Pendulum Motion & Approximations

Experiment Four

Physics 191
Michigan State University

Before Lab

- Read the theory section and **attempt all theory questions** (due at the beginning of lab)
- Review error analysis and statistics concepts in the Experiment One and Experiment Two lab guides

Experiment Overview

For this **two-week experiment**, you are expected to produce a professionally formatted lab report detailing all aspects of your research, experiments, and analyses.

- Study the periodic behavior of the simple pendulum
- Practice using mathematical approximations to solve complicated problems
- Develop a theoretical model of a simple pendulum
- Apply statistical methods developed in Experiment Two to improve measurement accuracy
- Test the validity of assumptions and approximations used in the theoretical model

Theory

The simple pendulum has played a significant role in the development of order within societies. The first documented scientific investigation of pendulum motion was carried out by Galileo Galilei around the year 1600[1]. The periodic behavior of the simple pendulum pioneered the first high-accuracy timekeeping instruments, a technology that would not be surpassed for nearly 300 years.

A simple pendulum is shown in Figure 1. A black mass (called a bob) is suspended by a string of length ℓ. The position of the bob can be described by an angle θ measured from the *equilibrium position.* Equilibrium refers to the resting position of the bob, which is shown in gray. The purpose of the subsequent analysis is to quantitatively describe the position of the bob at any moment in time. This turns out to be a surprisingly complicated problem, but many complications can be circumvented by applying a set of approximations:

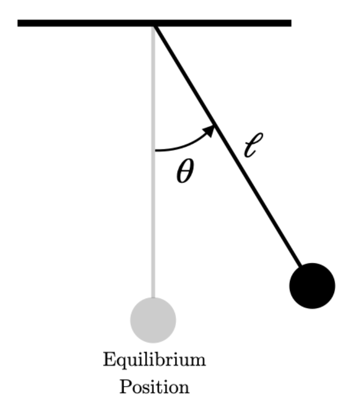


Figure 1: A simple pendulum of length ℓ at an angle θ with respect to equilibrium.

I. Approximate the string as massless
II. Approximate the pivot as frictionless and drag is negligible
III. Assume the string does not stretch
IV. Approximate the mass as a concentrated point at the end of the string
V. Restrict the motion of the pendulum to two dimensions

[1] You can read more about the history of the pendulum on Wikipedia

The purpose of making the aforementioned assumptions is to reduce the motion to its simplest form. This practice of decomposing a problem to expose only the fundamental elements is not limited to physics, but rather it applies to a host of situations. The logic is to start with a simplified theory and, once it is fully understood, reintroduce nuances to develop a more comprehensive model. In the case of the pendulum, the simplifying assumptions reduce the number and character of forces involved in the problem. There are only two remaining forces acting on the bob: gravity and tension. The tension in the string restricts the motion of the bob to an arc *with a fixed radius*.

The Period of Oscillation

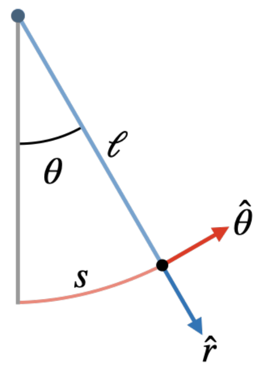


Figure 2

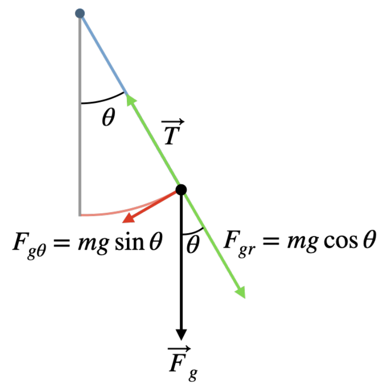


Figure 3

In order to describe how the position of the bob evolves in time, we must first decide on how to define position by choosing a coordinate system. Consider two choices:

Rectangular Coordinates: this is the familiar xy-plane in which the location of a point is described by coordinates (x, y). The unit vectors $\hat{x}$ and $\hat{y}$ are perpendicular and define the directions of the x and y axes.

Polar Coordinates: this is a system where a point is described by coordinates (r, θ) where r is the distance from the origin and θ is an angle that references a direction. The unit vectors $\hat{r}$ and $\hat{\theta}$ are perpendicular and are oriented as shown in Figure 2.

