PPKE - ITK Stipendium Hungaricum

Analog signals and systems

Random Processes

Analog and Digital Signal Processing 2020

Autocorrelation $R_{XX}(\tau)$

- The autocorrelation of X(t) is

$$R_{XX}(\tau) = E\left[X(t)X(t+\tau)\right]$$

- Properties of $R_{XX}(\tau)$:

1. $$R_{XX}(-\tau) = R_{XX}(\tau)$$
2. $$\left|R_{XX}(\tau)\right| \leq R_{XX}(0)$$
3. $$R_{XX}(0) = E\left[X^2(t)\right]$$

- Property 3 is easily obtained by setting $\tau = 0$ in the first. If we assume that X(t) is a voltage waveform across a 1-Ω resistor, then $E[X^2(t)]$ is the average value of power delivered to the 1-Ω resistor by X(t). Thus, $E[X^2(t)]$ is often called the average power of X(t).

Practice Problem 1.

Cross-correlation $R_{XY}(\tau)$

- The cross-correlation of X(t) and Y(t) is

$$R_{XY}(\tau) = E\left[X(t)Y(t+\tau)\right]$$

- Properties of $R_{XY}(\tau)$:

1. $R_{XY}(-\tau) = R_{YX}(\tau)$

2. $\left|R_{XY}(\tau)\right| \le \sqrt{R_{XX}(0)R_{YY}(0)}$

3. $\left|R_{XY}(\tau)\right| \le \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$

Practice Problem 2.

Power Spectral Density or Power Spectrum

- Let $R_{XX}(\tau)$ be the autocorrelation of X(t). Then the power spectral density (or power spectrum) of X(t) is defined by the Fourier transform of $R_{XX}(\tau)$ as

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

- Thus,

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

- The above equations are known as the Wiener-Khinchin relations.

- Properties of $S_{XX}(\omega)$:

  - $S_{XX}(\omega)$ is real and $S_{XX}(\omega) \geq 0$

  - $S_{XX}(-\omega) = S_{XX}(\omega)$

  - $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = R_{XX}(0) = E\left[X^2(t)\right]$

Cross-Power Spectral Densities

- The *cross-power spectral density* (or *cross-power spectrum*) $S_{XY}(\omega)$ of two continuous-time random processes X(t) and Y(t) is defined as the Fourier transform of $R_{XY}(\tau)$:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

- Thus, taking the inverse Fourier transform of $S_{XY}(\omega)$, we get

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

- Properties of $S_{XY}(\omega)$:
  Unlike $S_{XX}(\omega)$, which is a real-valued function of ω, $S_{XY}(\omega)$, in general, is a complex-valued function.

  - $$S_{XY}(\omega) = S_{YX}(-\omega)$$
  - $$S_{XY}(-\omega) = S^*_{XY}(\omega)$$

White Noise

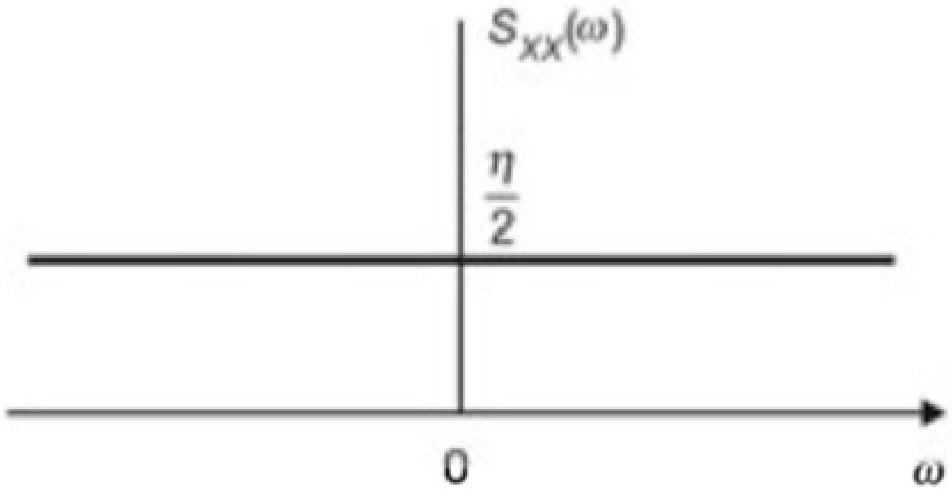
- A random process X(t) is called white noise if

