Lab #2

Theory

1 Theory Questions

1. (1 pt) Is high precision indicated by a small or a large standard deviation?

Small standard deviation

2. The goal of this problem is to prove the validity of the standard error as the uncertainty in the mean. Consider a set of N measurements of the time period of a clock $\{T_1 \pm \delta T_1,\ T_2 \pm \delta T_2, ...,\ T_N \pm \delta T_N\}$ where each measurement is subject to significant random error from human reaction time. The mean of the set of measurements is

$$\bar{T} = \frac{T_1 + T_2 + ... + T_N}{N} \tag{8}$$

and the standard deviation is σ. This is a function of N variables. The uncertainty propagation formula[3] can be applied to the mean to get an expression for the uncertainty,

$$\delta\bar{T} = \sqrt{\left(\frac{\partial\bar{T}}{\partial T_1}\cdot\delta T_1\right)^2 + \left(\frac{\partial\bar{T}}{\partial T_2}\cdot\delta T_2\right)^2 + ... + \left(\frac{\partial\bar{T}}{\partial T_N}\cdot\delta T_N\right)^2}. \tag{9}$$

where there are N total terms under the square root (one for each variable). In the following steps, your goal is to show that Equation 9 reduces to Equation 6. *Hint*: you do not need to explicitly evaluate any sums for this problem.

a. (1 pt) From the reading, the uncertainty in a single measurement of an arbitrary quantity x is $\delta x = \sigma$. What are the individual uncertainties $\delta T_1, \delta T_2, ..., \delta T_N$?

b. (1 pt) Determining the uncertainty involves evaluating N derivatives. What is the value of the first derivative, $\partial\bar{T}/\partial T_1$? *Hint:* take a derivative of the equation for $\bar{T}$ with respect to T_1, treating all other variables as constants (N is also a constant).

c. (1 pt) Show that

$$\frac{\partial\bar{T}}{\partial T_1} = \frac{\partial\bar{T}}{\partial T_2} = ... = \frac{\partial\bar{T}}{\partial T_N}.$$

Hint: if you get stuck, pretend there are only three variables ($N = 3$) and write out Equation 8 with just T_1, T_2, T_3. Then, take a derivate with respect to each variable.

d. (1 pt) Combine your results to show that $\delta\bar{T} = \sigma/\sqrt{N}$

a. Considering that the system we're dealing with is a singular clock, each δT_i should be the same standard deviation σ in clock measurement.

b.

$$\frac{\partial\bar{T}}{\partial T_1} = \frac{d}{dT_1}\frac{T_1}{N} = \frac{1}{N}$$

c. This is simple. Since $\bar{T}$ is just the sum of each term divided by N, each term has the same shape/operation to compute the partial derivative

$$\frac{\partial \bar{T}}{\partial T_i} = \frac{d}{dT_i}\frac{T_i}{N} = \frac{1}{N}$$

d. Using the above formula

$$\begin{aligned}\delta\bar{T} &= \sqrt{\left(\frac{1}{N}\sigma\right)^2 + \left(\frac{1}{N}\sigma\right)^2 + \cdots + \left(\frac{1}{N}\sigma\right)^2} \\ &= \sqrt{\frac{\sigma^2}{N^2} + \frac{\sigma^2}{N^2} + \cdots + \frac{\sigma^2}{N^2}} \\ &= \sqrt{N\frac{\sigma^2}{N^2}} = \sqrt{\frac{\sigma^2}{N}} = \frac{\sigma}{\sqrt{N}}\end{aligned}$$

3. Because of time constraints, we are often limited to very few measurements in this class. The goal of this problem is to check the validity of the standard error as the uncertainty in in the mean in a case of only two measurements ($N = 2$). Imagine making a measurement of a quantity x two times and obtaining values x_1 and x_2. We would like to determine an expression for $\delta\bar{x}$ in terms of x_1 and x_2.

   a. (1 pt) Write down the mean $\bar{x}$ in terms of x_1 and x_2.
   b. (1 pt) Use Equation 5 to express the standard deviation in terms of x_1, x_2, and $\bar{x}$
   c. (1 pt) Now, substitute $\bar{x}$ out of your expression for σ by plugging your result from 3a into the standard deviation obtained in 3b. Try to simplify your result. *Hint:* $(x_1 - x_2)^2 = (x_2 - x_1)^2$
   d. (1 pt) The final step is to plug your result from 3c into the equation $\delta\bar{x}$ in terms of x_1 and x_2.
   e. (1 pt) Sketch a number line and label x_1, x_2, and $\bar{x}$. Draw an error bar of length $\delta\bar{x}$ extending in both directions from $\bar{x}$ indicating the uncertainty in the mean. Does the error bar extend beyond the two points x_1 and x_2?

