Question 2 (3 credits)

Indicate for each pair of expressions (A,B) in the table below, whether A is the O, Ω or Θ of B. Show your work to get full points. You can fill up the table with yes or no at the end of your final answer.

A	B	O	Θ	Ω
$\lg(n!)$	$\lg(n^n)$	yes	yes	yes
$n^{\lg(c)}$	$c^{\lg(n)}$	yes	yes	yes

Question 2

for Θ

$0 \le c_1 g(n) \, f(n) \le c_2 g(n)$

$0 \le \lg(n!) \le c_2 \lg(n^n)$

$\log_b(x^a) = a \log_b x$

$\lg(n!) \le c_2 \lg(n^n)$

$\lg(n!) \le c_2 \cdot n \lg n$

using stirlings approximation

$\lg(n!) = \Theta(n \lg n)$

$\therefore \lg(n!) \le c_2 \cdot n \lg n$

and $c_1 g(n) \le \lg(n!)$

$$n^{\lg(c)} \qquad c^{\lg(n)}$$

$$0 \le c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n)$$

$$n^{\lg(c)} \le c_2 \cdot c^{\lg(n)}$$

$$+\lg \qquad +\lg$$

$$\lg\left(n^{\lg(c)}\right) \le \lg\left(c_2 \cdot c^{\lg(n)}\right)$$

$$\underbrace{\lg(n)}_{\lg n} \cdot \lg(c) \le \underbrace{\lg(c_2 \cdot c) \cdot \lg(n)}_{\lg n}$$

$$\lg(c) \le \lg(c_2 \cdot c)$$

$$\lg(c) \le \lg(c_2) + \lg(c)$$

$$-\lg(c) \qquad -\lg(c)$$

$$0 \le \lg(c_2)$$

$$\lg(c_2) \ge 0$$

$$c_2 \ge 1$$

$$\lg(c_1 \cdot c^{\lg(n)}) \leq \lg(n^{\lg(c)})$$

$$\lg(c_1 \cdot c) \cdot \lg(n) \leq \lg(n) \cdot \lg(c)$$

$$\lg n \qquad \lg n$$

$$\lg(c_1 \cdot c) \leq \lg(c)$$

$$\lg(c_1) + \lg(c) \leq \lg(c)$$

$$\lg(c_1) \leq 0$$

$$c_1 \leq 1/2$$

$$0 \leq \frac{1}{2} \cdot g(n) \leq f(n) \leq 1 \cdot g(n)$$

We can also say

$$c^{\log_b a} = a^{\log_b c}$$

$$\therefore \quad n^{\lg(c)} = c^{\lg(n)}$$

$$n^{\lg(c)} = n^{\lg(c)}$$

Question 3

show that

$T(n) = 2T(\frac{n}{2}) + n$ is $\Omega(n \lg n)$

- prove using substitution

$$\begin{array}{llll}
\text{lvl } 0 & cn & \rightarrow cn & \frac{n}{2^0} \\
\text{lvl } 1 & c\frac{n}{2} \quad c\frac{n}{2} & \rightarrow cn & \frac{n}{2^1} \\
\text{lvl } 2 & c\frac{n}{4} \quad c\frac{n}{4} \quad c\frac{n}{4} \quad c\frac{n}{4} & \rightarrow cn & \frac{n}{2^2}
\end{array}$$

$\therefore$ level $i = \frac{n}{2^i}$

Lets assume last level is x

$$\frac{n}{2^x} = 1$$

$$n = 2^x$$

- log both sides

$$\lg(n) = \lg(2^x)$$

$$\lg(n) = x \lg 2$$

$\lg n = x$

Next we take the number of nodes which is

$2^{\lg n} = n$

Thus, $T(n) = \Theta(n \lg n)$

and $\Omega(n \lg n)$ as well

proving by substitution

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$= 2c\,\frac{n}{2}\lg\frac{n}{2} + n$$

$$= cn\lg\frac{n}{2} + n$$

Goal: $\leq cn\lg n$

$= O(n \lg n)$

$$= cn(\lg n - \lg 2) + n$$

$$= cn\lg n - cn\lg 2 + n$$

$= cn\lg n - cn\lg 2 + n$

$= cn\lg n + \boxed{n(1-c)}$

gives negative value when $c \geq 1$

$\therefore$ if $c \geq 1$ $\Theta(n\lg n)$ as well as $\Omega(n\lg n)$ as lower bound

