Chapter 5

Ising model and phase transitions

5.1 Equilibrium statistical mechanics

GIBBS–BOLTZMANN DISTRIBUTION AND PARTITION FUNCTION

A spin system is described by placing a spin variable $\sigma_i \in \{-1, 1\}$ at every site i of a given lattice. A microstate is defined by a spin configuration $\{\sigma_i\}$, which specifies the values of the spins of every lattice site. The (canonical) probability to observe the system in the spin configuration $\{\sigma_i\}$ is given by the Gibbs–Boltzmann weight:

$$p_{\{\sigma_i\}} = \frac{e^{-\beta E_{\{\sigma_i\}}}}{Z} , \tag{5.1}$$

where $E_{\{\sigma_i\}}$ is the energy of the given configuration. The (canonical) partition function Z appearing in (5.1) is defined by:

$$Z(\beta, B) = \sum_{\{\sigma_i\}} e^{-\beta E_{\{\sigma_i\}}} , \tag{5.2}$$

and in general depends on the inverse temperature[1] $\beta = 1/T$ and the external magnetic field B. Averages are computed as usual:

$$\langle \mathcal{O} \rangle = \sum_{\{\sigma_i\}} p_{\{\sigma_i\}} \mathcal{O}_{\{\sigma_i\}} .$$

[1] We will measure the temperature in units of the Boltzmann constant, which can be reintroduced at any moment by the replacement $T \to k_B T$.

As we will see in a moment, the knowledge of the partition function is the key to all thermodynamical quantities.

We need now to specify the form of $E_{\{\sigma_i\}}$ for a given spin configuration. Nearest neighbor interactions are assumed[2] so that the energy of a given spin configuration is given by

DEFINITION OF THE ISING MODEL

$$E_{\{\sigma_i\}} = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j - \sum_i \sigma_i B_i \,. \tag{5.3}$$

J_{ij} is the spin interaction matrix strength and B_i is the external applied magnetic field (it most cases it can be considered constant $B_i = B$). If $J > 0$ the interaction is ferromagnetic while it is antiferromagnetic if $J < 0$. The energy (5.3) defines the model known as Ising model.

The free energy F can be obtained from the partition function (5.2) using the following relation:

FREE ENERGY, INTERNAL ENERGY, ENTROPY, SPECIFIC HEAT, MAGNETIZATION AND SUSCEPTIBILITY

$$F(\beta, B) = -\frac{1}{\beta}\log Z(\beta, B) \,. \tag{5.4}$$

The internal energy E and the entropy S are related to the free energy by the usual thermodynamic relation:

$$F = E - TS \,. \tag{5.5}$$

In particular, the internal energy and the specific heat can be obtained directly from the partition function (5.2) by employing the following relations:

$$E = -\frac{\partial}{\partial\beta}\log Z = \frac{\partial}{\partial\beta}(\beta F) \tag{5.6}$$

$$C = \frac{\partial E}{\partial T} = -\beta^2\frac{\partial E}{\partial\beta} \,. \tag{5.7}$$

where used $\frac{\partial}{\partial T} = -\beta^2\frac{\partial}{\partial\beta}$. The total magnetization

$$\langle M \rangle = \sum_i \langle\sigma_i\rangle = N\langle\sigma_i\rangle \equiv Nm$$

[2] They are actually the more general, if we add a constant term, since the spins are fermions!

can also be computed from the partition function:

$$\langle M\rangle = -\frac{\partial F}{\partial B}\,, \tag{5.8}$$

while the susceptibility is determined/defined by:

$$\chi = \frac{1}{N}\frac{\partial\langle M\rangle}{\partial B} = -\frac{1}{N}\frac{\partial^2 F}{\partial B^2}\,. \tag{5.9}$$

We will work with intensive variables, or equivalently, per spin variables:

$$f = \frac{F}{N} \qquad m = \frac{\langle M\rangle}{N} \qquad \epsilon = \frac{E}{N} \qquad c = \frac{C}{N}\,, \tag{5.10}$$

for which we have $f = -\frac{1}{\beta N}\log Z$ and

$$\epsilon = \frac{\partial}{\partial\beta}(\beta f) \qquad c = -\beta^2\frac{\partial\epsilon}{\partial\beta} \qquad m = -\frac{\partial f}{\partial B} \qquad \chi = \frac{\partial m}{\partial B}\,. \tag{5.11}$$

From (5.5) one can find the entropy per spin $s = \frac{\epsilon - f}{T}$. Using $\frac{\partial}{\partial\beta} = -T^2\frac{\partial}{\partial T}$ we can also write the internal energy and the specific heat as:

$$\epsilon = -T^2\frac{\partial}{\partial T}\left(\frac{1}{T}f\right) \qquad\qquad c = \frac{\partial\epsilon}{\partial T}\,. \tag{5.12}$$

The computation of f, ϵ, c, m, χ and s is the main goal of the study of equilibrium spin systems.

5.1.1 Fluctuations and correlations

SUSCEPTIBILITY AND SPECIFIC HEAT AS A MEASURE OF FLUCTUATIONS

It's useful to define the generating function

$$W(\beta, B) = \log Z(\beta, B) = -\beta F(\beta, B)\,,$$

from which we find:

$$\frac{\partial^2 W}{\partial B^2} = \beta^2\left[\langle M^2\rangle - \langle M\rangle^2\right] = -\beta\frac{\partial^2 F}{\partial^2 B} = -\beta N\frac{\partial^2 f}{\partial B^2}\,.$$

This last quantity is proportional to the magnetic susceptibility,

$$\chi = -\frac{\partial^2 f}{\partial B^2}\,.$$

which plays thus the role of magnetic variance[3] since:

$$\chi T = \frac{1}{N}\left[\left\langle M^2\right\rangle - \langle M\rangle^2\right] . \tag{5.13}$$

Thus the susceptibility measures the relative size of magnetization fluctuations around the average. The specific heat is instead the energy variance:

$$\frac{\partial^2 W}{\partial \beta^2} = \left\langle E^2\right\rangle - \langle E\rangle^2 = -\frac{\partial}{\partial \beta}\langle E\rangle = \frac{1}{\beta^2}C\,,$$

which measures the size of internal energy fluctuations around the average:

$$c\,T^2 = \frac{1}{N}\left[\left\langle E^2\right\rangle - \langle E\rangle^2\right] . \tag{5.14}$$

From relation (5.13) and (5.14) we expect that at a phase transition, where fluctuations are large and on all scales, both the susceptibility and the specific heat will diverge. We will soon see that this is indeed the case.

The two–point spin connected correlation function is: CORRELATION FUNCTION

$$C_{ij} = \langle \sigma_i \sigma_j\rangle - \langle \sigma_i\rangle\langle \sigma_j\rangle = \langle \sigma_i \sigma_j\rangle - m^2 = \langle (\sigma_i - \langle \sigma_i\rangle)(\sigma_j - \langle \sigma_j\rangle)\rangle\ . \tag{5.15}$$

This correlation function can be obtained from the generating function W if we promote the magnetic field to be space dependent $B \to B_i$:

$$C_{ij} = \frac{1}{\beta^2}\frac{\partial^2 W}{\partial B_i \partial B_j}\,. \tag{5.16}$$

Obviously W then becomes the generating function of spin connected correlation functions.

The two–point connected correlation is obviously related to the susceptibility since both are derived from W by derivatives with respect to the magnetic field. This relation is obtained as follows:

$$\begin{aligned}\sum_{ij} C_{ij} &= \sum_{ij}\left[\langle \sigma_i\sigma_j\rangle - \langle \sigma_i\rangle\langle \sigma_j\rangle\right]\\ &= \left\langle \sum_i \sigma_i \sum_j \sigma_j\right\rangle - \left\langle \sum_i \sigma_i\right\rangle\left\langle \sum_j \sigma_j\right\rangle\\ &= \left\langle M^2\right\rangle - \langle M\rangle^2\\ &= N\,T\,\chi\,. \end{aligned} \tag{5.17}$$

[3]The magnetic kurtosis is the so–called Binder cumulant.

Changing variables to $r = |i - j|$ gives $\sum_{ij} C_{ij} = N \sum_r C(r)$ and we find the sum rule:

$$\chi = \frac{1}{T} \sum_{\mathbf{r}} C(r) \,. \tag{5.18}$$

We will use relations (5.13) and (5.18) in the following to make precise quantitative statements about the strength/weakness of fluctuations and to derive scaling relations between critical exponents.

5.2 Mean field theory

MEAN FIELD THEORY $\equiv$ WEAK FLUCTUATIONS

The mean field approach consists in discarding fluctuations, and thus, as we will see it is quantitatively correct only above the upper critical dimension. The assumption of the mean field approach is that FLUCTUATIONS AROUND THE AVERAGE ARE SMALL. Still it often gives a qualitatively correct picture of the phase diagram. In the Ising model, the mean field approximation is implemented by linearizing the interaction between two near spins:

$$\begin{aligned}
\sigma_i \sigma_j &= [\sigma_i - \langle\sigma_i\rangle + \langle\sigma_i\rangle] [\sigma_j - \langle\sigma_j\rangle + \langle\sigma_j\rangle] \\
&= \langle\sigma_i\rangle \langle\sigma_j\rangle + [\sigma_i - \langle\sigma_i\rangle] \langle\sigma_j\rangle + [\sigma_j - \langle\sigma_j\rangle] \langle\sigma_i\rangle + \\
&\quad \underbrace{[\sigma_i - \langle\sigma_i\rangle] [\sigma_j - \langle\sigma_j\rangle]}_{\text{small}} \\
&\to m_i m_j + m_i [\sigma_j - m] + m_j [\sigma_i - m_i] \\
&= -m_i m_j + \sigma_i m_j + m_i \sigma_j \,,
\end{aligned} \tag{5.19}$$

where $\langle\sigma_i\rangle = m_i$ and we discarded the quadratic fluctuation term since we are assuming that the fluctuation are small. If the external magnetic field is constant then $m_i \equiv m$ for every i due to translation invariance and the the energy of a spin configuration becomes simply:

$$\begin{aligned}
E^{MF}_{\{\sigma_i\}} &= -J \sum_{\langle i,j\rangle} \left[-m^2 + m(\sigma_i + \sigma_j)\right] - B \sum_i \sigma_i \\
&= m^2 J \frac{Nz}{2} - (Jzm + B) \sum_i \sigma_i \,,
\end{aligned} \tag{5.20}$$

where we used the fact that there are $\frac{Nz}{2}$ nearest neighbors on a lattice (z is the coordination number) and that $\sum_{\langle i,j\rangle} \sigma_i = \frac{z}{2} \sum_i \sigma_i$. For a square lattice we have $z = 2d$ and in general the coordination number is proportional to the dimension. In this approximation the net effect of the spin interaction is to shift the external magnetic field to the value $Jmz + B$.

