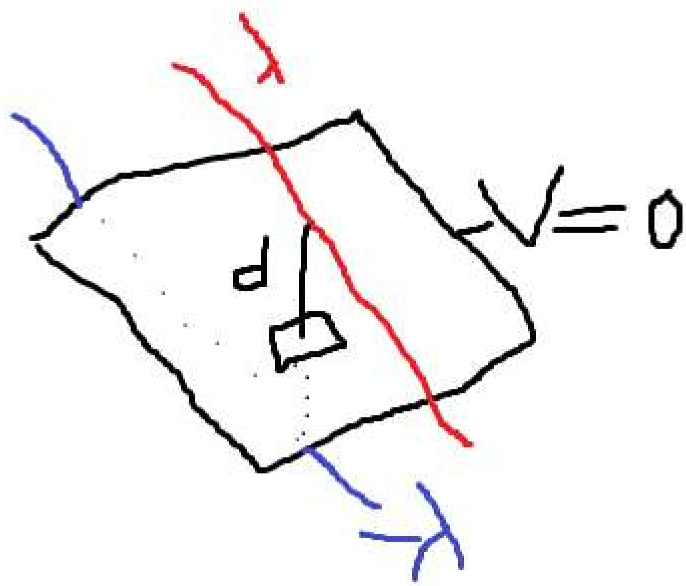
Phy 481 homework #7

1 The method of images

Griffiths Problem 3.10: A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy-plane.)

1. Find the potential in the region above the plane.
2. Find the charge density σ induced on the conducting plane.

Due to uniqueness, we can use the method of images to show how the grounded plane responds to the line charge density λ at height d above it. A line charge density of $-\lambda$ at $-d$ would counteract the electric potential made by this.



1. In this situation, $\frac{dV}{dx} = -E_x = 0$ since there the charge distribution is infinitely long in the x direction.

Using cylinder gaussian shells, we can find the electric field associated with any point due to a single charge distribution

$$\nabla \cdot E = \frac{Q_{enc}}{\epsilon_0}$$

$$2\pi r l E = \frac{\lambda l}{\epsilon_0}$$

$$E(r) = \frac{\lambda}{2\pi r \epsilon_0}\hat{r}$$

And the associated voltage is

$$\Delta V = -\int_O^r E(r)dr = -\frac{\lambda}{2\pi\epsilon_0}\int_O^r \frac{1}{r}dr = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{O}\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{O}{r}\right)$$

The term r here is distance from the charge distribution, while O is a point where voltage is zero. By superposition, we can represent each charge distribution as this ΔV if we find the distance from each line in terms of y and z. r_1 is for λ while r_2 is for $-\lambda$

$$r_1 = \sqrt{y^2 + (z-d)^2}$$
$$r_2 = \sqrt{y^2 + (z+d)^2}$$

$$\begin{aligned}
V(r_1, r_2) &= \Delta V_1 + \Delta V_2 = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{O}{r_1}\right) - \ln\left(\frac{O}{r_2}\right)\right] \\
&= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{O/r_1}{O/r_2}\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_2}{r_1}\right) \\
V(x, y) &= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{y^2 + (z+d)^2}}{\sqrt{y^2 + (z-d)^2}}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}\right)
\end{aligned}$$

2. Using $\hat{n} = \hat{z}$,

$$\sigma = -\epsilon_0\frac{\partial V}{\partial z}$$

Let's evaluate the derivative at $z = 0$

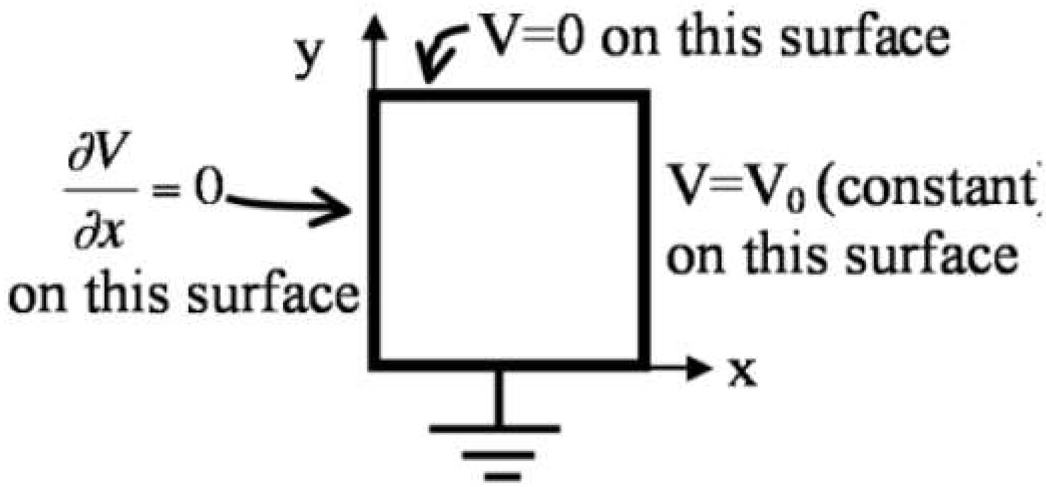
$$\sigma = -\epsilon_0\frac{\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\ln\left(\frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}\right)\right]_{z=0}$$

$$\begin{aligned}
&= \frac{\lambda}{4\pi}\left[\frac{2d}{y^2 + d^2} - \frac{-2d}{y^2 + d^2}\right] \\
&= -\frac{\lambda}{\pi}\frac{d}{y^2 + d^2}
\end{aligned}$$