- The vector $\hat{r}$ defines the *radial* component and points directly along the radius
- The vector $\hat{\theta}$ defines the *angular* or *tangential* component and always points along the circular path

You may not have realized that we already formulated this problem in polar coordinates. Figure 2 shows that the exact location of the mass can be entirely described by the coordinates (ℓ, θ). This elucidates the utility of polar coordinates for this problem. Describing the motion in rectangular coordinates requires *two* kinematic equations: $(x(t), y(t))$. In polar coordinates, the fixed string length allows for a complete description with only a *single* equation $\theta(t)$, which gives the angle at any point in time. The choice of polar coordinates reduces the problem to one dimension.

Figure 2 also shows the semicircular path labeled s traced out by the pendulum. This path is called the **arc length** and is related to the radius and angle by

$$s = \ell\theta \tag{1}$$

where the angle θ is necessarily in radians[2]. The dynamics of the pendulum can be determined by splitting the gravitational force vector into two components as shown in Figure 3. The tension T and the radial

[2]The radian is a dimensionless unit (a pure number) with 1 rad = 1. For this reason, it is only necessary to write "rad" when expressing an angle in radians if there is a possibility of ambiguity.

component of the of the gravitational force F_{gr} (shown in green) do not effect the motion. It is the angular part of the gravitational force $F_{g\theta}$ (red vector) that drives the motion because it always acts along the path of the bob. Newton's second law gives a differential relationship between the angle of the pendulum and time. The angular component of the force is

$$F_\theta = ma_\theta = -mg\sin\theta \tag{2}$$

where a_θ is the acceleration along the arc. The negative sign on the right side indicates that gravity always works to restore the pendulum to its equilibrium position. The mass in Equation 2 can be cancelled to determine the acceleration of the bob

$$a = -g\sin\theta. \tag{3}$$

	Linear	Rotational
Position	x	$s = \ell\theta$
Velocity	$v = \frac{dx}{dt}$	$v = \frac{ds}{dt} = \ell\frac{d\theta}{dt}$
Acceleration	$a = \frac{d^2x}{dt^2}$	$a_\theta = \frac{d^2s}{dt^2} = \ell\frac{d^2\theta}{dt^2}$

The final step is to find the relationship between the acceleration and the angle θ. The table above shows the correspondence between linear and rotational motion with a fixed radius. The rotational velocity and acceleration are determined by taking time derivatives of the arc length. The bottom right cell contains the expression for the acceleration in the equation above. By substituting the acceleration a_θ into Equation 3,

$$a_\theta = \ell\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta. \tag{4}$$

which gives the relationship between the angle θ and time. This is a nonlinear differential equation. It is very difficult to solve, so *we are forced to make an additional approximation!* The problem becomes more tractable under the **small-angle approximation**,

$$\sin\theta \approx \theta. \tag{5}$$

This type of approximation is extremely common in chemistry, physics, engineering, and astronomy. It means that the value of the sine function at small angles is approximately equal to the angle, itself. Figure 4 shows the consistency of the small-angle approximation. For small values of θ (left side of the plot), the two lines effectively overlap. Implementation of this approximation reduces Equation 4 to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta \tag{6}$$

which happens to describe **simple harmonic motion**. The broad scope of this equation extends from molecular vibrations to construction of skyscrapers and is identical to Hooke's law in form. The solution is the angle as a function of time

$$\theta(t) = \theta_0\cos\omega_0 t \tag{7}$$

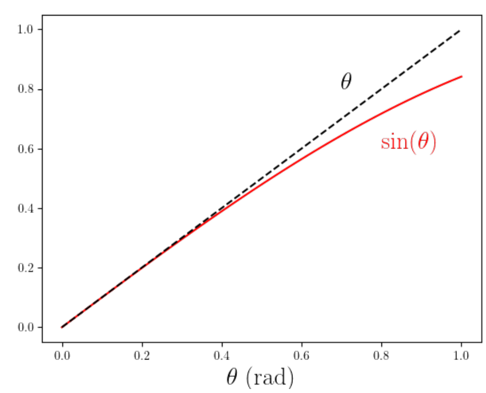


Figure 4: Plots of $\sin\theta$ and θ effectively overlap for small angles. Note that θ is a line of slope 1.

where θ_0 is the starting angle and ω_0 is the angular frequency[3]. After all of this work, we can use the relationship between angular frequency and the period of a single oscillation

$$\omega_0 = \frac{2\pi}{T_0} \tag{8}$$

[3]The symbol ω is a lowercase Greek omega.

to arrive at an expression for the **small-angle period** T_0, which is the amount of time required for the pendulum to complete one full oscillation

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}}. \tag{9}$$

It is important to note that this equation was derived under *many* approximations. It is only an accurate model under the specific conditions presented above. For this reason, we will denote the small-angle period with a subscript "0" and the measured period as T with no subscript.