The partition function is easily evaluated: MEAN FIELD FREE ENERGY

$$\begin{aligned} Z^{MF} &= \sum_{\{\sigma_i\}} e^{-\beta E^{MF}_{\{\sigma_i\}}} \\ &= e^{-\beta m^2 J \frac{Nz}{2}} \sum_{\{\sigma_i\}} e^{\beta(Jzm+B)\sum_i \sigma_i} \\ &= e^{-\beta m^2 J \frac{Nz}{2}} \sum_{\{\sigma_i\}} \prod_i e^{\beta(Jzm+B)\sigma_i} \\ &= e^{-\beta m^2 J \frac{Nz}{2}} \left\{2\cosh\left[\beta\left(Jzm+B\right)\right]\right\}^N . \end{aligned}$$

Thus the free energy per spin $f^{MF} = -\frac{1}{\beta N}\log Z^{MF}$ is:

$$f^{MF} = m^2 J \frac{z}{2} - \frac{1}{\beta}\log 2\cosh\left[\beta\left(Jzm+B\right)\right] . \tag{5.21}$$

Note that this relation is valid for any lattice in arbitrary dimension.

5.2.1 Mean field phase diagram

MEAN FIELD PHASE DIAGRAM: PHASE TRANSITION

To find the equilibrium value for the magnetization per spin we need to find the minimum of the free energy:

$$\frac{\partial f^{MF}}{\partial m} = 0 \qquad \Rightarrow \qquad m(T,B) = \tanh\left[\frac{T_c}{T} m(T,B) + \frac{B}{T}\right] , \tag{5.22}$$

where we defined the mean field critical temperature $T_c = Jz$. For $B = 0$ is easy to see that there is a phase transition at $T = T_c$ by studying the solutions of:

$$m = \tanh \frac{T_c}{T} m , \tag{5.23}$$

which can be found graphically as shown in Figure 5.1. Near T_c the magnetization is small and we can expand equation (5.20) to obtain[4]

$$m = \frac{T_c}{T} m - \frac{T_c^3}{3T^3} m^3 + O\left(m^4\right) ,$$

[4] $\tanh x = x - \frac{x^3}{3} + \ldots$

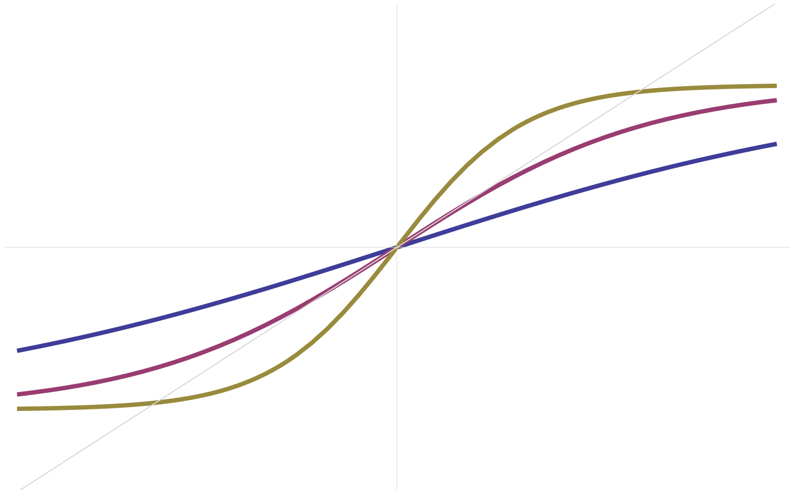

Figure 5.1: Graphical solution to equation (5.23) at $B = 0$. From top $T < T_c$, $T = T_c$ and $T > T_c$. For $T < T_c$ the system has a spontaneous magnetization.

and so

$$m_\pm = \begin{cases} 0 & T \geq T_c \\ \pm\sqrt{3\left(\frac{T}{T_c}\right)^3}\sqrt{\frac{T_c}{T} - 1} \sim \pm\sqrt{3}(-t)^{1/2} & T \to T_c^- \end{cases}$$

where we defined the reduced temperature $t = \frac{T}{T_c} - 1$. The order parameter is m and at the second order phase transition behaves thus as:

$$m_\pm \sim \pm(-t)^{1/2} \qquad T \to T_c^- \qquad t \to 0^- \,. \tag{5.24}$$

Thus the magnetization critical exponent is $\beta^{MF} = \frac{1}{2}$ in the mean field approximation.

At criticality the magnetization as a function of the external magnetic field can be written in the following way:

$$\begin{aligned} m(T_c, B) &= m(T_c, B) + \frac{B}{T_c} - \frac{1}{3}\left(m(T_c, B) + \frac{B}{T_c}\right)^3 + ... \\ &= m + \frac{B}{T_c} - \frac{1}{3}m^3 + ... \end{aligned}$$

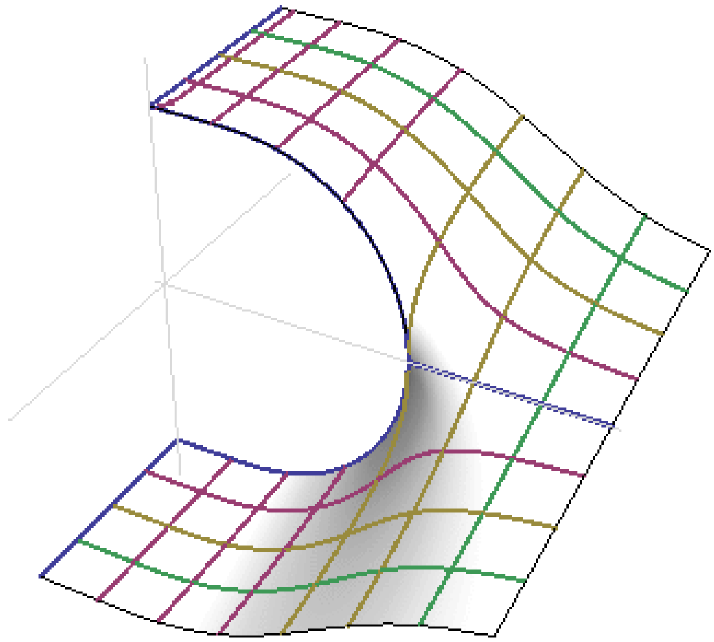

Figure 5.2: Mean field per spin magnetization for the Ising model.

from which we find:

$$m(T_c, B) = \mathrm{sign}(B)\left(\frac{3|B|}{T_c}\right)^{\frac{1}{3}} \sim \mathrm{sign}(B)|B|^{\frac{1}{\delta}} \,. \tag{5.25}$$

This gives the mean field magnetic exponent $\delta^{MF} = 3$.

It's easy now to compute the susceptibility:

$$\chi(T, B) = \frac{\partial m}{\partial B} = \frac{1}{T}\frac{1}{\cosh^2\left(m\frac{T_c}{T} + \frac{B}{T}\right) - \frac{T_c}{T}} \,. \tag{5.26}$$

Around the critical point we find using (5.24):

$$\chi_\pm(t, 0) = \begin{cases} 4\frac{1}{T-T_c} & T \to T_c^+ \\ 2\frac{1}{T_c-T} & T \to T_c^- \end{cases}$$

thus

$$\chi_\pm(t, 0) \sim \chi_\pm |t|^{-1} \qquad t \to 0^\pm \,. \tag{5.27}$$

This gives the mean field susceptibility critical exponent $\gamma^{MF} = 1$ and the mean field amplitude ratio $\chi_+/\chi_- = 2$. Note that the fact that the susceptibility diverges at the critical point means that the magnetic variance diverges.

Next we compute the energy per spin; we find simply:

$$\epsilon = \frac{\partial(\beta f)}{\partial \beta} = -\frac{Jz}{2}m^2 \,.$$

The specific heat per spin is not well defined in the mean field approximation (in particular it is not divergent at the critical point), but it usually assumed/defined that $\alpha_{MF} = 0$.

5.2.2 Correlation function

COMPUTING THE MEAN CORRELATION FUNCTION

If we want to compute the correlation function we need to relax the assumption of constant magnetic field in order to use (5.16). In this case the partition function turns out to be:

$$\begin{aligned} Z^{MF} &= \sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle i,j\rangle}[-m_i m_j + \sigma_i m_j + m_i \sigma_j] + \beta \sum_i B_i \sigma_i} \\ &= e^{-\beta J \sum_{\langle i,j\rangle} m_i m_j} \sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle i,j\rangle}[\sigma_i m_j + m_i \sigma_j] + \beta \sum_i B_i \sigma_i} \,. \end{aligned} \tag{5.28}$$

We need now to evaluate the sums:

$$\begin{aligned}\sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle i,j\rangle}[\sigma_i m_j + m_i \sigma_j] + \beta B \sum_i \sigma_i} &= \sum_{\{\sigma_i\}} e^{2\beta J \sum_{\langle i,j\rangle} \sigma_i m_j + \beta \sum B_i \sigma_i} \\ &= \sum_{\{\sigma_i\}} e^{\beta \sum_i \sigma_i \left[J \sum_{j(i)} m_j + B_i\right]} \\ &= \prod_i 2\cosh \beta \left(J \sum_{j(i)} m_j + B_i \right) ,\end{aligned}$$

where $j(i)$ are the nearest neighbors of i. The free energy is thus:

$$f^{MF} = \frac{J}{N} \sum_{\langle i,j\rangle} m_i m_j - \frac{1}{\beta N} \sum_i \log 2\cosh \left[\beta \left(J \sum_{j(i)} m_j + B_i \right) \right] . \qquad (5.29)$$