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

a. Since $N = 2$:

$$\bar{x} = \frac{x_1 + x_2}{2}$$

b. Equation 5:

$$\begin{aligned}\sigma &= \sqrt{\frac{1}{1}\sum_{i=1} 2(x_i - \bar{x})^2} \\ &= \sqrt{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2}\end{aligned}$$

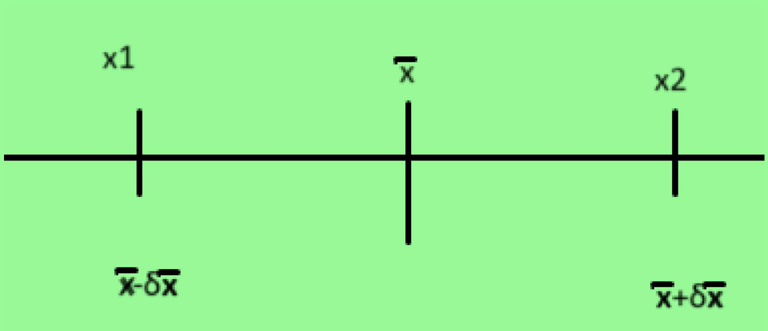
c.

$$\sigma = \sqrt{\left(x_1 - \frac{x_1 + x_2}{2}\right)^2 + \left(x_2 - \frac{x_1 + x_2}{2}\right)^2}$$
$$= \sqrt{\left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{x_2 - x_1}{2}\right)^2}$$
$$= \sqrt{2\left(\frac{x_1 - x_2}{2}\right)^2} = \frac{\sqrt{2}}{2}|x_1 - x_2|$$

d. $\delta\bar{x} = \dfrac{\sigma}{\sqrt{N}}$

$$\delta\bar{x} = \frac{\sigma}{\sqrt{2}} = \frac{1}{2}|x_1 - x_2|$$

e.



The error bar's ends coincide with the values x_1 and x_2

Lab

2. The Clock Experiment

Introduction

This experiment shows the difference between systematic error and random error by allowing participants to use common experimental methods to mitigate random error. Participants make several timing measurements that rely on human reaction time to accurately measure a counter's counting rate.

Observation

Before doing any measurements, the counter appears to be running a bit faster than one per second. My initial guess for random error (my own reaction time) is ~0.2 seconds.

3. Measurements

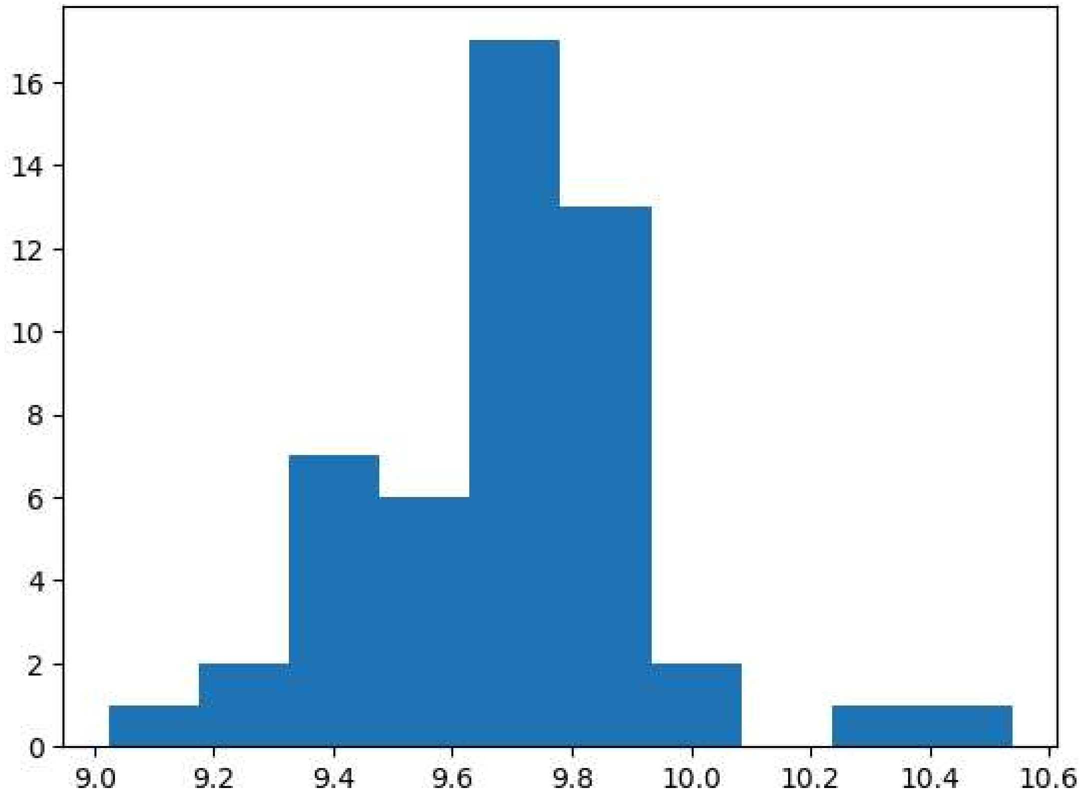
Our measurement data was loaded into a numpy array `measured_times` (see below)