Energy Conservation

A simple pendulum experiment is a good test of the conservation of energy, one of the most useful tools in physics. Energy conservation simply means that the total energy in a closed system is a constant–it does not change with time. A **closed system** is one in which no external forces, such as friction or drag, are present. Such forces are called *dissipative* as they dissipate the total energy. Most forms of energy can be categorized as one of two types:

Kinetic Energy: the energy associated with physical motion denoted by the symbol K

Potential Energy: the energy stored in a system as a result of relative position, electric charge, internal stresses, and various other sources and is denoted by the symbol U

The total energy E of a system is the sum of the two types,

$$E = K + U. \tag{10}$$

The values of K and U are free to change with time, so long as the total value of E remains constant. For the simple pendulum, the kinetic energy at any moment is

$$K = \frac{1}{2}mv^2 \tag{11}$$

where m is the mass and v is the velocity of the bob. The potential energy stored by a pendulum is purely a consequence of the gravitational force and is calculated by

$$U = mgy \tag{12}$$

where y is the vertical displacement from the equilibrium position. The previous three equations can be combined to get an expression for the total energy at any moment in time,

$$E = \frac{1}{2}mv^2 + mgy. \tag{13}$$

Equation Summary Table

Quantity	Equation
Arc Length	$s = \ell\theta$
Small-Angle Approximation	$\sin\theta \approx \theta$
Small-Angle Period	$T_0 = 2\pi\sqrt{\frac{\ell}{g}}$
Total Energy	$E = K + U$
Kinetic Energy	$K = \frac{1}{2}mv^2$
Potential Energy	$U = mgy$

Theory Questions

Note: no measured numbers are needed to solve any of the theory questions.

1. (1 pt) What is the arc length of a complete circle in terms of the diameter d? Is there a special word for this length?

2. (1 pt) What are the units of arc length?

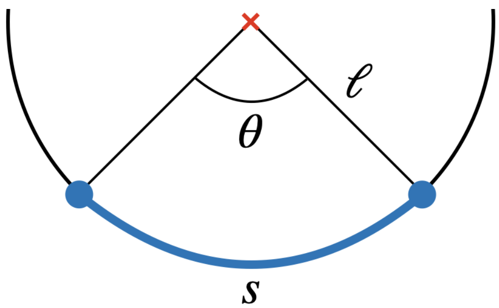
3. (1 pt) Explain why polar coordinates are the best choice for describing the motion of a pendulum. Think about the requirement that length ℓ is fixed.

4. (1 pt) Would a pendulum oscillate on the International Space Station? Explain your answer.

5. (1 pt) If a pendulum is set into motion on an elevator moving upward at a constant acceleration, would the period change? Explain your answer.

6. (1 pt) Your friend has an old grandfather clock that measures time using a simple pendulum. They complain about being late to class because the clock runs too slow. You notice that the pendulum has an adjustable length and a replacement bob with a larger mass. What can you do to correct the period of your friend's clock?

7. (2 pt) The picture below shows a bead that travels along a circular arc labeled s.



Imagine making physical measurements $\ell \pm \delta\ell$ and $\theta \pm \delta\theta$, which you then use to *calculate* the arc length, $s(\ell, \theta) = \ell\theta$. Your goal is to determine how the measurement uncertainties $\delta\ell$ and $\delta\theta$ contribute to the uncertainty in arc length, δs. Start with the general form for uncertainty in a two-variable function $f(x, y)$,

$$\delta f = \sqrt{\left(\frac{\partial f}{\partial x} \cdot \delta x\right)^2 + \left(\frac{\partial f}{\partial y} \cdot \delta y\right)^2},$$

and show that the uncertainty in arc length is $\delta s = \sqrt{(\theta \cdot \delta\ell)^2 + (\ell \cdot \delta\theta)^2}$.

