Enumerative combinatorics

Midterm exam

Andrey Chizhov

Problem 1. Find the number of ways to rearrange the letters of your last name so that the order of the vowels would not change.

Solution:

Number of permutations of the letters is 7!. My last name has 2 vowels, so to keep the order we should divide this number by 2!.

$$\frac{7!}{2!} = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 2520$$

Answer: 2520.

Problem 2. The sequence a_n is given by the recurrence relation

$$a_n = 2a_{n-2} + a_{n-3}$$

and initial conditions

$$a_0 = 3,\ \ a_1 = 0,\ \ a_2 = 4.$$

a) Find the generating function corresponding to the sequence.
b) Find an explicit formula for a_n.

Solution:

a) Let's write down the generating function for the sequence

$$A(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n + \ldots$$

$$A(q) = a_0 + a_1 q + a_2 q^2 + \sum_{i \geq 3} (2a_{i-2} + a_{i-3}) q^i$$

$$A(q) = a_0 + a_1 q + a_2 q^2 + 2 \sum_{i \geq 3} 2a_{i-2} q^i + \sum_{j \geq 3} a_{j-3} q^j$$

$$A(q) = a_0 + a_1 q + a_2 q^2 + 2q^2 (A(q) - a_0) + q^3 A(q)$$

Now considering first elements, we can compute the closed form

$$A(q) = 3 - 2q^2 + 2q^2 A(q) + q^3 A(q)$$

$$A(q) = \frac{3 - 2q^2}{1 - 2q^2 - q^3}$$

b) Characteristic equation is the following polynomial of degree 3

$$x^3 - 2x - 1 = 0$$

$$(x+1)(x^2 - x - 1) = 0$$

$$x_1 = -1,\, x_{2,3} = \frac{1 \pm \sqrt{5}}{2}$$

So an explicit formula is as follows

$$a_n = c_1(-1)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n + c_3\left(\frac{1+\sqrt{5}}{2}\right)^n$$

We can find c_1,c_2,c_3 by solving the system of 3 equations for the first elements of the sequence

$$\begin{cases} a_0 = 3 = c_1 + c_2 + c_3 \\ a_1 = 0 = -c_1 + c_2\frac{1-\sqrt{5}}{2} + c_3\frac{1+\sqrt{5}}{2} \\ a_2 = 4 = c_1 + c_2(\frac{1-\sqrt{5}}{2})^2 + c_3(\frac{1+\sqrt{5}}{2})^2 \end{cases}$$

Let's notice that $c_1 = c_2 = c_3 = 1$ is a solution. (one can show that there is no others)

$$\Rightarrow a_n = (-1)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Problem 3. Calculate $\sum\limits_{k=1}^{n} = (-1)^k k^2 \binom{n}{k}$, where $n \geq 1$.

Solution:

$$(1-x)^n = \sum_{k=0}^{n}(-1)^k x^k C_n^k$$

$$((1-x)^n)' = \sum_{k=1}^{n}(-1)^k k x^{k-1} C_n^k = (-n)(1-x)^{n-1} \quad | \cdot x$$

$$(-nx)(1-x)^{n-1} = \sum_{k=1}^{n}(-1)^k k x^k C_n^k$$

$$((-nx)(1-x)^{n-1})' = \boxed{\sum_{k=1}^{n}(-1)^k k^2 x^{k-1} C_n^k} = nx(n-1)(1-x)^{n-2}$$

To get the given sum, let's just say that $x = 1$ and it will give the answer.

$$\sum_{k=1}^{n} = (-1)^k k^2 C_n^k = n \cdot 0^{n-2}(n-1) = \begin{cases} 2, & n = 2 \\ 0, & n \neq 2 \end{cases}$$

Problem 4. Find the generating function for the sequence 5,0,5,0,5,0 . . . .

Solution:

Generating function is as follows

$$5 + 5q^2 + 5q^4 + \cdots = \sum_{n=0}^{\infty} 5q^{2n}$$

$$\sum_{n\geq 0} 5q^{2n} = \{q^2 = x\} = 5\sum_{n\geq 0} x^n = \frac{5}{1-x} = \frac{5}{1-q^2}$$