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Quant Project

QUESTION 1

A) The fundamental assumptions are:

1) There are no riskless arbitrage opportunities.
2) The asset price movements follow geometric Brownian motion with constant drift and constant volatility, so the random increments or price changes will be normally distributed with an infinitesimal variance.
Under this process the return to the holder of the stock in a small period of time is normally distributed and the returns in two nonoverlapping periods are independent. The value of the stock price at a future time follows a lognormal distribution.
3) It is possible to buy and sell stocks in any amount. The short selling of securities with full use of proceeds is permitted.
4) There are no transaction costs (the market is frictionless) or taxes. All securities are perfectly divisible.
5) There are no dividends during the life of the derivative.
6) Security trading is continuous.
7) It is possible to borrow and lend cash at the same constant risk-free interest rate, i.e. the risk-free rate of interest r is constant and the same for all maturities.
8) European options can only be exercised at expiration.
9) Market movements cannot be predicted.

<u>Many assumptions can be challenged. Let's discus a few.</u>

1) The **constant interest rates** assumption of is not very realistic.
2) The **frictionless market** Assumption is also not very realistic since trading has transaction costs: brokerage fees, commission, and others.
3) **The most important, perhaps, is the assumption that the price of the underlying asset S(t) is a geometric Brownian motion**, i.e. satisfies a stochastic differential equation of the form:
$dS/S = b\,dt + \sigma\,dW$ with the solution $S = S_0 \exp(b - \sigma^2 / 2)\, t + \sigma * W$, where W is the process of Brownian motion, and b and σ are the parameters of the drift and the standard deviation.
Nevertheless, the geometric Brownian motion (which was initially proposed to describe the behavior in the financial market in continuous time in the work of Samuelson [Samuelson, 1964]) does not very accurately model the change in value both in financial markets and in investment projects. In reality we can observe that fast (jump-like) changes, dependent or non-stationary

increments are characteristic of the processes observed in the financial market. This do not agree with the assumption that the increment in the logarithms of the change is a Brownian movement around some trend (drift). A number of papers have been published on condition softening and generalized Black-Scholz models (e.g., [Aase, 1988], [Barndorff-Nielsen, 1977]).

B) **$c = \exp(-r\,T) * (F_0\,N(d_1) - K\,N(d_2))$**

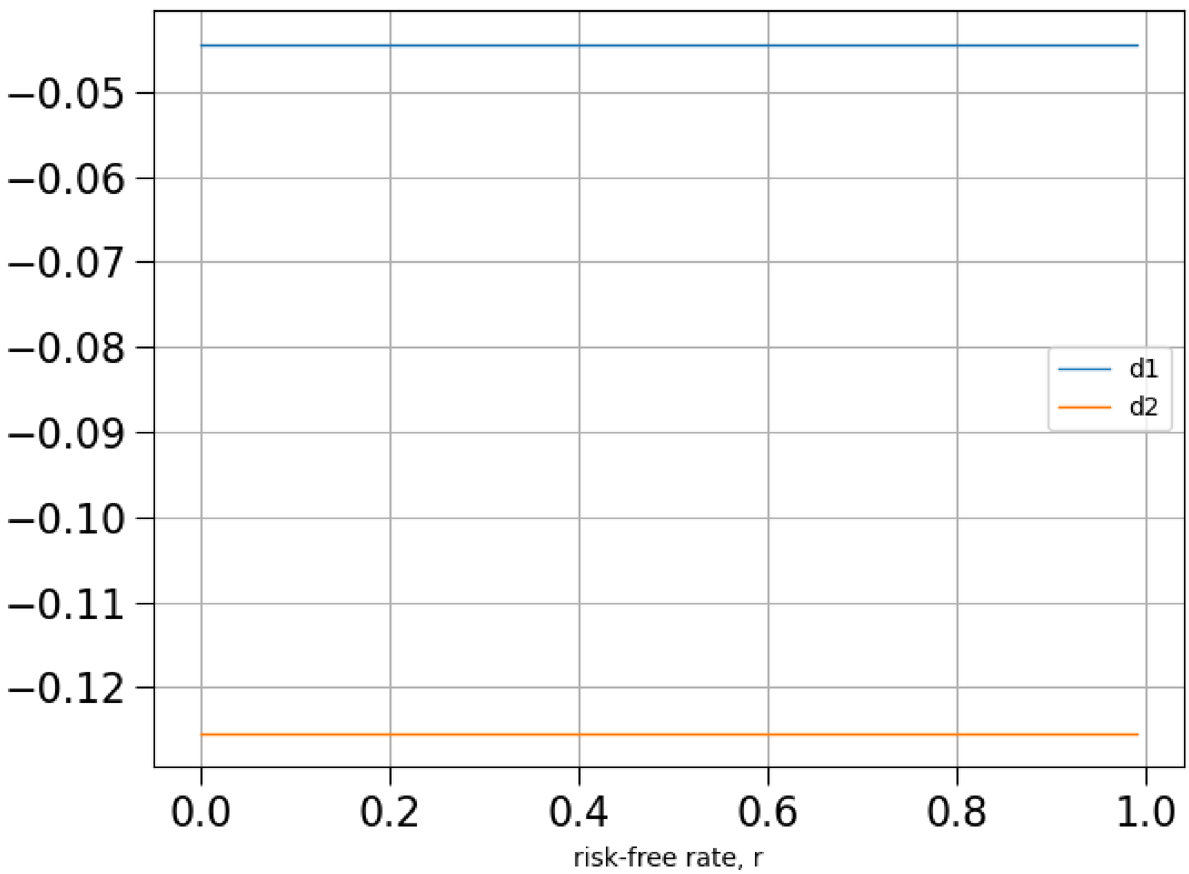
$p = \exp(-r\,T) * (K\,N(-d_2) - F_0\,N(-d_1))$

where

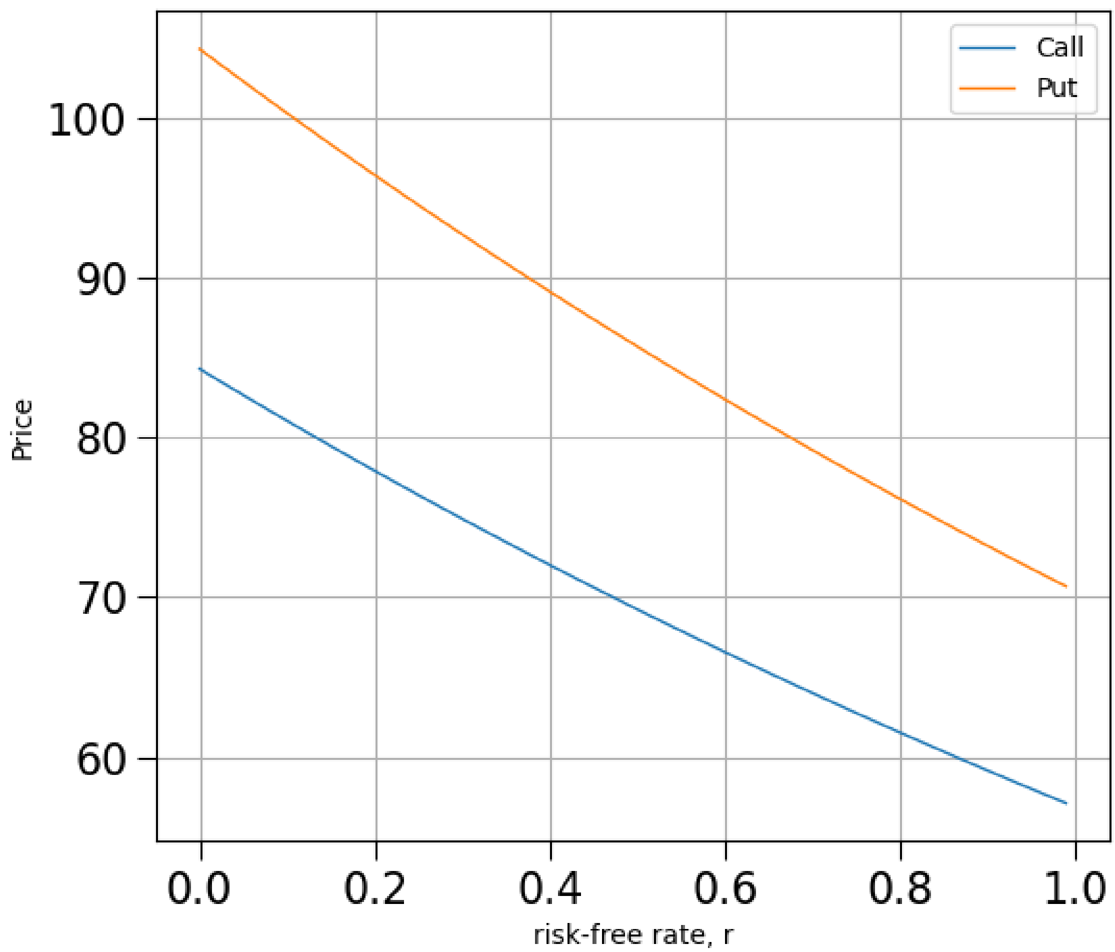
$d_1 = (\ln(F_0/K) + \sigma^2\,T / 2) / \sigma\,T^{0.5}$ and

