Введение

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Стохастическая математика:
стат. модели сл. проц. → УЧП → ЧМ → Проф.
+ экон. и фин. понимание

Asset Classes (of Tradable Assets)
(классы активов) можно продать

① Equity (акции)

public Equity — F, GM — ~ 40 трлн. долл.

private Equity ← более рисковано

огромная методологич. разница

"алхимия" — попытки предсказ. обычно почти не подкреп. (ML немного решает)

"наука" — более предсказ.

② IRP (Interest Rate Products) — инстр. проц. ставок
FI (Fixed Income)

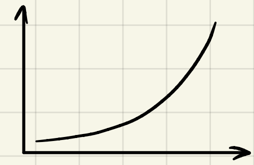
→ Bonds (облигации) (корпор., муницы, ...)
Chapter 11 — мягкая форма банкр. в Америке, компания может функц. дальше

→ LIBOR Futures — доллар за пред. США

→ SWAPs — фин. контр. между участн., долгосрочный, одна стор. платит фикс. % платежи, другая — плав. ставка
BIS ~ 200 трлн. долл.

Проц. ставки — не фин. инструмент;
экв. дисконтирования
откуда берутся?
Деньги — платежи, стоимость, накопление

$\exists x(t)$ — все реал. (не фин.) активы. $\dot{x}(t) \sim x(t)$



$\dot{x}(t) = \alpha x(t) \Rightarrow \alpha = const,\ x = x_0 e^{\alpha t}$
$\Rightarrow \alpha = \alpha(t),\ x = x_0 e^{\int_{t_0}^{t}\alpha(s)ds}$

=> должен расти и денежный объём для условой стаб.
=> появл. проц. ставки, т.е. для соотв. эксп. роста

$\bar{\alpha} = 0{,}04 \Rightarrow$ проц. ставка должна раст. ~ на $\bar{\alpha}$

На SWAPs сущ. очень большие предст. риски.

$10^{-2} = 1pp$ persent point
$10^{-4} = 1bp$ basis point

③ Currencies (FX)
④ Commodities

XAU XPT XAG XPD — драг. металлы

USD CHF
RUB JPY
GBP EUR

(2)-(4) — FICC

⑤ Credit Derivatives (CDO) (кредитный риск)

⑥* Digital Assets BTC ~ XBT

USD/RUB
A B

USD: $r_{USD} = 0.25\%$
$r_{RUB} = 4.25\%$
$X_t = (USD/RUB)_t \approx 75$
X_T $\bar{X}_T$

$t = 0$

$$\begin{array}{ccc} 1\,USD & \rightsquigarrow & 1 \cdot e^{r_{USD} \cdot T} \quad \longleftarrow \quad 1 \cdot e^{r_{USD}\cdot T}\,\bar{X}_T \\ \downarrow & & \downarrow \\ X_0\,RUB & \rightsquigarrow & X_0 e^{r_{RUB}\cdot T} \end{array}$$

// принцип отсутств. арбитража

$$X^T = X_0 e^{(r_{RUB} - r_{USD})T}$$

(1.1) - (1.2)

Ω 2^Ω - мн-во всех подмн-в
$\Sigma \subseteq 2^\Omega$

- $\varnothing \in \Sigma$, $\Omega \in \Sigma$
- $A \in \Sigma \Rightarrow \bar{A} \in \Sigma$
- $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, если $\forall A_n \in \Sigma$

$\mathbb{P}(A) \in [0,1]$ - мера на Σ, $\mathbb{P}(\Omega) = 1$

$\mathbb{P}\{\bigcup_{n=1}^{\infty} A_n\} = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$,
если $A_i \cap A_j = \varnothing$, $i \neq j$

$\mathbb{P}(\varnothing) = x \Rightarrow \mathbb{P}\{\bigcup_{n=1}^{\infty} \varnothing\} = \sum_{n=1}^{\infty} x$, $0 \le x \le 1$
$\Rightarrow x = 0$

$X : \Omega \to \mathbb{R}$ $X \in B$ B - борел. $\left\{ \omega : X(\omega) \in B \atop \omega \subset \Omega \right\} \in \Sigma \quad \forall B$

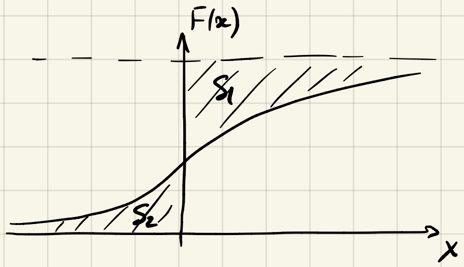
X измер. относ. Σ

мом. врем.
$s < t \Rightarrow \Sigma_s \subset \Sigma_t$

$\mathbb{E}X = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$

$\le$ - англ. версия

$F(x) = \mathrm{Prob}\{X \le x\}$



$\mathbb{E}[X] = S_1 - S_2 =$
$= \int_0^{+\infty} (1 - F(x))\,dx - \int_{-\infty}^{0} F(x)\,dx$

$A \in \Sigma$ $\mathbb{P}(A) \longrightarrow \tilde{\mathbb{P}}(A)$

$Z(\omega) \ge 0$
$\mathbb{E}Z = 1$ мера
$= \int_A \underbrace{Z(\omega)\,d\mathbb{P}(\omega)}_{\ge 0 \text{ almost surely}}$

по т. Радона - Никодима эта ф-ла для нов. меры явл. достат. и необх.

$X \ge 0$ almost surely
$\tilde{\mathbb{E}}[X] = \mathbb{E}[\underbrace{ZX}_{Y}]$

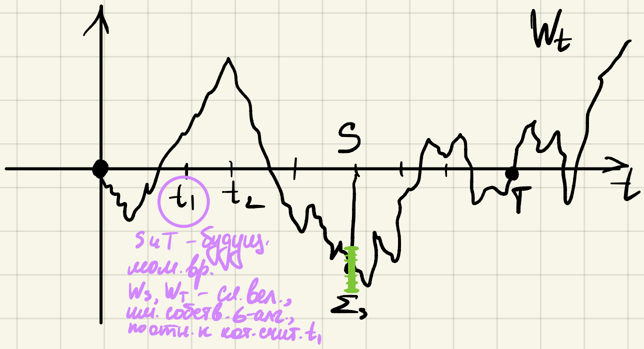
$Y \ge 0$ $\mathbb{E}[Y] = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right]$

$X \sim \mathcal{N}(0,1)$ $f(x) = dF(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
$Y = X + \theta \sim \mathcal{N}(\theta, 1)$

$Z(\underset{x}{\omega}) = e^{-X(\omega)\cdot\theta - \frac{\theta^2}{2}}$ $\mathbb{E}[Z] = \int_\Omega Z(\omega)\,dF(\omega) = \int_{-\infty}^{+\infty} e^{-\theta x - \frac{\theta^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2 + 2x\theta + \theta^2)} dx = 1$

$\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY] = \int_{-\infty}^{+\infty} (x+\theta) e^{-x\theta - \frac{\theta^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+\theta) e^{-\left(\frac{x+\theta}{2}\right)^2} dx = 0$

$X \qquad \Sigma \qquad Y = X|\Sigma$



Y : Σ-измерима. measurable

$\forall A \in \Sigma$

$$\int_A Y\,d\mathbb{P} = \int_A X\,d\mathbb{P}$$

условное м.о. вел. X

Усл. м.о. – использование промежуточных результатов.

$\forall t \quad \mathbb{E}[W_t] = 0 = \mathbb{E}[W_t / W_0 = 0]$

$t > S \quad \mathbb{E}[W_t | \Sigma_s] = W_s$ – измер. – сл. вел.

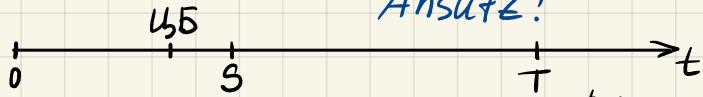
$T > S \quad \mathbb{E}[W_t | \Sigma_T] = W_T$

свойство именно броуновского движения

Обычное м.о. – фикс. величина

Условное м.о. – случ. величина

Ansatz!



$N = \$1000$

$V_0 = N \cdot D(0,T)$

коэф. дисконтирования

Посчитать present value N

$$D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_t\,dt}\right]$$

сл. вел.

пытаемся уменьшить недетерм.

„Переоценённый товар покупать глупо, поэтому надо его продавать"

$V_S = N \cdot D(S,T)$

$$D(S,T) = \mathbb{E}\left[e^{-\int_S^T r_t\,dt}\right] = \mathbb{E}\left[e^{-\int_S^T r_t\,dt} / \Sigma_s\right] = f(r_S, S)$$

мы признали, что для мом. S инф. дана

$\Sigma_S \subseteq \Sigma_t \subset \Sigma_T$

$$\mathbb{E}\left[\mathbb{E}[X_T | \Sigma_t] \,|\, \Sigma_S\right] = \mathbb{E}[X_T | \Sigma_S]$$

Tower rule (Rule of iterated condition)

$X, \varphi(X)$

$$\varphi(\mathbb{E}[X]) \le \mathbb{E}[\varphi(X)]$$

Option

$\dfrac{t=0}{S_0} \qquad \dfrac{t=T}{S_T}$

$S_T > K$
$\quad S_T - K$
$S_T \le K$
$\quad 0$

European Call Option

Underwriter $\quad$ Holder

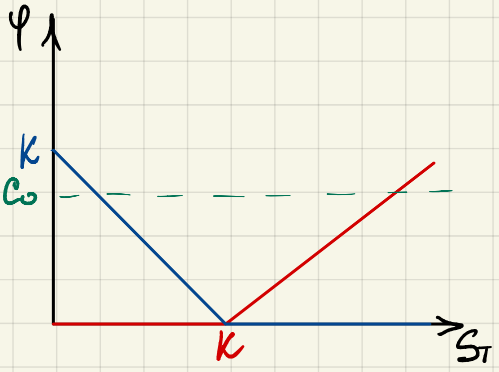
Right $\xleftarrow{\quad K \quad}$

Buy

$$\varphi(S_T) = \max(S_T - K, 0) = (S_T - K)^+$$

ф-ция платежа

Contingent claim



Eur. Call : $\varphi(S_T) = \max(S_T - K, 0) = (K - S_T)^+$

Eur. Put : $\varphi(S_T) = \max(K - S_T, 0) = (K - S_T)^+$

$$C_0 = D(0,T)\mathbb{E}[\varphi(S_T)] + \varepsilon_\varphi \geq D(0,T)\varphi(\mathbb{E}[S_T]) + \varepsilon_\varphi$$

$t_i = \tau \cdot i, \ i = 0, 1, \dots$

$x_i = x(t_i)$

$x_0 = 0$ с вер. 1

$$x_{i+1} = x_i + \underbrace{\Delta_{i+1}}_{\sim N(0,\sqrt{\tau})}$$

$\forall i \neq j \ \Delta_i, \Delta_j$ – independent

$$\underset{x(t_n)}{x_n} = \sum_{i=1}^{n} \Delta_i$$

Random Walk (случайное блуждание)

$$W_t = \lim_{\tau \to 0} x(t)$$

$$\mathbb{E}[X_n] = \sum \mathbb{E}[\Delta_i] = 0$$

$$Var[X_n] = \sum_{i=1}^{n} \underbrace{Var[\Delta_i]}_{\sigma^2} = n\tau = t_n$$

$$W_t \sim N(0, \sqrt{t})$$

$\Delta W_{ij} = W_{t_j} - W_{t_i}$

$\Delta W_{kl} = W_{t_l} - W_{t_k}$

independent if $[t_i, t_j] \cap [t_k, t_l] = \varnothing$

s: W_s, $t > s$

$\mathbb{E}[W_t | \Sigma_s]$ – ?

