Math 54 Notes & Key Sheet

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Chapter 1

Introduction

Lorem Ipsum

Chapter 2

Linear Equations

2.1 Linear Systems

Definition 1 (Linear Equations)**.** *Given variables $x_1, x_2, \ldots, x_n$, real or complex numbers b and coefficients $a_1, a_2, \ldots, a_n$, we introduce **Linear Equations** as an equation written in the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{2.1}$$

Definition 2 (Linear Systems)**.** *A **System of Linear Equations** or **Linear System** is a collection of m linear equations sharing n variables.*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n} = &\quad b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n} = &\quad b_2 \\ \vdots& \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn} = &\quad b_m \end{aligned} \tag{2.2}$$

Definition 3 (Solution Set)**.** *A **Solution** of a linear system is a list of numbers that satisfies each equation when substituted for $x_1, x_2, \ldots, x_n$. The **Solution Set** is the set of all possible solutions to the linear system. Also note that two linear systems are **equivalent** if they share the same solution set.*

Concept 1 (Existence/Uniqueness of Solutions)**.** *A linear system's solution set has a certain number of elements*

- *none* $\{\}$

- *one* $\{s\}$
- *infinitely many*

Definition 4 (Consistent)**.** *A linear system is* ***Consistent*** *is its solution set is nonempty. Otherwise, it is* ***Inconsistent****.*

Definition 5 (Matrix)**.** *We define a* $m \times n$ ***Matrix*** *as a rectangular array of objects. We can take the coefficients of a linear system as a* ***Coefficient Matrix*** *and a* ***Augmented Matrix*** *if we include the rightmost constants.*

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \tag{2.3}$$

Procedure 1 (Elementary Row Operations)**.** *We are allowed to preform three basic operations to create row equivalent augmented matrixes. Note that the new associated linear systems share the same solution set.*

Replacement *Replace row by sum of itself and a scaled version of another row*

Interchange *Swap two rows*

Scaling *Multiply a row by a nonzero constant*

Concept 2 (Fundamental Questions)**.** *We are faced with two fundamental questions when considering linear systems.*

- *Is a system consistent?*
- *If the system is consistent, is the solution unique?*

2.2 Row Reduction

Definition 6 (Echelon Form)**.** *A matrix is in* ***echelon form*** *if it has these three properties*

- *Nonzero rows are above any rows with all zeros*

- *Each leading entry in a row, its location called the* ***pivot position****, is in a column, called a* ***pivot column****, to the right of the leading entry of the row above it*

- *All entries in a column below a leading entry are zeros*

where leading entry refers to the first nonzero entry.
Example (where $\blacksquare$ *refers to a nonzero value and * refers to any value):*

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \tag{2.4}$$

A matrix is in ***reduced echelon form*** *is it satisfies*

- *Each nonzero row has only a leading entry of 1*

Note that each matrix is row equivalent to only one reduced echelon matrix.
Example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.5}$$

Procedure 2 (Row Reduction Algorithm)**.** *We use a set of steps in producing a matrix in first echelon, then reduced echelon form.*

1. *Identify the leftmost nonzero column as the first pivot column. Identify the topmost entry location as the pivot position*

2. *Identify a nonzero entry as the pivot and move it to the pivot position (if not already there)*

3. *Use row operations to turn all entries below the pivot in the pivot column into zeros*

4. *Ignoring the rows containing and above the pivot position just in question, and repeat steps 1-3 on the remaining submatrix. Repeat until there are no more nonzero rows to change*

   $\boxed{\textit{Steps 1-4 are called the forward phase. Echelon form has been achieved}}$

5. *Using row operations, scale pivots to one and create zeros above each pivot*

   Step 5 is called the backward pase. Row echelon form has been achieved

Definition 7 (Free/Basic variables)**.** *The variables corresponding to pivot columns are called **basic variables**. The other variables are called **free variables**. With solutions of linear systems, we want to represent basic variables as parametric descriptions with free variables as parameters.*

Concept 3 (Existence/Uniqueness Theorem)**.** *The echelon form of our augmented matrix offers some useful information on existence/uniqueness.*

Existence *A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column*

Uniqueness *If the linear system is consistent, then the solution set contains one or infinitely many solutions if they have*

One *No free variables*

Infinite *At least one free variable*

Procedure 3 (Methodology for Solving Linear Systems)**.** *In summary, the following is the general procedure for using row reduction to solve linear systems*