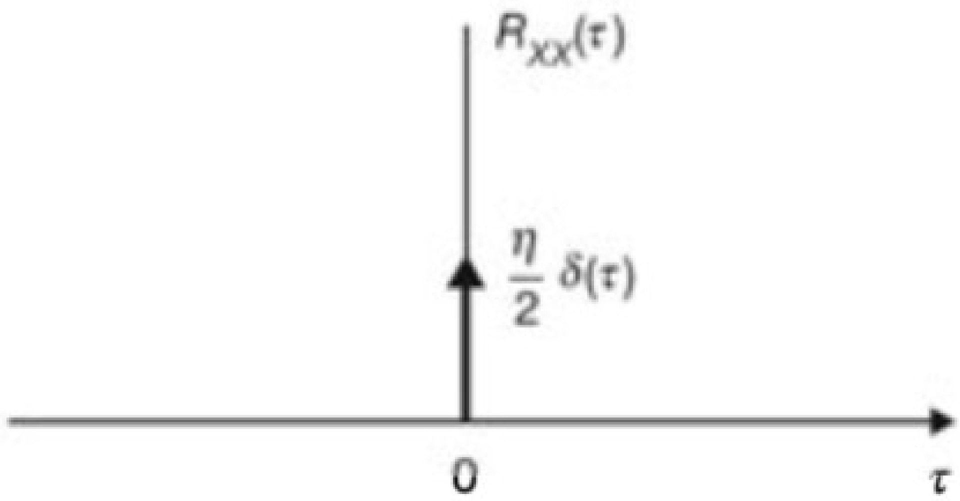
$$S_{XX}(\omega) = \frac{\eta}{2}$$

- Taking the inverse Fourier transform of the above equation, we have

$$R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$$

- It is usually assumed that the mean of white noise is zero.





Band-Limited White Noise

- A random process X(t) is called band-limited white noise if

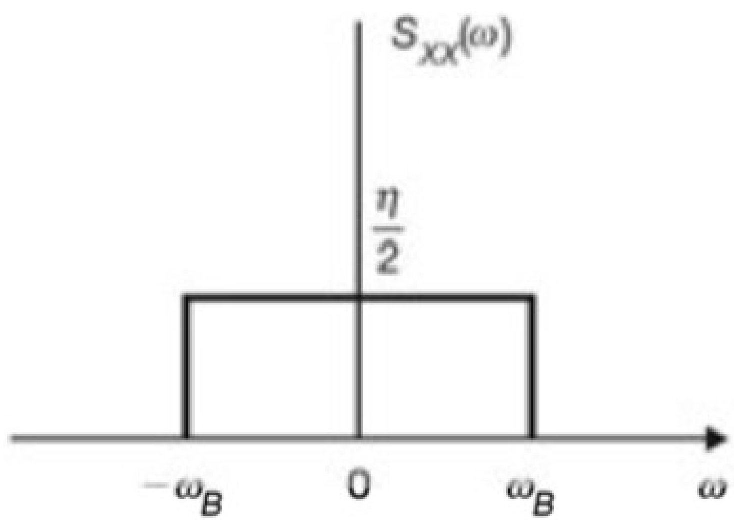
$$S_{XX}(\omega) = \begin{cases} \frac{\eta}{2} & |\omega| \leq \omega_B \\ 0 & |\omega| > \omega_B \end{cases}$$