Question 4

Can master method be applied to recurrence of the following

a. $T(n) = 4T\left(\frac{n}{2}\right) + n^2 \lg n$

$$n^{\log_b a} = n^{\log_4 2} = n^{\log_4 4^{1/2}} = n^{1/2}$$

$$f(n) = n^2 \lg n \cdot \Omega\left(n^{\log_b a + \epsilon}\right)$$

$$\epsilon = \frac{3}{2}$$

$$af\left(\frac{n}{b}\right) \leq c f(n)$$

$$a\left(\frac{n}{b}\right) \leq cn$$

$$2\left(\frac{n}{4}\right) \leq cn$$

$$\frac{1}{2} \leq c$$

$$\therefore T(n) = \Theta f(n)$$

$$= \Theta(n^2 \log n)$$

b. $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$

$n^{\log_b a} = n^{\log_4 2} = n^{\log_4 4^{1/2}} = n^{1/2}$

$f(n) = n^{1/2} = \Theta(n^{1/2} \log n)$

$\therefore \quad T(n) = \Theta(n^{1/2} \lg n)$

c. $T(n) = 2T\left(\frac{n}{4}\right) + 1$

$n^{\log_b a} = n^{\log_4 2} = n^{\log_4 4^{1/2}}$

$= n^{\frac{1}{2}\log_4 4} = n^{1/2}$

$f(n) = 1 = n^0 = O(n^{1/2 - \epsilon})$

$\epsilon = \frac{1}{2}$

$\therefore \quad T(n) = \Theta(n^{1/2})$

Question 5

use Strassen's algorithm to compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 11 & 12 \\ 21 & 22 \end{pmatrix}$$