$d_2 = (\ln(F_0/K) - \sigma^2\,T / 2) / \sigma\,T^{0.5} = d_1 - \sigma\,T^{0.5}$

C) **d_1 and d_2 do not depend on the risk-free rate, hence they are constants**.



The price of a call option decreases when r increases and the price of a put option also decreases when r increases. The partial derivatives with respect to r are the following:
$-T * \exp(-rT) * (F_0 N(d_1) - K N(d_2)) < 0$,
$-T * \exp(-rT) * (K N(-d_2) - F_0 N(-d_1)) < 0$.



D) First the formulas for the Greeks under consideration are:

$\partial C/\partial F$ = Delta(C) = $\exp(-r T) N(d_1)$

$\partial P/\partial F$ = Delta(P) = $-\exp(-r T) N(-d_1)$

$\partial C^2/\partial F^2$ = Gamma(C) = Gamma(P) = $\exp(-r T) / (F_0 \sigma T^{0.5}) * 1 / (2 \text{ pi})^{0.5} * \exp(-d_1^2 / 2)$

$\partial C/\partial \sigma$ = Vega (C) = Vega(P) = $F_0 \exp(-r T) (T / 2\text{pi})^{0.5} \exp(-d_1^2 / 2)$

$\partial C/\partial t$ = Theta(C) = $F_0 \sigma \exp(-r T) / 2(2 \text{ pi } T)^{0.5} \exp(-d_1^2 / 2) + r F_0 \exp(-r T) N(d_1) - r K \exp(-r T) N(d_2)$

$\partial P/\partial t$ = Theta(P) = $F_0 \sigma \exp(-r T) / 2(2 \text{ pi } T)^{0.5} \exp(-d_1^2 / 2) - r F_0 \exp(-r T) N(-d_1) + r K \exp(-r T) N(-d_2)$

a. First we will analyze the situation when the option expires in a week.

Gamma (and Delta) changes as the underlier price moves up or down from the initial price and the option moves ITM or OTM. The price of near-term options changes more significantly than the price of longer-term options. (Also: the price of ATM options will change more significantly than the price of ITM or OTM options with the same expiration).

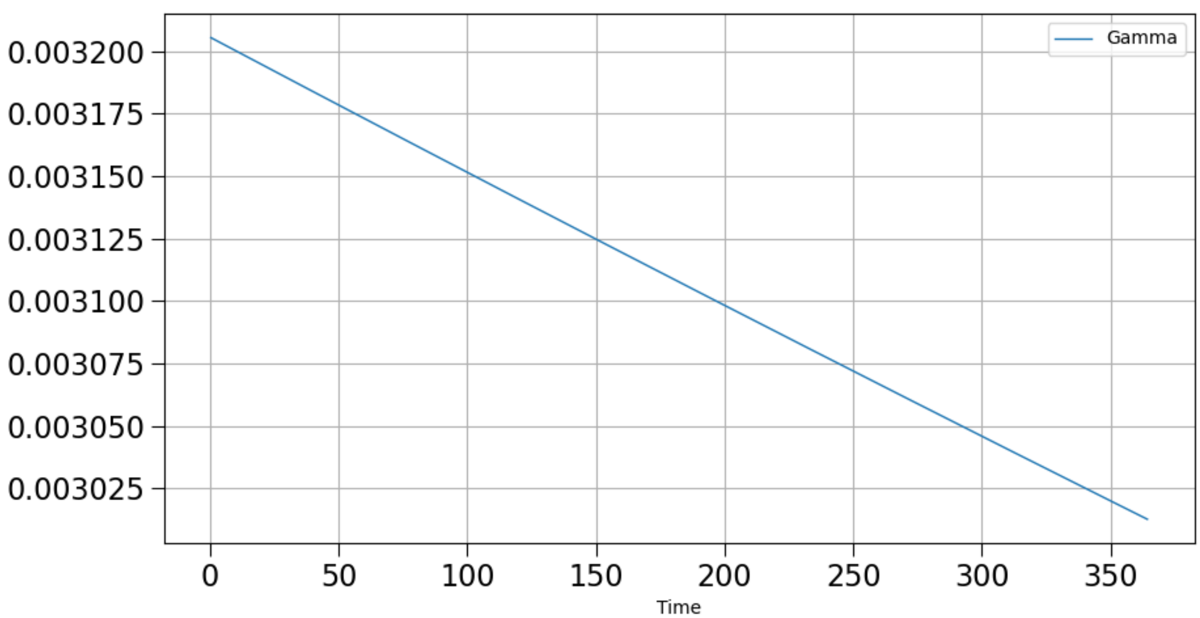
If we are going to buy an option, high Gamma is good if our forecast is right, because as the option moves ITM, Delta will approach 1.0 faster. But if the forecast is incorrect, Gamma is acting against us by quickly lowering Delta.

If we are selling an option and our forecast is incorrect then high gamma bad again: it can cause the position to work against us much faster as the option (that we have sold) moves ITM. But if our forecast was right, then high Gamma is good, because the value of the (sold by us) option will lose its value faster.

Vega is approaches zero when we approach expiration.

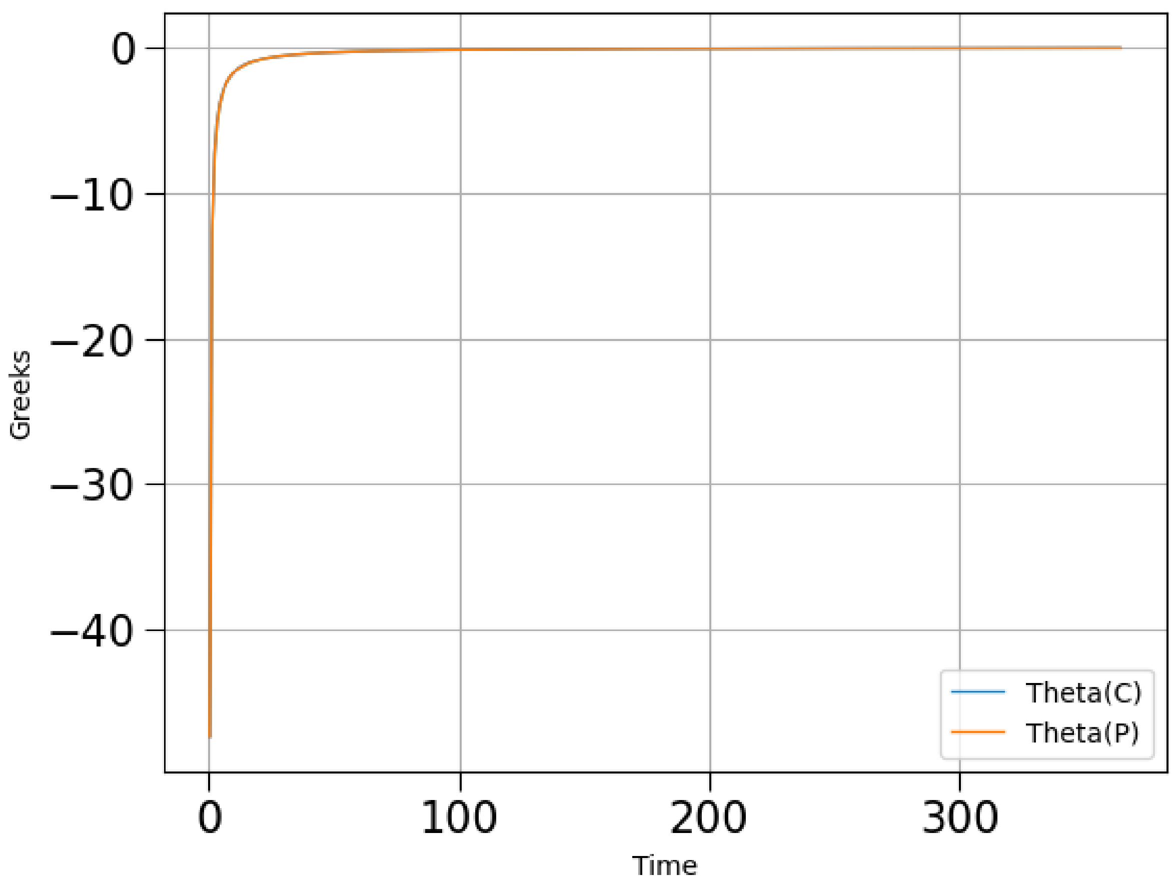
Theta is negative and increases in its absolute value as the expiration date approaches.

