

UNCERTAINTY ASSIGNMENT - SPH 3U3

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1.

2. (a)

$$\begin{aligned}f &= x_1 + x_2 \\&= (10.005 \pm .003) \text{ cm} + (20.06 \pm .03) \text{ cm} \\ \sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\&= \sqrt{(1)^2 (.003)^2 + (1)^2 (.03)^2} \\&= \sqrt{1(.000009) + 1(.0009)} \\&= \sqrt{.000009 + .0009} \\&= \sqrt{.0009} \\&= .03 \\f &= (10.005 \pm .003) \text{ cm} + (20.06 \pm .03) \text{ cm} \\&= (10.005 + 20.06 \pm \sigma_f) \text{ cm} \\&= (30.07 \pm .03) \text{ cm}\end{aligned}$$

(b)

$$\begin{aligned}f &= x_1 - x_2 \\&= (352.1 \pm .9) \text{ m} - (162.36 \pm .05) \text{ m} \\ \sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\&= \sqrt{(1)^2 (.9)^2 + (-1)^2 (.05)^2} \\&= \sqrt{1(.8) + 1(.003)} \\&= \sqrt{.8 + .003} \\&= \sqrt{.8} \\&= .9 \\f &= (352.1 \pm .9) \text{ m} - (162.36 \pm .05) \text{ m} \\&= (352.1 - 162.36 \pm \sigma_f) \text{ m} \\&= (189.7 \pm .9) \text{ m}\end{aligned}$$

(c)

$$\begin{aligned}f &= x_1 + x_2 \\&= (56.7 \pm .2) \text{ cm} + (93.48 \pm .01) \text{ m} \\ \sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\&= \sqrt{(1)^2 (.2)^2 + (1)^2 (.01)^2} \\&= \sqrt{1(.04) + 1(.01)} \\&= \sqrt{.04 + .01} \\&= \sqrt{.05} \\&= .2236 \\f &= (56.7 \pm .2) \text{ cm} + (93.48 \pm .01) \text{ m} \\&= (56.7 \pm .2) \text{ cm} + (9348 \pm 1) \text{ cm} \\&= (56.7 + 9348 \pm \sigma_f) \text{ cm} \\&= (9410 \pm 1) \text{ cm}\end{aligned}$$

(d)

$$\begin{aligned}f &= x_1 + x_2 - x_3 \\&= (14.5 \pm .2) \text{ mm} \pm (14.5 \pm .2) \text{ mm} - (23.1 \pm .1) \text{ mm} \\ \sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 \sigma_{x_3}^2} \\&= \sqrt{(1)^2 (.2)^2 + (1)^2 (.2)^2 + (-1)^2 (.1)^2} \\&= \sqrt{1(.04) + 1(.04) + 1(.01)} \\&= \sqrt{.04 + .04 + .01} \\&= \sqrt{.09} \\&= .3 \\f &= (14.5 \pm .2) \text{ mm} \pm (14.5 \pm .2) \text{ mm} - (23.1 \pm .1) \text{ mm} \\&= (14.5 + 14.5 - 23.1 \pm \sigma_f) \text{ mm} \\&= (5.9 \pm .3) \text{ mm}\end{aligned}$$

3. (a)

$$\begin{aligned}
f &= x_1 * x_2 \\
&= (23.56 \pm .05) \text{ kmh}^{-1} * (56.3 \pm .4) \text{ h} \\
\sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\
&= \sqrt{(x_2)^2 (.05)^2 + (x_1)^2 (.4)^2} \\
&= \sqrt{(56.3)^2 (.003) + (23.56)^2 (.2)} \\
&= \sqrt{3170(.003) + 555.1(.2)} \\
&= \sqrt{10 + 100} \\
&= \sqrt{100} \\
&= 10 \\
f &= (23.56 \pm .05) \text{ kmh}^{-1} * (56.3 \pm .4) \text{ h} \\
&= (23.56 * 56.3 \pm \sigma_f) \text{ km} \\
&= (1330 \pm 9) \text{ km}
\end{aligned}$$

(b)

$$\begin{aligned}f &= \frac{x_1}{x_2} \\&= \frac{(15.745 \pm .006) \text{ m}}{(0.36 \pm .05) \text{ m}} \\\sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\&= \sqrt{\left(\frac{1}{x_2}\right)^2 (.006)^2 + \left(\frac{-x_1}{x_2^2}\right)^2 (.05)^2} \\&= \sqrt{\left(\frac{1}{0.36}\right)^2 (.00004) + \left(\frac{-15.745}{0.36^2}\right)^2 (.003)} \\&= \sqrt{(2.8)^2 (.00004) + \left(\frac{-15.745}{0.13}\right)^2 (.003)} \\&= \sqrt{7.8(.00004) + (-120)^2 (.003)} \\&= \sqrt{0.0003 + 14000(.003)} \\&= \sqrt{0.0003 + 40} \\&= \sqrt{40} \\&= 6 \\f &= \frac{(15.745 \pm .006) \text{ m}}{(0.36 \pm .05) \text{ m}} \\&= \frac{15.745}{0.36} \pm \sigma_f \\&= 44 \pm 6\end{aligned}$$