$W_t = W_s + (W_t - W_s)$

$$\mathbb{E}[W_t | \Sigma_s] = \overset{\mathbb{E}[W_s|\Sigma_s]}{W_s} + \cancel{\mathbb{E}[W_t - W_s | \Sigma_s]}^{0}$$

$s < t$

$$Corr(W_s, W_t) = Corr(W_s, W_s + \Delta W_{st}) = \frac{Cov(W_s, W_s + \Delta W_{st})}{StdDev(W_s)\, StdDev(W_s + \Delta W_{st})} =$$

$$= \frac{\mathbb{E}[W_s(W_s + \Delta W_{st})]}{\sqrt{st}} = \frac{\mathbb{E}[W_s^2 + W_s \Delta W_{st}]}{\sqrt{st}} = \frac{\overset{\sqrt{s}}{s} + \cancel{\mathbb{E}[W_s]}^{0}\, \overset{\sqrt{t}}{\cancel{\mathbb{E}[\Delta W_{st}]}^{0}}}{\sqrt{ts}} = \sqrt{\frac{s}{t}} < 1$$

$$X_t = \mu t + \sigma W_t \sim \underbrace{N(\mu t, \sigma\sqrt{t})}$$

работает от 5 мин до часа (short term)

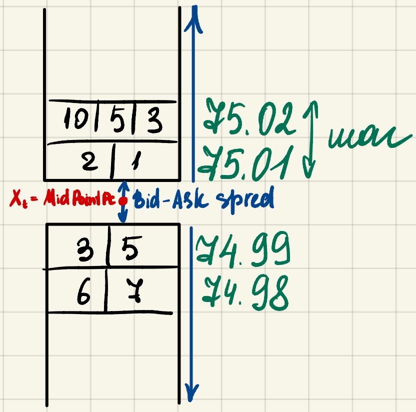
довольно тривиально для дист. врем.

USD/RUB
Order Book

Asks (Orders)

Bids

1 лот = $1000



$$VWAP(Ask, 10) = \frac{3}{10} \cdot 75{,}01 + \frac{7}{10} \cdot 75{,}02$$
$$VWAP(Bid, 10) = \frac{8}{10} \cdot 74{,}99 + \frac{2}{10} \cdot 74{,}98$$
Mid Point

Агрессор - участник, кот. хочет купить/продать прямо сейчас (структуру видно)

(1.3.) - (1.4.)

$X_t = \mu t + \sigma W_t$ $\quad$ $\mu, \sigma = const$, $\sigma > 0$

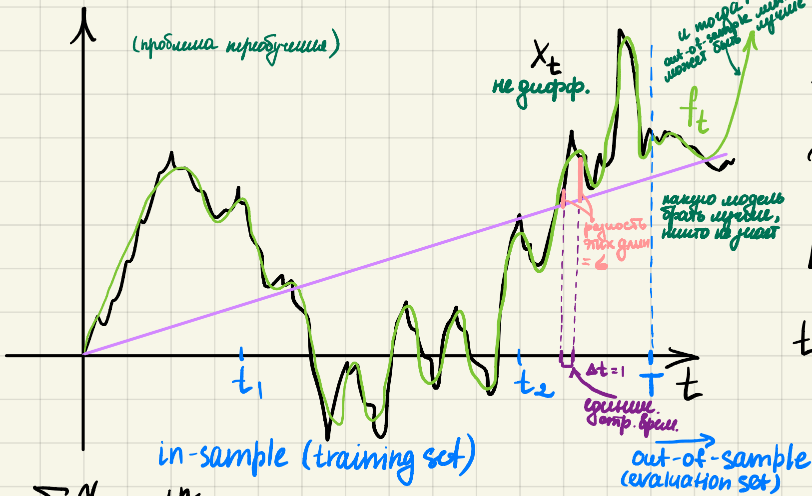
$= 0$ - не случ., < 0 - то же, что и > 0 по сути

Броуновское движение

μ: Trend / Drift
σ: Volatility

„All models are wrong, but some models are useful"

W_t: Brownian motion $\Rightarrow -W_t$: also (by symmetry)



f_t : кусочно-дифф.

$R_t = X_t - f_t$

$$\mathbb{E}\left[\int_{t_1}^{t_2} R_t \, dt\right] = 0$$

$t_1 < t_2$ - random
распределение любое
пусть будет равномерным

$f_t = \sum_{n=0}^{N} a_n t^n$
$\int_0^T (X_t - f_t)^2 dt \longrightarrow \min$

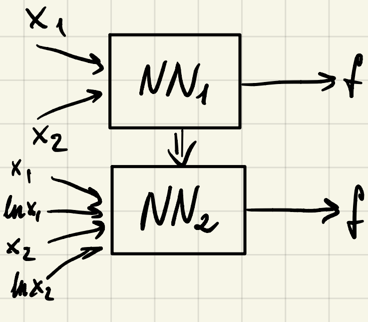
Феноменологический или непараметрический "мир" (ML)

$f_t = \mu t \Rightarrow X_t - \mu t = \sigma W_t$

$$\mathbb{E}\left[\sigma \int_{t_1}^{t_2} W_t \, dt\right] = \sigma \lim_{\tau \to 0} \mathbb{E}\left[\sum_{n=1}^{\frac{t_2 - t_1}{\tau}} W_n \cdot \tau\right] = 0$$

t_1 и t_2 уже фикс., W_t - сл. вел.

„Мир" параметрич. стохастич. моделей



$f(x_1, x_2)$ - ∞-diff.

$f = \frac{x_1}{x_2}$!

$dX_t = X_{t+dt} - X_t$ $\qquad dt \longrightarrow 0$

$dW_t = W_{t+dt} - W_t \sim N(0, \sqrt{dt})$

$$\boxed{dX_t = \underbrace{\mu dt}_{O(dt)} + \underbrace{\sigma dW_t}_{O(\sqrt{dt})}}$$

$$\mathbb{E}[|dW_t|] = \sqrt{\frac{2}{\pi}}\sqrt{dt}$$

$$\mathbb{E}[|X|] = \int_{-\infty}^{+\infty} |x|\varphi(x)dx = \int_{-\infty}^{0}(-x)\varphi(x)dx + \int_{0}^{+\infty} x\varphi(x)dx = \{y=-x\} =$$

$X \sim N(0,1)$, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$= \int_{+\infty}^{0} y\,\varphi(-y)(-dy) + \int_{0}^{+\infty} x\varphi(x)dx =$$

$$= 2\int_{0}^{+\infty} x\varphi(x)dx = \frac{2}{\sqrt{2\pi}}\int_{0}^{+\infty} x e^{-\frac{x^2}{2}}dx = \sqrt{\frac{2}{\pi}} = 0.8.$$

$$dX_t \cong \sigma dW_t$$

$$|dX_t| = |\mu| dt + \sigma|dW_t|$$

$$\mathbb{E}[|dX_t|] \leq |\mu|dt + \sigma\sqrt{\frac{2}{\pi}dt} = O(dt^{\frac{1}{2}})$$

Geometric Brownian Motion

GBM:
$$dX_t = \underset{\substack{\tilde\mu X_0 \\ \| \\ \mu}}{\tilde\mu} X_t dt + \underset{\substack{\tilde\sigma X_0 \\ \| \\ \sigma}}{\tilde\sigma} X_t dW_t$$

X_t [RUB] $\qquad \mu$ [RUB/y]

t [y]

W_t [$y^{1/2}$] $\qquad \sigma$ [RUB/$y^{1/2}$]

X_t = USD/RUB

$X_0 \cong 75$, $\tilde\sigma = 0.15$] $\sigma = 11.25$

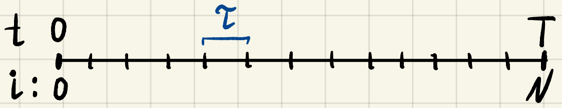
$\tilde\mu = 0.0425 - 0.0025 = 0.04$] $\mu = 3$

Обычно $\tilde\sigma > \tilde\mu$

$T = 1$ h (hour)

X_i, $i = \overline{0,N}$

$\tau = \frac{T}{N}$



X_t – не стац. процесс

ΔX_t : $\Delta X_i = X_i - X_{i-1}$, $i = 1, \dots, N$

$\Delta X_i = \mu(\underbrace{t_i - t_{i-1}}_{\tau}) + \sigma \Delta W_i$

$\Delta X_i \sim N(\mu\tau, \sigma\sqrt{\tau})$

ΔX_i : н.о.р.с.в.

$$\hat\mu = \frac{1}{\tau}\mathbb{E}[\Delta X] = \frac{1}{N\tau}\sum_{i=1}^{N}\Delta X_i = \frac{1}{T}\sum_{i=1}^{N}\Delta X_i = \frac{1}{T}(X_T - X_0)$$

$X_0 = 0$. $\quad \hat\mu = \frac{X_T}{T}$ – несмещ. оценка

$\mathbb{E}[\hat\mu] = \mu$

$$Var[\hat\mu] = \mathbb{E}[\hat\mu^2] - (\mathbb{E}[\hat\mu])^2 = \mathbb{E}[\hat\mu^2] - \mu^2 = \frac{1}{T^2}\mathbb{E}[X_T^2] - \mu^2 \;\circledcirc$$

$X_T^2 = \mu^2 T^2 - 2\mu\sigma T W_T + \sigma^2 W_T^2$

$\mathbb{E}[X_T^2] = \mu^2 T^2 + 0 + \sigma^2 T$

$$\circledcirc\; \frac{\mu^2 T^2 + \sigma^2 T}{T^2} - \mu^2 = \frac{\sigma^2}{T} \neq f(N)!$$

$\Delta X \sim N(\mu\tau, \sigma\sqrt{\tau})$ $\qquad \frac{X_T}{N} = \overline{\Delta X}$

$\hat{\sigma}^2\tau = Var[\Delta X] = \frac{1}{N-1}\sum_{i=1}^{N}\left(\Delta X_i - \frac{X_T}{N}\right)^2$ — верхняя несмещ. оценка

$\mathbb{E}[\hat{\sigma}^2] = \sigma^2$

$Var[\hat{\sigma}^2] = \{\text{TO DO}\} = O\left(\frac{1}{N}\right) \sim$ large Var

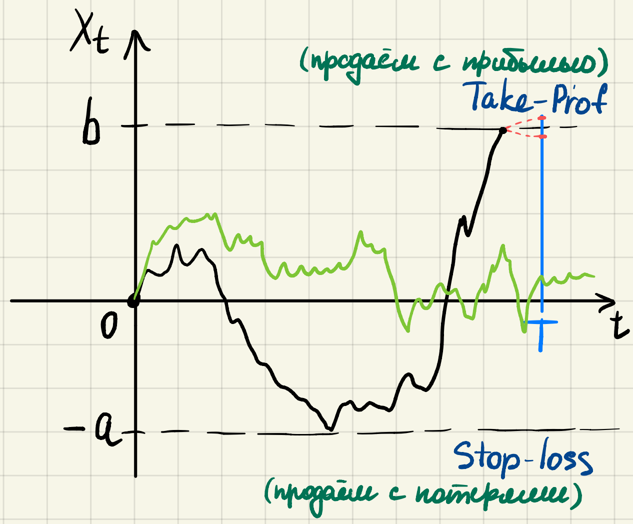
$\boxed{\hat{\sigma}^2 = \frac{1}{\underbrace{N\tau}_{\to T}}\sum_{i=1}^{N}\Delta X_i^2}$ — смещ. оценка, асимптот. смещ. $\to 0$

Realized Variance $\qquad$ smaller Var

$\mu = 0 \qquad X_t = \sigma W_t$ $\qquad$ мартингал

$\mathbb{E}_{s<t}[X_t | \Sigma_s] = X_s = \sigma W_s$

$\mathbb{E}[X_t] = \mathbb{E}[X_t | \Sigma_t] = 0$



First time:
Event t^*: $X_{t^*} = b$ or $X_{t^*} = -a$
$\exists\, t^*$ almost surely

$p = \mathbb{P}[X_{t^*} = b] \implies 1 - p = \mathbb{P}[X_{t^*} = -a]$

$0 = \mathbb{E}[X_{t^*}] = \{at\ t^*\} = pb - a(1-p) = p(a+b) - a \implies$ бесполезная стратегия

$p = \frac{a}{a+b}$

$\mathcal{B}: \exists\, 0 < t^* \leq T : X_{t^*} = b$ $\qquad$ используем принцип симметрии (метод зеркальности)

$p = \mathbb{P}[\mathcal{B}]$

$\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du.$ $\qquad X_t = \sigma W_t \sim N(0, \sigma\sqrt{t})$

$\mathbb{P}[X_T \leq x] = \mathbb{P}\Big[\underbrace{\frac{X_T}{\sigma\sqrt{T}}}_{\sim N(0,1)} \leq \frac{x}{\sigma\sqrt{T}}\Big] = \Phi\left(\frac{x}{\sigma\sqrt{T}}\right)$