All in all, **if an option expires in a week, then we should consider first of all its Gamma, and then also Theta (which has a negative value, and its absolute value grows as time to expiration decreases).**

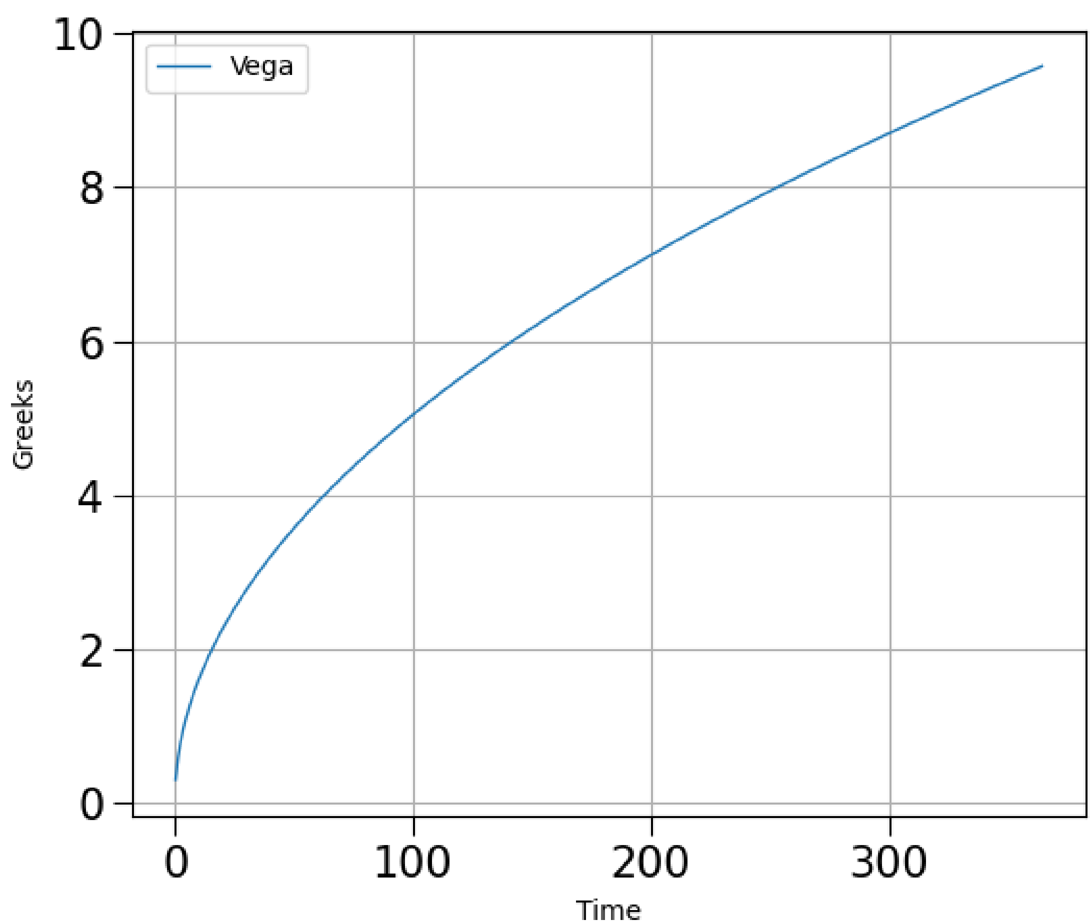


b. When the option expires **in a month we should start looking at Theta**, because the decay of the option's value significantly and not linearly accelerates when the time to expiration becomes less then a month. Theta is especially important for ATM options, since they have the highest volatility potential, and because of it impact of time decay is higher.

Theta is time decay, it shows the amount the price of calls and puts that will decrease when we get 1-day closer to the expiration. Time value disappears and does it at an accelerated rate as the expiration date approaches. ATM options will experience more significant dollar losses over time than ITM or OTM options (because ATM options have the most time value built into the premium). For OTM options Theta is smaller than for ATM options (because the dollar amount of time value is smaller), but the loss may be a bigger as a percentage for OTM options because of the smaller time value.

