1 QAM-16 Constellation

Let $M = (2N)^2$ be a number of constellation points $c_k = x_k + iz_k$, then the mean energy

$$E_{\text{av}} = \sum_{k=1}^{M} p_k |c_k|^2 = \frac{1}{M} \sum_{k=1}^{M} x_k^2 + z_k^2 = \frac{1}{M} \sum_{j=-N}^{N-1} \sum_{n=-N}^{N-1} x_j^2 + z_n^2 = \frac{a^2}{M} \sum_{j=-N}^{N-1} \sum_{n=-N}^{N-1} (2j+1)^2 + (2n+1)^2 =$$

$$= \frac{a^2}{M} \sum_{j=-N}^{N-1} \sum_{n=-N}^{N-1} 2 + 4(j+n) + 4(j^2+n^2) = \frac{a^2}{M} \left[8N^2 - 2 \cdot 8N^2 + 16N \left(-N^2 + 2 \sum_{n=1}^{N} n^2 \right) \right].$$

Using the equality $\sum\limits_{n=1}^{N} n^2 = \dfrac{N(N+1)(2N+1)}{6}$ we obtain

$$E_{\text{av}} = \frac{2a^2}{3} (2N-1)(2N+1).$$

$$PARP = \frac{2a^2(2N-1)^2}{E_{\text{av}}} = 3 \cdot \frac{2N-1}{2N+1}.$$

Given $M = 16$ we obtain $E_{\text{av}} = 10a^2$ and $PARP = 9/5$ respectively.

2 AQAM-16 Constellation

Let $N = M/4$ be a number of circles of four equidistant constellation points

$$c_{jn} = 2a(1 - n + \sqrt{2}n) \exp \frac{i\pi}{4} \left(2j + \mod (n, 2)\right),$$

then the mean energy is

$$E_{\text{av}} = \frac{1}{M} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} |c_{jn}|^2 = \frac{(2a)^2}{N} \sum_{n=0}^{N-1} (1 - n + \sqrt{2}n)^2 = \frac{4a^2}{N} \sum_{n=0}^{N-1} 1 + 2(\sqrt{2}-1)n + (3 - 2\sqrt{2})n^2 =$$

$$= 4a^2 \left(1 + (\sqrt{2}-1)(N-1) + \frac{3 - 2\sqrt{2}}{6} (N-1)(2N-1) \right).$$

$$PARP = \frac{(2a)^2 \left(1 + (N-1)(\sqrt{2}-1)\right)^2}{E_{\text{av}}} = \frac{\left(1 + (N-1)(\sqrt{2}-1)\right)^2}{1 + (\sqrt{2}-1)(N-1) + \frac{3-2\sqrt{2}}{6}(N-1)(2N-1)}$$

3 Mutual information evaluation

Let us introduce a 2D Gaussian probability density function (.cpp function **prob**) about the constellation point c_k:

$$p_k(y) = \frac{1}{2\pi\sigma^2} \exp \left(-\frac{|y - c_k|^2}{2\sigma^2} \right).$$

Probability density (.cpp function **pr**) of a random value y is therefore given by

$$p(y) = \frac{1}{M} \sum_{k=1}^{M} p_k(y).$$

Mutual information is generally defined as following

$$I(y;c) = H(y) + H(c) - H(y,c) = H(c) - H(c|y) = H(y) - H(y|c), \tag{1}$$

where $H(c) = -\sum\limits_{k=1}^{M} \frac{1}{M} \log_2 \frac{1}{M} = \log_2 M$; $H(y) = -\int d^2y\, p(y) \log_2 p(y)$; $H(y,c)$ is joint entropy, i.e.

$$H(y,c) = -\sum_{k=1}^{M} \int d^2y\, \frac{p_k(y)}{M} \log_2 \frac{p_k(y)}{M};$$

and $H(y|c)$ and $H(c|y)$ are conditional entropies:

$$H(y|c) = -\sum_{k=1}^{M} \frac{1}{M} \int d^2y\, p_k(y) \log_2 p_k(y),$$

$$H(c|y) = -\int d^2y\, p(y) \sum_{k=1}^{M} \frac{p_k(y)}{\sum\limits_{k=1}^{M} p_k(y)} \log_2 \frac{p_k(y)}{\sum\limits_{k=1}^{M} p_k(y)} = -\frac{1}{M} \sum_{k=1}^{M} \int p_k(y) \left(\log_2 \frac{p_k(y)}{M} - \log_2 p(y) \right) =$$

$$= H(y,c) - H(y).$$

Thus, after a little simplification one can see that all the three representations (1) for $I(y;c)$ come down to the following result

$$I(y;c) = \log_2 M - \int d^2y\, p(y) \log_2 Mp(y) + \frac{1}{M} \sum_{k=1}^{M} \int d^2y\, p_k(y) \log_2 p_k(y). \tag{2}$$