In [2]:

```
import matplotlib.pyplot as plt
import numpy as np

#recorded time in seconds
measured_times = np.array([
    9.267,9.863,9.391,9.869,
    9.703,10.025,9.441,9.852,
    9.417,9.659,9.435,9.675,
    9.851,9.776,10.023,9.895,
    9.666,9.781,9.407,9.531,
    9.871,9.349,9.792,9.744,
    9.267,9.768,9.896,9.794,
    9.724,9.599,9.024,10.539,
    9.352,9.749,9.591,9.826,
    9.887,10.273,9.613,9.657,
    9.481,9.839,9.652,9.714,
    9.656,9.525,9.817,9.724,
    9.657,9.667])

plt.hist(measured_times)
```

Out[2]:

```
(array([ 1.,  2.,  7.,  6., 17., 13.,  2.,  0.,  1.,  1.]),
 array([ 9.024 ,  9.1755,  9.327 ,  9.4785,  9.63  ,  9.7815,  9.933 ,
        10.0845, 10.236 , 10.3875, 10.539 ]),
 <BarContainer object of 10 artists>)
```



0
2
4
6
8
10
12
14
16
9.0
9.2
9.4
9.6
9.8
10.0
10.2
10.4
10.6

4. Data Analysis

1 Labeled Histogram: See above

2 Our mean and standard deviaton end up being 9.691 and 0.2527, respectively.

In [3]:
```
print(f'mean: {measured_times.mean()} sec')
print(f'standard deviation: {measured_times.std():.3f} sec')
```

```
mean: 9.691479999999999 sec
standard deviation: 0.253 sec
```

The uncertainty of the mean is found using the formula $\frac{\sigma}{\sqrt{N}}$, which is 0.0357 seconds

In [10]:
```
print(f'standard error: {measured_times.std()/np.sqrt(len(measured_times)):.6f}
```

```
standard error: 0.035740 sec
```

