Option pricing with fractional stochastic volatility and discontinuous payoff function of polynomial growth*

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Abstract. We consider the pricing problem related to payoffs that can have discontinuities of polynomial growth. The asset price dynamic is modeled within the Black and Scholes framework characterized by a stochastic volatility term driven by a fractional Ornstein-Uhlenbeck process. In order to solve the aforementioned problem, we consider three approaches. The first one consists in a suitable transformation of the initial value of the asset price, in order to eliminate possible discontinuities. Then we discretize both the Wiener process and the fractional Brownian motion and estimate the rate of convergence of the related discretized price to its real value, the latter one being impossible to be evaluated analytically. The second approach consists in considering the conditional expectation with respect to the entire trajectory of the fractional Brownian motion (fBm). Then we derive a closed formula which involves only integral functional depending on the fBm trajectory, to evaluate the price; finally we discretize the fBm and estimate the rate of convergence of the associated numerical scheme to the option price. In both cases the rate of convergence is the same and equals n^{-rH}, where n is a number of the points of discretization, H is the Hurst index of fBm, and r is the Hölder exponent of volatility function. The third method consists in calculating the density of the integral functional depending on the trajectory of the fBm via Malliavin calculus also providing the option price in terms of the associated probability density.

Key words. Option pricing, stochastic volatility, Black–Scholes model, Wiener process, fractional Brownian motion, discontinuous payoff function, polynomial growth, rate of convergence, discretization, conditioning, Malliavin calculus, stochastic derivative, Skorokhod integral

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1. Introduction. Starting with the pioneering works by Hull and White [14] and Heston [13], models of financial market whose asset prices include stochastic volatility have been the subject of an intensive research activity, which is still vibrant from analytical, computational and statistical points of view. Of course, option pricing is one of most relevant problems. In the latter context, stochastic volatility models are widely used because of their flexibility. Concerning the question how to model stochastic volatility, note that there are approaches in terms of Gaussian ([21], [25]), non-Gaussian ([3], [2]), jump-diffusion and Lèvy processes ([18], [10]), as well as time series ([6], [24], [26]). Our references are not in any way intended to be exhaustive or complete, we only illustrate the availability of different models. We would also like to mention the books [12], [15], [16] and references therein, as well as the paper [1]

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which in some sense was a starting point for our considerations. Furthermore, the models of financial market where the asset price includes stochastic volatility with long memory in the volatility process is a subject of extensive research acrivity, see, e.g., [4], where a wide class of fractionally integrated GARCH and EGARCH models for characterizing financial market volatility was studied, [9] for affine fractional stochastic volatility models and [8], where the Heston model with fractional Ornstein–Uhlenbeck stochastic volatility was studied. As was mentioned in [9], long memory included into the volatility model allows to explain some option pricing puzzles such as steep volatility smiles in long term options and co-movements between implied and realized volatility. Note also that as a rule, the option pricing in stochastic volatility models needs some approximation procedures including Monte–Carlo methods.

The present paper contains a comprehensive and diverse approach to the exact and approximate option pricing of the asset price model that is described by the linear model with stochastic volatility, and volatility is driven by fractional Ornstein–Uhlenbeck process with Hurst index $H > \frac{1}{2}$. For technical simplicity we assume that the Wiener process driving the asset price and the fractional Brownian motion driving stochastic volatility are independent. In these features, our model is similar to the model considered in [8]. However, the significant novelty of our approach is that we consider three possible levels of representation and approximation of option price, with the corresponding rate of convergence of discretized option price to the original one. Another novelty is that we can rigorously treat the class of discontinuous payoff functions of polynomial growth. As an example, our model allows to analyze linear combinations of digital and call options. Moreover, we provide, for the first time in literature to the best of our knowledge, rigorous estimates for the rates of convergence of option prices for polynomial discontinuous payoffs f and Hölder volatility coefficients, a crucial feature considering settings for which exact pricing is not possible.

The first level corresponds to the case when the price is presented as the functional of both driving stochastic processes, the Wiener process and the fBm, and we dicretize and simulate both the trajectories of the Wiener process and of the fBm (double discretization) and estimate the rate of convergence for the discretized model. In these settings we apply the elements of the Malliavin calculus, following [1], to transform the option price to the form that does not contain discontinuous functions. The second level corresponds to the case when we discretize and simulate only the trajectories of the fBm involved in Ornstein–Uhlenbeck stochastic volatility process (single discretization), basically conditioning on the stochastic volatility process, then calculating the corresponding option price as a functional of the trajectory of the fBm, and finally estimating the rate of convergence of the discretized price. This approach allows to simulate only the trajectories of the fBm. Corresponding simulations are presented and compared to those obtained by the first level. We conjecture that the single discretization gives better simulation results. The third level potentially permits to avoid simulations, because it is possible to provide an analytical expression for the option price, as an integral including the density of the functional which depends on stochastic volatility. Nevertheless, the density whose existence we can prove in the framework of Malliavin calculus, is rather complicated from the computational point of view, therefore this level is more of theoretical nature.

Taking into account previously mentioned approaches and techniques, our contribution is concerned with the treatment of a financial market, characterized by a finite maturity time T, and composed by a risk free bond, or bank account, $\beta = \{\beta_t, t \in [0,T]\}$, whose dynamic reads as $\beta_t = e^{\rho t}$, where $\rho \in \mathbb{R}^+$ represents the risk free interest rate, and a risky asset $S = \{S_t, t \in [0,T]\}$ whose stochastic price dynamic is defined, over the probability space $\{\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathrm{P}\}$, by the following system of stochastic differential equations

$$dS_t = bS_t dt + \sigma(Y_t)S_t dW_t, \tag{1}$$

$$dY_t = -\alpha Y_t dt + dB_t^H,\ t \in [0,T]. \tag{2}$$

Here $W = \{W_t, t \in [0,T]\}$ is a standard Wiener process, $b \in \mathbb{R}, \alpha \in \mathbb{R}^+$, are constants, while $Y = \{Y_t, t \in [0,T]\}$ characterizes the stochastic volatility term of our model, being the argument of the function σ. The process Y is Ornstein-Uhlenbeck, driven by a fractional Brownian motion $B^H = \{B_t^H, t \in [0,T]\}$, of Hurst parameter $H > \frac{1}{2}$, assumed to be independent of W. Let us recall that fBm is a centered Gaussian process with covariance function $\mathrm{E}B_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$. Moreover, due to the Kolmogorov theorem, any fBm has a modification such that its trajectories are almost surely Hölder function up to order H. In what follows we shall consider such modification. Moreover, it is well-known that for $H > \frac{1}{2}$, fBm has a long memory. This is suitable for stochastic volatility which represents the memory of the model. We would also like to remind that a market model as one described by the system of equations (1), (2), is incomplete because of two sources of uncertainty, wether or not it is arbitrage-free.

Therefore, in what follows we focus our attention on the so called physical, or *real world*, measure, instead of using an equivalent martingale one. Note, however, that in the case when the market is indeed arbitrage-free and there exists a minimal martingale measure, the stock prices evaluated w.r.t. the minimal martingale measure, resp. w.r.t. the objective measure, differ only to the non-random coefficient $e^{(b-\rho)t}$, as it happens in the standard Black–Scholes framework. For the discussion of conditions for the absence of arbitrage in the markets with stochastic volatility see, e.g., [17]. As concerns the payoff function, we consider a measurable one defined by $f : \mathbb{R}^+ \to \mathbb{R}^+$, and depending on the value S_T of the stock at maturity time T. It follows that our main goal is to calculate and approximate $\mathrm{E}f(S_T)$ with respect to the different aforementioned levels, also providing rigorous estimates for the corresponding rate of convergence for the first and second levels.

The paper is organized as follows: in section 2 we give additional assumptions on the components of the model and formulate auxiliary results; section 3 contains the necessary elements of the Malliavin calculus that will be used later; section 4 contains the main results on the rate of convergence of the discretized option pricing approach, when we simulate the trajectories of the Wiener process and of the fBm; section 5 contains the main results concerning the rate of convergence of the discretized option pricing problem when conditioning on the trajectories of the fBm, hence only simulating its trajectories; section 6 is devoted to the analytical derivation of the option price in terms of the density of the volatility functional, without trajectories simulations; the proofs are collected in section 7; finally, section 8 provides the computer simulations associated to the approaches described in section 4 and section 5.

2. Model of asset price and payoff function: additional assumptions, auxiliary properties. Throughout the paper we assume that payoff function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the

following conditions:
(**A**)

(i) f is a measurable function of polynomial growth,

$$f(x) \leq C_f(1 + x^p), \quad x \geq 0,$$

for some constants $C_f > 0$ and $p > 0$.
(ii) Function f is locally Riemann integrable, possibly, having discontinuities of the first kind.

Moreover we assume that the function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
(**B**) there exists $C_\sigma > 0$ such that

(i) σ is bounded away from 0, $\sigma(x) \geq \sigma_{\min} > 0$;
(ii) σ has moderate polynomial growth, i.e., there exists $q \in (0, 1)$ such that

$$\sigma(x) \leq C_\sigma(1 + |x|^q), \ x \in \mathbb{R};$$

(iii) σ is uniformly Hölder continuous, so that there exists $r \in (0, 1]$ such that

$$|\sigma(x) - \sigma(y)| \leq C_\sigma|x - y|^r, \ x, y \in \mathbb{R};$$

(iv) $\sigma \in C(\mathbb{R})$ is differentiable a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$, and its derivative is of polynomial growth: there exists $q' > 0$ such that

$$|\sigma'(x)| \leq C_\sigma(1 + |x|^{q'}),$$

a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$.

Remark 1. 1) Concerning the relations between properties (ii) and (iii), note that we allow $r = 1$ in (iii) whereas (ii) follows from (iii) only in the case $r < 1$.

2) Concerning the relations between properties (iii) and (iv), neither of these properties implies the other one unless $r = 1$. Indeed, on the one hand, a typical trajectory of a Wiener process is Hölder up to order $\frac{1}{2}$ but nowhere differentiable, on the other hand, even continuous differentiability does not imply the uniform Hölder property.

According to [22], fBm admits a compact interval representation via some Wiener process B, specifically,

$$B_t^H = \int_0^t k(t, s) dB_s, \quad k(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{1}{2}} (u - s)^{H - \frac{3}{2}} du \mathbb{1}_{s<t}, \tag{3}$$

with $c_H = (H - \frac{1}{2}) \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \right)^{1/2}$. Obviously, the processes B and W are independent. The next result is almost evident, however, we formulate it and even give a short proof for the reader's convenience.

Lemma 2. (i) *Equation* (2) *has a unique solution of the form*

$$Y_t = Y_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} dB_s^H. \tag{4}$$

Moreover, for any $\alpha > 0$ and any $\beta < 2$

$$\mathrm{E}\exp\{\alpha \sup_{t\in[0,T]} |Y_t|^{\beta}\} < \infty. \tag{5}$$

(ii) Equation (1) has a unique solution of the form

$$S_t = S_0 \exp\left\{bt + \int_0^t \sigma(Y_s)dW_s - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds\right\}. \tag{6}$$

Moreover, for any $m \in \mathbb{Z}$ we have $\mathrm{E}(S_T)^m < \infty$, and for any $m > 0$ it holds $\mathrm{E}(f(S_T))^m < \infty$.