2 Rectangular Pipe: Separation of Variables-Cartesian-2D

A square rectangular pipe (sides of length a) runs parallel to the z-axis (from $-\infty$ to ∞) The 4 sides are maintained with boundary conditions given in the figure. (Each of the 4 sides is insulated from the others at the corners).

1. Find the potential $V(x, y, z)$ at all points in this pipe.
2. Find the charge density $\sigma(x, y = 0, z)$ everywhere on the bottom conducting wall ($y = 0$). Check the units for your charge density (show us!).



1. Lets do it!

Can safely ignore z due to z invariance

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Using $V(x, y) = X(x)Y(y)$

$$\begin{gathered}
= Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial Y^2} \\
\rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 YV}{\partial Y^2} = 0 \\
\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 YV}{\partial Y^2}
\end{gathered}$$

Each term is constant as the other varies.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$$
$$\frac{1}{Y}\frac{\partial^2 YV}{\partial Y^2} = -k^2$$
$$X(x) = Ae^{kx} + Be^{-kx}$$
$$Y(y) = C\cos(ky) + D\sin(ky)$$

These are standard second order differential equations. We can actually swap which is sine/cosine dependant and which is exponential. I chose these since they fit well with the the left wall boundary condition.

Apply boundary conditions

$$Y(0) = 0 \rightarrow C = 0$$
$$Y(a) = 0 \rightarrow ka = n\pi \rightarrow k = \frac{n\pi}{a}$$
$$Y(y) = D\sin(ky) = D\sin\left(\frac{n\pi}{a}y\right)$$

Sine is zero at multiples of π. Next we'll apply the left wall boundary condition. Since it is dependant on x, Y will be constant.

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = 0 \rightarrow Yk\left[Ae^{k(0)} - Be^{-k(0)}\right] = 0 \rightarrow Yk[A - B] = 0 \rightarrow A = B$$
$$X(x) = Ae^{kx} + Ae^{-kx} = 2A\cosh(kx)$$
$$V(x,y) = \sum_n D_n \cosh\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)$$

Where we absorb $2A$ into D_n without loss of generality

To find the value of each D_n, we'll apply forier's trick using sine and integrating over the relevant length $[0, a]$. We'll also apply $V(x=a,y) = V_0 to help solve for the coefficients.

$$V(x=a,y) = V_0 = \sum_n D_n \cosh(n\pi)\sin\left(\frac{n\pi}{a}y\right)$$

Multiply by $\sin(\frac{m\pi}{a}y)$ and integrate

$$\int_0^a V_0 \sin\left(\frac{m\pi}{a}y\right)dy = \sum_n D_n \cosh(n\pi)\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}y\right)dy$$

LHS

$$\int_0^a V_0 \sin\left(\frac{m\pi}{a}y\right)dy = \frac{aV_0}{m\pi} - \frac{aV_0}{m\pi}\cos(m\pi)$$
$$= \begin{cases} 0 & m = 0 \pmod 2 \\ \frac{-2aV_0}{m\pi} & m = 1 \pmod 2 \end{cases}$$

RHS

$$\sum_n D_n \cosh(n\pi) \int_0^a \sin\left(\frac{n\pi}{a}y\right) \sin\left(\frac{m\pi}{a}y\right) dy = \frac{a}{2} C_m \cosh(m\pi)$$

$D_m = 0$ for even values of m, otherwise we have

$$\frac{-2aV_0}{m\pi} = \frac{a}{2} C_m \cosh(m\pi)$$
$$C_m = \frac{4V_0}{m\pi \cosh(m\pi)}$$

Now we have our voltage!

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5...} \frac{\cosh(\frac{n\pi}{a}x)}{\cosh(n\pi)} \frac{\sin(\frac{n\pi}{a}y)}{n}$$

Here's what this looks like:

In [34]:
```
import numpy as np
from numpy import cosh, pi, sin
import matplotlib.pyplot as plt
import seaborn as sns
from seaborn import cm

fourier_size = 80 # watch out for overflow
V0 = 1
a = 1

n2 = 2000
x = np.linspace(0, a, n2)
y = np.linspace(0, a, n2)
X, Y = np.meshgrid(x, y)

def phi(x, y, fourier_size, params):
    V0, a = params

    potential = 0

    for i in range(fourier_size):
        n = 2*i + 1
        potential += cosh(n*pi*x/a) * sin(n*pi*y/a)/ (n*cosh(n*pi))

    return 4*V0/pi * potential


params = (V0, a)

meshout = phi(X, Y, fourier_size, params)

plt.figure(figsize=(8, 6))
sns.heatmap(meshout, cmap=cm.rocket, xticklabels=False, yticklabels=False)
```

