# 1 Calculation of Hermite Polynomials $H_t^{112}(x,y)$ and $H_t^{221}(x,y)$

`we need to compute the third-order Hermite polynomials` $H_t^{112}(x,y)$ `and` $H_t^{221}(x,y)$ `of` $\Delta_t^1$
`please refer to the page` $65 \sim 66$ `of the PAPER`

## 1.1 Setup and Definitions

Given the diffusion matrix $\sigma(x)$:

$$\sigma(x) = \begin{pmatrix} s(x_2) & 0 \\ 0 & s(x_1) \end{pmatrix} = \operatorname{diag}\left(s(x_2), s(x_1)\right)$$

The covariance matrix $b(x)$ is defined as $b(x) = \sigma(x)^2$:

$$b(x) = \begin{pmatrix} s(x_2)^2 & 0 \\ 0 & s(x_1)^2 \end{pmatrix} = \operatorname{diag}\left(s(x_2)^2, s(x_1)^2\right)$$

Its inverse $b^{-1}(x)$ is:

$$b^{-1}(x) = \begin{pmatrix} \frac{1}{s(x_2)^2} & 0 \\ 0 & \frac{1}{s(x_1)^2} \end{pmatrix} = \operatorname{diag}\left(\frac{1}{s(x_2)^2}, \frac{1}{s(x_1)^2}\right)$$

We denote the components of the inverse matrix as $(b^{-1})_{ij}$. Thus:

$$(b^{-1})_{11} = \frac{1}{s(x_2)^2}, \quad (b^{-1})_{22} = \frac{1}{s(x_1)^2}, \quad (b^{-1})_{12} = (b^{-1})_{21} = 0$$

## 1.2 Centralized Variables

Let $r$ be the centered spatial increment (accounting for drift $a(x)$):

$$r = y - x - a(x)t \implies \begin{cases} r_1 = y_1 - x_1 - a_1(x)t \\ r_2 = y_2 - x_2 - a_2(x)t \end{cases}$$

We define the standardized variable $u = b^{-1}(x)r$. In component form:

$$u_i = \sum_{j=1}^{d} (b^{-1})_{ij} r_j$$

Substituting our specific diagonal $b^{-1}$:

$$u_1 = (b^{-1})_{11} r_1 = \frac{r_1}{s(x_2)^2}, \quad u_2 = (b^{-1})_{22} r_2 = \frac{r_2}{s(x_1)^2}$$

## 1.3 General Formula for Third-Order Hermite Polynomials

According to page 53 of the PAPER, the general formula for the third-order multi-index Hermite polynomial $H_t^{ijk}(x,y)$ is:

$$H_t^{ijk}(x,y) = \frac{1}{t^3} u_i u_j u_k - \frac{1}{t^2}\left((b^{-1})_{ij} u_k + (b^{-1})_{ik} u_j + (b^{-1})_{jk} u_i\right)$$

## 1.4 Calculation of $H_t^{112}(x, y)$

We set indices $(i, j, k) = (1, 1, 2)$.

**Term$_1$ of $H_t^{112}$ :**

$$\text{Term}_1 = \frac{1}{t^3} u_1 u_1 u_2 = \frac{1}{t^3} (u_1)^2 u_2$$

Substituting $u_1$ and $u_2$:

$$\text{Term}_1 = \frac{1}{t^3} \left( \frac{r_1}{s(x_2)^2} \right)^2 \left( \frac{r_2}{s(x_1)^2} \right) = \frac{r_1^2 r_2}{t^3 s(x_2)^4 s(x_1)^2}$$

**Term$_2$ of $H_t^{112}$ :**

We need to evaluate the sum $S = (b^{-1})_{11} u_2 + (b^{-1})_{12} u_1 + (b^{-1})_{12} u_1$. Using $(b^{-1})_{12} = (b^{-1})_{21} = 0$:

$$S = (b^{-1})_{11} u_2 + 0 + 0 = \frac{1}{s(x_2)^2} \cdot \frac{r_2}{s(x_1)^2} = \frac{r_2}{s(x_2)^2 s(x_1)^2}$$

Thus, the second part of the formula is:

$$\text{Term}_2 = -\frac{1}{t^2} S = -\frac{r_2}{t^2 s(x_2)^2 s(x_1)^2}$$

**Final Expression for $H_t^{112}$** Combining Term$_1$ and Term$_2$:

$$\boxed{H_t^{112}(x, y) = \frac{\left(y_1 - x_1 - a_1(x)t\right)^2 \left(y_2 - x_2 - a_2(x)t\right)}{t^3 s(x_2)^4 s(x_1)^2} - \frac{y_2 - x_2 - a_2(x)t}{t^2 s(x_2)^2 s(x_1)^2}}$$

## 1.5 Calculation of $H_t^{221}(x, y)$

We set indices $(i, j, k) = (2, 2, 1)$.