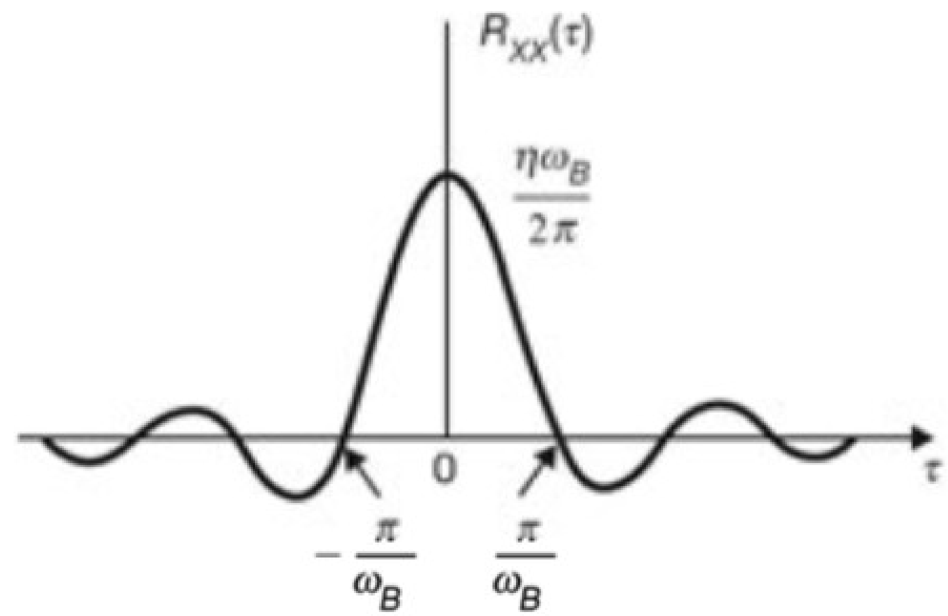
- Then

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} \frac{\eta}{2} e^{j\omega\tau} d\omega = \frac{\eta\omega_B}{2\pi}\frac{sin\omega_B\tau}{\omega_B\tau}$$

- And $S_{XX}(\omega)$ and $R_{XX}(\tau)$ of band-limited white noise can be seen on the diagram.

- Note that the term white or band-limited white refers to the spectral shape of the process X(t) only, and these terms do not imply that the distribution associated with X(t) is Gaussian.





Narrowband Random Process

- Suppose that X(t) is a WSS process with zero mean and its power spectral density $S_{XX}(\omega)$ is nonzero only in some narrow frequency band of width 2W that is very small compared to a center frequency ω_C. Then the process X(t) is called a narrowband random process.

- In many communication systems, a narrowband process (or noise) is produced when white noise (or broadband noise) is passed through a narrowband linear filter.

- When a sample function of the narrowband process is viewed on an oscilloscope, the observed waveform appears as a sinusoid of random amplitude and phase. For this reason, the narrowband noise X(t) is conveniently represented by the expression

$$X(t) = V(t) cos \left[\omega_C t + \phi(t)\right]$$

Response of Linear System to Random Input

- A system is a mathematical model for a physical process that relates the input (or excitation) signal x to the output (or response) y, and the system is viewed as a transformation (or mapping) of x into y. This transformation is represented by the operator T as

$$y = \mathbf{T}x$$

- For a continuous-time linear time-invariant (LTI) system,

$$y(t) = \int_{-\infty}^{\infty} h(\alpha)x(t-\alpha)d\alpha = h(t) * x(t)$$

- where h(t) is the impulse response of a continuous-time LTI system.

Response of a Continuous-Time Linear System to Random Input

- When the input to a continuous-time linear system represented by y=**T**x is a random process {X(t), t ∈ T_X), then the output will also be a random process {Y(t), t ∈ T_Y}; that is,

$$\mathbf{T}\left\{X(t), t \in T_x\right\} = \left\{Y(t), t \in T_y\right\}$$

- For any input sample function x_i(t), the corresponding output sample function is

$$y_i(t) = \mathbf{T}\left\{x_i(t)\right\}$$

- If the system is LTI, then by the convolution equation, we can write

$$Y(t) = \int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha = h(t) * X(t)$$

- The above equation is stochastic integral, then

$$\mu_Y(t) = E\left[Y(t)\right] = E\left[\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha\right] =$$

$$= \int_{-\infty}^{\infty} h(\alpha)E\left[X(t-\alpha)\right]d\alpha = \int_{-\infty}^{\infty} h(\alpha)\mu_X(t-\alpha)d\alpha = h(t) * \mu_X(t)$$

$$R_{YY}(t_1, t_2) = E\left[Y(t_1)Y(t_2)\right] = E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)X(t_1-\alpha)h(\beta)X(t_2-\beta)d\alpha d\beta\right] =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)E\left[X(t_1-\alpha)X(t_2-\beta)\right]d\alpha d\beta =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(t_1-\alpha, t_2-\beta)d\alpha d\beta$$

- If the input X(t) is WSS, then from the convolution we have

$$E\left[Y(t)\right] = \int_{-\infty}^{\infty} h(\alpha)\mu_X d\alpha = \mu_X \int_{-\infty}^{\infty} h(\alpha)d\alpha = \mu_X H(0)$$

- where H(0) is the frequency response of the linear system at ω = 0. Thus, the mean of the output is a constant.

Practice Problem 3.