$S_1 = B_{12} - B_{22} = 6$ $\quad S_6 = B_{11} + B_{22} = 8$

$S_2 = A_{11} + A_{12} = 4$ $\quad S_7 = A_{12} - A_{22} = -2$

$S_3 = A_{21} + A_{22} = 12$ $\quad S_8 = B_{21} + B_{22} = 6$

$S_4 = B_{21} - B_{11} = -2$ $\quad S_9 = A_{11} - A_{21} = -6$

$S_5 = A_{11} + A_{22} = 6$ $\quad S_{10} = B_{11} + B_{12} = 14$

$$P_1 = A_{11} \cdot S_1 = 6$$

$$P_2 = S_2 \cdot B_{22} = 8$$

$$P_3 = S_3 \cdot B_{11} = 72$$

$$P_4 = A_{22} \cdot S_4 = -10$$

$$P_5 = S_5 \cdot S_6 = 48$$

$$P_6 = S_7 \cdot S_8 = -12$$

$$P_7 = S_9 \cdot S_{10} = -84$$

$$C_{11} = P_5 + P_4 - P_2 + P_6 = 18$$

$$C_{12} = P_1 + P_2 = 14$$

$$C_{21} = P_3 + P_4 = 62$$

$$C_{22} = P_5 + P_1 - P_3 - P_7 = 66$$

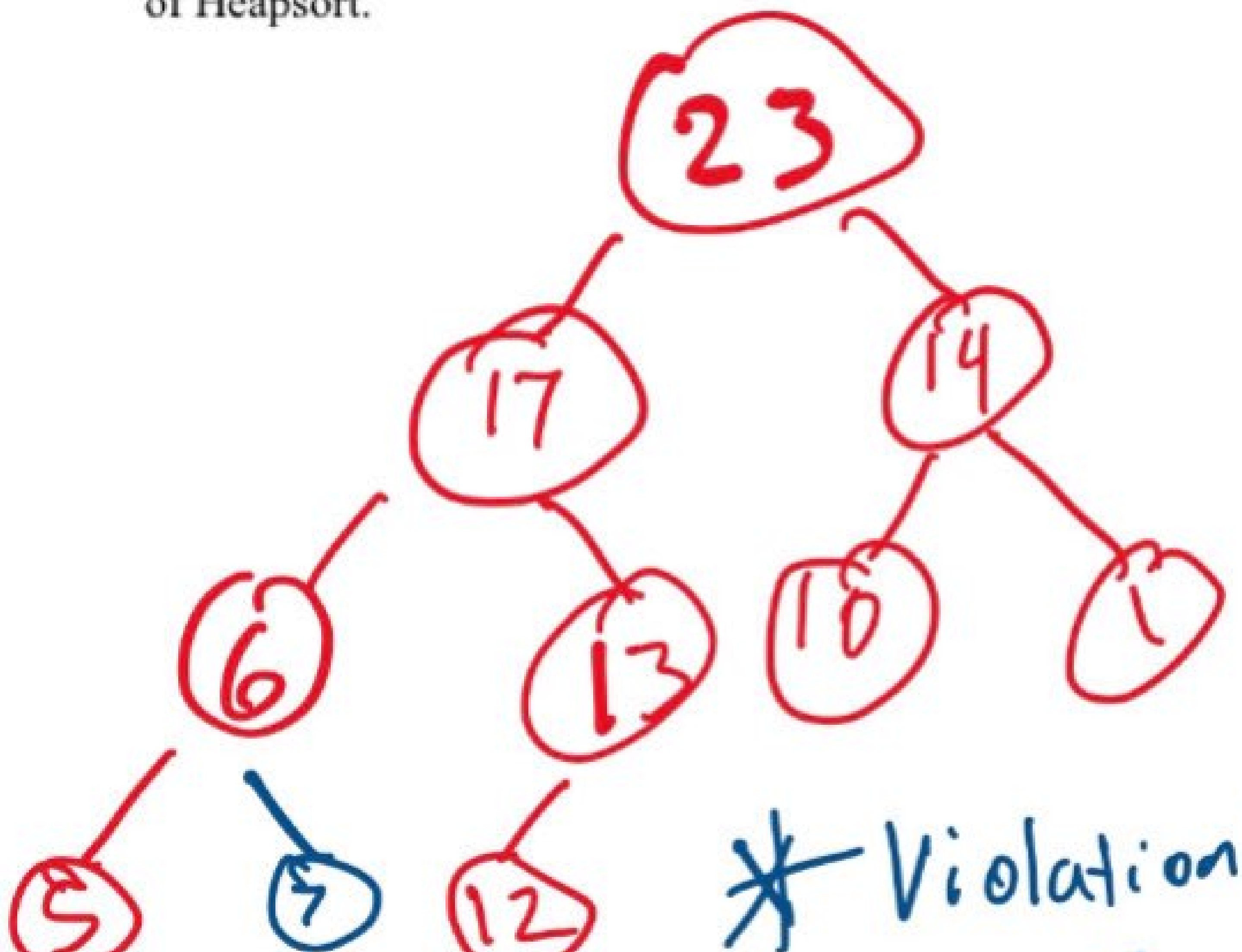
Question 6 (3 points)

a) Is the array with values ⟨23,17,14,6,13,10,1,5,7,12⟩ a max-heap? Draw the tree-like plot, and identify the nodes that violate the max-heap property.

b) Using the plots on Heapsort slide as a model, illustrate the final outcome of calling MAX-HEAPIFY(A, 3) on the input array: $A = \langle 27,17,3,16,13,10,1,5,7,12,4,8,9,0 \rangle$. (Just need to draw two heap plots, one before the calling and one after.)

c) Call HEAPSORT on the input array $A = \langle 5,13,2,25,7,17,20,8,4 \rangle$. First, draw the resulting max-heap of calling BUILD-MAX-HEAP on A (line 1); then illustrate the elements in A after the first 2 iterations of the **for** loop (line 2 to 5) respectively, as shown in lecture slides of Heapsort.

a)