1. *Create augmented matrix*
2. *Use row reduction algorithm to reduce to echelon form. If system is consistent, continue. Otherwise, return no solution.*
3. *Continue reducing to reduced echelon form.*
4. *Write parametrized solutions for each basic variable is expressed in terms of the free variables.*

2.3 Vectors

Definition 8 (Vector)**.** *A* ***vector*** *is defined as an element of a vector space. In the next few sections we're only going to deal with vectors which are just ordered collections of numbers.*

Definition 9 (Vector Operations)**.** *For scalar k and vectors in $\mathbb{R}^n$*

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

vector addition

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \tag{2.6}$$

scalar multiplication

$$k\mathbf{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix} \tag{2.7}$$

Property 1 (Algebraic properties of vectors in $\mathbb{R}^n$)**.** *Given $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$ and scalars c and d:*

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
3. $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$
4. $\boldsymbol{u} + (-\boldsymbol{u})$
5. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$

6. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$

7. $1\boldsymbol{u} = \boldsymbol{u}$

Definition 10 (Linear Combinations). *Given vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_p$ (called weights), we define the **linear combination***

$$\boldsymbol{y} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_p\boldsymbol{v_p} \tag{2.8}$$

Definition 11 (Span). *For vectors $bmv_1, bmv_2, \ldots, bmv_p$ in $\mathbb{R}^n$, Span$\{bmv_1, bmv_2, \ldots, \boldsymbol{v_p}\}$ is the collection of all vectors that may be represented in the form:*

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_p\boldsymbol{v_p} \tag{2.9}$$

2.4 Matrix Equation

Definition 12 (Ax=b). *Given matrix $A_{m\times n}$ and vector $\boldsymbol{x}$ in $\mathbb{R}^n$:*

$$A\boldsymbol{x} = \begin{bmatrix}\boldsymbol{a_1} \ \boldsymbol{a_2} \ \cdots \ \boldsymbol{a_n}\end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \begin{bmatrix} x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn} \end{bmatrix} \tag{2.10}$$

Note that the row-vector rule for $A\boldsymbol{x}$ means the i-th entry in $A\boldsymbol{x}$ is the dot product of the row i in A and the vector x.

Concept 4. *Important! We now have enough definitions to establish a relationship between three concepts.*

A vector equation of the form

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_p\boldsymbol{v_p} = \boldsymbol{b} \tag{2.11}$$

has the same solution set as the L.S with the augmented matrix

$$\begin{bmatrix}\boldsymbol{a_1} \ \boldsymbol{a_2} \ \cdots \ \boldsymbol{a_n} \ \boldsymbol{b}\end{bmatrix} \tag{2.12}$$

which has the same solution set as the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b} \tag{2.13}$$

A nice observation from this is that the equation $\boldsymbol{x} = \boldsymbol{b}$ has a solution iff $\boldsymbol{b}$ is a linear combination of the column vectors of A.

Theorem 1. *Given $A_{m\times n}$, the follwing statements are logically equivalent.*

1. *For each $\boldsymbol{b}$ in $\mathbb{R}^m$ the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution (is consistent)*
2. *Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the column vector of A*
3. *The column vectors of A span $\mathbb{R}^m$*
4. *A has a pivot position in every row*

Definition 13 (Identity Matrix)**.** *The I_n matrix is the matrix such that $I_n\boldsymbol{x} = \boldsymbol{x}$ for every x in $\mathbb{R}^n$. It is an $n \times n$ square matrix with 1s on every row position and 0s everywhere else.*

Property 2 (Matrix Operations)**.** *Given matrix $A_{m\times n}$:*

1. $A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$
2. $A(c\boldsymbol{u}) = c(A\boldsymbol{u})$

2.5 Solution Sets

Definition 14 (Homogenous Linear Systems)**.** *A linear system is* ***homogenous*** *if it can be represented in the form $A\boldsymbol{x} = \boldsymbol{0}$. This always has the trivial solution, where $\boldsymbol{x} = \boldsymbol{0}$. This also has a nontrivial solution if there exists a nonzero vector $\boldsymbol{x}$ that also satisfies the equation.*

Definition 15 (Homogenous Linear System)**.** *A linear system is* ***homogenous*** *if it can be represented in the form $A\boldsymbol{x} = \boldsymbol{0}$. This always has the trivial solution, where $\boldsymbol{x} = \boldsymbol{0}$. This also has a nontrivial solution if there exists a nonzero vector $\boldsymbol{x}$ that also satisfies the equation.*

Theorem 2 (Nonhomogenous vs Homogenous Solution Sets)**.** *Given nonhomogenous equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent with some solution $\boldsymbol{p}$, its solution set is a translation of the solution set of the homogenous equation $A\boldsymbol{x} = \boldsymbol{0}$ (which passes through the origin) with any solution v_h where we can write each solution as $w = p + v_h$.*

Note, to parameterize the solution set of a system, we write a typical solution x according to the free variables as parameters.