$\mathbb{P}[X_T \leq b] = \Phi\left(\frac{b}{\sigma\sqrt{T}}\right) = \underbrace{\cancel{\mathbb{P}[X_T \leq b | \mathcal{B}]}}_{0.5\ \text{(зерк-ть)}}p + \underbrace{\cancel{\mathbb{P}[X_T \leq b | \overline{\mathcal{B}}]}}_{1}(1-p) = \frac{1}{2}p + 1 - p = 1 - \frac{p}{2}$

$\underbrace{p}_{0 \leq \ \leq 1} = 2\Big[1 - \underbrace{\Phi\left(\frac{b}{\sigma\sqrt{T}}\right)}_{0,5 \leq \ \leq 1}\Big]$

$PnL = \begin{cases} b, & \mathbb{P} = p \\ X_T, & \mathbb{P} = 1 - p \end{cases}$
Profit & Loss

$\mathbb{E}[PnL] = bp + (1-p)\,\mathbb{E}[X_T | \overline{\mathcal{B}}]$

подумать, как получить
$\overline{\mathcal{B}} : \max_{0 \leq t \leq T} X_t < b$

Как подобрать b и T, чтобы макс. PnL

$$dX_t = \mu dt + \sigma dW_t \Rightarrow X_t = \mu t + \sigma W_t$$

$$dX_t = \mu_t dt + \underset{\uparrow \text{determ.}}{\sigma_t} W_t \Rightarrow X_t = \int_0^t \mu_s ds + ?$$

$$\int_0^t \sigma_s dW_s \quad ?$$

Ito Integral

похоже на инт. Стилт.
но в данном инт. ф-ция, по кот. инт-ем, не явл. детермин.

$$\int_0^t \sigma_s dW_s = \lim_{\substack{\delta \to 0 \\ N \to \infty}} \sum_{i=1}^N \sigma(t_{i-1})(W_{t_i} - W_{t_{i-1}}) = I_t$$

Also: σ_t - Random
measurable Σ_t
dW_t independent of σ_t

$$\mathbb{E}[I_t] = 0$$

$$Var[I_t] = \lim \Big(\sum_{i=1}^N \underbrace{\mathbb{E}[\sigma_i^2 \Delta W_i^2]}_{\mathbb{E}[\sigma_i^2]\Delta t_i} + \sum_{\substack{i,j=1 \\ i \neq j}}^N \mathbb{E}[\sigma_{i-1}\sigma_{j-1}\Delta W_i \Delta W_j]\Big) = \lim \sum_{i=1}^N \mathbb{E}[\sigma_i^2]\Delta t_i = \int_0^t \mathbb{E}[\sigma_s^2] ds$$

$\underbrace{\sigma_{i-1}\Delta W_i}_{\text{indep.}}$ $\underbrace{\sigma_{j-1}\Delta W_j}_{\text{indep.}}$

$$\mathbb{E}[\sigma_{i-1}\sigma_{j-1}\Delta W_i]\cancel{\mathbb{E}[\Delta W_j]} \to 0$$

σ_t - determ: $$\mathbb{E}[I_t] = \lim_{\substack{\delta \to 0 \\ N \to \infty}} \sum_{i=1}^N \sigma(t_{i-1}) \cancel{\mathbb{E}[\Delta W_i]}^{\to 0} = 0$$

$$Var[I_t] = \lim_{\substack{\delta \to 0 \\ N \to \infty}} \sum_{i=1}^N Var[\sigma(t_{i-1})\Delta W_i] = \lim_{\substack{\delta \to 0 \\ N \to \infty}} \sum_{i=1}^N \sigma^2(t_{i-1})(t_i - t_{i-1})$$

$$Var[\sigma(t_{i-1})\Delta W_i] = \sigma^2(t_{i-1}) Var[\Delta W_i] = \sigma^2(t_{i-1})(t_i - t_{i-1})$$

$$\lim_{\substack{\delta \to 0 \\ N \to \infty}} \sum_{i=1}^N \sigma^2(t_{i-1})(t_i - t_{i-1}) = \int_0^t \sigma^2(s) ds$$

Интеграл Ито является мартингалом

$$\mathbb{E}[I_t | \Sigma_s]_{s<t} = \mathbb{E}\Big[\int_0^s \sigma_u dW_u \Big| \Sigma_s\Big] + \cancel{\mathbb{E}\Big[\int_s^t \sigma_u dW_u \Big| \Sigma_s\Big]}^{\to 0} = \int_0^s \sigma_u dW_u = I_s$$

$$X_t = \mu_t dt + \sigma_t dW_t \iff X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

$\mu_t = \mu(X_t, t)$ $\quad \Sigma_t$

$Y_t = f(X_t, t)$

Itô - Döblin
1942 1940

$$dY_t = \frac{\partial f}{\partial x} dX_t + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

$$(dX_t)^2 = \underbrace{\cancel{\mu_t^2 dt^2}}_{\to 0} + \underbrace{\cancel{2\mu_t \sigma_t dt dW_t}}_{o(dt^{3/2}) \to 0} + \sigma_t^2 dW_t^2$$

$$\mathbb{E}[dW_t^2] = dt \qquad Var[dW_t^2] = \mathbb{E}[dW_t^4] - \mathbb{E}[dW_t^2]^2 =$$
$$= 3dt^2 - dt^2 = 2dt^2$$
$$StdDev[dW_t^2] = \sqrt{2}dt$$

$$\sigma_t^2 dt$$

$$\mathbb{E}\Big[\int_0^T dW_s^2\Big] = ? = T$$

$$\lim_{\substack{\tau \to 0 \\ N \to \infty}} \sum_{i=1}^N (W_{t_i} - W_{t_{i-1}})^2$$

$t_i = \tau \cdot i$, $T = \tau \cdot N$

$$Var\Big[\int_0^T dW_s^2\Big] = \mathbb{E}\Big[\Big(\int_0^T dW_s^2\Big)^2\Big] - T^2 =$$
$$= \lim \Big(\sum_{i=1}^N \mathbb{E}[(\Delta W_i^2)^2] + 2\sum_{i=1}^N \sum_{j=i+1}^N \mathbb{E}[\cancel{\Delta W_i^2} \cdot \cancel{\Delta W_j^2}]\Big) - T^2 =$$
$$= \lim \Big(\cancel{3\tau^2 N}^{3\tau T} + 2\tau^2 \frac{N(N-1)}{2}\Big) - T^2 = \lim\big(\tau^2(3N + N^2 - N)\big) - T^2 =$$
$$= \lim\big(2N\tau^2 + \cancel{N^2\tau^2}^{T}\big) - T^2 = \lim_{\substack{\tau \to 0 \\ N \to \infty}} (2T\tau) = 0$$

Rewriting Rule: $dW_t^2 \longrightarrow dt$

$$dY_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dt$$

$$\boxed{dY_t = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x} dW_t}$$

цена акции

Equity (стартовая цена)

$$dS_t = \underbrace{\overset{\sim 0.1}{\tilde{\mu}_t}}_{\text{determ.}} S_t dt + \underbrace{\overset{\sim 0.2}{\tilde{\sigma}_t}}_{\substack{\text{determ.}\\ \sigma_t > 0}} S_t dW_t$$

стохаст. дифф. ур-ние

$S_t \geq 0$ a.s.

$$\frac{dS_t}{S_t} = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t$$

$\cancel{X_t = \ln S_t}$ $\quad X_t = \ln\frac{S_t}{S_0} = f(S_t)$

$X_0 = 0$

$$dX_t = \left(\tilde{\mu}_t \cancel{S_t}\frac{1}{\cancel{S_t}} + \frac{\tilde{\sigma}_t^2 \cancel{S_t^2}}{2}\left(-\frac{1}{\cancel{S_t^2}}\right)\right)dt + \tilde{\sigma}_t \cancel{S_t}\cdot\frac{1}{\cancel{S_t}} dW_t = \left(\tilde{\mu}_t - \frac{\tilde{\sigma}_t^2}{2}\right)dt + \tilde{\sigma}_t dW_t$$

$$X_t = \int_0^t \left(\tilde{\mu}_s - \frac{\tilde{\sigma}_s^2}{2}\right)ds + \int_0^t \tilde{\sigma}_s dW_s$$

$\underline{S_t = S_0 e^{X_t}}$ > 0 a.s.

[illegible] просто

$\sigma_t = \tilde{\sigma}_t S_t$: GBM

$\tilde{\sigma}_t S_t^{\beta},\ 0 < \beta < 1$: CEV

$\sigma_0 + \sigma_1 S_t + \sigma_2 S_t^2$: Lipton Hyperbolic Vol.

не очень хор., т.к. допускает отриц. цену.

должен быть полож. опред. $\forall S$

$\rightarrow \exists \beta > 1$

$$dS_t = \sigma S_t^{\beta} dW_t$$

$\bar{S}_t = \mathbb{E}[S_t]$

$d\bar{S}_t = d\mathbb{E}[S_t] = \mathbb{E}[S_{t+dt}] - \mathbb{E}[S_t] = \mathbb{E}[S_{t+dt} - S_t] = \mathbb{E}[dS_t] = 0$

$\bar{S}_t = const = S_0$

$S_t = S_0 + \sigma\int_0^t S_s^{\beta} dW_s \Rightarrow$ мартингал

$X_t = f(S_t)$: $dX_t = \ldots \sigma dW_t$

$\sigma S_t^{\beta}\frac{\partial f}{\partial S} = \sigma \Rightarrow \frac{\partial f}{\partial S} = S^{-\beta}$

$$\Rightarrow f = \int_{S^*}^{S}\frac{dx}{S^{\beta}} = \frac{x^{1-\beta}}{1-\beta}\Bigg|_{S^*}^{S} = -\frac{1}{\beta-1}\frac{1}{x^{\beta-1}}\Bigg|_{S^*}^{S}$$

$$= -\frac{1}{\beta-1}\left(\frac{1}{S^{\beta-1}} - \frac{1}{S^{*\beta-1}}\right)$$

$S^* > 0$

$+\infty$ просто берём значение

$$dX_t = -\frac{\beta\sigma^2}{2}S_t^{\beta-1}dt + \sigma dW_t$$

$$X_t = f(S_t) = -\frac{1}{\beta-1}\left(\frac{1}{S^{\beta-1}}\right)$$

$$S_t^{\beta-1} = -\frac{1}{(\beta-1)X_t}$$

$\beta > 1$ – не очень хорошо, т.к. мы можем получ. бескон. зн. за кон. время

(1.5) - (1.7)

Dividends:
→ Continuous
→ Relative to curr stock peice

$D_t S_t dt$
dividend rate

если начали платить дивиденты, то рост стоимости акций уменьшается (скорость создания новой стоим. компании μ уменьшилось)

$$dS_t = (\mu_t - D_t) S_t dt + \underbrace{\sigma_t S_t}_{\sigma(S_t, t)} dW_t$$

FX: $P_x(\overset{\text{Asset}}{A} / B) =$ # of B units equiv. by value to 1 unit of A

Base currency Ccy / Quote Ccy
Settlement Ccy

EUR GBP USD RUB JPY
AUD CHF

Asset
AAPL / USD
Instrument

A / B

$D_t = r_A$
$\mu_t = r_B$

$S_t = P_x(A/B)$

$$dS_t = (r_B - r_A) S_t dt + \underbrace{\sigma_t S_t}_{\text{or more complex}} dW_t$$

S_t [RUB]
r_A, r_B [$year^{-1}$]
dt [year]
dW_t [$year^{1/2}$]
$\Rightarrow \sigma_t$ [$year^{-1/2}$]

как и в физике, полезно проверять размерности

$ACT/365 = \dfrac{\Delta t \text{ (days)}}{365}$ Fixed Leg

$ACT/360 = \ldots$ (для упрощ. – 12 мес. по 30 дн.)

Среднее по веку – 365.2425

$ACT/365.25 = \ldots$ 365.25 – $\frac{3}{400}$

С учётом тропич. года: 365.2423..