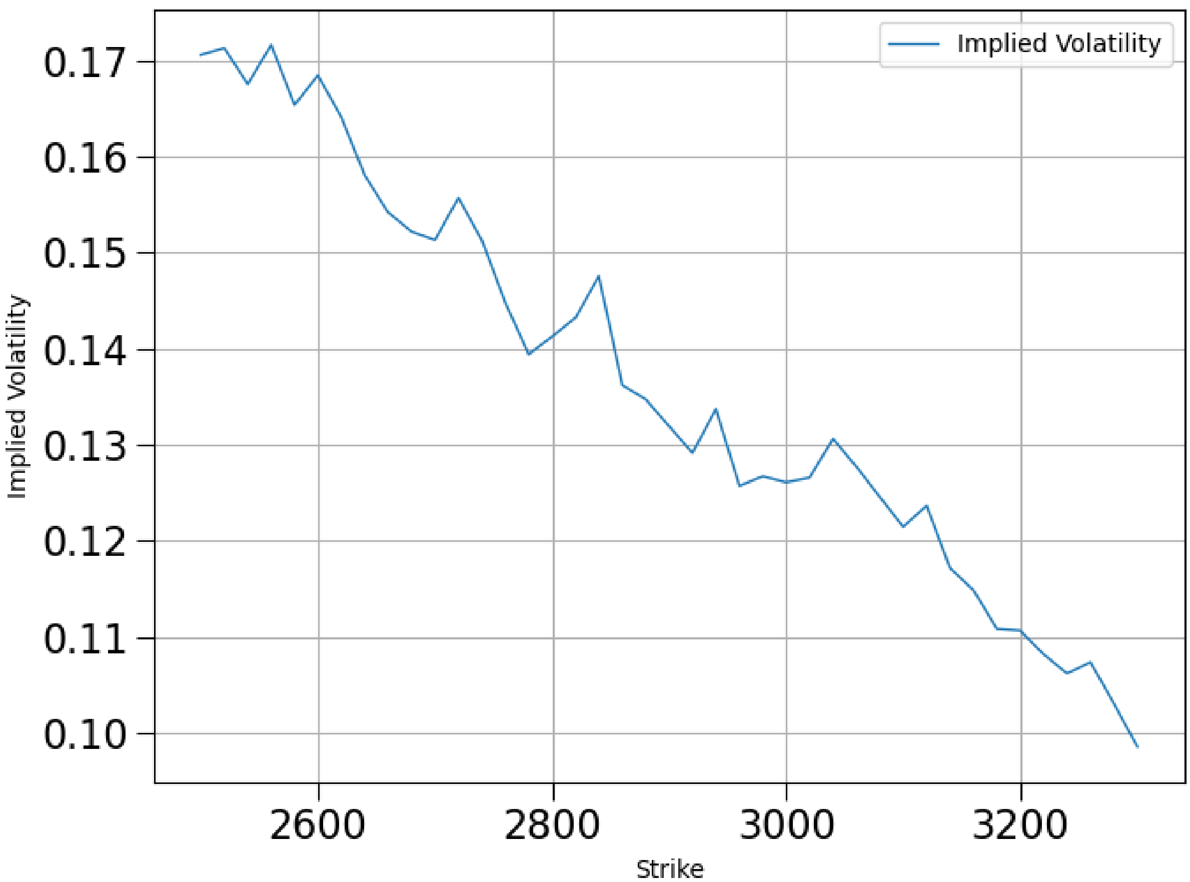


c. When the option expires in a year we see from the formulas and from the graphs that Theta approaches 0, Vega increases, and Gamma decreases. Intuitively when it is a year till the expiration it is a lot time left and an increase in implied volatility leads to a significantly increased range of potential movement for the underlier (since there is a lot of time left).
Vega is the most important when the option expires in a year.

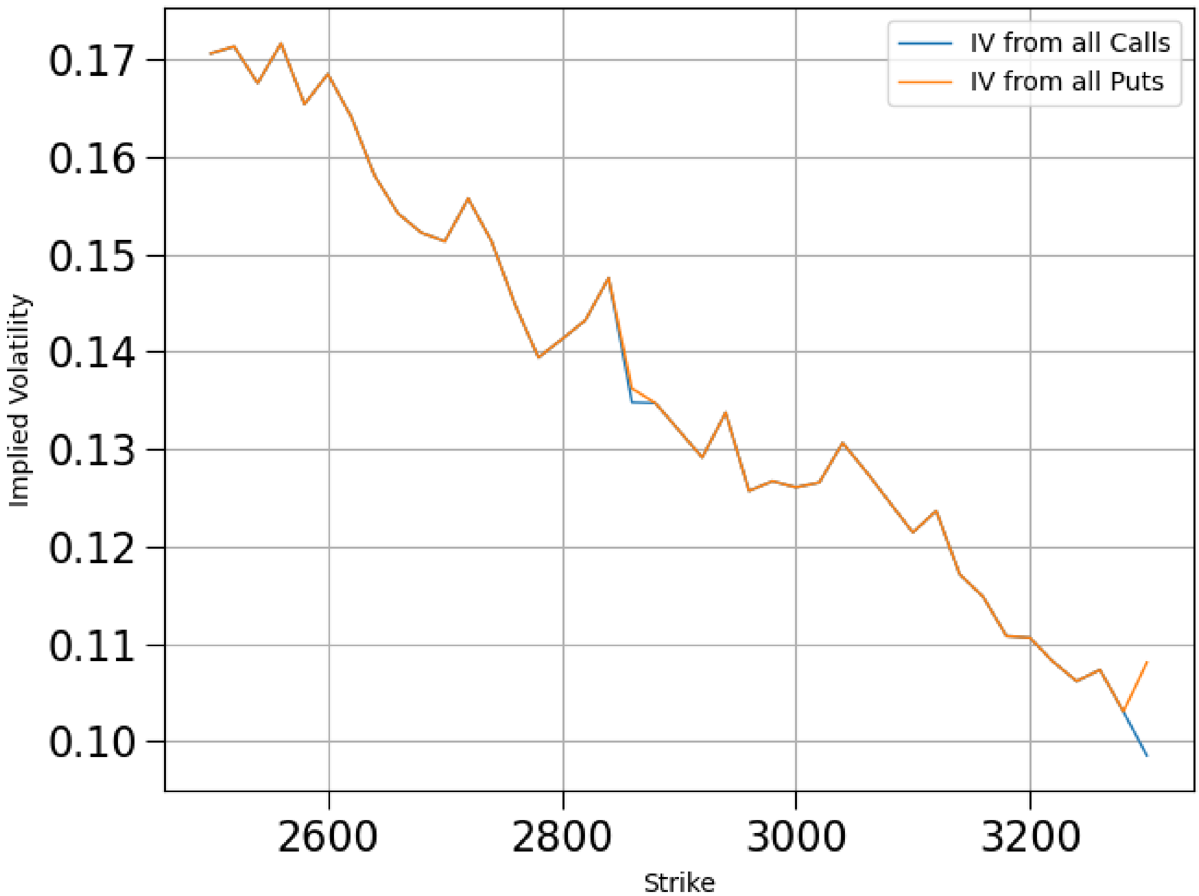


QUESTION 2

A) First we need to compute implied volatility (IV) for each data point. Since we want to use Out-of-the-Money options for strikes less than 2900 we will be using put option prices and for strikes greater than 2900 we will be using call option prices. Since there is no closed formula for expressing IV as a function of F , K, r, T, and c or p, we will use a numerical solution. An iterative search procedure can be used to find the IV. Using the fact that the price of an option is an increasing function of IV and applying Bisection method we can iteratively (by halving the range for IV at each iteration) calculated the correct value of IV to any required accuracy. Alternatively we can use py_vollib.black.implied_volatility.implied_volatility_of_discounted_option_price function from py_vollib.black package: http://vollib.org/documentation/python/1.0.2/apidoc/py_vollib.black.html



We can also plot on one plot two IV curves, one generated just from all Put options and one from all Call options. Theoretically these two curves should be exactly the same. But From the plot we notice that they are different at two points: for strike 2860 and 3300.