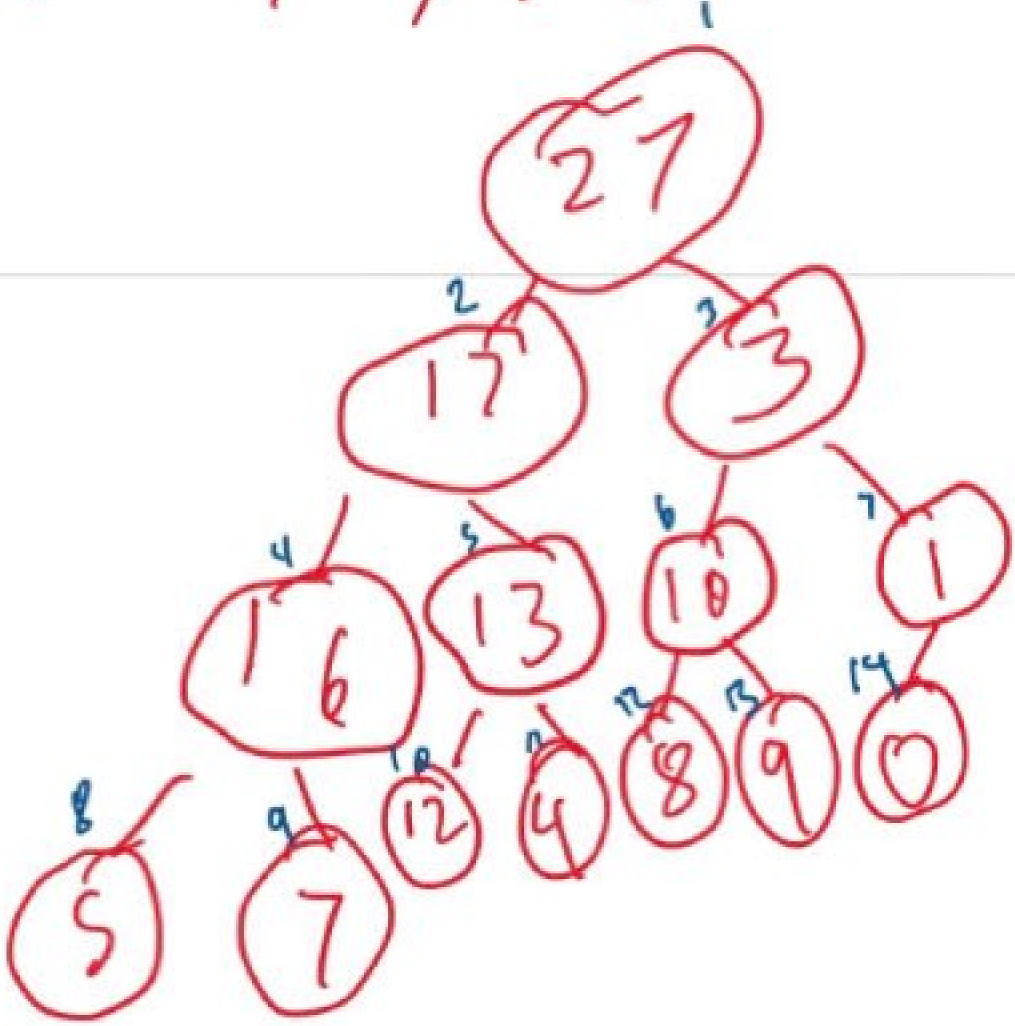
*Violation is 7 in blue

6 > 7 is false

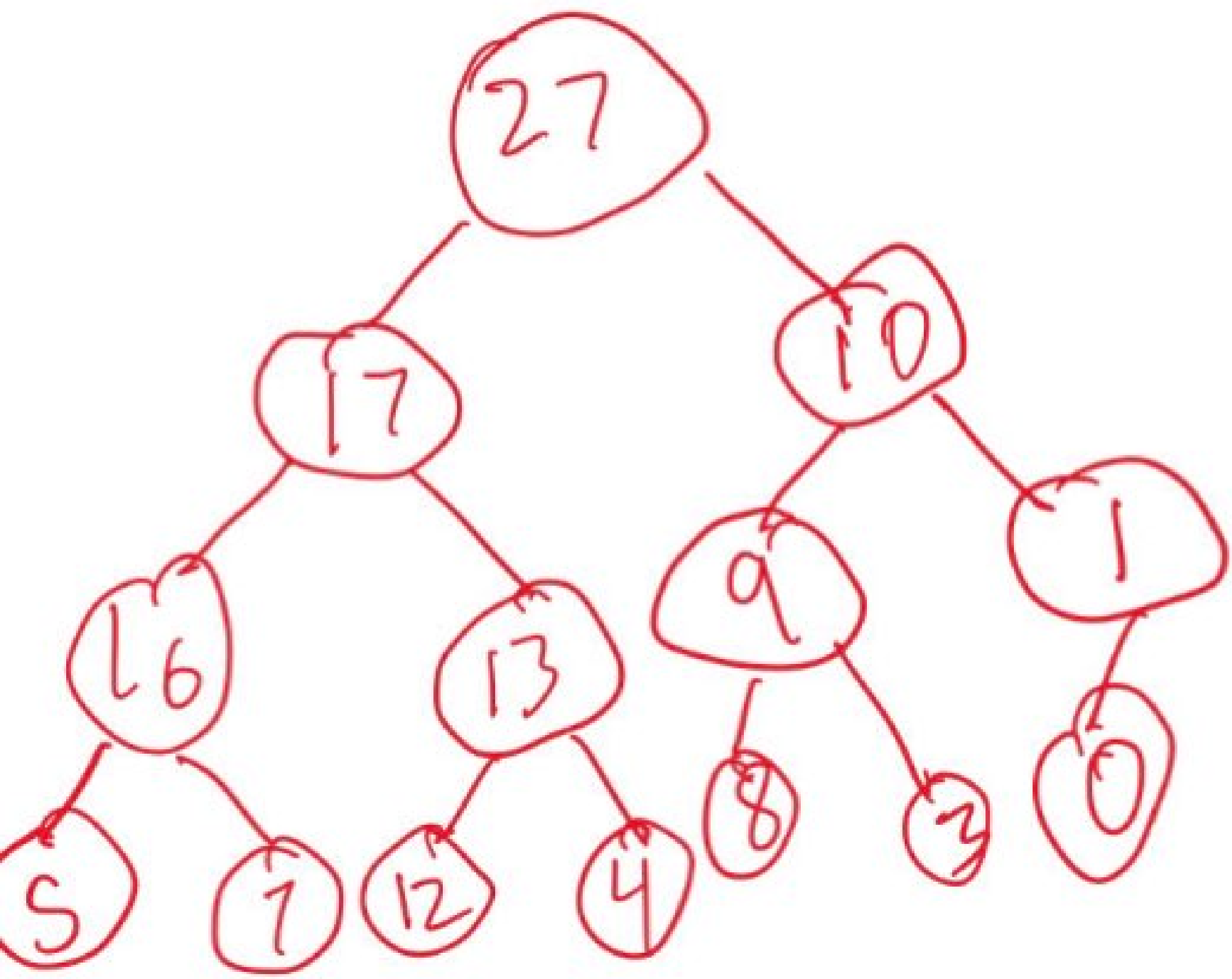
∴ Not a max heap

6)

A = <27, 17, 3, 16, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0>



Call method maxheapify(A, 3)