рабочие дни США

$ACT/352 = \ldots$ EQ

$$VOL = 15\% / year^{1/2} = \frac{15\%}{(252\,bd)^{1/2}} \cong \frac{1\%}{bd^{1/2}}$$

business day

Професcии:

Options Vols	QA	Quantitative Analyst
Quant/Stock Equity Portfolio	QR	Quantitative Researcher
Data engineer	RA	Research Analyst

$$dS_t = \overset{QR}{\mu_t} S_t dt + \overset{QA}{\sigma_t} S_t dW_t$$

S_t: commodity price
$S_t > 0$

$$dS_t = \varkappa(\theta - S_t) dt + \sigma_t S_t dW_t$$
$$\sigma_t S_t^{\beta} dW_t$$

Эта модель аналит. решения не имеет. Но явные решения не обязательны. Гораздо важнее научить плотность распр. S_t

могут быть ф-ями времени

$$dS_t = \varkappa(\theta - S_t e^{-\alpha(t - t_0)}) dt + \ldots$$

Откуда брать конст.? – использов. опционной инф., цены commodities

$q = \begin{bmatrix} \varkappa \\ \theta \\ \alpha \\ \sigma \\ \beta \end{bmatrix} \in \mathbb{R}^5$ $\quad t_0 \overset{\tau}{\ldots} t_N = T$ $\quad T = 3m$ $\quad 288 \cdot 65 = 18720 = N$

$S_0 .. S_i .. S_N$ $\quad \tau = 5\,min$ $\quad$ торговые дни

Исполь. мет. заморож. коэф.

$dS_i \approx S_i - S_{i-1}$; $S_i \cong S_{i-1} + \varkappa(\theta - S_{i-1}e^{-\alpha(i-1)\tau})\tau + \sigma S_{i-1}^{\beta}\Delta W_i$, $\Delta W_i \to N(0,\sqrt{\tau})$

Метод макс. правдоподобия

плотность

$p(S_1,t_1|S_0)\Delta$ Марковский процесс

$p(S_2,t_2|\cancel{S_0},S_1)\Delta$ пл-ти не зав. от "истории"

$\vdots$

$p(S_i,t_i|\cancel{S_0},\dots,S_{i-1})\Delta$

$S_{t_i} \sim N(S_{i-1}+\varkappa(\dots)\tau,\ \sigma S_{i-1}^{\beta}\sqrt{\tau})$

$p(S_i,t_i|S_{i-1}) =$

$p(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. $F(S_i)=\Phi\left(\frac{S_i-M}{\sigma}\right)$
$f(S_i)=F'(S_i)=\frac{1}{\sigma}\varphi\left(\frac{S_i-M}{\sigma}\right)$

$= \varphi\left(\frac{S_i-S_{i-1}-\varkappa(\dots)\tau}{\sigma S_{i-1}^{\beta}\sqrt{\tau}}\right)\frac{1}{\sigma S_{i-1}^{\beta}\sqrt{\tau}}$

$L=\prod_{i=1}^{N}\left(p(S_i,t_i|S_{i-1})\Delta\right)=\Delta^N\prod_{i=1}^{N}p(S_i,t_i|S_{i-1})$

$\mathcal{L}=\ln L - N\ln\Delta=\sum_{i=1}^{N}\ln p(S_i,t_i|S_{i-1}) \to \max_{q}$ решение можно получить только численно

$\varkappa>0$

Откалибровали модель $\longmapsto$ значение $\{\Delta W_i\}$ $\longmapsto$ стат. тест на броун. дв.

$\sum_{i=k}^{l}\Delta W_i^2 \to t_l-t_k=\tau(l-k)$

Замораживали коэф. в волат-ти дает большую погр.
Сделаем замену, чтобы эти коэф. не замораживать

$X_t=f(S_t)$

$\frac{df}{dS}=\frac{1}{\sigma S_t^{\beta}}>0$

$dS_t=\tilde{\mu}(S_t,t)dt+\tilde{\sigma}(S_t)dW_t$

$dX_t=\frac{\partial f}{\partial S}dS_t+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\tilde{\sigma}^2=\frac{1}{\sigma S_t^{\beta}}\left(\tilde{\mu}(S_t,t)dt+\sigma S_t^{\beta}dW_t\right)+\frac{1}{2}\frac{1}{\sigma}(-\beta)\frac{1}{S_t^{\beta+1}}\sigma^2 S_t^{2\beta}dt=$

$=\left(\frac{\tilde{\mu}(S_t,t)}{\sigma S_t^{\beta}}-\frac{\sigma\beta}{2}S_t^{\beta-1}\right)dt+1\cdot dW_t$

замен. на 1 за счёт услож. тренда

$\hat{\mu}(X_t)=(\dots)f^{-1}(X_t) \longmapsto$ метод замор. коэф.

$\mathcal{L}=\sum_{i=1}^{N}\ln\hat{p}(X_i,t_i|X_{i-1})\to\max$

$\hat{p}(X_i,t_i|X_{i-1})=\varphi(X_i-X_{i-1}-\hat{\mu}(X_{i-1})\tau)$

$\mathcal{L}=\sum_{i=1}^{N}\ln\left(\frac{\hat{p}(X_i,t_i|X_{i-1})}{\tilde{\sigma}(S_i)}\right)$ правильно!

$t_i=i\tau$	
i	$0\dots i\dots N$
S_i	$S_0\dots S_i\dots S_N$
$X_i(S_i,\bar{q})$	

Но есть ошибка!

$dS_t=\tilde{\mu}(S_t,t;\bar{q})dt+\tilde{\sigma}(S_t;\bar{q})dW_t$

$X_t=f(S_t,\bar{q})$: $\frac{df}{dS}=\tilde{\sigma}(S_t;\bar{q})$

$\Downarrow$

$dX_t=\hat{\mu}(X_t,t;\bar{q})dt+1\cdot dW_t$

мы максимизир. другую ф-цию правдоподобия!

Максим. в новом пространстве знач. не относ. изб. S_i

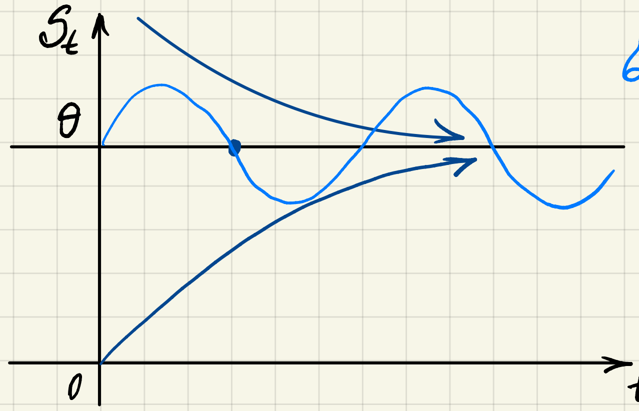
$$\mathbb{P}\{S_{t_i} \in (S_i - \Delta/2, S_i + \Delta/2)\} = \mathbb{P}\{f(S_{t_i}, \bar{q}) \in (f(S_i - \Delta/2, \bar{q}), f(S_i + \Delta/2, \bar{q}))\} =$$
$$= \mathbb{P}\{X_{t_i} \in (f(S_i,\bar{q}) - f'(S_i,\bar{q})\tfrac{\Delta}{2}, f(S_i,\bar{q}) + f'(S_i,\bar{q})\tfrac{\Delta}{2})\} = p(X_i, t_i \mid X_{i-1}) \cdot \underbrace{f'(S_i,\bar{q})}_{\frac{1}{\sigma(S_i,\bar{q})}}\Delta$$

Неправильный интервал

$A < \theta$ $\quad \underset{\text{годовая сезонность}}{\theta + A\sin(2\pi t + \varphi_0)}$ ← ввели эффект сезонности

$$\underline{dS_t = \varkappa(\theta - S_t e^{-\alpha(t-t_0)})\,dt} + \sigma S_t^{\beta}\, dW_t$$

$\sigma \to B\sin(2\pi t + \varphi_0)$, $B < \sigma$



$\bar{S}_t = \mathbb{E}[S_t]$

$$d\bar{S}_t = \varkappa(\theta - \bar{S}_t e^{-\alpha(t-t_0)})\,dt$$

Попробуем решить без стохастич. компоненты dW_t

$$d\bar{S}_t = -\varkappa e^{-\alpha(t-t_0)} \bar{S}_t\, dt \qquad t \geq t_0$$

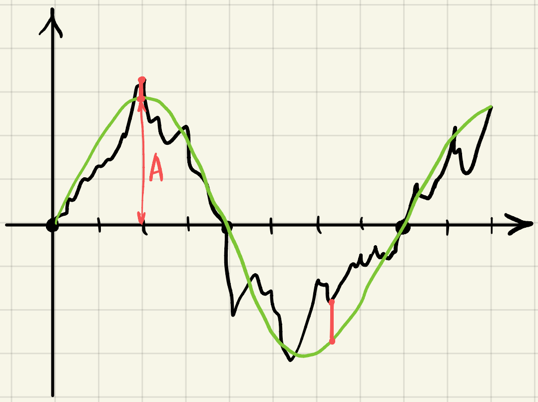
$$\frac{d\bar{S}_t}{\bar{S}_t} = -\varkappa e^{-\alpha(t-t_0)}\, dt \qquad \tau = t - t_0$$

$$\ln \bar{S}_t - \ln C = \frac{\varkappa}{\alpha}(e^{-\alpha\tau} - 1) = -\frac{\varkappa}{\alpha}(1 - e^{-\alpha\tau})$$

$$\bar{S}_t = Ce^{-\frac{\varkappa}{\alpha}(1 - e^{\alpha\tau})} \qquad C = C(\tau) \Rightarrow \bar{S}(\tau)$$

доделать

$$X_t = \underline{A\sin\omega t} + \sigma W_t$$



Ornstein-Uhlenbeck

$$dS_t = \varkappa(\theta - S_t)\,dt + \sigma\, dW_t$$

$p(S_t \mid S_0)$ $\qquad S_t = X_t e^{-\varkappa t}$

$dS_t = -\varkappa S_t\, dt$ $\qquad X_t = S_t e^{\varkappa t} = f(S_t, t)$

$$dX_t = S_t e^{\varkappa t}\varkappa\, dt + e^{\varkappa t} dS_t = e^{\varkappa t}(S_t \varkappa\, dt + \varkappa(\theta - S_t)\,dt +$$
$$+ \sigma\, dW_t = e^{\varkappa t}(\varkappa\theta\, dt + \sigma\, dW_t)$$

$$X_t = \overset{=S_0}{X_0} + \varkappa\theta\frac{1}{\varkappa}\left(e^{\varkappa t}-1\right) + \sigma\int_0^t e^{\varkappa s}\,dW_s$$

$$S_t = S_0e^{-\varkappa t} + \theta\left(1-e^{-\varkappa t}\right) + \underbrace{\sigma\int_0^t e^{\varkappa(t-s)}\,dW_s}_{\text{инт. Ито}} \sim N\left(0,\sigma\sqrt{\frac{1-e^{-2\varkappa t}}{2\varkappa}}\right)$$

$$Var = \sigma^2\int_0^t e^{-2\varkappa(t-s)}\,ds = \left(\frac{\sigma^2}{-2\varkappa}\right)e^{-2\varkappa(t-s)}\Big|_0^t = \frac{\sigma^2}{2\varkappa}\left(1-e^{-2\varkappa t}\right)$$

$$\mathbb{E}[S_t] = S_0e^{-\varkappa t} + \theta\left(1-e^{-\varkappa t}\right)$$

$$\theta = \lim_{t\to+\infty}\mathbb{E}[S_t]$$

$$\theta = \lim_{t\to+\infty}\mathbb{E}[S_t]$$

$$\frac{\sigma^2}{2\varkappa} = \lim_{t\to\infty} Var[S_t]$$

$$S_t \sim N\left(S_0e^{-\varkappa t}+\theta\left(1-e^{-\varkappa t}\right),\ \sigma\sqrt{\frac{1-e^{-2\varkappa t}}{2\varkappa}}\right)$$

$$S_t|S_{i-1} \sim N\left(S_{i-1}e^{-\varkappa(t-t_{i-1})}+\theta\left(1-e^{-\varkappa(t-t_{i-1})}\right),\ \sigma\sqrt{\frac{1-e^{-2\varkappa(t-t_{i-1})}}{2\varkappa}}\right)$$

$$p(S_i|S_{i+1}) = \varphi\left(\frac{S_i-S_{i-1}e^{-\varkappa\tau}-\theta\left(1-e^{-\varkappa\tau}\right)}{\tilde\sigma}\right)$$

$$\tilde\sigma = \sigma\sqrt{\frac{1-e^{-2\varkappa\tau}}{2\varkappa}},\quad \tau = t_i - t_{i-1}$$

$$\mathcal{L} = \sum_{i=1}^N \ln p(S_i|S_{i-1}) \longrightarrow \max_q,\quad q = \begin{bmatrix}\varkappa\\ \theta\\ \sigma\end{bmatrix}$$

$$\varphi(x) = const\cdot e^{-\frac{x^2}{2}}$$

$$\ln\varphi = const - \frac{x^2}{2}$$

$$-\frac{1}{2}\sum_{i=1}^N\left(\frac{S_i-S_{i-1}e^{-\varkappa\tau}-\theta\left(1-e^{-\varkappa\tau}\right)}{\tilde\sigma}\right)^2 - N\ln\tilde\sigma \longrightarrow \max_q$$

$$N\ln\tilde\sigma + \frac{1}{2\tilde\sigma^2}\sum_{i=1}^N\left(S_i-S_{i-1}e^{-\varkappa\tau}-\theta\left(1-e^{-\varkappa\tau}\right)\right)^2 \longrightarrow \min_q$$

$\mathbb{E}[\hat\varkappa] \neq \varkappa \quad \mathbb{E}[\hat\varkappa] \not\xrightarrow[N\to\infty]{} \varkappa$ – смещ. и ас. смещ.