Minimizing the free energy leads to:

$$m_i = \tanh \left[\beta \left(J \sum_{j(i)} m_j + B_i \right) \right] , \qquad (5.30)$$

where we used:

$$\sum_{j(i)} \delta_{jk} = \delta_{ik}$$

and

$$\frac{\partial}{\partial m_k} \sum_{\langle i,j\rangle} m_i m_j = 2 \sum_{\langle i,j\rangle} m_i \delta_{jk} = \sum_i m_i \sum_{j(i)} \delta_{jk} = m_k \, .$$

The average magnetic field felt by a spin σ_i is now $J \sum_{j(i)} m_j + B_i$; equation (5.30) reduces to (5.22) when m_i is constant since $\sum_{j(i)} 1 = z$. To compute the correlation function using (5.16) we need to determine the solutions of (5.30) to linear order in the external magnetic field. Since we are interested in the critical region, we can assume that also the magnetization is small and expand the hyperbolic tangent to first order:

$$-\beta J \sum_{j(i)} m_j + m_i = \beta B_i \, . \qquad (5.31)$$

To solve (5.31) we perform a Fourier transform:

$$-\beta J m_{\mathbf{k}} \sum_a 2\cos k_a + m_{\mathbf{k}} = \beta B_{\mathbf{k}}\,, \tag{5.32}$$

where $a = 1, ..., d$ runs over the basis vectors of the lattice. We find:

$$m_{\mathbf{k}} = \frac{1}{T - J\sum_a 2\cos k_a} B_{\mathbf{k}}\,,$$

which gives[5]:

$$m_i = \Omega \int_{BZ} \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{x}_i\cdot\mathbf{k}}}{T - J\sum_a 2\cos k_a} B_{\mathbf{k}}\,.$$

This expression is IR divergent when $T = T_c$ since:

$$J\sum_a 2\cos k_a = Jz + J\sum_a k_a^2 + ... = T_c + J\sum_a k_a^2 + ...$$

Finally, since $C_{ij} = \frac{\partial m_i}{\partial B_j}$, the correlator near the critical point can be written as (on a hyper-cubical lattice):

$$C(r = |\mathbf{r}|) = \frac{1}{T_c} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{r}\cdot\mathbf{k}}}{t + \frac{1}{z}\sum_a 2(1-\cos k_a)}\,, \tag{5.33}$$

where $t = \frac{T}{T_c} - 1$ is the reduced temperature. At large r we can write[6]:

$$\begin{aligned} C(r) &\approx \frac{1}{T_c}\int \frac{d^d k}{(2\pi)^d}\frac{e^{-i\mathbf{r}\cdot\mathbf{k}}}{t+k^2} \\ &= \frac{1}{T_c}\frac{1}{(2\pi)^d}\left(\frac{\sqrt{t}}{r}\right)^{d-2} K_{\frac{d}{2}-1}(\sqrt{t}r) \\ &\approx \frac{1}{T_c}\frac{1}{(2\pi)^d}\left(\frac{\sqrt{t}}{r}\right)^{d-2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{\sqrt{t}r}}e^{-\sqrt{t}r}\,. \end{aligned} \tag{5.34}$$

The correlation length is the inverse lattice mass $\xi = 1/m$ and thus from (5.34) we see that at the critical point it diverges like:

$$\xi = t^{-1/2} \equiv t^{-\nu}\,. \tag{5.35}$$

[5] $m_i \equiv m(\mathbf{x}_i)$.

[6] $K_n(x)$ is a Bessel function.

The mean field correlation length critical exponent is thus $\nu_{MF} = \frac{1}{2}$.

Exactly at the critical point we find instead:

$$C(r) \approx \frac{1}{r^{d-2}} \equiv \frac{1}{r^{d-2+\eta}}\,, \tag{5.36}$$

telling us that the mean field anomalous dimension is $\eta_{MF} = 0$.

Finally the mean field critical exponents are:

MEAN FIELD CRITICAL EXPONENTS

$$\begin{aligned} \alpha^{MF} &= 0 \qquad \beta^{MF} = \frac{1}{2} \qquad \gamma^{MF} = 1 \\ \delta^{MF} &= 3 \qquad \nu^{MF} = \frac{1}{2} \qquad \eta^{MF} = 0\,. \end{aligned} \tag{5.37}$$

To understand the quantitative validity of the mean field approximation we need to understand the quantitative meaning of weak fluctuations assumption.

5.2.3 Ginzburg criterion

THE GINZBURG CRITERION AND UPPER CRITICAL DIMENSION

The Ginzburg criterion establishes when the mean field analysis is valid, i.e. under which conditions fluctuations are small. We start from:

$$\sqrt{\langle M^2\rangle - \langle M\rangle^2} \ll \langle M\rangle\ ,$$

which is equivalent to

$$NT\chi \ll \langle M\rangle^2\ .$$

At criticality fluctuations are on all scales and we have (for $t \to 0^-$):

$$\xi \sim |t|^{-\nu} \qquad\qquad \langle M\rangle^2 \sim (-t)^{2\beta} L^{2d} \qquad\qquad N\chi \sim (-t)^{-\gamma} L^d\,.$$

Thus, setting $\xi = L$, we find:

$$T(-t)^{-\gamma}\xi^d \ll (-t)^{2\beta}\xi^{2d}\,,$$

or

$$T(-t)^{-\gamma-2\beta+\nu d} \ll 1\,,$$

which may happen when $-\gamma - 2\beta + \nu d > 0$ or:

$$\frac{\gamma + 2\beta}{\nu} < d\,,$$

which is the Ginzburg criterion. If we insert the mean field exponents we find $d > 4$. Thus the mean field predictions are valid in dimensions greater than four, which for this reason is called upper critical dimension $d_c = 4$.

5.3 Solving the Ising model

In this section we explore the Ising model in dimensions below the upper critical dimension $d_c = 4$, where the Ginzburg criterion shows that the mean field approach is unable to describe the phase transition quantitatively. Fortunately in both $d = 1$ and $d = 2$ it is possible to solve the Ising model exactly. These solutions are invaluable models characterized by non–mean field phase transitions.

5.3.1 $1d$ Ising model

THE TRANSFER MATRIX APPROACH

In one dimension the partition function for the Ising model can be rewritten as follows

$$\begin{aligned} Z &= \sum_{\{\sigma_i\}} e^{\beta J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} + \beta B \sum_{i=1}^{N} \frac{1}{2}(\sigma_i + \sigma_{i+1})} \\ &= \sum_{\{\sigma_i\}} \prod_{i=1}^{N} e^{\beta\left[J\sigma_i\sigma_{i+1} + \frac{B}{2}(\sigma_i+\sigma_{i+1})\right]} \\ &= \sum_{\{\sigma_i\}} T_{\sigma_1\sigma_2} \ldots T_{\sigma_N\sigma_1} \\ &= \operatorname{tr}\mathbf{T}^N\,, \end{aligned} \tag{5.38}$$

where we introduced the transfer matrix

$$\mathbf{T} = \begin{pmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-B)} \end{pmatrix}. \tag{5.39}$$

By diagonalizing (5.39) we can write the partition function in terms of the eigenvalues of the transfer matrix:

$$Z = \operatorname{tr}\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}^N = \lambda_+^N + \lambda_-^N\,,$$

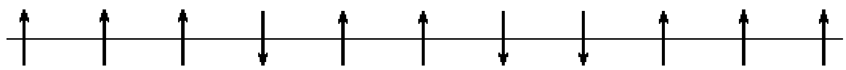

Figure 5.3: Ising spins on a one dimensional infinite lattice.

where

$$\lambda_\pm = e^{\beta J}\left(\cosh\beta B \pm \sqrt{\sinh^2\beta B + e^{-4\beta J}}\right) .$$

Note that we have $\lambda_+ > \lambda_-$. The free energy is thus given by the following expression:

$$F = -\frac{1}{\beta}\log Z = -\frac{1}{\beta}\left\{N\,\log\lambda_+ + \log\left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right]\right\} ,$$

in the thermodynamic limit $N \to \infty$ only the contribution from λ_+ survives and we find the following free energy per spin:

$$f(T,B) = -J - T\,\log\left(\cosh\frac{B}{T} + \sqrt{\sinh^2\frac{B}{T} + e^{-4J/T}}\right) .$$

In the case of zero magnetic field we have simply

$$f(T,0) = -J + T\,\log\left(1 + e^{-2J/T}\right) = -T\,\log 2 - T\,\log\cosh\frac{2}{T} .$$

We can easily calculate the magnetization per spin:

$$m(T,B) = -\frac{\partial f}{\partial B} = \frac{\sinh\frac{B}{T}}{\sqrt{\sinh^2\frac{B}{T} + e^{-4J/T}}} .$$

