PHY 481 - Fall 2024
Homework 08

Due Saturday November 2, 2024

Preface

Homework 08 focuses on Laplace's equation and solving it using infinite series solution in spherical and cylindrical coordinates. You should become comfortable with setting boundary conditions for PDE problems like this and develop a sense of the process for solving these problems analytically. We also will plot a potential in 3D using Python. In addition, we will start investigating the multipole expansion.

1 Sphere with a known potential

We have a sphere (radius, R) where we have glued charges to the outside such that the electric potential at the surface of the sphere is given by:

$$V_0 = k\cos 3\theta$$

where k is some constant.

You are going to find the potential inside and outside the sphere (there are no charges other than those at the surface of the sphere) as well as the charge density $\sigma(\theta)$ on the surface of the sphere. Each part of this problem is meant to walk you through the process for solving these kinds of boundary-value problems.

1. Rewrite the potential at the surface using Legendre polynomials. *You will need to dust off some trigonometric identities to do this.* You can find a listing of Legendre polynomials online.

2. Using this boundary condition and the knowledge that V should be finite inside the sphere, find the electric potential, $V(r,\theta)$, inside this sphere. You do not have to re-derive the general solution to Laplace's equation, just use the result:

$$V(r,\theta) = \sum_{\ell}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

3. Using the same boundary condition and the knowledge that V should vanish far from the sphere, find the electric potential, $V(r,\theta)$, outside this sphere.

4. Show explicitly that your solutions to parts 2 and 3 match at the surface of the sphere.

5. Take the "normal" derivative of each of your solutions ($\partial V/\partial r$) and use their difference at the surface to find the charge on the surface:

$$\left(\frac{\partial V_{\text{outside}}}{\partial r} - \frac{\partial V_{\text{inside}}}{\partial r}\right) = -\frac{\sigma}{\varepsilon_0}$$

6. What is the total charge on the sphere? *Hint: Note that* $P_0(\cos(\theta)) = 1$ *and use orthogonality: Eqn. 3.68 of Griffiths!*

2 Separation of Variables in Cylindrical Coordinates

We have gone through how to solve Laplace's equation in Cartesian and spherical coordinates. In both cases, finding a separable and general solution was possible. In fact, there are a number of possible coordinate systems where we can do this, but the most relevant to this class (besides Cartesian and spherical) is cylindrical coordinates.

In this problem, you will develop the general solution to Laplace's equation in cylindrical coordinates where there is no dependence on the z coordinate (i.e., where we have cylindrical symmetry).

1. Starting from Laplace's equation in Cylindrical coordinates, use the ansatz $V(s, \phi) = S(s)\Phi(\phi)$ to convert the problem from one partial differential equation to two 2nd order ordinary differential equations – one for $S(s)$ and one for $\Phi(\phi)$.

2. As we have argued twice, each of those differential equations is equal to a constant. Which constant is positive and which is negative? Explain your choice. *Think about what happens when you rotate your problem by* 2π *in the* ϕ *direction, should the physics care that you've done that?* Going forward, choose the positive constant to be $+k^2$ and the negative one to be $-k^2$.

3. Solve the differential equation for $\Phi(\phi)$ to obtain the general solution for $\Phi(\phi)$. Hint: $\Phi(\phi) = \Phi(\phi + 2\pi)$ so this puts an additional condition on k that it must be an integer with $k \geq 0$.

4. Armed with this information about k, solve the differential equation for $S(s)$ to obtain the general solution for $S(s)$. *Be careful to treat* $k = 0$ *separately as that generates an additional and completely physical solution!*

5. Combine your solutions to Parts 3 and 4 to generate the complete general solution $V(s, \phi) = S(s)\Phi(\phi)$.

6. The potential at a distance s away from an infinite line charge (which should be captured by this solution) is: $V(s) = \dfrac{2\lambda}{4\pi\varepsilon_0}\log(s)$ + constant, which terms in general solution vanish to capture this solution?