8. (2 pt) Sketch a graph of $y = Ax^B$ with $A = 1$ for the following values of B:

   (a) $B = -1$

   (b) $B = 0$

   (c) $B = 0.5$

   (d) $B = 2$

1 Small-Angle Period Analysis

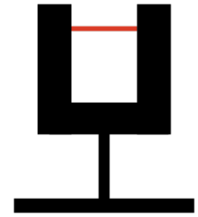


Figure 5

In this part of the lab, you will test the theoretical model of the period under the small-angle approximation. At your bench, you have a pendulum attached to a pivot point with a protractor for measuring angles. All of the parts of your setup have knobs that allow for adjustments and alignment. You can loosen a knob to move something, and then tighten the knob to lock the position. You also have a device called a *photogate timer* that allows for very precise time measurements. It operates like the safety mechanism on a garage door, where a focused invisible light is pointed at a sensor. At the instant the beam is blocked, the device will begin recording time. The timer has several modes for different types of operation. A side view of the sensor is shown to the right with the infrared light shown in red.

- Familiarize yourself with the experimental setup. Move the pivot point up and down to adjust the string length. **Start with a relatively large length** and lock the pivot in place, then hang a weight from the string.
- Align the protractor with the string as precisely as possible. Once you are satisfied with the alignment, lock the protractor and leave it in the same position throughout the experiment.
- Place the photogate timer at the equilibrium position of the pendulum. Pay attention to the alignment of the components.
- Set the timer to ”PEND” mode. This will measure the period of one complete oscillation.
- Practice swinging the pendulum through the photogate timer at various angles. Start by moving it through the gate slowly by hand, try to understand exactly how the device works.

1.1 Determine Measurement Uncertainties

This lab consists of four different experiments examining different cause-and-effect relationships related to the period of a pendulum. You will find that there are many sources of random error in the measurement process. Recall that **random errors** are equally likely to affect the accuracy of a measurement in either direction. As such, random effects are minimized by averaging over repeated measurements. In all of the following experiments, you should repeat each period measurement three times. As you learned in the statistics lab, the uncertainty in the mean is called the **standard error** (or standard deviation of the mean). It will help to organize your data as shown below, where ”X” represents the independent variable in question. The spreadsheet functions for average and standard error are shown in the corresponding columns.

X	T_1 (s)	T_2 (s)	T_3 (s)	$\bar{T}$ (s)	$\delta\bar{T}$ (s)
				=AVERAGE(...)	=STDEV.S(...)/SQRT(3)

Answer the follow questions to prepare for your measurements:

1. (1 pt) Hold the pendulum at some angle and move your head from side to side to see how perspective skews the observed angle. Keep this in mind and determine you uncertainty $\delta\theta$ based on your ability to read the protractor. Record this number including units.
2. (1 pt) Try measuring the length of the string. Consider the same effect of perspective and determine your uncertainty in measuring the string length $\delta\ell$. Record this number including units.
3. (1 pt) The photogate timer begins counting the instant the beam is blocked. Is it better to use a weight with a small or large diameter? Think about approximation IV on the first page of the theory section.

1.2 Small-Angle Criterion

Before making measurements, you should determine what is considered a ”small” angle by examining the relative error in the small-angle approximation,

$$e_\theta = \left| \frac{\sin\theta - \theta}{\sin\theta} \right| \tag{14}$$

where θ is necessarily in radians. Figure 6 shows how the relative error grows with the angle θ. To obtain accurate results, you want a starting angle large enough to maximize the speed at the bottom of the swing, but small enough to maintain the small-angle approximation. A good starting point is to aim for a relative error less than 0.5%.

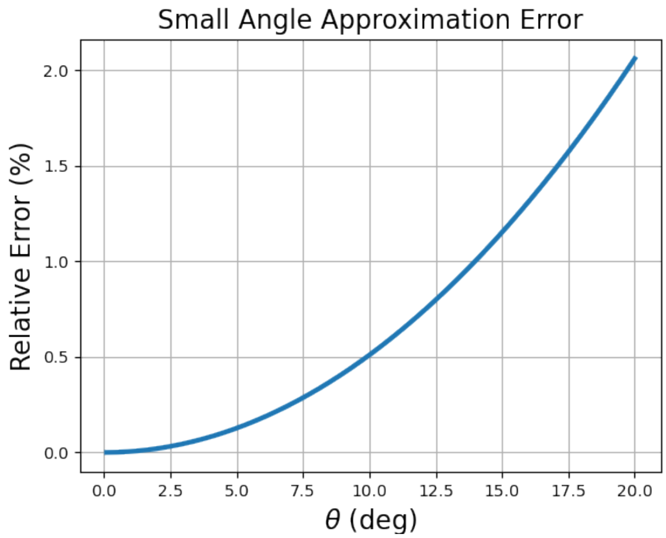


Figure 6

1. (1 pt) Use the error plot in Figure 6 to determine a starting angle with an acceptable relative error. You will use this angle for the mass and length dependance experiments. Explain how you determined this value.