Remark 3. We can generalize the last conclusion of Lemma 2 to the following one: for any function $\psi = \psi(x) : \mathbb{R} \to \mathbb{R}$ of polynomial growth $\sup_{t\in[0,T]} \mathrm{E}(|\psi(S_t)|) < \infty$.

3. Elements of Malliavin calculus and application to option pricing. In what follows, we recall some basic definitions and results about Malliavin calculus, doing that we mainly refer to [23]. Let $W = \{W(t), t \in [0,T]\}$ be a Wiener process on the standard probability space $\{\Omega, \mathcal{F}, \mathrm{F} = \{\mathcal{F}_t^W\}, t \in [0,T], \mathrm{P}\}$, where $\Omega = C([0,T], \mathbb{R})$. Denote by $\widehat{C}^{\infty}(R)$ the set of all infinitely differentiable functions with the derivatives of polynomial growth at infinity.

Definition 4. *Random variables ξ of the form $\xi = h(W(t_1), \dots, W(t_n))$,*

$$h = h(x^1, \dots, x^n) \in \widehat{C}^{\infty}(\mathbb{R}^n),\ t_1, \dots t_n \in [0,T],\ n \geq 1$$

are called smooth. Denote by $\mathcal{S}$ the class of smooth random variables.

Definition 5. *Let $\xi \in \mathcal{S}$. The stochastic derivative of ξ at t is the random variable*

$$D_t\xi = \sum_{i=1}^{n} \frac{\partial h}{\partial x^i}(W(t_1), \dots, W(t_n))\mathbb{1}_{t\in[0,t_i]}, \quad t \in [0,T].$$

Considered as an operator from $L^2(\Omega)$ to $L^2(\Omega; L^2[0,T])$, D is a closable operator. We use the same notation D for its closure. D is known as the Malliavin derivative, or the stochastic derivative. The domain of the operator of the stochastic derivative is a Hilbert space $\mathrm{D}^{1,2}$ of random variables, on which the inner product (which coincides with the operator norm) is given by

$$\langle \xi, \eta\rangle_{1,2} = \mathrm{E}(\xi\eta) + \mathrm{E}(\langle D\xi, D\eta\rangle_H), \quad H = L^2([0,T], \mathbb{R}).$$

Thus, the operator of stochastic derivative D is closed, unbounded and defined on a dense subset of the space $L^2(\Omega)$ (see [23]). The following statement is known as the chain rule.

Proposition 6. ([23, Proposition 1.2.3]). Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $\xi = (\xi_1, ..., \xi_m)$ is a random vector whose components belong to $\mathrm{D}^{1,2}$. Then $\varphi(\xi) \in \mathrm{D}^{1,2}$ and

$$D\varphi(\xi) = \sum_{i=1}^{m} \partial_i\varphi(\xi)D\xi_i.$$

Denote by δ the operator adjoint to D and by Dom δ its domain. The operator δ is unbounded in H with values in $L^2(\Omega)$ and such that

(i) Dom δ consists of square-integrable random processes $u \in H$, satisfying

$$|\mathrm{E}\left(\langle D\xi, u\rangle_H\right)| \leq C(\mathrm{E}(\xi^2))^{1/2},$$

for any $\xi \in \mathrm{D}^{1,2}$, where C is a constant depending on u;

(ii) If u belongs to Dom δ, then $\delta(u)$ is an element of $L^2(\Omega)$ and

$$\mathrm{E}\left(\xi\delta(u)\right) = \mathrm{E}\left(\langle D\xi, u\rangle_H\right)$$

for any $\xi \in \mathrm{D}^{1,2}$.

The operator δ is closed. Consider the space $L^{1,2} = L^2([0,T], \mathrm{D}^{1,2})$ with the norm $||\cdot||_{L^{1,2}}$, where

$$||u||^2_{L^{1,2}} = \mathrm{E}\left(\int_0^T u_t^2 dt + \int_0^T \int_0^T \left(D_s u_t\right)^2 dtds\right).$$

If $u \in L^{1,2}$, then the integral $\delta(u)$ is correctly defined and

$$\mathrm{E}\left(\int_0^T u_t dW_t\right)^2 \leq ||u||^2_{L^{1,2}}$$

(see [23]). In this case operator $\delta(u)$ is called the Skorokhod integral of the process u and is denoted by

$$\delta(u) = \int_0^T u_t dW_t.$$

To apply Malliavin calculus to the asset price S, note that we have a two-dimensional case with two independent Wiener processes (W, B). With evident modifications, denote by (D^W, D^B) the stochastic derivative with respect to the two-dimensional Wiener process (W, B). Denote also

$$X(t) = \log S(t) = \log S_0 + bt - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds + \int_0^t \sigma(Y_s)dW_s.$$

Lemma 7. (i) *The stochastic derivatives of the fBm B^H equal to*

$$D_u^W B_t^H = 0, \quad D_u^B B_t^H = k(t,u).$$

(ii) *The stochastic derivatives of Y equal to*

$$D_u^W Y_t = 0, \quad D_u^B Y_t = c_H e^{-\alpha t} u^{1/2-H} \int_u^t e^{\alpha s} s^{H-1/2}(s-u)^{H-3/2} ds \mathbb{1}_{u<t}. \tag{7}$$

(iii) The stochastic derivatives of X equal to

$$D_u^W X_t = \sigma(Y_u)\mathbb{1}_{u<t},\ D_u^B X_t = \left(-\int_0^t \sigma(Y_s)\sigma'(Y_s) D_u^B Y_s ds + \int_0^t \sigma'(Y_s) D_u^B Y_s dW_s \right) \mathbb{1}_{u<t}. \tag{8}$$

Lemma 8. *The laws of S_T and X_T are absolutely continuous with respect to the Lebesgue measure.*

From now on, we denote C any constant whose value is not important and can change from line to line and even inside the same line. Throughout the paper, C cannot depend on n, t, s, but can depend on $\sigma, H, T, Y_0, S_0, \alpha, b, p, r, q, q', f$ and other parameters specified in the problem. In what follows we need the statement contained in the next remark.

Remark 9. The chain rule of stochastic differentiation can be extended to the wider class of functions in the following way. Applying Proposition 1.2.4 from [23] and the related remark, we get that in the case when the function φ is Lipschitz and has a derivative a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$, and the law of r.v. ξ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then $\varphi(\xi)$ has a stochastic derivative and $D\varphi(\xi) = \varphi'(\xi) D\xi$ a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$. Now, consider the stochastic differentiation of the functions of Ornstein–Uhlenbeck process Y. Let φ be locally Lipschitz, with both and φ and φ' being of polynomial growth. Then $\varphi_n(x) = \varphi(x)\mathbb{1}_{|x|\le n} + (\varphi(-n))\mathbb{1}_{x<-n} + \varphi(n)\mathbb{1}_{x>n}$ is Lipschitz, has a derivative φ_n' a.e. w.r.t. the Lebesgue measure, and moreover, there exists a polynomial $\bar\varphi$ with non-negative coefficients such that

$$|\varphi_n(x)| + |\varphi_n'(x)| \le \bar\varphi(|x|),\ x \in \mathbb{R},$$

which implies that

$$\mathrm{E}|\varphi(Y_s) - \varphi_n(Y_s)|^2 \le 4\mathrm{E}\left(\bar\varphi^2(|Y_s|)\mathbb{1}_{Y_s \notin [-n,n]}\right) \to 0.$$

Furthermore, it easily follows from (7) that in fact $D^B Y_s$ is in $L^2([0,T])$. Indeed,

$$0 \le D_u^B Y_s \le C u^{1/2-H}(s-u)^{H-1/2}\mathbb{1}_{u<s}. \tag{9}$$

Furthermore,

$$\mathrm{E}\left(\max_{s\in[0,T]} \bar\varphi^2(|Y_s|)\right) < \infty,$$

due to the fact that $\max\limits_{s\in[0,T]} |Y_s|$ has exponential moments. Therefore,

$$\mathrm{E}\left(\int_0^T (\varphi'(Y_s) D_u^B Y_s - D_u^B \varphi_n(Y_s))^2 du\right)$$
$$\le 4\mathrm{E}\left(\max_{s\in[0,T]} \bar\varphi^2(|Y_s|)\mathbb{1}_{\max\limits_{s\in[0,T]} Y_s \notin [-n,n]} \int_0^s (D_u^B Y_s)^2 du\right) \to 0.$$

Previous results, together with the fact that D is closed, imply that $D_u^B \varphi(Y_s) = \varphi'(Y_s) D_u^B Y_s$.

Let us introduce the following notations: $g(y) = f(e^y)$, $F(x) = \int\limits_0^x f(z)dz$ and let $G(y) = \int\limits_0^y g(z)dz$, $x \geq 0$, $y \in \mathbb{R}$. Also, let

$$Z_T = \int\limits_0^T \sigma^{-1}(Y_u)dW_u. \tag{10}$$

Note that Z_T is well defined because of condition (**B**), (i). Now, analogously to [1], we are in position to transform the option price in such a way that it does not contain discontinuous functions.

Lemma 10. *Under conditions* (**A**) *and* (**B**) *the option price* $\mathrm{E}f(S_T) = \mathrm{E}g(X_T)$ *can be represented as*

$$\mathrm{E}f(S_T) = \mathrm{E}\left(\frac{F(S_T)}{S_T}\left(1 + \frac{Z_T}{T}\right)\right). \tag{11}$$

Alternatively,

$$\mathrm{E}g(X_T) = \frac{1}{T}\mathrm{E}\left(G(X_T)Z_T\right). \tag{12}$$

4. The rate of convergence of approximate option prices in the case when both Wiener process and fractional Brownian motions are discretized. In the present section we provide our first approach (first level) to the numerical approximation of the solution for the option pricing problem. In particular, we are going to provide a double discretization procedure, with related simulations, with respect to both the Wiener process and the fBm, also estimating the rate of convergence for the corresponding approximated option prices to the real value given by $\mathrm{E}f(S_T)$.

To pursue latter aim, let us introduce the following notation. For any $n \in \mathbb{N}$ consider equidistant partition of the interval $[0,T]$: $t_i = t_i(n) = \frac{iT}{n}$, $i = 0, 1, 2, ..., n$. Then we define the discretizations of Wiener process W and fractional Brownian motion B^H:

$$\Delta W_i = W\left(t_{i+1}\right) - W\left(t_i\right),$$

$$\Delta B_i^H = B^H\left(t_{i+1}\right) - B^H\left(t_i\right), i = 0, 1, 2, ..., n.$$

Discretized processes Y and X, corresponding to a given partition have the form

$$Y_{t_j}^n = Y_0 e^{-\alpha t_j} + e^{-\alpha t_{j-1}} \sum_{i=0}^{j-1} e^{\alpha t_i} \Delta B_i^H,$$

$$X_{t_j}^n = X_0 + bt_j - \frac{1}{2n}\sum_{i=0}^{j-1}\sigma^2\left(Y_{t_i}^n\right) + \sum_{i=0}^{j-1}\sigma\left(Y_{t_i}^n\right)\Delta W_i$$

$$= X_0 + bt_j - \frac{1}{2}\int_0^{t_j} \sigma^2(Y_s^n)ds + \int_0^{t_j} \sigma(Y_s^n)dW_s, \quad j = 0, ..., n,$$

where we put $Y_s^n = Y_{t_i}^n$ for $s \in [t_i, t_{i+1})$. Concerning the discretizaion of the term Z_T from (10), it has a form $Z_T^n = \int_0^T \frac{1}{\sigma(Y_s^n)} dW_s$. Eventually we define $S_{t_j}^n = \exp\{X_{t_j}^n\}$. Three lemmas below contain all auxiliary bounds that are necessary in order to establish the main result.

