonical basis
  - e.g. [0 0], [1 0], [0 1]
- **Seems backward but bears thinking about**

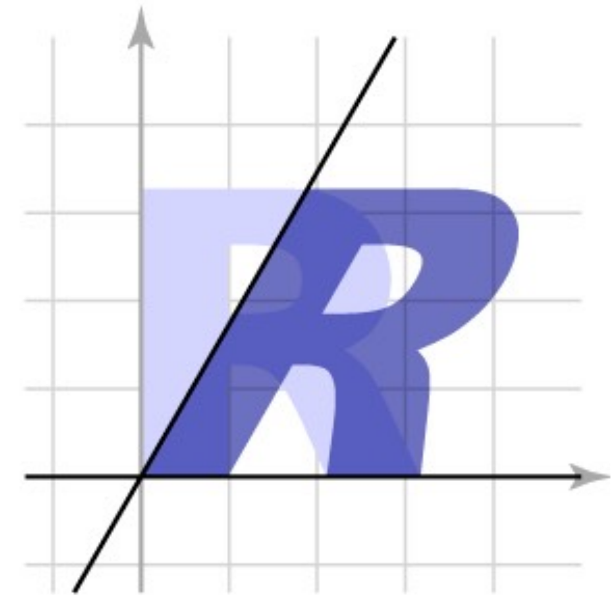


$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

Affine change of coordinates

- **A new way to "read off" the matrix**
  - e.g. shear from earlier
  - can look at picture, see effect on basis vectors, write down matrix

$$\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- **Also an easy way to construct transforms**
  - e. g. scale by 2 across direction (1,2)

Affine change of coordinates

- **When we move an object to the canonical frame to apply a transformation, we are changing coordinates**
  - the transformation is easy to express in object's frame
  - so define it there and transform it

$$T_e = F T_F F^{-1}$$

  - $T_{\mathbf{e}}$ is the transformation expressed wrt. $\{\mathbf{e}_1, \mathbf{e}_2\}$
  - T_F is the transformation expressed in natural frame
  - F is the frame-to-canonical matrix $[u\ v\ p]$
- **This is a *similarity transformation***

Building general rotations

- **Using elementary transforms you need three**
  - translate axis to pass through origin
  - rotate about y to get into x-y plane
  - rotate about z to align with x axis

- **Alternative: construct frame and change coordinates**
  - choose p, u, v, w to be orthonormal frame with p and u matching the rotation axis
  - apply similarity transform $T = F\, R_x(\theta)\, F^{-1}$

Coordinate frame summary

- **Frame = point plus basis**
- **Frame matrix (frame-to-canonical) is**

$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

- **Move points to and from frame by multiplying with *F***

$$p_e = F p_F \quad p_F = F^{-1} p_e$$

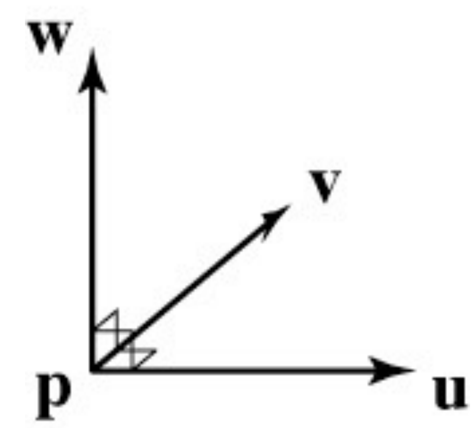
- **Move transformations using similarity transforms**

$$T_e = F T_F F^{-1} \quad T_F = F^{-1} T_e F$$

Orthonormal frames in 3D

- **Useful tools for constructing transformations**
- **Recall rigid motions**
  - affine transforms with pure rotation
  - columns (and rows) form right handed ONB
    - that is, an **o**rtho**n**ormal **b**asis

$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Building 3D frames

- **Given a vector a and a secondary vector b**
  - The **u** axis should be parallel to **a**; the **u**–**v** plane should contain **b**
    - $\mathbf{u} = \mathbf{u} \,/\, ||\mathbf{u}||$
    - $\mathbf{w} = \mathbf{u} \times \mathbf{b}$; $\mathbf{w} = \mathbf{w} \,/\, ||w||$
    - $\mathbf{v} = \mathbf{w} \times \mathbf{u}$
- **Given just a vector a**
  - The **u** axis should be parallel to **a**; don't care about orientation about that axis
    - Same process but choose arbitrary **b** first
    - Good choice is not near **a**: e.g. set smallest entry to 1

Building transforms from points

- **2D affine transformation has 6 degrees of freedom (DOFs)**
  - this is the number of "knobs" we have to set to define one
- **So, 6 constraints suffice to define the transformation**
  - handy kind of constraint: point $\mathbf{p}$ maps to point $\mathbf{q}$ (2 constraints at once)
  - three point constraints add up to constrain all 6 DOFs (i.e. can map any triangle to any other triangle)
- **3D affine transformation has 12 degrees of freedom**
  - count them from the matrix entries we're allowed to change
- **So, 12 constraints suffice to define the transformation**
  - in 3D, this is 4 point constraints (i.e. can map any tetrahedron to any other tetrahedron)