**Term$_1$ of $H_t^{221}$ :**

$$\text{Term}_1 = \frac{1}{t^3} u_2 u_2 u_1 = \frac{1}{t^3} (u_2)^2 u_1$$

Substituting $u_1$ and $u_2$:

$$\text{Term}_1 = \frac{1}{t^3} \left( \frac{r_2}{s(x_1)^2} \right)^2 \left( \frac{r_1}{s(x_2)^2} \right) = \frac{r_2^2 r_1}{t^3 s(x_1)^4 s(x_2)^2}$$

**Term$_2$ of $H_t^{221}$ :**

We evaluate the sum $S = (b^{-1})_{22} u_1 + (b^{-1})_{21} u_2 + (b^{-1})_{21} u_2$. Using $(b^{-1})_{21} = (b^{-1})_{12} = 0$:

$$S = (b^{-1})_{22} u_1 + 0 + 0 = \frac{1}{s(x_1)^2} \cdot \frac{r_1}{s(x_2)^2} = \frac{r_1}{s(x_1)^2 s(x_2)^2}$$

Thus, the second part of the formula is:

$$\text{Term}_2 = -\frac{1}{t^2} S = -\frac{r_1}{t^2 s(x_1)^2 s(x_2)^2}$$

## 1.6 Final Expression for $H_t^{221}$

Combining Term$_1$ and Term$_2$:

$$\boxed{H_t^{221}(x, y) = \frac{\left(y_2 - x_2 - a_2(x)t\right)^2 \left(y_1 - x_1 - a_1(x)t\right)}{t^3 s(x_1)^4 s(x_2)^2} - \frac{y_1 - x_1 - a_1(x)t}{t^2 s(x_1)^2 s(x_2)^2}}$$

# 2 Calculation of Hermite Polynomials of Orders 2 and 4

we need to compute the third-order Hermite polynomials $H_t^{11}(x,y), H_t^{12}(x,y), H_t^{22}(x,y), H_t^{1111}(x,y), H_t^{1112}(x,y), H_t^{1}$
of $\Delta_t^2$
please refer to the page $65 \sim 66$ of the PAPER

## 2.1 Two-Order Hermite Polynomials

### 2.1.1 Calculation of $H_t^{11}$

Setting $(i,j) = (1,1)$:

$$H_t^{11}(x,y) = \frac{1}{t^2} u_1^2 - \frac{1}{t}(b^{-1})_{11}.$$

Substituting $u_1 = \frac{r_1}{s(x_2)^2}$ and $(b^{-1})_{11} = \frac{1}{s(x_2)^2}$:

$$\boxed{H_t^{11}(x,y) = \frac{r_1^2}{t^2 s(x_2)^4} - \frac{1}{ts(x_2)^2} = \frac{\big(y_1 - x_1 - a_1(x)t\big)^2}{t^2\, s(x_2)^4} - \frac{1}{t\, s(x_2)^2}}$$

### 2.1.2 Calculation of $H_t^{22}$

Setting $(i,j) = (2,2)$:

$$H_t^{22}(x,y) = \frac{1}{t^2} u_2^2 - \frac{1}{t}(b^{-1})_{22}.$$

Substituting $u_2 = \frac{r_2}{s(x_1)^2}$ and $(b^{-1})_{22} = \frac{1}{s(x_1)^2}$:

$$\boxed{H_t^{22}(x,y) = \frac{r_2^2}{t^2 s(x_1)^4} - \frac{1}{ts(x_1)^2} = \frac{\big(y_2 - x_2 - a_2(x)t\big)^2}{t^2\, s(x_1)^4} - \frac{1}{t\, s(x_1)^2}}$$

### 2.1.3 Calculation of $H_t^{12}$

Setting $(i,j) = (1,2)$:

$$H_t^{12}(x,y) = \frac{1}{t^2} u_1 u_2 - \frac{1}{t}(b^{-1})_{12}.$$

Since $(b^{-1})_{12} = 0$:

$$\boxed{H_t^{12}(x,y) = \frac{r_1 r_2}{t^2 s(x_2)^2 s(x_1)^2} \frac{\big(y_1 - x_1 - a_1(x)t\big)\big(y_2 - x_2 - a_2(x)t\big)}{t^2\, s(x_2)^2\, s(x_1)^2}}$$