Lemma 11. (i) *For any $\theta > 0$ there exists a constant C depending on θ such that for any $s, t \in [0, T]$*

$$\mathrm{E}|Y_t - Y_s|^\theta \le C\,|t-s|^{\theta H}.$$

(ii) *For any $\theta > 0$ there exists a constant C depending on θ such that for any $0 \le j \le n$*

$$\mathrm{E}\left|Y_{t_j} - Y_{t_j}^n\right|^\theta \le Cn^{-\theta}.$$

(iii) *For any $\theta > 0$ there exists a constant C depending on θ such that for any $s \in [0, T]$*

$$\mathrm{E}|Y_s - Y_s^n|^\theta = \mathrm{E}|Y_s - Y_{t_i}^n|^\theta \le Cn^{-\theta H}.$$

(iv) *Approximating process has uniformly bounded moments: for any $\theta > 0$*

$$\sup_{s\in[0,T]} \mathrm{E}|Y_s^n|^\theta < \infty. \tag{13}$$

Remark 12. Using (13) and the fact that the approximating process Y^n is Gaussian, we can prove similarly to Lemma 2 and Remark 3 that for any $m \in \mathbb{Z}$

$$\sup_{n\ge1}\sup_{0\le j\le n} \mathrm{E}\left(S_{t_j}^n\right)^m < \infty.$$

Lemma 13. *There exists a constant $C > 0$ such that for any $n \ge 1$*

$$\mathrm{E}(X_T - X_T^n)^2 \le Cn^{-2rH}, \tag{14}$$

and

$$\mathrm{E}(Z_T - Z_T^n)^2 \le Cn^{-2rH}. \tag{15}$$

Lemma 14. *Under conditions* $(\mathbf{A})$ *and* $(\mathbf{B})$ *we have the following upper bound: there exists a constant C_F such that*

$$\mathrm{E}\left|\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right|^2 \le C_F \cdot n^{-2rH}.$$

Using previous lemmas, we are now in position to state the main result of this section, namely to provide the rate of convergence of discretized option prices to the exact one represented by $\mathrm{E}f(S_T)$, under double discretization.

Theorem 15. *Let conditions* $(\mathbf{A})$ *and* $(\mathbf{B})$ *hold. There exists a constant C not depending on n such that*

$$\left|\mathrm{E}f(S_T)-\mathrm{E}\left(\frac{F(S_T^n)}{S_T^n}\left(1+\frac{Z_T^n}{T}\right)\right)\right|\le Cn^{-rH}.$$

5. The rate of convergence of approximate option prices in the case when only fractional Brownian motion is discretized. The present section is devoted to the implementation of the second approach (second level) to approximate the option price. It is based on the fact that in the case when W and B are independent, logarithm of asset price is conditionally Gaussian under the fixed trajectory of fractional Brownian motion. It allows to exclude Wiener process W from the consideration and to calculate the option price explicitly in terms of the trajectory of fBm B^H. Respectively, we can discretize and simulate only the trajectories of B^H (single discretization). Theorem 17 gives the explicit option pricing formula as the functional of the trajectory of fBm B^H, and Theorem 18 gives the rate of convergence. Comparing to Theorem 15, we see that the rate of convergence admits the same bound, influenced by the behavior of volatility.

Let us introduce the following notations: let the covariance matrix reads as follows

$$C_{X,Z}=\begin{pmatrix}\sigma_Y^2 & T\\ T & \sigma_Z^2\end{pmatrix},$$

and let

$$\sigma_Y^2=\int_0^T\sigma^2(Y_s)ds,\, m_Y=X_0+bT-\frac{1}{2}\sigma_Y^2,\, \sigma_Z^2=\int_0^T\sigma^{-2}(Y_s)ds,\, \Delta=|C_{X,Z}|=\sigma_Y^2\sigma_Z^2-T^2.$$

Evidently, $\Delta\ge 0$. We assume additionally that the following assumption is fulfilled.

$(\mathbf{C})$ $\Delta=\sigma_Y^2\sigma_Z^2-T^2>0$ with probability 1.

Note that the random vector

$$(X_T,Z_T)=\left(X_0+bT-\frac{1}{2}\int_0^T\sigma^2(Y_s)ds+\int_0^t\sigma(Y_s)dW_s,\ \int_0^T\sigma^{-1}(Y_s)dW_s\right)$$

is Gaussian conditionally to the given trajectory $\{Y_t,t\in[0,T]\}$. The conditional covariance matrix is $C_{X,Z}$. Next lemma presents common conditional density of (X_T,Z_T). Note that under assumption $(\mathbf{C})$ the distribution of (X_T,Z_T) is non-degenerate in $\mathbb{R}^2$.

Lemma 16. *Let assumption* (**C**) *hold. Then the common conditional density* $p_{X,Z}(x,z)$ *of* (X_T, Z_T)*, conditionally to the given trajectory* $\{Y_t, t \in [0,T]\}$*, equals*

$$p_{X,Z}(x,z) = \frac{1}{2\pi\Delta^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2\Delta}\left(\sigma_Z^2(x-m_Y)^2 + \sigma_Y^2 z^2 - 2T(x-m_Y)z\right)\right\}. \tag{16}$$

The next result states that option price can be presented as the functional of σ_Y^2 only.

Theorem 17. *Under conditions* (**A**)–(**C**) *the following equality holds:*

$$\begin{aligned} \mathrm{E}g(X_T) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} G(x)\mathrm{E}\left(\frac{(x-m_Y)}{\sigma_Y^3}\exp\left\{-\frac{(x-m_Y)^2}{2\sigma_Y^2}\right\}\right)dx \\ &= (2\pi)^{-\frac{1}{2}}\mathrm{E}\left((\sigma_Y)^{-1}\int_{\mathbb{R}} G((x+m_Y)\sigma_Y)xe^{-\frac{x^2}{2}}dx\right). \end{aligned} \tag{17}$$

In order to state the main result of the present section, et us define the following quantities

$$\sigma_{Y,n} = \int_0^T \sigma^2(Y_s^n)ds, \quad m_{Y,n} = X_0 + bT - \frac{1}{2}\sigma_{Y,n}^2.$$

Theorem 18. *Under conditions* (**A**), (**B**), *and* (**C**) *we have*

$$\left|\mathrm{E}g(X_T) - (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} G(x)\mathrm{E}\left(\frac{(x-m_{Y,n})}{\sigma_{Y,n}^3}\exp\left\{-\frac{(x-m_{Y,n})^2}{2\sigma_{Y,n}^2}\right\}\right)dx\right| \le Cn^{-rH}.$$

6. Option price in terms of density of the integrated stochastic volatility. Applying Theorem 17 and equality (17), we clearly see that the option price depends on the random variable $\sigma_Y^2 = \int_0^T \sigma^2(Y_s)ds$. Therefore it is natural to derive the density of this random variable. Since σ_Y^2 depends on the whole trajectory of the fBm B^H on $[0,T]$, we apply Malliavin calculus in an attempt to find the density. First, establish some auxiliary results. For any $\varepsilon > 0$ and $\delta > 0$ introduce the stopping times $\tau_\varepsilon = \inf\{t > 0 : |B_t^H| \ge \varepsilon\}$ and $\nu_\delta = \inf\{t > 0 : |Y_t - Y_0| \ge \delta\}$.

Lemma 19. *For any* $l > 0$ *negative moments are well defined:* $\mathrm{E}(\nu_\delta)^{-l} < \infty$.

Now we introduce additional assumptions on the function σ.

(**D**) The function $\sigma \in C^{(2)}(\mathbb{R})$, its derivative σ' is strictly nonnegative, $\sigma'(x) > 0,\ x \in \mathbb{R}$, and σ', σ'' are of polynomial growth.

Lemma 20. *Under assumptions* (**B**) *and* (**D**) *the stochastic process*

$$\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2} = \left\{\frac{D_t^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2},\ t\in[0,T]\right\}$$

belongs to the domain Dom δ of the Skorokhod integral δ.

Denote $\eta = (||D^B\sigma_Y^2||_H^2)^{-1}$, $l(u,s) = c_H e^{-\alpha s}\int\limits_u^s e^{\alpha v} v^{H-1/2}(v-u)^{H-3/2}dv$, $\kappa(y) = \sigma(y)\sigma'(y)$.

Theorem 21. *(i) The density $p_{\sigma_Y^2}$ of the random variable σ_Y^2 is bounded, continuous and given by the following formulas*

$$p_{\sigma_Y^2}(u) = \mathrm{E}\left[\mathbb{1}_{\sigma_Y^2 > u}\,\delta\left(\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2}\right)\right], \tag{18}$$

where the Skorokhod integral is in fact reduced to a Wiener integral,

$$\delta\left(\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2}\right) = 2\eta\int_0^T \kappa(Y_s)\left(\int_0^s u^{1/2-H} l(u,s)dB_u\right)ds - \int_0^T D_u^B\eta D_u^B(\sigma_Y^2)du.$$

(ii) The option price $\mathrm{E}g(X_T)$ can be represented as the integral with respect to the density $p_{\sigma_Y^2}(u)$ defined by (18) as follows:

$$\mathrm{E}g(X_T)$$
$$= (2\pi)^{-\frac{1}{2}}T\int\limits_{\mathbb{R}} G(x)\int_{\mathbb{R}}\frac{(x+u/2-X_0-bT)}{u^3}\exp\left\{-\frac{(x+u/2-X_0-bT)^2}{2u^2}\right\}p_{\sigma_Y^2}(u)du.$$

7. Proofs. Proof of Lemma 2. (i) The representation (4) for the fractional Ornstein-Uhlenbeck process Y is well known, see, e.g., [7]. It is a continuous Gaussian process with $\sup_{t\in[0,T]}\mathrm{E}(Y_t)^2 < \infty$. Then the boundedness of any exponential moments of the form (5) follows from [11] and [27].