Since the hyperbolic sine is zero in the origin there is no spontaneous magnetization at $B = 0$. The susceptibility is

$$\chi(T,B) = \frac{\partial m}{\partial B} = \frac{e^{-4J/T}\cosh\frac{B}{T}}{T\left(\sinh^2\frac{B}{T} + e^{-4J/T}\right)^{3/2}} ,$$

while the energy per spin and the specific heat are:

$$\epsilon(T,B) = \frac{\partial(\beta f)}{\partial\beta} = -J\tanh\frac{J}{T}$$

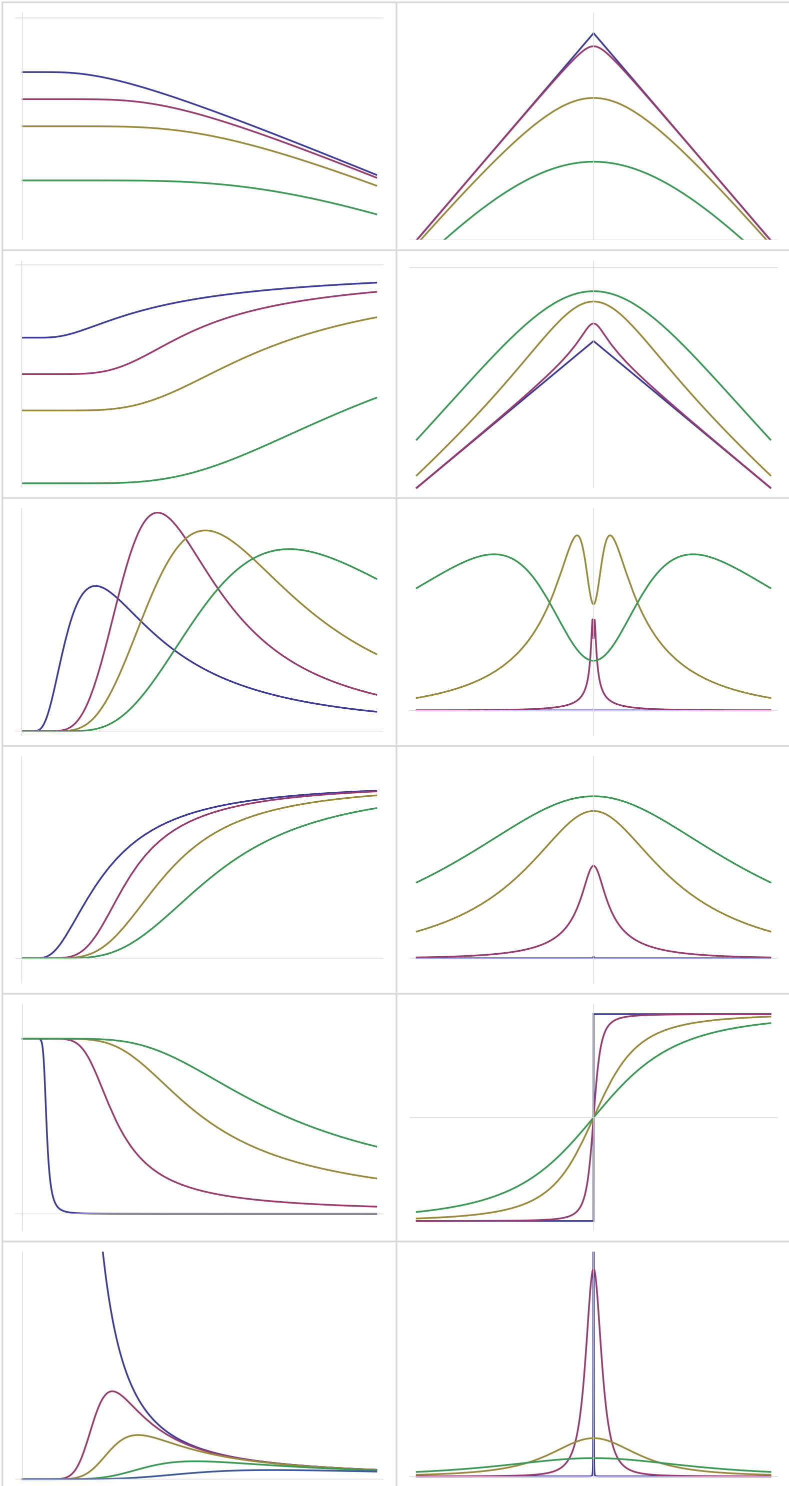

Figure 5.4: Exact (per spin) thermodynamic functions for the $1d$ Ising model. From top: $f, \epsilon, c, s, m, \chi$; left column as a function of T, right column as a function of B.

$$c(T,B) = -\frac{1}{T^2}\frac{\partial\epsilon}{\partial\beta} = \frac{J^2\mathrm{sech}^2\frac{J}{T}}{T^2}\,.$$

We can also calculate correlation functions. The magnetization is:

$$\begin{aligned}
\langle\sigma_i\rangle &= \frac{1}{Z}\sum_{\{\sigma_i\}}\sigma_i e^{\beta E_{\{\sigma_i\}}} \\
&= \frac{1}{Z}\sum_{\{\sigma_i\}} T_{\sigma_1\sigma_2}\cdots T_{\sigma_{i-1}\sigma_i}\sigma_i T_{\sigma_i\sigma_{i+1}}\cdots T_{\sigma_N\sigma_1} \\
&= \frac{1}{Z}\mathrm{tr}\mathbf{T}^{i-1}\sigma_z\mathbf{T}^{N-i+1} \\
&= \frac{1}{Z}\mathrm{tr}\mathbf{T}^{N}\sigma_z\,,
\end{aligned}$$

where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

When $B = 0$ the diagonalizing matrix is $U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and we find:

$$\langle\sigma_i\rangle = \frac{1}{Z}\mathrm{tr}\begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{Z}\mathrm{tr}\begin{pmatrix} 0 & \lambda_+^N \\ \lambda_-^N & 0 \end{pmatrix} = 0\,,$$

as expected. The correlation function can be calculated in a similar way:

$$\begin{aligned}
\langle\sigma_i\sigma_{i+r}\rangle &= \frac{1}{Z}\mathrm{tr}\mathbf{T}^{N-r}\sigma_z\mathbf{T}^r\sigma_z \\
&= \frac{1}{Z}\mathrm{tr}\begin{pmatrix} \lambda_+^{N-r} & 0 \\ 0 & \lambda_-^{N-r} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_+^{r} & 0 \\ 0 & \lambda_-^{r} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{\lambda_+^{N-r}\lambda_-^r + \lambda_+^r\lambda_-^{N-r}}{\lambda_+^N + \lambda_-^N} \\
&\to \left(\frac{\lambda_-}{\lambda_+}\right)^r .
\end{aligned}$$

Thus we have:

$$C(r) = \langle\sigma_i\sigma_{i+r}\rangle = e^{-r/\xi}\,,$$

where the correlation length is:

$$\xi(T,0) = \frac{1}{\log\tanh\beta J}\,.$$

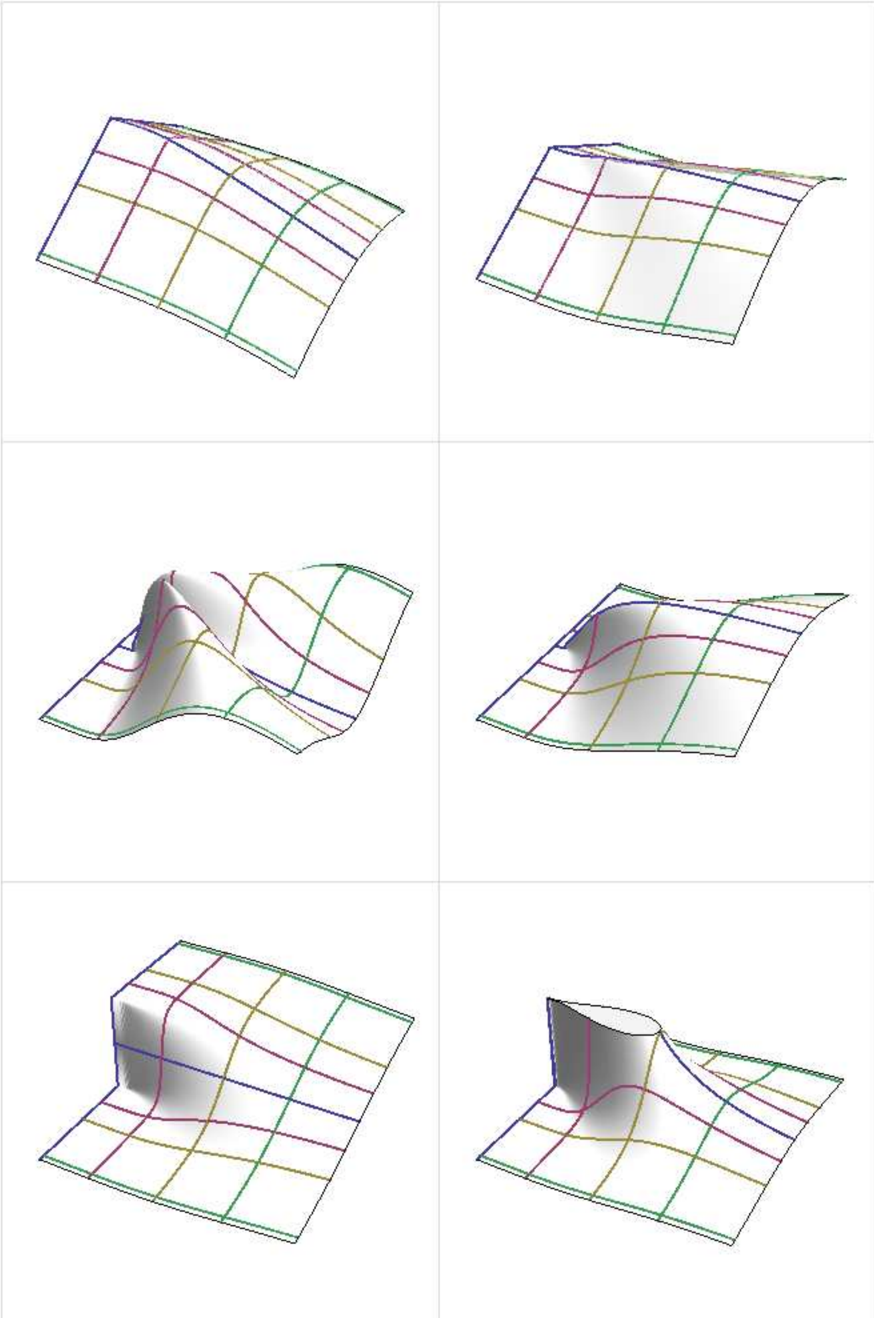

Figure 5.5: Exact (per spin) thermodynamic functions for the $1d$ Ising model. From top: $f, \epsilon, c, s, m, \chi$; the color code is consistent with the previous figure.

We can also consider the limit $r \to \infty$ where

$$C(r) \to 0\,,$$

showing again that there is no spontaneous magnetization.

The critical exponents for the one dimensional Ising model are: CRITICAL EXPONENTS FOR THE $d = 1$ ISING MODEL

$$\alpha^{d=1} = 1 \qquad \beta^{d=1} = 0 \qquad \gamma^{d=1} = 1$$
$$\delta^{d=1} = \infty \qquad \nu^{d=1} = 1 \qquad \eta^{d=1} = 1\,.$$

These clearly differ from the mean field ones. Note that the Ginzburg inequality is violated but saturated since $\frac{\gamma+2\beta}{\nu} = 1$.

