Monte Carlo Illumination

CS 4620 Lecture 22

Monte Carlo Integration

- **Monte Carlo idea: design a random experiment whose average outcome is the answer we want**
- **Integration:**

$$I = \int_a^b f(x)dx$$

- **want to define an "estimator"** *g*(*x*) **such that**

$$E\{g(x)\} = I$$ for random values of *x*

  - **that is, the expected value of g is the answer we seek when *x* is chosen randomly.**

Uniform sampling

- **If *x* is chosen uniformly at random from [*a*, *b*]:**

$$E\{f(x)\} = \frac{1}{b-a}\int_a^b f(x)dx$$

- so, to get the desired answer, set

$$g(x) = (b-a)f(x)$$

- then

$$E\{g(x)\} = \int_a^b f(x)dx = I \qquad \text{for } x \text{ uniform in } [a, b]$$

Aside: probability density functions

- **Probability distribution: familiar notion in the discrete case**
  - a distribution divides up one unit of probability among the elements of a *probability space*.
  - e.g. roll two dice; probability space is $\Omega = \{1, \ldots, 6\}^2$
  - each possible roll is equally likely: $p((i,j)) = \frac{1}{36}$
  - probability distribution p has to be normalized: $\sum_{x \in \Omega} p(x) = 1$
  - a random variable is a function on Ω
  - e.g. sum of the two dice: $S((i,j)) = i + j$
  - values of S are distributed over $\{2, \ldots, 12\}$

  $$S \sim p_S \quad \text{where} \quad p_S(n) = \Pr\{S(x) = n\}$$

Aside: probability density functions

- **Probability distribution can also be over a continuous set**
  - e.g. spin a spinner from 0 to 6; probability space is $\Omega = [0, 6)$
  - each possible spin is equally likely: $p(x_0) = \frac{1}{6} = \frac{\Pr\{x_0 < x < x_0 + dx\}}{dx}$
  - probability density p has to be normalized: $\int_\Omega p(x) dx = 1$
  - a random variable is a function on Ω
  - e.g. sum of two spins: $S : \Omega^2 \to \mathbb{R} : S(x, y) = x + y$
  - values of S are distributed over $[0, 12)$

$$S \sim p_S \quad \text{where} \quad p_S(z)\, dz = \Pr\{z < S(x, y) < z + dz\}$$

$$p(0) = 0; \quad p(1) = \frac{1}{36}; \quad p(6) = \frac{1}{6}; \quad p(12) = 0$$

Expectation

- **Discrete case**

$$\text{when } x \sim p(x),\ E\{f(x)\} = \sum_{x \in \Omega} f(x) p(x)$$

- **Continuous case**

$$\text{when } x \sim p(x),\ E\{f(x)\} = \int_{\Omega} f(x) p(x)\, dx$$

Uniform sampling revisited

- **Choosing points uniformly from [a, b] is sampling from a pdf that has density 1 / (b – a).**
  - if we use an estimator g with uniformly sampled x:

$$E\{g(x)\} = \int_a^b g(x)p(x)\,dx = \frac{1}{b-a}\int_a^b g(x)dx$$

  - so if f is the desired integrand, the correct estimator is

$$g(x) = (b-a)f(x)$$

Nonuniform sampling

- **Choosing points instead from some other distribution over the interval [a, b] also works just as well**
  - if we use an estimator g with $x \sim p(x)$

$$E\{g(x)\} = \int_a^b g(x)p(x)\,dx$$

  - so if f is the desired integrand, the correct estimator is

$$g(x) = \frac{f(x)}{p(x)}$$

$$E\{g(x)\} = \int_a^b \frac{f(x)}{p(x)}p(x)\,dx = \int_a^b f(x)dx$$

as long as p(x) is not zero!