(ii) To establish the representation (6) for S, we need only to prove that the integrals $\int_0^t \sigma(Y_s)dW_s$ and $\int_0^t \sigma(Y_s)S_s dW_s$ are well defined, while the form of the representation is obvious. To what concerns $\int_0^t \sigma(Y_s)dW_s$, it follows from (5) and condition (**B**), (i) that $\int_0^t \mathrm{E}\sigma^2(Y_s)ds \le C\int_0^t \mathrm{E}(1+|Y_s|^{2q})ds < \infty$. Moreover, obviously, the moments of any order are finite: $\sup_{t\in[0,T]}\mathrm{E}\sigma^{2n}(Y_t) < \infty$. Furthermore, the conditional distribution of the integral $\int_0^t \sigma(Y_s)dW_s$ given the whole trajectory $Y = \{Y_t, t\in[0,T]\}$ is Gaussian. Since we have $q<1$ together with exponential inequality (5), then for any $n\in\mathbb{Z}$, the following holds

$$\begin{aligned}&\sup_{t\in[0,T]}\mathrm{E}S_t^n \le C\sup_{t\in[0,T]}\mathrm{E}\exp\left\{n\int_0^t\sigma(Y_s)dW_s\right\} = C\sup_{t\in[0,T]}\mathrm{E}\exp\left\{\frac{n^2}{2}\int_0^t\sigma^2(Y_s)ds\right\}\\ &\le C\sup_{t\in[0,T]}\mathrm{E}\exp\left\{C_\sigma\frac{n^2}{2}\int_0^t(1+|Y_s|^{2q})ds\right\} \le C\mathrm{E}\exp\left\{\frac{C_\sigma T n^2}{2}\sup_{s\in[0,T]}|Y_s|^{2q}\right\} < \infty.\end{aligned} \tag{19}$$

In particular,

$$\int_0^T \mathrm{E}(\sigma^2(Y_s)S_s^2)ds \le T\left(\sup_{t\in[0,T]}\mathrm{E}S_t^4\sup_{t\in[0,T]}\mathrm{E}\sigma^4(Y_t)\right)^{\frac{1}{2}} < \infty$$

whence the proof follows.

Proof of Lemma 7. Statement (i) follows directly from the definition of stochastic derivative and from the fact that B and W are independent. Similarly, the first equality in (7) is obvious since Y is independent of W. Furthermore, integrating by parts (4) and taking into account representation (3), we get following equalities

$$Y_t = Y_0 e^{-\alpha t} + B_t^H - \alpha e^{-\alpha t} \int_0^t e^{\alpha s} B_s^H ds, \tag{20}$$

whence

$$D_u^B Y_t = k(t,u) - \alpha e^{-\alpha t} \int_u^t e^{\alpha s} k(s,u) ds,$$

where the kernel k was introduced in (3). Note that the derivative k'_s of the kernel k equals

$$k'_s(s,u) = c_H u^{\frac{1}{2}-H} s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} \mathbb{1}_{u<s}.$$

It is an integrable function, therefore we can integrate by parts once again and get that

$$D_u^B Y_t = e^{-\alpha t} \int_u^t e^{\alpha s} k'_s(s,u) ds = c_H e^{-\alpha t} u^{\frac{1}{2}-H} \int\limits_u^t e^{\alpha s} s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} ds \mathbb{1}_{u<t}.$$

The first equation from (8) follows from the definition of stochastic derivative. Further,

$$D_u^B X_t = \left(-\frac{1}{2} \int\limits_0^t D_u^B(\sigma^2(Y_s)) ds + \int\limits_0^t D_u^B(\sigma(Y_s)) dW_s \right) \mathbb{1}_{u<t}.$$

Note that Remark 9, together with Gaussian distribution of Y_s, allows to apply the chain rule to the continuous and a.e. differentiable function σ taken at point Y_s. Moreover, the result can be written in the standard form, so that $D_u^B(\sigma(Y_s)) = \sigma'(Y_s) D_u^B(Y_s)$, and

$$D_u^B(\sigma^2(Y_s)) = 2\sigma(Y_s)\sigma'(Y_s) D_u^B(Y_s) \tag{21}$$

a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$. Besides, similarly to proof of Lemma 2, we can apply properties (**B**), (ii) and (iv), which, together with the upper bound (9) ensure that the integrals in (8) exist, whence the proof follows.

Proof of Lemma 8. Conditionally on the trajectory $Y = \{Y_t, t \in [0,T]\}$, X_T is a Gaussian random variable. Therefore, for any Borel set $A \subset \mathbb{R}$ of zero Lebesgue measure, we have

$$\mathrm{P}\{X_T \in A\} = \mathrm{E}(\mathbb{1}_{X_T \in A}) = \mathrm{E}\left(\mathrm{E}\left(\mathbb{1}_{X_T \in A} | \{Y_t, t \in [0,T]\}\right)\right) = 0.$$

The absolute continuity of the law of S_T follows from that of X_T since $S_T = \exp\{X_T\}$.

Proof of Lemma 10. Let the function H be locally Lipschitz and $H'(x) = h(x)$ a.e. with respect to the Lebesgue measure. Assume additionally that h is of exponential growth. Hence it follows from Remark 9 that

$$D_u^W H(X_T) = h(X_T) D_u^W X_T.$$

Establish now that $H(X_T) \in \mathrm{D}^{1,2}$, where we consider stochastic differentiation w.r.t. W, i.e., $\mathrm{D}^{1,2} = \mathrm{D}^{W,1,2}$. Indeed, h is of exponential growth,

$$h(x) \leq C_h(1 + e^{p_h|x|}),$$

and

$$H(x) = \int_0^x h(y)dy \leq C_h|x|(1 + e^{p_h|x|}) \leq C_h(1 + e^{(p_h+1)|x|}).$$

Furthermore,

$$e^{(p_h+1)|X_T|} = (S_T)^{p_h+1} \vee (S_T)^{-p_h-1},$$

we get from (19) that $\mathrm{E}H^2(X_T) < \infty$. Additionally,

$$\mathrm{E}\int_0^T \left(h(X_T) D_u^W X_T\right)^2 du = \mathrm{E}\left(h^2(X_T)\int_0^T \sigma^2(Y_u)du\right) =$$

$$\leq C\left(\mathrm{E}h^4(X_T)\int_0^T \mathrm{E}\sigma^4(Y_u)du\right)^{1/2} < \infty.$$

Therefore

$$H(X_T) \in \mathrm{D}^{1,2}. \tag{22}$$

Having established both the existence and the form of the stochastic derivative, together with (22), we can proceed as in the proof of Proposition 4.1 [1]. Namely, the Skorokhod integral is the adjoint operator to the Malliavin derivative, therefore

$$\begin{aligned}\mathrm{E}h(X_T) &= \frac{1}{T}\mathrm{E}\left(\int_0^T h(X_T) D_u^W X_T \frac{1}{D_u^W X_T} du\right) = \frac{1}{T}\mathrm{E}\left(\int_0^T D_u^W H(X_T)\frac{1}{\sigma(Y_u)}du\right) \\ &= \frac{1}{T}\mathrm{E}\left(H(X_T)\int_0^T \frac{1}{\sigma(Y_u)}dW_u\right) = \frac{1}{T}\mathrm{E}\left(H(X_T)Z(T)\right).\end{aligned} \tag{23}$$

In particular, the function G is locally Lipschitz and $G'(x) = g(x)$ a.e. with respect to the Lebesgue measure. Moreover, g is of exponential growth, namely,

$$g(x) \leq C_f(1 + e^{p|x|}),$$

therefore (12) follows directly from (23).

To establish (11), we start with the identity

$$G(x) = \frac{F(e^x)}{e^x} + \int_0^x \frac{F(e^y)}{e^y} dy - F(1),$$

then we rewrite it, applying (12), as follows:

$$\mathrm{E}f(S_T) = \mathrm{E}g(X_T) = \frac{1}{T}\mathrm{E}\left(G(X_T)Z_T\right)$$
$$= \frac{1}{T}\mathrm{E}\left(\left(\frac{F(S_T)}{S_T} + \int_0^{X_T} \frac{F(e^y)}{e^y} dy - F(1)\right) Z_T\right) = \frac{1}{T}\mathrm{E}\left(\frac{F(S_T)}{S_T} Z_T\right)$$
$$+ \frac{1}{T}\mathrm{E}\left(\int_0^{X_T} \frac{F(e^y)}{e^y} dy Z_T\right) - \frac{1}{T}\mathrm{E}(F(1)Z_T)$$
$$= \frac{1}{T}\mathrm{E}\left(\frac{F(S_T)}{S_T} Z_T\right) + \frac{1}{T}\mathrm{E}\left(\int_0^{X_T} \frac{F(e^y)}{e^y} dy Z_T\right).$$

Applying equation (23) to $h(x) = \frac{F(e^x)}{e^x}$, we get that

$$\mathrm{E}\left(\frac{F(S_T)}{S_T}\right) = \frac{1}{T}\mathrm{E}\left(\int_0^{X_T} \frac{F(e^y)}{e^y} dy Z_T\right).$$

Hence

$$\mathrm{E}f(S_T) = \frac{1}{T}\mathrm{E}\left(\frac{F(S_T)}{S_T} Z_T\right) + \mathrm{E}\left(\frac{F(S_T)}{S_T}\right) = \mathrm{E}\left(\frac{F(S_T)}{S_T}\left(1 + \frac{Z_T}{T}\right)\right).$$

Proof of Lemma 11. (i) Since the process Y is Gaussian, it is sufficient to consider $\theta = 2$. According to inequality (1.9.2) from [20], there exists a constant C_H such that for any function $f \in L_{\frac{1}{H}}[0,T]$

$$\mathrm{E}\left(\int_0^T f(s) dB_s^H\right)^2 \le C_H \|f\|^2_{L_{\frac{1}{H}}[0,T]}. \tag{24}$$

Now, the increment of Y can be presented as

$$Y_t - Y_s = \int_0^t h_{t,s}(\tau) dB^H(\tau) + Y_0(e^{-\alpha t} - e^{-\alpha s}),$$

where

$$h_{t,s}(\tau) = \mathbb{1}_{(s,t]}(u)e^{-\alpha(t-u)} + \mathbb{1}_{[0,s]}(u)(e^{-\alpha(t-u)} - e^{-\alpha(s-u)}).$$

Note that

$$|e^{-\alpha(t-u)} - e^{-\alpha(s-u)}| \le \alpha(t-s), \quad |e^{-\alpha t} - e^{-\alpha s}| \le \alpha(t-s),$$

and these simple inequalities imply, in particular, that

$$\int_0^t |h_{t,s}(u)|^{1/H} du = \int_0^s |e^{-\alpha(t-u)} - e^{-\alpha(s-u)}|^{1/H} du + \int_s^t e^{-\frac{\alpha(t-u)}{H}} du \le C|t-s|,$$

and (i) follows from (24).
(ii) Again, since Y and Y^n both are Gaussian processes, it is sufficient to consider only $\theta = 2$. Define the approximation $e_n(s) = e^{-\alpha t_i}, \quad s \in [t_i, t_{i+1}),\ 0 \le i \le n-1$. Then it follows from (24) that

$$\mathrm{E}\left(Y_{t_j} - Y^n_{t_j}\right)^2 = \mathrm{E}\left[\int_0^{t_j} \left(e^{-\alpha(t_j-s)} - e_n(t_j-s)\right) dB^H_s\right]^2$$

$$\le C_H \left(\int_0^{t_j} \left(e^{-\alpha(t_j-s)} - e_n(t_j-s)\right)^{1/H} ds\right)^{2H}$$

$$\le C_H \left(\int_0^{t_j} \alpha n^{-1/H} ds\right)^{2H} = C_H \alpha \left(t_j n^{-1/H}\right)^{2H} = C n^{-2}.$$

(iii) Now, let $s \in [t_i, t_{i+1})$ and $\theta \ge 1$. Then it follows from (i) and (ii) that

$$\begin{aligned} \mathrm{E}|Y_s - Y^n_s|^\theta &= \mathrm{E}|Y_s - Y^n_{t_i}|^\theta \le C\mathrm{E}|Y_s - Y_{t_i}|^\theta + C\mathrm{E}|Y_{t_i} - Y^n_{t_i}|^\theta \\ &\le C\left(n^{-\theta H} + n^{-\theta}\right) \le C n^{-\theta H}. \end{aligned}$$

Statement (iv) follows immediately from (24), since

$$Y^n_{t_j} = \int_0^{t_j} e_n(t_j - s) dB^H_s,$$

and functions e_n are uniformly bounded in n.