5.3.2 $2d$ Ising model

ONSAGER'S SOLUTION

The exact zero–field free energy for the square lattice Ising model was found for the first time by L.Onsager using algebraic methods. The zero–field free energy can be exactly computed for square, triangular and hexagonal lattices; the result is:

$$f(T,0) = -Tp\log 2 - \frac{T}{2}\int \frac{d^2k}{(2\pi)^2}\log P(T,\mathbf{k})\,, \tag{5.40}$$

where the argument of the logarithms is:

$$\begin{aligned}
P_S(T,\mathbf{k}) &= \cosh^2\frac{2J}{T} - \sinh\frac{2J}{T}(\cos k_1 + \cos k_2)\\
P_T(T,\mathbf{k}) &= \cosh^3\frac{2J}{T} + \sinh^3\frac{2J}{T} - \sinh\frac{2J}{T}(\cos k_1 + \cos k_2 + \cos k_3)\\
P_H(T,\mathbf{k}) &= \frac{1}{2}\left[1 + \cosh^3\frac{2J}{T} - \sinh^2\frac{2J}{T}(\cos k_1 + \cos k_2 + \cos k_3)\right]\,,
\end{aligned}$$

for, respectively, square, triangular and hexagonal lattices. p is the number of sites in a unit cell of the lattice, which is one for the square and triangular lattices and two for the hexagonal lattice.

The argument of the logarithm is always positive except for $\mathbf{k} = 0$ in which a singularity occurs. This singularity corresponds to the phase transition, thus the critical temperature is the solution of: CRITICAL TEMPERATURE

$$P(T_c, 0) = 0\,.$$

We find, respectively, the following equations:

$$\begin{aligned}
\cosh^2 \frac{2J}{T_c^S} - 2\sinh \frac{2J}{T_c^S} &= 0 \\
\cosh^3 \frac{2J}{T_c^T} + \sinh^3 \frac{2J}{T_c^T} - 3\sinh \frac{2J}{T_c^T} &= 0 \\
1 + \cosh^3 \frac{2J}{T_c^H} - 3\sinh^2 \frac{2J}{T_c^H} &= 0\,,
\end{aligned}$$

which give the following critical temperatures:

$$T_c^S = \tfrac{2}{\log(1+\sqrt{2})} J = 2.26919\, J \tag{5.41}$$

$$T_c^T = \tfrac{2}{\log\sqrt{3}} J = 3.64096\, J \tag{5.42}$$

$$T_c^H = \tfrac{2}{\log(2+\sqrt{3})} J = 1.51865\, J\,, \tag{5.43}$$

which may be compared with the mean field result $T_c^{MF} = z\,J$. The critical temperature (5.41) was first found by Kramer and Wannier using duality arguments. Since the free energy diverges logarithmically we have $\alpha^{d=2} = 0$ in all three cases.

We can now compute the per spin internal energy and the specific heat using equation (5.12): INTERNAL ENERGY AND SPECIFIC HEAT

$$\epsilon(T,0) = \frac{T^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{P(T,\mathbf{k})} \frac{\partial P(T,\mathbf{k})}{\partial T} \tag{5.44}$$

and

$$c(T,0) = \frac{\epsilon(T,0)}{T^2} + \frac{T^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{P(T,\mathbf{k})} \left[\frac{\partial^2 P(T,\mathbf{k})}{\partial T^2} - \frac{1}{P(T,\mathbf{k})} \frac{\partial P(T,\mathbf{k})}{\partial T} \right]. \tag{5.45}$$

The three thermodynamic functions f, ϵ, c are shown in Figure 5.6.

Onsager and Yang were also able to determine the spontaneous magnetization exactly: MAGNETIZATION

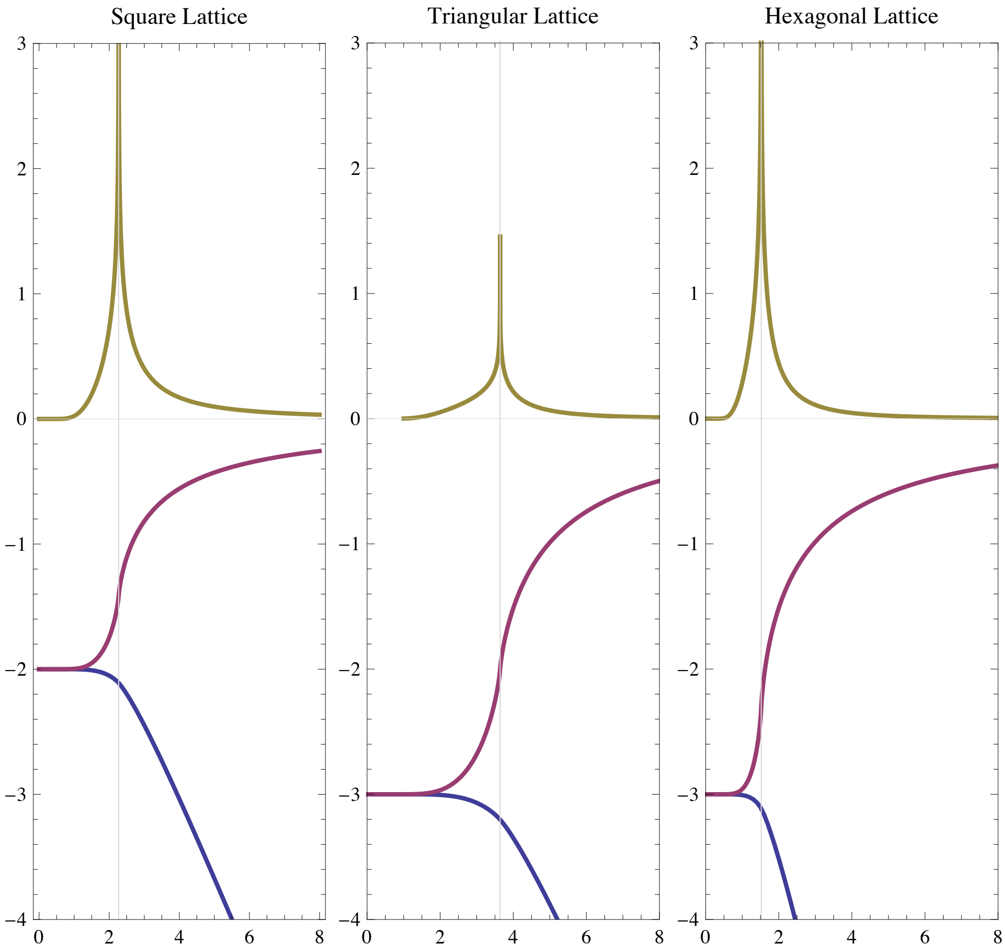

Figure 5.6: Per spin free energy (bottom), internal energy (middle) and specific heat (top) for the $2d$ Ising model on a square, triangular and hexagonal lattices.

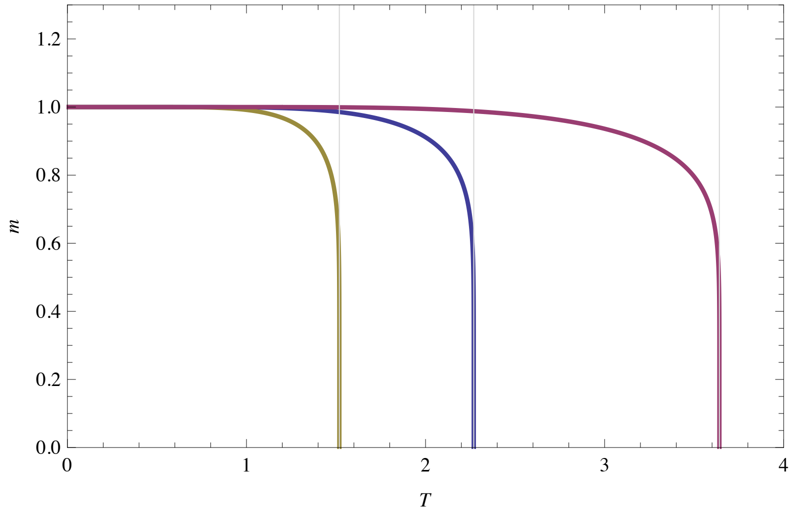

Figure 5.7: Spontaneous magnetization per spin for the $2d$ Ising model on a square (middle), triangular (right) and hexagonal (left) lattices.

$$\begin{aligned} m_S(T,0) &= \left[\frac{\left(1+e^{4J/T}\right)^2\left(1-6e^{4J/T}+e^{8J/T}\right)}{(1-e^{4J/T})^4}\right]^{\frac{1}{8}} \\ m_T(T,0) &= \left[\frac{\left(3-e^{4J/T}\right)\left(1+e^{4J/T}\right)^3}{(1-e^{4J/T})^3\,(3+e^{4J/T})}\right]^{\frac{1}{8}} \\ m_H(T,0) &= \left[\frac{\left(1+e^{4J/T}\right)^3\left(1+e^{4J/T}-4e^{2J/T}\right)}{(1-e^{2J/T})^6\,(1+e^{2J/T})^2}\right]^{\frac{1}{8}} . \end{aligned}$$

These expressions are valid for $T < T_c$ and are shown in Figure 5.7. In the case of the square lattice we also have the equivalent simple form:

$$m_S(T,0) = \left[1 - \frac{1}{\sinh^4 \frac{2J}{T}}\right]^{\frac{1}{8}} , \tag{5.46}$$

first derived by Yang.