2.6 Linear Dependence/Independence

Definition 16 (Linear Independence/Dependence)**.** *A set of vectors* $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ *in* $\mathbb{R}^n$ *is* ***linearly independent*** *if there exists only the trivial solution to vector equation*

$$x_1\boldsymbol{v_1} + x_2\boldsymbol{v_2} + \ldots + x_p\boldsymbol{v_p} = \mathbf{0} \tag{2.14}$$

Else, if there exists nontrivial solutions, it is called ***linearly dependent****. We can represent the above homogenous equation as an augmented matrix and solve for solutions. The vectors are linear independent iff from this we only find the trivial solution.*

Theorem 3. *A set* $S = \boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ *is linearly dependent iff any vector is a linear combination of the other vectors.*

Observation 1. *Given a set* $S = \boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ *in* $\mathbb{R}^n$*:*

1. *S is linearly dependent if* $p > n$
2. *S is linearly dependent if it contains the zero vector*

2.7 Linear/Matrix Transformations

Definition 17 (Transformation)**.** *A* ***transformation*** *(also known as a* ***function*** *or* ***mapping*** T *from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *assigns some vector* $T(x)$ *(called the* ***image*** *of* x*) in* $\mathbb{R}^m$ *to each vector* x *in* $\mathbb{R}^n$*.*

The ***domain*** *is the set of input values for which T is defined. The* ***codomain*** *is the set within which the output values may lie. The* ***range*** *is the set of all images.*

Definition 18 (Linear Transformation)**.** ***Linear transformations*** *must follow the following conditions:*

1. $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ *for all* $\boldsymbol{u}$*,* $\boldsymbol{v}$ *in the domain of T*
2. $T(c\boldsymbol{u}) = cT()$ *for all scalars c and* $\boldsymbol{u}$ *in the domain of T*

   From this, we can also draw the conclusions that
3. $T(\mathbf{0}) = \mathbf{0}$

4. $T(c_1\boldsymbol{v_1} + \ldots + c_p\boldsymbol{v_p} = c_1T(\boldsymbol{v_1}) + \ldots + dT(\boldsymbol{v_p})$

Definition 19 (Matrix Transformations)**.** ***Matrix Multiplications****, denoted by* $\boldsymbol{x} \mapsto A\boldsymbol{x}$ *are transformations given by* $T(x) = A\boldsymbol{x}$ *for some matrix* $A_{m\times n}$*. The domain and codomain are* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively, and the range is the span of the columns vectors of* A*.*

2.8 Standard Matrix

Theorem 4. *First, note that every linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ *is a matrix transformation with* ***standard matrix*** $A_{m\times n}$ *given by*

$$A = \begin{bmatrix} T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix} \tag{2.15}$$

Concept 5 (Onto and One-to-one)**.** *Given mapping* $T : \mathbb{R}^n \to \mathbb{R}^m$*:*

	Onto (surjective)	*One-to-one (injective)*
Definition	*Each* $\boldsymbol{b}$ *in the codomain is the image of at least one* $\boldsymbol{x}$ *in the domain*	*Each* $\boldsymbol{b}$ *in the range is the image of at most one* $\boldsymbol{x}$ *in the domain*
Relation to Ax=b	*Existence of solution*	*Uniqueness of solution*
Conditions (iff)	*1. columns of* A *span* $\mathbb{R}^m$	*1.* $T(\boldsymbol{x}) = \boldsymbol{0}$ *has only trivial solution* *2. columns of* A *are linearly independent*