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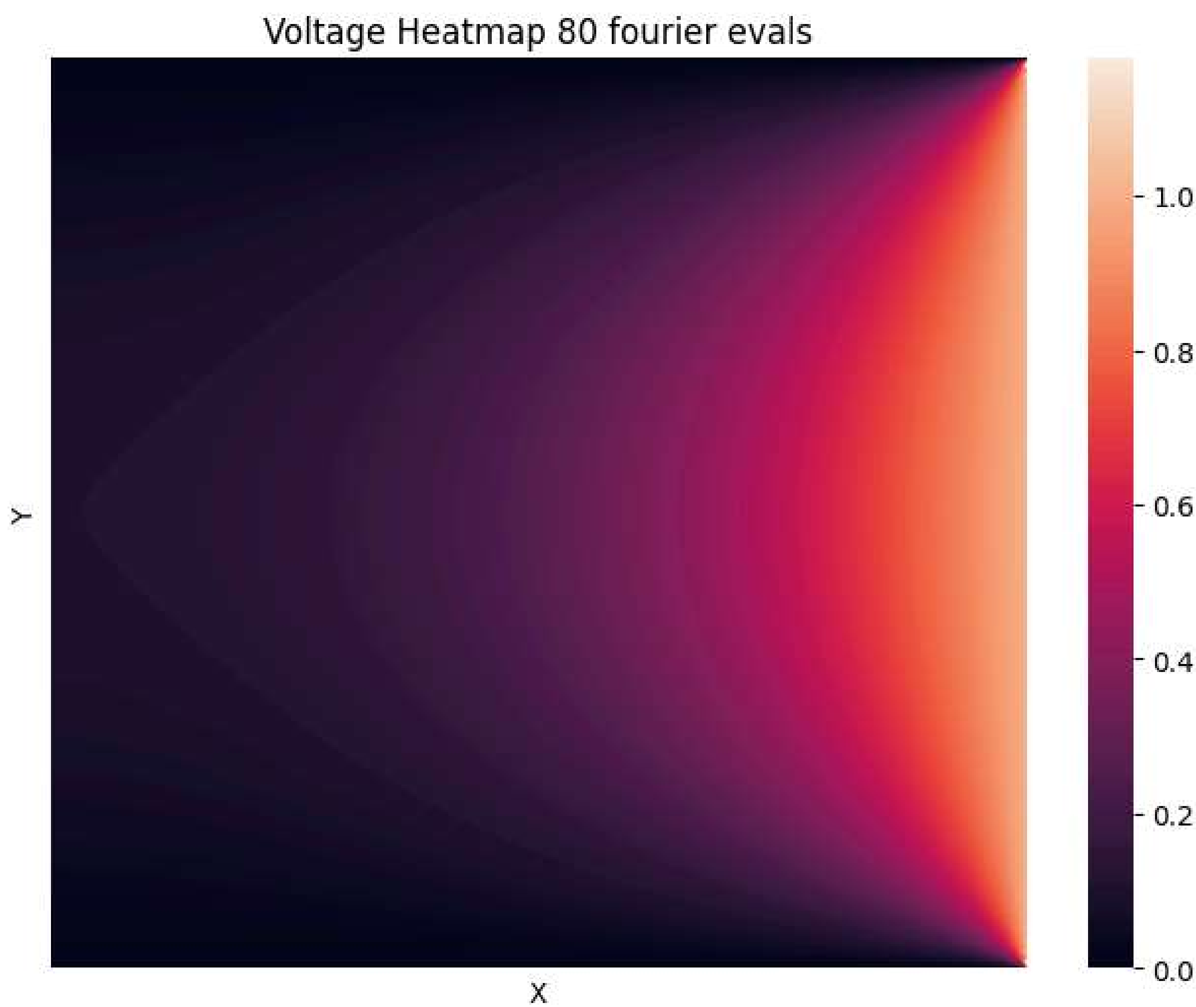
plt.title(f'Voltage Heatmap {fourier_size} fourier evals')
plt.xlabel('X')
plt.ylabel('Y')

plt.show()
```



2. $\hat{n} = \hat{x}$

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial x}\bigg|_{y=0}$$

$$= -\epsilon_0 \frac{4V_0}{\pi} \sum_{n=1,3,5...} \frac{\sin(0)}{n\cosh(n\pi)} \frac{\partial}{\partial x} \cosh(\frac{n\pi}{a} x)$$

Since $\sin(0) = 0$, the result is zero.

Units:

$$[\sigma] = [C \cdot m^{-2}]$$
$$[\epsilon_0] = [C^2 \cdot N^{-1} \cdot m^{-2}]$$
$$[V] = [N \cdot m \cdot C^{-1}]$$

For derivative, use $[m^{-1}]$

$$[\epsilon_0]\left[\frac{\partial}{\partial n}\right][V] = [C^2 N^{-1} m^{-2} N m C^{-1} m^{-1}] = [C m^{-2}] = [\sigma]$$

3 Pvthon: Method of Relaxation in 2D

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