Phy 481 - HW 2

1. Line (or path) integral

The work done on a path is simply an integral of the dot with the path. This is one of the easier ones.

$$\vec{F} = y^3\hat{x} - 2x^2\hat{y}$$
$$dl = dx\hat{x} + dy\hat{y}$$
$$W = \int_l \vec{F} \cdot d\vec{l}$$
$$= \int_l y^3 dx - 2x^2 dy$$

To make the path itself convert to one variable

$$y = x^2 + 1$$
$$dy = 2xdx$$
$$W = \int_0^2 (x^2+1)^3 dx - 2x^2(2xdx) = \int_0^2 \left[(x^2+1)^3 - 4x^3\right] dx$$

Wolfram reports that this is

$$\left[\frac{1}{7}x^7 + \frac{3}{5}x^5 - x^4 + x^3 + x\right]\Bigg|_0^2 = \frac{1102}{35}$$

$\vec{F}$ is path-independent if $\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & -2x^2 & 0 \end{vmatrix} = \hat{z}\left(-4x - 3y^2\right)$$

So $\vec{F}$ is not path independent

2. Surface Integrals

We only look at the components of $\vec{v}$ parallel to the surface, and integrate twice since it is area.

$$\vec{v} = 3zx\hat{x} + 5x\hat{y} + 2y\hat{z}$$
$$d\vec{A} = dxdz\hat{y}$$
$$\vec{v} \cdot d\vec{A} = 5xdxdz$$
$$\int_S \vec{v} \cdot d\vec{A} = \int_0^3 \int_0^2 5xdxdz$$
$$= \int_0^3 10dz = 30$$

The result is positive since the two important vectors are $\hat{y}$ (from the surface integral) and $5x\hat{y}$ (from the flux vector). Since x is positive on the surface, this yields two positive vectors being dotted together, and a positive result

3. Volume integrals

Here we will simply calculate the volume integral. These are radially/azimuthally invariant, so we can actually just do a simple integral

$$dV = \rho^2 \sin(\theta) d\rho d\theta d\phi$$
$$\int_0^\pi \int_0^{2\pi} \sin(\phi) d\theta d\phi = 4\pi$$
$$\int_V p_0 dv = 4\pi \int_0^R \rho_0 \rho^2 d\rho = \frac{4}{3}\pi R^3 \rho_0$$

Which makes me realize I could have used the volume formula for this part

$$\int_V \frac{4\rho_0}{5R} \rho dv = 4\pi \frac{4\rho_0}{5R} \int_0^R \rho^3 d\rho = \frac{4\pi}{5} R^3 \rho_0$$

$4/3 > 4/5$ so the first sphere is heavier than the second

4. Some vector proofs

Part 1: show $\nabla \times \nabla T = 0$

Gradient theorem:

$$\oint_C \nabla T \cdot d\vec{l} = 0$$

Stokes Theorem states the integral of the curl of a vector across a surface is equal to the integral of a path around the surface with the vector. Since ∇T the gradient of a scalar function is a vector, stokes states that

$$\int_S \vec{\nabla} \times \vec{\nabla T} \cdot d\vec{a} = \oint_C \nabla T \cdot d\vec{l} = 0$$

Part 2: show $\int \nabla \cdot \left(\vec{\nabla} \times \vec{v}\right) dV = 0$

Same as before, we use the divergence theorem to convert the desired expression into one that equals zero.

Stokes: $\oint_S \vec{\nabla} \times \vec{v} \cdot d\vec{a} = 0$

Since the curl of a vector is a vector itself, we can use stokes theorem to convert from a volume intergral of a divergence to a closed area integral of a vector

$$\int \nabla \cdot \left(\vec{\nabla} \times \vec{v}\right) dV = \oint_S \vec{\nabla} \times \vec{v} \cdot d\vec{a} = 0$$

5. Test stokes' theorem

Do the integrals!

$$\vec{v} = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$$
$$\oint d\vec{l} = \int_{y=0}^{y=2} dy\hat{y} + \int_{y=2}^{z=2} [dy\hat{y} + dz\hat{z}] + \int_{z=2}^{z=0} dz\hat{z}$$

First let's see all the dot products

$$\vec{v} \cdot dy\hat{y} = 2yzdy$$
$$\vec{v} \cdot [dy\hat{y} + dz\hat{z}] = 2yzdy + 3zxdz$$
$$\vec{v} \cdot dz\hat{z} = 3zxdz$$

For the first curve, z=0 so the integral is zero. For the second and third curve, x = 0, so the third integral is zero and the second reduces to $2yzdy$ Using $z = 2 - y$, this gives us

$$\int_2^0 2y(2-y)dy = 2y^2 - \frac{2}{3}y^3\Big|_2^0 = -\frac{8}{3}$$

Lets verify against the curl-area integral

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = -2y\hat{x} - 3z\hat{y} - x\hat{z}$$

For the integral, we'll use the same bounds as before with $y = 2 - z$ and $y = 0$ as our first integral. Using $d\vec{a} = dydz\hat{x}$ we can use the $\hat{x}$ portion of the curl.

$$\int_0^2 \int_0^{2-z} -2ydydz = -\int_0^2 (2-z)^2 dz = -\frac{8}{3}$$

As such our test of stokes is succesful

6. Python: An odd charge distribution

See: Koren, Andrew - HW02.ipynb