CP#2 – Numerical Programming

Modeling Climate Feedback via Temperature–Ice Albedo Coupling
Implicit Euler with Fixed-Point Iteration and Newton–Gauss–Seidel

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Abstract

This project studies a simplified climate feedback system that couples global mean temperature and ice coverage through the ice–albedo feedback mechanism. The objective is not realistic climate prediction, but a numerical comparison of nonlinear solvers used within an implicit time-stepping method. A system of two nonlinear ordinary differential equations is formulated and solved using the implicit Euler method. Fixed-Point Iteration and Newton–Gauss–Seidel are implemented to solve the nonlinear system arising at each time step and are compared in terms of convergence speed, iteration count, and runtime. External forcing is included to study the response of the system to gradual warming.

1 Introduction

Climate systems often exhibit feedback mechanisms that amplify or dampen external influences. One important example is the **ice–albedo feedback**. Ice and snow reflect more incoming solar radiation than ocean or land. As ice coverage increases, planetary albedo increases, absorbed solar energy decreases, and temperature tends to decrease, which can further increase ice coverage. Conversely, warming reduces ice, lowers albedo, and increases absorbed energy.

The purpose of this project is to investigate such feedback behavior using a deliberately simplified conceptual model. The emphasis is on numerical techniques rather than physical realism. The resulting nonlinear system provides a suitable test case for comparing implicit numerical methods and nonlinear solvers.

2 Mathematical Model

2.1 State variables

The model consists of two state variables:

- $T(t)$: global mean temperature anomaly (in °C),
- $x(t)$: ice fraction, where $0 \leq x \leq 1$.

2.2 Albedo model

The effective planetary albedo is modeled as a linear function of ice fraction:

$$\alpha(x) = \alpha_{\text{ocean}} + (\alpha_{\text{ice}} - \alpha_{\text{ocean}})\, x. \tag{1}$$

2.3 External forcing

An external forcing term $F(t)$ is introduced to represent a controlled warming input independent of the ice–albedo feedback. In this work, a ramp forcing is used:

$$F(t) = \min(F_{\text{start}} + rt,\ F_{\max}), \tag{2}$$

which models gradual forcing increase capped at a maximum value.

2.4 Temperature equation

Temperature evolution is described by an energy-balance-inspired equation:

$$\dot{T} = \frac{S(1 - \alpha(x)) - (A + BT) + F(t)}{\tau_T}. \tag{3}$$

2.5 Ice equation

Ice fraction evolves toward a temperature-dependent equilibrium:

$$\dot{x} = \frac{x_{\text{eq}}(T) - x}{\tau_x}, \tag{4}$$

where

$$x_{\text{eq}}(T) = \frac{1}{1 + \exp\left(\frac{T - T_m}{w}\right)}. \tag{5}$$

3 Numerical Method

3.1 Implicit Euler discretization

Time is discretized with step size h. The implicit Euler method is applied:

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1}). \tag{6}$$

Applied to the present model, this yields the coupled nonlinear system:

$$T_{n+1} = T_n + h\,F_T(T_{n+1}, x_{n+1}, t_{n+1}), \tag{7}$$
$$x_{n+1} = x_n + h\,F_x(T_{n+1}, x_{n+1}). \tag{8}$$

Because the unknowns appear on both sides, a nonlinear solver is required at each time step.

4 Nonlinear Solvers

4.1 Fixed-Point Iteration

The implicit equations are rearranged into a fixed-point map:

$$T^{(k+1)} = T_n + h\,F_T(T^{(k)}, x^{(k)}, t_{n+1}), \tag{9}$$
$$x^{(k+1)} = x_n + h\,F_x(T^{(k+1)}, x^{(k)}). \tag{10}$$

Iterations continue until the maximum change between successive iterates falls below a prescribed tolerance.

4.2 Newton–Gauss–Seidel

Residuals are defined as:

$$R_1(T,x) = T - T_n - h\,F_T(T,x,t_{n+1}), \tag{11}$$
$$R_2(T,x) = x - x_n - h\,F_x(T,x). \tag{12}$$

Newton–Gauss–Seidel updates are performed by:

1. updating T using a one-dimensional Newton step with x fixed,
2. updating x using a one-dimensional Newton step with the updated T.

This method exploits derivative information and typically converges in fewer iterations.

5 Experiments

Simulations were performed on the interval $[0, 200]$ with step size $h = 0.5$. Both solvers were applied using identical parameters and initial conditions ($T_0 = -5$, $x_0 = 0.9$).

6 Results

6.1 Solver comparison

Method	h	Avg iters/step	Max iters	Runtime (s)	Failed steps	$(T_{\text{end}}, x_{\text{end}})$
Implicit Euler + FPI	0.5	8.28	11	0.0069	0	$(-13.25,\ 1.00)$
Implicit Euler + NGS	0.5	2.74	5	0.0034	0	$(-13.25,\ 1.00)$

Table 1: Comparison of nonlinear solvers used within implicit Euler.

7 Discussion

Both solvers converge to the same numerical solution, confirming correctness of the implementation. However, Newton–Gauss–Seidel requires significantly fewer iterations per time step and achieves lower runtime compared to fixed-point iteration. This behavior is expected for nonlinear implicit problems and highlights the efficiency of Newton-type solvers.

8 Conclusion

A simplified temperature–ice albedo feedback model was formulated and solved using implicit Euler time integration. Two nonlinear solvers were implemented and compared. Fixed-point iteration proved robust but computationally expensive, while Newton–Gauss–Seidel converged faster and required fewer iterations. The results demonstrate the practical advantage of Newton-type methods for solving nonlinear systems arising from implicit numerical schemes.