$\mathbb{E}[\hat\theta] = \theta \quad Var[\hat\theta]$: large, slowly $\xrightarrow[N\to\infty]{} 0$

$\mathbb{E}[\hat\sigma^2] = \sigma^2 \quad Var[\hat\sigma^2]$ small $\xrightarrow[N\to\infty]{} 0$

Model: $\vec{q}\in\mathbb{R}^k$

(1) Generate $\{\bar q_j\}$

(2) For each $\vec{q}_j \longrightarrow$ Using Monte-Carlo, generate $\{[S_1^{m,j},\dots,S_L^{m,j}]\}$ M times

(3) Using MLE estimator estimate param. $\{\hat q_{jm}^{est}\}$ M times

(4) Get the means $\bar q_j = \sum_{m=1}^N \hat q_{jm}$

(5) Construct the inverse: $\bar q_j \xrightarrow{NN} q_j$

Neural Set

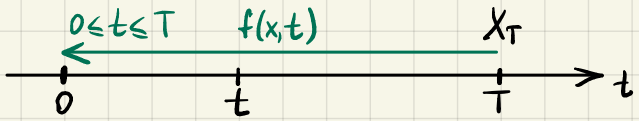
но ещё нет

Feynman-Kac PDE

$$dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t$$

$g(X_T)$

$g(x)\ \forall x \geq 0$



$\mathbb{E}[g(X_T)]$ $\qquad$ $g(X_T)$

Ansatz

$$\mathbb{E}[g(X_T)\,|\,\Sigma_t] = f(X_t,t)$$

$$\mathbb{E}[f(X_T,t)\,|\,\Sigma_s] = \mathbb{E}[\,\mathbb{E}[g(X_T)|\Sigma_t]\,|\,\Sigma_s] = \mathbb{E}[g(X_T)\,|\,\Sigma_s] = f(X_s,s)$$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 dt = \underbrace{\left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)}_{0,\ \text{т.к. } f - \text{март.}}dt + \frac{\partial f}{\partial x}\sigma\,dW_t$$

Martingale!

$f(x,t)$ - ?

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial t} + \underbrace{\mu(x,t)\dfrac{\partial f}{\partial x}}_{\text{convective term}} + \underbrace{\underbrace{\dfrac{\sigma^2(x,t)}{2}}_{>0}\dfrac{\partial^2 f}{\partial x^2}}_{\text{diffusion term}} = 0 \\ f(x,T) = g(x),\ 0 \leq t \leq T \end{cases}$$

ур-ние Феймана-Каца

(== обратное ур-ние Колмогорова)

$f(0,t) = g(0)$ $\quad \leftarrow$ Dirichlet

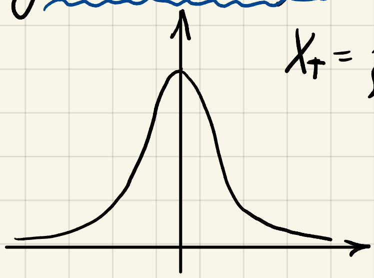
$\lim\limits_{x\to+\infty} f'_x(x,t) = \lim\limits_{x\to+\infty} g'(x)$ $\quad \leftarrow$ Neumann

$$\frac{\partial f}{\partial t} = -L_2 f \qquad L_2 = \mu\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}$$

дельта-ф-ция Дирака

$g(x) = \delta(x-\xi)$

$X_T = \xi$



$$f(x,t) = \frac{1}{\sqrt{2\pi(T-t)}\,\sigma}\, e^{-\frac{(x-\xi)^2}{2(T-t)\sigma^2}}$$

$\mu(x,t) = 0$

$\sigma(x,t) = \sigma = const$

Fokker-Planck

$$\begin{cases} dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t \\ X(0) = X_0 \text{ a.s.} \end{cases}$$

$p(x,t)$: PDF X_t

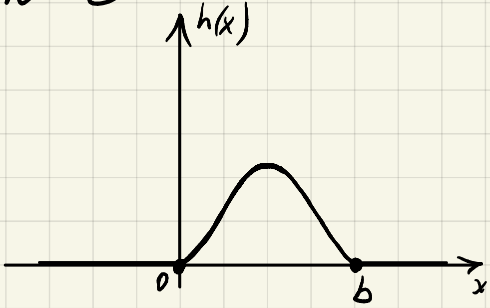
$t = T$

$h(x \leq 0) = 0$

$h(x \geq b) = 0$

$h(0 < x < b) > 0$

$h \in C^2$



$$h(X_T) = ? = \int_0^T dh(X_t) + h(X_0)$$

$$dh(X_T) = h'(X_t)\,dX_t + h''(X_t)/2\,(dX_t)^2 =$$

$$= \underline{h'(X_t)\mu(X_t,t)\,dt + \frac{h''(X_t)}{2}\sigma^2(X_t,t)\,dt} +$$

$$+ h'(X_t)\sigma(X_t,t)\,dW_t$$

$\mathbb{E}[dh(X_t)|\Sigma_t]$

$\hookrightarrow \mathbb{E}[dh(X_t)\,|\,\Sigma_0] = \int_0^b p(x,t)\,dh(X_t)$

$$\mathbb{E}[h(X_T)] = \int_0^T \mathbb{E}[dh(X_t)] + h(X_0)$$
$$\Vert$$
$$\int_0^b p(u,T)\,h(u)\,du$$

$$\Rightarrow \int_0^b p(u,T)\,h(u)\,du = h(x_0) + \int_0^T \left\{ \int_0^b p(x,t)\left[h'(x)\,\mu(x,t) + \frac{h''(x)}{2}\sigma^2(x,t)\right]dx \right\} dt =$$
$$= h(x_0) + \int_0^T (I_1 + I_2)\,dt \;=$$

$$I_1 = \int_0^b p(x,t)\,h'(x)\,\mu(x,t)\,dx = h(x)\,p(x,t)\,\mu(x,t)\Big|_0^b - \int_0^b h(x)\frac{\partial}{\partial x}\big(p(x,t)\,\mu(x,t)\big)dx$$

$$I_2 = \frac{1}{2}\int_0^b p(x,t)\,h''(x)\,\sigma^2(x,t)\,dx = \frac{1}{2}\left[h'(x)\,p(x,t)\,\sigma^2(x,t)\Big|_0^b - \int_0^b h'(x)\frac{\partial}{\partial x}\big(p(x,t)\,\sigma^2(x,t)\big)dx\right] =$$
$$= -\frac{1}{2}\left(h(x)\frac{\partial}{\partial x}(\ldots)\Big|_0^b - \int_0^b h(x)\frac{\partial^2}{\partial x^2}\big(p(x,t)\,\sigma^2(x,t)\big)dx\right)$$

$$= h(x_0) + \int_0^T\int_0^b \left(-h(x)\frac{\partial}{\partial x}(p\mu) + \frac{1}{2}h(x)\frac{\partial^2}{\partial x^2}(p\sigma^2)\right)dx\,dt \;\Big|\; \frac{\partial}{\partial t}$$

$$\int_0^b \frac{\partial p(u,T)}{\partial T}h(u)\,du = \int_0^b \left(-h(x)\frac{\partial}{\partial x}(p\mu) + \frac{1}{2}h(x)\frac{\partial^2}{\partial x^2}(p\sigma^2)\right)dx =$$
$$= \int_0^b h(x)\left\{\frac{\partial p(x,T)}{\partial T} + \frac{\partial}{\partial x}\big(p(x,T)\,\mu(x,T)\big) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(p(x,T)\,\sigma^2(x,T)\big)\right\}dx = 0$$

$T \mapsto t$

$$\begin{cases} \dfrac{\partial p}{\partial t} = -\dfrac{\partial}{\partial x}\big(\mu(x,t)\,p\big) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\big(\sigma^2(x,t)\,p\big) \\ p(x,0) = \delta(x-x_0) \end{cases}$$

ур-ние Фоккера-Планка
(== прямое ур-ние Колмогорова)

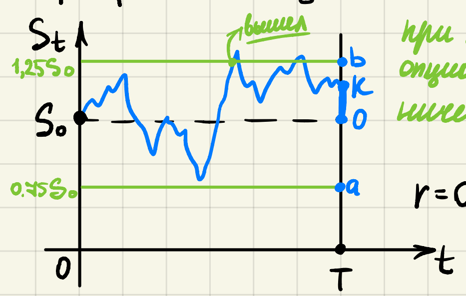
$L_1 p \qquad L_1 = -\frac{\partial}{\partial x}(\mu\,\cdot) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2\,\cdot)$

FP: $\frac{\partial p}{\partial t} = L_1 p$

FK: $\frac{\partial f}{\partial t} = -L_2 f$

$L_1 = L_2^+$!

Барьерный опцион



при пересечении опцион обнуляется и ничего не выплачивается

$r = 0$

из ур. Ф.-П.:

$p(S,T,a,b)\,dS =$

$= \mathbb{P}\{S_T \in (S; S+dS)\ \&\ \max_{0-T} S_t \le b\ \&\ \min_{0-T} S_t \ge a\}$

$B_0 = 0.01K$

$\frac{\text{Payoff}(T) - B_0}{\text{PnL}}$

$F(x) = \mathbb{P}\{\overbrace{\text{Return}}^{\text{относит.}} \le x\} =$

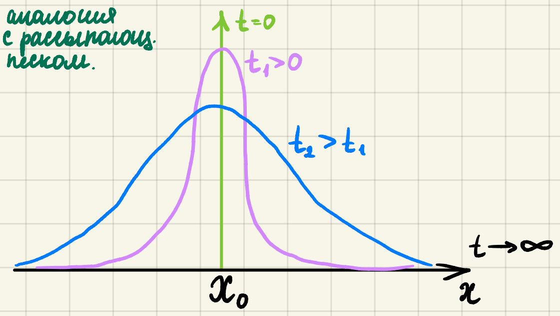
$= \mathbb{P}\left\{\frac{\text{Payoff}(T) - B_0}{B_0} \le x\right\} = \mathbb{P}\left\{\frac{\text{Payoff}(T)}{B_0} - 1 \le x\right\} =$

$= \mathbb{P}\{\overbrace{\text{Payoff}(T)}^{(S_T-K)^+} \le \overbrace{(1+x)B_0}^{y}\} = \mathbb{P}\{(S_T-K)^+ \le y\} =$

$\Rightarrow \begin{cases} (1)\ \ y = 0 & \text{потеряли всё (knock-out)}\ \ x = -1 \\ (2)\ \ y > 0 & \end{cases}$

(1.8) – (1.10)

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\mu p) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 p), \quad p(x,0) = \delta(x - x_0)$$



$$\mu(x,t) = 0$$
$$\sigma(x,t) = 1$$
$$dX_t = dW_t$$
$$X_t = X_0 + W_t$$

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x_0)^2}{2t}} \xrightarrow[t\to 0]{} \delta(x - x_0)$$