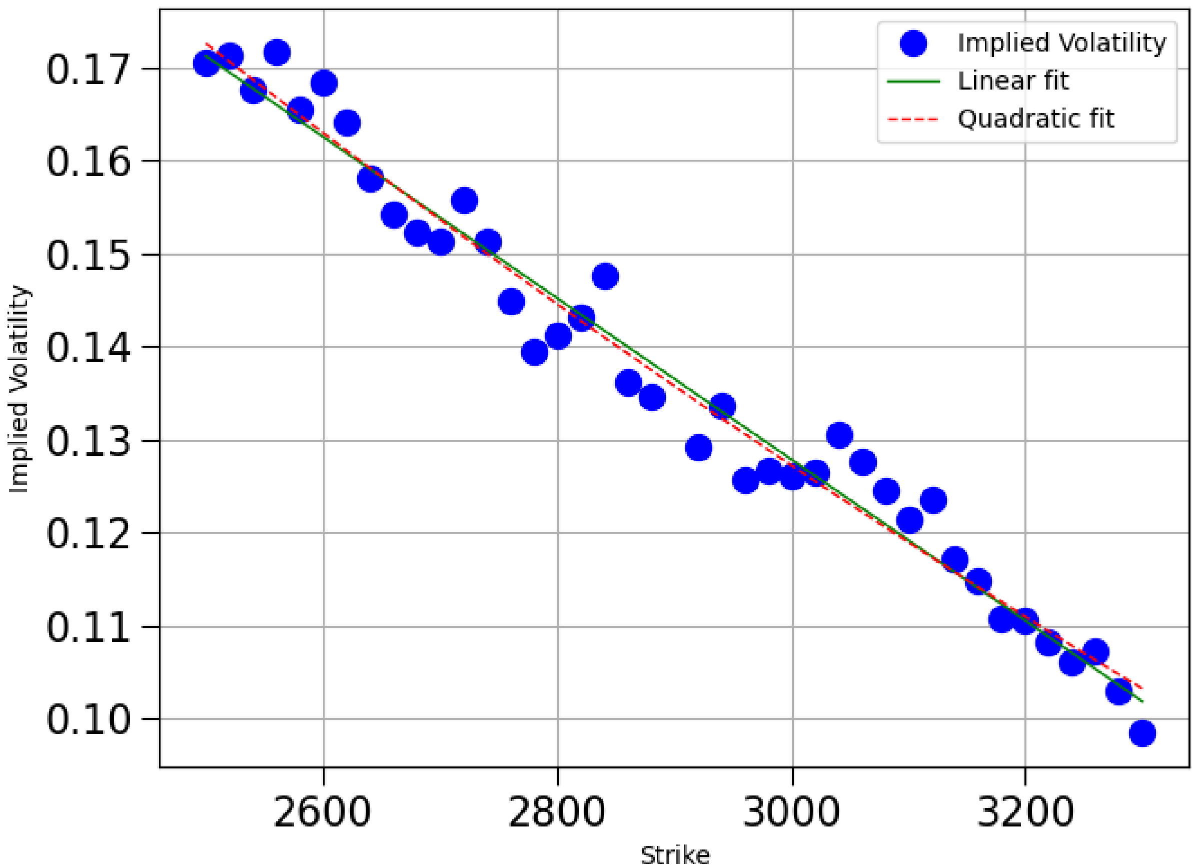


Now we want to find an analytical function IV = f (strike) which would approximate our data. We want this function to have a constant convexity, so we will be looking for a polynomial of a degree one or two. We will check both options.
For the numerical solution we will use polyfit function from pylab module.
For the linear approximation (a degree one polynomial) the solution is:

$$IV = -8.66886234e\text{-}05 \text{ strike} + 3.87943037e\text{-}01.$$

For the quadratic approximation (a degree two polynomial) the solution is:

$$IV = 1.33025617e\text{-}08 \text{ strike}^2 - 1.63843481e\text{-}04 \text{ strike} + 4.99054014e\text{-}01.$$



B) From the plotted volatility curve and the fitted polynomials we can observe that IV is high for the low strike values and then drops as the strike values increase. This picture is sometimes called Volatility smile/smirk. This kind of a behavior suggests that OTM puts (and ITM calls) are more expensive compared to ITM puts (and OTM calls).

One explanation for this phenomenon can be the following. Investors are generally worried about market crashes and buy puts for protection. There is an evidence supporting this point of view: this patter, which is sometimes called “reverse skew”, did not show up for equity options until after the Crash of 1987. So, it can lead to a higher demand on OTM, i.e. with the lower strike values, puts. Hence, we can observe higher prices for them and hence higher IV.

Another explanation is that ITM calls are considered as alternatives to outright stock purchases, because they offer leverage (and therefore a higher ROI). This can lead to a greater demand for ITM calls and consequently higher prices, and hence higher IV for the lower strike values.

One more explanation is that this Volatility smirk is the market's view of what is going to happen to the realized volatility of the underlying asset (in our case the December ES Mini Future) if it falls/rises to a particular strike value. In the case of downside (i.e. for lower strike values), the market is rationally building in an expectation that a rapid fall in the underlying asset can foretell a rising risk of credit default. And in the case of upside, i.e. if the price of the underlying asset will rise (i.e. what we see for the higher strike values), the realized volatility is expected to drop, because the risk drops, because e.g. the risk of credit default is falling.

C) Let's remember the explanation from (B) via "investors are buying protective puts". It is sometimes said that "mutual funds are being cautious about current market conditions and expect deterioration". It is hard for mutual funds to just sell shares quickly, because there can be not enough liquidity.

In the normal situation we should observe volatility smile on the market (or a constant IV from the theory). But from our data we observe Smirk, because there are forced buyers and forced sellers on the market. Funds are forced to buy options with lower strike values (puts for protection and calls for investing). Because of this the prices of the options with low strike values are too high (people are "overpaying") and with high strike values are too low. So, we can play against them. This means that I offer to **sell the options with the highest IV (corresponding to the strike = 2560) and buy the options with the with the lowest IV (corresponding to the strike = 3300).**

D) Let's check the put–call parity relationship for our European futures options: c + K exp(-rT) = p + F_0 exp(-rT). Checking it for all values of strike we find that it is violated for strikes equal to 2860 and 3300. In both cases we observe that
C + K exp(-rt) + 1 = p + F_0 exp(-rt), i.e. the corresponding calls are cheap and puts are expensive. Hence there is an arbitrage opportunity: **we can exploit it by selling the put (e.g. with strike 3300) and buying call (with the same strike)**. Here is the exact strategy. We should create a portfolio consisting of:
Short European futures put option, short futures contract (it costs zero), long European futures call option, borrow an amount of cash equal to F_0 exp(-rT), and lend an amount of cash equal to K exp(-rT). Creating this portfolio leaves us with +1, e.g. with a positive sum of money, which we use to make a cash deposit, which will be exp(rT)> at the moment of expiration. At the expiration the value of this portfolio in any scenario (whether F_T>K or F_T≤K) is exactly zero. Hence, we have created an arbitrage.

E) Vega = the derivative of the portfolio price by the volatility. Vega measures an option price's value relative to changes in implied volatility of an underlying asset. Options that are long have positive Vega, and options that are short have negative Vega.

For long Vega portfolio we with limited Gamma risk we can construct the following portfolio: **long call**.
(Note: Gamma risk is minimal for deep OTM and deep ITM options. From the actual computations we can find that Gamma is minimal for the calls with strike 3300. But delta is also declining for deep OTM and deep ITM options.) **Let's choose a call with strike 2900**. (We could have chosen any other value of strike, e.g. 3100).

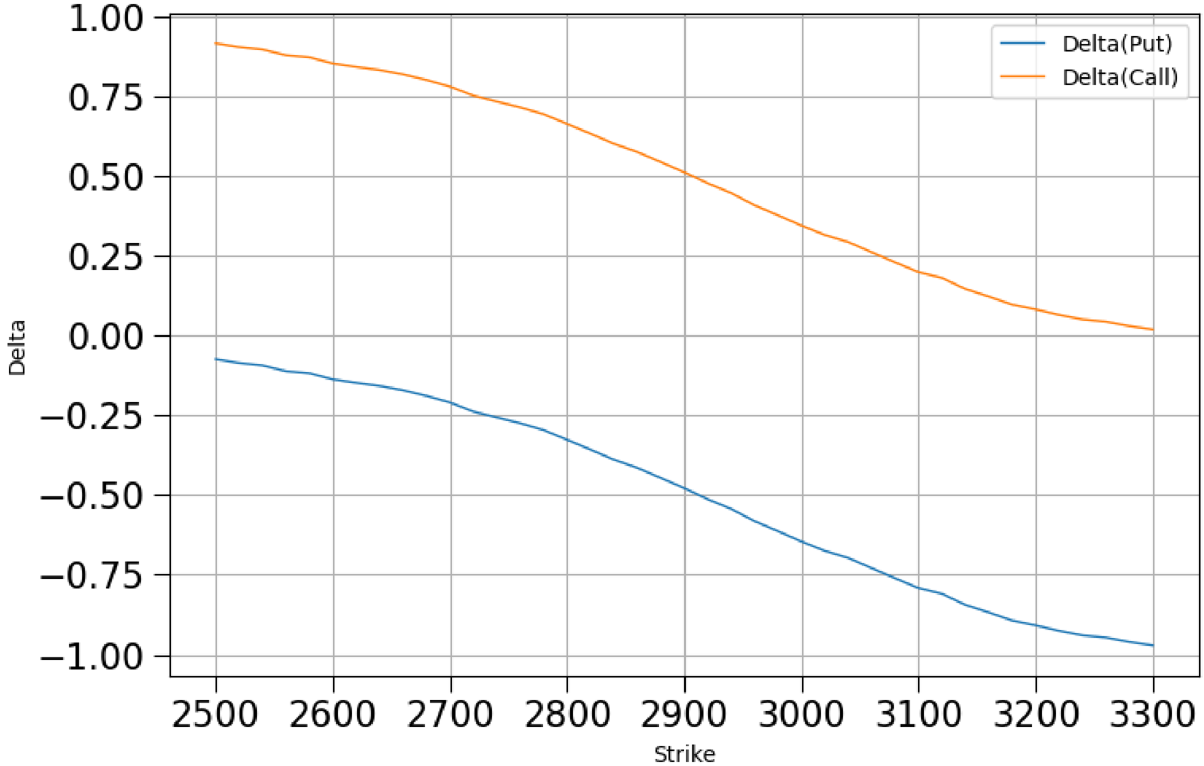
There are many other possible choices for our portfolio. For example: long call and short futures; or long put and long futures.