## 2.2 Fourth-Order Hermite Polynomials

We use the recurrence relation (The page 53, Eq. A.5 or A.7 of the PAPER) to derive higher orders. For any multi-index $\alpha = (i_1, \ldots, i_n)$, adding an index $j$ yields:

$$H_t^{j,\alpha}(x,y) = \frac{1}{t} u_j H_t^{\alpha}(x,y) - \sum_{k=1}^{n} (b^{-1})_{j,i_k} H_t^{\alpha \setminus \{i_k\}}(x,y) \quad (k \in (1, \ldots, n))$$

Since $b$ is diagonal, the sum only survives when $i_k = j$.

### 2.2.1 Calculation of $H_t^{1111}$

This is effectively the 1D Hermite polynomial of degree 4 in the variable $r_1$. From the recurrence on $H_t^{111}$ (derived similarly):

$$H_t^{111} = \frac{u_1^3}{t^3} - \frac{3(b^{-1})_{11}u_1}{t^2}.$$

Applying the recurrence for $H_t^{1111}$ (adding index 1 to $H_t^{111}$):

$$H_t^{1111} = \frac{u_1}{t}H_t^{111} - \frac{3(b^{-1})_{11}}{t}H_t^{11}.$$

Substituting the expressions:

$$\begin{aligned}
H_t^{1111} &= \frac{u_1}{t}\left(\frac{u_1^3}{t^3} - \frac{3(b^{-1})_{11}u_1}{t^2}\right) - \frac{3(b^{-1})_{11}}{t}\left(\frac{u_1^2}{t^2} - \frac{(b^{-1})_{11}}{t}\right) \\
&= \frac{u_1^4}{t^4} - \frac{3(b^{-1})_{11}u_1^2}{t^3} - \frac{3(b^{-1})_{11}u_1^2}{t^3} + \frac{3(b^{-1})_{11}^2}{t^2} \\
&= \frac{u_1^4}{t^4} - \frac{6(b^{-1})_{11}u_1^2}{t^3} + \frac{3(b^{-1})_{11}^2}{t^2}.
\end{aligned}$$

Substituting $u_1 = \dfrac{r_1}{s(x_2)^2}$ and $(b^{-1})_{11} = \dfrac{1}{s(x_2)^2}$:

$$\begin{aligned}
H_t^{1111}(x,y) &= \frac{r_1^4}{t^4 s(x_2)^8} - \frac{6r_1^2}{t^3 s(x_2)^6} + \frac{3}{t^2 s(x_2)^4} \\
&= \boxed{\frac{\big(y_1 - x_1 - a_1(x)t\big)^4}{t^4\, s(x_2)^8} - \frac{6\big(y_1 - x_1 - a_1(x)t\big)^2}{t^3\, s(x_2)^6} + \frac{3}{t^2\, s(x_2)^4}}
\end{aligned}$$

### 2.2.2 Calculation of $H_t^{2222}$

By symmetry with $H_t^{1111}$, replacing index 1 with 2:

$$\begin{aligned}
H_t^{2222}(x,y) &= \frac{r_2^4}{t^4 s(x_1)^8} - \frac{6r_2^2}{t^3 s(x_1)^6} + \frac{3}{t^2 s(x_1)^4} \\
&= \boxed{\frac{\big(y_2 - x_2 - a_2(x)t\big)^4}{t^4\, s(x_1)^8} - \frac{6\big(y_2 - x_2 - a_2(x)t\big)^2}{t^3\, s(x_1)^6} + \frac{3}{t^2\, s(x_1)^4}}
\end{aligned}$$

### 2.2.3 Calculation of $H_t^{1112}$

We add index 2 to $H_t^{111}$. Since $(b^{-1})_{21} = 0$, the correction terms in the recurrence vanish:

$$H_t^{1112} = \frac{u_2}{t}H_t^{111} - \sum_{k=1}^{3}(b^{-1})_{21}H_t^{11} = \frac{u_2}{t}H_t^{111}.$$

Using $H_t^{111} = \frac{u_1^3}{t^3} - \frac{3(b^{-1})_{11}u_1}{t^2}$:

$$H_t^{1112} = \frac{u_2}{t}\left(\frac{u_1^3}{t^3} - \frac{3(b^{-1})_{11}u_1}{t^2}\right) = \frac{u_1^3u_2}{t^4} - \frac{3(b^{-1})_{11}u_1u_2}{t^3}.$$