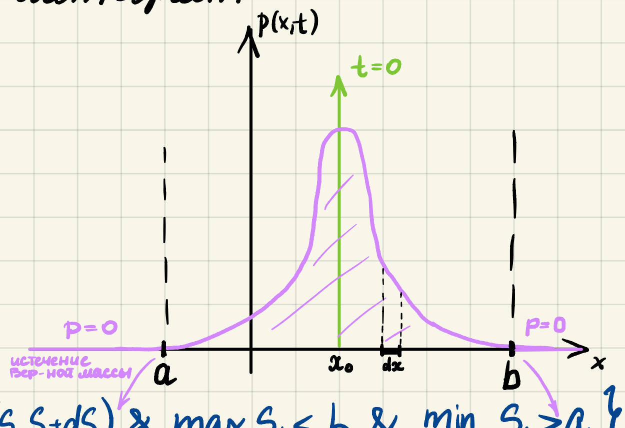
$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}$$ – удовлетворяет

$$\forall t > 0 \int_{-\infty}^{+\infty} p(x,t)\,dx = 1$$

$$Y_t = f(X_t) \qquad \frac{\partial f}{\partial t} = \frac{1}{\sigma}$$

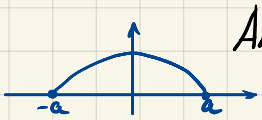
$$dY_t = \tilde{\mu}(Y_t, t)dt + 1 \cdot dW_t \overset{\text{Girsanov Theorem}}{\Longrightarrow} dY_t = d\tilde{W}_t$$

$$p(a,t) = p(b,t) = 0 \qquad a \le x \le b \qquad \text{Dirichlet}$$
$$p(x,0) = f(x - x_0)$$



$$\Rightarrow p(S,T,a,b)dS = \mathbb{P}\{S_T \in (S, S+dS) \;\&\; \max_{t\in[0,T]} S_t \le b \;\&\; \min_{t\in[0,T]} S_t \ge a\}$$

$$X_t = W_t \qquad p(x,0) = \delta(x)$$
$$X_0 = 0 \qquad p(\pm a, t) = 0$$



Ansatz

$$p(x,t) = p_1(x)\,p_2(t)$$ сепарабельное решение

$$\mu = 0 \qquad \frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}$$
$$\sigma = 1$$

$$p_1(x)\,\dot{p}_2(t) = \frac{1}{2}p_1''(x)\,p_2(t)$$

$$\frac{\dot{p}_2(t)}{p_2(t)} = \frac{1}{2}\frac{p_1''(x)}{p_1(x)} = const = \lambda$$

$$p_1''(x) = 2\lambda p_1(x)$$

$$\lambda \ge 0 \text{ невозм.} \Rightarrow \lambda < 0$$

$$\begin{cases} p_1''(x) - \underbrace{2\lambda}_{-\omega^2} p_1(x) = 0 \\ p_1(\pm a) = 0 \end{cases}$$

$$\begin{cases} p_1''(x) + \omega^2 p_1(x) = 0 \\ p_1(\pm a) = 0 \end{cases}$$

$$p_1(x) = A\cos\omega x$$
$$\cos\omega a = 0$$
$$\omega a = \frac{\pi}{2} + \pi n, \quad n = 0, 1, \ldots$$
$$\omega = \omega_n = \frac{\pi}{a}\left(n + \frac{1}{2}\right)$$
$$\lambda = -\frac{\omega^2}{2} \Rightarrow \lambda_n = -\frac{\pi^2}{2a^2}\left(n + \frac{1}{2}\right)^2$$
$$p_1(x) = A_n\cos\omega_n x = A_n\cos\left(\frac{\pi x}{a}\left(n + \frac{1}{2}\right)\right)$$

$$\frac{\dot{p}_2(t)}{p_2(t)} = \lambda_n \Rightarrow p_2(t) = B_n e^{\lambda_n t} = B_n e^{-\frac{\pi^2}{2a^2}\left(n+\frac{1}{2}\right)^2 t}$$

$$C_n = B_n \cdot A_n \Rightarrow p_n(x,t) = C_n e^{-\frac{\pi^2 t}{2a^2}\left(n+\frac{1}{2}\right)^2}\cos\left(\frac{\pi x}{a}\left(n + \frac{1}{2}\right)\right)$$

$$p(x,t) = \sum_{n=0}^{\infty} p_n(x,t)$$

$$\sum_{n=0}^{\infty} C_n\cos\left(\frac{\pi x}{a}\left(n + \frac{1}{2}\right)\right) = \delta(x)$$

$$C_m \underbrace{\int_{-a}^{+a}\cos^2\left(\frac{\pi x}{a}\left(m + \frac{1}{2}\right)\right)dx}_{a} = \int_{-\infty}^{+\infty}\cos\left(\frac{\pi x}{a}\left(m + \frac{1}{2}\right)\right)\delta(x)\,dx \Rightarrow C_m = \frac{1}{a} \quad \forall m \ge 0$$

$$p(x,t,-a,a) = \frac{1}{a}\sum_{n=0}^{+\infty} e^{-\frac{\pi^2 t}{2a^2}\left(n+\frac{1}{2}\right)^2}\cos\left(\frac{\pi x}{a}\left(n+\frac{1}{2}\right)\right)$$

$$\frac{4a(-1)^n}{(2n+1)\pi}$$

$$\int_{-a}^{a} p(x,t,-a,a) = \frac{1}{a}\sum_{n=0}^{+\infty} e^{-\frac{\pi^2 t}{2a^2}\left(n+\frac{1}{2}\right)^2}\int_{-a}^{a}\cos\left(\frac{\pi x}{a}\left(n+\frac{1}{2}\right)\right)dx = \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}e^{-\frac{\pi^2 t}{2a^2}\left(n+\frac{1}{2}\right)^2}$$

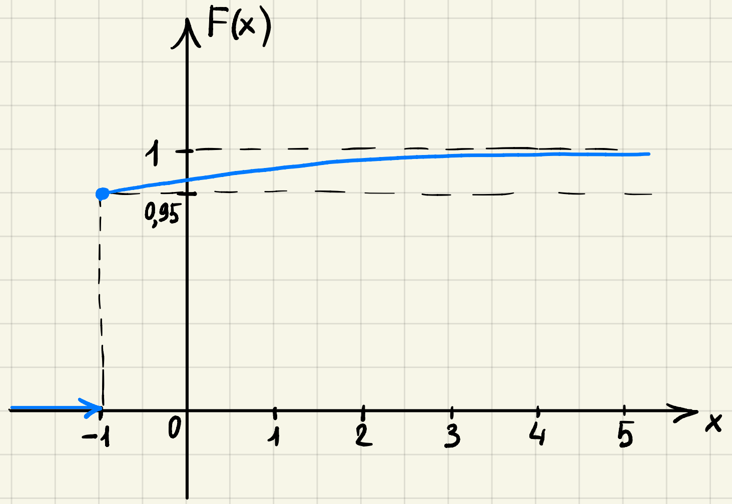
"$N(t)$

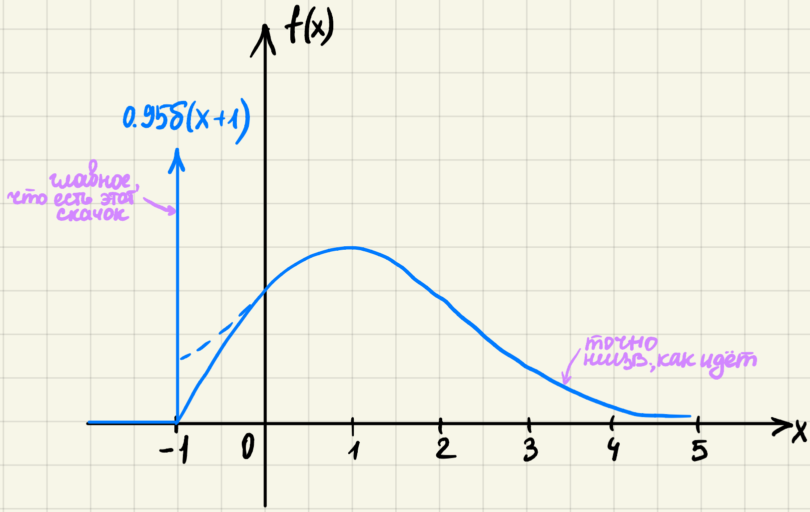
$t=0 \Rightarrow \;=1$
$t>0 \Rightarrow \;<1$

$$p(s,T,a,b) = p_{Tab}(s)$$

$$\mathbb{P}\{0 < \overset{S_T-K}{Payoff(T)} \le \overset{>0}{(1+X)B_0}\} = P\{K < S_T \le (1+X)B_0 + K\} = \int_K^{K+(1+X)B_0} p_{Tab}(s)\,ds$$

$$P\{Payoff(T) \ge \overset{X>-1}{(1+X)B_0}\} = \int_{K+(1+X)B_0}^{+\infty} p_{Tab}(s)\,ds$$





$N \longrightarrow$
$D(0,1)$
$D_{01}N$ - - - - ⟶ N Risk-free account
$(1-D_{01})N$ - - 1 - - ⟶ Barrier Option

$D_{01} = e^{-rT}$, $T = 1y$, $r = 0.0425$

FP: $\frac{\partial p}{\partial t} = L_1 p$, $L_1 = -\frac{\partial}{\partial x}(\mu\cdot) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2\cdot)$

FC: $\frac{\partial f}{\partial t} = -L_2 p$, $L_2 = \mu\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}$

$L_1 = L_2^+$ -? у сопряж. опер. матр. явл. сопряж., а для вещ. матр. это просто транспон.

$$\langle L_1 p, f\rangle = \langle p, L_2 f\rangle$$
$$\langle \tilde{x}, \tilde{y}\rangle = \int_{-\infty}^{+\infty}\tilde{x}(x)\tilde{y}(x)\,dx$$

$$\int_{-\infty}^{+\infty} L_1 p\cdot f\,dx \overset{?}{=} \int_{-\infty}^{+\infty} p\cdot L_2 f\,dx \iff \int_{-\infty}^{+\infty}\frac{\partial p}{\partial t}f\,dx = -\int_{-\infty}^{+\infty}p\frac{\partial f}{\partial t}dx \iff$$

$$\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}(pf)\,dx = 0 \iff \frac{\partial}{\partial t}\int_{-\infty}^{+\infty}pf\,dx = 0$$

$f(X_t,t) = \mathbb{E}[g(X_T)\,|\,\Sigma_t]$ $\quad 0 \le t \le T$
$p(x,t)$: PDF(X_T)

$$\int_{-\infty}^{+\infty}pf\,dx = const(t) \iff \mathbb{E}[f(X_t)] = const(t) \iff \mathbb{E}[\mathbb{E}[g(X_T)|\Sigma_t]] = const(t)$$

$$\iff \mathbb{E}[g(X_T)] = const(t)\;!$$

	$t=0$	t (r_t)	$t+dt$
Money account	$C_0-\Delta_0 S_0$	M_t	$M_{t+dt}=M_t e^{r_t dt}\cong M_t(1+r_t dt)=M_t+M_t r_t dt$
Undelying	$\Delta_0\cdot S_0$	$\Delta_t\cdot S_t$	$\Delta_t S_{t+dt}=\Delta_t(S_t+dS_t)=\Delta_t(S_t+\mu_t dt+\sigma_t dW_t)$
Option	$-C_0$	$-C_t$	$-C_{t+dt}=f(S_t,t)$
	$\Pi_0=0$	$\Pi_t=M_t+\Delta_t S_t-C_t$	

option price in t

для нейтрализации своей опционной позиции

(1) Ansatz — To be determined!