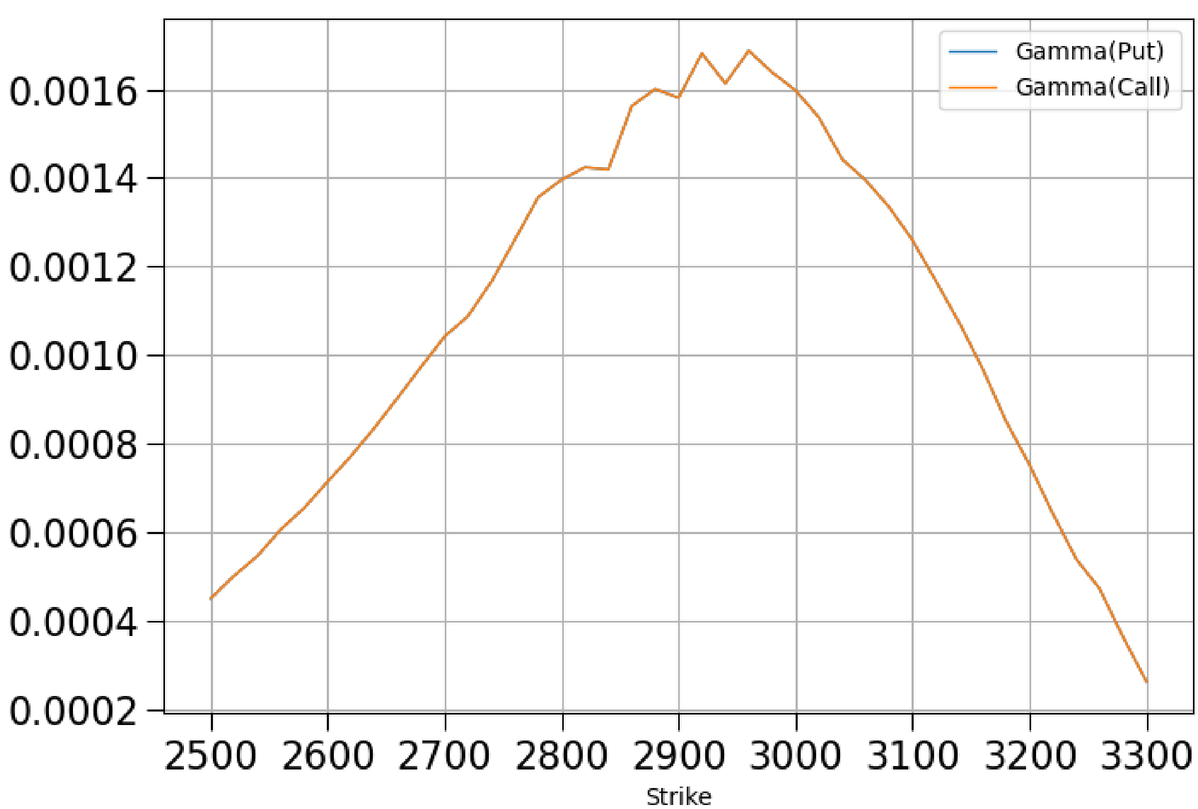
F) To eliminate exposure to directional underlying moves we need to make delta of the portfolio zero.
For this we create **Long Straddle** by buying ATM call and buying ATM put (e.g. both with the same strike of 2900) and the same expiration (Dec20). This is a long Vega portfolio with limited Gamma risk and zero Delta, because call and put have Deltas of different signs and the same absolute value, so the portfolio is delta-neutral at initiation of the strategy.
Here we should make a note: this was just an approximation. If we want to be precise than we should say that the delta of ATMF call is not exactly 0.5, but it is a little bigger due to the fact that the volatility is positive, so Δ(ATM call) > 0.5, we can use the first order approximation: Δ(ATM call) $\approx$0.5 + 0.2 σ sqrt(time to maturity) = 0.5 + 0.2 * 0.137 * 0.627 = 0.517. Without approximations the exact Δ(ATM call) = 0.512. Δ(ATM put) = -0.478. This means that for exact Delta-neutral portfolio we take 1 long ATM call and 1.07 long ATM put. If we need integer number of options, we **buy 100 ATM calls and buy 107 ATM puts**.

Another option, e.g., is **Long Strangle**, which means that we are: long OTM call and long OTM put option. The call and put strikes we choose to be equidistant from 2900, hence e.g. call with strike 3100 and put with strike 2700. Their deltas have different signs and almost the same absolute values: Δ(put with strike 2700) = -0.21, Δ(call with strike 3100) = 0.199. So for 1 puts we need 1.05 calls, hence we **buy 100 puts with strike 2700 and 105 calls with strike 3100** and we get a delta-neutral portfolio. This portfolio generates a positive return when the price of an underlying asset moves significantly in either direction, so we do not need to forecast the trend of the market, but just bet on the volatility. This is a long Vega portfolio with limited Gamma risk and zero Delta.

If we have chosen in (E) long ATM call (strike 2900) and short futures, then for delta neutral portfolio we need to balance the delta of the short futures, i.e. – 1, and the delta of our long call, which is 0.512. For this for every futures we need 1.95 calls. Hence, we can **sell 100 futures and buy 195 ATM calls** to achieve delta-neutrality.

If we have chosen in (E) long ATM put (strike 2900) and long futures, then for delta neutral portfolio we need to balance the delta of the long futures, i.e. + 1, and the delta of our long put, which is -0.478. For this for every futures we need 2.09 puts. Hence, we can **buy 100 futures and 209 ATM puts** to achieve delta-neutrality.


Delta(Put)
Delta(Call)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Delta
2500
2600
2700
2800
2900
3000
3100
3200
3300
Strike


Gamma(Put)
Gamma(Call)
0.0016
0.0014
0.0012
0.0010
0.0008
0.0006
0.0004
0.0002
2500
2600
2700
2800
2900
3000
3100
3200
3300
Strike

G) We can use, e.g. a **short straddle** and so the following: sell ATM call (strike 2900), sell ATM put (strike 2900). A short straddle leads to the obligation to sell the futures at strike price and the obligation to buy the futures at strike price if the options are executed. It is used when we hope that the underlying asset will not move significantly higher or lower over the lives of the options contracts. Normally realized volatility is smaller than implied volatility, so shorting volatility makes money. Until it does not is a situation of a big unpredicted price move (like a crash/crisis). So, such strategy has significant costs, because it leads to an unlimited risk on the upside and substantial downside risk (shorting volatility is "like picking up pennies in front of a stream roller").

H) We enter into a short vol position by selling call (strike K) and selling put (strike K).
To choose the best strike we do the following simulation. For each strike K we first find the delta-neutral strategy by finding the coefficient m that gives zero Delta to the portfolio of 1 short call (with strike K) and m short puts (with strike K). Then we run a Monte Carlo simulation 10 000 times. On each iteration we simulate the futures price at expiration:
$F_T = F_0 * \exp((r - 0.5 * \sigma^2) * T + \sigma \text{ sqrt}(T) * \text{gauss}(0, 1))$, where σ is taken equal to the current implied volatility. Then we calculate the corresponding C_1, price of the call, and P_1, price of the put option. Then we compute the corresponding PNL at maturity as $(C_0 + m P_0) \exp(r t) - C_1 - m P_1$ - commission, where C_0 and P_0 are the prices of the call and put options when we sell them on Dec1, and commission = (1 + m)*\$0.85, since selling each option has commission of \$0.85. Then we average all 10 000 simulations and find the best K. For this K we find how much per contract it costs to enter the position. The best K was found to be K = 2720, m = 3.16, so we **sell 100 calls with strike 2720 and 316 puts with strike 2720**.