Substituting variables:

$$\begin{aligned}
H_t^{1112}(x,y) &= \frac{r_1^3r_2}{t^4 s(x_2)^6 s(x_1)^2} - \frac{3r_1r_2}{t^3 s(x_2)^4 s(x_1)^2} \\
&= \boxed{{}_t^{1112}(x,y) = \frac{\big(y_1 - x_1 - a_1(x)t\big)^3\big(y_2 - x_2 - a_2(x)t\big)}{t^4\, s(x_2)^6\, s(x_1)^2} - \frac{3\big(y_1 - x_1 - a_1(x)t\big)\big(y_2 - x_2 - a_2(x)t\big)}{t^3\, s(x_2)^4\, s(x_1)^2}}
\end{aligned}$$

### 2.2.4 Calculation of $H_t^{1222}$

By symmetry with $H_t^{1112}$, swapping indices 1 and 2:

$$
\begin{aligned}
H_t^{1222}(x,y) &= \frac{r_2^3 r_1}{t^4 s(x_1)^6 s(x_2)^2} - \frac{3 r_2 r_1}{t^3 s(x_1)^4 s(x_2)^2} \\
&= \boxed{\frac{\big(y_1 - x_1 - a_1(x)t\big)\big(y_2 - x_2 - a_2(x)t\big)^3}{t^4\, s(x_2)^2\, s(x_1)^6} - \frac{3\big(y_1 - x_1 - a_1(x)t\big)\big(y_2 - x_2 - a_2(x)t\big)}{t^3\, s(x_2)^2\, s(x_1)^4}}
\end{aligned}
$$

### 2.2.5 Calculation of $H_t^{1122}$

We add index 2 to $H_t^{112}$. Recall (from previous derivation) that:

$$
H_t^{112} = \frac{u_1^2 u_2}{t^3} - \frac{(b^{-1})_{11} u_2}{t^2}.
$$

Using the recurrence to add index 2:

$$
\begin{aligned}
H_t^{1122} &= \frac{u_2}{t} H_t^{112} - \big((b^{-1})_{21} H_t^{12} + (b^{-1})_{21} H_t^{12} + (b^{-1})_{22} H_t^{11}\big) \\
&= \frac{u_2}{t} H_t^{112} - \frac{(b^{-1})_{22}}{t} H_t^{11}.
\end{aligned}
$$

Where,$(b^{-1})_{21} = 0$
The sum over indices in $H^{112}$ (indices 1,1,2) involves $(b^{-1})_{21}, (b^{-1})_{21}, (b^{-1})_{22}$. Only the last term survives.
Substituting expressions:

$$
\begin{aligned}
H_t^{1122} &= \frac{u_2}{t} \left( \frac{u_1^2 u_2}{t^3} - \frac{(b^{-1})_{11} u_2}{t^2} \right) - \frac{(b^{-1})_{22}}{t} \left( \frac{u_1^2}{t^2} - \frac{(b^{-1})_{11}}{t} \right) \\
&= \frac{u_1^2 u_2^2}{t^4} - \frac{(b^{-1})_{11} u_2^2}{t^3} - \frac{(b^{-1})_{22} u_1^2}{t^3} + \frac{(b^{-1})_{11}(b^{-1})_{22}}{t^2}.
\end{aligned}
$$

Substituting variables:

$$
\begin{aligned}
H_t^{1122}(x,y) &= \frac{r_1^2 r_2^2}{t^4 s(x_2)^4 s(x_1)^4} - \frac{r_2^2}{t^3 s(x_2)^2 s(x_1)^4} - \frac{r_1^2}{t^3 s(x_1)^2 s(x_2)^4} + \frac{1}{t^2 s(x_2)^2 s(x_1)^2} \\
&= \boxed{\frac{\big(y_1 - x_1 - a_1(x)t\big)^2 \big(y_2 - x_2 - a_2(x)t\big)^2}{t^4\, s(x_2)^4\, s(x_1)^4} - \frac{\big(y_1 - x_1 - a_1(x)t\big)^2}{t^3\, s(x_2)^4\, s(x_1)^2} - \frac{\big(y_2 - x_2 - a_2(x)t\big)^2}{t^3\, s(x_2)^2\, s(x_1)^4} + \frac{1}{t^2\, s(x_2)^2\, s(x_1)^2}}
\end{aligned}
$$