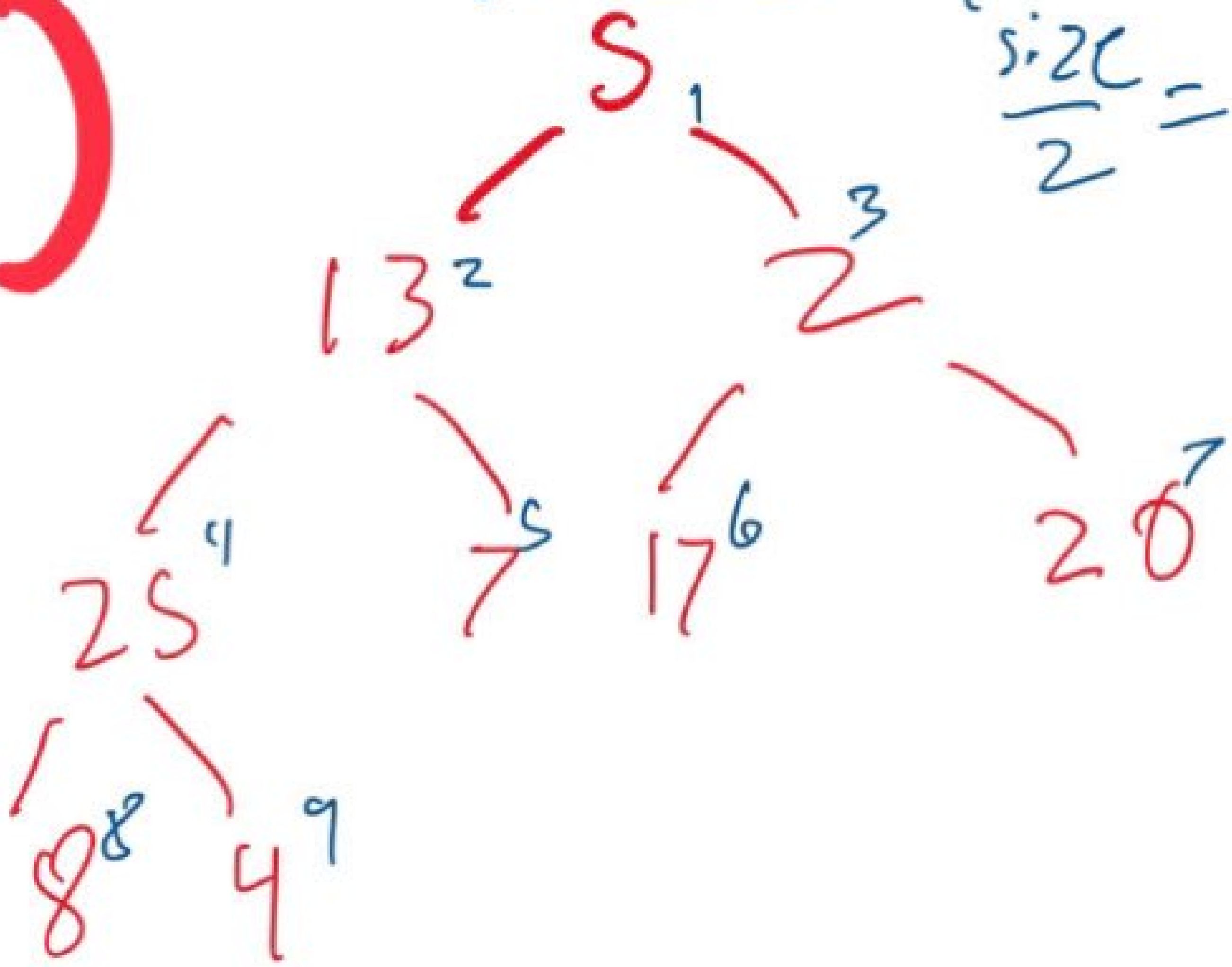