The correlation length is the inverse of the lattice mass. $P(T,\mathbf{k})$ plays the role of the lattice propagator (it is the argument of the log in the free energy) CORRELATION LENGTH

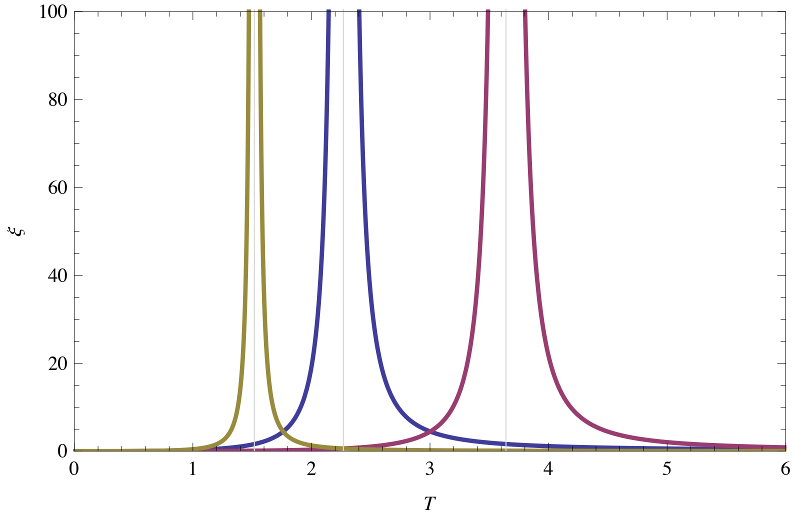

Figure 5.8: Correlation length for the $2d$ Ising model on a square (middle), triangular (right) and hexagonal (left) lattices.

as in the mean field case, thus the mass squared can be defined by writing:

$$P(T,\mathbf{k}) = Z(T)\left[m^2(T) + \frac{1}{z}\sum_a 2(1-\cos k_a)\right], \tag{5.47}$$

where $Z(T)$ is a wave function renormalization. From this we can write:

$$m^2(T) = \frac{P(T,0,0)}{P(T,\pi/2,\pi/2,)-P(T,0,0)} \tag{5.48}$$

or

$$\xi(T) = \sqrt{\frac{P(T,\pi/2,\pi/2,)-P(T,0,0)}{P(T,0,0)}}\,. \tag{5.49}$$

For the lattices we are considering we find the following results:

$$\xi_S(T,0) = \frac{1}{\log\frac{\sqrt{2}-1}{\frac{1-e^{-2J/T}}{1+e^{-2J/T}}}}\,.$$

Note that the argument of the log is $e^{-2K_D}\tanh 2K$ ($=1$ at the critical point) or something like this. At the critical point the correlation length diverges in the following way:

$$\begin{aligned}
\xi_S\left(T_c^S(1+t),0\right) &= \frac{1}{4\log\left(1+\sqrt{2}\right)}\,|t|^{-1}+\text{regular} \\
&\simeq 0.321825\,|t|^{-1}+\text{regular}\,. \\
\xi_T\left(T_c^T(1+t),0\right) &= \frac{1}{72\text{Arctanh}\left(2-\sqrt{3}\right)}\,|t|^{-1}+\text{regular} \\
&\simeq 0.184119\,|t|^{-1}+\text{regular} \\
\xi_H\left(T_c^H(1+t),0\right) &= \frac{1}{24\text{Arctanh}\frac{1}{\sqrt{3}}}\,|t|^{-1}+\text{regular} \\
&\simeq 0.0960959\,|t|^{-1}+\text{regular}\,,
\end{aligned}$$

giving $\nu^{d=2}=1$.

We can expand the magnetization around the critical point finding:

$$\begin{aligned}
m_S\left(T_c^S(1+t),0\right) &= \left[2\left(\sqrt{2}-1\right)\right]^{\frac{1}{4}}\left[\left(4+3\sqrt{2}\right)\log\left(1+\sqrt{2}\right)\right]^{\frac{1}{8}}(-t)^{\frac{1}{8}}+\text{regular} \\
&\simeq 1.22241\,(-t)^{\frac{1}{8}}+\text{regular}\,. \\
m_T\left(T_c^T(1+t),0\right) &= \sqrt{2}\left[\text{Arctanh}\left(2-\sqrt{3}\right)\right]^{\frac{1}{8}}(-t)^{\frac{1}{8}}+\text{regular} \\
&\simeq 1.20327\,(-t)^{\frac{1}{8}}+\text{regular} \\
m_H\left(T_c^H(1+t),0\right) &= \sqrt{2}\left(2+\sqrt{3}\right)^{\frac{1}{4}}\left[\left(\frac{7}{\sqrt{3}}-4\right)\text{Arccoth}\sqrt{3}\right]^{\frac{1}{8}}(-t)^{\frac{1}{8}}+\text{regular} \\
&\simeq 1.25318\,(-t)^{\frac{1}{8}}+\text{regular}\,.
\end{aligned}$$

We thus see that independently of the lattice the order parameter critical exponent is $\beta^{d=2}=\frac{1}{8}$. This is an instance of universality.

The knowledge of the two critical exponents α and β allows the determination of all the other critical exponents if we assume the validity of the scaling relations (to be proved in the next section):

CRITICAL EXPONENTS FOR THE $d=2$ ISING MODEL

$$\begin{array}{lll}
\alpha^{d=2}=0 & \beta^{d=2}=\dfrac{1}{8} & \gamma^{d=2}=\dfrac{7}{4} \\
\delta^{d=2}=15 & \nu^{d=2}=1 & \eta^{d=2}=\dfrac{1}{4}\,.
\end{array}$$

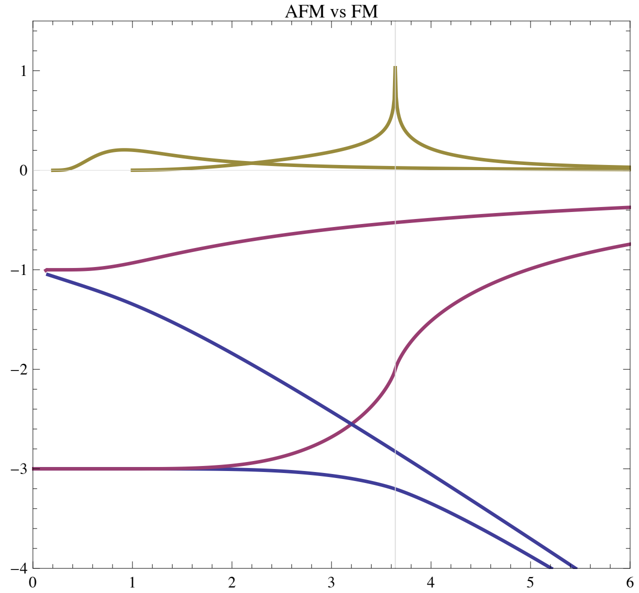

Figure 5.9: Comparison between ferromagnetic and antiferromagnetic thermodynamic quantities for the triangular lattice Ising model.

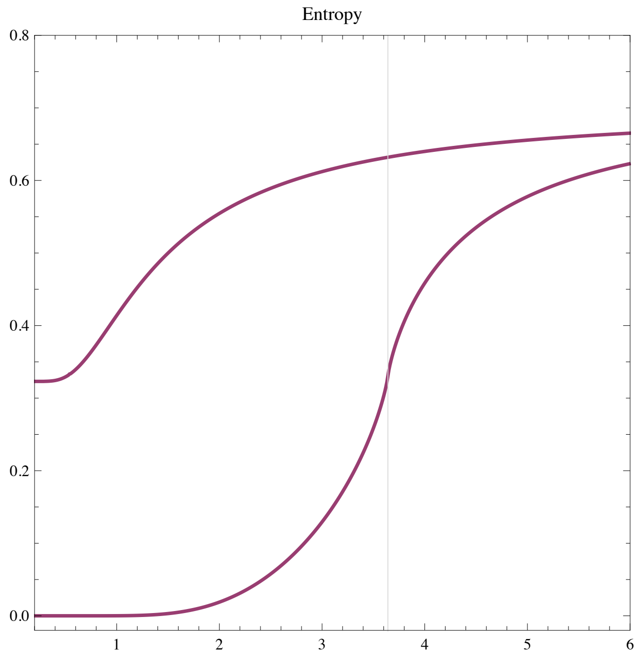

Figure 5.10: The frustrated antiferromagnetic ground state of the triangular lattice Ising model has non–zero entropy at $T = 0$.

These clearly differ from the mean field ones, in accord with the fact that the Ginzburg inequality is violated (but saturated) since $\frac{\gamma+2\beta}{\nu} = 2$.

5.3.3 Anti–ferromagnetism and frustration

ANTI–FERROMAGNET AND FRUSTRATION

For antiferromagnetic coupling $J < 0$ the triangular lattice Ising model becomes very interesting since it represents the simplest example of frustrated system. There is no phase transition at finite temperature and the model is always disordered, even at $T = 0$ where the ground state is frustrated and the system has a non–zero entropy. The thermodynamic quantities and the entropy per spin are shown in Figure 5.10.

5.4 Scaling hypothesis

HOMOGENEOUS FUNCTIONS

A function of one variable is said to be homogeneous if there exist a scaling exponent a such that

$$f(\lambda^a x) = \lambda f(x) \qquad \lambda > 0\,.$$

It is easy to see that a necessary and sufficient condition for $f(x)$ to be an homogeneous functions that it is a power law. Taking $\lambda = |x|^{-1/a}$ we find:

$$f(x) = f(\pm 1)|x|^{1/a}\,,$$

as discussed in the first lecture.

