PHY 481 - Fall 2024
Homework 06

Due Saturday October 12, 2024

Preface

Homework 05 starts with the conductor problem, which requires new methods to approach finding V or $\vec{E}$ as we are often unable to find ρ a priori. In the presence of an external electric field, charges will shift in a conductor, thus complicating matters of finding ρ. Gauss's law can be used to calculate the E-field in cases of very high symmetry. Once the E-field is known, then the potential can be calculated by performing the appropriate path integral. We will also use this to calculate capacitance of coaxial conductors. In addition, we will prove the "second uniqueness theorem".

1 Gauss' Law and Cavities

A **metal** sphere of radius R, carrying a charge $+q$, is surrounded by a thick concentric **metal** shell (inner radius a, outer radius b). The shell carries no net charge. Where requested, please explain your reasoning.

1. Sketch the charge distribution everywhere. If the charge is zero anywhere, indicate that explicitly.
2. From part 1, you probably noticed the charge distributes in some way on the metals. Determine the surface charge density σ at R, at a, and at b.
3. Find and sketch the electric field everywhere; explain how you know the field you have drawn is correct (by looking at the discontinuities above and below surfaces). If the field is zero anywhere, indicate that explicitly.
4. Find and sketch the potential everywhere, use $r \to \infty$ as your reference point for $V = 0$. Is the potential continuous as expected?

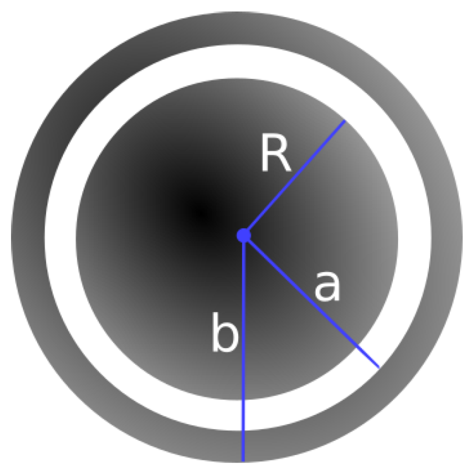

Figure 1: Concentric Metal Shells

2 Coax capacitors

Consider a coaxial cable with an inner conducting cylinder has radius a and the outer conducting cylindrical shell has inner radius b. It is physically easy to set up any fixed potential difference ΔV between the inner and outer conductors. In practice, the cable is always electrically neutral.

1. Assuming charge per length $+\lambda$ and $-\lambda$ on the inner and outer cylinders, derive a formula for the voltage difference ΔV between the cylinders.

2. Assuming infinitely long cylinders, find the **energy stored per length** (W/L) inside this capacitor. *Notice we are asking for the energy per unit length, the answer is not infinity!* Let's do it two ways so we can check: **First Method:** find the capacitance per length (C/L) of this system, and then use stored energy $W = \frac{1}{2}C(\Delta V)^2$.

3. **Second Method:** Integrate the energy density stored in the E field in order to obtain the **energy stored per unit length**.

3 Proving Uniqueness

For this homework problem, you will prove the "second uniqueness theorem" yourself, using a slightly different method than what Griffiths does (though you may find some common "pieces" are involved!) It will really help to review/read the section on the **Second Uniqueness Theorem** as you work through this problem. For the record, the **Second Uniqueness Theorem** states:

> In a volume $\mathcal{V}$ surrounded by a conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given.

Prove it like this:

1. Green's Identity (a direct consequence of the divergence theorem) is true for ANY choice of T and U, so let the functions T and U in that identity both be the SAME function:
$$\int_V \left(T\nabla^2 U + \nabla T \cdot \nabla U\right) d\tau = \oint_S (T\nabla U) \cdot d\mathbf{A} \tag{1}$$

2. Specifically, you should set them both equal to $V_3 = V_1 - V_2$ where V_1 and V_2 represent different solutions to the same boundary value problem ($\nabla^2 V = -\rho/\epsilon_0$ with boundary conditions).

3. Then, using Green's Identity (along with some arguments about what happens at the boundaries, rather like Griffith's uses in his proof) should let you quickly show that E_3, which is defined to be the negative gradient of V_3 (as usual), must vanish everywhere throughout the volume. QED.

Work to understand the game. We are checking if there are two different potential functions, V_1 *and* V_2*, each of which satisfies Laplace's equation throughout the region we're considering. You construct (define)* V_3 *to be the difference of these, and you prove that* V_3 *(or in this case,* $\mathbf{E}_3$*) must vanish everywhere in the region. This means there really is only one unique E-field throughout the region after all! This is another one of those "formal manipulation" problems, giving you a chance to practice with the divergence theorem and think about boundary conditions.*