It will cost us $- 100 C_0 - 316 P_0 = - 35\,175.94$ to enter into this position (meaning that, since we are shorting, we will get paid for entering this into position). For the commission per contract lets use the information from Interactive Brokers (https://www.interactivebrokers.com/en/index.php?f=1590&p=futures1), which says that the commission is \$0.85, so we need to pay 416*\$0.85 = \$353.60, since we are selling 100 calls and 316 puts. **In total we get \$34 822.24**.

For shorting Call and shorting Put sale we will need to put on the margin account the following sum of money (let's use the information from Interactive Brokers https://www.interactivebrokers.com/en/index.php?f=26660&hm=us&ex=us&rgt=1&rsk=0&pm=1&rst=101004100808):
Initial margin short put = put price + max{15% * underlying price – OTM amount), 10% * strike price}.
Initial margin short call = call price + max{15% * underlying price – OTM amount), 10% * underlying price}.
Initial/RegT End of Day Margin is found as:
if Initial margin short put > Initial margin short call, then Initial margin short put + price short call,
else if Initial margin short call >= Initial margin short put, then Initial margin short call + price short put.
We are shorting 100 calls and 316 puts, so we get Initial Margin Short Put = 991.65, Initial Margin Short Call = 654.94, so
Initial/RegT End of Day Margin = **\$1 211.58, which we need to put on the margin account**.

I) For our strategy (sell 100 call with strike 2720 and sell 316 puts with strike 2720). The results are the following:

If the futures settles **at 2500 then PNL = –34 670.38** USD (such a big loss is due to the coefficient m = 3.16).

If the futures settles **at 2800 then PNL = 26 874.63** USD.

If the futures settles **at 2900 then PNL = 16 874.63** USD.

If the futures settles **at 3000 then PNL = 6 874.63** USD.

If the futures settles **at 3300 then PNL = –23 125.3**7 USD.

QUESTION 3

For all calculations in this question we will use the ACT/ACT convention which calculates actual days in a time period, over the actual number of days in a year.

A) Since the yield curve is flat, we do not expect any changes in the risk-free rate. So as a big picture we expect ES Minis price to be changing in parallel with S&P 500 and with SPY. But there are several reasons for futures to be the better option.

- SPY as any ETF has Operating expenses.
- SPY as any ETF has tracking error, so the hedge is not ideal.
- The commission is usually higher for ETFs (for 30 000 SPY than for 60 ES mini as I will write in (E)).
- For ES Minis investor can usually use bigger margin and it will leave him with more free cash.

Although, typically Bid/Ask spread it higher for ETFs, but since we assumed extremely high liquidity, it is not applicable for our situation.

For non-US investors, futures can be cheaper than ETFs, even if roll yields are rich, because of withholding taxes (WHT) on dividend income. There are some strategies to reduce/avoid WHT, but if an investor cannot fully avoid WHT then ETFs will be subject to WHT of up to 30%, and this will lower its returns by up to 70 basis points. Futures do not pay dividends, so there is nothing to withhold. Futures prices already includes the expected dividend income.

All in all, when roll costs are cheap futures are more cost effective than ETFs, so **I think ES Minis are better**.

B) Each 30 Delta SPX Call Options has delta 0.30. Since we bought 100 such potions and each has a 100 multiplier we get 100 * 100 * 0.30 = 3000.
Let x be the number of shares of SPY that we need to be trade is found via the following computation:
x * 0.1 = 3000, hence x = 3000 / 0.1 = 30 000, so <u>**30 000 shares of SPY**</u> need to be traded.
Let y be the number of ES Minis that we need to be trade is found via the following computation:
y * 50 = 3000, hence y = 3000 / 50 = 60, so <u>**60 ES Minis**</u> need to be traded.

C) I would suggest that the trader <u>***either sells 30 000 shares of SPY or 60 ES Minis***</u> to hedge her 100 30 Delta SPX Call Options. Then if the market moves a little during in the period until the expiration, then the movements of the 30 Delta SPX Call Options and of the SPY or ES Minis will have the opposite signs (they will move on the opposite directions), but the equal absolute values (since the deltas are the same). I.e. the value of the new portfolio (100 30 Delta SPX Call Options long and 30 000 SPY or 60 ES Minis short) is not sensitive to small changes in the price of the underlying asset.

I would also note that for taxable US investors, <u>**ES Minis can be more tax efficient**</u>, since 60% of gains will be treated as long-term capital gains, and hence taxed at a lower rate than the 40% of gains that will be treated as short-term capital gains. At the same time shorter-term SPY trades would be treated entirely (i.e. 100%) as short term gains.

D) We will use the data from https://www.multpl.com/s-p-500-dividend-yield which says that Current S&P 500 Dividend Yield is 1.8% = 0.018.
There are 80 days between October, 1st and December 20th.
Futures price = Current S&P Cash Value * exp((0.026-0.018)*(80/365)).
From here we find that: Current S&P Cash Value = $2920 / exp((0.026-0.018)*(80/365)),
i.e. **Current S&P Cash Value** = **$2914.88**. Hence the **fair value of SPY is $291.49**.

E) According to https://www.interactivebrokers.com/ Commission per Contract for CME Group E-mini is $0.50, which together with the other fees will be = IBKR Execution Fee USD 0.5 + Exchange Fee USD 0.85 + Clearing Fee USD 0.00 + Regulatory Fee USD 0.02 = USD 1.37, hence for 60 contracts we will need to pay $1.37 * 60 = **$82.2** (just for comparison for 30 000 SPY we would need to pay 30 000 * $0.0035 = $105).
I should note here that there are additional expenses associated with it. First, we must put money into to margin account (according to https://www.cmegroup.com/education/courses/introduction-to-futures/margin-know-what-is-needed.html it is typically 3-12% per futures contract). If we assume that we should put 10% it will mean 60 * 50 * 2920 * 5% = **$438 000**.
Altogether it will be **$438 082.2 to execute the hedge immediately**. Of course, the sum money that we put now on the margin account will be changing over time, and after the hedge is not needed any more we can close the account and get what is there back.
And we also will have to keep adding money to the margin account is case there are margin calls. If at any point during the time between now and the next month (when options expire and hence the hedge is no longer necessary) there is a margin call and we are not able to provide the money necessary to keep your position going the broker is going to cancel the position and all the benefits from the future position will no longer be available.

F) Hedging with SPY will mean selling 30 000 SPY which will give us 30 000 * 291.49 = **$8 744 700**, which we deposit on a risk-free account.
Commission $0.0035 per share, we are buying 30 000 shares, so the **total commission will be** 30 000 * $0.0035 = **$105.**
Altogether we will get $8 744 595 from execute the hedge immediately. After the hedge is not needed anymore, we buy the SPY shares back, for which we again pay the total commission of **$105**.

G) It is still October 1, 2019.
There are 45 days between Oct 1 and Nov 15; lets denote the current Nov 15, 2019 spot rate as x.
There are 35 days between Nov 15 and Dec 20; Nov 15, Dec 20 forward rate is 0.0300.
There are 80 days between Oct 1 and Dec 20; Dec 20 spot rate is 0.0260.
From no arbitrage condition we have: exp(x * 45) * exp (0.03 * 35) = exp(0.026 * 80), hence x * 45 + 0.03 * 35 = 0.026 * 80, so x = (0.026 * 80 - 0.03 * 35) / 45 = 0.0229, i.e. **the current Nov 15, 2019 spot rate equals 229 basis points**.

H) If the ES Future remains at 2920 and the Dec 20, 2019 spot rate is still 260 basis points, the new SPY price is the same as before, i.e. it is **$291.49.**

It can be found from the following equation:
2920 = Current S&P Cash Value * exp((0.026-0.018)*(80/365)),
or equivalently from 2920 = Current S&P Cash Value * exp((0.0229-0.018)*(45/365)) * exp((0.0300-0.018)*(35/365)).

I) We can assume that market expects N&P 500 and SPY to change according to the average total return of 10%.
If we assume this then $1.1 = (1+x)^{365}$, hence $x = 1.1^{1/365} - 1 = 0.00026$.

