

## 1 QAM-16 Constellation

Let  $M = (2N)^2$  be a number of constellation points  $c_k = x_k + iz_k$ , then the mean energy

$$\begin{aligned} E_{av} &= \sum_{k=1}^M p_k |c_k|^2 = \frac{1}{M} \sum_{k=1}^M x_k^2 + z_k^2 = \frac{1}{M} \sum_{j=-N}^{N-1} \sum_{n=-N}^{N-1} x_j^2 + z_n^2 = \frac{a^2}{M} \sum_{j=-N}^{N-1} \sum_{n=-N}^{N-1} (2j+1)^2 + (2n+1)^2 = \\ &= \frac{a^2}{M} \sum_{j=-N}^{N-1} \sum_{n=-N}^{N-1} 2 + 4(j+n) + 4(j^2 + n^2) = \frac{a^2}{M} \left[ 8N^2 - 2 \cdot 8N^2 + 16N \left( -N^2 + 2 \sum_{n=1}^N n^2 \right) \right]. \end{aligned}$$

Using the equality  $\sum_{n=1}^N n^2 = \frac{N(N+1)(2N+1)}{6}$  we obtain

$$E_{av} = \frac{2a^2}{3} (2N-1)(2N+1).$$

$$PARP = \frac{2a^2(2N-1)^2}{E_{av}} = 3 \cdot \frac{2N-1}{2N+1}.$$

Given  $M = 16$  we obtain  $E_{av} = 10a^2$  and  $PARP = 9/5$  respectively.

## 2 AQAM-16 Constellation

Let  $N = M/4$  be a number of circles of four equidistant constellation points

$$c_{jn} = 2a(1 - n + \sqrt{2}n) \exp \frac{i\pi}{4} (2j + \text{mod}(n, 2)),$$

then the mean energy is

$$\begin{aligned} E_{av} &= \frac{1}{M} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} |c_{jn}|^2 = \frac{(2a)^2}{N} \sum_{n=0}^{N-1} (1 - n + \sqrt{2}n)^2 = \frac{4a^2}{N} \sum_{n=0}^{N-1} 1 + 2(\sqrt{2} - 1)n + (3 - 2\sqrt{2})n^2 = \\ &= 4a^2 \left( 1 + (\sqrt{2} - 1)(N-1) + \frac{3 - 2\sqrt{2}}{6} (N-1)(2N-1) \right). \end{aligned}$$

$$PARP = \frac{(2a)^2 (1 + (N-1)(\sqrt{2} - 1))^2}{E_{av}} = \frac{(1 + (N-1)(\sqrt{2} - 1))^2}{1 + (\sqrt{2} - 1)(N-1) + \frac{3-2\sqrt{2}}{6} (N-1)(2N-1)}$$

## 3 Mutual information evaluation

Let us introduce a 2D Gaussian probability density function (.cpp function **prob**) about the constellation point  $c_k$ :

$$p_k(y) = \frac{1}{2\pi\sigma^2} \exp \left( -\frac{|y - c_k|^2}{2\sigma^2} \right).$$

Probability density (.cpp function **pr**) of a random value  $y$  is therefore given by

$$p(y) = \frac{1}{M} \sum_{k=1}^M p_k(y).$$

Mutual information is generally defined as following

$$I(y; c) = H(y) + H(c) - H(y, c) = H(c) - H(c|y) = H(y) - H(y|c), \quad (1)$$

where  $H(c) = -\sum_{k=1}^M \frac{1}{M} \log_2 \frac{1}{M} = \log_2 M$ ;  $H(y) = -\int d^2y p(y) \log_2 p(y)$ ;  $H(y, c)$  is joint entropy, i.e.

$$H(y, c) = -\sum_{k=1}^M \int d^2y \frac{p_k(y)}{M} \log_2 \frac{p_k(y)}{M};$$

and  $H(y|c)$  and  $H(c|y)$  are conditional entropies:

$$H(y|c) = -\sum_{k=1}^M \frac{1}{M} \int d^2y p_k(y) \log_2 p_k(y),$$

$$\begin{aligned} H(c|y) &= -\int d^2y p(y) \sum_{k=1}^M \frac{p_k(y)}{\sum_{k=1}^M p_k(y)} \log_2 \frac{p_k(y)}{\sum_{k=1}^M p_k(y)} = -\frac{1}{M} \sum_{k=1}^M \int p_k(y) \left( \log_2 \frac{p_k(y)}{M} - \log_2 p(y) \right) = \\ &= H(y, c) - H(y). \end{aligned}$$

Thus, after a little simplification one can see that all the three representations (1) for  $I(y; c)$  come down to the following result

$$I(y; c) = \log_2 M - \int d^2y p(y) \log_2 M p(y) + \frac{1}{M} \sum_{k=1}^M \int d^2y p_k(y) \log_2 p_k(y). \quad (2)$$

There are two integrals in the formula (2): the first one is to be computed numerically (.cpp function **entropy**), while the second one can be evaluated analytically. Indeed, let us see the following

$$\begin{aligned} \int d^2y p_k(y) \log_2 p_k(y) &= [r = |y - c_k|] = \frac{1}{2\pi\sigma^2 \ln 2} \int_0^{2\pi} d\varphi \int_0^\infty r \exp\left(-\frac{r^2}{2\sigma^2}\right) \left(-\frac{r^2}{2\sigma^2} - \ln(2\pi\sigma^2)\right) dr = \\ &= \left[t = \frac{r^2}{2\sigma^2}\right] = -\frac{1}{\ln 2} \int_0^\infty t e^t dt - \frac{\ln(2\pi\sigma^2)}{\ln 2} \int_0^\infty e^t dt = -\log_2(2\pi e\sigma^2). \end{aligned}$$

Finally we have

$$I(y; c) = \log_2 M - \log_2(2\pi e\sigma^2) - \int d^2y p(y) \log_2 M p(y).$$

This result is implemented in the final loop of .cpp **main**.

## 4 Figures

Below are the plots of capacities  $C(\text{SNR})$  for both constellations considered. One can easily see that both capacities do not exceed Shannon limit  $\log_2(1 + \text{SNR})$ , although approach that as SNR decreases, as expected. On the other hand, both capacities also never exceed a level of 4 B/s, although approach that as SNR increases, as expected.

