$$H_r = -\frac{\hbar^2}{2\mu}\frac{P_r^2}{\hbar^2} + \frac{\hbar^2 L^2}{2\mu r^2} - \frac{Ze^2}{4\pi\varepsilon_0 r}$$

$$[H_r, L^2] = 0$$

→ complete set of mutual eigenstates of H_r and L^2

looking for $|E, \ell\rangle$

$$L^2 |E,\ell\rangle = \ell(\ell+1)|E,\ell\rangle$$

$$H_r |E,\ell\rangle = E|E,\ell\rangle$$

$$H_\ell = \frac{P_r^2}{2\mu} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} - \frac{Ze^2}{4\pi\varepsilon_0 r}$$

ladder operator:

$$A_\ell = \frac{a_0}{\sqrt{2}}\left(\frac{iP_r}{\hbar} - \frac{\ell+1}{r} + \frac{Z}{(\ell+1)a_0}\right)$$

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{\mu e^2} \quad \text{(Bohr radius)}$$

$$\frac{e^2}{4\pi\varepsilon_0 a_0} = \frac{\hbar^2}{\mu a_0^2}$$

$$A_\ell^+ A_\ell = \frac{a_0^2}{2}\left(\frac{-i p_r}{\hbar} - \frac{\ell+1}{r} + \frac{z}{(\ell+1)a_0}\right)\left(\frac{i p_r}{\hbar} - \frac{\ell+1}{r}\right.$$

$$\left. + \frac{z}{(\ell+1)a_0}\right)$$

$$= \frac{a_0^2}{2}\left(\frac{p_r^2}{\hbar^2} + \left(\frac{\ell+1}{r}\right)^2 + \left(\frac{z}{(\ell+1)a_0}\right)^2 - \frac{2z}{a_0 r}\right.$$

$$\left. - \frac{i(\ell+1)}{\hbar}\left[p_r, \frac{1}{r}\right]\right)$$

$$= \frac{a_0^2}{2}\left(\frac{p_r^2}{\hbar^2} + \frac{\ell(\ell+1)}{r^2} - \frac{2z}{a_0 r}\right) + \frac{z^2}{2(\ell+1)^2}$$

$$= \frac{a_0^2 \mu}{\hbar} H_\ell + \frac{z^2}{2(\ell+1)^2}$$

$$H_\ell = \frac{\hbar^2}{a_0^2 \mu} \left(A_\ell^+ A_\ell - \frac{z^2}{2(\ell+1)^2} \right)$$

$$[A_\ell , A_\ell^+] = \frac{a_0 (\ell+1)}{r^2}$$

we're given : $H_\ell | E, \ell \rangle = E | E, \ell \rangle$

$$E A_\ell | E, \ell \rangle = A_\ell H_\ell | E, \ell \rangle$$

$$= (H_\ell A_\ell + \underbrace{[A_\ell , H_\ell]}_{(H_{\ell+1} - H_\ell) A_\ell}) | E, \ell \rangle$$

$$= H_{\ell+1} (A_\ell | E, \ell \rangle)$$

$$= H_{\ell+1} (\alpha | E, \ell+1 \rangle)$$

by applying A_ℓ , we shift energy from radial to angular component

after applying A_ℓ a couple of times, no radial component left

$\rightarrow$ must exist $\underline{\text{max. } \ell(E)} = \mathcal{L}$

$$|A_{\mathcal{L}} |E, \mathcal{L}\rangle|^2 = 0$$

$$\langle E, \mathcal{L} | A_{\mathcal{L}}^{+} A_{\mathcal{L}} | E, \mathcal{L}\rangle = 0$$

$$\Rightarrow \frac{a_0^2 \mu}{\hbar^2} E = -\frac{z^2}{2(\mathcal{L}+1)^2}$$

$n \equiv \mathcal{L}+1$ is integer (principal quantum number)

$$E = \frac{z^2 \hbar^2}{2 a_0^2 \mu} \frac{1}{n^2} = -\frac{z^2 R}{n^2}$$

with $R = \dfrac{\hbar^2}{2 a_0^2 \mu} = \dfrac{e^2}{8\pi \varepsilon_0 a_0}$ (Rydberg constant)

$$= 13.6 \text{ eV}$$

Spectroscopic lines

for $n \to n'$

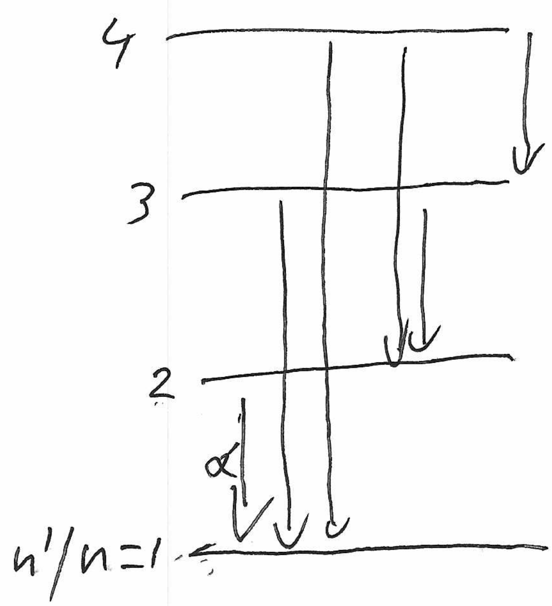
$$\nu = \frac{\Delta E}{h} = \frac{Z^2 R}{h}\left(\frac{1}{n'^2} - \frac{1}{n^2}\right)$$

Series of spectral lines for fixed n'

Lyman series $n = 2,3,4 \to n' = 1$

Balmer series $n = 3,4,5 \to n' = 2$

Paschen series $n = 4,5 \to n' = 3$



$Ly(\alpha) = 121\,nm$

H_α (Balmer α) $= 656.2\,nm$

Paschen α $1875\,nm$