There are two integrals in the formula (2): the first one is to be computed numerically (.cpp function **entropy**), while the second one can be evaluated analytically. Indeed, let us see the following

$$\int d^2y\, p_k(y) \log_2 p_k(y) = \big[r = |y - c_k| \big] = \frac{1}{2\pi\sigma^2 \ln 2} \int\limits_0^{2\pi} d\varphi \int\limits_0^{\infty} r \exp\left(-\frac{r^2}{2\sigma^2}\right) \left(-\frac{r^2}{2\sigma^2} - \ln(2\pi\sigma^2)\right) dr =$$

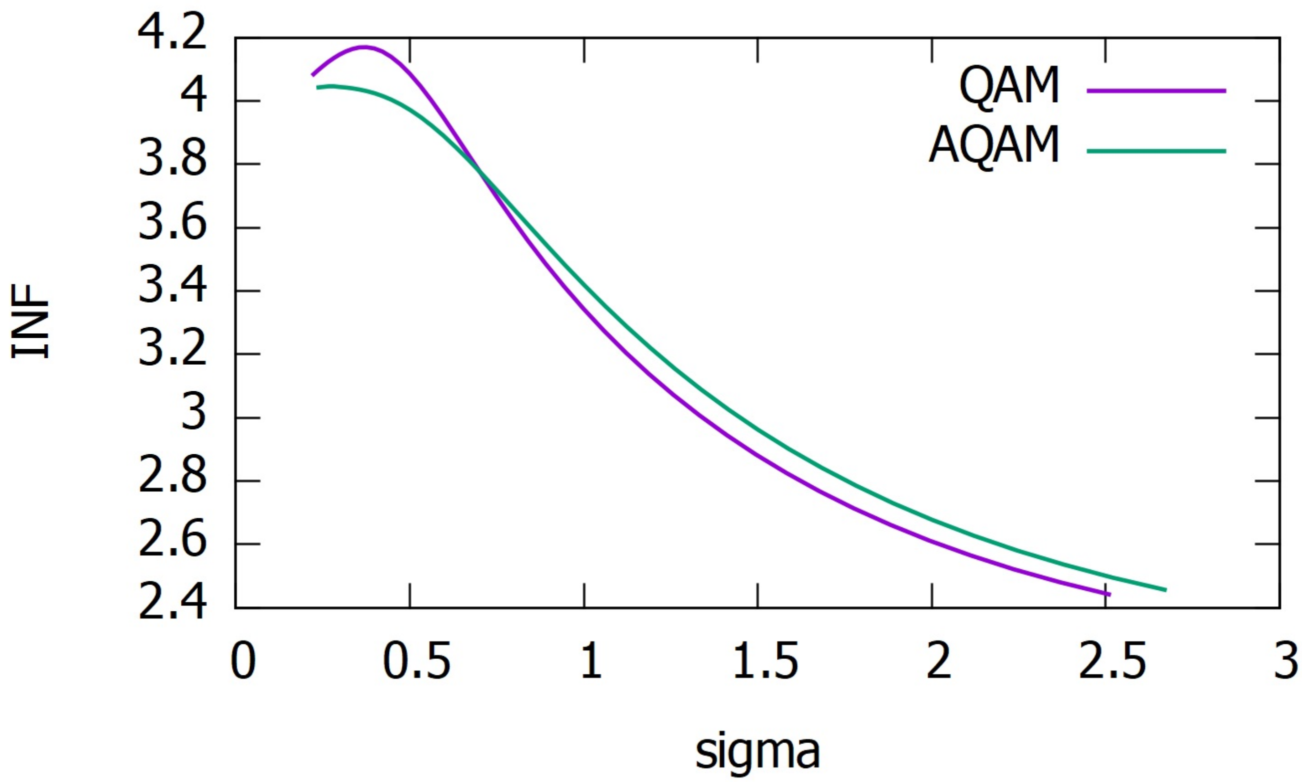
$$= \left[t = \frac{r^2}{2\sigma^2} \right] = -\frac{1}{\ln 2} \int\limits_0^{\infty} te^t\, dt - \frac{\ln(2\pi\sigma^2)}{\ln 2} \int\limits_0^{\infty} e^t\, dt = -\log_2(2\pi e\sigma^2).$$

Finally we have

$$I(y;c) = \log_2 M - \log_2(2\pi e\sigma^2) - \int d^2y\, p(y) \log_2 Mp(y).$$

This result is implemented in the final loop of .cpp **main**.

Below are the two plots for $I(y;c)$



INF
sigma
QAM
AQAM
4.2
4
3.8
3.6
3.4
3.2
3
2.8
2.6
2.4
0
0.5
1
1.5
2
2.5
3