(c)

$$\begin{aligned}f &= 2x_1 \\&= 2(1.63 \pm .03) \text{ mm} \\\sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2} \\&= 2(.03) \\&= .06 \\f &= 2(1.63 \pm .03) \text{ mm} \\&= 2(1.63) \pm \sigma_f \text{ mm} \\&= 3.26 \pm .06 \text{ mm}\end{aligned}$$

(d)

$$\begin{aligned} f &= \frac{x_1 x_2}{x_3} \\ &= \frac{(1.23 \pm .02) \text{ ms}^{-1} * (2.637 \pm .003) \text{ ms}^{-1}}{(5.6 \pm .1) \text{ ms}^{-1}} \\ \sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 \sigma_{x_3}^2} \\ &= \sqrt{\left(\frac{x_2}{x_3}\right)^2 (.02)^2 + \left(\frac{x_1}{x_3}\right)^2 (.003)^2 + \left(\frac{-x_1 x_2}{x_3^2}\right)^2 (.1)^2} \\ &= \sqrt{\left(\frac{2.637}{5.6}\right)^2 (.0004) + \left(\frac{1.23}{5.6}\right)^2 (.000009) + \left(\frac{-1.23(2.637)}{5.6^2}\right)^2 (.01)} \\ &= \sqrt{(.47)^2 (.0004) + (.22)^2 (.000009) + \left(\frac{-3.24}{31.36}\right)^2 (.01)} \\ &= \sqrt{.22(.0004) + .048(.000009) + (-.103)^2 (.01)} \\ &= \sqrt{.00009 + .0000004 + .0107} (.01) \\ &= \sqrt{.00009 + .0001} \\ &= \sqrt{.0002} \\ &= .01 \\ f &= \frac{(1.23 \pm .02) \text{ ms}^{-1} * (2.637 \pm .003) \text{ ms}^{-1}}{(5.6 \pm .1) \text{ ms}^{-1}} \\ &= \left(\frac{1.23 * 2.637}{5.6} \pm \sigma_f\right) \text{ ms}^{-1} \\ &= (.58 \pm .01) \text{ ms}^{-1} \end{aligned}$$

4. (a)

$$\begin{aligned}
f &= x_1 + \frac{2x_2}{x_3} \\
&= (61.12 \pm .05) \text{ cm} + \frac{2(45.23 \pm .06) \text{ cm}^2}{(1.03 \pm .04) \text{ cm}} \\
\sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 \sigma_{x_3}^2} \\
&= \sqrt{(1)^2 (.05)^2 + \left(\frac{2}{x_3}\right)^2 (.06)^2 + \left(\frac{-2x_2}{x_3^2}\right)^2 (.04)^2} \\
&= \sqrt{1(.003) + \left(\frac{2}{1.03}\right)^2 (.004) + \left(\frac{-2(45.23)}{1.03^2}\right)^2 (.002)} \\
&= \sqrt{.003 + \left(\frac{2}{1.03}\right)^2 (.004) + \left(\frac{-2(45.23)}{1.03^2}\right)^2 (.002)} \\
&= \sqrt{.003 + (2)^2 (.004) + \left(\frac{-90.46}{1.06}\right)^2 (.002)} \\
&= \sqrt{.003 + 4(.004) + (-85.3)^2 (.002)} \\
&= \sqrt{.003 + .02 + 7280(.002)} \\
&= \sqrt{.02 + 15} \\
&= \sqrt{15} \\
&= 4 \\
f &= (61.12 \pm .05) \text{ cm} + \frac{2(45.23 \pm .06) \text{ cm}^2}{(1.03 \pm .04) \text{ cm}} \\
&= (61.12 + \frac{2(45.23)}{1.03} \pm \sigma_f) \text{ cm} \\
&= (61.12 + \frac{90.46}{1.03} \pm 3) \text{ cm} \\
&= (61.21 + 87.8252 \pm 3) \text{ cm} \\
&= (148.9 \pm 3) \text{ cm} \\
&= (149 \pm 3) \text{ cm}
\end{aligned}$$

(b)

$$\begin{aligned}f &= x_1 x_2 - x_3 \\&= (1005.1 \pm .2) \text{ kmh}^{-1} * (3.93 \pm .02) \text{ kmh}^{-1} - (583.68 \pm .06) \text{ km}^2\text{h}^{-2} \\ \sigma_f &= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 \sigma_{x_3}^2} \\&= \sqrt{(x_2)^2 (.2)^2 + (x_1)^2 (.02)^2 + (-1)^2 (.06)^2} \\&= \sqrt{(3.93)^2 (.2)^2 + (1005.1)^2 (.02)^2 + 1(.0004)} \\&= \sqrt{15.4(.04) + 1010200(.0004) + .0004} \\&= \sqrt{.6 + 400 + .0004} \\&= \sqrt{400} \\&= 20 \\f &= (1005.1 \pm .2) \text{ kmh}^{-1} * (3.93 \pm .02) \text{ kmh}^{-1} - (583.68 \pm .06) \text{ km}^2\text{h}^{-2} \\&= (1005.1 * 3.93 - 583.68 \pm \sigma_f) \text{ km}^2\text{h}^{-2} \\&= (3370 \pm 20) \text{ km}^2\text{h}^{-2}\end{aligned}$$