Proof of Lemma 13. Let us start with (14). Taking into account condition (**B**), (ii)

and (iii), we can write

$$
\begin{aligned}
\mathrm{E}(X_T - X_T^n)^2 &= \mathrm{E}\left[-\frac{1}{2}\int_0^T \sigma^2(Y_s)ds + \int_0^T \sigma(Y_s)dW_s + \frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s\right]^2 \\
&\le 2\mathrm{E}\left[\frac{1}{2}\int_0^T (\sigma^2(Y_s) - \sigma^2(Y_s^n))ds\right]^2 + 2\mathrm{E}\left[\int_0^T (\sigma(Y_s) - \sigma(Y_s^n))dW_s\right]^2 \\
&\le \frac{T}{2}\int_0^T \mathrm{E}(\sigma^2(Y_s) - \sigma^2(Y_s^n))^2 ds + 2\int_0^T \mathrm{E}(\sigma(Y_s) - \sigma(Y_s^n))^2 ds \qquad (25) \\
&= \frac{T}{2}\int_0^T \mathrm{E}\left[|\sigma(Y_s) - \sigma(Y_s^n)|^2 |\sigma(Y_s) + \sigma(Y_s^n)|^2\right] ds + 2\int_0^T \mathrm{E}(\sigma(Y_s) - \sigma(Y_s^n))^2 ds \\
&\le C\int_0^T \Big(\mathrm{E}\left(|Y_s - Y_s^n|^{2r}\left(C + |Y_s|^{2q} + |Y_s^n|^{2q}\right)\right) \\
&\qquad + \mathrm{E}\,|Y_s - Y_s^n|^{2r}\Big)ds \\
&\le C\int_0^T \left(\mathrm{E}\,|Y_s - Y_s^n|^{4r}\,\mathrm{E}\left(C + |Y_s|^{4q} + |Y_s^n|^{4q}\right)\right)^{1/2} ds.
\end{aligned}
$$

Lemma 2 (i), and Lemma 11 (ii) imply that for any $\theta \ge 1$

$$
\sup_{n\in\mathbb{N}, s\in[0,T]} \mathrm{E}\left(|Y_s|^\theta + |Y_s^n|^\theta\right) < \infty. \tag{26}
$$

Moreover, it follows from Lemma 11 that for any $s \in [0,T]$ and $\theta \ge 1$

$$
\mathrm{E}|Y_s - Y_s^n|^{\theta r} \le C n^{-\theta r H}. \tag{27}
$$

Put $\theta = 4q$ in (26) and $\theta = 4$ in (27) and substitute the result into the right-hand side of (25):

$$
\mathrm{E}(X_T - X_T^n)^2 \le C\int_0^T \left(\mathrm{E}\,(Y_s - Y_s^n)^{4r}\right)^{\frac{1}{2}} ds \le C n^{-2rH},
$$

so that (14) is proved. Now continue with (15). Taking into account condition $(\mathbf{B})$, (i), we get that

$$
\left|\frac{1}{\sigma(x)} - \frac{1}{\sigma(y)}\right| \le \frac{|\sigma(x) - \sigma(y)|}{\sigma(x)\sigma(y)} \le \frac{|\sigma(x) - \sigma(y)|}{\sigma^2_{\min}},
$$

whence

$$\mathrm{E}(Z_T - Z_T^n)^2 = \int_0^T \left(\frac{1}{\sigma(Y)} - \frac{1}{\sigma(Y_s^n)}\right)^2 ds$$

$$\leq \frac{1}{\sigma_{\min}^2} C_\sigma \int_0^T \mathrm{E}\left(Y_s - Y_s^n\right)^{2r} ds.$$

We can apply (27) with $\theta = 2$ to the latter inequality and conclude this part of the proof exactly as it was done for (14).

Proof of Lemma 14. We can write

$$\mathrm{E}\left|\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right|^2 \leq 2\mathrm{E}\left|\frac{F(S_T)}{S_T} - \frac{F(S_T)}{S_T^n}\right|^2 + 2\mathrm{E}\left|\frac{F(S_T)}{S_T^n} - \frac{F(S_T^n)}{S_T^n}\right|^2 := 2I_1 + 2I_2. \tag{28}$$

Now we estimate the right-hand side of (28) term by term. For I_1 we have that

$$I_1 = \mathrm{E}\left(F(S_T)\big((S_T)^{-1} - (S_T^n)^{-1}\big)\right)^2 \leq \Big(\mathrm{E}(F(S_T))^4 \mathrm{E}\big((S_T)^{-1} - (S_T^n)^{-1}\big)^4\Big)^{1/2}. \tag{29}$$

On one hand, since f consequently F both have a polynomial growth, $\mathrm{E}(F(S_T))^4 < \infty$ according to Remark 3. On the other hand,

$$\mathrm{E}\big((S_T)^{-1} - (S_T^n)^{-1}\big)^4 = S_0^{-4} e^{-4bT} \tag{30}$$

$$\times \mathrm{E}\left(\exp\left\{\frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s\right\} - \exp\left\{\frac{1}{2}\int_0^T \sigma^2(Y_s)ds - \int_0^T \sigma(Y_s)dW_s\right\}\right)^4.$$

Using the inequalities

$$|e^x - e^y| \leq (e^x + e^y)|x - y|, \quad x, y \in \mathbb{R},$$

$$(x+y)^{2n} \leq C(n)(x^{2n} + y^{2n}), \quad x, y \in \mathbb{R}, n \in \mathbb{N},$$

along with results outlined in Remark 3 and Remark 12, the Burkholder–Gundy and Hölder inequalities, condition (**B**), (ii) and (iii), and relation (26) with $\nu = 16q$, we get from (30) that

$$
\begin{aligned}
&\mathrm{E}\big((S_T)^{-1}-(S_T^n)^{-1}\big)^4\\
&\le C\mathrm{E}\left(\exp\left\{\frac{1}{2}\int_0^T\sigma^2(Y_s^n)ds-\int_0^T\sigma(Y_s^n)dW_s\right\}-\exp\left\{\frac{1}{2}\int_0^T\sigma^2(Y_s)ds-\int_0^T\sigma(Y_s)dW_s\right\}\right)^4\\
&\le C\mathrm{E}\Bigg(\left(\exp\left\{2\int_0^T\sigma^2(Y_s^n)ds-4\int_0^T\sigma(Y_s^n)dW_s\right\}+\exp\left\{2\int_0^T\sigma^2(Y_s)ds-4\int_0^T\sigma(Y_s)dW_s\right\}\right)\\
&\qquad\times\left(\frac{1}{2}\int_0^T\sigma^2(Y_s^n)ds-\int_0^T\sigma(Y_s^n)dW_s-\frac{1}{2}\int_0^T\sigma^2(Y_s)ds+\int_0^T\sigma(Y_s)dW_s\right)^4\Bigg)\\
&\le C\Bigg(\mathrm{E}\Bigg(\exp\left\{4\int_0^T\sigma^2(Y_s^n)ds-8\int_0^T\sigma(Y_s^n)dW_s\right\}\\
&\qquad+\exp\left\{4\int_0^T\sigma^2(Y_s)ds-8\int_0^T\sigma(Y_s)dW_s\right\}\Bigg)\Bigg)^{1/2}\\
&\qquad\times\left[\mathrm{E}\left(\frac{1}{2}\int_0^T\sigma^2(Y_s^n)ds-\int_0^T\sigma(Y_s^n)dW_s-\frac{1}{2}\int_0^T\sigma^2(Y_s)ds+\int_0^T\sigma(Y_s)dW_s\right)^8\right]^{1/2}\\
&\le C\left[\mathrm{E}\left(\frac{1}{2}\int_0^T\sigma^2(Y_s^n)ds-\int_0^T\sigma(Y_s^n)dW_s-\frac{1}{2}\int_0^T\sigma^2(Y_s)ds+\int_0^T\sigma(Y_s)dW_s\right)^8\right]^{1/2}\\
&\le C\left[\mathrm{E}\left(\int_0^T\sigma^2(Y_s^n)ds-\int_0^T\sigma^2(Y_s)ds\right)^8+\mathrm{E}\left(\int_0^T\sigma(Y_s^n)dW_s-\int_0^T\sigma(Y_s)dW_s\right)^8\right]^{1/2}\\
&\le C\left[T^7\mathrm{E}\left(\int_0^T\left(\sigma^2(Y_s^n)-\sigma^2(Y_s)\right)^8ds\right)+C\mathrm{E}\left(\int_0^T\left(\sigma(Y_s^n)-\sigma(Y_s)\right)^2ds\right)^4\right]^{1/2}\\
&=C\Bigg[T^7\left(\int_0^T\mathrm{E}\left\{(\sigma(Y_s^n)-\sigma(Y_s))(\sigma(Y_s^n)+\sigma(Y_s))\right\}^8ds\right)\\
&\qquad+CT^3\mathrm{E}\left(\int_0^T\left(\sigma(Y_s^n)-\sigma(Y_s)\right)^8ds\right)\Bigg]^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left[\mathrm{E}\left(\int_0^T \left(\mathrm{E}|Y_s-Y_s^n|^{16r}\mathrm{E}\left(C+|Y_s|^{16q}+|Y_s^n|^{16q}\right)\right)^{1/2}ds\right)\right.\\
&\qquad\left.+\,\mathrm{E}\left(\int_0^T |Y_s-Y_s^n|^{8r}\,ds\right)\right]^{1/2} \qquad (31)\\
&\leq C\left[\int_0^T \left(\left(\mathrm{E}\,|Y_s-Y_s^n|^{16r}\right)^{\frac12}+\mathrm{E}\,|Y_s-Y_s^n|^{8r}\right)ds\right]^{1/2}.
\end{aligned}
$$

Applying (27) consequently with $\theta = 8$ and $\theta = 16$ we get that the last expression in (31) does not exceed $C\left(\frac{1}{n}\right)^{4rH}$, thus from (29) we obtain

$$I_1 \leq Cn^{-2rH}. \qquad (32)$$

Now we continue with I_2 from the relation (28):

$$I_2 \leq \left[\mathrm{E}(F(S_T)-F(S_T^n))^4\right]^{1/2}\left[\mathrm{E}(S_T^n)^{-4}\right]^{1/2}.$$

The second multiplier is bounded according to Remark 12, therefore it follows from condition (A), (i), that

$$
\begin{aligned}
I_2 &\leq C\left[\mathrm{E}(F(S_T)-F(S_T^n))^4\right]^{1/2} = C\left[\mathrm{E}\left(\int_{S_T\wedge S_T^n}^{S_T\vee S_T^n} f(x)dx\right)^4\right]^{1/2}\\
&\leq C(C_f)^2\left[\mathrm{E}\left(|S_T-S_T^n|^4(1+S_T^p+(S_T^n)^p)^4\right)\right]^{1/2} \leq C\left[\mathrm{E}|S_T-S_T^n|^8\mathrm{E}(1+S_T^p+(S_T^n)^p)^8\right]^{1/4}.
\end{aligned}
$$

According to Lemma 2 and Remark 12,

$$\sup_{n\in\mathbb{N}}\mathrm{E}(1+S_T^p+(S_T^n)^p)^8<\infty,$$

whence we get that

$$I_2 \leq C\left[\mathrm{E}|S_T-S_T^n|^8\right]^{1/4}.$$

To evaluate the right-hand side of this inequality, we can proceed as in the proof of (31) and subsequent inequalities, because neither the opposite sign of the exponents nor the 8th power instead of the 4th lead to serious discrepancies in the estimations. Therefore we get

$$I_2 \leq C\left[\int_0^T\left(\left(\mathrm{E}\,|Y_s-Y_s^n|^{32r}\right)^{\frac12}+\mathrm{E}\,|Y_s-Y_s^n|^{16r}\right)ds\right]^{\frac18} \leq C\left(n^{-16rH}\right)^{1/8} = Cn^{-2rH}. \qquad (33)$$

Bounds (32) and (33) complete the proof.