1.3 Mass Dependance

Your goal in this section is to determine the relationship between the period and the mass. Your results may indicate a source of **systematic error**. You have bobs made out of four different materials: plastic, wood, aluminum, and brass. By measuring the period for each material, you can determine if the mass of the bob is an important factor.

1. (1 pt) Write a brief introduction to this part of the lab. Describe the cause-and-effect relationship you are testing: what are the dependent and independent variables?
2. State the values of $\theta \pm \delta\theta$ and $\ell \pm \delta\ell$ you plan to use for this part of the experiment. Explain why you should use the same angle and string length for all measurements in this section.
3. (1 pt) Look at the equation for the small-angle period, $T_0 = 2\pi\sqrt{\frac{\ell}{g}}$. According to the theory, how does the period change with mass?
4. Perform your measurements using the averaging procedure described in Section 1.1.
5. (1 pt) Make a scatter plot of the dependent vs. independent variables. This can be done in your spreadsheet, curve fitting is not necessary.
6. (1 pt) Does your plot indicate a relationship between between the period and mass? Does this agree with your answer to Question 3?
7. (1 pt) Why might the period be different for different weights? Think back to the simplifying assumptions made in the derivation of the model.

8. (1 pt) Quantify this systematic effect by calculating the relative error using the minimum and maximum mean periods.

9. Choose a bob to use for the remaining experiments. To help with your decision, consider which bob would swing the longest before stopping.

1.4 Length Dependance

Your goal for this experiment is to examine the relationship between the period and the length of the string. You will make a series of measurements and compare your data with the theoretical model by testing three methods of analysis.

1. (1 pt) Write a brief introduction to this part of the lab. Describe the cause-and-effect relationship you are testing: what are the dependent and independent variables?

2. Determine your starting length and state as $\ell \pm \delta\ell$, and state the value of $\theta \pm \delta\theta$ you intend to use for this experiment.

3. Measure the period using the averaging procedure for lengths ℓ, $\ell/2$, $\ell/3$, $\ell/4$, $\ell/5$, and $\ell/6$ (measure the actual lengths). Be sure your initial value of ℓ is sufficiently large, otherwise a measurement of $\ell/6$ will not be possible.

4. **Power Law Method**: according to the theory, the small-angle period is related to the length by $\bar{T} = 2\pi\,(\ell/g)^{1/2}$. Define $y = \bar{T}$ and $x = \ell$ to express this equation in the form of the "power law" fit equation, $y = Ax^B$.

   (a) (1 pt) How is the parameter A related to the gravitational acceleration, g? What do you expect for the parameter B?

   (b) (1 pt) Fit your data to the "power law" fit type using curve.fit. Include horizontal and vertical error bars.

   (c) (1 pt) Put the model to the test by comparing your value of B to the theoretical value. Calculate the relative error in B and the statistical t-test (see Appendix for equations). Are your results in good agreement with the theoretical model?

   (d) (1 pt) Calculate $g \pm \delta g$ from your fit parameter A. Use uncertainty propagation to compute the uncertainty:

   $$\delta g = \left| \frac{dg}{dA} \cdot \delta A \right|.$$

   Your results should show that the relationship between $\bar{T}$ and ℓ is nonlinear (i.e. $B \neq 1$). In the next two problems, you will plot modified representations of your data to display a linear trend. This is an alternative form of analysis called **linearization**. The shape of the plot is changed, but the fundamental relationship is preserved. This method is useful because linear trends are easier to visualize than more complicated relationships.

5. (1 pt) Show that $\bar{T} = 2\pi\sqrt{\ell/g}$ can be expressed as

   $$\bar{T} = \frac{2\pi}{\sqrt{g}}\ell^{1/2} \tag{15}$$

   Recall that the equation for a line is $y = mx + b$. The trick to linearization is defining y and x such that substitution into the theoretical equation results in a linear relationship. In the next questions, you will plot two different representations of your data that produce a linear trend.