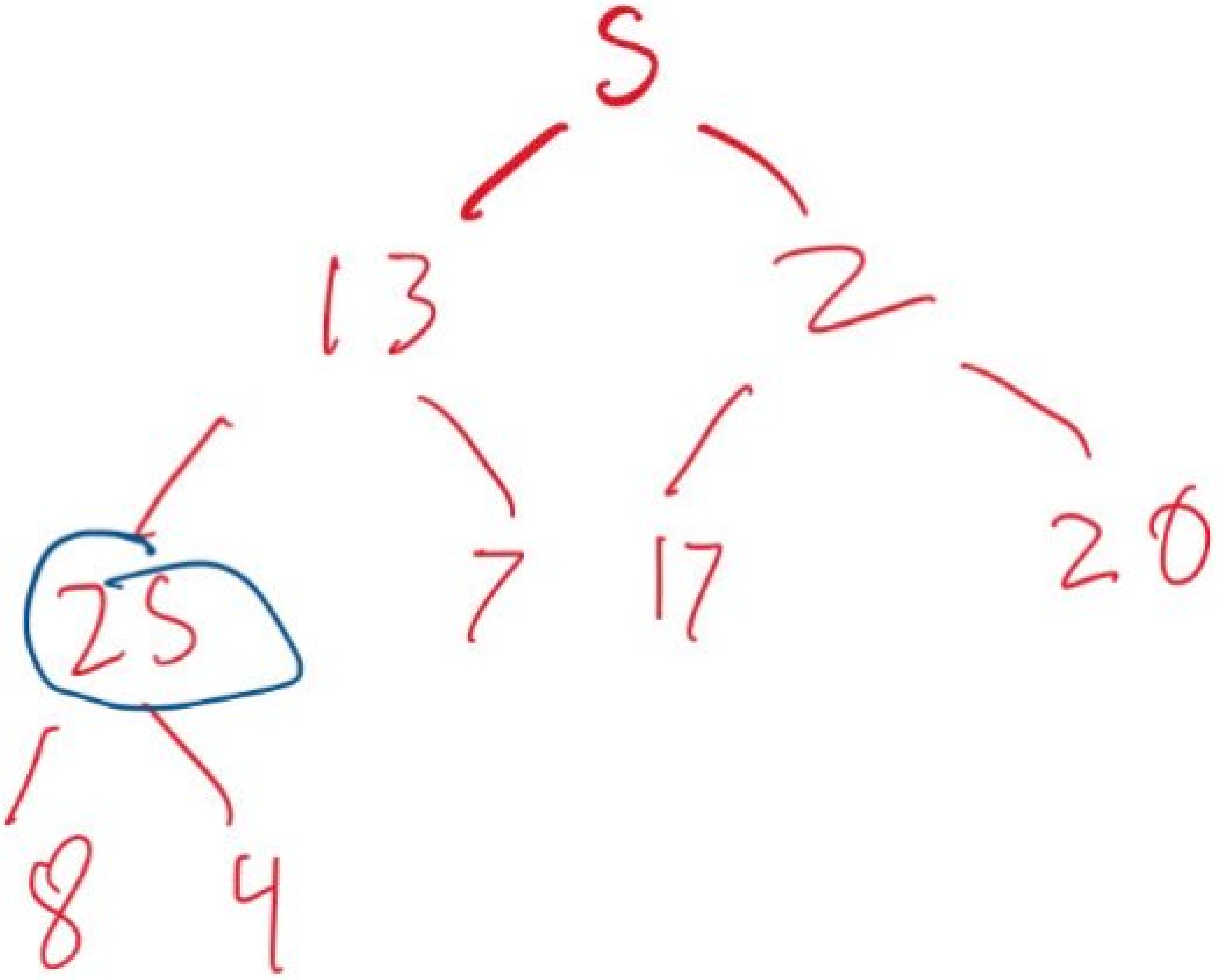
of Heapsort.