Proof of Theorem 15. By Lemma 10 we can write

$$\left|\mathrm{E}f(X_T) - \mathrm{E}\left(\frac{F(S_T^n)}{S_T^n}\left(1+\frac{Z_T^n}{T}\right)\right)\right| = \mathrm{E}\left|\left(\frac{F(S_T)}{S_T}\left(1+\frac{Z_T}{T}\right)\right) - \left(\frac{F(S_T^n)}{S_T^n}\left(1+\frac{Z_T^n}{T}\right)\right)\right|$$

$$\leq \frac{1}{T}\mathrm{E}\left|\frac{F(S_T)}{S_T}\left(Z_T - Z_T^n\right)\right| + \mathrm{E}\left|\left(1+\frac{Z_T^n}{T}\right)\left(\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right)\right|$$

$$\leq \frac{1}{T}\left[\mathrm{E}\left(\frac{F(S_T)}{S_T}\right)^2 \mathrm{E}\left(Z_T - Z_T^n\right)^2\right]^{1/2} + \left[\mathrm{E}\left(\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right)^2 \mathrm{E}\left(1+\frac{Z_T^n}{T}\right)^2\right]^{1/2}.$$

According to Lemma 2, Remark 3 and Cauchy–Schwartz inequality, $\mathrm{E}\left(\frac{F(S_T)}{S_T}\right)^2 < \infty$. Obviously, $\sup_{n\geq 1}\mathrm{E}\left(Z_T^n\right)^2 < \frac{T}{\sigma_{min}^2}$. Now the proof follows from Lemma 13 and Lemma 14.

Proof of Lemma 16. Proof immediately follows from the general formula for the density of k-dimensional Gaussian vector:

$$p(\bar{x}) = (2\pi)^{-\frac{k}{2}}|C|^{-1}\exp\{-(C^{-1}(\bar{x}-\bar{a}), \bar{x}-\bar{a}\}, \tag{34}$$

where $\bar{x} \in \mathbb{R}^k$, $\bar{a}$ is a vector of expectations, C is a covariance matrix. In our case covariance matrix equals

$$C = C_{X,Z} = \begin{pmatrix} \sigma_Y^2 & T \\ T & \sigma_Z^2 \end{pmatrix},$$

$$k = 2,\ |C_{X,Z}| = \Delta = \sigma_Y^2\sigma_Z^2 - T^2, \bar{a} = (m_X, 0) = (\log S_0 + bT - \frac{1}{2}\sigma_Y^2, 0),$$

and (16) follows immediately from (34).

Proof of Theorem 17. Applying conditioning on Y, (12), and Lemma 16, we get that

(35)
$$T\mathrm{E}g(X_T) = \mathrm{E}\left(G(X_T)\int_0^T \frac{1}{\sigma(Y_u)}dW_u\right) = \mathrm{E}\left(\mathrm{E}\left(G(X_T)\int_0^T \frac{1}{\sigma(Y_u)}dW_u \middle| \{Y_s, s\in[0,T]\}\right)\right)$$

$$= \mathrm{E}\left(\mathrm{E}\left(\int_{\mathbb{R}^2} G(x)zp_{X,Z}(x,z)dxdz \middle| \{Y_s, s\in[0,T]\}\right)\right) = \mathrm{E}\int_{\mathbb{R}^2} G(x)zp_{X,Z}(x,z)dxdz$$

$$= \mathrm{E}\int_{\mathbb{R}} G(x)\left(\int_{\mathbb{R}} zp_{X,Z}(x,z)dz\right)dx.$$

The inner integral can be significantly simplified. Indeed, denote $\tilde{x} = x - m_Y$. Then

$$\int_{\mathbb{R}} zp_{X,Z}(x,z)dz = \frac{1}{2\pi\Delta^{\frac{1}{2}}}\int_{\mathbb{R}} z\exp\left\{-\frac{1}{2\Delta}\left(\sigma_Z^2\tilde{x}^2 + \sigma_Y^2 z^2 - 2T\tilde{x}z\right)\right\}dz$$

$$= \frac{1}{2\pi\Delta^{\frac{1}{2}}} \int_{\mathbb{R}} z \exp\left\{ - \frac{1}{2\Delta} \left(\left(\sigma_Y z - \frac{T\tilde{x}}{\sigma_Y} \right)^2 - \frac{T^2 \tilde{x}^2}{\sigma_Y^2} + \sigma_Z^2 \tilde{x}^2 \right) \right\} dz$$

$$= \frac{1}{2\pi\Delta^{\frac{1}{2}}} \exp\left\{ - \frac{\tilde{x}^2}{2\Delta} \frac{\sigma_Y^2 \sigma_Z^2 - T^2}{\sigma_Y^2} \right\} \int_{\mathbb{R}} z \exp\left\{ - \frac{1}{2\Delta} \left(\sigma_Y z - \frac{T\tilde{x}}{\sigma_Y} \right)^2 \right\} dz$$

$$= \frac{1}{2\pi\Delta^{\frac{1}{2}}} \exp\left\{ - \frac{\tilde{x}^2}{2\sigma_Y^2} \right\} \int_{\mathbb{R}} z \exp\left\{ - \left(\frac{\sigma_Y}{\sqrt{2}\Delta^{\frac{1}{2}}} z - \frac{T\tilde{x}}{\sqrt{2}\Delta^{\frac{1}{2}} \sigma_Y} \right)^2 \right\} dz.$$

Since

$$\int_{\mathbb{R}} x e^{-(ax-b)^2} dx = \frac{b}{a^2} \sqrt{\pi},$$

we obtain

$$\int_{\mathbb{R}} z p_{X,Z}(x,z) dz = \frac{T\tilde{x}}{\sigma_Y^3 \sqrt{2\pi}} \exp\left\{ - \frac{\tilde{x}^2}{2\sigma_Y^2} \right\}. \tag{36}$$

Combining (35) and (36), we get the proof.

Proof of Theorem 18. To simplify notations, without loss of generality, let us assume that $X_0 + bT = 0$. Then, using (17), we get that

$$\left| \mathrm{E}g(X_T) - (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} G(x) \mathrm{E}\left(\frac{(x - m_{Y,n})}{\sigma_{Y,n}^3} \exp\left\{ - \frac{(x - m_{Y,n})^2}{2\sigma_{Y,n}^2} \right\} \right) dx \right|$$
$$= (2\pi)^{-\frac{1}{2}} \left| \int_{\mathbb{R}} G(x) \mathrm{E}\left(\frac{(x - m_Y)}{\sigma_Y^3} \exp\left\{ - \frac{(x - m_Y)^2}{2\sigma_Y^2} \right\} - \right. \right.$$
$$\left. \left. \frac{(x - m_{Y,n})}{\sigma_{Y,n}^3} \exp\left\{ - \frac{(x - m_{Y,n})^2}{2\sigma_{Y,n}^2} \right\} \right) dx \right|$$
$$\le (2\pi)^{-\frac{1}{2}} \left(\int_{\mathbb{R}} G(x) \left(\mathrm{E}\left(\left| \frac{x - m_Y}{\sigma_Y^3} - \frac{x - m_{Y,n}}{\sigma_{Y,n}^3} \right| \exp\left\{ - \frac{(x - m_Y)^2}{2\sigma_Y^2} \right\} \right) \right. \right.$$
$$\left. \left. + \mathrm{E}\left| \frac{(x - m_{Y,n})}{\sigma_{Y,n}^3} \left(\exp\left\{ - \frac{(x - m_Y)^2}{2\sigma_Y^2} \right\} - \exp\left\{ - \frac{(x - m_{Y,n})^2}{2\sigma_{Y,n}^2} \right\} \right) \right| \right) dx \right)$$
$$:= (2\pi)^{-\frac{1}{2}} \left(\int_{\mathbb{R}} G(x) (J_1(x) + J_2(x)) dx \right).$$

To estimate $J_1(x)$, denote $E_{exp}(x) = \left(\mathrm{E} \exp\left\{ - \frac{(x - m_Y)^2}{\sigma_Y^2} \right\} \right)^{1/2}$ and notice that

$$\frac{x-m_{Y}}{\sigma^3_{Y}}-\frac{x-m_{Y,n}}{\sigma^3_{Y,n}}=\sigma_{Y}^{-3}(m_{Y,n}-m_{Y})+(x-m_{Y,n})(\sigma_{Y}^{-3}-\sigma_{Y,n}^{-3}).$$

Hence

$$J_1(x)\le C\left(\mathrm{E}\left(\sigma_{Y}^{-3}(m_{Y,n}-m_{Y})\right)^2+\mathrm{E}\left(\left(x-m_{Y,n})(\sigma_{Y}^{-3}-\sigma_{Y,n}^{-3})\right)^2\right)^{1/2}E_{exp}(x). \tag{37}$$

Since $|a_1^3-a_2^3|\le|a_1^2-a_2^2|(a_1+a_2)$, $a_1,a_2>0$, and also the lower bounds $\sigma^2_{Y}\ge T\sigma^2_{\min}$, $\sigma^2_{Y,n}\ge T\sigma^2_{\min}$ hold, one can conclude that

$$\begin{aligned}\left|\sigma_{Y}^{-3}-\sigma_{Y,n}^{-3}\right|&\le\left|\sigma_{Y}^{-2}-\sigma_{Y,n}^{-2}\right|\left(\sigma_{Y,n}^{-1}+\sigma_{Y}^{-1}\right)=\frac{\left|\sigma^2_{Y}-\sigma^2_{Y,n}\right|}{\sigma^2_{Y}\sigma^2_{Y,n}}\left(\sigma_{Y,n}^{-1}+\sigma_{Y}^{-1}\right)\\&\le\frac{2\left|\sigma^2_{Y}-\sigma^2_{Y,n}\right|}{\sigma^2_{Y,n}T^{\frac{3}{2}}\sigma^3_{\min}}.\end{aligned} \tag{38}$$