6. **Linearization 1**:

   (a) (1 pt) Define $y = \bar{T}$ and $x = \ell^{1/2}$ and substitute into the small-angle period in Equation 15. You should arrive at an equation with the form $y = mx + b$ where m is the slope. Write the slope as a function of g.

(b) (1 pt) Plot $y = \bar{T}$ vs. $x = \ell^{1/2}$ in curve.fit. Choose the appropriate fit type. Include axes labels with units and a good title. Error bars are not necessary. *Important*: pay close attention to the units. The x-axis units are modified as a result of plotting $x = l^{1/2}$.

(c) (1 pt) Calculate $g \pm \delta g$ using the slope fit parameter, m. Use uncertainty propagation to compute the uncertainty:

$$\delta g = \left| \frac{dg}{dm} \cdot \delta m \right| .$$

7. **Linearization 2**:

(a) (1 pt) Show that squaring both sides of equation 15 and defining $y = \bar{T}^2$ and $x = \ell$ results in a linear relationship. Write the slope as a function of g.

(b) (1 pt) Plot y vs. x for this method in curve.fit. Choose the appropriate fit type. Include axes labels with units and a good title. Error bars are not necessary.

(c) (1 pt) Calculate $g \pm \delta g$ using the *slope* fit parameter, m. Use uncertainty propagation to compute the uncertainty:

$$\delta g = \left| \frac{dg}{dm} \cdot \delta m \right| .$$

8. (1 pt) Compare the accuracy of each method by calculating the relative error in g using the standard value $g_0 = 980.7$ cm/s^2. Which method produced the most accurate result?

2 Angular Dependance

Up to this point, all measurements have been made with a small angles to account for the assumptions made when developing the theoretical model. In this section of the experiment, you will examine a corrected model of the period which can better describe the period for larger angles[4]. The small-angle period has no angular dependance, which is evident by the lack of a θ in the equation.

If the starting angle of the pendulum is large, the bob must travel a longer distance so, intuitively, it should take more time to complete a full swing. The true dependance of T on θ comes in the form of an infinite polynomial. The small-angle period T_0 is the first term in the polynomial. For our correction, we will also include the second term which *does* depend on θ. The modified equation is

$$T^* = T_0 + \frac{1}{16} T_0 \cdot \theta^2$$

where we are calling the corrected model T^*. Note that the small-angle period $T_0 = 2\pi\sqrt{\frac{\ell}{g}}$ is in both terms, so it is natural to factorize the equation as

$$\bar{T}^* = T_0 \left(1 + \frac{1}{16} \cdot \theta^2 \right) \tag{16}$$

where $\bar{T}^*$ denotes your three averaged measurements for each angle. The objective for this section is to test the modified model by examining the period at larger angles.

1. (1 pt) Write a brief introduction to this part of the lab. Describe the cause-and-effect relationship you are testing: what are the dependent and independent variables?

2. (1 pt) Determine the best string length for this experiment and record the value. To determine what is best, think about minimizing the fractional uncertainty $\delta\ell/\ell$.

3. Format a table to organize your measurements, and follow the same averaging procedure as in previous sections. Start with a value of $\theta = 5°$ and work your way up to $\theta = 35°$ in $5°$ increments. Be sure to convert the angles to radians before any calculations or plotting.

[4] More detail can be found on the pendulum mechanics Wikipedia page.

The best way to examine the relationship is to plot the data and fit a curve representing T^*. The analysis is simplified by dividing both side of Equation 16 by T_0 so the units cancel and both sides are dimensionless,

$$\frac{\bar{T}^*}{T_0} = 1 + \frac{1}{16} \cdot \theta^2. \tag{17}$$

4. (1 pt) Calculate the small-angle period $T_0 = 2\pi\sqrt{\frac{\ell}{g}}$ using $g = 980.7$ cm/s^2 and your measured value of ℓ (in cm).

5. (1 pt) Make a scatter plot with T_0 vs. θ and your measured values of $\bar{T}^*$ vs. θ on the same plot. Around what angle does $\bar{T}^*$ deviate from the small angle period?