This problem is tough. But here's a little help. The general solution for the electric potential in cylindrical coordinates (with cylindrical symmetry) is:

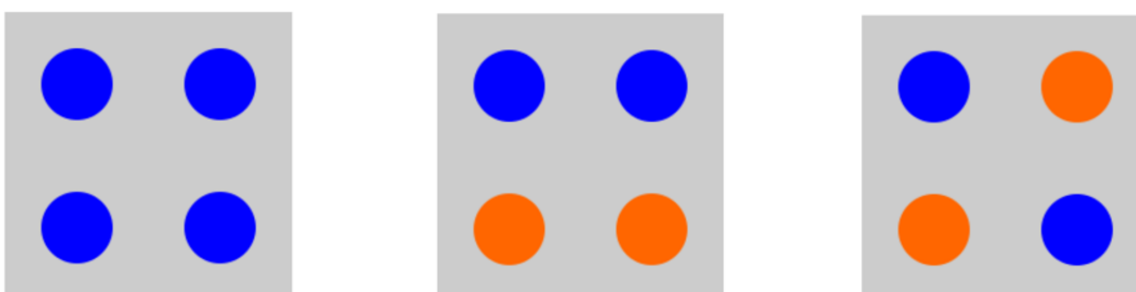
$$V(s, \phi) = a_0 + b_0 \log s + \sum_{k=1}^{\infty} \left[s^k (a_k \cos k\phi + b_k \sin k\phi) + s^{-k} (c_k \cos k\phi + d_k \sin k\phi) \right]$$

You will not get full credit for this problem unless your work clearly shows how you this solution is developed.

3 Developing Intuition for Multipole Expansion

Developing intuition about the dominant contribution to the field that you are looking at will serve you very well in the future. In this problem, you will look at a few charge distributions (blue - positive charge; orange - negative charge) and discuss what the dominant contribution (monopole, dipole, quadrupole) to the field would be far from the distribution (as $r \to \infty$).

For each distribution below, discuss which contribution to the multipole expansion dominates at large r. Explain how you can tell this is the dominant contribution (use equations, pictures, and words as you see fit).



Distribution 1 Distribution 2 Distribution 3

Figure 1: Charge distributions

4 Python: Visualizing Solutions to the Laplace Equation

In Example 3.3 of the book, Griffiths solves the problem of a U-shaped infinite slot with the base in the yz-plane, see below.

Griffiths goes on to solve the problem for the specific case where the potential $V_0(y)$ at the base ($x = 0$) is just a constant $V_0(y) = V_0 = \text{constant}$. This solution was analytic, but it

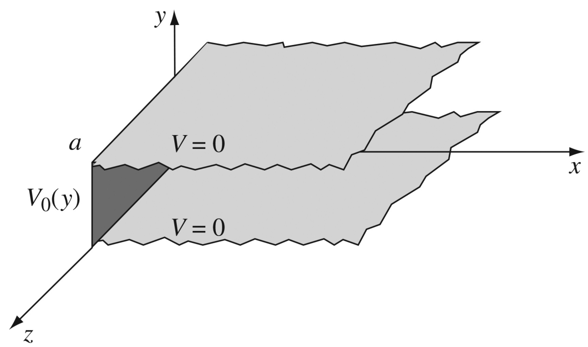


Figure 2: U-shaped Slot

contained an infinite series:

$$V(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_0}{\pi n} \sin\left(\frac{n\pi y}{a}\right) \exp\left[-\frac{n\pi x}{a}\right] \tag{1}$$

While perfectly analytic, this solution is hard to visualize. What does that solution look like? Take $V_0 = 10$ V and $a = 1$ m.

1. Plot the approximate solution in 3D space using Python's "mplot3D" for just the first term in the sum (i.e., only for $n = 1$). Download this Jupyter notebook: HW08_3dPotentialPlot.ipynb from D2L, which walks you through how to plot in 3D.

2. Plot the approximate solution in 3D space for the sum of first 5 terms. What do you notice about the boundary where $V = V_0$?

3. Test the plot by summing different number of terms to see how the plot starts to look constant $V = V_0$ at the boundary. Plot the approximate solution for the sum of first 100 terms such that the boundary where $V = V_0$ looks very close to constant.