Chapter 3

Matrix Algebra

3.1 Matrix Operations

Before we begin, note that the entries in a matrix $A_{m\times n}$ with m rows and n columns are denoted a_{ij} with i and j referring to the row and column number respectively. Two matrices are equal if their entries are all equal. Addition of two matrices involves adding their corresponding entires. Scalar multiplication involves scaling each entry by the scalar.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & & & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{i(n-1)} & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{(m-1)1} & a_{(m-1)2} & \cdots & a_{(m-1)j} & \cdots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{m(n-1)} & a_{mn} \end{bmatrix} \tag{3.1}$$

Concept 6 (Types of matrices)**.** *A* ***diagonal matrix*** *$A_{m\times n}$ is a square matrix whose non**diagonal entries** are zero. A special case is the **identity matrix** $I_{n\times n}$ which has all diagonal entries as 0. The **zero matrix** has all entries as 0.*

Property 3 (Properties of matrix operations)**.** *Given A, B, C are matrices of the same size and r & s are scalar:*

Commutative $A + B = B + A$

Associative $(A+B)+C = A+(B+C)$

Identity $A+0 = A$

Scalar Distributive $r(A+B) = rA + rB$

1. $(r+s)A = rA + sA$

Scalar associative $r(sA) = (rs)A$

Concept 7 (Matrix Multiplication)**.** *Given*

$$A_{m\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

and

$$B_{n\times p} = \begin{bmatrix} b_{11} & \cdots & b_{1l} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kl} & \cdots & b_{1l} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nl} & \cdots & b_{np} \end{bmatrix}$$

:

We have

$$AB = \begin{bmatrix} a_{11}b_{11}+\ldots+a_{1n}b_{n1} & \cdots & a_{11}b_{1l}+\ldots+a_{1n}b_{nl} & \cdots & a_{11}b_{1p}+\ldots+a_{1n}b_{np} \\ \vdots & & \vdots & & \vdots \\ a_{i1}b_{11}+\ldots+a_{in}b_{n1} & \cdots & a_{i1}b_{1l}+\ldots+a_{in}b_{nl} & \cdots & a_{i1}b_{1p}+\ldots+a_{in}b_{np} \\ \vdots & & \vdots & & \vdots \\ a_{m1}b_{11}+\ldots+a_{mn}b_{n1} & \cdots & a_{m1}b_{1l}+\ldots+a_{mn}b_{nl} & \cdots & a_{m1}b_{1p}+\ldots+a_{mn}b_{np} \end{bmatrix} \tag{3.2}$$

Note that we must have the number of columns of A and the number of rows of B be equal. Also, the (i, j) entry of AB comes out to be:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{i(n-1)}b_{(n-1)j} + a_{in}b_{nj} \tag{3.3}$$

Property 4 (Properties of matrix multiplication). *Given $A_{m\times n}$, $B_{n\times p}$, $C_{p\times q}$:*

Associative $A(BC) = (AB)C$

Distributive $A(B + C) = AB + AC$

Distributive $(B + C)A = BA + CA$

- $r(AB) = (rA)B = A(rB)$

Identity $I_m A = A = AI_n$

Definition 20 (Transpose). *The* ***transpose*** *of $A_{m\times n}$ is some matrix $(A^T)_{n\times m}$ where each column is formed from the corresponding row of A. For example, the nth column of A^T is the nth row of A.*

Property 5 (Properties of transpositions). *We obtain the following properties for transpositions.*

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB \dots C)^T = B^T A^T \dots C^T$

3.2 Matrix Multiplicative Inverse

Definition 21 (Inverse). *We define the* ***inverse*** *of a square matrix A to be the matrix A^{-1} such that*

$$A^{-1}A = AA^{-1} = I \tag{3.4}$$

If such a matrix can be found, then A is called invertible. If not, then it is called singular.

Property 6 (Properties of Operations with inverses). *Given A is an invertible matrix:*

1. $(A^{-1})^{-1} = A$

2. $(AB \dots C)^{-1} = C^{-1}B^{-1} \dots A^{-1}$

3. $(A^T)^{-1} = (A^{-1})^T$

Definition 22 (Elementary Matrix)**.** *An **elementary matrix** is created by performing one row operation on an identity matrix. Multiplying a matrix A by it performs the same operation on A.*

Procedure 4 (Finding the inverse)**.** *Row reduce the augmented matrix $\begin{bmatrix} A\ I \end{bmatrix}$ until we obtain $\begin{bmatrix} I\ A^{-1} \end{bmatrix}$. If we can't row reduce A into I, then A is not invertible.*