A function of n variables $x, y, z, ...$ is homogeneous if there exist scaling exponents $a, b, c, ..$ such that:

$$f(\lambda^a x, \lambda^b y, \lambda^c z, ...) = \lambda f(x, y, z, ...)\,.$$

In this case we can eliminate one variable by introducing the scaling functions $\mathcal{F}_\pm$ defined as before by setting $\lambda = |x|^{-1/a}$:

$$\begin{aligned} f(x, y, z, ...) &= |x|^{1/a} f\left(\pm 1, \frac{y}{|x|^{b/a}}, \frac{z}{|x|^{c/a}}, ...\right) \\ &\equiv |x|^{1/a}\mathcal{F}_\pm\left(\frac{y}{|x|^{b/a}}, \frac{z}{|x|^{c/a}}, ...\right)\,. \end{aligned}$$

In the case of a two variable function the scaling function is a function of one rescaled variable: by plotting $|x|^{-1/a} f(x,y)$ as a function of $y/|x|^{b/a}$ all data "collapses" to the graph of the scaling function $\mathcal{F}_\pm$. DATA COLLAPSE

Near the critical point the various thermodynamic functions exhibit a scaling behavior. The scaling hypothesis assumes that near to the phase transition the (non-analytical part of the) free energy per spin is an homogeneous function: SCALING HYPOTHESIS

$$f(t,b) = |t|^{2-\alpha} \mathcal{F}_\pm \left(\frac{b}{|t|^\Delta} \right) \qquad t \to 0^\pm \qquad b \to 0\,, \tag{5.50}$$

where $t = \frac{T-T_c}{T_c}$ is the reduced temperature, $b = \beta B$ and the exponents $\alpha < 2$ and Δ are the first two critical exponents. Since derivatives of homogeneous functions are again homogeneous, the scaling ansatz (5.50) implies that all thermodynamic quantities are so near the phase transition. The magnetization per spin scales as:

$$\begin{aligned} m(t,b) &= -\beta \frac{\partial f}{\partial b} \\ &= -\beta |t|^{2-\alpha-\Delta} \mathcal{F}'_\pm \left(\frac{b}{|t|^\Delta} \right) \\ &\equiv |t|^{2-\alpha-\Delta} \mathcal{M}_\pm \left(\frac{b}{|t|^\Delta} \right) . \end{aligned} \tag{5.51}$$

The exponents β and δ where defined by the relations:

$$m(t,0) \sim \pm |t|^\beta \qquad m(0,b) \sim \mathrm{sign}(b) |b|^{1/\delta} \,. \tag{5.52}$$

Comparing (5.51) to (5.52) gives the first two scaling relations:

$$\beta = 2 - \alpha - \Delta \qquad \Delta = \beta\delta \,. \tag{5.53}$$

The susceptibility scales as follows:

$$\begin{aligned} \chi(t,b) &= \beta \frac{\partial m}{\partial b} \\ &= -\beta^2 |t|^{2-\alpha-2\Delta} \mathcal{F}''_\pm \left(\frac{b}{|t|^\Delta} \right) \\ &\equiv |t|^{2-\alpha-2\Delta} \chi_\pm \left(\frac{b}{|t|^\Delta} \right) . \end{aligned} \tag{5.54}$$

The exponent γ is defined by the relation $\chi(t,0) \sim \chi_\pm |t|^{-\gamma}$ and from (5.54) we find the scaling relation:

$$\gamma = 2 - \alpha - 2\Delta \tag{5.55}$$

and the relation $\chi_\pm = \chi_\pm(0)$. The singular part of the specific heat scales as:

$$\begin{aligned} c(t,b) &= -T\frac{\partial^2 f}{\partial T^2} \\ &= -\frac{T}{T_c}\frac{(2-\alpha)(1-\alpha)}{T_c}|t|^{-\alpha}\mathcal{F}_\pm\left(\frac{b}{|t|^\Delta}\right) \\ &\equiv |t|^{-\alpha}\mathcal{C}_\pm\left(\frac{b}{|t|^\Delta}\right) \; ; \end{aligned} \tag{5.56}$$

thus at the phase transition the specific heat diverges like $c(t,0) \sim \mathcal{C}_\pm |t|^{-\alpha}$ with $\mathcal{C}_\pm = \mathcal{C}_\pm(0)$.

5.4.1 Scaling relations

SCALING RELATIONS

Combining the relations (5.53) with (5.55) we can eliminate Δ to obtain

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \beta + \gamma &= \beta\delta \, . \end{aligned} \tag{5.57}$$

ANOMALOUS DIMENSION

One also finds the scaling relation:

$$\gamma = \nu(2-\eta)\, . \tag{5.58}$$

SCALING RELATION

To derive (5.58) we start from (5.18) and transform the sum in an integral:

$$\chi \sim \int_0^L dr\, r^{d-1} C(r) \sim \int_0^L dr\, r^{d-1} r^{-(d-2+\eta)} = \int_0^L dr\, r^{1-\eta} = \frac{L^{2-\eta}}{2-\eta}\, , \tag{5.59}$$

where we used (5.36) to express the correlation function at criticality. At the critical point $\xi = L$ and thus $\chi \sim \xi^{2-\eta}$. Recalling the scaling behavior of both the susceptibility and the correlation length this implies $|t|^{-\gamma} \sim |t|^{-\nu(2-\eta)}$, which immediately gives (5.58). Note that this scaling relation is the same we found when we studied percolation. Note also that since $\chi > 0$ we must have $\eta < 2$, which is thus a restriction on the possible values assumed by the anomalous dimension.

The hyper-scaling relation is:

HYPER-SCALING RELATION

$$2 - \alpha = \nu d\,, \tag{5.60}$$

which depends explicitly on d. To see why (5.60) holds consider that the free energy per spin is an intensive variable and thus scales as $f(t,0) \sim N^{-1} \sim L^{-d}$. But from the scaling ansatz (5.50) we also know that $f(t,0) \sim |t|^{2-\alpha}$, thus at criticality where $\xi = L$ we must have $|t|^{-\nu d} \sim |t|^{2-\alpha}$and so we get(5.60). Note that we have already met this relation when we studied the random walk; we called $\alpha = \frac{3}{2}$ the Hurst exponent and we had $d = 1$ and $d_f = \frac{1}{\nu} = 2$.

Note that the mean field critical exponents (5.37) and the one and two dimensional Ising model critical exponents indeed satisfy the scaling relations. Note that in the mean field case the hyper-scaling relation implies $2 = d/2$ or $d = 4$.

Combining the previous relations we find:

THERE ARE ONLY TWO INDEPENDENT CRITICAL EXPONENTS

$$\begin{aligned}
\alpha &= 2 - \nu d \\
\beta &= \nu\frac{d-2+\eta}{2} \\
\gamma &= \nu(2-\eta) \\
\delta &= \frac{d+2-\eta}{d-2+\eta}\,,
\end{aligned} \tag{5.61}$$

with only ν and η independent.

5.5 RG for lattice models

We discuss the RG approach to lattice models developed mostly by Kadanoff and Wilson.

5.5.1 RG for the 1d Ising model

We construct the RG transformation by grouping the spins in pairs and by applying the decimation procedure where we sum over the second spin in

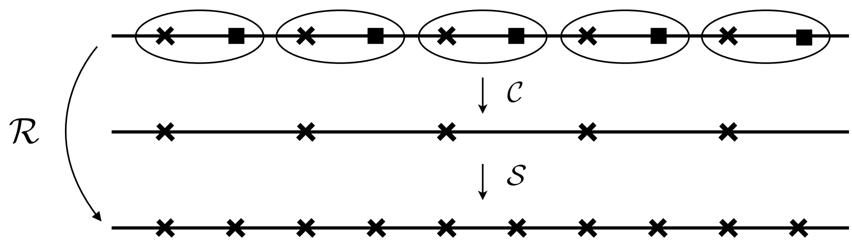

Figure 5.11: RG on a one dimensional lattice.

each pair.

RG TRANSFORMATION FOR THE $d = 1$ ISING MODEL

By defining the coupling $K_1 = \beta B$ and $K_2 = \beta J$ we have that the partition function becomes:

$$Z = \sum_{\{\sigma_i\}} e^{K_1 \sum_i \sigma_i + K_2 \sum_i \sigma_i \sigma_{i+1}} . \tag{5.62}$$

To perform the decimation, consider the sum over the second spin:

$$\begin{aligned} e^{\frac{1}{2}K_1(\sigma_1+\sigma_3)} \sum_{\sigma_2=\pm} e^{K_1\sigma_2 + K_2\sigma_2(\sigma_1+\sigma_3)} &= e^{\frac{1}{2}K_1(\sigma_1+\sigma_2)} 2\cosh\left[K_1 + K_2(\sigma_1+\sigma_3)\right] . \\ &\equiv e^{K_0' + \frac{1}{2}K_1'(\sigma_1+\sigma_3) + K_2'\sigma_1\sigma_3} , \end{aligned} \tag{5.63}$$

where we defined the renormalized couplings so to preserve the functional form of the partition function. From (5.63) we can extract three independent relations by evaluating it for the various configuration of the two spins, i.e. for $\sigma_1 = \sigma_3 = 1$, for $\sigma_1 = -\sigma_3 = 1$ and for $\sigma_1 = \sigma_3 = -1$, which give respectively:

$$\begin{aligned} e^{K_1} 2\cosh(K_1 + 2K_2) &= e^{K_0' + K_1' + K_2'} \\ 2\cosh K_1 &= e^{K_0' - K_2'} \\ e^{-K_1} 2\cosh(K_1 - 2K_2) &= e^{K_0' - K_1' + K_2'} . \end{aligned} \tag{5.64}$$

Equation (5.64) defines (implicitly) the RG transformation

$$\mathcal{R} \begin{pmatrix} K_0 \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} K_0' \\ K_1' \\ K_2' \end{pmatrix} \tag{5.65}$$

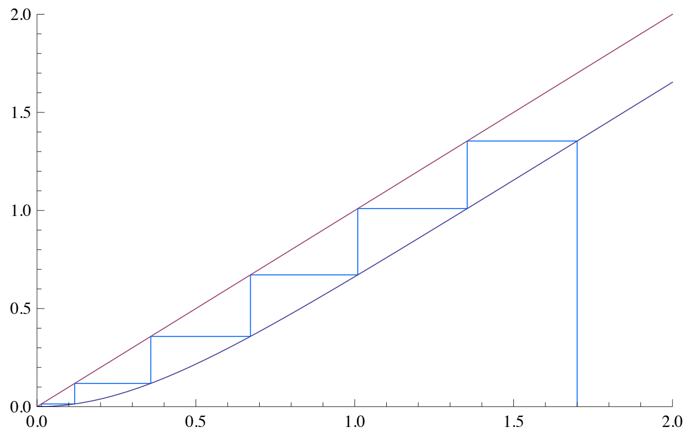

Figure 5.12: The RG flow for the coupling K_2.

in the three dimensional theory space of all the free energies of the form:

$$-\beta F(K_0, K_1, K_2) = K_0 + K_1 \sum_i \sigma_i + K_2 \sum_i \sigma_i \sigma_{i+1} \, .$$

The fact that theory space is finite dimensional is a peculiarity of the one dimensional model, note that this implies that (5.65) is an exact RG.

