PHY 481 - Fall 2024
Homework 04

Due Saturday, September 28, 2024

1 Spherical charge distributions are special

As you might have picked up by now, spherically symmetric charge distributions are very special. We have a number of theoretical tools we can bring to bear on them and the results we produce are often quite simple in a mathematical sense. In this problem, you will explore these distributions a bit more and gain intuition about these spherically symmetric distributions of charge.

Consider a sphere of radius R, centered on the origin, with a radially symmetric charge distribution $\rho(r)$:

1. What $\rho(r)$ is required for the electric field **in the sphere** to have the power law form $\mathbf{E} = cr^n\hat{\mathbf{r}}$, where c and n are constants? *Hint: Use the differential form of the Gauss' law and divergence of a vector in spherical coordinates (in Griffiths flyleaf)* .
2. You will see that the case of $n = -2$ is special. How so? Calculate $\nabla \cdot \mathbf{E}$ for $n = -2$ to find $\rho(r)$. *Hint:* $\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta^3(\mathbf{r})$
3. What kind of charge distribution is required for the radial E-field inside the sphere to be of constant magnitude; that is, what $\rho(r)$ produces $E(r) =$ constant (inside only)?

2 A rod with a hole drilled in it

Gauss's Law can be useful in situations where you want to determine the electric field in conceptually different physical spaces. In this problem, you will explore this using the example of a uniformly charged rod with a hole drilled through it.

Consider a rod of length L and radius b that has a hole drilled down the center of it (along it's length-wise axis) with a radius of a. The rod is very long compared to it's radius, so that Gauss's Law can be used to find the *approximate* electric field near the middle of the rod (far from the ends). The rod has a uniform charge distribution ρ. You will determine the electric field "everywhere" - meaning everywhere near the middle of the rod. Use Griffith's convention for a cylindrical coordinate system (s, ϕ, z).

1. Find the electric field inside the hole ($s < a$).
2. Find the electric field outside the rod ($s > b$).
3. Find the electric field between the hole and outer surface of the rod ($a < s < b$).
4. Compare the value of the electric field right at the material boundaries ($s = a$ and $s = b$), do the values match? As we will find later, this matching has important implications for bound charge on material surfaces.

3 Cube with a hole

What happens when you have problems were the symmetries are mixed? How do you tackle a problem with two different geometries? In this problem, you will explore how to deal with situations where they are two "competing" geometries for the problem. Sometimes you will need to bring two (or more!) aspects of your theoretical toolbox to bear on a problem.

Consider a cube (edge length a) with a uniform charge distributed throughout its volume (ρ). We carve a spherical cavity out of it of radius d, such that the cavity is centered at the center of the cube:

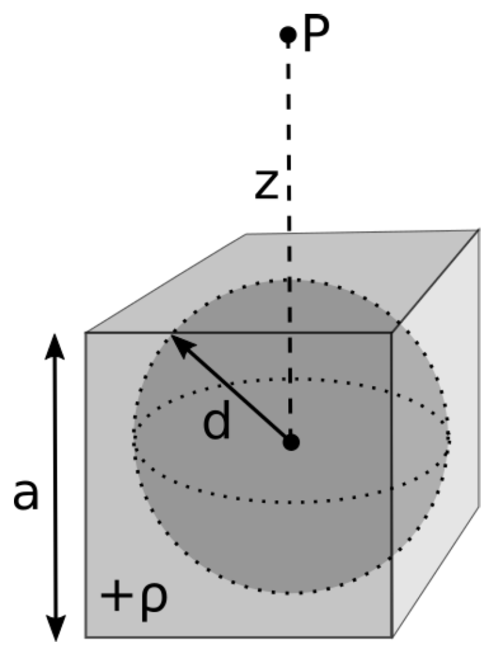

Figure 1: Cube with a hole

1. Does Gauss's Law hold for this problem? Can Gauss's Law be used on this problem? If so, what surface do you use? If not, why?

2. Let the center of the cube (and thus the center of the cavity) be located at the origin $\langle 0, 0, 0\rangle$. **Explain** how you would determine the electric field at point P a distance z from the center of the cube. If there is a direct integral involved, then show how you would set it up - but don't do it! If Gauss's Law can be used, then apply it.

4 Connecting potential, electric field, and charge

It is common in theoretical physics to describe the interactions of a system in terms of a scalar field (i.e. its potential). It is a compact description and you can (if you are careful) derive other important aspects of the system (e.g. how its sources are configured) from that scalar field if there is a rule for doing so. In this problem, you will do this work for a negative point charge. The understanding you draw from this problem will be used in future problems where the electric field and charge density might not be obvious.

Consider the potential of a point charge at the origin:

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r} \tag{1}$$

1. Determine the electric field of this charge by calculating the gradient ($\mathbf{E} = -\nabla V$). Show your work.

2. Calculate the charge density from the electric field by using Gauss's Law directly ($\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$). Do this 2 ways: (a) Use the definition of the divergence from the front fly leaf of Griffiths in spherical coordinates (what do you get?) and (b) by performing a "coordinate-free calculation" (*A "coordinate-free" calculation is one that makes no specific reference to any particular coordinate system. In this specific context, it means that you should use one or more of the identities given in the problem as Eqns. (11)-(13).*) (is your answer the same?).

3. How do your two answers from part 2 compare? Which one is correct? What does this tell you about computing charge densities from electric potentials?

For part 2b, the following vector identities might be helpful:

$$\nabla \cdot (f(\mathbf{r})\mathbf{A}) = \nabla f(\mathbf{r}) \cdot \mathbf{A} + f(\mathbf{r})\nabla \cdot \mathbf{A} \tag{2}$$

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\mathbf{r}) \tag{3}$$

$$\nabla \cdot \frac{\hat{r}}{r} = \frac{1}{r^2} \tag{4}$$

5 Python: Electric field of a line charge

As discussed in the book, you can break up a distribution of charge into chunks - each a point charge - and add up the contributions to the electric field of each chunk. This process forms the basis of numerical superposition, which you began to explore in the last homework. In this problem, you will extend that work to a line of charge. You will solve this problem using a Jupyter notebook. You can download it from D2L.

Using numerical superposition, adding up the contributions to the electric field due to each chunk, you will solve the following problems:

1. We want to compute and represent the electric field of the charge at a distance of $0.01\ \mathrm{m}$ from the line charge along the y-axis. Do this.

2. The analytical formula for the electric field of the rod at that location is:

$$E_{\mathrm{rod}} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{y\sqrt{L^2/4 + y^2}} \tag{5}$$

   Compare the value of the electric field at that location for different values of "Nchunks," say for 10, 20, 50, and 100 chunks. How close do you get with 100 chunks? How many chunks do you need to get within 1% of the analytical solution?

3. Using what you have built to find the electric field at this location, find the electric field at a variety of points around the the line charge and represent them with arrows. You can choose the locations, but be systematic.