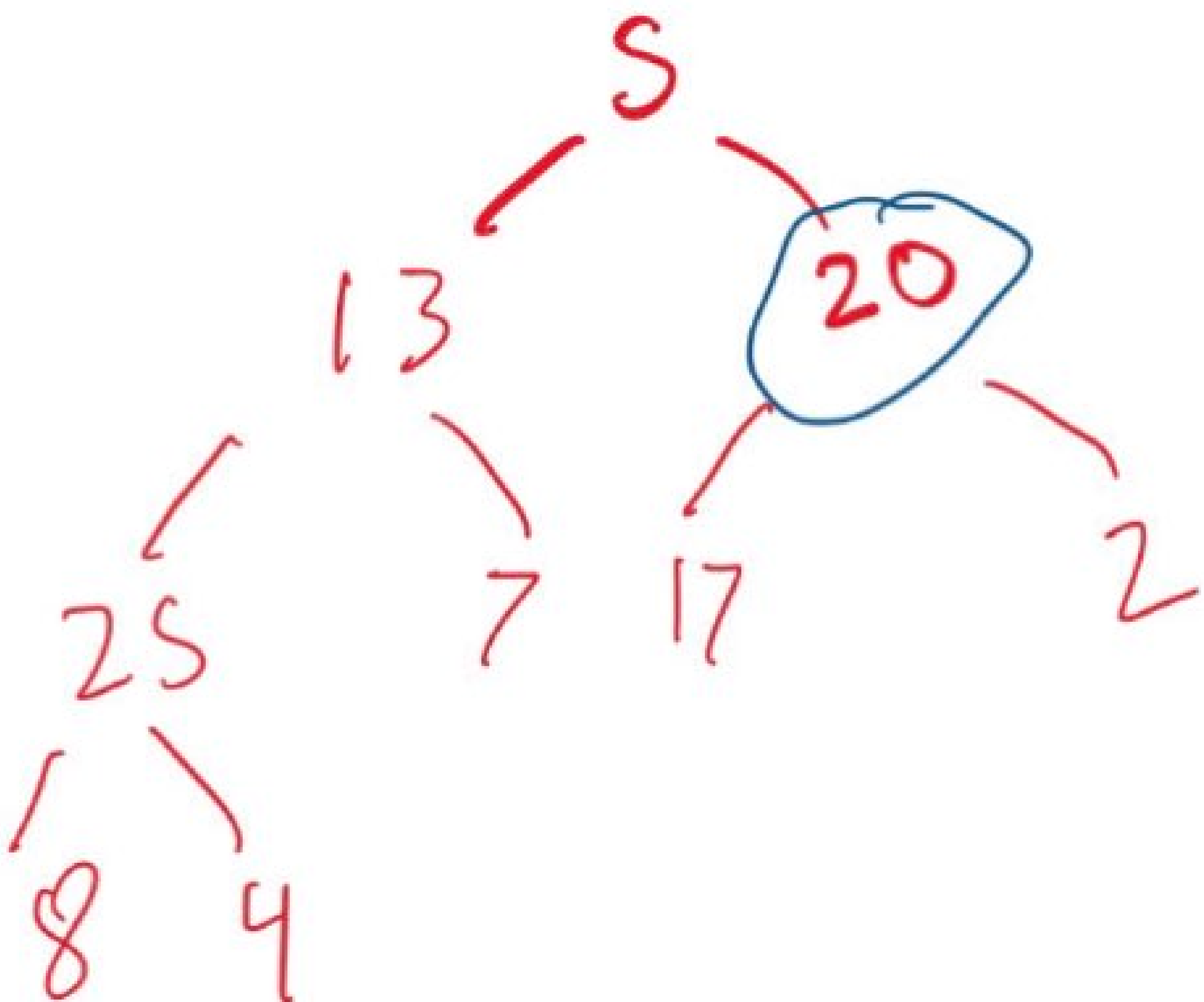
c)