короткая позиц. по опциону — получил деньги, но взял на себя обязанности

(3) Self-Financing

$t=T$

$$M'_{t+dt}=M_{t+dt}-(\Delta_{t+dt}-\Delta_t)S_{t+dt}$$

Rebalancing

$$-C_T=-\varphi(S_T)$$

Payoff (Europe)

$$C_{t+dt}=C_t+dC_t=C_t+\frac{\partial f}{\partial t}+\frac{\partial f}{\partial S}(\mu dt+\sigma dW_t)+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 dt=$$
$$=C_t+\left(\frac{\partial f}{\partial t}+\mu\frac{\partial f}{\partial S}+\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\right)dt+\frac{\partial f}{\partial S}\sigma dW_t$$

$$\Pi_{t+dt}=M_t+M_t r_t dt+\Delta_t(S_t+\mu dt+\sigma dW_t)-C_t-(\ldots)dt-\frac{\partial f}{\partial S}\sigma dW_t=$$
$$=\Pi_t+\underbrace{M_t r_t dt+\Delta_t\mu dt+\Delta_t\sigma dW_t-(\ldots)dt-\frac{\partial f}{\partial S}\sigma dW_t}_{d\Pi_t}$$

Δ_t нах-ся под нашим контролем

$$d\Pi_t=\Big[M_t r_t+\Delta_t\mu-\frac{\partial f}{\partial t}-\mu\frac{\partial f}{\partial S}-\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\Big]dt+\sigma\left(\Delta_t-\frac{\partial f}{\partial S}\right)dW_t$$

$\Delta_t = \frac{\partial f}{\partial S}$ => No Trend!

$$\Delta_t=\frac{\partial f}{\partial S}$$

(2) Delta-Hedging

$$d\Pi_t=\Big[M_t r_t-\frac{\partial f}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\Big]dt=\Pi_t(e^{r_t dt}-1)\approx\Pi_t r_t dt$$

(4) No arbitrage

$$M_t r_t-\frac{\partial f}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}-\Pi_t r_t=\left(M_t-\frac{\partial f}{\partial S}S_t-C_t\right)r_t$$

$$-\frac{\partial f}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}=r_t\frac{\partial f}{\partial S}S_t-r_t \overset{f}{C_t}$$

$$\frac{\partial f}{\partial t}+r_t\frac{\partial f}{\partial S}S_t+\frac{\sigma^2}{2}(S_t,t)\frac{\partial^2 f}{\partial S^2}-r_t f=0$$

Convective term ($r_t S_t=\mu(S_t,t)$) — Diffusion term — Reactive term

(ур-ние Блэка-Шоулса-Мертона)

BSM

$f(S_T,T)=\varphi(S_T)$: European Payoff

Trivial Solutions:

- $f=S$ — Stock
- $f=M_0 e^{\int_0^t r_s ds}$ — Money account
- $f=B_T e^{-\int_0^T r_s ds}$ — Bond

$F_t<S_T$: seller of loss
$F_t>S_T$: buyer of loss

t — T

F_t → S_T+F_t

Seller: S_t → S_T

$M_t=-S_t$ → $\underbrace{M_T=-S_t e^{\int_0^T r_s ds}}_{\text{debt}}$

$$f=F_t=S_t e^{\int_0^T r_s ds}$$
$$\frac{\partial f}{\partial t}=-r_t f$$
$$-r_t f+r_t\underbrace{S_t e}_{f}-r_t f\neq 0$$

Такое не удовл. ур-нию

$$S_t-Ke^{-\int_t^T r_s ds}$$

экон. смысл инструмента?

С дивидендами: $dS = (\mu_t - D_t S_t)dt + \sigma_t dW_t$

$$M_{t+dt} = M_t e^{r_t dt} + \Delta_t S_t D_t dt \cong M_t(1 + r_t dt) + \Delta_t S_t D_t dt = M_t + (r_t M_t + \Delta_t S_t D_t)dt$$
$$\Delta_t \cdot S_{t+dt} = \Delta_t(S_t + dS_t) = \Delta_t(S_t + (\mu_t - S_t D_t)dt + \sigma_t dW_t) = \Delta_t S_t + \Delta_t(\mu_t - S_t D_t)dt + \Delta_t \sigma_t dW_t$$
$$C_{t+dt} = C_t + dC_t = C_t + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}((\mu_t - S_t D_t)dt + \sigma_t dW_t) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}dt$$

$$\Pi_{t+dt} = \Pi_t + (r_t M_t + \Delta_t S_t D_t)dt + \Delta_t(\mu_t - S_t D_t)dt + \Delta_t \sigma_t dW_t - \frac{\partial f}{\partial t}dt - \frac{\partial f}{\partial S}((\mu_t - S_t D_t)dt + \sigma_t dW_t) - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}dt =$$
$$= \Pi_t + [r_t M_t + \cancel{\Delta_t S_t D_t} + \cancel{\Delta_t \mu_t} - \cancel{\Delta_t D_t S_t} - \frac{\partial f}{\partial t} - \cancel{\frac{\partial f}{\partial S}\mu_t} + \frac{\partial f}{\partial S}D_t S_t - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}]dt +$$
$$+ \sigma_t \cancel{\left(\Delta_t - \frac{\partial f}{\partial S}\right)} dW_t =$$
$$= \Pi_t + [r_t M_t + \frac{\partial f}{\partial S}D_t S_t - \frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}]dt \overset{\text{No Arb}}{=} \Pi_t e^{r_t dt} \cong \Pi_t(1 - r_t dt)$$

$$r_t M_t + \frac{\partial f}{\partial S}S_t D_t - \frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2} = \Pi_t r_t$$
$$\frac{\partial f}{\partial S}S_t D_t - \frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2} = r_t \frac{\partial f}{\partial S}S_t - r_t f \qquad \left(M_t'' + \frac{\partial f}{\partial S}S_t - f\right)$$
$$\frac{\partial f}{\partial t} + \underbrace{[r_t - D_t]}_{\text{from MM (Money Market)}} S\frac{\partial f}{\partial S} + \frac{\sigma^2(S,t)}{2}\frac{\partial^2 f}{\partial S^2} - r_t f = 0 \quad \text{BSM with dividends}$$

from portfolio

$$f(S,T) = \varphi(S) \longleftarrow T$$

Stock — уже не удовлетв.
Dividend Paying Stock: $f = Se^{\int_0^t D_s ds}$ V
Bond V
Money account V

Зануление волатильности — очень плохо, т.к. если σ стан. слишком малым, что можно пренебречь, то ур-ние стан. УЧП гиперб. типа. Диффуз. член "сглаживает" скачки. Также зануление $(\Delta_t - \frac{\partial f}{\partial S})$ при dW_t перестает быть верным, а, следовательно, и сущ-ние решения. Волатильность — риск.

Analitical solution to BSMD PDE (УЧП)

Assume that $\sigma(S,t) = \sigma_t S_t$ (только от времени)

$$\frac{\partial f}{\partial t} + (r_t - D_t)S\frac{\partial f}{\partial S} + \frac{\sigma_t^2}{2}S^2\frac{\partial^2 f}{\partial S^2} - r_t f = 0$$

(1) $\tau = T - t$
(2) Eliminate the reactive turm => Get Feynmann-Kac

(1) $f(S,t) = f(S, T-\tau) = f_1(S,\tau)$. $\frac{\partial f_1}{\partial \tau} = (\tilde{r}_\tau - \tilde{D}_\tau)S\frac{\partial f_1}{\partial S} + \frac{\tilde{\sigma}_\tau^2}{2}S^2\frac{\partial^2 f_1}{\partial S^2} - f_1\tilde{r}_\tau$

$\tilde{r}_\tau = r_{T-\tau}$, $\tilde{\sigma}_\tau = \sigma_{T-\tau}$, $\tilde{D}_\tau = D_{T-\tau}$

(2) $f_2(S,\tau) = f_1(S,\tau)e^{\int_0^\tau \tilde{r}_u du}$
$\Rightarrow f_1(S,\tau) = f_2(S,\tau)e^{-\int_0^\tau \tilde{r}_u du}$

$$f_2 e^{-\int_0^\tau \tilde{r}_u du}(-\tilde{r}_\tau) = (\tilde{r}_\tau - \tilde{D}_\tau)S\frac{\partial f_2}{\partial S}e^{-\int_0^\tau \tilde{r}_u du} + \frac{\partial f_2}{\partial \tau}e^{-\int_0^\tau \tilde{r}_u du} + \frac{\tilde{\sigma}_\tau^2}{2}S^2\frac{\partial^2 f_2}{\partial S^2}e^{-\int_0^\tau \tilde{r}_u du} - \tilde{r}_\tau f_2 e^{-\int_0^\tau \tilde{r}_u du}$$
$$\frac{\partial f_2}{\partial \tau} = (\tilde{r}_\tau - D_\tau)S\frac{\partial f_2}{\partial S} + \frac{\partial^2 f}{\partial S^2}\cdot\frac{\sigma^2}{2}S^2$$

(3) Eliminate S, S^2:

Работает только для геом. броун. движ.

(3) $x = \ln\frac{S}{1}$ (избавиться от разм-ти) $S = 1e^x$ $f_2(S,\tau) = f_2(e^x,\tau) = f_3(x,\tau) = f_3(\ln S, \tau)$

$$\frac{\partial f_3}{\partial x} = \frac{\partial f_2}{\partial S}\cdot\frac{dS}{dx} = \frac{\partial f_2}{\partial S}S$$

$$\frac{\partial^2 f_2}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial f_2}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{1}{s}\cdot\frac{\partial f_3}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{s}\frac{\partial f_3}{\partial x}\right)\frac{dx}{ds} = \left(-\frac{1}{s^2}\frac{ds}{dx}\frac{\partial f_3}{\partial x} + \frac{1}{s}\frac{\partial^2 f_3}{\partial x^2}\right)\frac{1}{s}$$

$$= \frac{1}{s^2}\left(\frac{\partial^2 f_3}{\partial x^2} - \frac{\partial f_3}{\partial x}\right) \Rightarrow s^2\frac{\partial^2 f_2}{\partial s^2} = \frac{\partial^2 f_3}{\partial x^2} - \frac{\partial f_3}{\partial x}$$

$$\frac{\partial f_3}{\partial \tau} = (\tilde r_\tau - \tilde D_\tau)\frac{\partial f_3}{\partial x} + \frac{\tilde\sigma_\tau^2}{2}\left(\frac{\partial^2 f_3}{\partial x^2} - \frac{\partial f_3}{\partial x}\right)$$

(4) Eliminate the conv. term

$$\frac{\partial f_3}{\partial \tau} = \left(r_\tau - D_\tau - \frac{\sigma_\tau^2}{2}\right)\frac{\partial f_3}{\partial x} + \frac{\tilde\sigma_\tau^2}{2}\frac{\partial^2 f_3}{\partial x^2}$$

$$\left\{ \xi = x + \int_0^\tau \underbrace{\tilde b_u}_{\tilde r_u - \tilde D_u - \frac{\tilde\sigma_u^2}{2}} du \right\}$$

$$f_3(x,\tau) = f_3\left(\xi - \int_0^\tau \tilde b_u du, \tau\right) = f_4(\xi,\tau) \quad (4)$$

$$\frac{\partial f_4}{\partial \xi} = \frac{\partial f_3}{\partial x}; \quad \frac{\partial f_4}{\partial \tau} = \frac{\partial f_3}{\partial \tau} + \frac{\partial f_3}{\partial x}\underbrace{\frac{\partial x}{\partial \tau}}_{-\tilde b_\tau} = -\tilde b_\tau\frac{\partial f_3}{\partial x} + \frac{\partial f_3}{\partial \tau}$$

$$\frac{\partial f_3}{\partial \tau} = \frac{\partial f_4}{\partial \tau} + \tilde b_\tau\frac{\partial f_3}{\partial x} = \tilde b_\tau \underbrace{\frac{\partial f_3}{\partial x}}_{\frac{\partial f_4}{\partial \xi}} + \frac{\tilde\sigma_\tau^2}{2}\underbrace{\frac{\partial^2 f_3}{\partial x^2}}_{\frac{\partial^2 f_4}{\partial \xi^2}} \Rightarrow \frac{\partial f_4}{\partial \tau} = \frac{\tilde\sigma_\tau^2}{2}\frac{\partial^2 f_4}{\partial \xi^2}$$

(5) Eliminate $\tilde\sigma_\tau^2$

$$y(\tau) = \int_0^\tau \tilde\sigma_u^2 du \Rightarrow \tau(y)$$

$$(5)\quad f_4(\xi,\tau) = f_4(\xi,\tau(y)) = f_5(\xi,y)$$

$$\frac{\partial f_5}{\partial y} = \frac{\partial f_4}{\partial \tau}\frac{d\tau}{dy} = \frac{1}{\tilde\sigma_\tau^2}\frac{\partial f_4}{\partial \tau} \Rightarrow \frac{\partial f_5}{\partial \tau} = \tilde\sigma_\tau^2\frac{\partial f_5}{\partial y}; \quad \frac{\partial f_5}{\partial y} = \frac{1}{2}\frac{\partial^2 f_5}{\partial \xi^2}$$ — тепловое УЧП

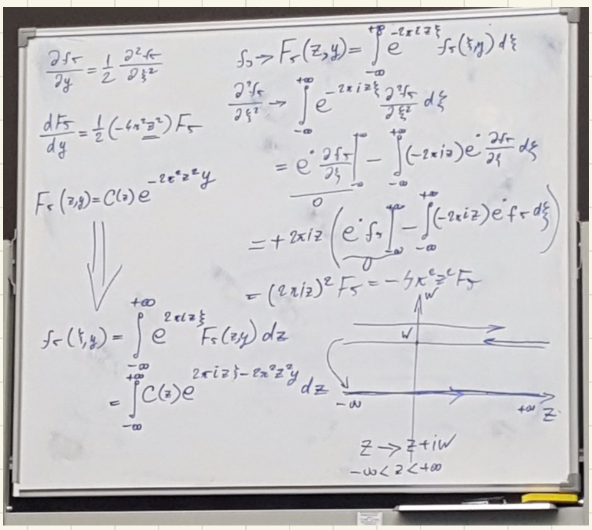
$$f_5(\xi,y) = \frac{1}{\sqrt{2\pi y}}e^{-\frac{(\xi-a)^2}{2y}} \quad \forall a\in\mathbb{R}$$