Then tomorrow's S&P 500 is expected to be: S&P 500 today * (1 + x) = $2914.88 * (1+0.00026) = 2915.6378688.
Tomorrow's SPY is expected to be: SPY today * (1 + x) = 291.57.
(Nevertheless, some people may say that we cannot apply the long term forecast as the tomorrow's expectation, they may also say that we should just assume that SPY tomorrow will be the same as today, hence 291.49.)
Let's assume we chose the first approach and **expect SPY to grow in value by $0.08 and cost 291.57.**

From here we can find the expected price of Dec 20, 2019 ES Minis tomorrow:
F_1 = S&P 500 tomorrow * exp((0.0229 – 0.018) * 44/365) * exp((0.0300 – 0.018) * 35/365) = 2920.72, i.e. due to $0.25 increments **the price of Dec, 20 ES Mini on Oct 2 is 2920.75.**
Let's denote the October 1st price of Dec 20, 2019 ES Mini is F_0, i.e. F_0 = 2920.
Hence the value of the forward will be: f = (F_1 - F_0) * d(Oct1, Dec 20) = **0.72**.

J) We will still use the ES Minis, but the cashflow will be different.
We sell short Dec 20, 2019 ES Minis on Oct1. Their value f_0=0 (and their price is F_0=2920). And then we buy them on Nov1, their value f_1 = (F_1 - F_0) * d(Nov1, Dec20), where F_1 is their price on Nov1. Let's denote the price of the S&P 500 on Nov1 as S_1, and let's assume that the dividend yield is 1.8%.

Scenario 1: The yield curve is flat, risk-free rate is 260 basis points. Then F_1=S_1exp((0.026 – 0.018)*49/365).
Hence f_1 = (S_1 exp((0.026-0.018) *49/365) - 2920) / exp(0.026-49/365).

Scenario 2: The Dec 20 spot rate is 260 basis points, the Nov15 Dec20 forward rate is 300 basis points, Nov15 spot rate is 229 basis points. In this scenario F_1 = S_1 exp((0.0229 – 0.018)*14/365) exp((0.030 – 0.018)*35/365),
d(Nov1, Dec20)=1/ (exp(0.0229*14/365)* exp(0.0300*35/365)). So all together we have:
f_1 = (S_1 exp((0.0229 – 0.018)*14/365) exp((0.030 – 0.018)*35/365) – 2920) / (exp(0.0229*14/365)* exp(0.0300*35/365)).

On Nov1 we need to buy ES Minis back, paying f1 (see above) for each, and we need to buy 60 of them (see (B)).

If we decided to hedge by selling 30 000 shares of SPY, the price today would be the same in Scenario 1 and 2. Then we could deposit the money that we got from selling short on a bank account with a risk-free rate (0.0260 in the Scenario 1, and 0.0229 in the Scenario 2). It is also possible that the price of SPY on Nov1 would be the same in both scenarios, so that it would cost us the same to buy them back on Nov1 in both scenarios.

K) First, let's research the situation when we enter the contract on Oct 1.
Let's make the following notations:
the price of Dec, 20 ES Mini on Oct 1 we denote as F_0 (and it is the same for both scenarios and equals 2920),
the price of Dec, 20 ES Mini on Nov 1 we denote as F_1,
the Nov1-Dec20 discount rate as d(Nov1, Dec20),
the S&P 500 spot price on Nov 1 we denote as S_0 (we know it since we found it above, it is 2914.88),
the S&P 500 spot price on Nov 1 we denote as S_1 and let's denote it as $S_1 = S_0 * x$, x shows how much the price is going to change (the reasonable value could be somewhere between 0.5 and 2.0).
the value of the forward position on Nov 1 we denote as f_1.
The value of the forward position on Oct 1 (when we are entering the contract) is equal to zero.
It is known that $f_1 = (F_1-F_0) * d(Nov1, Dec20)$. F_1 and d(Nov1, Dec20) depend on the scenario.

<u>For scenario 1:</u> the yield curve is flat, the current risk-free rate is 260 basis points. The dividend yield is 1.8%.
Then $F_1 = S_1 * \exp((0.0260 - 0.018)*(49/365))$, $d(Nov1, Dec20) = 1/\exp(0.0260*49/365)$, so
$f_1 = (F_1-F_0) * d(Nov1, Dec20) = (x * 2914.88 * \exp((0.0260 - 0.018)*(49/365)) - 2920) / \exp(0.0260*49/365)$.
<u>For scenario 2:</u> the Nov 15, 2019, Dec 20, 2019 forward rate is now 300 basis points, as we found earlier the current Nov 15, 2019 spot rate equals 229 basis points.
Then $F_1 = x * 2914.88 * \exp((0.0229 - 0.018)*(14/365)) * \exp((0.0300 - 0.018)*(35/365))$, $F_0=2920$, $d(Nov1, Dec20)=1/\exp(0.0229*14/365)/\exp(0.0300*35/365)$, and $f_1 = (F_1-F_0) * d(Nov1, Dec20)$.

For any x > 0 (i.e. for any possible x) the value f_1 in scenario1 is strictly less than the value f_1 in scenario 2.
We sell ES Minis on Oct 1, we want to close the position on Nov 1, so we want to buy ES Minis, and their value in scenario 1 is less than in scenario 2, hence <u>**in scenario 1 it is cheaper to hedge than in scenario 2.**</u>

Now let's research the situation when we enter the contract (sell short) on Nov 15 and close it (buy) on Dec 15.
Let's make the following notations:
the price of Dec, 20 ES Mini on Nov 15 we denote as F_0 (it will be different for different scenarios),
the price of Dec, 20 ES Mini on Dec 15 we denote as F_1,
the Dec15-Dec20 discount rate we denote as d(Nov1, Dec20),
the S&P 500 spot price on Nov 15 we denote as S_0 (we know it since we found it above, it is 2914.88),
the S&P 500 spot price on Dec 15 as S_1 and let's denote is as $S_1 = S_0 * x$, x shows how much the price is going to change (the reasonable value could be somewhere between 0.5 and 2.0).
the value of the forward position on Dec 15 we denote as f_1.
The value of the forward position on Nov 15 (when we are entering the contract) is equal to zero.

It is known that $f_1 = (F_1-F_0) * d(Dec15, Dec20)$. Here F_0, F_1 and d(Dec15, Dec20) depend on the scenario.

<u>For scenario 1:</u> the yield curve is flat, the current risk-free rate is 260 basis points. The dividend yield is 1.8%. Then:
$F_0 = S_0 * \exp((0.0260 - 0.018)*(35/365))$, $F_1 = S_1 * \exp((0.0260 - 0.018)*(5/365))$, $d(Dec15, Dec20) = 1/\exp(0.0260*5/365)$, so
$f_1 = (F_1-F_0) * d(Dec\ 15, Dec\ 20) = S_0*(x*\exp((0.0260 - 0.018)*(5/365)) - \exp((0.0260 - 0.018)*(35/365))) / \exp(0.0260*5/365)$.
<u>For scenario 2:</u> the yield curve between Nov 15 and Dec 15 is flat, the risk-free rate is 300 basis points. The dividend yield is 1.8%.
$F_0 = S_0 * \exp((0.0300 - 0.018)*(35/365))$, $F_1 = S_1 * \exp((0.0300 - 0.018)*(5/365))$, $d(Dec15, Dec20) = 1/\exp(0.0300*5/365)$, so
$f_1 = (F_1-F_0) * d(Dec\ 15, Dec\ 20) = S_0*(x*\exp((0.0300 - 0.018)*(5/365)) - \exp((0.0300 - 0.018)*(35/365))) / \exp(0.0300*5/365)$.

For any x the value f_1 in scenario1 is strictly bigger than in scenario 2.
We sell ES Minis on Nov 15, now we want to close the position, so we want to buy ES Minis, and their value in scenario 1 is bigger than in scenario 2., so **<u>in scenario 1 it is more expansive to hedge than in scenario 2.</u>**