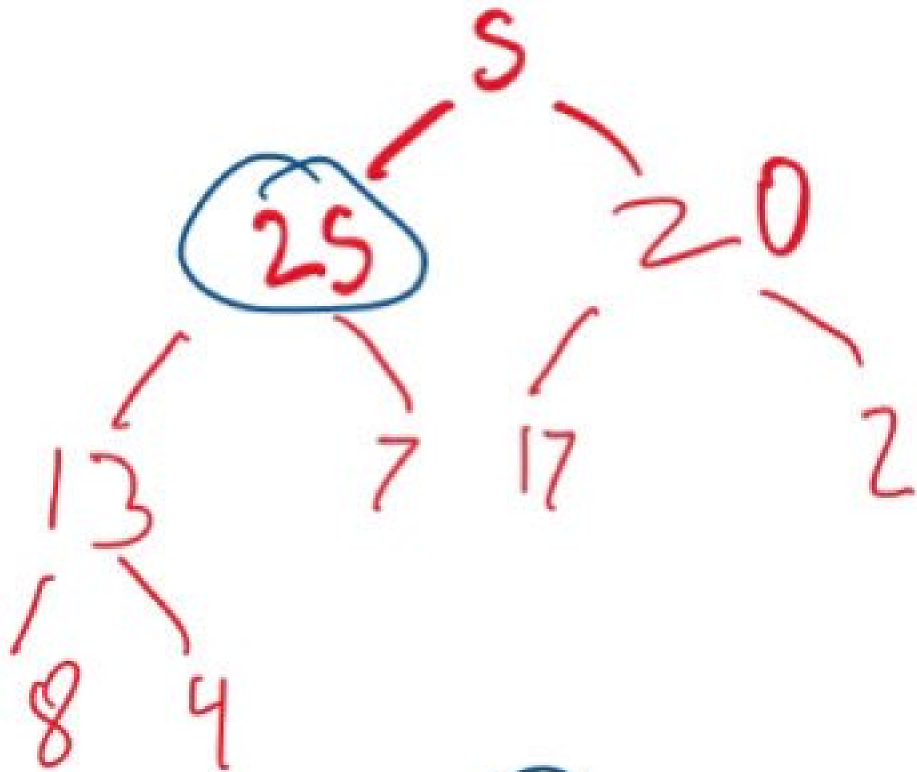
size = 9

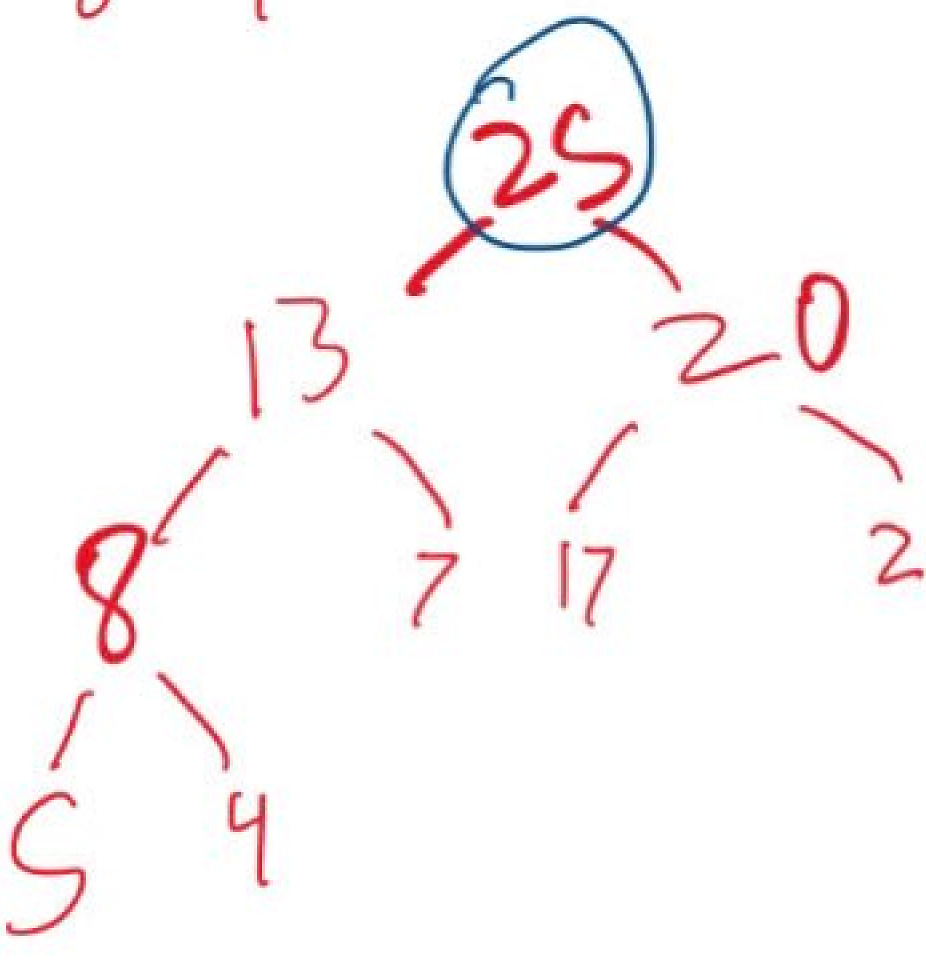
$\frac{size}{2} = 4$








5
25
20
13
7
17
2
8
4


25
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