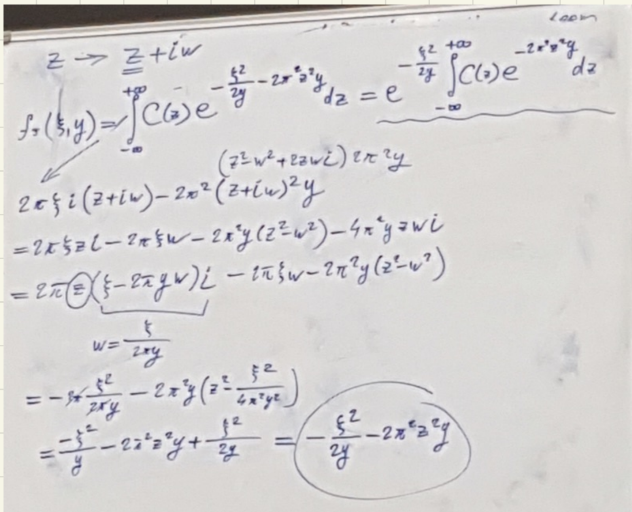
$$f_5(\xi,y) = \frac{1}{\sqrt{2\pi y}}\int_{-\infty}^{+\infty} g(a)e^{-\frac{(\xi-a)^2}{2y}}da$$

$$f_5(\xi,y(\tau)) = f_4(\xi,\tau) = \frac{1}{\sqrt{2\pi y(\tau)}}\int_{-\infty}^{+\infty} g(a)e^{-\frac{(\xi-a)^2}{2y}}da$$

(ответ на чей-то вопрос) слишком сложное обоснование ф-ти решения (лучше посмотреть в урматах)

Смысл дельта-хеджирования — заменить stocks на процесс, близкий к безрисковому





(1.10) - (1.11)

$dS_t = (r_t - D_t) S_t dt$ Risk-neutral equities

$dS_t = (r_A - r_B) S_t dt$ FX

$$f_5(\xi, y(\tau)) = f_4(\xi, \tau) = \frac{1}{\sqrt{2\pi y(\tau)}} \int_{-\infty}^{+\infty} g(a) e^{-\frac{(\xi-a)^2}{2y}} da$$

При $y \to 0$ $\frac{1}{\sqrt{2\pi y}} e^{-\frac{(\xi-a)^2}{2y}} \longrightarrow \delta(a-\xi) \Rightarrow f_4(\xi, 0) = g(\xi)$

Гран. усл. – регулярность на ∞

$$f_3(x, \tau) = \frac{1}{\sqrt{2\pi y(\tau)}} \int_{-\infty}^{+\infty} g(a) e^{-\frac{(x + \int_0^\tau b_u du - a)^2}{2y(\tau)}} da$$

$$f_2(S, \tau) = \frac{1}{\sqrt{2\pi y(\tau)}} \int_{-\infty}^{+\infty} g(a) e^{-\frac{(\ln S + \int_0^\tau b_u du - a)^2}{2y(\tau)}} da$$

$$f_1(S, \tau) = \frac{e^{-\int_0^\tau r_u du}}{\sqrt{2\pi y(\tau)}} \int_{-\infty}^{+\infty} g(a) e^{-\frac{(\ln S + \int_0^\tau b_u du - a)^2}{2y(\tau)}} da \qquad \{ y(\tau) \Rightarrow V(t) = \int_t^T \sigma_u^2 du$$

time to expiration (истечение срока)

$$f(S, t) = \frac{e^{-\int_t^T r_u du}}{\sqrt{2\pi V(t)}} \int_{-\infty}^{+\infty} g(a) e^{-\frac{(\ln S + \int_t^T (r_u - D_u - \frac{\sigma_u^2}{2}) du - a)^2}{2V(t)}} da, \quad V(t) = V_t = \int_t^T \sigma_u^2 du, \quad r_t - D_t - \frac{\sigma_t^2}{2} = b_t$$

По всем заменам смотрим начальные условия: $f_5(\xi, 0) = \varphi(e^\xi)$

$f_5(\xi, 0) = \varphi(e^\xi) = g(\xi) \Rightarrow g(a) = \varphi(e^a)$

$$f(S, t) = \frac{e^{-\int_t^T r_u du}}{\sqrt{2\pi \int_t^T \sigma_u^2 du}} \int_{-\infty}^{+\infty} \varphi(e^a) e^{-\frac{(\ln S + \int_t^T (r_u - D_u - \frac{\sigma_u^2}{2}) du - a)^2}{2\int_t^T \sigma_u^2 du}} da$$

физ. смысл – S_T ($z \sim S_T$)

$e^a = z, \; da = \frac{dz}{z}$

$$\boxed{f(S, t) = \frac{e^{-\int_t^T r_u du}}{\sqrt{2\pi \int_t^T \sigma_u^2 du}} \int_0^{+\infty} \varphi(z) \cdot e^{-\frac{(\ln \frac{S}{z} + \int_t^T (r_u - D_u - \frac{\sigma_u^2}{2}) du)^2}{2\int_t^T \sigma_u^2 du}} \frac{dz}{z}}$$

$C_t = f(S_t, t)$

European Call: $\varphi(z) = (z - K)^+ \quad z \geq K$

$$f_c(S, t) = \frac{D(t, T)}{\sqrt{2\pi\tau}\,\sigma} \int_K^{+\infty} \left(1 - \frac{K}{z}\right) e^{-\frac{(\ln \frac{S}{z} + B)^2}{2\sigma^2\tau}} dz$$

$r_t = r = const$
$D_t = D = const$
$\sigma_t = \sigma = const$
$\tau = T - t$ (time to expiration)
$B = (r - D - \frac{\sigma^2}{2})\tau$

предположение для сокращения записи

$x = \ln \frac{S}{z} + B$
$dx = -\frac{dz}{z}$

$$f_c(S, t) = \frac{D(t, T)}{\sqrt{2\pi\tau}\,\sigma} \int_{-\infty}^{\ln \frac{S}{K} + B} (S e^{B - x} - K) e^{-\frac{x^2}{2\sigma^2\tau}} dx =$$

$$= Se^{-D\tau}N_1 - Ke^{-r\tau}N_2$$

формула Блэка-Шоулса-Мертона

$$N_{1,2} = \Phi\left(\frac{\ln\frac{S}{K} + (r - D \pm \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}dx$$

General Case:

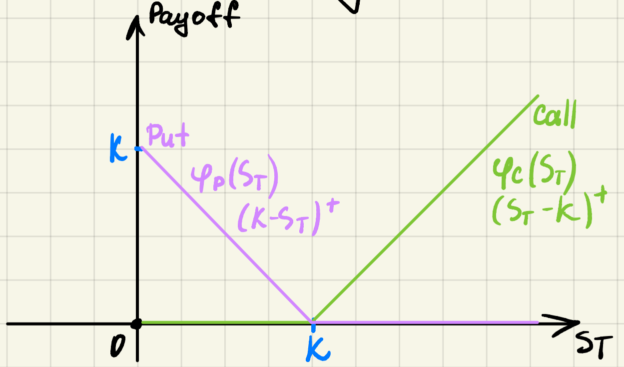
$$(r - D \pm \frac{\sigma^2}{2})\tau \longrightarrow \int_t^T \left(r_u - D_u \pm \frac{\sigma_u^2}{2}\right)du$$

$$e^{-D\tau} \longrightarrow e^{-\int_t^T D_u du}$$

$$e^{-r\tau} \longrightarrow e^{-\int_t^T r_u du} = D(t,T)$$

$$\sigma\sqrt{\tau} \longrightarrow \sqrt{\int_t^T \sigma_u^2 du}$$

Put-Call-Parity (Model-free)



$$\varphi_c(S_T) - \varphi_p(S_T) = S_T - K$$

$t < T$

$S_t, \; \overbrace{-Ke^{-\int_t^T r_u du}}^{D(t,T)}$

Long Call
Short Put

OR:

Long Stock
Short Bond

OR:

Borrow $K \cdot D(t,T)$

простейшие репликации (для создания безрискового портфеля)

$$\underset{\substack{K\\T}}{Call\_price_t} - \underset{\substack{K\\T}}{Put\_price_t} = S_t - KD(t,T)$$

Дифференциальные риски опционов ("Greeks")

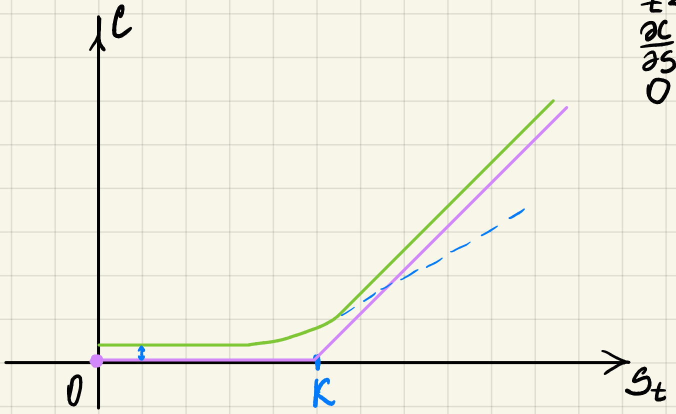
$$Call\ Px = C(\underset{\text{наблюд. переменная}}{S_t}, t, \underbrace{K, T}_{\text{зависят от контракта}}, \underset{\text{дет. ф-ция времени}}{r}, \underset{\text{почти конст.}}{D}, \sigma)$$

σ — самая ненаблюдаемая переменная

$t < T$

$\frac{\partial C}{\partial S} = \Delta$

$0 < \Delta < 1$



$t=0$

$\tau = T - t = T$

Money $C_0 = f_c(S_0, 0)$ Mo $\left(\!\left(C_0 = \Delta_0 S_0 - \dots \right.\right.$

$C_0 - \Delta_0 S_0 < 0$

Hedge Δ-hedge $\Delta_0 = \frac{\partial f_c}{\partial S}\Big|_0 \in (0,1)$ $\Delta_0 S_0$

Option Pos. $-C_0$

$\Pi_0 = 0$

t

M_t

$\Delta_t S_t$

$-C_t$

$\Pi_t = 0$

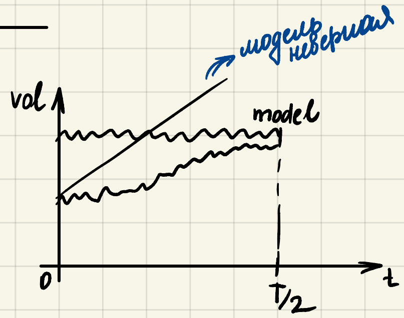
Call: $\Delta = \frac{\partial C}{\partial S} = N_1 = \varphi\left(\frac{\ln\frac{S}{K} + (r - D + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right)$

$0 < \Delta < 1$

$S > K$: $\Delta \xrightarrow{t\to\infty} 1$

$S < K$: $\Delta \xrightarrow{t\to\infty} 0$

$S = K$: Δ not defined $(\to \frac{1}{2})$



Покупаем недооценённое рынком

Хеджирование

σ_{impl} : Mkt

$\hat{\sigma}_{mod} = \frac{1}{\sqrt{\tau}} \sqrt{\int_\tau^T \sigma^2_{mod}(u)\,du}$

$S_0 = \$100 \rightsquigarrow S_T = \$120$

(1) Buy stock $\rightsquigarrow$ Ret = 20%

(2) Buy option (Call):

- at the money $K = 100$

$C \cong \underbrace{0.4}_{\sim}\,\underbrace{\sigma}_{\sim 0.25}\,\underbrace{\sqrt{\tau}}_{1}\,K$

$0.1K$

$C_0 = \$10 \Rightarrow$ Payoff = \$20

RnL = \$10

Ret = 100%

Barrier Opt:

$B_0 = \$1 \Rightarrow$ Payoff = \$20

PnL = \$19

Ret = 1900%

(3-5% случаев)