Let's consider for a moment the case where the magnetic field is zero. Thus there are only parity invariant interactions, i.e. only the couplings K_0 and K_2:

$$\begin{aligned} K_0' &= \log 2 + \frac{1}{2} \log\cosh 2K_2 \\ K_2' &= \frac{1}{2} \log\cosh 2K_2 \, . \end{aligned} \tag{5.66}$$

The RG transformation (5.66) has a fixed point for $K_2^* = 0$ and $K_2^* = \infty$. The second is attractive and corresponds to the ordered phase transition while the first represent the disordered high temperature phase. Note that K_0 converges in the IR to the free energy, since at every RG step we are integrating out some degree of freedom (spins).

In terms of the variables $u = e^{-K_2}$, $v = e^{-K_1}$, $w = e^{-K_0}$ the solution of

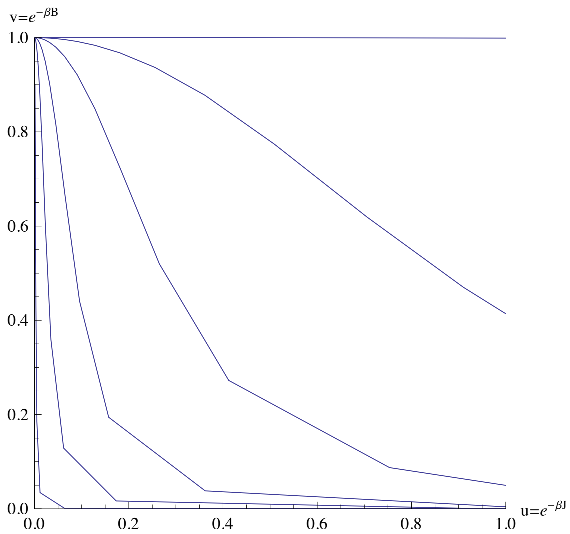

Figure 5.13: RG flow for the $1d$ Ising model in the (u,v) plane.

(5.64) can be written as:

$$
\begin{aligned}
u' &= \frac{\left(v+\frac{1}{v}\right)^{1/2}}{\left(u^4+\frac{1}{u^4}+v^2+\frac{1}{v^2}\right)^{1/4}} \\
v' &= \left(\frac{u^4+v^2}{u^4+\frac{1}{v^2}}\right)^{1/2} \\
w' &= \left(v+\frac{1}{v}\right)^{1/2}\left(u^4+\frac{1}{u^4}+v^2+\frac{1}{v^2}\right)^{1/4},
\end{aligned}
\tag{5.67}
$$

and the exact RG transformation becomes:

$$
\mathcal{R}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}.
$$

In the (u,v) plane there is a repulsive fixed point $(0,B)$, a mixed fixed point $(0,\infty)$ and a line of fixed points (∞,B), as can be seen in Figure 1.13.

5.5.2 General RG analysis

The general case is much more complicated. The coarse–graining is still implemented by decimation but for $d>1$ this procedure introduces new

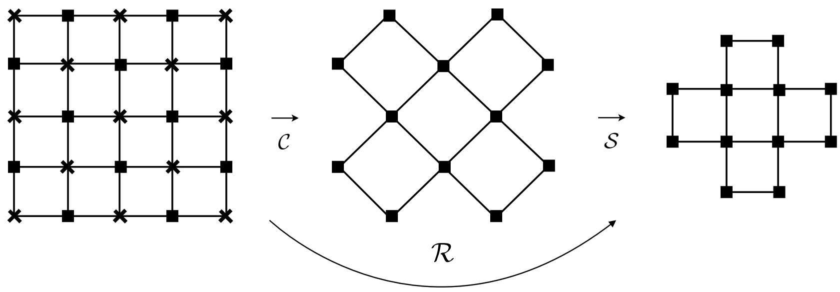

Figure 5.14: RG transformation for the two dimensional Ising model. The coarse–graining is performed by summing over any second spin, while the similarity has scale factor $\lambda = \frac{1}{\sqrt{2}}$ and a rotation of $\frac{\pi}{4}$.

couplings at every iteration of the RG transformation.

RG FOR THE $d = 2$ ISING MODEL

The full complexity is already present in the two dimensional square lattice case. The RG transformation is illustrated in the Figure. Since now

$$\sum_{\sigma_5=\pm 1} e^{K_2\sigma_5(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} = 2\cosh\left[K_2\left(\sigma_1+\sigma_2+\sigma_3+\sigma_4\right)\right]$$

$$= e^{K_0'+\frac{1}{2}K_2'(\sigma_1\sigma_2+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_3\sigma_4)+K_3'(\sigma_1\sigma_3+\sigma_2\sigma_4)+K_4'\sigma_1\sigma_2\sigma_3\sigma_4}\,, \tag{5.68}$$

new next–nearest–neighbor interactions K_3 and quadruple interactions K_4 are generated already after the first RG step. At every iteration the number of coupling constants grows and finally the RG flow takes place in the infinite dimensional theory space parametrized by the couplings $\mathbf{K} = (K_0, K_1, K_2, K_3, K_4, ...)$:

$$\mathcal{R}\mathbf{K} = \mathbf{K}'\,. \tag{5.69}$$

The most general free energy has now the form:

$$-\beta F(\mathbf{K}) = K_0 + K_1\sum_i \sigma_i + K_2\sum_{\langle i,j\rangle}\sigma_i\sigma_j + K_3\sum_{\langle\langle i,j\rangle\rangle}\sigma_i\sigma_j + K_4\sum_{\Box}\sigma_i\sigma_j\sigma_k\sigma_l + ... \tag{5.70}$$

We go on with the theory.

FIXED POINTS

Phase transitions correspond to fixed points:

$$\mathcal{R}\mathbf{K}_* = \mathbf{K}_*\,, \tag{5.71}$$

and their domain of attraction define a universality class.

Linear stability. We can linearize the RG transformation around a fixed point: LINEAR STABILITY

$$\mathcal{R}\mathbf{K} = \mathcal{R}\mathbf{K}_* + \mathcal{L}(\mathbf{K} - \mathbf{K}_*) + ... \tag{5.72}$$

where the linear RG transformation is now the stability matrix:

$$\mathcal{L}_{ij} = \left.\frac{\partial K'_i}{\partial K_j}\right|_* . \tag{5.73}$$

In general there is no reason why$\mathcal{L}$ should be symmetric, but for ease of discussion we will assume it. Thus the eigenvalues and eigenvectors are all real:

$$\mathcal{L}\mathbf{v}_n = \lambda_n \mathbf{v}_n \qquad\qquad \lambda_n = \lambda^{-\theta_n} , \tag{5.74}$$

where we introduced, as customary, the RG eigenvalues θ_n. The couplings in the direction of the eigenvectors are called the scaling fields or couplings and denoted with s_i. Since we are interested in the Ising fixed point which has two relevant directions we will set $t = s_1$ and $b = s_2$.

We have seen the the Ginzburg criterium tells us that the mean field critical exponents are correct in $d > 4$. We have the critical exponents in $d = 1, 2$ from the exact solutions. We now see that we can use the RG to compute the critical exponents for every d. Note that by the RG decimation procedure we are not changing the partition function (since we are considering the most general for the energy or free energy): DERIVING THE SCALING FORM AND THE CRITICAL EXPONENTS

$$\begin{aligned} Z(\mathbf{K}) &= \sum_{\{\sigma_i\}} e^{-\beta E_{\{\sigma_i\}}(\mathbf{K})} \\ &= \sum_{\{\sigma_i\}} \sum_{\{\sigma_i^{dec}\}} e^{-\beta E_{\{\sigma_i, \sigma_i^{dec}\}}(\mathbf{K})} \\ &= \sum_{\{\sigma_i\}} e^{-\beta E_{\{\sigma_i\}}(\mathbf{K}')} \\ &= Z(\mathbf{K}') , \end{aligned} \tag{5.75}$$

thus the free energy transforms as:

$$\begin{aligned} f(\mathbf{K}) &= -\frac{1}{\beta N}\log Z(\mathbf{K}) \\ &= -\frac{1}{\beta N}\log Z(\mathbf{K}') \\ &= \frac{N'}{N} f(\mathbf{K}')\,. \end{aligned} \tag{5.76}$$

Since $N' = \lambda^d N$ we find:

$$f(\mathbf{K}) = \lambda^d f(\mathcal{R}\mathbf{K})\,. \tag{5.77}$$

As we said, the coupling K_0 converges to the total free energy since at every step we integrate out half of the spins. Around a fixed point we can diagonalize the stability matrix and express the coupling vector in terms of the scaling fields$\mathbf{K} = (t, b, s_3, s_4, ...)$ and we find:

$$\begin{aligned} f(\mathbf{K}) &= \lambda^{nd} f(\mathcal{R}^n \mathbf{K}) \\ &= \lambda^{nd} f(\lambda^{-n\theta_t} t, \lambda^{-n\theta_b} b, \lambda^{-n\theta_3} s_3, \lambda^{-n\theta_4} s_4, ...) \\ &\underset{n\to\infty}{\to} \lambda^{nd} f(\lambda^{-n\theta_t} t, \lambda^{-n\theta_b} b, 0, 0, ...)\,, \end{aligned} \tag{5.78}$$

if we now choose $\lambda = |t|^{1/n\theta_t}$, we find the scaling form for the free energy:

$$f(t,b) \sim |t|^{d/\theta_t} f\left(\pm, \frac{b}{|t|^{\theta_b/\theta_t}}, 0, 0, ...\right) \qquad t \to 0 \qquad b \to 0\,, \tag{5.79}$$

with:

$$2 - \alpha = \frac{d}{\theta_t} \qquad \Delta = \frac{\theta_b}{\theta_t} \qquad \mathcal{F}_\pm(x) = f(\pm 1, x, ...)\,. \tag{5.80}$$

The RG analysis has thus proved the scaling hypothesis and showed that the critical exponents are related to the RG eigenvalues! The first relation is just the hyperscaling relation (5.60) and thus

$$\nu = \frac{1}{\theta_t}\,, \tag{5.81}$$

which immediately tells us how to compute the correlation length critical exponent. Spin block transformations as defined in this section are not an exact way to implement the coarse–grain $\mathcal{C}$ and are thus of of limited practical/quantitative use, even if conceptually very illuminating. In the next lecture we will develop a better formalism to construct the transformation $\mathcal{R}$ exactly.

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