3.3 More on Invertible Matrices

Theorem 5 (Invertible Matrix Theorem)**.** *Given $A_{m\times n}$, the following statements are equivelent:*

1. *A is invertible*
2. *There exists inverse A^{-1} such that $A^{-1}A = AA^{-1} = I$. Note that A and A^{-1} are both inverses of each other*
3. *A is row equivalent to $I_{n\times n}$*
4. *A has n pivot positions*
5. *$A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution*
6. *The column vectors of A are linearly independent*
7. *The linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ from $\mathbb{R}^n \mapsto \mathbb{R}^m$ is bijective (one-to-one and onto)*
8. *The equation $A\boldsymbol{x} = \boldsymbol{b}$has at least one solution for each $\boldsymbol{b}$ in $\mathbb{R}^n$*
9. *The column vectors of A span $\mathbb{R}^n$*
10. *A^T is also an invertible matrix*
11. *The columns of A form a basis of $\mathbb{R}^n$*
12. *$Col(A) = \mathbb{R}^n$*

13. $rank(A) = dim(Col(A) = n$

14. $Nul(A) = \{\mathbf{0}\}$

15. $dim(Nul(A)) = 0$

Theorem 6. *Linear Transformation T with standard matrix A is invertible if A is invertible. In that case, its inverse is the transformation given by standard matrix A^{-1}*

3.4 Subspaces

Definition 23 (Subspace)**.** *A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ with three properties:*

1. *H contains the zero vector*
2. *For each $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, the sum $\approxeq + \underset{\approx}{\succsim}$ is in H*
3. *For each $\boldsymbol{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H*

Definition 24 (Column Space)**.** *The **column space** of a matrix A is the set Col(A) of all linear combinations of the column vectors of A.*

$$ColA = Span\{a_1, \ldots, a_n\} \tag{3.5}$$

Definition 25 (Null Space)**.** *The **null space** of a matrix A is the set Nul(A) of all solutions of $A\boldsymbol{x} = \mathbf{0}$*

$$NulA = \{\boldsymbol{x} : \boldsymbol{x}\, is\, in\, \mathbb{R}^n\, and\, A\boldsymbol{x} = \mathbf{0}\} \tag{3.6}$$

Also note that Nul($A_{m\times n}$) is a subspace of $\mathbb{R}^n$

Definition 26 (Basis)**.** *A **basis** for a subspace H is a linearly independent set in H that spans H*

*The set of vectors $e_1, \ldots, e_n$ is called the **standard basis** for $\mathbb{R}^n$ where the the jth entry in e_j is 1, all else being 0*

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Note that the pivot columns of A form a basis for the column space of A. The free variable columns of A form a basis for the null space of A.

3.5 Dimension and Rank

Concept 8 (Coordinate Vectors)**.** *Given a basis* $B = b_1, \ldots, b_p$ *for subspace H, for each* $\boldsymbol{x}$ *in H, the coordinates of* $\boldsymbol{x}$ *relative to B are the weights* $c_1, \ldots, c_p$ *such that* $\boldsymbol{x} = c_1\boldsymbol{b_1} + \ldots + c_p\boldsymbol{b_p}$*.*

We call the B-coordinate vector:

$$[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

Definition 27 (Dimension)**.** *The* ***dimension*** *of subspac, dim(H), is the number of vectors in the basis for H*

Definition 28 (Rank)**.** *The* ***rank*** *of a matrix A, rank(A), is dim(col(A))*

Theorem 7. *Rank Theorem For matrix A with n columns, we have* $rank(A) + dim(Nul(A)) = n$

Theorem 8 (Basis Theorem)**.** *Given H is a p-dimensional subspace of* $\mathbb{R}^n$*, any linearly dependent set of p elements in H is a basis for H. In addition, any set of p elements that span H are a basis as well.*

Chapter 4

Vector Spaces

4.1 Vector Spaces and Subspaces

Definition 29 (Vector Spaces)**.** *A **vector space** is a nonempty set V of objects, called vectors, for which we define two operations: vector addition (input: 2 vectors, output: 1 vector) and scalar multiplication (input: 1 vector 1 scalar, output: 1 vector). Given vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, and scalars and d, the following ten axioms most hold:*

osed under vector addition $\boldsymbol{u} + \boldsymbol{v}$ is in V

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

3. *We can define a zero vector $\boldsymbol{0}$ s.t* $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$

4. *We can define an additive inverse $-\boldsymbol{u}$ for each $\boldsymbol{u}$ in V s.t* $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$

under scalar multiplication $c\boldsymbol{u}$ is in V

5. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$

6. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$

7. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$

8. $1\boldsymbol{u} = \boldsymbol{u}$

Definition 30 (Subspace). *A* ***subspace*** *of a vector space V is a subset H of V that obeys the following three properties:*

1. *The zero vector of V is in H*
2. *H is closed under vector addition*
3. *H is closed under scalar multiplication*

Note that if $v_1, \dots, v_p$ are in a vector space V, then $Span v_1, \dots, v_p$ is a subspace of V.