6. Choose a method for your analysis of the data. You have two options: linearization or a quadratic fit.
   (a) (1 pt) State your method of choice. Define your variables for the y and x axes.
   (b) (1 pt) Choose the appropriate fit type, and fit a curve to your data using curve.fit. Label your axes. Uncertainty analysis and error bars are not necessary.
   (c) (1 pt) Compare your experimental coefficient of θ^2 with the expected value of 1/16 by calculating the relative error.
   (d) (1 pt) Are your data in good agreement with Equation 17? Use the statistical t-test to quantify your conclusion.

3 Energy Conservation

For the final part of the experiment, you will investigate the conservation of energy. Repeated measurements of the time interval for each angle are not necessary for this part of the lab. Your objective is to test the dependance of the maximum velocity on the starting angle θ. In order to calculate the maximum velocity, you will need to switch the photogate timer to "GATE" mode. In this setting, the timer will count whenever the beam is blocked by the bob. The average velocity at the bottom of the swing can be calculated using

$$\bar{v} = \frac{\Delta x}{\Delta t}. \tag{18}$$

The change in position is shown in Figure 7, which is exactly the diameter of the bob. It takes an interval in time Δt for the bob to traverse the photogate sensor. With the diameter and your measurement from the timer, you can calculate the velocity.

Under the approximations from the theory section, the total energy of the swinging pendulum should remain constant with time. The total energy of the system flows between gravitational potential and kinetic energy. Energy conservation provides a shortcut for determining important relationships without a complicated analysis of forces.

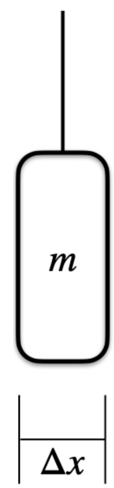


Figure 7: Δx is the diameter of the bob.

The following questions will help you obtain a relationship between starting angle and the bob's maximum velocity using the principle of conservation of energy. Recall from the theory that the total energy of the simple pendulum is

$$E = \frac{1}{2}mv^2 + mgy$$

where v is the velocity and y is the vertical displacement from equilibrium at a given time.

1. (1 pt) The pendulum reaches its maximum height at the turning point, defined as $y_{max} = h$. What is the velocity of the bob at the turning point? Write the total energy at this point in terms of h, g, and m.

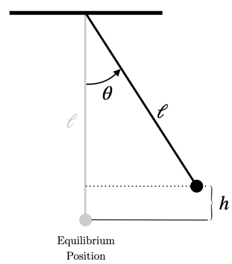
2. (1 pt) At the bottom of the swing, the potential energy has been converted entirely to kinetic energy. At this point, $y = 0$ and $v = v_{max}$. Write the total energy at this point in terms of m and v_{max}.

3. (1 pt) Because the total energy is constant, your results from the previous two questions should be equivalent. Determine a relationship between v_{max} and h. Your answer should depend on m, g, h, and v_{max}.

4. (1 pt) It is not possible to directly measure h accurately with the equipment in the lab, but starting angle measurements are simple. Use trigonometry to determine a relationship between the maximum height and the angle θ. The diagram below should help you visualize the geometry.

5. (1 pt) Substitute your results from the previous question into your equation for the total energy.

6. (1 pt) Now you have a relationship between a variable you can control and another that can be measured. What is the independent variable? What is the dependent variable?



7. (1 pt) Write a brief introduction to this part of the lab. Describe the cause-and-effect relationship you are testing: what are the dependent and independent variables?

8. (1 pt) For three different starting angles, determine the difference in energy at the top and bottom of the swing, $\Delta E = |E_{top} - E_{bottom}|$. Use statistics to test the conservation of energy, and explain your conclusions. *Hint*: calculate the mean and standard deviation of your values of ΔE.

9. (1 pt) Do you think the assumptions made in the theoretical model allow for an accurate description of a simple pendulum?

4 Appendix: Analysis Equations

The **relative error** in a quantity x with a reference (expected) value x_{ref} is

$$e_x = \left| \frac{x - x_{ref}}{x_{ref}} \right|. \tag{19}$$

The statistical **t-test** gives the size of a discrepancy in terms of the uncertainty,

$$t = \frac{|x - x_{ref}|}{\delta x}. \tag{20}$$

A discrepancy is typically considered different when $|x - x_{ref}| \geq 2 \cdot \delta x$. A small value of t indicates that the measured value x is in good agreement with x_{ref}.