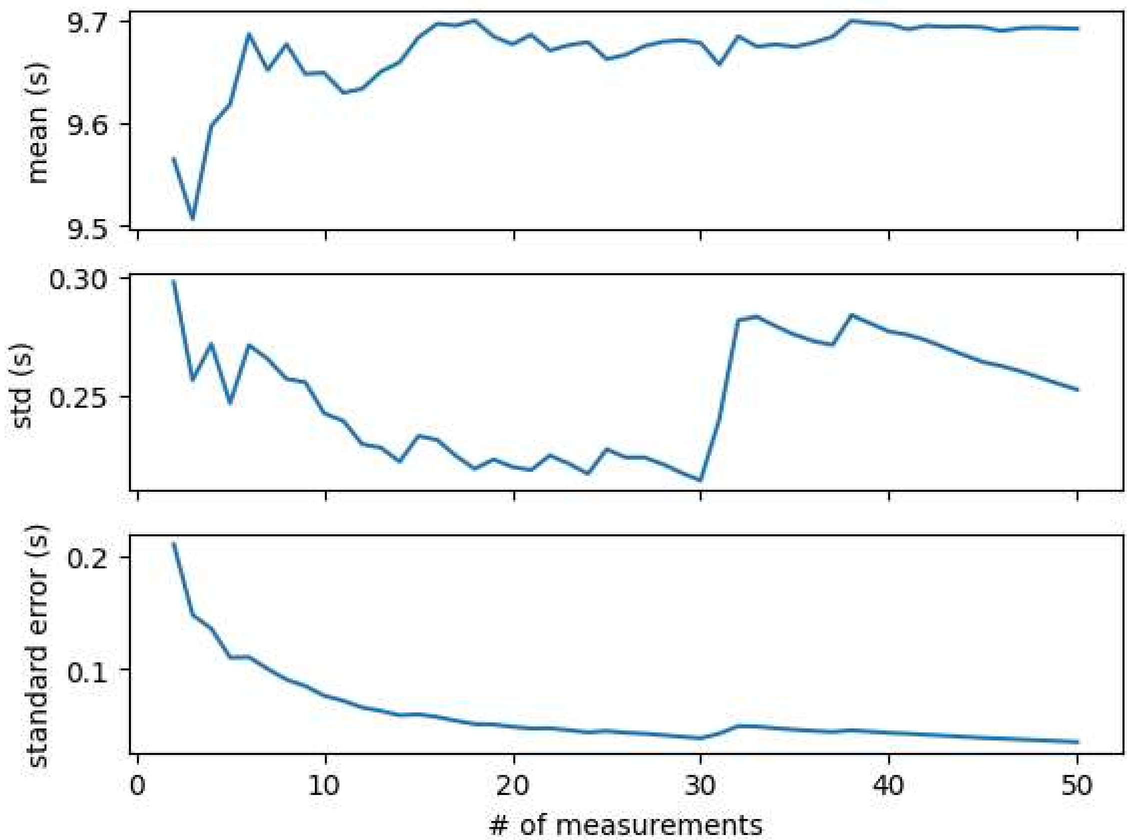
As such, we apply our sig fig rules and report a standard error of 0.04 and mean of 9.69

3 My best estimate of the actual mean is 9.69 ± 0.04

4, 5, 6 see plots below. I expect $\bar{T}$ and σ to zig zag and eventually become equal to the final values I calculated above. I expect $\delta\bar{T}$ to decrease as N increases.

Also, why sketch the graph when I can plot it all! (see below)

In [12]:
```
current_mean = []
current_std = []
current_st_err = []

for i, result in enumerate(measured_times):
    current_mean.append(measured_times[:i+1].mean())
    current_std.append(measured_times[:i+1].std())
    current_st_err.append(measured_times[:i+1].std()/np.sqrt(i+1))

fig, axs = plt.subplots(3, 1, sharex=True)
axs[0].plot(range(2, 51), current_mean[1:])
axs[0].set_ylabel('mean (s)')

axs[1].plot(range(2, 51), current_std[1:])
axs[1].set_ylabel('std (s)')

axs[2].plot(range(2, 51), current_st_err[1:])
axs[2].set_xlabel('# of measurements')
axs[2].set_ylabel('standard error (s)')

plt.show()
```



7 See table below

In [13]:
```
import pandas as pd

N = [3, 5, 10, 20, 30, 40, 50]

df = pd.DataFrame(columns=['N', 'T(s)', 'std (s)', 'dT(s)'])


for i in N:

    dmean = f'{current_st_err[i-1]:.1g}'
    mean = f'{round(current_mean[i-1], len(dmean)-2)}'

    std = f'{current_std[i-1]:.3f}'

    df.loc[len(df)] = [str(i), mean, std, dmean]

df
```

Out[13]:

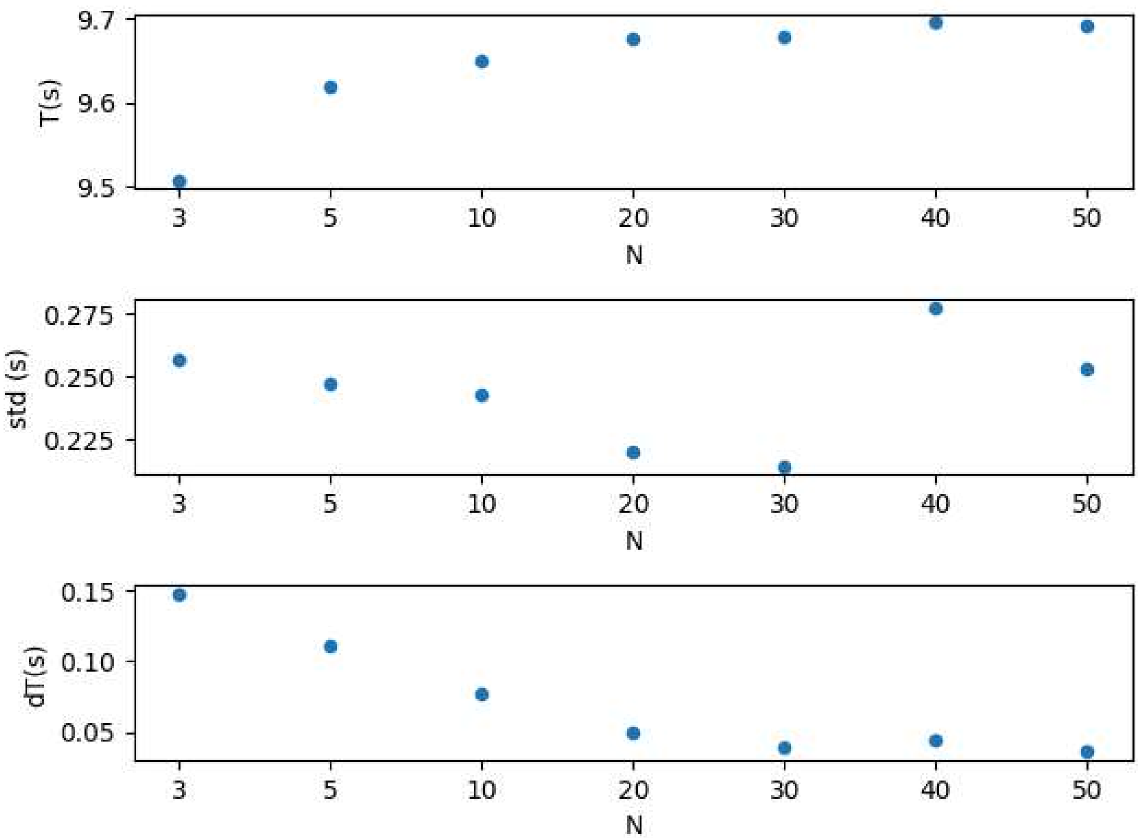
	N	T(s)	std (s)	dT(s)
0	3	9.5	0.257	0.1
1	5	9.6	0.247	0.1
2	10	9.65	0.243	0.08
3	20	9.68	0.220	0.05
4	30	9.68	0.214	0.04
5	40	9.7	0.277	0.04
6	50	9.69	0.253	0.04

8 Scatter plot

Kind of made redundant, but here it is:

In [14]:

```
df = pd.DataFrame(columns=['N', 'T(s)', 'std (s)', 'dT(s)'])

for i in N:
    dmean = current_st_err[i-1]
    mean = current_mean[i-1]
    std = current_std[i-1]
    df.loc[len(df)] = [str(i), mean, std, dmean]

fig, axs = plt.subplots(3, 1)
for i, col in enumerate(df.columns[1:]):
    df.plot(x='N', y=col, kind='scatter', ax=axs[i])

plt.tight_layout()
```



9 This is what I described above

5. Drawing Conclusions

1 Count rate estimate

We should be able to simply divide our mean time by 10 to get our sec/count.

$$\frac{9.69}{10} = 0.969 \text{ sec/count}$$

Standard error in a mean depends on the standard deviation. If we divide each measurement by 10, the new standard deviation is 0.0243

In [15]:
```
measured_1_count = measured_times / 10
measured_1_count.std()
```

Out[15]:
```
0.025272010129785884
```

It makes sense that this is 1/10 the standard error previously found. It also yields a reduced standard error of 0.004 instead of 0.04

In [16]:
```
measured_1_count.std() / np.sqrt(len(measured_1_count))
```

Out[16]:
```
0.003574001947397344
```

Our resulting mean is 0.969 ± 0.004. The relative error is the same, which makes sense since our measurements are the same

2 t-test

$$\begin{aligned} t = \frac{|R_{ref} - \bar{R}}{\delta \bar{R}} &= \frac{1 - 0.969}{0.004} \\ &= \frac{0.031}{0.004} = 7.75 \end{aligned}$$

3 Our results indicate that the clock has a systematic error, and a large one. It is not likely that one count is one second