4.2 Null and Column Spaces

Concept 9 (Comparison of Nul A and Col A for $A_{m\times n}$). *The following table may provide some useful insights on the Null Space and Column Space.*

Nul A	
Nul A is a subspace of $\mathbb{R}^n$	
Each vector $\boldsymbol{v}$ in Nul A satisfies $A\boldsymbol{v} = \boldsymbol{0}$.	*Ea*
Implicitly defined. You must take time to find vectors that satisfy a given condition.	*Explici*
$Nul A = \{\boldsymbol{0}\}$ iff the L.T $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is one-to-one.	

Sidenote that when applied to some linear transformation T, the null space of said transformation is also called the kernel of T.

4.3 Bases

Theorem 9 (Spanning Set Theorem). *Let $S = \{v_1, \dots, v_p\}$ be a set in V, and let $H = Span\{v_1, \dots, v_p\}$, then the following statements hold true:*

1. *If some vector, v_k, is a L.C of other vectors in S, then the set formed by removing v_k from S still spans H.*
2. *If $H \neq \{\boldsymbol{0}\}$, then there exists some subset of S that is a basis for H.*

Observation 2 (Finding Bases for Col A and Null A). *The pivot columns of matrix A form a basis of Col A. The columns corresponding to free variables in form a basis of Null A.*

4.4 Coordinate systems

Definition 31 (Coordinates based on Basis)**.** *Given basis for V, $\beta = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n}$, and vector $\boldsymbol{x}$ in V, the coordinates of x relative to the basis β are the weights $c_1, \ldots, c_2$ such that $\boldsymbol{x} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2} + \ldots + c_n\boldsymbol{b_n}$. We also define the coordinate vector to be*

$$[\boldsymbol{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \tag{4.1}$$

and coordinate mapping to be $\boldsymbol{x} \mapsto [\boldsymbol{x}]_\beta$. Note that this is a one-to-one coordinate mapping from V to $\mathbb{R}^n$. This is referred to as an isomorphism.

Also, given the matrix with the set of bases as its column vectors,

$$P_\beta = [\boldsymbol{b}_1 \; \boldsymbol{b}_n] \tag{4.2}$$

we have the vector equation

$$\boldsymbol{x} = P_\beta[\boldsymbol{x}]_\beta \tag{4.3}$$

4.5 Dimension

Definition 32 (Dimension)**.** *If vector space V is spanned by a finite set (where V is referred to as finite-dimensional), the* ***dimension*** *of V, dim V, is the number of vectors in the basis of V. If V cannot be spanned by a finite set, then V is infinite-dimensional.*

Note that any set in V with more than dim V vectors must be linearly dependent. Also, every basis in V must contain exactly dim V vectors.

Theorem 10. *Let H be a subspace of finite dimensional vector space V. We can build a basis for H from any linearly independent set in H by adding in more linearly independent vectors. We also know that*

$$dimH \leq dimV \tag{4.4}$$

Theorem 11 (Basis Theorem)**.** *Given p-dimensional vector space V, any set of p vectors that is linearly independent or spanning V is a basis for V.*

Definition 33 (Row Space)**.** *The* ***Row Space*** *of a matrix is the span of its row vectors.*

Definition 34 (Rank)**.** $\boldsymbol{rank} = dim(Col(A))$

Theorem 12 (Rank Theorem)**.** *Given* $A\boldsymbol{x} = \boldsymbol{b}Amn$

$$rankA + dimNullA = n \tag{4.5}$$

4.6 Change of Basis

Theorem 13. *Let* $\beta = [b_1, \ldots, b_n$ *and* $C = [c_1, \ldots, c_n$ *be bases of a vector space V. Then there exists a unique* $n/ctimesn$ *matrix* $\underset{C \leftarrow B}{P}$*, such that*

$$[\boldsymbol{x}]_C = \underset{C \leftarrow B}{P}[\boldsymbol{x}]_\beta \tag{4.6}$$