Therefore

$$|(x-m_{Y,n})(\sigma_{Y}^{-3}-\sigma_{Y,n}^{-3})|\le\frac{2|x+\frac{1}{2}\sigma^2_{Y,n}||\sigma^2_{Y}-\sigma^2_{Y,n}|}{\sigma_{Y,n}^2T^{\frac{3}{2}}\sigma^3_{\min}}\le C(1+|x|)\left|\sigma^2_{Y}-\sigma^2_{Y,n}\right|.$$

Since $m_{Y,n}-m_{Y}=-\frac{1}{2}(\sigma^2_{Y,n}-\sigma^2_{Y})$, we get from (37) and (38) that

$$J_1(x)\le C(1+|x|)(\mathrm{E}(\sigma^2_{Y,n}-\sigma^2_{Y})^2)^{1/2}E_{exp}(x). \tag{39}$$

Similarly to (25) and (31), we get, applying condition (**B**), Lemma 11, (iii) and (iv), together with the standard Hölder's inequality, that

(40)

$$\begin{aligned}\mathrm{E}(\sigma^2_{Y,n}-\sigma^2_{Y})^2&=\mathrm{E}\left(\int_0^T(\sigma^2(Y^n_s)-\sigma^2(Y_s))ds\right)^2\le T\mathrm{E}\int_0^T(\sigma^2(Y^n_s)-\sigma^2(Y_s))^2ds\\&\le C_\sigma C\int_0^T\left[\mathrm{E}(Y^n_s-Y_s)^{4r}\mathrm{E}(\sigma(Y^n_s)+\sigma(Y_s))^4\right]^{1/2}ds\le C\int_0^T\left[\mathrm{E}(Y^n_s-Y_s)^{4r}\right]^{1/2}ds\le Cn^{-2rH}.\end{aligned}$$

Combining the latter inequality with (39) we get that

$$J_1(x)\le Cn^{-rH}(1+|x|)E_{exp}(x), \tag{41}$$

and consequently

$$\int_{\mathbb{R}}G(x)J_1(x)dx\le Cn^{-rH}\int_{\mathbb{R}}G(x)(1+|x|)E_{exp}(x)dx. \tag{42}$$

Let us show that the integral in the right–hand side of (42) is finite. Applying the standard Hölder inequality together with polynomial growth of $G(x)$, we get that

$$
\begin{aligned}
\int_{\mathbb{R}} G(x)(1+|x|)E_{exp}(x)dx &\le \left(\int_{\mathbb{R}} G^2(x)(1+|x|)^2 e^{-(2p+1)|x|}dx\right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}} e^{(2p+1)|x|}\mathrm{E}\exp\left\{-\frac{(x-m_Y)^2}{\sigma_Y^2}\right\}dx\right)^{1/2} \\
&\le C\left(2\int_0^\infty e^{(2p+2)x}\mathrm{E}\exp\left\{-\frac{x^2}{\sigma_Y^2}-\frac{\sigma_Y^2}{4}\right\}dx\right)^{1/2} \\
&= C\left(2\mathrm{E}\int_0^\infty \exp\left\{-\left(\frac{x}{\sigma_Y}-(p+1)\sigma_Y\right)^2+(p+1)^2\sigma_Y^2-\frac{\sigma_Y^2}{4}\right\}dx\right)^{1/2} \qquad (43) \\
&= C\left(2\mathrm{E}\left[\exp\left\{(p+1)^2\sigma_Y^2-\frac{\sigma_Y^2}{4}\right\}\int_0^\infty \exp\left\{-\left(\frac{x}{\sigma_Y}-(p+1)\sigma_Y\right)^2\right\}dx\right]\right)^{1/2} \\
&= C\left(2\mathrm{E}\left[\exp\left\{(p+1)^2\sigma_Y^2-\frac{\sigma_Y^2}{4}\right\}\int_0^\infty \exp\left\{-\left(\frac{x}{\sigma_Y}\right)^2\right\}dx\right]\right)^{1/2} \\
&\le C\left(\mathrm{E}\left[\sigma_Y\exp\left\{(p+1)^2\sigma_Y^2-\frac{\sigma_Y^2}{4}\right\}\right]\right)^{1/2}.
\end{aligned}
$$

The finiteness of the expectation on the last line follows from condition (**B**) and Lemma 2, formula (5), since

$$\sigma_Y^2 \le C(1+\sup_{t\in[0,T]}|Y_t|^{2q}).$$

Construction of upper bound for $J_2(x)$ is similar. Indeed,

$$|\exp\{-u^2\}-\exp\{-v^2\}| \le 2(|u|+|v|)(\exp\{-u^2\}+\exp\{-v^2\})|u-v|.$$

In our case $|u| = \left|\frac{x-m_Y}{\sigma_Y}\right| \le C(1+|x|)\sigma_Y$, $|v| = \left|\frac{x-m_{Y_n}}{\sigma_{Y_n}}\right| \le C(1+|x|)\sigma_{Y_n}$, and $\left|\frac{(x-m_{Y,n})}{\sigma_{Y,n}^3}\right| \le C(1+|x|)$, therefore

$$
\begin{aligned}
J_2(x) &\le C(1+|x|)^2 \mathrm{E}\Bigg((\sigma_{Y,n}+\sigma_Y)\Bigg(\exp\Bigg\{-\frac{(x-m_Y)^2}{2\sigma_Y^2}\Bigg\}+\exp\Bigg\{-\frac{(x-m_{Y,n})^2}{2\sigma_{Y,n}^2}\Bigg\}\Bigg) \\
&\times \left|\frac{x-m_Y}{\sigma_Y}-\frac{x-m_{Y,n}}{\sigma_{Y,n}}\right|\Bigg) \le C(1+|x|)^2\left(\mathrm{E}\left(\frac{x-m_Y}{\sigma_Y}-\frac{x-m_{Y,n}}{\sigma_{Y,n}}\right)^2\right)^{\frac12} \\
&\times \left(\mathrm{E}\left((\sigma_Y+\sigma_{Y,n})\left(\exp\left\{-\frac{(x-m_Y)^2}{2\sigma_Y^2}\right\}+\exp\left\{-\frac{(x-m_{Y,n})^2}{2\sigma_{Y,n}^2}\right\}\right)\right)^2\right)^{\frac12}.
\end{aligned} \tag{44}
$$

Now, taking into account that $\sigma_Y \wedge \sigma_{Y,n} \ge T^{\frac12}\sigma_{min}$, and $\sigma_Y\sigma_{Y,n}(\sigma_Y+\sigma_{Y,n}) \ge C\sigma_{Y,n}^2$, we get

$$
\begin{aligned}
\left|\frac{x-m_Y}{\sigma_Y}-\frac{x-m_{Y,n}}{\sigma_{Y,n}}\right| &\le C|\sigma_{Y,n}^2-\sigma_{Y,n}^2| + \frac{|x-m_{Y,n}|}{\sigma_Y\sigma_{Y,n}(\sigma_Y+\sigma_{Y,n})}|\sigma_Y^2-\sigma_{Y,n}^2| \\
&\le C(1+|x|)|\sigma_Y^2-\sigma_{Y,n}^2| \le C(1+|x|)|\sigma_Y^2-\sigma_{Y,n}^2|,
\end{aligned}
$$

and from (40) we deduce that $\left(\mathrm{E}\left(\frac{x-m_Y}{\sigma_Y}-\frac{x-m_{Y,n}}{\sigma_{Y,n}}\right)^2\right)^{\frac12} \le C(1+|x|)n^{-rH}$. Together with (44) this implies that

$$
\begin{aligned}
\int_{\mathbb{R}} G(x)J_2(x)dx &\le Cn^{-rH}\int_{\mathbb{R}}(1+|x|)^3 G(x)\Bigg(\mathrm{E}\Bigg((\sigma_Y+\sigma_{Y,n}) \\
&\times \left(\exp\left\{-\frac{(x-m_Y)^2}{2\sigma_Y^2}\right\}+\exp\left\{-\frac{(x-m_{Y,n})^2}{2\sigma_{Y,n}^2}\right\}\right)\Bigg)^2\Bigg)^{\frac12}dx.
\end{aligned}
$$

The fact that the integral in the right-hand side is finite, can be established via the same approach as applied to the integral $\int_{\mathbb{R}} G(x)(1+|x|)E_{exp}(x)dx$ in (43).

Proof of Lemma 19. For any $\delta > 0$ choose $\varepsilon = \varepsilon(\delta)$ in such a way that $\varepsilon(2+\alpha|Y_0|) < \delta$. Then we get from the representation (20) that for $0 \le t \le \tau_\varepsilon \wedge \varepsilon$

$$
\begin{aligned}
|Y_t - Y_0| &\le |Y_0|(1-e^{-\alpha t}) + |B_t^H| + \alpha e^{-\alpha t}\int_0^t e^{\alpha s}|B_s^H|ds \le |Y_0|\alpha\varepsilon + \varepsilon + \varepsilon e^{-\alpha t}(e^{\alpha t}-1) \\
&\le \varepsilon(2+\alpha|Y_0|) < \delta.
\end{aligned}
$$

Therefore for $\varepsilon < \frac{\delta}{2+\alpha|Y_0|}$ we have that $\nu_\delta > \tau_\varepsilon \wedge \varepsilon$. So, it is sufficient to prove that for any $\varepsilon > 0$ and any $l > 0$

$$
\mathrm{E}(\tau_\varepsilon \wedge \varepsilon)^{-l} < \infty.
$$

Now, for $v < \varepsilon$

$$
\mathrm{P}\{\tau_\varepsilon \wedge \varepsilon < v\} = \mathrm{P}\{\tau_\varepsilon < v\} = \mathrm{P}\{\sup_{0\le t\le v}|B_t^H| \ge \varepsilon\}.
$$

Furthermore, it follows from self-similarity and symmetry of the fBm that $\mathrm{P}\{\sup_{0\le t\le v}|B_t^H|\ge \varepsilon\} \le 2\mathrm{P}\{\sup_{0\le t\le 1} B_t^H \ge \frac{\varepsilon}{v^H}\}$. Moreover, denote $\vartheta = \mathrm{E}\sup_{0\le t\le 1} B_t^H$. Then, according to inequality (2.2) from [27] that for $\frac{\varepsilon}{v^H} > \vartheta$

$$\mathrm{P}\{\sup_{0\le t\le v} B_t^H \ge \frac{\varepsilon}{v^H}\} \le \exp\left\{-\frac{\left(\frac{\varepsilon}{v^H}-\vartheta\right)^2}{2}\right\} = \exp\left\{-\frac{(\varepsilon-\vartheta v^H)^2}{2v^{2H}}\right\},$$

whence the proof immediately follows.

Remark 22. Exponential bounds for the distribution of τ_ε allow to prove that $\mathrm{E}(\tau_\varepsilon\wedge\varepsilon\wedge a)^{-l}<\infty$ for any $a,l>0$.