5. (a)

$$\begin{aligned}
y &= x_1^2 \sin(x_1 x_2) \\
&= (1.23 \pm .04)^2 \sin((1.23 \pm .04) * (1.99 \pm .01)) \\
\sigma_y &= \sqrt{\left(\frac{\partial y}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial y}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\
&= \sqrt{(2x_1 \sin(x_1 x_2) + x_1^2 x_2 \cos(x_1 x_2))^2 (.04)^2 + (x_1^3 \cos(x_1 x_2))^2 (.01)^2} \\
&= ((2(1.23) \sin(1.23(1.99)) + .002(1.23)^2(1.99) \cos(1.23(1.99)))^2 + \\
&\quad + .0001((1.23)^3 \cos(1.23(1.99)))^2)^{\frac{1}{2}} \\
&= \sqrt{(2.46 \sin(2.45) + 1.51(.004) \cos(2.45))^2 + .0001(1.86 \cos(2.45))^2} \\
&= \sqrt{(2.46(.638) + .006(-.770))^2 + .0001(1.86(-.770))^2} \\
&= \sqrt{(1.57 - .005)^2 + .0001(-1.43)^2} \\
&= \sqrt{(2)^2 + .0001(2.04)} \\
&= \sqrt{4 + .0002} \\
&= \sqrt{4} \\
&= 2 \\
y &= (1.23 \pm .04)^2 \sin((1.23 \pm .04) * (1.99 \pm .01)) \\
&= (1.23)^2 \sin(1.23(1.99)) \pm \sigma_y \\
&= 1.51 \sin(2.45) \pm 2 \\
&= 1.51(.638) \pm 2 \\
&= 0.963 \pm 2 \\
&= 1 \pm 2
\end{aligned}$$

(b)

$$\begin{aligned}
y &= x_1^2 x_2^3 e^{x_1 x_2} + x_1^2 x_2 \\
&= (1.23 \pm .04)^2 * (1.99 \pm .01)^3 e^{(1.23 \pm .04) * (1.99 \pm .01)} + (1.23 \pm .04)^2 * (1.99 \pm .01) \\
\sigma_y &= \sqrt{\left(\frac{\partial y}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial y}{\partial x_2}\right)^2 \sigma_{x_2}^2} \\
&= \sqrt{(2x_1 x_2^3 e^{x_1 x_2} + x_1^2 x_2^4 e^{x_1 x_2} + 2x_1 x_2)^2 (.04)^2 + (3x_1^2 x_2^2 e^{x_1 x_2} + x_1^3 x_2^3 e^{x_1 x_2} + x_1^2)^2 (.01)^2} \\
&= ((2(1.23)(1.99)^3 e^{1.23(1.99)} + (1.23)^2 (1.99)^4 e^{1.23(1.99)} + 2(1.23)(1.99))^2 (.002) + \\
&\quad + (3(1.23)^2 (1.99)^2 e^{1.23(1.99)} + (1.23)^3 (1.99)^3 e^{1.23(1.99)} + (1.23)^2)^2 (.0001))^{\frac{1}{2}} \\
&= ((2.46(7.88)e^{2.45} + 1.51(15.7)e^{2.45} + 4.90)^2 (.002) + (3(1.51)(3.96)e^{2.45} + \\
&\quad + 1.86(7.88)e^{2.45} + 1.51)^2 (.0001))^{\frac{1}{2}} \\
&= \sqrt{(19.4(11.6) + 23.7(11.6) + 4.90)^2 (.002) + (17.9(11.6) + 14.7(11.6) + 1.51)^2 (.0001)} \\
&= \sqrt{(225 + 275 + 4.90)^2 (.002) + (208 + 171 + 1.51)^2 (.0001)} \\
&= \sqrt{(505)^2 (.002) + (381)^2 (.0001)} \\
&= \sqrt{255000(.002) + 145000(.0001)} \\
&= \sqrt{500 + 10} \\
&= \sqrt{500} \\
&= 20 \\
y &= (1.23 \pm .04)^2 * (1.99 \pm .01)^3 e^{(1.23 \pm .04) * (1.99 \pm .01)} + (1.23 \pm .04)^2 * (1.99 \pm .01) \\
&= (1.23)^2 (1.99)^3 e^{1.23(1.99)} + 1.99(1.23)^2 \pm \sigma_y \\
&= 1.51(7.88)e^{2.45} + 1.99(1.51) \pm 20 \\
&= 138 + 3 \pm 20 \\
&= 141 \pm 20 \\
&= 140 \pm 20
\end{aligned}$$