We know that

$$\underset{C \leftarrow B}{P} = \left[[\boldsymbol{b_1}]_C \; [\boldsymbol{b_2}]_C \ldots [\boldsymbol{b_n}]_C\right] \tag{4.7}$$

Also note that $(\underset{C \leftarrow B}{P})^{-1} = \underset{B \leftarrow C}{P}$

Chapter 5

Determinants

5.1 Intro to Determinants

Definition 35 (Recursive definition of Determinant)**.** *The **determinant** of matrix $A_{n\times n}$, where A_{ij} refers to the matrix A without row i and column j, is*

$$detA = a_{11}detA_{11} - a_{12}detA_{12} + \cdots + (-1)^{1+n}detA_{1n} = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}detA_{1j} \tag{5.1}$$

If we define the (i, j)-cofactor of A (where i, j refers to the row, column) as the number

$$C_{ij} = (-1)^{i+j}detA_{ij} \tag{5.2}$$

we can also write the determinant of $A_{n\times n}$ as the cofactor expansion across the ith row or down the jth column.

$$detA = a_{i1}C_{i1} + a_{i2} + \cdots + a_{in}C_{in} \tag{5.3}$$

and

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{5.4}$$

5.2 Properties of Determinants

Property 7 (Row Operations). *For matrix $A_{n\times n}$, we can perform the following row operations to obtain matrix $B_{n\times n}$ s.t*

1. *Adding a multiple of one row to another row gives us $detB = detA$*
2. *Interchanging two rows gives us $detB = -detA$*
3. *Multiplying one row by a factor of k gives us $detB = k \cdot detA$*

Procedure 5 (Row Reduction for Determinant). *If we row reduce matrix $A_{n\times n}$ to row echelon form $\boldsymbol{U}$ using* r *and a total of* K *scaling factors, we know that*

$$detA = \begin{cases} (-1)^r \cdot K \cdot \prod(pivots\ in\ U) & \text{if } A \text{ is invertible (triangular)} \\ 0 & \text{if } A \text{ is not invertible} \end{cases} \tag{5.5}$$

Property 8 (More Properties). *Given matrix $A_{n\times n}$ and $B_{n\times n}$, we know that*

1. $detA^T = detA$
2. $detAB = (detA)(detB)$

Observation 3. *Interesting side note. If we frame det A as a function of the n column vectors of A, by holding the other columns fixed, we can define another function on some ith column x such that the function is linear.*

5.3 Cramer's Rule

Theorem 14 (Cramer's Rule). *Given matrix $A_{n\times n}$ and any $\boldsymbol{b}$ in $\mathbb{R}^n$, let $A_i(\boldsymbol{b}) = [\boldsymbol{a}_1 \ldots \boldsymbol{b} \ldots \boldsymbol{a}_n]$ where $\boldsymbol{b}$ is located in column i. Then, Cramer's Rule states that the solution $\boldsymbol{x}$ in $A\boldsymbol{x} = \boldsymbol{b}$ has entries given by*

$$x_i = \frac{detA_i(\boldsymbol{b})}{detA} \text{for } i \text{ in } \{1,\ 2,\ \ldots,\ n\} \tag{5.6}$$

Theorem 15 (Inverse Formula from Determinants)**.** *Given matrix $A_{n\times n}$, let $C_{n\times n}$ be its cofactor matrix and the (j, i)th cofactor be C_{ji}. Let the adjugate of A be*

$$adj A = C^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{21} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} \tag{5.7}$$

then we know the inverse of A is given by

$$A^{-1} = \frac{1}{det A} adj A \tag{5.8}$$

Theorem 16 (Geometric Interpretations)**.** *If A is a 2×2 matrix, then the area of the parallelogram determined by its column vectors is det A. If A is a 3×3 matrix, then the volume of the parallelepiped determined by its column vectors is det A.*

If we define a linear transformation, T, using A as the standard matrix:

1. If A is a 2×2 matrix and S is a parallelogram in $\mathbb{R}^2$ then
   - area of T(S) = $det A \cdot$ area of S
2. If A is a 3×3 matrix and S is a parallelepiped in $\mathbb{R}^3$ then
   - volume of T(S) = $det A \cdot$ volume of S