Proof of Lemma 20. As it follows from Proposition 2.1.1 and Exercise 2.1.1 in [23], it is sufficient to show that

$$\sigma_Y^2 \in \mathrm{D}^{2,4} \tag{45}$$

and that

$$\mathrm{E}\left(||D^B\sigma_Y^2||_H\right)^{-8} < \infty. \tag{46}$$

Recall that $\kappa(x)=\sigma(x)\sigma'(x)$. It follows from conditions (**B**) and (**D**) that κ and κ' are functions of polynomial growth, $\kappa(x)>0$. Recall the notation $l(u,s)=c_He^{-\alpha s}\int_u^s e^{\alpha v}v^{H-1/2}(v-u)^{H-3/2}dv$. Taking into account (21) and (7), we write the stochastic derivative as

$$\begin{aligned}D_u^B(\sigma_Y^2) &= D_u^B(\int_0^T\sigma^2(Y_s)ds) = 2\int_0^T\kappa(Y_s)D_u^BY_sds\\ &= 2c_Hu^{1/2-H}\int_u^T\kappa(Y_s)e^{-\alpha s}\int_u^s e^{\alpha v}v^{H-1/2}(v-u)^{H-3/2}dvds\\ &= 2u^{1/2-H}\int_u^T\kappa(Y_s)l(u,s)ds.\end{aligned}$$

Therefore, the iterated derivative equals

$$\begin{aligned}&D_z^B(D_u^B(\sigma_Y^2))\\ &= 2u^{1/2-H}z^{1/2-H}\int_{u\vee z}^T\kappa'(Y_s)l(z,s)l(u,s)ds.\end{aligned} \tag{47}$$

Obviously, the right-hand side of (47) is in $H\otimes H$, and the corresponding integral has moments of any order, due to polynomial growth of κ', which implies (45).

To prove (46), note that

$$D_u^B(\sigma_Y^2) \geq C \int_u^T \kappa(Y_s)(s-u)^{H-1/2} ds,$$

whence

$$||D^B\sigma_Y^2||_H^2 = \int_0^T \left(D_u^B\sigma_Y^2\right)^2 du \geq C \int_0^T du \left(\int_u^T \kappa(Y_s)(s-u)^{H-1/2} ds \right)^2.$$

Now, let $\sigma'(Y_0) = \sigma_0 > 0$. Choose $\delta > 0$ so that for $y \in [Y_0 - \delta, Y_0 + \delta]$ to provide on this interval the lower bound $\sigma'(y) > \frac{\sigma_0}{2}$. Then choose $\varepsilon = \varepsilon(\delta)$, as it was mentioned in the proof of Lemma 19, and put $\zeta = \tau_\varepsilon \wedge \varepsilon \wedge \frac{T}{2}$.

Then

$$\int_0^T du \left(\int_u^T \kappa(Y_s)(s-u)^{H-1/2} ds \right)^2 \geq C \int_0^{\frac{1}{3}\zeta} du \left(\int_{\frac{2}{3}\zeta}^{\zeta} \kappa(Y_s)(s-u)^{H-1/2} ds \right)^2$$

$$\geq C \int_0^{\frac{1}{3}\zeta} du \left(\int_{\frac{2}{3}\zeta}^{\zeta} \sigma_{\min}\sigma_0 \left(\frac{1}{3}\zeta\right)^{H-1/2} ds \right)^2 = C\zeta^{2+2H}.$$

It follows immediately from Lemma 19 and Remark 22 that

$$\mathrm{E}\left(||D^B\sigma_Y^2||_H\right)^{-8} \leq C\mathrm{E}\zeta^{-8-8H} \leq C\mathrm{E}\left(\tau_\varepsilon \wedge \varepsilon \wedge \frac{T}{2}\right)^{-8-8H} < \infty.$$

Proof of Theorem 21.

From Lemma 20 and Proposition 2.1.1, [23] we get the first part of equality (18):

$$p_{\sigma_Y^2}(u) = \mathrm{E}\left[\mathbb{1}_{\sigma_Y^2 > u} \delta\left(\frac{D^B\sigma_Y^2}{||D\sigma_Y^2||_H^2} \right) \right].$$

To get the second part, note that $\eta := \left(||D\sigma_Y^2||_H\right)^{-2}$ admits stochastic derivative and, according to Proposition 1.3.3 from [23], the following holds

$$\delta\left(\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2} \right) = \int_0^T \eta D_u^B(\sigma_Y^2) dB_u = \eta \int_0^T D_u^B(\sigma_Y^2) dB_u$$

$$- \int_0^T D_u^B \eta D_u^B(\sigma_Y^2) du = 2\eta \int_0^T u^{1/2-H} \int_u^T \kappa(Y_s) l(u,s) ds dB_u - \int_0^T D_u^B \eta D_u^B(\sigma_Y^2) du.$$

According to Lemma 2.10 from [19], we can apply the Fubini theorem for the Skorokhod integral. Then

$$\int_0^T u^{1/2-H} \int_u^T \kappa(Y_s) l(u,s) ds dB_u = \int_0^T \kappa(Y_s) (\int_0^s u^{1/2-H} l(u,s) dB_u) ds,$$

where the interior integral is a Wiener one.

Finally, taking into account that $m_Y = X_0 + bT - \frac{1}{2}\sigma_Y^2$, we get

$$\mathrm{E}\frac{(x-m_Y)}{\sigma_Y^3\sqrt{2\pi}} \exp\left\{ -\frac{(x-m_Y)^2}{\sigma_Y^2} \right\}$$

$$= \int_{\mathbb{R}} \frac{(x+u/2-X_0-bT)}{u^3\sqrt{2\pi}} \exp\left\{ -\frac{(x+u/2-X_0-bT)^2}{u^2} \right\} p_{\sigma_Y^2}(u) du.$$

Combining this with (17), we get the proof.

8. Simulations.

In this section we use the discretization schemes proposed in section 4 and section 5 to simulate the option price. We treat double and single discretization, respectively.

The values of b, α and T are the same in all simulations, and equal $b = 0.2, \alpha = 0.6, T = 1$. In Table 1 and Table 2 we give the results of simulations based on (11) (double discretization) for different n and σ. The functions f and the values of H are given in the table headers.

Table 1

double discretization, $f(s) = (s-1)_+ + \mathbb{1}_{s>1}$, $H = 0.6$

n	125	250	500	1000	2000	4000	8000
$\sigma(y) = \sqrt{\lvert y\rvert + 0.1}$	0.95124	0.92149	0.93664	0.89628	0.88124	0.89717	0.92390
$\sigma(y) = \lvert y\rvert + 0.1$	0.92121	0.95820	0.94733	1.02572	0.90530	0.92062	0.97430
$\sigma(y) = \sqrt{y^2+1}$	0.93357	1.01340	0.99205	0.95801	0.97705	0.96882	0.92312
$\sigma(y) = \sin^2(y)$ $+0.05$	0.87957	0.87842	0.94525	0.93053	0.91097	0.89368	1.00256

Table 2

double discretization, $f(s) = (s-1.5)_+ + \mathbb{1}_{s>2}$, $H = 0.8$

n	125	250	500	1000	2000	4000	8000
$\sigma(y) = \sqrt{\lvert y\rvert + 0.1}$	0.48073	0.46643	0.53185	0.53124	0.53128	0.50020	0.57804
$\sigma(y) = \lvert y\rvert + 0.1$	0.51931	0.48957	0.50875	0.48999	0.46509	0.48525	0.55298
$\sigma(y) = \sqrt{y^2+1}$	0.66845	0.81584	0.66681	0.64368	0.65982	0.76746	0.74945
$\sigma(y) = \sin^2(y)$ $+0.05$	0.31951	0.30628	0.28770	0.29487	0.30395	0.29509	0.30490

Table 3 and Table 4 present the results of simulations for the same parameters as in Table 1 and Table 2, respectively, but the price is computed using (17), which corresponds to single discretization.

Table 3
single discretization, $f(s) = (s-1)_+ + \mathbb{1}_{s>1}$, $H = 0.6$

n	125	250	500	1000	2000	4000	8000
$\sigma(y) = \sqrt{\lvert y\rvert + 0.1}$	0.92266	0.92257	0.92249	0.92232	0.92219	0.92263	0.92262
$\sigma(y) = \lvert y\rvert + 0.1$	0.92985	0.92910	0.92980	0.92937	0.92955	0.92958	0.92916
$\sigma(y) = \sqrt{y^2+1}$	0.96127	0.96115	0.96124	0.96114	0.96155	0.96107	0.96093
$\sigma(y) = \sin^2(y)$ $+0.05$	0.92171	0.92143	0.92148	0.92136	0.92086	0.92092	0.92131

Table 4
single discretization, $f(s) = (s-1.5)_+ + \mathbb{1}_{s>2}$, $H = 0.8$

n	125	250	500	1000	2000	4000	8000
$\sigma(y) = \sqrt{\lvert y\rvert + 0.1}$	0.51543	0.51558	0.51415	0.51461	0.51592	0.51498	0.51525
$\sigma(y) = \lvert y\rvert + 0.1$	0.50348	0.50743	0.50659	0.50460	0.50591	0.50450	0.50641
$\sigma(y) = \sqrt{y^2+1}$	0.67837	0.67881	0.67782	0.67843	0.67846	0.67848	0.67825
$\sigma(y) = \sin^2(y)$ $+0.05$	0.31314	0.31239	0.30948	0.31214	0.30948	0.30658	0.30887

Note that the results of simulations related to a single discretization appear to be much more consistent in n. This is maybe due to the fact that, unlike in (11), the value under the sign of expectation in (17) is bounded, and thus the average over 10^4 trials gives a good approximation. The "geometric" nature of S is probably also a factor. Indeed, let us have a look at the 10 largest values of S_T in 10^4 trials with the $\sigma(y) = |y| + 0.1$, $H = 0.8$, and $n = 400$:

89.4301 36.2412 34.7761 34.7639 34.7330 30.7971 27.0400 25.7752 24.2836 23.0231.

Taking into account that F exhibits quadratic growth, we see that the result of a single trial can influence the average over all 10^4 trials. The consistency of single discretization is also confirmed by the results in the Table 5, where we also represent the results of simulation of the direct average $\mathrm{E} f(S_T)$, without reducing to continuous functions, over the same number of trials. Note that we take the same realizations for double discretization, single discretization and the direct average, observing the evident correlation between the results for double discretization and for the direct average.

Table 5
$f(s) = (s-1)_+ + \mathbb{1}_{s>1}$, $\sigma(y) = |y| + 0.2$, $H = 0.75$

n	125	250	500	1000	2000	4000	8000
level one	0.96675	0.90599	0.90634	0.9483	1.0206	0.91574	0.92130
level two	0.93501	0.93429	0.93415	0.93449	0.93471	0.93402	0.93415
direct average	0.9602	0.9047	0.91156	0.94228	1.0023	0.92594	0.91595

We thus conjecture that in practice single discretization gives a good approximation for the expectation $\mathrm{E} f(S_T)$, better than double discretization or the direct averaging. The simulations are provided in MATLAB. To simulate the fBm, we use the algorithm "Fractional Brownian motion generator" by Zdravko Botev which can be found at [5]. For simulations related to double discretization we take the average of the value under the expectation in the right hand side of (11) over 10^4 trials. For simulations related to single discretization, we replace infinite interval of integration in the right hand side of (17) with a finite one, making sure that the integral over the complement is small. Then we discretize the finite interval; the partition size is 2500. For each x from the partition, we take the average over 10^4 trials of the value under the expectation in (17). The trials are common for all x, i.e. it is not necessary to generate 10^4 trials for every x from the partition.

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