- The autocorrelation of the output $R_{YY}(t_1, t_2)$ becomes

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(t_2 - t_1 + \alpha - \beta)d\alpha d\beta$$

- which indicates that $R_{YY}(t_1, t_2)$ is a function of the time difference $\tau = t_2 - t_1$. Hence,

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(\tau + \alpha - \beta)d\alpha d\beta$$

- Thus, we conclude that if the input X(t) is WSS, the output Y(t) is also WSS.

- The cross-correlation function between input X(t) and Y(t) is given by

$$R_{XY}(t_1, t_2) = E\left[X(t_1)Y(t_2)\right] = E\left[X(t_1)\int_{-\infty}^{\infty} h(\alpha)X(t_2-\alpha)d\alpha\right] =$$

$$= \int_{-\infty}^{\infty} h(\alpha)E\left[X(t_1)X(t_2-\alpha)\right]d\alpha = \int_{-\infty}^{\infty} h(\alpha)R_{XX}(t_1, t_2-\alpha)d\alpha$$

- When input X(t) is WSS, it becomes

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha)R_{XX}(t_2-t_1-\alpha)d\alpha$$

- which indicates that $R_{XY}(t_1, t_2)$ is a function of the time difference $\tau = t_2 - t_1$. Hence,

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(\alpha)R_{XX}(\tau-\alpha)d\alpha = h(\tau) * R_{XX}(\tau)$$

- Thus, we conclude that if the input X(t) to an LTI system is WSS, the output Y(t) is also WSS. Moreover, the input X(t) and output Y(t) are jointly WSS.

- In a similar manner, it can be shown that

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} h(-\alpha)R_{XY}(\tau-\alpha)d\alpha = h(-\tau) * R_{XY}(\tau)$$

- Substituting $R_{XY}(\tau)$ equation into the above equation, we have

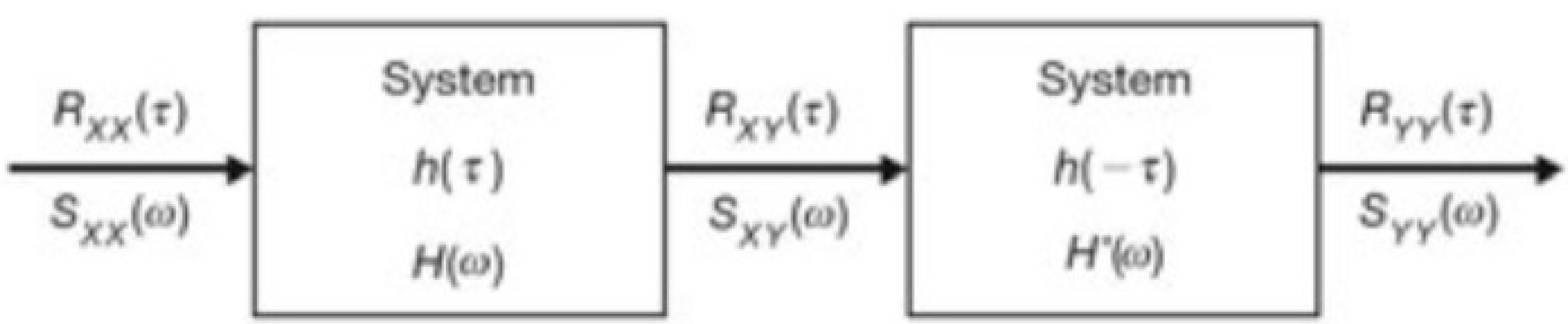
$$R_{YY}(\tau) = h(-\tau) * h(\tau) * R_{XX}(\tau)$$

- Now taking Fourier transforms of previous equations, and using convolution property of Fourier transform, we obtain

$$S_{XY}(\omega) = H(\omega)S_{XX}(\omega)$$

$$S_{YY}(\omega) = H^*(\omega)S_{XY}(\omega)$$

$$S_{YY}(\omega) = H^*(\omega)H(\omega)S_{XX}(\omega) = \left|H(\omega)\right|^2 S_{XX}(\omega)$$



- The last equation indicates the important result that the power spectral density of the output is the product of the power spectral density of the input and the magnitude squared of the frequency response of the system.

- When the autocorrelation of the output $R_{YY}(\tau)$ is desired, it is easier to determine the power spectral density $S_{YY}(\omega)$ and then to evaluate the inverse Fourier transform. Thus,

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{XX}(\omega)e^{j\omega t}d\omega$$

- The average power in the output Y(t) is

$$E\left[Y^2(t)\right] = R_{YY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{XX}(\omega)d\omega$$

Practice Problem 4.